To link to this article: DOI:10.1016/j.euromechsol.2012.06.001
http://dx.doi.org/10.1016/j.euromechsol.2012.06.001

To cite this version:
Goidescu, Cristina and Welemane, Hélène and Kondo, Djimédo and Gruescu, Cosmin *Microcracks closure effects in initially orthotropic materials.* (2013) European Journal of Mechanics - A/Solids, vol. 37 . pp. 172-184. ISSN 0997-7538

Microcracks closure effects in initially orthotropic materials

Cristina Goidescu [a], Hélène Welemane [a,*], Djimédo Kondo [b], Cosmin Gruescu [c]

[a] Université de Toulouse, Ecole Nationale d'Ingénieurs de Tarbes – INP/ENIT, LGP EA 1905, 47 avenue d'Azereix, F-65013 Tarbes, France
[b] Université de Paris VI, Institut d'Alembert, UMR CNRS 7190, 4 place Jussieu, F-75252 Paris, France
[c] Université de Lille I, LML, UMR CNRS 8107, Boulevard Paul Langevin, F-59655 Villeneuve d'Ascq, France

Keywords:
Anisotropic materials
Microcracking
Micromechanics
Induced anisotropy
Unilateral effects

A B S T R A C T

Microcracking is one of the basic mechanisms of inelastic deformation for a large class of anisotropic materials such as brittle matrix composites. Even at fixed microcracks density, the macroscopic behavior of these materials is very complex due to the combination of two specific features of such deteriorating phenomenon. First, the oriented nature of microcracks induces an evolution of the material symmetry (interaction between the initial anisotropy and the microcracks induced one). Secondly, a change in the elastic response of the material is expected, based on whether microcracks are open or closed in response to specific loading situations (the so-called "unilateral effect"). The present paper is devoted to a continuum micromechanics-based investigation of the resulting – generally fully – anisotropic multilinear response of orthotropic materials containing microcracks. The procedure leads to the proposal of a closed-form expression of the macroscopic free energy corresponding to 2D initially orthotropic materials weakened by arbitrarily oriented microcracks systems. The established results provide a complete quantification of both coupling effects of anisotropies and elastic moduli recovery phenomena induced by microcracks closure. A particular emphasis is put on the importance of Hill lemma for the derivation of these results which constitute a basis to the micro-macro modeling of damage process in initially orthotropic media.

1. Introduction

For a large class of engineering materials, diffuse microcracking plays a crucial role in their macroscopic mechanical behavior. In the particular case of anisotropic materials such as composites, sedimentary rocks or some metals, the modeling of microcracks-induced effects at fixed density of microcracks is of large interest in view of two specific features which characterize the involving process. If the question of anisotropic behavior related to microcracks existence is itself a quite difficult task for initially isotropic materials, it becomes even more complicated in the case of initially anisotropic materials. This is mainly due to the strong interaction between the primary (structural) material anisotropy and the microcracks-induced one: the presence of the related oriented defects may indeed lead to a very complex overall response. In addition to this aspect, microcracks can be either open or closed according to the loading and affect differently the weakened elastic properties of the material. This unilateral effect leads to a multilinear macroscopic response when microcracks change from opening to closure state, with generally a partial recovery of properties at the closure of some defects (Chaboche, 1992, 1993; Welemane and Cormery, 2002).

Experimental studies are often too restrictive to provide exhaustive data on such aspects, especially due to the difficulties to investigate elastic properties in various directions of the space and also to separate each of these effects. The interaction between structural and microcracks-induced anisotropies has been partly investigated through ultrasonic measures on ceramic matrix composites (Baste and Aristégui, 1998; Baste and El Bouazzaoui, 1996). These authors provide data which confirm the stiffness modifications due to degradation process, both on the amplitude (when loading axes correspond to initial material axes) and on the type of resulting material symmetry, especially in the case of off-axis loadings. They confirm also the major influence of microcracks in the direction of their normal (which corresponds in fact to the loading direction). Concerning the unilateral effect, some authors have put in evidence the partial recovery of elastic properties at the closure of microcracks (Allix et al., 1993; Morvan and Baste, 1998) but these studies are often limited to axial properties or to defects configurations coinciding with the structural anisotropy of the material (debonding mechanisms governed by reinforcements or loading in the material principal axes). In terms of representation, the simultaneous account of microcracks-induced

* Corresponding author. Tel.: +33 (0)5 62 44 29 47; fax: +33 (0)5 62 44 27 08.
E-mail address: Helene.Welemane@enit.fr (H. Welemane).

http://dx.doi.org/10.1016/j.euromechsol.2012.06.001

anisotropy and opening–closure effects often leads to serious difficulties. Even in the isotropic context, at constant microcracks density and under frictionless conditions, mathematical or thermodynamical inconsistencies have been pointed out in existing formulations, such as discontinuities of the stress–strain response or non-uniqueness of the thermodynamic potential (Carol and Willam, 1996; Chaboche, 1992; Challamel et al., 2006; Cormery and Welemane, 2002).

In order to provide a fundamental basis for damage modelings, micromechanical investigations provide some interesting and proper issues to account for such specific features (see for instance Cormery and Welemane, 2010; Costanzo et al., 1996; Halm and Dragon, 1996; Krajcinovic, 1996; Murakami and Kamiya, 1997; Pensée et al., 2002; Zhu et al., 2008). If many works have been devoted to the analysis of the overall elastic properties of microcracked materials, most of them deal with initially isotropic materials (Dormieux and Kondo, 2009; Hashin, 1988; Kachanov, 1993; Nemat-Nasser and Hori, 1993; Ponte Castañeda and Willis, 1995; Welemane and Cormery, 2002; etc.) and only few publications consider the case of anisotropic materials (Horii and Nemat-Nasser, 1983; Mauge and Kachanov, 1994; Santare et al., 1995; Tsukrov and Kachanov, 2000; Wang et al., 2009). Moreover, it must be emphasized that a general energy-based analysis of the multilinear behavior due to cracks closure effects in the presence of matrix anisotropy has not been treated in literature. This motivates the present study devoted to a homogenization-based investigation of initial orthotropy containing arbitrarily oriented microcracks which can be open or closed. The considered problem is two-dimensional in order to derive a closed-form expression of the overall anisotropy and of the stiffness recovery conditions.

To address such issues, one can adopt an Eshelby-like formalism in which microcracks are modeled as elliptical voids in the 2D orthotropic present context (Laws, 1977, 1983; Laws and Brockenbrough, 1987). Despite the interest of the above approach, the main difficulty lies in the determination of the Eshelby tensor corresponding to arbitrarily oriented microcracks within an orthotropic matrix. This can be done by taking advantage of results recently established by Gruescu et al. (2005). However in such case, the analysis of closed cracks remains a difficult task, especially the definition of the local stiffness tensor corresponding to closed cracks phase (see Deudé et al., 2002 or Dormieux and Kondo, 2009 in the context of an isotropic matrix). Consequently, we have focused on a direct approach by making use of fracture-mechanics based solutions as already considered by Andrieux et al. (1986) or Pensée et al. (2002) for isotropic matrix, and by Mauge and Kachanov (1994) (see also Horii and Nemat-Nasser, 1983; Tsukrov and Kachanov, 2000) in the context of anisotropic matrix. Compared to these analyses, the main contribution of the present work is to simultaneously account, within a strain energy-based framework, for the microcracks-induced anisotropy and for the defects opening to closure transition. Again, this is primarily required for the formulation of a rigorous and complete unilateral damage model of initially orthotropic materials.

The study is developed under the assumptions of non interacting frictionless microcracks. It provides a proper representation of the anisotropic multilinear response of weakened materials and constitutes a first step to address in the future more complex situations such as interactions between defects or dissipative sliding-based behavior. Let us emphasize also that the results have been established owing to a careful application of the Hill lemma in the case of anisotropic fissured media. This is crucial for arbitrarily oriented closed microcracks embedded in an orthotropic matrix.

The paper is organized as follows. In Section 2, we provide the general background of the micromechanical analysis, as already exposed by Andrieux et al. (1986) in the context of initially isotropic materials. Then, in Sections 3 and 4, original closed-form expressions of the elastic free energy are derived in the more general case of the 2D orthotropic matrix weakened by arbitrarily oriented microcracks; a particular attention is paid to the importance of the proper formulation of the Hill lemma in this context. We then analyze and discuss in Section 5 the interaction between primary and microcracks-induced anisotropies through a comparison of the different terms appearing in the expression of the macroscopic free energy. The resulting material symmetry is also analyzed through the directional distribution of overall anisotropic elastic properties of the microcracked material. In this last part, the focus is finally on the influence of the opening/closure status of the microcracks according to their orientation (with respect to symmetry axes of the uncracked material).

