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A stress-based macroscopic approach for microcracks unilateral effect

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A B S T R A C T

The question of the nonlinear response of brittle materials undergoing elastic damage is investigated here. Owing to the specific nature of microcracking, the macroscopic behaviour of these materials is complex, generally anisotropic owing to the possible preferential orientation of defects and multilinear because of the unilateral effect due to the transition between open and closed state of microcracks. A new three-dimensional macroscopic model outlined by Welemane and Cormery [1] has been proposed to account simultaneously for these both aspects. This paper intends to present in details the principles of such approach and to demonstrate its applicability to a stress-based framework. Based on a fabric tensor representation of the damage density distribution, the model provides a continuum and rigorous description of the contribution of defaults which avoids classical spectral decompositions and related inconsistencies. The model is also strongly micromechanically motivated, especially to handle the elastic moduli recovery that occurs at the closure of microcracks. The macroscopic theoretical framework proposed constitutes a general approach that leads in particular to predictions of a class of micromechanical models. The capacities of the approach are illustrated and discussed on various cases of damage configurations and opening–closure states, with a special attention to the differences with the strain-based framework and to the influence of the damage variables order.

Keywords:
Damage
Microcracks
Anisotropy
Unilateral effect

1. Introduction

The mechanical behaviour of brittle materials (rocks, concrete, ceramics, etc.) is mainly governed by microscopic damage mechanisms, namely the existence, growth of microcracks and possible friction on their lips. Among specific features of such degradation process, two aspects induce serious difficulties for the formulation of the constitutive model of these materials. First, they usually exhibit a damage induced anisotropy in their macroscopic response due to the oriented character of the microcracks growth under non-hydrostatic loads [2,3]. Besides, microcracks can be opened or closed according to the applied loading and then affect the mechanical properties of the materials differently: the elastic moduli recovery at the closure of microcracks is typical of this so-called damage unilateral effect [4–6].

If the question of induced anisotropy has been widely treated within the framework of Continuum Damage Mechanics (CDM), its association with closure effects still represents a serious and open challenge. Even at constant damage state and under frictionless conditions, various mathematical or thermodynamical inconsistencies have been pointed out in existing formulations: discontinuities of the stress–strain response, non-uniqueness of the thermodynamic potential or non-symmetry of the elastic operator or Hessian tensor [7–9]. As shown by [10], such problems can largely be attributed to the use in the recovery procedures of spectral decomposition, either of the strain/stress tensor or damage tensors. In contrast to formulations still based on this concept that try to correct *a posteriori* encountered deficiencies (for example [11,12]), we propose to develop at the very beginning of the formulation some new and proper tools to account for unilateral conditions in anisotropic damage modelling.

Built within the consistent thermodynamics framework with internal variables, the present approach suggests the large introduction of physical motivation within constitutive equations to allow a better insight into the considered phenomenon:

- First, through the selection of damage parameter(s); if this question is still largely a matter of convenience, it remains however a crucial point for the description of closure effects [13]; here the key idea is to refer to a fine description of the material microstructure through the classical concept of orientation distribution functions (see [14] for a review).
- Then, when experimental data are lacking, by justifying the macroscopic choices (especially regarding unilateral effect) from simple micromechanical analyses rather than from purely phenomenological assumptions, as inspired by works of [15–17].

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In this sense, we pursue the major challenge of micromechanically-based modelling expected by many authors [18–20], that is to formulate macroscopic constitutive laws for materials while accounting for underlying physical specificities of their microstructure revealed by micro–macro considerations.

The present concern is the elastic behaviour of microcracked materials, dissipative mechanisms such as microcracks growth or frictional sliding on the closed microcracks lips are not considered. This paper describes in complete and detailed form the fundamentals and building steps of the three-dimensional theory outlined by [1]. Besides, we develop here a stress-based formulation as we intend to show the general character of the modelling method and underline at the same time the related differences with strain-based framework. Accordingly, the model aims at representing the anisotropic (due to defects directionality) and multilinear (owing to unilateral effect) response of microcracked materials at fixed damage state. For simplicity, we restrict the discussion to materials that are initially isotropic and subjected to small perturbations; rate independent and isothermal conditions are considered.

After the presentation of notations used in this paper (Section 2), we first provide a physical interpretation of damage variables describing the orientation and extent of decohesion microsurfaces (Section 3). Then, the thermodynamic potential that defines the elastic energy of damaged medium is derived from physically-based hypotheses, including microcracks opening and closure effects (Section 4). In Section 5 are finally investigated and discussed the capacities of the proposed formulation for various cases of damage distribution and opening–closure states of microcracks.

2. Notations

All tensors used in this study are defined on the euclidean vectorial space $\mathbb{R}^3$ for which the set of vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ constitutes an orthonormal basis. Especially, T^k represents the vectorial space of tensors of order k $(k \in \mathbb{N})$.

Usual intrinsic notations are employed throughout. In order to make clear expressions in the paper, we just recall the definition of the main operators. The inner products are labelled as follows: $a \cdot b,\ \forall (a,b) \in (T^1 = \mathbb{R}^3)^2$; $a : b,\ \forall (a,b) \in (T^2)^2$; $a :: b,\ \forall (a,b) \in (T^4)^2$; $a \bullet b,\ \forall (a,b) \in (T^k)^2$ for any integer $k \geqslant 6$. Additionally, the tensor products of two second-order tensors a and b are defined by:

$$\begin{cases} [a \otimes b] : x = (b : x)a; & [a \underline{\otimes} b] : x = a \cdot x \cdot b^T \\ [a \overline{\otimes} b] : x = a \cdot x^T \cdot b^T; & a \underline{\overline{\otimes}} b = \frac{1}{2}(a \underline{\otimes} b + a \overline{\otimes} b) \end{cases} \tag{1}$$

for any tensor $x \in T^2$. In particular, the term $a^{\otimes i} = a \otimes a \ldots \otimes a$ represents the ith tensor product of a tensor a $(i \in \mathbb{N})$.

We finally denote by $\lfloor b \rfloor$ the irreducible part of any tensor b. A tensor T of T^k is said to be irreducible if it is completely symmetric and traceless, *i.e.*:

$$\begin{cases} T_{i_1 i_2 i_3 \ldots i_k} = T_{i_2 i_1 i_3 \ldots i_k} = T_{i_3 i_2 i_1 \ldots i_k} = \cdots = T_{i_k i_2 i_3 \ldots i_1} \\ \sum_{i_1=1}^3 T_{i_1 i_1 i_3 \ldots i_k} = (0_{k-2})_{i_3 \ldots i_k} \end{cases} \tag{2}$$

where $(T_{i_1 i_2 \ldots i_k})_{i_1, i_2, \ldots, i_k \in [1,3]^k}$ are the components of T in any basis of $\mathbb{R}^3$ and 0_k is the zero tensor of T^k. $\mathbf{I}$ represents the second-order identity tensor, $\mathscr{S} = \{\mathbf{n} \in \mathbb{R}^3,\ \mathbf{n} \cdot \mathbf{n} = 1\}$ denotes the unit sphere of $\mathbb{R}^3$ and ds the infinitesimal surface element on $\mathscr{S}$.

3. Damage description

Let consider a representative volume element (RVE) V of a material weakened by flat microcracks with various shapes (penny-shaped, elliptic, etc.). The degradation state of this material is then entirely defined by the spatial arrangement, orientation and geometry of such defects. Owing to the complexity involved, a complete description of these aspects is clearly not compatible with an efficient constitutive modelling. Therefore, we restrict ourselves to the morphological information that we consider as the most representative feature of the damage state of the material, namely the microcrack density distribution.

The damage density related to direction of unit vector $\mathbf{n}$ refers to a scalar property without dimension that can be defined as follows [21]:

$$\rho_n = \frac{2}{\pi V} \sum\nolimits_l \frac{S_{nl}^2}{P_{nl}} \tag{3}$$

with S_{nl} and P_{nl} respectively the surface and perimeter of the lth microcrack with unit normal $\mathbf{n}$. The extension to all space directions allows to introduce a scalar orientation function ρ such that [22]:

$$\begin{aligned} \rho :\quad & \mathscr{S} \to \mathbb{R} \\ & \mathbf{n} \mapsto \rho(\mathbf{n}) = \rho_n \end{aligned} \tag{4}$$

that describes the directional dependence of the microcrack distribution within the material [23–26].

Owing to the property $\rho(\mathbf{n}) = \rho(-\mathbf{n})$ and assuming square-integrability of ρ, it can be expanded in a convergent Fourier series [27]:

$$\rho(\mathbf{n}) = d_0 + d_2 : \lfloor \mathbf{n}^{\otimes 2} \rfloor + d_4 :: \lfloor \mathbf{n}^{\otimes 4} \rfloor + \cdots, \quad \forall \mathbf{n} \in \mathscr{S} \tag{5}$$

where irreductible tensors $(1, \lfloor \mathbf{n}^{\otimes 2} \rfloor, \lfloor \mathbf{n}^{\otimes 4} \rfloor, \cdots)$ form a complete orthogonal basis for the square-integrable functions on $\mathscr{S}$ (detailed expressions of these tensors are given in Appendix A). Hence, the corresponding irreductible even ranked fabric tensors $(d_0, d_2, d_4, \cdots)$ defined by:

$$d_k = \frac{(1+2k)!}{2^{k+2}(k!)^2 \pi} \int_{\mathscr{S}} \rho(\mathbf{n}) \lfloor \mathbf{n}^{\otimes k} \rfloor \, ds, \quad \forall k = 0, 2, 4, \cdots \tag{6}$$

fully characterize the microcrack density distribution ρ. In particular, the zero order coefficient

$$d_0 = \frac{1}{4\pi} \int_{\mathscr{S}} \rho(\mathbf{n}) \, ds \tag{7}$$

refers to the density of all microcracks within the representative volume.

For simplicity, we assume that the microcrack distribution can be approximated by the orientation function $\hat{\rho}$ obtained by a finite truncation within the expansion (5):

$$\begin{aligned} \hat{\rho}(\mathbf{n}, \mathbf{d}) = d_0 + d_2 : \lfloor \mathbf{n}^{\otimes 2} \rfloor + d_4 :: \lfloor \mathbf{n}^{\otimes 4} \rfloor + \cdots + d_p \bullet \lfloor \mathbf{n}^{\otimes p} \rfloor, \\ \forall \mathbf{n} \in \mathscr{S} \end{aligned} \tag{8}$$

with p an even integer. The set of irreductible tensors $\mathbf{d} = (d_0, d_2, d_4, \ldots, d_p)$ allows then to characterize the damage density in any direction of the space and constitutes appropriate internal damage variables. Actually, such mathematical representation fulfils many requirements for the selection of damage parameter [13]: variables $\mathbf{d}$ are first directly related to the microstructural changes involved, then allow to capture the directional aspects of damage in invariant form and finally can be identified accurately using stereological methods [22,23] or other appropriate techniques (for instance X-ray tomography). Although defined from crack density parameters which refer to microscale, these damage variables constitute macroscopic variables through the summation of these quantities over all defects orientation (Eq. (6)).

An interesting issue is to specify the approximation order p that provides the more relevant description of damage. Obviously, greater values of p clearly improve the geometrical estimations of the microcrack distribution and taking into account a preferential orientation of the defects requires at least that $p \geqslant 2$ (see analyses of [23,26]). As an example, let consider the following density function:

$$\rho(\mathbf{n}) = \rho(\phi) = \begin{cases} 0, & \text{if } \phi \in \left[\frac{\pi}{8}, \frac{7\pi}{8}\right] \\ 4\cos(4\phi), & \text{else} \end{cases} \tag{9}$$

where unit vector $\mathbf{n}$ is represented by spherical angles θ and ϕ in the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Fig. 1 presents the graphical representations of exact distribution ρ and associated approximations $\hat{\rho}$ respectively obtained for orders $p = 0, 2, 4, 6$. Nevertheless, a better accuracy may lead at the same time to important computational costs. As such a choice depends on users applications, precision required and computational capacities, we will preserve in what follows the generality of the formulation and develop constitutive laws for any even order p.