Standard notations are employed throughout. The inner products are labeled as follows: $a \cdot b$ for two vectors and $a:b$ for two second-order tensors. Additionally, the tensor products of two second-order tensors a and b are defined by:

$$[a \otimes b]_{ijkl} = a_{ij}b_{kl}; \quad (a \,\underline{\overline{\otimes}}\, b)_{ijkl} = \frac{1}{2}\left(a_{ik}b_{jl} + a_{il}b_{jk}\right) \tag{1}$$

2. General background

We follow here the micromechanical approach proposed by Andrieux et al. (1986) and leading to a closed-form expression of the macroscopic free energy of a 2D isotropic elastic medium weakened by arbitrarily oriented microcracks. The later were assumed open or frictionless closed, and in dilute concentration. The proposed approach allows also to derive the opening–closure transition criterion.

Let us consider a Representative Volume Element (RVE) in the form of a square cell area $\mathcal{A}$ (with boundary denoted $\partial\mathcal{A}$). This RVE is made up of an orthotropic linear elastic matrix with symmetry axes corresponding to the orthonormal basis ($\mathbf{e}_1$, $\mathbf{e}_2$). The stiffness tensor of the virgin matrix is denoted $\mathbb{C}^0$ and reads:

$$\mathbb{C}^0 = a_1 \mathbf{I} \otimes \mathbf{I} + a_2 \mathbf{I} \,\underline{\overline{\otimes}}\, \mathbf{I} + a_3 \mathbf{A} \otimes \mathbf{A} + a_4 (\mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}) \tag{2}$$

in which $\mathbf{I}$ denotes the second-order unit tensor and $\mathbf{A} = \mathbf{e}_1 \otimes \mathbf{e}_1$; constant coefficients $\{a_i\}_{i=1,4}$ can be related to the stiffness components in the basis associated to the material axes ($\mathbf{e}_1$, $\mathbf{e}_2$) as follows:

$$\begin{aligned} a_1 &= \mathbb{C}^0_{2222} - 2\mathbb{C}^0_{1212}, & a_2 &= 2\mathbb{C}^0_{1212} \\ a_3 &= \mathbb{C}^0_{1111} + \mathbb{C}^0_{2222} - 2\mathbb{C}^0_{1122} - 4\mathbb{C}^0_{1212}, & a_4 &= \mathbb{C}^0_{1122} - \mathbb{C}^0_{2222} + 2\mathbb{C}^0_{1212} \end{aligned} \tag{3}$$

Note that above expressions depend on the engineering moduli of the material and can be written either for plane stress or for plane strain conditions.

This matrix is weakened by an array of N families of flat microcracks with arbitrary orientation relative to orthotropic axes (Fig. 1). Microcracks of the ith family are characterized by their mean length $2l_i$. According to the classical scale separation conditions, the defects size is assumed to be very small in comparison with the size of the RVE but much larger than the scale of the microstructure or planes of symmetry inducing the primary anisotropy. The unit vector $\mathbf{n}_i$ normal to these cracks and $\mathbf{t}_i$ their unit tangent vector are such that ($\mathbf{n}_i$, $\mathbf{t}_i$) forms an orthonormal direct basis; ω_i^+ (respectively ω_i^-) designates their upper face (resp. the lower face) such that $\mathbf{n}_i$ is oriented from ω_i^- toward ω_i^+; $\omega_i = \omega_i^+ \cup \omega_i^-$ corresponds to the domain occupied by these cracks. We denote by $[[\mathbf{u}(\mathbf{x})]]_i = \mathbf{u}(\mathbf{x} \in \omega_i^+) - \mathbf{u}(\mathbf{x} \in \omega_i^-)$ the displacement jump between the two faces ω_i^+ and ω_i^- for any point $\mathbf{x} \in \omega_i$ and

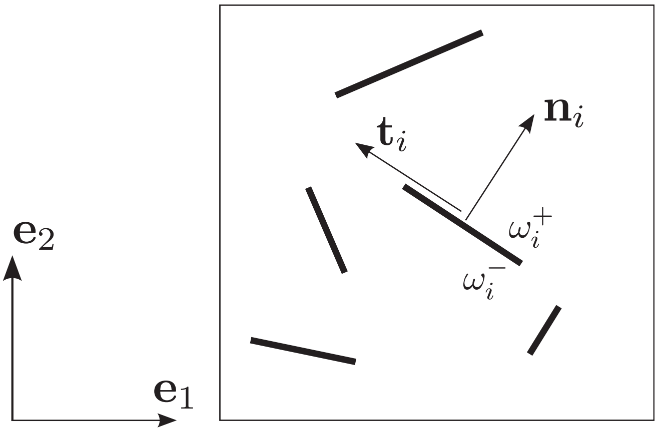

Fig. 1. Representative Volume Element in the two-dimensional case.

classical unilateral conditions are retained (see for instance Leguillon and Sanchez-Palencia, 1982):

$$\begin{cases} [\![u_n(\mathbf{x})]\!]_i = [\![\mathbf{u}(\mathbf{x})]\!]_i \cdot \mathbf{n}_i & \geq 0 \\ \mathbf{n}_i \cdot \boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{n}_i & \leq 0 \ , \forall \mathbf{n}_i \\ [\![u_n(\mathbf{x})]\!]_i (\mathbf{n}_i \cdot \boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{n}_i) & = 0 \end{cases} \tag{4}$$

in which $\boldsymbol{\sigma}$ is the local equilibrated stress field within the RVE. As classically, small perturbations assumption, rate independent and isothermal conditions are considered in the whole study.

Assuming dilute concentration of microcracks, the solution of the homogenization problem comes to sum up the contributions of each family of parallel microcracks. Therefore, the analysis can classically be reduced to the study of an elementary cell weakened by a single set of microcracks having a same unit normal $\mathbf{n}$ and a mean length $2l$ (Fig. 2). $\mathcal{N}$ denotes the number of defects per unit surface and ω represents the domain occupied by the cracks.

As a general recall (Nemat-Nasser and Hori, 1993; Zaoui, 2002), the macroscopic stress $\boldsymbol{\Sigma}$ and strain $\mathbf{E}$ tensors and the macroscopic free energy W on a cell $\mathcal{A}$ are respectively defined as average values of microscopic stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ fields and local free energy, namely:

$$\boldsymbol{\Sigma} = \langle \boldsymbol{\sigma} \rangle_{\mathcal{A}} \tag{5}$$

$$\mathbf{E} = \langle \varepsilon \rangle_{\mathcal{A}} \tag{6}$$

and

$$W = \frac{1}{2} \left\langle \varepsilon : \mathbb{C}^0 : \varepsilon \right\rangle_{\mathcal{A}} \tag{7}$$

with the integral operator $\langle \cdot \rangle_\Omega = (1/|\mathcal{A}|) \int_\Omega (\cdot) \mathrm{d}S$ and $\mathrm{d}S$ the surface element. Still, particular attention should be paid to such

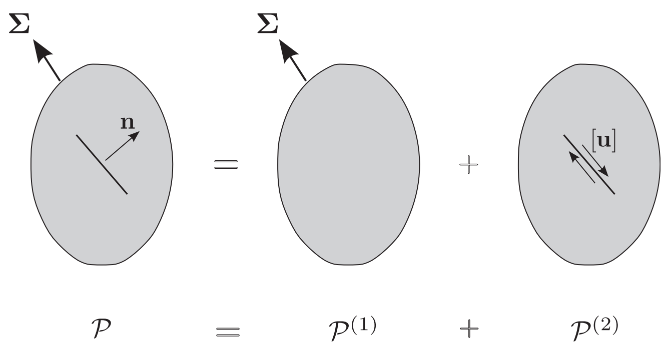

Fig. 2. Decomposition of the considered homogenization problem.

definitions in the specific context of cracked bodies due to the presence of discontinuity surfaces (Suquet, 1982). Indeed, let denote by $\overline{\mathcal{A}} = \mathcal{A} - \omega$ the area of the matrix phase, $\mathbf{v}(\mathbf{x})$ the outward unit normal to ω and $\mathbf{T}(\mathbf{x}, \mathbf{v}(\mathbf{x}))$ the traction along the crack faces for any point $\mathbf{x} \in \omega$. Decomposition of local fields over the cell and application of the divergence theorem allow to relate macroscopic and microscopic quantities. Typically, for the macroscopic stress, one has ($\mathrm{d}x$ the length element):

$$\boldsymbol{\Sigma} = \langle \boldsymbol{\sigma} \rangle_{\overline{\mathcal{A}}} + \frac{\mathcal{N}}{2} \int_\omega (\mathbf{T}(\mathbf{x}, \mathbf{v}(\mathbf{x})) \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{T}(\mathbf{x}, \mathbf{v}(\mathbf{x}))) \mathrm{d}x = \langle \boldsymbol{\sigma} \rangle_{\overline{\mathcal{A}}} \tag{8}$$

which appears to be trivial for open microcracks (for which $\mathbf{T}(\mathbf{x}, \mathbf{v}(\mathbf{x})) = 0$ on any point $\mathbf{x} \in \omega$) but is also satisfied for closed microcracks since $\mathbf{T}(\mathbf{x} \in \omega^+, \mathbf{v}(\mathbf{x})) = \boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{n} = -\mathbf{T}(\mathbf{x} \in \omega^-, \mathbf{v}(\mathbf{x})) = \boldsymbol{\sigma}(\mathbf{x}) \cdot (-\mathbf{n})$ for any point $\mathbf{x} \in \omega$. On the other hand, the macroscopic strain $\mathbf{E}$ reads (Hashin, 1988):

$$\begin{aligned} \mathbf{E} &= \langle \varepsilon \rangle_{\overline{\mathcal{A}}} + \frac{\mathcal{N}}{2} \int_\omega (\mathbf{u}(\mathbf{x}) \otimes \mathbf{v}(\mathbf{x}) + \mathbf{v}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x})) \ \mathrm{d}x \\ &= \langle \varepsilon \rangle_{\overline{\mathcal{A}}} + \frac{\mathcal{N}}{2} \int_{\omega^+} ([\![\mathbf{u}(\mathbf{x})]\!] \otimes \mathbf{n} + \mathbf{n} \otimes [\![\mathbf{u}(\mathbf{x})]\!]) \mathrm{d}x \end{aligned} \tag{9}$$