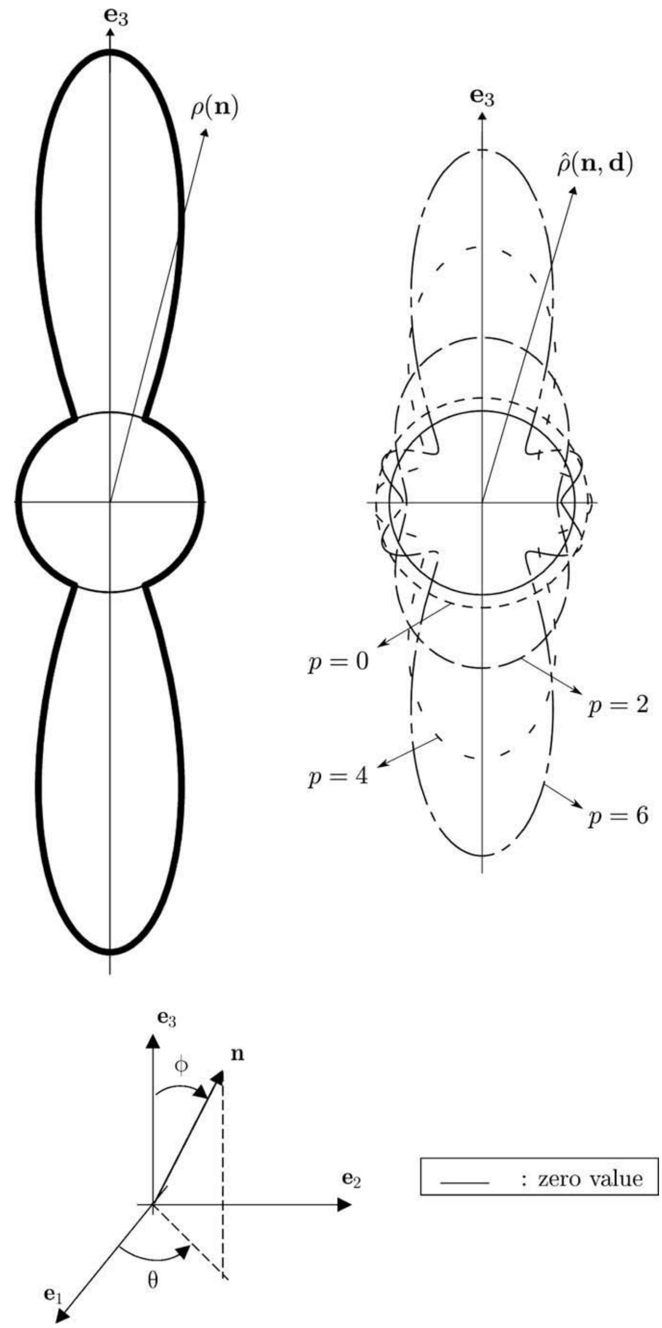

Fig. 1. Influence of approximation order p.

4. Thermodynamic potential

In the present work, we adopt a stress-based formulation (potential written in term of the stress tensor $\boldsymbol{\sigma}$). Such choice enables indeed direct connections with laboratory tests and with some elementary micromechanical solutions.

4.1. General hypotheses

We assume the existence of a thermodynamic potential, namely the free enthalpy W per unit volume, depending on the state variables $(\boldsymbol{\sigma}, \mathbf{d})$ and verifying the following properties:

- in order to satisfy the fundamental principle of space isotropy, the application W is an isotropic invariant with respect to its arguments:

  $$W(\mathscr{T}(\boldsymbol{\sigma}, \mathbf{d})) = W(\boldsymbol{\sigma}, \mathbf{d}) \tag{10}$$

  under any orthogonal transformation $\mathscr{T}$ (see for instance [28]);
- function W is of class C^1, which ensures the continuity of W, the existence and continuity of the strain $\boldsymbol{\varepsilon}$:

  $$\boldsymbol{\varepsilon} = \frac{\partial W}{\partial \boldsymbol{\sigma}} \tag{11}$$

  and the existence and continuity of the conjugate thermodynamic forces associated to damage:

  $$\mathbf{F}^d = (F^{d_0}, F^{d_2}, F^{d_4}, \ldots, F^{d_p}) = \left(-\frac{\partial W}{\partial d_0}, -\frac{\partial W}{\partial d_2}, -\frac{\partial W}{\partial d_4}, \ldots, -\frac{\partial W}{\partial d_p}\right) \tag{12}$$

  The latter are not developed since the question of damage evolution is not treated in what follows;
- W is finally positively homogenous of degree two with respect to $\boldsymbol{\sigma}$, that is:

  $$\forall \mathbf{d}, \quad W(\lambda\boldsymbol{\sigma}, \mathbf{d}) = \lambda^2\, W(\boldsymbol{\sigma}, \mathbf{d}), \quad \forall \lambda \geqslant 0 \tag{13}$$

  This condition implies that the strain defined in Eq. (11) is positively homogenous of degree one with respect to $\boldsymbol{\sigma}$:

  $$\forall \mathbf{d}, \quad \boldsymbol{\varepsilon}(\lambda\boldsymbol{\sigma}, \mathbf{d}) = \lambda \boldsymbol{\varepsilon}(\boldsymbol{\sigma}, \mathbf{d}), \quad \forall \lambda \geqslant 0 \tag{14}$$

  Consequently, the stress–strain response of the material will be linear during a reversible process, with possible asymmetrical behaviours between tensile and compressive loading [29].

In agreement with these assumptions, we propose the following expression for the thermodynamic potential:

$$W(\boldsymbol{\sigma}, \mathbf{d}) = W_0(\boldsymbol{\sigma}) + W_d(\boldsymbol{\sigma}, \mathbf{d}) \tag{15}$$

where $W_0(\boldsymbol{\sigma})$ denotes the free enthalpy of the undamaged material, assumed to be isotropic and linear elastic (Young modulus E_0 and Poisson ratio ν_0):

$$W_0(\boldsymbol{\sigma}) = \frac{1 + \nu_0}{2E_0}\ tr(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}) - \frac{\nu_0}{2E_0}\ tr^2\boldsymbol{\sigma} \tag{16}$$

For a current state $(\boldsymbol{\sigma}, \mathbf{d})$, the latter term of Eq. (15) represents the modification induced by damage and is defined as the sum of elementary contributions on the unit sphere:

$$W_d(\boldsymbol{\sigma}, \mathbf{d}) = \frac{1}{4\pi} \int_{\mathscr{S}} w(\hat{\rho}(\mathbf{n}, \mathbf{d}), \mathbf{n}, \boldsymbol{\sigma})\ ds \tag{17}$$

where function w characterizes the enthalpy modification induced by each set of parallel microcracks. According to Eq. (17), the definition of the thermodynamic potential W reduces therefore to the determination of the elementary enthalpy function w. The objective of the following section is then to formulate an elementary contribution able to capture the salient features of the elastic response of microcracked materials.

4.2. Elementary enthalpy and unilateral effect

The elementary enthalpy function w is taken linear in the density $\hat{\rho}(\mathbf{n},\mathbf{d})$:

$$w(\hat{\rho}(\mathbf{n},\mathbf{d}),\mathbf{n},\boldsymbol{\sigma}) = \hat{\rho}(\mathbf{n},\mathbf{d})\ h(\mathbf{n},\boldsymbol{\sigma}) \tag{18}$$

Within the present stress-based framework, the linearity assumption is representative of a large class of microcracked material behaviours insofar as it allows even for example to account for some defects interactions [30–32]. Moreover, it leads here to great simplifications in further calculations (continuity and recovery conditions, identification procedure, etc.). If required continuity conditions on w are still satisfied (see below), note that it is possible to relax this assumption, by introducing for instance higher order terms in the density $\hat{\rho}(\mathbf{n},\mathbf{d})$ (as [33,34] for instance), and pursue the next modelling steps. Yet, it has to be said that the expression (17), which separates the elementary contribution of each set of parallel microcracks, does not allow to deal with strongly interacting cracks (such as in [35]).

In Eq. (18), the function h specifies the nature of the contribution to the energy of microcracks in a given direction of the space. As microcracks affect the mechanical response of brittle materials differently when they are open or closed, such contribution should then differ according to this status. At this stage, the formulation in stress allows to rely on direct micromechanical results such as those developed by Kachanov [31,36] in the context of uniform stress boundary conditions. For an undamaged isotropic material weakened by non-interacting microcracks, it is classically shown that the opening–closure of microcracks is controlled by their normal macroscopic stress. Accordingly, we introduce the following characteristic function g:

$$g(\mathbf{n},\boldsymbol{\sigma}) = \mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n} \tag{19}$$

that indicate the transition from the opening status (when $g(\mathbf{n},\boldsymbol{\sigma}) > 0$) to the crack closure (when $g(\mathbf{n},\boldsymbol{\sigma}) \leqslant 0$) for microdefects with normal $\mathbf{n}$. As indicated before, the opportunity to refer to such elementary micromechanical solution clearly simplifies this building step of the model since application of g can be defined from the outset. For the strain-based formulation indeed, only a general expression of the opening–closure function is deduced from continuity conditions [1]. The use of equivalent micromechanical results in strain to define precisely its expression would require in this case additional hypotheses for the derivation of the Legendre transformation (for example the small microcrack density assumption [37]). From the scalar application g defined in Eq. (19), one can thus account for the damage unilateral effect through expression of h:

$$h(\mathbf{n},\boldsymbol{\sigma}) = \begin{cases} h_1(\mathbf{n},\boldsymbol{\sigma}), & \text{if } g(\mathbf{n},\boldsymbol{\sigma}) > 0 \\ h_2(\mathbf{n},\boldsymbol{\sigma}), & \text{if } g(\mathbf{n},\boldsymbol{\sigma}) \leqslant 0 \end{cases} \tag{20}$$

Functions $(h_i)_{i=1,2}$ allow then to characterize the elementary enthalpy of microcracks whether they are respectively open (h_1) or closed (h_2).

From the mathematical point of view first, function w must be isotropic with respect to its arguments, of class C^1, radially symmetric with respect to variable $\mathbf{n}$ (that is $w(\cdot,\mathbf{n},\cdot) = w(\cdot,-\mathbf{n},\cdot)$, $\forall\mathbf{n}\in\mathscr{S}$) and positively homogenous of degree two with respect to $\boldsymbol{\sigma}$ in order to fulfil previous conditions imposed to W. According to (18) and properties of $\hat{\rho}$ and g, then scalar applications h_1 and h_2 should be isotropic with respect to their arguments $(\mathbf{n},\boldsymbol{\sigma})$ and, at the same time, radially symmetric with respect to unit vector $\mathbf{n}$. The representation theory of tensorial functions [28,38] imposes consequently h_1 and h_2 to be isotropic invariants of $(\mathbf{n}^{\otimes 2},\boldsymbol{\sigma})$ that can be expressed by a combination of basic invariants. As functions $(h_i)_{i=1,2}$ must likewise be positively homogenous of degree two with respect to $\boldsymbol{\sigma}$, one obtains the general form whatever $i = 1,2$:

$$\begin{aligned} h_i(\mathbf{n},\boldsymbol{\sigma}) = {} & \alpha_i\ tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) + \beta_i\ tr^2\boldsymbol{\sigma} + \chi_i tr\boldsymbol{\sigma} tr(\boldsymbol{\sigma}\cdot\mathbf{n}^{\otimes 2}) \\ & + \varphi_i\ tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}^{\otimes 2}) + \psi_i tr^2(\boldsymbol{\sigma}\cdot\mathbf{n}^{\otimes 2}) \end{aligned} \tag{21}$$

where the scalar coefficients $(\alpha_i,\beta_i,\chi_i,\varphi_i,\psi_i)_{i=1,2}$ are constant.

In view of Eqs. (18), (20) and (21), the elementary enthalpy of microcracks with normal $\mathbf{n}$ depends on two sets of five parameters: $(\alpha_1,\beta_1,\chi_1,\varphi_1,\psi_1)$ and $(\alpha_2,\beta_2,\chi_2,\varphi_2,\psi_2)$ related respectively to the open and closed states. However, these coefficients cannot be completely independent owing to continuity conditions on w. Indeed, since $\hat{\rho}$ is of class C^1, function h defined by (20) must be also of class C^1. A generalization of the works of [39] on the continuity of multilinear functions shows that this property is satisfied if and only if [40]:

$$\forall(\mathbf{n},\boldsymbol{\sigma}),\quad g(\mathbf{n},\boldsymbol{\sigma}) = 0,\quad \frac{\partial^2[h_1 - h_2]}{\partial\boldsymbol{\sigma}^2}(\mathbf{n},\boldsymbol{\sigma}) = s\frac{\partial g}{\partial\boldsymbol{\sigma}}(\mathbf{n},\boldsymbol{\sigma})\otimes\frac{\partial g}{\partial\boldsymbol{\sigma}}(\mathbf{n},\boldsymbol{\sigma}) \tag{22}$$

where s is here a constant due to homogeneity properties of functions $(h_i)_{i=1,2}$ and g (details are presented in Appendix B). Combining (21) with relation (22) leads to the following conditions:

$$\begin{cases} \alpha_1 = \alpha_2 \\ \beta_1 = \beta_2 \\ \chi_1 = \chi_2 \\ \varphi_1 = \varphi_2 \end{cases} \tag{23}$$

which reduce to six the number of constitutive parameters relative to damage effect, namely $(\alpha_1,\beta_1,\chi_1,\varphi_1,\psi_1,\psi_2)$.

Strictly speaking, the identification of above coefficients would require the evaluation of damage effect on elastic properties through some mechanical experiments. Especially, these tests have to deal with open and closed configurations of microcracks in order to quantify the unilateral effect and the related recovery phenomenon. Yet, the main difficulty at this stage concerns the lack of exhaustive experimental studies concerning this aspect. Some authors let this question open-ended and introduce an additional scalar crack closure coefficient to characterize the damage deactivation (for example [12,41–43]), others postulate *a priori* specific modifications in the elastic tensor (for example [16,44]). In the present model, we have chosen an alternative approach that refers to micromechanical considerations. Indeed, simple homogenization schemes can provide interesting informations to quantify microcracks closure–reopening consequences and, in this way, stands as a judicious guide for macroscopic modelling. Precisely, the recovery condition adopted in this work is strongly motivated by the results outlined by [17].

Recovery condition: We assume that a set of closed microcracks with unit normal $\mathbf{m}$ does not contribute to the degradation of the material Young modulus $E(\mathbf{m})$ related to their normal direction, neither to the material volumetric modulus $\gamma(\mathbf{v})$ related to any direction of unit vector $\mathbf{v}$.

Consider a pure tension test $\boldsymbol{\sigma} = \bar{\sigma}\ \mathbf{v}^{\otimes 2}$ in the direction of unit vector $\mathbf{v}$, then the Young modulus $E(\mathbf{v})$ and the volumetric modulus $\gamma(\mathbf{v})$ of the material related to this direction are defined by [45,46]:

$$E(\mathbf{v}) = \frac{\mathbf{v}\cdot\boldsymbol{\sigma}\cdot\mathbf{v}}{\mathbf{v}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{v}},\quad \gamma(\mathbf{v}) = \frac{tr\boldsymbol{\sigma}}{tr\boldsymbol{\varepsilon}} \tag{24}$$

Presently, relevant calculations give rise to the following expressions:

$$E(\mathbf{v}) = \left[E_0^{-1} + \frac{1}{4\pi\bar{\sigma}} \int_{\mathscr{S}} \hat{\rho}(\mathbf{n},\mathbf{d})\mathbf{v} \cdot \frac{\partial h}{\partial \boldsymbol{\sigma}}(\mathbf{n}, \bar{\sigma}\mathbf{v}^{\otimes 2}) \cdot \mathbf{v} ds\right]^{-1} \tag{25}$$

$$\gamma(\mathbf{v}) = \left[\gamma_0^{-1} + \frac{1}{4\pi\bar{\sigma}} \int_{\mathscr{S}} \hat{\rho}(\mathbf{n},\mathbf{d}) tr\left(\frac{\partial h}{\partial \boldsymbol{\sigma}}(\mathbf{n}, \bar{\sigma}\mathbf{v}^{\otimes 2})\right) ds\right]^{-1} \tag{26}$$

with $\gamma_0 = E_0/(1-2\nu_0)$ the volumetric modulus for the undamaged material. The above-mentioned recovery assumption is then satisfied if and only if:

$$\mathbf{m} \cdot \frac{\partial h_2}{\partial \boldsymbol{\sigma}}(\mathbf{m}, \bar{\sigma}\mathbf{m}^{\otimes 2}) \cdot \mathbf{m} = 0 \tag{27}$$

and

$$tr\left(\frac{\partial h_2}{\partial \boldsymbol{\sigma}}(\mathbf{m}, \bar{\sigma}\mathbf{v}^{\otimes 2})\right) = 0, \quad \forall \mathbf{v} \in \mathscr{S} \tag{28}$$

which imposes thereby:

$$\begin{cases} \alpha_2 + \beta_2 + \chi_2 + \varphi_2 + \psi_2 = 0 \\ 2\alpha_2 + 6\beta_2 + \chi_2 = 0 \\ 3\chi_2 + 2\varphi_2 + 2\psi_2 = 0 \end{cases} \tag{29}$$

Putting together continuity (23) and recovery (29) conditions and retaining the three parameters $(\alpha_1, \varphi_1, \psi_1)$ as independent coefficients in order to obtain a linear system, one can deduce the expression of the principal unknowns:

$$\begin{cases} \alpha_2 = \alpha_1 \\ \beta_2 = \beta_1 = -\frac{\alpha_1}{2} \\ \chi_2 = \chi_1 = \alpha_1 \\ \varphi_2 = \varphi_1 \\ \psi_2 = -\frac{3}{2}\alpha_1 - \varphi_1 \end{cases} \tag{30}$$

As a result, we can note that the introduction of the recovery postulate allows again a reduction in the number of the constitutive parameters of the model (eventually only three for damage description including unilateral effect).

4.3. Final expression of the potential

Let denote by $\mathscr{S}_1(\boldsymbol{\sigma})$ and $\mathscr{S}_2(\boldsymbol{\sigma})$ the sets of unit vectors that correspond for a given stress $\boldsymbol{\sigma}$ to normals of microcracks respectively open and closed, *i.e*:

$$\begin{cases} \mathscr{S}_1(\boldsymbol{\sigma}) = \{\mathbf{n} \in S,\ g(\mathbf{n},\boldsymbol{\sigma}) = \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} > 0\} \\ \mathscr{S}_2(\boldsymbol{\sigma}) = \{\mathbf{n} \in S,\ g(\mathbf{n},\boldsymbol{\sigma}) = \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \leqslant 0\} \end{cases} \tag{31}$$

thus

$$\forall \boldsymbol{\sigma}, \quad \begin{cases} \mathscr{S}_1(\boldsymbol{\sigma}) \cup \mathscr{S}_2(\boldsymbol{\sigma}) = \mathscr{S} \\ \mathscr{S}_1(\boldsymbol{\sigma}) \cap \mathscr{S}_2(\boldsymbol{\sigma}) = \emptyset \end{cases} \tag{32}$$

According to the damage contribution (17) to the material enthalpy and relations (18) and (20), one obtains:

$$W_d(\boldsymbol{\sigma},\mathbf{d}) = \frac{1}{4\pi}\int_{\mathscr{S}_1(\boldsymbol{\sigma})} \hat{\rho}(\mathbf{n},\mathbf{d}) h_1(\mathbf{n},\boldsymbol{\sigma}) ds + \frac{1}{4\pi}\int_{\mathscr{S}_2(\boldsymbol{\sigma})} \hat{\rho}(\mathbf{n},\mathbf{d}) h_2(\mathbf{n},\boldsymbol{\sigma}) ds \tag{33}$$

or, in the same way:

$$W_d(\boldsymbol{\sigma},\mathbf{d}) = \frac{1}{4\pi}\int_{\mathscr{S}} \hat{\rho}(\mathbf{n},\mathbf{d}) h_1(\mathbf{n},\boldsymbol{\sigma}) ds - \frac{1}{4\pi}\int_{\mathscr{S}_2(\boldsymbol{\sigma})} \hat{\rho}(\mathbf{n},\mathbf{d}) [h_1 - h_2](\mathbf{n},\boldsymbol{\sigma}) ds \tag{34}$$

From expressions (21) of functions $(h_i)_{i=1,2}$, conditions (30) on parameters and properties of integration on the unit sphere $\mathscr{S}$ (see Appendix C), we can derive the general form of the thermodynamic potential:

$$\begin{aligned} W(\boldsymbol{\sigma},\mathbf{d}) = {} & W_0(\boldsymbol{\sigma}) + \alpha tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) d_0 + \beta tr^2\boldsymbol{\sigma}\ d_0 \\ & + \chi tr\boldsymbol{\sigma} tr(\boldsymbol{\sigma}\cdot d_2) + \varphi\ tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot d_2) \\ & + \psi\ \boldsymbol{\sigma} : d_4 : \boldsymbol{\sigma} - \frac{s}{2}\ \boldsymbol{\sigma} : \frac{1}{4\pi}\int_{\mathscr{S}_2(\boldsymbol{\sigma})} \hat{\rho}(\mathbf{n},\mathbf{d})\ \mathbf{n}^{\otimes 4}\ ds : \boldsymbol{\sigma} \end{aligned} \tag{35}$$

where dependent parameters $(\alpha, \beta, \chi, \varphi, \psi, s)$ are given in the Table 1. Since following relations can be established between these coefficients:

$$\begin{cases} s = 6(\alpha + 3\beta), \quad \forall p \\ \chi = \frac{2}{15}[2(\alpha+3\beta) - 5\varphi], \quad \forall p \geqslant 2 \\ \varphi = \frac{1}{10}[4(\alpha+3\beta) - 15\chi], \quad \forall p \geqslant 2 \\ \psi = \frac{1}{18}(4\beta + 5\chi), \quad \forall p \geqslant 4 \end{cases} \tag{36}$$

the free enthalpy proposed in the model can finally be rewritten according to the approximation order p:

- for $p = 0$, with only two independent material constants (α, β) relative to damage:

$$\begin{aligned} W(\boldsymbol{\sigma},\mathbf{d}) = {} & W_0(\boldsymbol{\sigma}) + \left[\alpha\ tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) + \beta tr^2\boldsymbol{\sigma}\right] d_0 \\ & - 3(\alpha+3\beta)\boldsymbol{\sigma} : \frac{1}{4\pi}\int_{\mathscr{S}_2(\boldsymbol{\sigma})} \hat{\rho}(\mathbf{n},\mathbf{d})\mathbf{n}^{\otimes 4} ds : \boldsymbol{\sigma} \end{aligned} \tag{37}$$

  where $\hat{\rho}(\mathbf{n},\mathbf{d}) = d_0$,
- for $p = 2$, with only three independent material constants (α, β, χ) (or equivalently (α, β, φ)) relative to damage:

$$\begin{aligned} W(\boldsymbol{\sigma},\mathbf{d}) = {} & W_0(\boldsymbol{\sigma}) + \left[\alpha\ tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) + \beta tr^2\boldsymbol{\sigma}\right] d_0 \\ & + \chi tr\boldsymbol{\sigma} tr(\boldsymbol{\sigma}\cdot d_2) + \tfrac{1}{10}[4(\alpha+3\beta) - 15\chi]\ tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot d_2) \\ & - 3(\alpha+3\beta)\boldsymbol{\sigma} : \tfrac{1}{4\pi}\int_{\mathscr{S}_2(\boldsymbol{\sigma})} \hat{\rho}(\mathbf{n},\mathbf{d})\mathbf{n}^{\otimes 4} ds : \boldsymbol{\sigma} \end{aligned} \tag{38}$$

  where $\hat{\rho}(\mathbf{n},\mathbf{d}) = d_0 + d_2 : \lfloor \mathbf{n}^{\otimes 2} \rfloor$,
- and whatever $p \geqslant 4$, with as well only three independent material constants (α, β, χ) (or equivalently (α, β, φ)) relative to damage:

$$\begin{aligned} W(\boldsymbol{\sigma},\mathbf{d}) = {} & W_0(\boldsymbol{\sigma}) + [\alpha\ tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) + \beta tr^2\boldsymbol{\sigma}] d_0 + \chi tr\boldsymbol{\sigma} tr(\boldsymbol{\sigma}\cdot d_2) \\ & + \frac{1}{10}[4(\alpha+3\beta) - 15\chi]\ tr(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot d_2) \\ & + \frac{1}{18}(4\beta + 5\chi)\boldsymbol{\sigma} : d_4 : \boldsymbol{\sigma} \\ & - 3(\alpha+3\beta)\boldsymbol{\sigma} : \frac{1}{4\pi}\int_{\mathscr{S}_2(\boldsymbol{\sigma})} \hat{\rho}(\mathbf{n},\mathbf{d})\mathbf{n}^{\otimes 4} ds : \boldsymbol{\sigma} \end{aligned} \tag{39}$$

  where $\hat{\rho}(\mathbf{n},\mathbf{d}) = d_0 + d_2 : \lfloor \mathbf{n}^{\otimes 2} \rfloor + d_4 :: \lfloor \mathbf{n}^{\otimes 4} \rfloor + \cdots + d_p \bullet \lfloor \mathbf{n}^{\otimes p} \rfloor$.

  Important comments should be done from these results:

- The first line of Eq. (37), the first two lines of Eq. (38) and the first three lines of (39) respectively define at order $p = 0$, $p = 2$ and $p \geqslant 4$ the enthalpy of the microcracked material when defects are all opened. The macroscopic material sym-

Table 1
Expressions of the coefficients $(\alpha, \beta, \chi, \varphi, \psi, s)$ according to approximation order p and coefficients $(\alpha_1, \varphi_1, \psi_1)$.

	$p = 0$	$p = 2$	$p \geqslant 4$
α	$\alpha_1 + \frac{1}{3}\varphi_1 + \frac{2}{15}\psi_1$	$\alpha_1 + \frac{1}{3}\varphi_1 + \frac{2}{15}\psi_1$	$\alpha_1 + \frac{1}{3}\varphi_1 + \frac{2}{15}\psi_1$
β	$-\frac{1}{6}\alpha_1 + \frac{1}{15}\psi_1$	$-\frac{1}{6}\alpha_1 + \frac{1}{15}\psi_1$	$-\frac{1}{6}\alpha_1 + \frac{1}{15}\psi_1$
χ	0	$\frac{2}{105}(7\alpha_1 + 2\psi_1)$	$\frac{2}{105}(7\alpha_1 + 2\psi_1)$
φ	0	$\frac{2}{105}(7\varphi_1 + 4\psi_1)$	$\frac{2}{105}(7\varphi_1 + 4\psi_1)$
ψ	0	0	$\frac{8}{315}\psi_1$
s	$3\alpha_1 + 2\varphi_1 + 2\psi_1$	$3\alpha_1 + 2\varphi_1 + 2\psi_1$	$3\alpha_1 + 2\varphi_1 + 2\psi_1$

metry described in such "all opened" state thus corresponds for $p = 0$ to isotropy, for $p = 2$ to orthotropy coinciding with axes of d_2, and for $p \geqslant 4$ to general anisotropy whose axes depend on tensors $(d_2, d_4, \ldots, d_p)$.

- On the other hand, each last integral term of three Eqs. (37)–(39) accounts for the modification induced by the possible closure of some of the defects. In comparison with existing formulations which account for unilateral behaviour only in the principal directions of the state variables, the present approach leads to an enriched continuum representation since all directions of the space are checked and contribution of all closed microcracks are taken into account. Moreover, this additional term introduces a perturbation of the previous "all opened" material symmetry that may lead whatever order p to a general anisotropic resulting behaviour according to the domain $\mathscr{S}_2(\boldsymbol{\sigma})$ of normals to closed microcracks. Concerning practical aspects, the mathematical evaluation of integrals terms on the truncated unit sphere $\mathscr{S}_2(\boldsymbol{\sigma})$ is not a difficult task and can be simply implemented within any simulation tool [47].
- It is finally interesting to note that the formulation for $p \geqslant 4$ reduces to quite simple forms when all microcracks exhibit the same status, namely:
  - when microcracks are all opened (*i.e.* $\mathscr{S}_2(\boldsymbol{\sigma}) = \emptyset$), potential (39) comes to:

$$W(\boldsymbol{\sigma}, \mathbf{d}) = W_0(\boldsymbol{\sigma}) + [\alpha\ tr(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}) + \beta tr^2 \boldsymbol{\sigma}]\ d_0 + \chi tr\boldsymbol{\sigma} tr(\boldsymbol{\sigma} \cdot d_2) + \frac{1}{10}[4(\alpha + 3\ \beta) - 15\ \chi]\ tr(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} \cdot d_2) + \frac{1}{18}(4\beta + 5\chi)\ \boldsymbol{\sigma} : d_4 : \boldsymbol{\sigma} \quad (40)$$

  - when microcracks are all closed (*i.e.* $\mathscr{S}_2(\boldsymbol{\sigma}) = \mathscr{S}$), one obtains:

$$W(\boldsymbol{\sigma}, \mathbf{d}) = W_0(\boldsymbol{\sigma}) + \frac{1}{5}(\alpha - 2\beta)\left[3tr(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}) - tr^2\boldsymbol{\sigma}\right]d_0 + [\chi - \frac{4}{35}(\alpha + 3\beta)]\left[tr\ \boldsymbol{\sigma} tr(\boldsymbol{\sigma} \cdot d_2) - \frac{3}{2}tr(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma} \cdot d_2)\right] + \frac{1}{630}(175\chi - 48\alpha - 4\beta)\boldsymbol{\sigma} : d_4 : \boldsymbol{\sigma} \quad (41)$$

In accordance with micromechanical results [31,48], tensors $d_0,\ d_2$ and d_4 are then sufficient to describe the enthalpy degradation for these both simple damage configurations. On the other hand for mixed states, that is if the material is concerned at the same time by opened and closed defects ($\mathscr{S}_2(\boldsymbol{\sigma}) \neq \{\mathscr{S}, \emptyset\}$), the whole tensors $(d_0, d_2, d_4, \ldots, d_p)$ emerge through the last term of (39).

4.4. Identification

In its final form, the model requires the identification of the two elastic coefficients of the material (E_0 and v_0), which is rather classical, and in the more complex case of three constants (for instance (α, β, χ)) related to the microcracks contribution to energy. As a matter of fact, the determination of these latter coefficients requires the indication for a given damage distribution of both:

- the microcrack density distribution,
- the effective compliance of the material when all microcracks are opened.

As an illustration, we recall below the main steps of the identification procedure developed by Welemane [40] for the case $p \geqslant 4$:

(1) Firstly, testing specimens are affected by an axisymmetric (around axis $\mathbf{m}$) damage state by means for example of uniaxial tension or compression tests (in the direction of $\mathbf{m}$). The maximum load applied should induce a significant damage extent and, at the same time, stand quite far from the stress–strain peak to avoid disturbances linked to localization phenomena. After unloading, several measures of the microcrack density $(\rho_1, \rho_2, \ldots, \rho_N)$ in directions of respective unit vectors $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_N$ (Fig. 2) need to be performed, as in works of [2,3]. From this, one can determine the best approximation $\hat{\rho}$ of such real distribution, that is tensors $\mathbf{d} = (d_0, d_2, d_4)$ that appear in the compliance expression when defects are all opened, by solving the following minimization problem:

$$\min_{\mathbf{d}} \left\{ \sum_{i=1}^{N} [\rho_i - \hat{\rho}(\mathbf{n}_i, \mathbf{d})]^2 \right\} \quad (42)$$

(2) Knowing the damage variable $\mathbf{d}$, reversible sollicitations (without damage growth) giving rise to the material compliance tensor $\mathbb{S}^{exp}$ components are then applied to the damaged specimens such that all microcracks remain in the opened state. This can be done for example by means of uniaxial tension tests [49] or uniaxial elongation tests [50]. Again, the identification of model constants required the resolution of a minimization problem:

$$\min_{(\alpha,\beta,\chi)} \left\{ (\|\mathbb{S}^{exp}\| - \|\mathbb{S}(\alpha, \beta, \chi)\|)^2 \right\} \quad (43)$$

where $\|a\| = \sqrt{a :: a}$ denotes the norm of the fourth-order tensor a, and $\mathbb{S}$ refers to the compliance tensor provided by the model. Namely, derivation of potential (39) in the case of all opened microcracks leads to:

$$\mathbb{S} = \frac{\partial^2 W}{\partial \boldsymbol{\sigma}^2} = \left(\frac{1 + v_0}{E_0} + 2\alpha\ d_0\right)\mathbf{I} \,\overline{\underline{\otimes}}\, \mathbf{I} + \left(-\frac{v_0}{E_0} + 2\beta\ d_0\right)\mathbf{I} \otimes \mathbf{I} + \chi(\mathbf{I} \otimes d_2 + d_2 \otimes \mathbf{I}) + \frac{1}{10}[4(\alpha + 3\beta) - 15\chi](\mathbf{I} \,\overline{\underline{\otimes}}\, d_2 + d_2 \,\overline{\underline{\otimes}}\, \mathbf{I}) + \frac{1}{9}(4\beta + 5\chi)d_4 \quad (44)$$

In order to obtain a representative result, such identification procedure should obviously be repeated many times, even for different damage distributions, in view of the difficulty and systematic scattering associated to the study of microcracking [51].