The average strain field on the solid part $\langle \varepsilon \rangle_{\overline{\mathcal{A}}}$ is therefore not sufficient to describe $\mathbf{E}$, the contribution of displacements jump on the cracks must be taken into account in its expression. At last, the macroscopic free energy of the material is a finite quantity exclusively defined on the matrix part of the material, that is:

$$W = \frac{1}{2} \langle \varepsilon : \mathbb{C}^0 : \varepsilon \rangle_{\overline{\mathcal{A}}} \tag{10}$$

Again, it can be shown that for cracked media, W is given by (see for instance Telega, 1990):

$$W = \frac{1}{2} \int_{\partial \overline{\mathcal{A}}} \mathbf{u}(\mathbf{x}) \cdot \boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \ \mathrm{d}x = \frac{1}{2} \boldsymbol{\Sigma} : \mathbf{E} - \frac{\mathcal{N}}{2} \int_{\omega^+} [\![\mathbf{u}(\mathbf{x})]\!] \cdot \boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{n} \mathrm{d}x \tag{11}$$

with $\partial \overline{\mathcal{A}} = \partial \mathcal{A} \cup \omega$ the boundary of the solid matrix (according to the integration domain, one should consider for $\mathbf{v}(\mathbf{x})$ in (11) the outward unit normal to $\overline{\mathcal{A}}$, that is $\mathbf{v}(\mathbf{x}) = -\mathbf{n}$ for ω^+ and $\mathbf{v}(\mathbf{x}) = \mathbf{n}$ for ω^-). For open microcracks (load free), expression (11) reduces to the elastic energy $(1/2)\boldsymbol{\Sigma}:\mathbf{E}$ corresponding to the classical form of the Hill lemma for continuous media. In what follows (Section 3.4), the importance of the second term of (11) will be emphasized, in particular for the case of closed microcracks embedded in the orthotropic matrix.

Uniform boundary conditions applied on the boundary $\partial \mathcal{A}$ can be given either by stresses or displacements. In the present case, we consider stress boundary conditions on any point $\mathbf{x}$ of $\partial \mathcal{A}$ with outward unit normal $\mathbf{v}$, namely $\boldsymbol{\sigma}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = \boldsymbol{\Sigma} \cdot \mathbf{v}(\mathbf{x})$, $\forall \mathbf{x} \in \partial \mathcal{A}$, in order to take advantage in what follows of fracture-mechanics based solutions.

Since main definitions and boundary conditions are clearly put forward, the objective is now to derive various local fields involved and to determine the effective microcracks contribution. In this way, similarly to Andrieux et al. (1986), the problem $\mathcal{P}$ is decomposed into two sub-problems, as shown in Fig. 2:

- in the sub-problem $\mathcal{P}^{(1)}$, the displacement field $\mathbf{u}^{(1)}$ corresponds to that of the homogeneous virgin material subjected to uniform stress conditions; accordingly the related local stress $\boldsymbol{\sigma}^{(1)}$ and strain $\varepsilon^{(1)}$ fields are uniform and must comply with the average stress rule $\langle \boldsymbol{\sigma}^{(1)} \rangle_{\mathcal{A}} = \boldsymbol{\Sigma}$ and $\mathbf{E}^{(1)} = \langle \varepsilon^{(1)} \rangle_{\mathcal{A}} = [\mathbb{C}^0]^{-1} : \boldsymbol{\Sigma}$ from (6),

- for the sub-problem $\mathcal{P}^{(2)}$, the displacement field $\mathbf{u}^{(2)}$ is induced by the displacement jump $[\mathbf{u}]$ between the crack faces; the related local stress $\boldsymbol{\sigma}^{(2)}$ is in this case self-equilibrated, i.e. $\langle\boldsymbol{\sigma}^{(2)}\rangle_{\overline{\mathcal{A}}} = 0$ from (8); on the other hand, since $\langle\varepsilon^{(2)}\rangle_{\overline{\mathcal{A}}} = [\mathbb{C}^0]^{-1} : \langle\boldsymbol{\sigma}^{(2)}\rangle_{\overline{\mathcal{A}}} = 0$, the macroscopic strain reads from (9):

$$\mathbf{E}^{(2)} = \frac{\mathcal{N}}{2}\int_{\omega^+}(\llbracket\mathbf{u}(\mathbf{x})\rrbracket\otimes\mathbf{n} + \mathbf{n}\otimes\llbracket\mathbf{u}(\mathbf{x})\rrbracket)\mathrm{d}x \tag{12}$$

Introducing two scalar variables β and γ related to the normal $\llbracket u_n(\mathbf{x})\rrbracket = \llbracket\mathbf{u}(\mathbf{x})\rrbracket\cdot\mathbf{n}$ and tangential $\llbracket u_t(\mathbf{x})\rrbracket = \llbracket\mathbf{u}(\mathbf{x})\rrbracket\cdot\mathbf{t}$ average displacement jump components on the cracks faces:

$$\beta = \mathcal{N}\int_{\omega^+}\llbracket u_n(\mathbf{x})\rrbracket\mathrm{d}x, \quad \gamma = \mathcal{N}\int_{\omega^+}\llbracket u_t(\mathbf{x})\rrbracket\mathrm{d}x \tag{13}$$

the macroscopic strain in $\mathcal{P}^{(2)}$ can be expressed as follows:

$$\mathbf{E}^{(2)} = \beta\mathbf{n}\otimes\mathbf{n} + \frac{\gamma}{2}(\mathbf{n}\otimes\mathbf{t} + \mathbf{t}\otimes\mathbf{n}) \tag{14}$$

Due to the above decomposition, the overall strain of the problem $\mathcal{P}$ is then given by

$$\mathbf{E} = \mathbf{E}^{(1)} + \mathbf{E}^{(2)} \tag{15}$$

On the other hand, the overall free energy per unit surface W defined by (10) with $\varepsilon = \varepsilon(\mathbf{u}^{(1)} + \mathbf{u}^{(2)})$ can be expressed as the sum of two contributions (Andrieux et al., 1986):

$$W = \frac{1}{2}\left\langle\left(\varepsilon^{(1)}+\varepsilon^{(2)}\right):\mathbb{C}^0:\left(\varepsilon^{(1)}+\varepsilon^{(2)}\right)\right\rangle_{\overline{\mathcal{A}}} = W^{(1)} + W^{(2)} \tag{16}$$

for which have been taken into account the uniformity of $\boldsymbol{\varepsilon}^{(1)}$ and the property $\langle\varepsilon^{(2)}\rangle_{\overline{\mathcal{A}}} = 0$. $W^{(1)}$ is the free energy of the virgin material related to the problem $\mathcal{P}^{(1)}$:

$$W^{(1)} = \frac{1}{2}\left\langle\varepsilon^{(1)} : \mathbb{C}^0 : \varepsilon^{(1)}\right\rangle_{\mathcal{A}} = \frac{1}{2}\mathbf{E}^{(1)} : \mathbb{C}^0 : \mathbf{E}^{(1)} \tag{17}$$

and $W^{(2)}$ corresponds to the contribution of displacement discontinuities in problem $\mathcal{P}^{(2)}$. It follows from (11) that:

$$\begin{aligned} W^{(2)} &= \frac{1}{2}\left\langle\varepsilon^{(2)} : \mathbb{C}^0 : \varepsilon^{(2)}\right\rangle_{\overline{\mathcal{A}}} = -\frac{\mathcal{N}}{2}\int_{\omega^+}\llbracket\mathbf{u}(\mathbf{x})\rrbracket\cdot\boldsymbol{\sigma}^{(2)}(\mathbf{x})\cdot\mathbf{n}\mathrm{d}x \\ &= -\frac{\mathcal{N}}{2}\int_{\omega^+}\left(\llbracket u_n(\mathbf{x})\rrbracket\mathbf{n}\cdot\boldsymbol{\sigma}^{(2)}(\mathbf{x})\cdot\mathbf{n} + \llbracket u_t(\mathbf{x})\rrbracket\mathbf{n}\cdot\boldsymbol{\sigma}^{(2)}(\mathbf{x})\cdot\mathbf{t}\right)\mathrm{d}x \\ &= -\frac{1}{2}\left(\beta\mathbf{n}\cdot\boldsymbol{\sigma}^{(2)}\cdot\mathbf{n} + \gamma\mathbf{n}\cdot\boldsymbol{\sigma}^{(2)}\cdot\mathbf{t}\right) \end{aligned} \tag{18}$$

for which use has been done of the fact that the microscopic stress $\boldsymbol{\sigma}^{(2)}$ remains constant on ω under the assumption of small microcracks density ($\boldsymbol{\sigma}^{(2)}(\mathbf{x}) = \boldsymbol{\sigma}^{(2)}, \forall\mathbf{x}\in\omega$). Note that these general results (Eq. (16)–(18)) are valid for any linear elastic solid matrix and may be particularly useful in the case of the orthotropic matrix considered in the study.