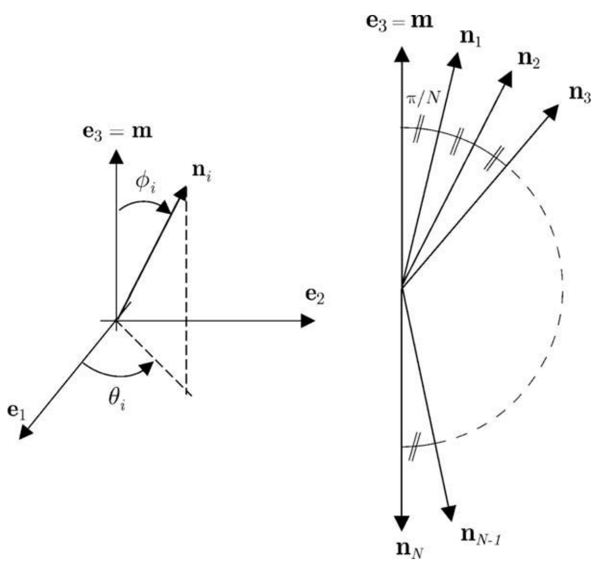

Fig. 2. Microcrack density measures directions.

Yet, the problem at the moment is that experimental data needed (damage distribution and elastic properties) exist separately but they are not available for the same material. Accordingly, we have chosen to follow the strategy adopted for example by [33,52] that is to specify the model parameters by using micromechanical estimations of elastic properties.

Consider a representative volume element V of an homogenous isotropic elastic linear matrix (elastic properties E_0 and ν_0), weakened by an array of parallel flat penny-shaped microcracks (unit normal $\mathbf{m}$ and total density $\varpi = \frac{1}{V}\sum_i a_i^3$, where a_i denotes their radii). Assuming non-interacting microcracks in the opened state, homogenization results lead to the following expression of the material effective compliance [24,31]:

$$\mathbb{S} = \frac{1+\nu_0}{E_0}\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} - \frac{\nu_0}{E_0}\mathbf{I}\otimes\mathbf{I} + \frac{16(1-\nu_0^2)}{3(2-\nu_0)E_0}\varpi\left[\mathbf{m}^{\otimes 2}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{m}^{\otimes 2} - \nu_0\mathbf{m}^{\otimes 4}\right] \quad (45)$$

Regarding the model, we retain the most general case of an approximation order $p \geqslant 4$ in order to be consistent with the order of tensorial terms appearing in the micromechanical expression (45). Besides, for such damage configuration, damage variables d_0, d_2 and d_4 reduce to [23]:

$$d_0 = \varpi, \quad d_2 = \frac{15}{2}\varpi\lfloor\mathbf{m}^{\otimes 2}\rfloor, \quad d_4 = \frac{315}{8}\,\varpi\,\lfloor\mathbf{m}^{\otimes 4}\rfloor \quad (46)$$

Identifying (44) to micromechanics (45) leads finally to the unique solution for parameters (α, β, χ):

$$\begin{cases} \alpha = \frac{16(1-\nu_0^2)(5-\nu_0)}{45E_0(2-\nu_0)} \\ \beta = -\frac{8\nu_0(1-\nu_0^2)}{45E_0(2-\nu_0)} \\ \chi = -\frac{32\nu_0(1-\nu_0^2)}{315E_0(2-\nu_0)} \end{cases} \quad (47)$$

According to this identification methodology, the determination of the model parameters reduces to the knowledge of elastic parameters E_0 and ν_0.

As a conclusion to this presentation of the modelling approach, it has to be said that the new formulation presented in this paper clearly belongs to the class of macroscopic physically-based models. Its macroscopic character comes from the definition of damage variables associated with a Fourier expansion of the microcracks density and also the use of representation theorems to derive the energy expression. In view of the current lack of experimental results, some of its assumptions are supported by micromechanical considerations, which provides a deeper physical justification than purely phenomenological approaches and allows a reduction in the model material parameters. Yet, a strong interest of this theoretical framework is that it leads to the predictions of a large class of models, and especially some micromechanical models that are recovered as a particular case (see above identification procedure). Accordingly, it allows to go beyond results predicted by these approaches as it is demonstrated on following applications.

5. Applications and discussion

This section aims at illustrating the ability of the formulation to represent the anisotropic behaviour of brittle materials and unilateral effects. As an example, all further numerical investigations will be applied to the case of a concrete with elastic parameters E_0 = 35 GPa and $\nu_0 = 0.2$ [5].

The two first applications refer to the same isotropic damage configuration studied in the strain-based model presented by [1] in order to show the differences inherent to the stress-based formulation. Since the crack density does not depend on the orientation, the zero order fabric tensor is sufficient to account for the damage distribution, so we retain $\mathbf{d} = (d_0)$.

5.1. Hydrostatic tests

Let examine first the bulk modulus K of the material, defined by:

$$K = \frac{tr\boldsymbol{\sigma}}{3tr\boldsymbol{\varepsilon}} \quad (48)$$

with $\boldsymbol{\varepsilon}$ the strain induced by an hydrostatic test $\boldsymbol{\sigma} = \sigma_0\mathbf{I}$. As the opening–closure function (19) comes here to:

$$g(\mathbf{n}, \boldsymbol{\sigma}) = \sigma_0, \quad \forall\mathbf{n} \in \mathscr{S} \quad (49)$$

then microcracks are all opened during a tension test ($\sigma_0 > 0$), and on the contrary, all closed for a compressive sollicitation ($\sigma_0 < 0$). According to the model, the bulk modulus takes the form:

$$K = \begin{cases} \left[K_0^{-1} + 6(\alpha + 3\beta)d_0\right]^{-1}, & \text{if } \sigma_0 > 0(\text{“all opened” state}) \\ K_0, & \text{if } \sigma_0 < 0(\text{“all closed” state}) \end{cases}, \forall d_0 \quad (50)$$

where $K_0 = E_0/(3 - 6\nu_0)$ is the bulk modulus of the virgin material. Even if the bulk modulus expression in the opened state is slightly different from the one derived from the strain-based formulation, the model in stress exhibits the same advantage. Indeed, since total density $d_0 \geqslant 0$ and

$$\alpha + 3\beta = \frac{8(1-\nu_0^2)}{9E_0} > 0 \quad (51)$$

it is possible to account for the damage degradation of the bulk modulus when microcracks are opened ($K \leqslant K_0$), and the total recovery of this elastic property at their closure ($K = K_0$). This last result, which stands in direct accordance with experimental observations on brittle materials [53], is independent of the identification procedure of the model (it is obtained for any set of model parameters) and of the microcracks distribution (such recovery also arises for an anisotropic damage configuration).

5.2. Uniaxial tension and compression

We keep the same isotropic damage distribution and submit now the material to reversible uniaxial tests in the direction of unit vector $\mathbf{v}$, then $\boldsymbol{\sigma} = \bar{\sigma}\,\mathbf{v}^{\otimes 2}$ and d_0 remains constant. As before, such loading paths correspond to complete states of opening–closure. Indeed,

$$g(\mathbf{n}, \boldsymbol{\sigma}) = \bar{\sigma}(\mathbf{n}\cdot\mathbf{v})^2, \quad \forall\mathbf{n} \in \mathscr{S} \quad (52)$$

then $\mathscr{S}_2(\boldsymbol{\sigma}) = \emptyset$ for uniaxial tension ($\bar{\sigma} > 0$) and $\mathscr{S}_2(\boldsymbol{\sigma}) = \mathscr{S}$ for the compressive case ($\bar{\sigma} < 0$). Accordingly, the Young modulus related to axial direction $\mathbf{v}$ is expressed as follows:

$$E(\mathbf{v}) = \begin{cases} \left[E_0^{-1} + 2(\alpha + \beta)d_0\right]^{-1}, & \text{if } \bar{\sigma} > 0(\text{“all opened” state}) \\ \left[E_0^{-1} + \frac{4}{5}(\alpha - 2\beta)d_0\right]^{-1}, & \text{if } \bar{\sigma} < 0(\text{“all closed” state}) \end{cases}, \forall d_0 \quad (53)$$

Fig. 3 illustrates thereby the uniaxial stress–strain response ($\bar{\sigma}$, $\bar{\varepsilon} = \mathbf{v}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{v}$) of the studied concrete for these both tests. Note first the continuity of the multilinear stress–strain response between these two different states of microcracks. Then, as the microcracks closure leads to a certain recovery of the axial Young modulus $E(\mathbf{v})$, the model allows to represent the different behaviour of brittle materials in tension and in compression. It is also

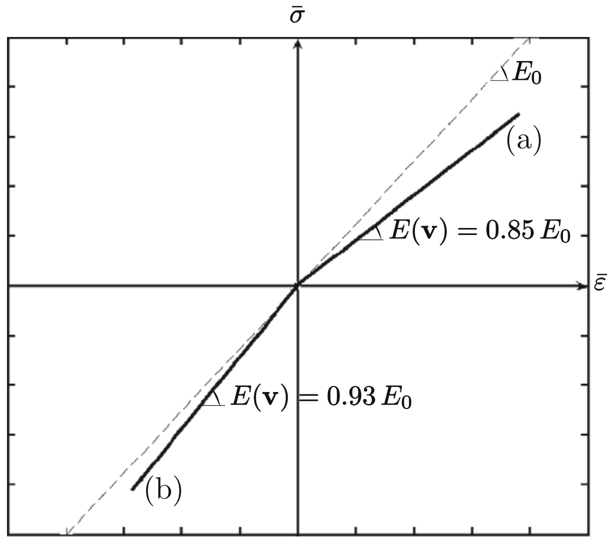

Fig. 3. Axial stress–strain responses of a concrete weakened by an isotropic distribution of microcracks ($d_0 = 0.1$) submitted to reversible uniaxial tension (a) and compression (b) tests.

interesting to observe that this property remains degraded in comparison with its value for the undamaged material ($E(\mathbf{v}) \leqslant E_0$) in both states, so even when all microcracks are closed. Such partial character of the recovery, which can be observed experimentally [4] and pointed out from micromechanical analyses [17], comes from the recovery condition selected: a set of closed microcracks with unit normal **m** still contributes to the degradation of the Young moduli $E(\mathbf{t})$ in directions of unit vectors **t** such that $\mathbf{t} \neq \mathbf{m}$.

Moreover, if we compare to the same simulation (same loading path and damage density) provided in the strain-based framework [1], we can underline various points:

- Elastic moduli intensity obtained for the two formulations are slightly the same for the both sollicitations; this comes from the identification procedure which has been done with quite equivalent micromechanical stress and strain-based results established under the assumption of small microcracks density [37,40]; another identification procedure and another set of model parameters would have led obviously to notably different results for the two approaches.
- The stress-based framework facilitates much more the derivation of the model response for classical loading paths (triaxial tension or compression) since opening–closure domains just correspond either to "all opened" (tension sollicitations) or "all closed" (compression sollicitations) states of microcracks; for the strain approach, uniaxial compression path belongs to a mixed state where some microcracks are opened and some are closed (see Fig. 4); the latter formulation requires then a specific integration within a simulation tool to get the model responses [47].
- Besides, this last difference on opening–closure domains clearly shows that stress and strain-based formulations are not equivalent; for example, as thermodynamic forces associated to damage (12) depend on the opening–closure domain (similarly to the stress–strain response), the damage evolution which is governed by such forces will not be the same according to the framework chosen; besides, if one intends to account for the friction on the lips of closed microcracks during compressive tests, the closed defects orientations may differ between the two formulations which can affect notably the stress–strain responses and the elastic moduli.

5.3. Influence of damage variables order

This part aims at investigating the ability of the formulation to account simultaneously for directional aspects relative to damage (induced anisotropy) and also the unilateral contribution of damage according to the opening–closure state of microcracks. In this way, we consider the damage distribution ρ studied in the first section (defined by Eq. (9)) and we examine the consequences on elastic properties of the damage variables order and of the microcrack opening–closure state.