3. Free energy of an orthotropic medium weakened by arbitrarily oriented microcracks

3.1. Expression of the free energy as function of microcracks displacements jump

The assumption of microcracks dilute concentration allows to apply basic solutions of anisotropic elasticity theory to derive expressions of the crack displacements jump and then the free energy. Such determination of the displacements solutions has been investigated through complex potential theory. For any point x along an arbitrarily oriented crack, the displacement jump takes the form (Lekhnitskii, 1963; Horii and Nemat-Nasser, 1983; Mauge and Kachanov, 1994 or Tsukrov and Kachanov, 2000):

$$\llbracket\mathbf{u}(x)\rrbracket = \frac{4}{\pi}\sqrt{l^2 - x^2}\mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{H}, \quad \forall x\in[-l, +l] \tag{19}$$

Expression (19) is based on classical fracture mechanics solutions; sign $(-)$ is introduced to be consistent with the stress field $\boldsymbol{\sigma}^{(2)}$ considered in sub-problem $\mathcal{P}^{(2)}$ that should ensure the decomposition $\mathcal{P} = \mathcal{P}^{(1)} + \mathcal{P}^{(2)}$ under frictionless conditions. The symmetric second-order tensor $\mathbf{H}$ takes the following form in the crack coordinate system $(\mathbf{n}, \mathbf{t})$:

$$\mathbf{H} = H_{nn}\mathbf{n}\otimes\mathbf{n} + H_{nt}(\mathbf{n}\otimes\mathbf{t} + \mathbf{t}\otimes\mathbf{n}) + H_{tt}\mathbf{t}\otimes\mathbf{t} \tag{20}$$

whose components depend on the virgin matrix properties and on the crack orientation with initial symmetry axes (angle $\phi = (\mathbf{e}_1, \mathbf{n})$, Fig. 3):

$$H_{nn} = C(1 - D\cos 2\phi),\ H_{nt} = CD\sin 2\phi,\ H_{tt} = C(1 + D\cos 2\phi) \tag{21}$$

with scalars C and D related to the initial stiffness components:

$$\begin{aligned} C &= \frac{\pi}{4}\frac{\sqrt{\mathbb{C}^0_{1111}} + \sqrt{\mathbb{C}^0_{2222}}}{\sqrt{\mathbb{C}^0_{1111}\mathbb{C}^0_{2222} - (\mathbb{C}^0_{1122})^2}} \\ &\quad\times\sqrt{\frac{1}{\mathbb{C}^0_{1212}} + 2\left(\frac{\sqrt{\mathbb{C}^0_{1111}\mathbb{C}^0_{2222}} - \mathbb{C}^0_{1122}}{\mathbb{C}^0_{1111}\mathbb{C}^0_{2222} - (\mathbb{C}^0_{1122})^2}\right)} \\ D &= \frac{\sqrt{\mathbb{C}^0_{1111}} - \sqrt{\mathbb{C}^0_{2222}}}{\sqrt{\mathbb{C}^0_{1111}} + \sqrt{\mathbb{C}^0_{2222}}} \end{aligned} \tag{22}$$

Note that the above constants characterize the amount of the structural anisotropy. In particular, for cubic symmetry or isotropy (for which $\mathbb{C}^0_{1111} = \mathbb{C}^0_{2222}$), one gets $D = 0$.

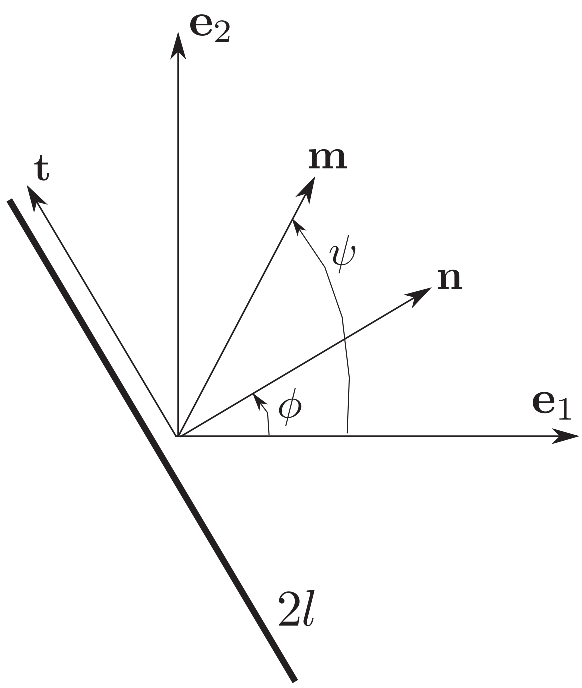

Fig. 3. Geometrical representation of the crack coordinate system.

Introduction of (20) within (19) leads to the following expressions of the normal and tangential components of the displacement jump $[\![\mathbf{u}(\mathbf{x})]\!]$ between the crack faces:

$$
\begin{aligned}
[\![u_n(x)]\!] &= \frac{4}{\pi}\sqrt{l^2-x^2}\left[H_{nn}\mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{n}+H_{nt}\mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{t}\right] \\
[\![u_t(x)]\!] &= \frac{4}{\pi}\sqrt{l^2-x^2}\left[H_{nt}\mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{n}+H_{tt}\mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{t}\right], \\
&\forall x \in [-l,+l]
\end{aligned}
\tag{23}
$$

An important remark concerns the existence of a coupling between cracks mode related to the anisotropic context. Indeed, for the two following cases:

- the virgin material is such that $\mathbb{C}^0_{1111} \neq \mathbb{C}^0_{2222}$ (so $D \neq 0$),
- the crack orientation does not coincide with the orthotropy axes ($\mathbf{n} \neq \{\mathbf{e}_1, \mathbf{e}_2\}$ so $\phi \neq \{0, (\pi/2)\}$),

a direct coupling between the normal and tangential cracking modes is noted since tensor $\mathbf{H}$ is a non diagonal one ($H_{nt} \neq 0$): a normal (resp. tangential) stress $\mathbf{n}\cdot\boldsymbol{\sigma}^{(2)}\cdot\mathbf{n} \neq 0$ (resp. $\mathbf{n}\cdot\boldsymbol{\sigma}^{(2)}\cdot\mathbf{t} \neq 0$) on the crack faces induces both normal ($[\![u_n(x)]\!] \neq 0$) and tangential ($[\![u_t(x)]\!] \neq 0$) crack opening displacements. On the other hand, as already noted by Mauge and Kachanov (1994) and Tsukrov and Kachanov (2000), the above mentioned coupling between cracks modes obviously disappears

- for cubic or isotropic cases (so $D = 0$),
- or, if the crack is parallel to one of the orthotropy axes ($\mathbf{n} = \{\mathbf{e}_1, \mathbf{e}_2\}$ so $\phi = \{0, (\pi/2)\}$),

since both situations induce a diagonal expression for $\mathbf{H}$ (see Eq. (21)).