According to expression (39), one can derive the general expression of the Young modulus in any direction of unit vector **v**:

$$E(\mathbf{v}) = \begin{bmatrix} E_0^{-1} + 2(\alpha+\beta)d_0 + \frac{2}{10}[4(\alpha+3\beta) - 5\chi]d_2 : \mathbf{v}^{\otimes 2} \\ +\frac{1}{9}(4\beta + 5\chi)d_4 :: \mathbf{v}^{\otimes 4} \\ -6(\alpha+3\beta)\frac{1}{4\pi}\int_{\mathscr{S}_2(\sigma)} \hat{\rho}(\mathbf{n},\mathbf{d})\mathbf{n}^{\otimes 4} ds :: \mathbf{v}^{\otimes 4} \end{bmatrix}^{-1} \quad (54)$$

where $(d_0, d_2, d_4, \ldots, d_p)$ and $\hat{\rho}(\mathbf{n}, \mathbf{d})$ are respectively given by Eqs. (6) and (8). For example, the total microcrack density related to the distribution (9) is thus equal to:

$$d_0 = \frac{1}{4\pi}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \rho(\phi)\sin\phi \; d\theta \; d\phi = 0.14 \quad (55)$$

with (θ, ϕ) the spherical angles of unit vector **n** in the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ (see Fig. 1). In order to account for the anisotropic

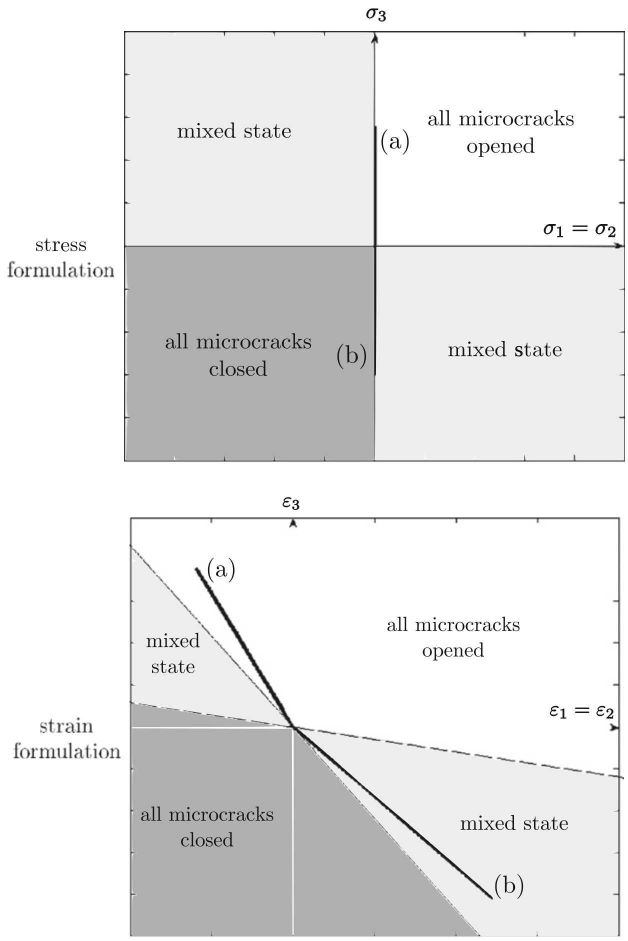

Fig. 4. Uniaxial tension (a) and compression (b) loading paths and opening–closure domains of microcracks in the axisymmetric space of observable variables (concrete weakened by a fixed isotropic damage distribution, $d_0 = 0.1$).

character of the damage distribution, one retains formulations based at least on the second-order damage tensor ($p \geqslant 2$).

5.3.1. Microcracks all opened or all closed

As demonstrated before through Eqs. (40) and (41), representations for $\mathbf{d} = (d_0, d_2, d_4, \ldots, d_p)$ are strictly identical whatever $p \geqslant 4$ for "all opened" and "all closed" states of microcracks. Focusing first on the configuration of all opened microcracks ($\mathscr{S}_2(\boldsymbol{\sigma}) = \emptyset$), Fig. 5 of the roses of the concrete Young moduli shows that representations obtained with $\mathbf{d} = (d_0, d_2)$ and $\mathbf{d} = (d_0, d_2, d_4, \ldots, d_p)$ are quite identical. The effect of the fourth-order is then small in this case, which agrees the statement of [31] derived from micromechanical considerations. Precisely, the model provides an anisotropic distribution of the Young modulus with a maximum degradation for $E(\mathbf{e}_3)$, namely:

$$\text{if } \mathscr{S}_2(\boldsymbol{\sigma}) = \emptyset, \quad E(\mathbf{e}_3) = 0.59\ E_0, \quad \forall \mathbf{d} \tag{56}$$

This stands in agreement with the damage distribution ρ for which most of microcrack surfaces stand in the direction orthogonal to vector $\mathbf{e}_3$ (Fig. 1). If we examine now the "all closed" state ($\mathscr{S}_2(\boldsymbol{\sigma}) = \mathscr{S}$), we note on Fig. 5 that the account of unilateral behaviour through the damage variable $\mathbf{d} = (d_0, d_2)$ leads to a weak resulting anisotropy whereas the model with $\mathbf{d} = (d_0, d_2, d_4, \ldots, d_p)$ exhibits more marked directional dependence. Still in line with [31], that shows the significant impact of the fourth-order damage tensor on the representation of closure effects. Moreover, the main consequences are observed in both cases in the direction orthogonal to the majority of closed microcracks, that corresponds here to the majority of microcracks, namely the direction of $\mathbf{e}_3$:

$$\text{if } \mathscr{S}_2(\boldsymbol{\sigma}) = \mathscr{S}, \quad \begin{cases} E(\mathbf{e}_3) = 0.85\ E_0, & \text{for } \mathbf{d} = (d_0, d_2) \\ E(\mathbf{e}_3) = 0.96\ E_0, & \text{for } \mathbf{d} = (d_0, d_2, d_4, \ldots, d_p) \end{cases} \tag{57}$$

Nevertheless, even if the closure effects induce a clear recovery of this elastic property, damage is still active for the material behaviour since $E(\mathbf{v})$ for unit vectors $\mathbf{v} \neq \mathbf{e}_3$ remains degraded (partial damage deactivation), according once again to the recovery condition of the model.

5.3.2. Mixed states

Let consider finally mixed states of microcrack opening–closure, with some microcracks opened and others closed ($\mathscr{S}_2(\boldsymbol{\sigma}) \neq \{\mathscr{S}, \emptyset\}$). This time, the response provided by the model depends directly on the order $p \geqslant 4$ chosen, through the approximated density $\hat{\rho}(\mathbf{n}, \mathbf{d})$ entering the integral term in Eq. (54). We have studied here the model with $\mathbf{d} = (d_0, d_2, d_4)$ and with $\mathbf{d} = (d_0, d_2, d_4, d_6)$. Moreover, we have chosen a fixed closure domain defined through spherical angles:

$$\mathscr{S}_2(\boldsymbol{\sigma}) = \mathscr{S}_2 = \{(\theta, \phi) \in [0, 2\pi] \times \mathscr{S}_2(\phi)\} \tag{58}$$

and two cases have been considered for the closure domain $\mathscr{S}_2(\phi)$ related to angle $\phi = (\mathbf{e}_3, \mathbf{n})$:

$$\mathscr{S}_2(\phi) = \mathscr{S}_2^1 = \left[0, m\ \frac{\pi}{2}\right] \cup \left[(2-m)\ \frac{\pi}{2}, \pi\right] \quad \text{and}$$
$$\mathscr{S}_2(\phi) = \mathscr{S}_2^2 = [0, \pi] - \mathscr{S}_2^1 \tag{59}$$

with $m = 0.16$ (Fig. 6). In all cases, the Young moduli roses related to the mixed states are well included between the "all opened" and "all closed" roses (Fig. 7). Yet, one should note the notable difference in the material response provided by a mixed state compared with these two extremes configurations. The impact of partial opening–closure depends obviously on the extent of the closure domain and is of greater amount for configurations such that:

- $\mathscr{S}_2$ corresponds to a domain very distinct from $\emptyset$ (which tends to the "all opened" state) or from $\mathscr{S}$ (which tends to the "all closed" state),
- $\mathscr{S}_2$ includes the directions of major values of the damage density $\hat{\rho}(\mathbf{n}, \mathbf{d})$; as shown by Fig. 7, the integral term of Eq. (54) has a more important value for the case $\mathscr{S}_2^1$, for which vector $\mathbf{e}_3$ belongs to the closure domain $\mathscr{S}_2$, than for $\mathscr{S}_2^2$; precisely:

$$\text{if } \mathscr{S}_2(\phi) = \mathscr{S}_2^1, \quad \begin{cases} E(\mathbf{e}_3) = 0.69\ E_0, & \text{for } \mathbf{d} = (d_0, d_2, d_4) \\ E(\mathbf{e}_3) = 0.75\ E_0, & \text{for } \mathbf{d} = (d_0, d_2, d_4, d_6) \end{cases} \tag{60}$$

$$\text{if } \mathscr{S}_2(\phi) = \mathscr{S}_2^2, \quad \begin{cases} E(\mathbf{e}_3) = 0.65\ E_0, & \text{for } \mathbf{d} = (d_0, d_2, d_4) \\ E(\mathbf{e}_3) = 0.60\ E_0, & \text{for } \mathbf{d} = (d_0, d_2, d_4, d_6) \end{cases} \tag{61}$$

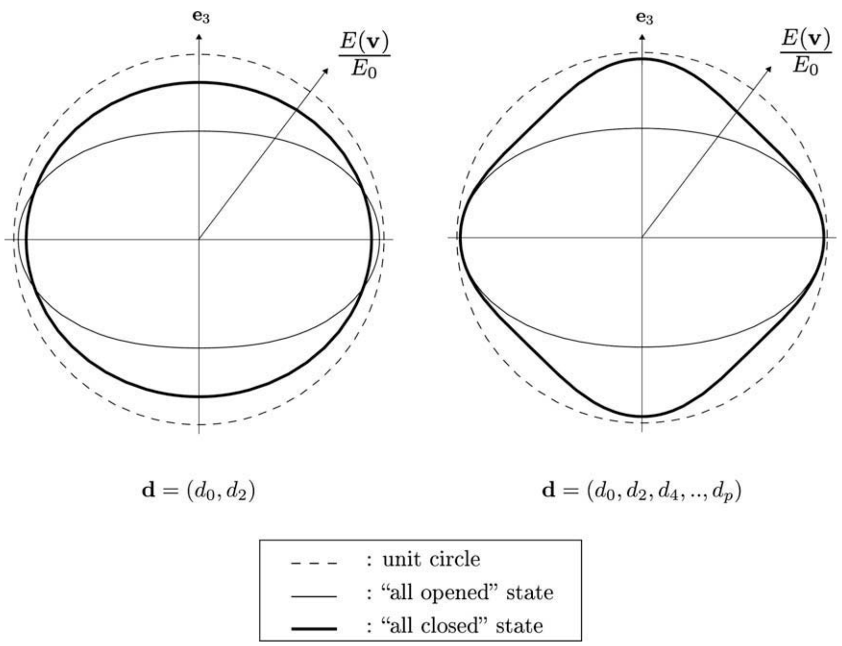

Fig. 5. Young moduli of a concrete according to the damage variables for "all opened" and "all closed" states of microcracks.