Integrating (23) along the crack faces gives rise to the expressions of variables β and γ as function of the local stress:

$$
\begin{aligned}
\beta &= 2d\left[H_{nn}\mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{n}+H_{nt}\mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{t}\right] \\
\gamma &= 2d\left[H_{nt}\mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{n}+H_{tt}\mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{t}\right]
\end{aligned}
\tag{24}
$$

with $d = \mathcal{N} l^2$ the crack density parameter (Bristow, 1960; see also Budiansky and O'Connel, 1976). This corresponds to results by Mauge and Kachanov (1994) and Tsukrov and Kachanov (2000). In other terms, the normal and tangential local stresses on the crack depend as follows on β and α:

$$
\mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{n} = \frac{1}{2d}[G_{11}\beta+G_{12}\gamma], \quad \mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{t} = \frac{1}{2d}[G_{21}\beta+G_{22}\gamma]
\tag{25}
$$

with

$$
\begin{aligned}
G_{11} &= \frac{H_{tt}}{H_{nn}H_{tt}-H_{nt}^2}, \quad G_{12} = -\frac{H_{nt}}{H_{nn}H_{tt}-H_{nt}^2} = G_{21}, \\
G_{22} &= \frac{H_{nn}}{H_{nn}H_{tt}-H_{nt}^2}
\end{aligned}
\tag{26}
$$

Taking account (25) in (18) gives

$$
W^{(2)} = \frac{1}{4d}\left[G_{11}\beta^2+2G_{12}\beta\gamma+G_{22}\gamma^2\right]
\tag{27}
$$

and reporting in (16) together with (17) leads to the following expression of the overall free energy of the orthotropic material weakened by the considered microcracks system as function of the displacements jump variables:

$$
W = \frac{1}{2}\mathbf{E}^{(1)}:\mathbb{C}^0:\mathbf{E}^{(1)}+\frac{1}{4d}\left[G_{11}\beta^2+2G_{12}\beta\gamma+G_{22}\gamma^2\right]
\tag{28}
$$

Since crack opening displacements explicitly appear in (28), this free energy expression accounts for both open and closed microcracks. Analysis of microcracks closure effects which constitutes the main originality of the present study will be examined later in Subsection 3.3 and fully exploited in Section 4.

3.2. Expression of displacements jump variables as function of macroscopic quantities

The free energy of the material describes the elastic behavior of the microcracked material for a given density d. In this way, we intend to express W as a function of the macroscopic strain $\mathbf{E}$ uniquely, the microcrack density parameter d being fixed($\dot{d} = 0$). To this end, one may express β and γ as function of $\mathbf{E}$. Following Andrieux et al. (1986) and Pensée et al. (2002), we propose an analysis of reversible conditions for which the dissipation $\mathcal{D}$ must cancell:

$$
\mathcal{D} = \boldsymbol{\Sigma}:\dot{\mathbf{E}}-\dot{W} = 0
\tag{29}
$$

Combining $\mathcal{P}^{(1)}$ (in which $\boldsymbol{\Sigma} = \mathbb{C}^0:\mathbf{E}^{(1)}$) with Eqs. (14) and (15) yields to the expression of the macroscopic stress as a function of variables $\mathbf{E}$, β and γ:

$$
\boldsymbol{\Sigma} = \mathbb{C}^0:\left(\mathbf{E}-\beta\mathbf{n}\otimes\mathbf{n}-\frac{\gamma}{2}(\mathbf{n}\otimes\mathbf{t}+\mathbf{t}\otimes\mathbf{n})\right)
\tag{30}
$$

Such remark is valid also for the free energy from relation (28). In this latter case, the time derivation of W then leads to:

$$
\mathcal{D} = -\frac{\partial W}{\partial\beta}\dot{\beta}-\frac{\partial W}{\partial\gamma}\dot{\gamma} = 0, \quad \forall\left(\dot{\beta},\dot{\gamma}\right)
\tag{31}
$$

for open or for frictionless closed microcracks as considered in the whole study. It follows that:

$$
\begin{cases}
\dfrac{\partial W}{\partial\beta} = 0 \\
\dfrac{\partial W}{\partial\gamma} = 0
\end{cases}
\Leftrightarrow
\begin{cases}
\mathbf{n}\cdot\boldsymbol{\Sigma}\cdot\mathbf{n} = \dfrac{1}{2d}[G_{11}\beta+G_{12}\gamma] = \mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{n} \\
\mathbf{n}\cdot\boldsymbol{\Sigma}\cdot\mathbf{t} = \dfrac{1}{2d}[G_{12}\beta+G_{22}\gamma] = \mathbf{n}\cdot\left(-\boldsymbol{\sigma}^{(2)}\right)\cdot\mathbf{t}
\end{cases}
\tag{32}
$$

Accordingly, Eq. (32) clearly shows the stress transfer from macroscale to microscale, in particular on the crack faces, in agreement with the dilute concentration hypothesis. Moreover, note also that (24) relates the displacements jump variables β and γ to the overall stress $\boldsymbol{\Sigma}$:

$$
\begin{aligned}
\beta &= 2d[H_{nn}\mathbf{n}\cdot\boldsymbol{\Sigma}\cdot\mathbf{n}+H_{nt}\mathbf{n}\cdot\boldsymbol{\Sigma}\cdot\mathbf{t}], \\
\gamma &= 2d[H_{nt}\mathbf{n}\cdot\boldsymbol{\Sigma}\cdot\mathbf{n}+H_{tt}\mathbf{n}\cdot\boldsymbol{\Sigma}\cdot\mathbf{t}]
\end{aligned}
\tag{33}
$$

At this stage, the derivation of a closed-form expression of the macroscopic free energy as function of the macroscopic strain $\mathbf{E}$ requires to express β and γ as function of $\mathbf{E}$ (and d) instead of $\boldsymbol{\Sigma}$. This is done by taking advantage of the dilute concentration assumption for which one can make the approximation $\boldsymbol{\Sigma} \approx \mathbb{C}^0:\mathbf{E}$ (or equivalently to neglect second-order terms in d). So that, introducing Eq. (30) within (33), one has the following approximate expressions:

$$\beta = 2d[H_{nn}\mathbf{N}:\mathbf{E}+H_{nt}\mathbf{T}:\mathbf{E}], \quad \gamma = 2d[H_{nt}\mathbf{N}:\mathbf{E}+H_{tt}\mathbf{T}:\mathbf{E}] \tag{34}$$

where $\mathbf{N}$ and $\mathbf{T}$ denote second order symmetric tensors defined by:

$$\begin{aligned}\mathbf{N} &= \mathbb{C}^0:\mathbf{n}\otimes\mathbf{n}\\ &= a_1\mathbf{I}+a_2\mathbf{n}\otimes\mathbf{n}+a_3(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})\mathbf{A}+a_4[(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})\mathbf{I}+\mathbf{A}]\\ \mathbf{T} &= \mathbb{C}^0:\frac{1}{2}(\mathbf{n}\otimes\mathbf{t}+\mathbf{t}\otimes\mathbf{n})\\ &= \frac{a_2}{2}(\mathbf{n}\otimes\mathbf{t}+\mathbf{t}\otimes\mathbf{n})+a_3(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{t})\mathbf{A}+a_4(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{t})\mathbf{I}\end{aligned} \tag{35}$$

Equation (34) appear then as strain localization relations which determine $\mathbf{E}^{(2)}$ as a function of $\mathbf{E}$ from (14).

3.3. Opening–closure criterion

These above expressions of the average crack displacements jump allow to distinguish the microcrack status. Indeed, in accordance with unilateral conditions (4), microcracks with normal $\mathbf{n}$ are:

- open when the normal average crack displacement is strictly positive ($\beta > 0$),
- frictionless closed when such component comes to zero ($\beta = 0$),

whatever the tangential value γ. In view of previous developments, we can thus introduce a strain-based opening–closure criterion in the context of orthotropic media weakened by arbitrarily oriented cracks.

Considering expressions (34), the following opening–closure characteristic function is defined:

$$g(\mathbf{E},\mathbf{n}) = H_{nn}\mathbf{N}:\mathbf{E}+H_{nt}\mathbf{T}:\mathbf{E} = \mathbf{M}:\mathbf{E} \tag{36}$$

in which $\mathbf{M}$ is a symmetric second-order tensor which reads:

$$\mathbf{M} = H_{nn}\mathbf{N}+H_{nt}\mathbf{T} \tag{37}$$

For each set of parallel cracks set with normal $\mathbf{n}$, the transition from crack opening to crack closure occurs when $g(\mathbf{E}, \mathbf{n})$ cancels, such that:

- microcracks are open if $g(\mathbf{E}, \mathbf{n}) > 0$,
- microcracks are closed if $g(\mathbf{E}, \mathbf{n}) \le 0$.

It must be emphasized that, even for isotropic matrix or crack orientation coinciding with the orthotropy axes, the formulation (36) involves both normal and shear strain on the crack faces. Indeed, for an isotropic matrix such that $\mathbb{C}^0 = \lambda_0\mathbf{I}\otimes\mathbf{I}+2\mu_0\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$ (with λ_0 and μ_0 the Lamé constants), Eq. (36) reads $g(\mathbf{E},\mathbf{n}) = 2\mu_0 tr(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n})+\lambda_0 tr\,\mathbf{E}$, which has been obtained by Andrieux et al. (1986) and Pensée et al. (2002). In this way, it appears as much more general than the phenomenological criterion $\mathbf{n}\cdot\mathbf{E}\cdot\mathbf{n} = 0$ employed by different authors and restricted to the normal strain (for instance Chaboche et al., 1996; Chaboche and Maire, 2001, 2002; Halm et al., 2002; Maire and Chaboche, 1997; Maire and Lesne, 1998).

3.4. Importance of the Hill lemma for anisotropic cracked media

For continuous media (without cavities), the Hill lemma allows to express the work equivalence between the micro and the macro-scales over the cell $\mathcal{A}$ (Christensen, 1979; Hashin, 1983; Hazanov, 1998; Hill, 1965):

$$\langle\boldsymbol{\sigma}\rangle_{\mathcal{A}}:\langle\varepsilon\rangle_{\mathcal{A}} = \langle\boldsymbol{\sigma}:\varepsilon\rangle_{\mathcal{A}} \tag{38}$$

In such a case, this leads to the correspondence between the average of the free energy over $\mathcal{A}$ and the macroscopic elastic energy ($W = (1/2)\boldsymbol{\Sigma}:\mathbf{E}$). As underlined by Eq. (11), the context of cracked bodies requires an extended formulation of the Hill lemma (see for instance discussions by Suquet, 1982 or Telega, 1990). For illustration purpose, let us compute in the present case of orthotropic matrix with arbitrarily oriented microcracks the amount of the following quantity:

$$\Delta = W-\frac{1}{2}\boldsymbol{\Sigma}:\mathbf{E} = -\frac{\mathcal{N}}{2}\int_{\omega^+}[\![\mathbf{u}(\mathbf{x})]\!]\cdot\boldsymbol{\sigma}(\mathbf{x})\cdot\mathbf{n}dx \tag{39}$$

According to the decomposition of the problem $\mathcal{P}$, it is readily seen that:

$$\Delta = -\frac{\mathcal{N}}{2}\int_{\omega^+}[\![\mathbf{u}(\mathbf{x})]\!]\cdot\left(\boldsymbol{\sigma}^{(1)}(\mathbf{x})+\boldsymbol{\sigma}^{(2)}(\mathbf{x})\right)\cdot\mathbf{n}dx \tag{40}$$

where the displacements jump vector $[\![\mathbf{u}(\mathbf{x})]\!]$ for any point $\mathbf{x}$ on the crack corresponds to sub-problem $\mathcal{P}^{(2)}$ (no discontinuity in $\mathcal{P}^{(1)}$) and the stress field $\boldsymbol{\sigma}^{(1)}$ of sub-problem $\mathcal{P}^{(1)}$ is homogeneous so that $\boldsymbol{\sigma}^{(1)}(\mathbf{x}) = \boldsymbol{\Sigma}$, $\forall\mathbf{x}\in\mathcal{A}$. By introducing the already defined variables β and γ, it comes:

$$\Delta = -\frac{1}{2}\boldsymbol{\Sigma}:\left[\beta\mathbf{n}\otimes\mathbf{n}+\frac{\gamma}{2}(\mathbf{n}\otimes\mathbf{t}+\mathbf{t}\otimes\mathbf{n})\right]+W^{(2)} \tag{41}$$

with $W^{(2)}$ given by (27). Moreover, taking into account the approximation $\boldsymbol{\Sigma}\approx\mathbb{C}^0:\mathbf{E}$ (justified by the assumption of microcracks dilute concentration) and the expressions of $\mathbf{N}$ and $\mathbf{T}$ defined by (35), one gets finally:

$$\Delta = -\frac{\beta}{2}\mathbf{N}:\mathbf{E}-\frac{\gamma}{2}\mathbf{T}:\mathbf{E}+\frac{1}{4d}\left[G_{11}\beta^2+2G_{12}\beta\gamma+G_{22}\gamma^2\right] \tag{42}$$

By the help of (26) and of expressions (34) of β and γ, and taking into account the unilateral conditions, one can express the quantity Δ according to the microcracks opening/closure status:

- if microcracks are open (i.e. $g(\mathbf{E},\mathbf{n}) > 0$):

$$\Delta = 0 \tag{43}$$

- if microcracks are closed (i.e. $g(\mathbf{E},\mathbf{n}) \le 0$):

$$\begin{aligned}\Delta &= \frac{\gamma H_{nt}}{2\left(H_{nn}H_{tt}-H_{nt}^2\right)}(H_{nn}\mathbf{N}:\mathbf{E}+H_{nt}\mathbf{T}:\mathbf{E})\\ &= \frac{\gamma H_{nt}}{2\left(H_{nn}H_{tt}-H_{nt}^2\right)}g(\mathbf{E},\mathbf{n})\end{aligned} \tag{44}$$

In the open state of microcracks, Δ vanishes systematically; indeed, as mentioned before, microcracks surfaces are in this case free of traction and the classical form $W = (1/2)\boldsymbol{\Sigma}:\mathbf{E}$ of the Hill lemma is obviously still valid. In contrast, if the microcracks are closed, the only situations for which $\Delta = 0$ are the following ones such, that $H_{nt} = 0$ (see Section 3.1 for the definition of H_{nt}):

- the matrix has a cubic symmetry or is isotropic; this result therefore justifies the approach of Kachanov (1993) (consisting of computing the overall free energy by means of

$W = (1/2)\boldsymbol{\Sigma} : \mathbf{E}$) for isotropic materials but could not be used in the case of orthotropic matrix,
- microcracks are parallel to the orthotropic axes.

In the most general case of an orthotropic medium weakened by microcracks having arbitrary orientation, (44) readily proves that the free energy differs from $(1/2)\boldsymbol{\Sigma} : \mathbf{E}$, pointing out the importance of the extended form (11) of the Hill lemma for microcracked media.

4. Final expression of the overall energy

By combining (14 and 15) and (28) together with the assumption of microcracks dilute concentration, one obtains the overall free energy:

$$W = W^{(1)} + W^{(2)} = W_0 + W_d \tag{45}$$

with W_0 the free energy of the virgin material

$$\begin{aligned} W_0 &= \frac{1}{2}\mathbf{E} : \mathbb{C}^0 : \mathbf{E} \\ &= \frac{a_1}{2}tr^2\mathbf{E} + \frac{a_2}{2}tr(\mathbf{E}\cdot\mathbf{E}) + \frac{a_3}{2}tr^2(\mathbf{E}\cdot\mathbf{A}) + a_4 tr\,\mathbf{E}\, tr(\mathbf{E}\cdot\mathbf{A}) \end{aligned} \tag{46}$$

and W_d the contribution of microcracks:

$$\begin{aligned} W_d &= -\mathbf{E} : \mathbb{C}^0 : \mathbf{E}^{(2)} + W^{(2)} = -\beta\mathbf{N} : \mathbf{E} \\ &\quad - \gamma\mathbf{T} : \mathbf{E} + \frac{1}{4d}\left[G_{11}\beta^2 + 2G_{12}\beta\gamma + G_{22}\gamma^2\right] \end{aligned} \tag{47}$$

From (26) and (34) and introducing unilateral conditions, the free energy can be written for both microcracks states, namely:

- if cracks are open ($g(\mathbf{E}, \mathbf{n}) > 0$):

$$W = W^{open} = W_0 - d\left[H_{nn}(\mathbf{N}:\mathbf{E})^2 + 2H_{nt}(\mathbf{N}:\mathbf{E})(\mathbf{T}:\mathbf{E}) + H_{tt}(\mathbf{T}:\mathbf{E})^2\right] \tag{48}$$

- if cracks are closed ($g(\mathbf{E}, \mathbf{n}) \le 0$):

$$\begin{aligned} W = W^{\text{clos}} &= W_0 + \frac{d}{H_{nn}H_{tt} - H_{nt}^2}\Big[H_{nn}H_{nt}^2(\mathbf{N}:\mathbf{E})^2 \\ &\quad + 2H_{nt}^3(\mathbf{N}:\mathbf{E})(\mathbf{T}:\mathbf{E}) + H_{tt}\left(2H_{nt}^2 - H_{nn}H_{tt}\right)(\mathbf{T}:\mathbf{E})^2\Big] \end{aligned} \tag{49}$$

It is interesting to emphasize that, for any given orthotropic matrix described by the stiffness $\mathbb{C}^0$ and any microcrack density parameter d, expressions (48 and 49) remarkably satisfy the following form at the transition between the microcracks opening and closure states:

$$\begin{aligned} \forall(\mathbf{E},\mathbf{n}), g(\mathbf{E},\mathbf{n}) = 0, \quad \frac{\partial^2\left[W^{\text{open}} - W^{\text{clos}}\right]}{\partial\mathbf{E}^2} &= s\left(\mathbb{C}^0, d, \mathbf{n}\right)\,\mathbf{M}\otimes\mathbf{M} \\ &= s\left(\mathbb{C}^0, d, \mathbf{n}\right)\,\frac{\partial g}{\partial\mathbf{E}}\otimes\frac{\partial g}{\partial\mathbf{E}} \end{aligned} \tag{50}$$

with $\mathbf{M}$ the second-order tensor expressed in Eq. (37) and s a scalar function of the solid matrix elastic properties and of the crack density parameter d as well as cracks orientation:

$$s\left(\mathbb{C}^0, d, \mathbf{n}\right) = -2dG_{11} = -2d\frac{H_{tt}}{H_{nn}H_{tt} - H_{nt}^2} \tag{51}$$

As demonstrated by Curnier et al. (1995) and then by Cormery and Welemane (2010), the quadratic form (50) ensures in all cases that the free energy function W is continuously differentiable, or equivalently that the stress-strain response is continuous at the microcrack deactivation point, that is on the hypersurface $g(\mathbf{E}, \mathbf{n}) = \mathbf{M}{:}\mathbf{E} = 0$ between the open ($g(\mathbf{E}, \mathbf{n}) > 0$) and closed ($g(\mathbf{E}, \mathbf{n}) > 0$) strain-based domains.