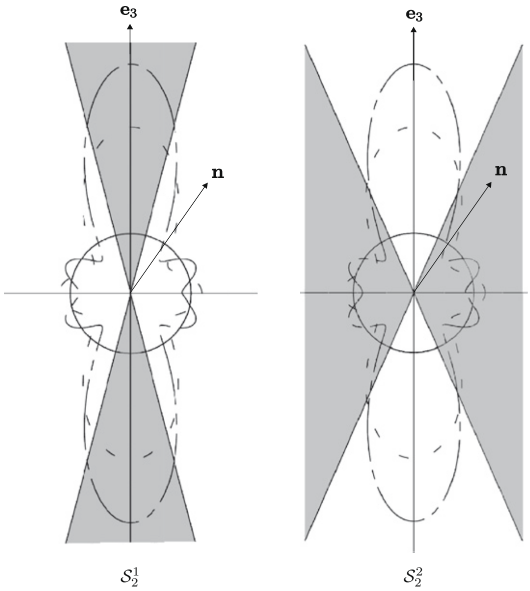

Fig. 6. Approximated density $\hat{\rho}$ for $\mathbf{d} = (d_0, d_2, d_4)$ and $\mathbf{d} = (d_0, d_2, d_4, d_6)$ (as in Fig. 1) and closure domains $\mathscr{S}_2(\phi)$ (in grey) for the two studied mixed states of microcrack opening–closure.

This example illustrates also the results obtained by representations including fourth-order tensor d_4 and sixth order tensor d_6. Owing to the convergence of the development (5), the introduction of tensor d_6 gives a result closer to the exact response (that is for $p \to \infty$) since the truncation at the sixth order provides a better approximation $\hat{\rho}$ of the damage distribution ρ. In particular, the closure domain $\mathscr{S}_2^2$ almost corresponds to the directions $\mathbf{n}$ such that $\rho(\mathbf{n}) = 0$ (see Figs. 1 and 6), this mixed state is then very close to the "all opened" situation. Accordingly, the best approximation of $\hat{\rho}$ given by $\mathbf{d} = (d_0, d_2, d_4, d_6)$ tends to such result for $E(\mathbf{v})$, whereas $\mathbf{d} = (d_0, d_2, d_4)$ which overvalues $\hat{\rho}(\mathbf{n}, \mathbf{d})$ within $\mathscr{S}_2^2$ induces a deviation from this configuration (Eq. (56) and (61), Fig. 7). Nevertheless, the difference between models with $\mathbf{d} = (d_0, d_2, d_4)$ and with $\mathbf{d} = (d_0, d_2, d_4, d_6)$, which is obviously much more sensible in the direction $\mathbf{e}_3$ normal to the preferential orientation of microcracks, is not yet of great amount regarding the Young moduli $E(\mathbf{v})$ in all others directions of the space (Fig. 7). Others various damage configurations have been tested, either with sharp peaks or with mild maxima, and this example is quite representative of the conclusions obtained: in all cases, the overall contribution of tensor d_6 to this elastic property is quite negligible. Moreover, for any relevant set of model parameters, such prediction is qualitatively similar to the one obtained here with a particular micromechanically-based identification. In view also of the complexity associated to the use of tensors of order six or more, one can conclude that a fourth-order truncation in the development (5) remains sufficient to provide a satisfactory representation of microcracks unilateral effect even for mixed states.

6. Conclusion

The work presented here introduces an alternative model for microcracked materials concerned by damage induced anisotropy and unilateral effect. The objective was to capture within a macroscopic continuum approach the salient features of such behaviour and to develop a rigorous formulation in the context of multilinear elasticity. The introduction of micromechanical considerations allows to cover up the lack of exhaustive experimental studies on microcracks closure consequences and provides more generally a deep physical justification to the macroscopic constitutive laws established. In this way, the proposed representation is able to capture the main closure effects observed on the elastic properties of damaged materials while avoiding the spectral decompositions generally used in CDM models.

In addition to the detailed presentation of the buildings steps of the approach, this paper demonstrates its feasible application to the stress-based framework. Compared to the strain-based model, the formulation in stress allows direct connections with microme-

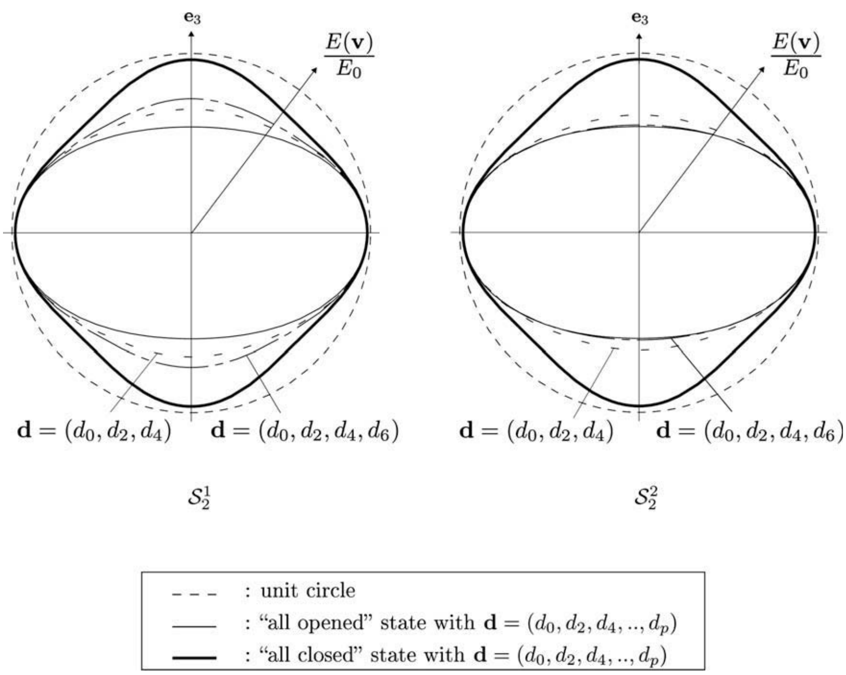

Fig. 7. Young moduli of a concrete according to the damage variables for the two studied mixed states of microcrack opening–closure.

chanical results that simplifies the global expressions of the model (in particular the opening–closure function and then the thermodynamic potential). We have shown also that the representation of the damage unilateral effect differs between these two approaches, in particular the microcracks opening–closure domains. Such aspects may lead to notable differences in the material responses and should be considered for future developments of the model.

The damage characterization used in the model is related to the microcrack distribution. Such mathematical representation confers a more rational understanding of the relationship between damage variables and the material microstructure and constitutes an appropriate choice to account for closure effects as the contribution of each set of parallel microcracks can be distinguished between the open and the closed state. Through a series expansion, it allows moreover the study of the influence of damage variables order on the representation provided, and especially for various opening–closure states of microcracks including mixed states. For anisotropic damage distributions, results obtained confirm the significant role of the fourth-order tensor damage variable and demonstrate the weak influence of higher order tensors.

Further investigations must now be conducted in order to complete the identification and validation procedures of the formulation, which requires mainly to perform experimental tests on brittle materials with various damage distributions and various opening–closure states of microcracks. Finally, in order to achieve the constitutive modelling, we need to study the question of damage evolution in the same spirit, that is to develop a thermodynamically admissible law in relation with the physical mechanisms involved.

Appendix A. Irreductible part of $\mathbf{n}^{\otimes m}$

Let $\langle T\rangle$ be the sum of all possible different permutations of a tensor T. For example:

$$\begin{aligned}&\langle \mathbf{I}^{\otimes 2}\rangle = \mathbf{I}\otimes\mathbf{I} + 2\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}\\ &\langle \mathbf{I}\otimes\mathbf{n}^{\otimes 2}\rangle = \mathbf{n}^{\otimes 2}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{n}^{\otimes 2} + 2\left(\mathbf{n}^{\otimes 2}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{n}^{\otimes 2}\right)\end{aligned} \tag{A.1}$$

The irreductible part $\lfloor \mathbf{n}^{\otimes m}\rfloor$ of the even order tensor $\mathbf{n}^{\otimes m}$ can be expressed by [14]:

$$\lfloor \mathbf{n}^{\otimes m}\rfloor = \sum_{r=0}^{m/2}(-1)^r\beta_r(m)\langle \mathbf{I}^{\otimes r}\otimes\mathbf{n}^{\otimes m-2r}\rangle \tag{A.2}$$

with

$$\beta_0(m) = 1,\quad \beta_r(m) = \frac{\beta_{r-1}(m)}{2\left(\frac{1}{2}+m-r\right)} \tag{A.3}$$

in the three-dimensional framework. Thereby, one obtains for example the following irreductible tensors:

$$\begin{cases}\lfloor 1\rfloor = 1\\ \lfloor \mathbf{n}^{\otimes 2}\rfloor = \mathbf{n}^{\otimes 2} - \frac{1}{3}\mathbf{I}\\ \lfloor \mathbf{n}^{\otimes 4}\rfloor = \mathbf{n}^{\otimes 4} - \frac{1}{7}\left[\mathbf{n}^{\otimes 2}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{n}^{\otimes 2} + 2\left(\mathbf{n}^{\otimes 2}\,\underline{\overline{\otimes}}\,\mathbf{I} + \mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{n}^{\otimes 2}\right)\right]\\ \quad + \frac{1}{35}\left(\mathbf{I}\otimes\mathbf{I} + 2\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}\right)\end{cases} \tag{A.4}$$

Appendix B. Continuity conditions

In this part, we recall the main result concerning the continuity of multilinear functions demonstrated by [40]. This work is an extension of the study by [39] to the case of functions of many variables.

B.1. General background

Consider an euclidean normed vectorial space F. Let $\mathscr{A}$ be a domain of F divided in two subdomains $\mathscr{A}_1$ and $\mathscr{A}_2$ by means of an interface ϕ characterized by a function $g : \mathscr{A} \to \mathbb{R}$:

$$\begin{cases}\mathscr{A}_1 = \{x\in\mathscr{A},\ g(x) > 0\}\\ \mathscr{A}_2 = \{x\in\mathscr{A},\ g(x) < 0\}\\ \phi = \{x\in\mathscr{A},\ g(x) = 0\}\end{cases} \tag{B.1}$$

with the properties:

$$\begin{cases}\mathscr{A}_1\cup\mathscr{A}_2\cup\phi = \mathscr{A}\\ \mathscr{A}_1\cap\mathscr{A}_2 = \emptyset\end{cases} \tag{B.2}$$

We assume that function g satisfies all the conditions necessary for the subdivision to be valid. In particular, ϕ must be such that $\mathscr{A}_1$ and $\mathscr{A}_2$ are simply connected [39].

Assume a function $h : \mathscr{A} \to \mathbb{R}$ defined by:

$$\forall x\in\mathscr{A},\quad h(x) = \begin{cases}h_1(x), & \text{if } g(x) > 0\\ h_2(x), & \text{if } g(x) \leqslant 0\end{cases} \tag{B.3}$$

where $h_1 : \mathscr{A} \to \mathbb{R}$ and $h_2 : \mathscr{A} \to \mathbb{R}$ are two twice continuously differentiable functions on $\mathscr{A}$ (class C^2), and $g : \mathscr{A} \to \mathbb{R}$ is a continuously differentiable function on $\mathscr{A}$ (class C^1).

B.2. Proposition

The function h defined in (B.3) is continuously differentiable on $\mathscr{A}$ if and only if there is a point x_0 of ϕ such that

$$\begin{cases}h_1(x_0) = h_2(x_0)\\ Dh_1(x_0) = Dh_2(x_0)\end{cases} \tag{B.4}$$

and if, in any point x of ϕ,

$$D^2[h_1 - h_2](x)(x') = s(x)Dg(x)(x')\ Dg(x),\quad \forall x'\in F \tag{B.5}$$

where $s : F \to \mathbb{R}$ is a continuous function on F (class C^0), and symbols D and D^2 before a function denote (when they exist) respectively its first and second Frechet differentials.