In order to get more explicit expressions that put in evidence the coupling between primary anisotropy (through tensor $\mathbf{A}$) and crack-induced one (through vector $\mathbf{n}$), we introduce the notations $k_1 = C$ and $k_2 = CD$, so that components of tensor $\mathbf{H}$ in the crack coordinate system ($\mathbf{n}$, $\mathbf{t}$) come to:

$$\begin{aligned} H_{nn} &= k_1 + k_2 - 2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}),\ H_{nt} = -2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{t}),\ H_{tt} \\ &= k_1 - k_2 + 2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}) \end{aligned} \tag{52}$$

From this, one can first express the opening–closure characteristic function as a function of $\mathbf{E}$, $\mathbf{n}$ and $\mathbf{A}$:

$$\begin{aligned} g(\mathbf{E},\mathbf{n}) = g(\mathbf{E},\mathbf{n},\mathbf{A}) &= \eta_1 tr(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}) + \eta_2 tr\mathbf{E} + \eta_3 tr(\mathbf{E}\cdot\mathbf{A}) \\ &\quad + \eta_4 tr(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}\cdot\mathbf{A}) \end{aligned} \tag{53}$$

with coefficients $\{\eta_p(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,4}$ depending on $\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}$, as described in Appendix A (Eq. (A.1)). As an example for an isotropic matrix, one recovers $\eta_1 = 2\mu_0$, $\eta_2 = \lambda_0$ and $\eta_3 = \eta_4 = 0$, as mentioned previously. The anisotropic context induces consequently an explicit coupling between structural and induced anisotropies even in the definition of the microcracks opening/closure status.

Moreover, we can deduce also the global form of the potential:

$$W = W_0 + d\begin{bmatrix} c_1 tr^2\mathbf{E} + c_2 tr^2(\mathbf{E}\cdot\mathbf{A}) + c_3 tr\,\mathbf{E}\, tr(\mathbf{E}\cdot\mathbf{A}) \\ + c_4 tr^2(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}) + c_5 tr\,\mathbf{E}\, tr(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}) + c_6 tr(\mathbf{E}\cdot\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}) \\ + c_7 tr\,\mathbf{E}\, tr(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}\cdot\mathbf{A}) + c_8 tr(\mathbf{E}\cdot\mathbf{A}) tr(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}) \\ + c_9 tr(\mathbf{E}\cdot\mathbf{A}) tr(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}\cdot\mathbf{A}) \\ + c_{10} tr(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}) tr(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}\cdot\mathbf{A}) \end{bmatrix} \tag{54}$$

This clearly demonstrates that W is positively homogeneous of degree two with respect to $\mathbf{E}$, linear in the crack density d, radially symmetric with respect to $\mathbf{n}$ and depends on the orientation $\mathbf{n}$ of the cracks with respect to symmetry axes described by tensor. Coefficients $\{c_p(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,10}$ in (54), which depend also on $\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}$, obviously take expressions depending on the microcracks status, that is (see detailed expressions in Appendix A):

- $\{c_p(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,10} = \{c_p^{\text{open}}(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,10}$ in the open state ($g(\mathbf{E}, \mathbf{n}, \mathbf{A}) > 0$, Eq. (A.2)),
- $\{c_p(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,10} = \{c_p^{\text{clos}}(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,10}$ in the closed state ($g(\mathbf{E}, \mathbf{n}, \mathbf{A}) \le 0$, Eq. (A.3)).

5. Discussion and applications

We propose in this last section to highlight some of the above results derived from the micromechanical approach. Both general considerations on the resulting anisotropy and numerical applications illustrating the effective elastic properties are presented.

5.1. Interaction between primary and microcracks-induced anisotropies

The overall anisotropy of the microcracked material results from:

- the primary orthotropy of the matrix (uncracked material),
- the microcracks-induced anisotropy governed by the orientation of the defects,

- the opening–closure state of microcracks; at fixed microcracks density, a specific anisotropy may be induced by the opening/closure configuration of the microcracks systems. This has been already demonstrated by Welemane and Goidescu (2010) in the context of initially isotropic materials with isotropic microcracks density distribution. The result for this case is contained in (74) provided coefficients $\{c_p^{(i)}(\mathbb{C}^0, \mathbf{n}_i, \mathbf{A})\}_{p=1,10}$ are defined according to the microcracks status, but will not be detailed here.

The above micromechanical derivations allow to address the crucial question of interaction between anisotropies. Indeed, expression (54) provides explicit contribution of orientational effects of the considered family of microcracks to the material overall behavior. Note that the behavior of the solid matrix is described by means of the free energy W_0 expressed in (46) through the combined invariants of the macroscopic strain $\mathbf{E}$ and structural tensor $\mathbf{A}$. On the other hand, the induced anisotropy depends on $\mathbf{n}\otimes\mathbf{n}$ and is represented within W_d through the invariants of combinations of $\mathbf{E}$, $\mathbf{A}$ and $\mathbf{n}\otimes\mathbf{n}$. Detailing such latter contribution, one could distinguish three kinds of coupling between the initial anisotropy and the microcracks-induced one:

- isotropic-like coupling, that preserves the initial orthotropy of the material (see 46); in Eq. (54), this feature is represented by following invariants of $\mathbf{E}$ and $\mathbf{A}$ (namely terms of coefficients c_1, c_2 and c_3):

$$tr^2\mathbf{E},\; tr^2(\mathbf{E}\cdot\mathbf{A}),\, tr\,\mathbf{E}\; tr(\mathbf{E}\cdot\mathbf{A}) \tag{55}$$

whose derivation leads to stiffness tensorial generators that characterize orthotropy (see later Eq. (61)):

$$\mathbf{I}\otimes\mathbf{I},\; \mathbf{A}\otimes\mathbf{A},\; \mathbf{I}\otimes\mathbf{A}+\mathbf{A}\otimes\mathbf{I} \tag{56}$$

Note that coefficients c_1, c_2 and c_3 (see Appendix A) involve both a constant part (function of the virgin properties of the material) and a scalar dependence of the microcrack orientation with the material symmetry axes (function also of $\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}$);

$$\mathbb{C} = \mathbb{C}^0 + d\begin{bmatrix} 2c_1\mathbf{I}\otimes\mathbf{I}+2c_2\mathbf{A}\otimes\mathbf{A}+c_3(\mathbf{I}\otimes\mathbf{A}+\mathbf{A}\otimes\mathbf{I}) \\ +2c_4\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}+c_5(\mathbf{I}\otimes\mathbf{n}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{I}) \\ +c_6(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{n}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{n}\,\overline{\underline{\otimes}}\,\mathbf{I}) \\ +\frac{c_7}{2}[(\mathbf{n}\otimes\mathbf{n}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{n}\otimes\mathbf{n})\otimes\mathbf{I}+\mathbf{I}\otimes(\mathbf{n}\otimes\mathbf{n}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{n}\otimes\mathbf{n})] \\ +c_8(\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{A}+\mathbf{A}\otimes\mathbf{n}\otimes\mathbf{n}) \\ +\frac{c_9}{2}[(\mathbf{n}\otimes\mathbf{n}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{n}\otimes\mathbf{n})\otimes\mathbf{A}+\mathbf{A}\otimes(\mathbf{n}\otimes\mathbf{n}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{n}\otimes\mathbf{n})] \\ +\frac{c_{10}}{2}[(\mathbf{n}\otimes\mathbf{n}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{n}\otimes\mathbf{n})\otimes\mathbf{n}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{n}\otimes(\mathbf{n}\otimes\mathbf{n}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{n}\otimes\mathbf{n})] \end{bmatrix} \tag{61}$$

- weak anisotropic coupling, that accounts for the loss of material orthotropy through a directional dependence identical to the isotropic context (see Andrieux et al., 1986); this concerns terms c_4, c_5 and c_6 related to combined invariants of $\mathbf{E}$ and $\mathbf{n}\otimes\mathbf{n}$

$$tr^2(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}),\, tr\,\mathbf{E}\; tr(\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}),\; tr(\mathbf{E}\cdot\mathbf{E}\cdot\mathbf{n}\otimes\mathbf{n}) \tag{57}$$

whose derivation leads to following stiffness tensorial generators

$$\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n},\; \mathbf{I}\otimes\mathbf{n}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{I},\; \mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{n}\otimes\mathbf{n}+\mathbf{n}\otimes\mathbf{n}\,\overline{\underline{\otimes}}\,\mathbf{I} \tag{58}$$

Still, one should note that the present case differs from isotropy since the amplitude of constants c_4, c_5 and c_6 are affected by the primary orthotropy through a scalar dependence in $\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}$ (see Appendix A);

- strong anisotropic coupling, accounted by all others terms (c_7–c_{10}), which involves combinations of orientational effects of $\mathbf{E}$, $\mathbf{A}$ and $\mathbf{n}\otimes\mathbf{n}$ and leads to a complex resulting anisotropy.