B.3. Application to function h defined in Eq. (20)

The vectorial space of the study corresponds here to $F = \mathbb{R}^3 \times T^2$ and $\mathscr{A} = \mathscr{S} \times T^{2S}$ (T^{2S} denotes the set of second-order symmetric tensors). We recall that if a function $h : \mathscr{A} \to \mathbb{R}$ is differentiable at the point $(\mathbf{n},\boldsymbol{\sigma})$ of $\mathscr{A}$, its first differential at that point is defined as follows:

$$\forall(\mathbf{n}',\boldsymbol{\sigma}'),\quad Dh(\mathbf{n},\boldsymbol{\sigma})(\mathbf{n}',\boldsymbol{\sigma}') = \frac{\partial h}{\partial\mathbf{n}}(\mathbf{n},\boldsymbol{\sigma})\cdot\mathbf{n}' + \frac{\partial h}{\partial\boldsymbol{\sigma}}(\mathbf{n},\boldsymbol{\sigma}) : \boldsymbol{\sigma}' \tag{B.6}$$

where $\frac{\partial h}{\partial\mathbf{n}}(\mathbf{n},\boldsymbol{\sigma})$ and $\frac{\partial h}{\partial\boldsymbol{\sigma}}(\mathbf{n},\boldsymbol{\sigma})$ represent the first partial derivatives of h at $(\mathbf{n},\boldsymbol{\sigma})$. As well, if this function is twice differentiable at the point $(\mathbf{n},\boldsymbol{\sigma})$ of $\mathscr{A}$, its second differential at that point is given by:

$$\begin{aligned}&\forall(\mathbf{n}',\boldsymbol{\sigma}'),\ (\mathbf{n}'',\boldsymbol{\sigma}''),\\ &D^2h(\mathbf{n},\boldsymbol{\sigma})(\mathbf{n}',\boldsymbol{\sigma}')(\mathbf{n}'',\boldsymbol{\sigma}'') = \mathbf{n}''\cdot\frac{\partial^2 h}{\partial\mathbf{n}^2}(\mathbf{n},\boldsymbol{\sigma})\cdot\mathbf{n}' + \mathbf{n}''\cdot\frac{\partial^2 h}{\partial\mathbf{n}\partial\boldsymbol{\sigma}}(\mathbf{n},\boldsymbol{\sigma}) : \boldsymbol{\sigma}'\\ &\qquad + \boldsymbol{\sigma}'' : \frac{\partial^2 h}{\partial\boldsymbol{\sigma}\partial\mathbf{n}}(\mathbf{n},\boldsymbol{\sigma})\cdot\mathbf{n}' + \boldsymbol{\sigma}'' : \frac{\partial^2 h}{\partial\boldsymbol{\sigma}^2}(\mathbf{n},\boldsymbol{\sigma}) : \boldsymbol{\sigma}'\end{aligned} \tag{B.7}$$

where $\frac{\partial^2 h}{\partial\mathbf{n}^2}(\mathbf{n},\boldsymbol{\sigma})$, $\frac{\partial^2 h}{\partial\mathbf{n}\partial\boldsymbol{\sigma}}(\mathbf{n},\boldsymbol{\sigma})$, $\frac{\partial^2 h}{\partial\boldsymbol{\sigma}\partial\mathbf{n}}(\mathbf{n},\boldsymbol{\sigma})$ and $\frac{\partial^2 h}{\partial\boldsymbol{\sigma}^2}(\mathbf{n},\boldsymbol{\sigma})$ denote the second partial derivatives of h at $(\mathbf{n},\boldsymbol{\sigma})$.

Considering function $h : \mathscr{A} \to \mathbb{R}$ such that

$$h(\mathbf{n},\boldsymbol{\sigma}) = \begin{cases}h_1(\mathbf{n},\boldsymbol{\sigma}), & \text{if } g(\mathbf{n},\boldsymbol{\sigma}) > 0\\ h_2(\mathbf{n},\boldsymbol{\sigma}), & \text{if } g(\mathbf{n},\boldsymbol{\sigma}) \leqslant 0\end{cases} \tag{B.8}$$

where $h_1 : \mathscr{A} \to \mathbb{R}$ and $h_2 : \mathscr{A} \to \mathbb{R}$ are two twice continuously differentiable functions on $\mathscr{A}$, and $g : \mathscr{A} \to \mathbb{R}$ is a continuously differentiable function on $\mathscr{A}$, the above proposition induces thereby that h is continuously differentiable on $\mathscr{A}$ if and only if there is a point $(\mathbf{n}_0, \boldsymbol{\sigma}_0)$ such that $g(\mathbf{n}_0, \boldsymbol{\sigma}_0) = 0$ and

$$\begin{cases} h_1(\mathbf{n}_0, \boldsymbol{\sigma}_0) = h_2(\mathbf{n}_0, \boldsymbol{\sigma}_0) \\ \frac{\partial h_1}{\partial \mathbf{n}}(\mathbf{n}_0, \boldsymbol{\sigma}_0) = \frac{\partial h_2}{\partial \mathbf{n}}(\mathbf{n}_0, \boldsymbol{\sigma}_0) \\ \frac{\partial h_1}{\partial \boldsymbol{\sigma}}(\mathbf{n}_0, \boldsymbol{\sigma}_0) = \frac{\partial h_2}{\partial \boldsymbol{\sigma}}(\mathbf{n}_0, \boldsymbol{\sigma}_0) \end{cases} \tag{B.9}$$

and if, at any point $(\mathbf{n}, \boldsymbol{\sigma})$ such that $g(\mathbf{n}, \boldsymbol{\sigma}) = 0$,

$$\begin{cases} \frac{\partial^2 [h_1 - h_2]}{\partial \mathbf{n}^2}(\mathbf{n}, \boldsymbol{\sigma}) = s(\mathbf{n}, \boldsymbol{\sigma}) \frac{\partial g}{\partial \mathbf{n}}(\mathbf{n}, \boldsymbol{\sigma}) \otimes \frac{\partial g}{\partial \mathbf{n}}(\mathbf{n}, \boldsymbol{\sigma}) \\ \frac{\partial^2 [h_1 - h_2]}{\partial \boldsymbol{\sigma} \partial \mathbf{n}}(\mathbf{n}, \boldsymbol{\sigma}) = s(\mathbf{n}, \boldsymbol{\sigma}) \frac{\partial g}{\partial \boldsymbol{\sigma}}(\mathbf{n}, \boldsymbol{\sigma}) \otimes \frac{\partial g}{\partial \mathbf{n}}(\mathbf{n}, \boldsymbol{\sigma}) \\ \frac{\partial^2 [h_1 - h_2]}{\partial \mathbf{n} \partial \boldsymbol{\sigma}}(\mathbf{n}, \boldsymbol{\sigma}) = s(\mathbf{n}, \boldsymbol{\sigma}) \frac{\partial g}{\partial \mathbf{n}}(\mathbf{n}, \boldsymbol{\sigma}) \otimes \frac{\partial g}{\partial \boldsymbol{\sigma}}(\mathbf{n}, \boldsymbol{\sigma}) \\ \frac{\partial^2 [h_1 - h_2]}{\partial \boldsymbol{\sigma}^2}(\mathbf{n}, \boldsymbol{\sigma}) = s(\mathbf{n}, \boldsymbol{\sigma}) \frac{\partial g}{\partial \boldsymbol{\sigma}}(\mathbf{n}, \boldsymbol{\sigma}) \otimes \frac{\partial g}{\partial \boldsymbol{\sigma}}(\mathbf{n}, \boldsymbol{\sigma}) \end{cases} \tag{B.10}$$

where $s : F \to \mathbb{R}$ is a continuous function on F.

According to expressions (21), conditions (B.9) are clearly satisfied at any point $(\mathbf{n}, 0_2)$ (with 0_2 the second-order zero tensor). Moreover, due to homogeneity properties with respect to $\boldsymbol{\sigma}$, functions $h_1,\ h_2$ and g are such that:

$$\begin{cases} \frac{\partial [h_1 - h_2]}{\partial \boldsymbol{\sigma}}(\mathbf{n}, \boldsymbol{\sigma}) : \boldsymbol{\sigma} = 2[h_1 - h_2](\mathbf{n}, \boldsymbol{\sigma}) \\ \frac{\partial^2 [h_1 - h_2]}{\partial \boldsymbol{\sigma}^2}(\mathbf{n}, \boldsymbol{\sigma}) : \boldsymbol{\sigma} = \frac{\partial [h_1 - h_2]}{\partial \boldsymbol{\sigma}}(\mathbf{n}, \boldsymbol{\sigma}) \quad , \quad \forall \mathbf{n} \in S \\ \frac{\partial g}{\partial \boldsymbol{\sigma}}(\mathbf{n}, \boldsymbol{\sigma}) : \boldsymbol{\sigma} = g(\mathbf{n}, \boldsymbol{\sigma}) \end{cases} \tag{B.11}$$

As a consequence, the verification of the fourth relation of Eq. (B.10) implies the satisfaction of the first three ones, and continuity conditions reduce at last to Eq. (22).

Appendix C. Integration over the unit sphere $\mathcal{S}$

Let denote by $(I_{ij})_{i,j \in [1,3]^2}$ and $(n_i)_{i \in [1,3]}$ the respective components of the second-order identity tensor $\mathbf{I}$ and of the unit vector $\mathbf{n}$ in the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^3$.

The integration on the unit sphere $\mathscr{S}$ of the tensor products of $\mathbf{n}$ leads to following expressions [22,54]:

$$\begin{cases} \frac{1}{4\pi} \int_{\mathscr{S}} ds = 1, \quad \frac{1}{4\pi} \int_{\mathscr{S}} n_i n_j ds = \frac{1}{3} I_{ij}, \quad \frac{1}{4\pi} \int_{\mathscr{S}} n_i n_j n_k n_l ds = \frac{1}{5} J_{ijkl} \\ \frac{1}{4\pi} \int_{\mathscr{S}} n_i n_j n_k n_l n_m n_n ds = \frac{1}{7} J_{ijklmn} \\ \frac{1}{4\pi} \int_{\mathscr{S}} n_i n_j n_k n_l n_m n_n n_p n_q ds = \frac{1}{7} J_{ijklmnpq} \end{cases} \tag{C.1}$$

with

$$\begin{cases} J_{ijkl} = \frac{1}{3} \left(I_{ij} I_{kl} + I_{ik} I_{jl} + I_{il} I_{jk} \right) \\ J_{ijklmn} = \frac{1}{5} \left(I_{ij} J_{klmn} + I_{ik} J_{jlmn} + I_{il} J_{jkmn} + I_{im} J_{jkln} + I_{in} J_{jklm} \right) \\ J_{ijklmnpq} = \frac{1}{7} (I_{ij} J_{klmnpq} + I_{ik} J_{jlmnpq} + I_{il} J_{jkmnpq} + I_{im} J_{jklnpq} + I_{in} J_{jklmpq} \\ \qquad + I_{ip} J_{jklmnq} + I_{iq} J_{jklmnp}) \end{cases} \tag{C.2}$$

Moreover, it can be demonstrated that [14]:

$$X_i \bullet \int_{\mathscr{S}} \mathbf{n}^{\otimes(i+j)} ds = 0_j, \quad \forall j = 0, 2, 4, \ldots, i-2 \tag{C.3}$$

for any irreductible tensor X_i of even order i.

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