5.2. Effective moduli

Expression (54) also reveals the intricate effects in the material response of the orientation of microcracks and their individual unilateral behavior. In order to demonstrate such influence, we propose to analyze the macroscopic elastic properties of orthotropic materials containing a dilute concentration of microcracks, with account of opening–closure effects.

To this end, we consider the variations of the elongation L($\mathbf{m}$) and bulk $\kappa(\mathbf{m})$ moduli related to each direction of the space of unit vector $\mathbf{m}$ that fully characterize the elasticity of the material (He and Curnier, 1995). Consider a pure elongation test $\mathbf{E} = \bar{E}\mathbf{m}\otimes\mathbf{m}$ in the direction of unit vector $\mathbf{m}$, these moduli are defined by:

$$L(\mathbf{m}) = \frac{\mathbf{m}\cdot\boldsymbol{\Sigma}\cdot\mathbf{m}}{\mathbf{m}\cdot\mathbf{E}\cdot\mathbf{m}},\quad \kappa(\mathbf{m}) = \frac{tr\boldsymbol{\Sigma}}{tr\mathbf{E}} \tag{59}$$

with $\boldsymbol{\Sigma}$ the macroscopic stress related to the macrostrain $\mathbf{E}$ by $\boldsymbol{\Sigma} = \mathbb{C} : \mathbf{E}$, where the effective stiffness tensor of the microcracked material is given by $\mathbb{C} = \partial^2 W/\partial\mathbf{E}^2$. One has:

$$L(\mathbf{m}) = \mathbf{m}\otimes\mathbf{m} : \mathbb{C} : \mathbf{m}\otimes\mathbf{m},\; \kappa(\mathbf{m}) = \mathbf{I} : \mathbb{C} : \mathbf{m}\otimes\mathbf{m} \tag{60}$$

The derivation of the free energy (54) gives rise to the overall stiffness tensor:

By definition of multilinear functions (Curnier et al., 1995; Cormery and Welemane, 2010), such stiffness tensor exists only on strict domains of opening (i.e. $g(\mathbf{E}, \mathbf{n}, \mathbf{A}) > 0$) or closure (i.e. $g(\mathbf{E}, \mathbf{n}, \mathbf{A}) < 0$) of microcracks ($W$ is of class C^1).

Introducing angle $\psi = (\mathbf{e}_1, \mathbf{m})$ (Fig. 3), relevant calculations from Eqs. (2), (60) and (61) lead to the following expressions:

$$L(\mathbf{m}) = L(\psi,\phi) = L^0(\mathbf{m}) + 2d\begin{bmatrix} c_1 + c_2\cos^4\psi + c_3\cos^2\psi + c_4\cos^4(\psi-\phi) \\ +c_5\cos^2(\psi-\phi) + c_6\cos^2(\psi-\phi) \\ +c_7\cos\phi\cos\psi\cos(\psi-\phi) + c_8\cos^2\psi\cos^2(\psi-\phi) \\ +c_9\cos\phi\cos^3\psi\cos(\psi-\phi) \\ +c_{10}\cos\phi\cos\psi\cos^3(\psi-\phi) \end{bmatrix} \tag{62}$$

$$\kappa(\mathbf{m}) = \kappa(\psi,\phi) = \kappa^0(\mathbf{m}) + d\begin{bmatrix} 6c_1 + 2c_2cos^2\psi + c_3\left(1+3\cos^2\psi\right) + 2c_4\cos^2(\psi-\phi) \\ +c_5\left[1+3\cos^2(\psi-\phi)\right] + 2c_6\cos^2(\psi-\phi) \\ +c_7\cos\phi[\cos\phi + 3\cos\psi\cos(\psi-\phi)] \\ +c_8\left[\cos^2\psi + \cos^2(\psi-\phi)\right] \\ +c_9\cos\phi\cos\psi[\cos\phi\cos\psi + \cos(\psi-\phi)] \\ +c_{10}\cos\phi\cos(\psi-\phi)[\cos\phi\cos(\psi-\phi) + \cos\psi] \end{bmatrix} \tag{63}$$

where $L^0(\mathbf{m})$ and $\kappa^0(\mathbf{m})$ represent the elongation and bulk moduli of the virgin orthotropic matrix:

$$L^0(\mathbf{m}) = L^0(\psi) = a_1 + a_2 + 2a_4\cos^2\psi + a_3\cos^4\psi \tag{64}$$

$$\kappa^0(\mathbf{m}) = \kappa^0(\psi) = 3a_1 + a_2 + a_4 + (a_3 + 3a_4)\cos^2\psi \tag{65}$$

In order to illustrate the consequences of microcracks on these elastic properties, let consider the plane-stress context of a 2D SiC–SiC composite characterized by Aubard (1995). Coefficients $\{a_i\}_{i=1,4}$ of the stiffness tensor (2) are obtained as:

$$\begin{aligned} a_1 &= \frac{E_2}{1-\nu_{12}\nu_{21}} - 2G_{12}, a_2 = 2G_{12}, \\ a_3 &= \frac{(1-2\nu_{21})E_1 + E_2}{1-\nu_{12}\nu_{21}} - 4G_{12}, a_4 = \frac{E_2(\nu_{12}-1)}{1-\nu_{12}\nu_{21}} + 2G_{12} \end{aligned} \tag{66}$$

with the following engineering constants in the orthotropic axes: Young moduli $E_1 = 320$ GPa in the direction of $\mathbf{e}_1$ and $E_2 = 170$ GPa in the direction of $\mathbf{e}_2$, Poisson ratio $\nu_{12} = 0.18$ and shear modulus $G_{12} = 90$ GPa relative to directions $\mathbf{e}_1$ and $\mathbf{e}_2$. Note that expressions of the constants C and D given by (22) read:

$$C = \frac{\pi}{4}\frac{\sqrt{E_1}+\sqrt{E_2}}{\sqrt{E_1E_2}}\sqrt{\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1} + \frac{2}{\sqrt{E_1E_2}}}, D = \frac{\sqrt{E_1}-\sqrt{E_2}}{\sqrt{E_1}+\sqrt{E_2}} \tag{67}$$

Figures (4) and (5) show the roses (also known as direction curves) of generalized moduli $L(\mathbf{m})$ and $\kappa(\mathbf{m})$ of such material weakened by a set of parallel microcracks with density $d = 0.1$ and two different orientations of their unit normal $\mathbf{n}$; the anisotropic virgin properties are depicted as a reference.

Several important comments can be done in view of previous results. First, concerning the interaction between initial orthotropy and microcracks-induced one, Fig. (4-a) and (5-a) show that microcracking along initial symmetry axes (for example cracks with normal $\mathbf{n} = \mathbf{e}_2$ so $\phi = \pi/2$) preserves the structural orthotropy of the material (isotropic-like effect) since it induces an orthotropic perturbation in the same axes of the uncracked material. Indeed, for open or closed microcracks, equations (62) and (63) come to:

$$\begin{aligned} L(\mathbf{m}) &= L\left(\psi, \phi = \frac{\pi}{2}\right) \\ &= L^0(\psi) + 2d\begin{bmatrix} c_1 + c_4 + c_5 + c_6 \\ +(c_3 - 2c_4 - c_5 - c_6 + c_8)\cos^2\psi \\ +(c_2 + c_4 - c_8)\cos^4\psi \end{bmatrix} \end{aligned} \tag{68}$$

and

$$\begin{aligned} \kappa(\mathbf{m}) &= \kappa\left(\psi, \phi = \frac{\pi}{2}\right) \\ &= \kappa^0(\psi) + d\begin{bmatrix} 6c_1 + c_3 + 2c_4 + 4c_5 + 2c_6 + c_8 \\ +[2c_2 + 3c_3 - 2c_4 - 3c_5 - 2c_6]cos^2\psi \end{bmatrix} \end{aligned} \tag{69}$$

that are of the same form as $L^0(\mathbf{m})$ and $\kappa^0(\mathbf{m})$ given by (64) and (65). Otherwise, an arbitrary orientation of the microcracks induces an overall complex material anisotropy, as shown by Fig. (4-b) and (5-b) for $(\mathbf{e}_1, \mathbf{n}) = \phi = \pi/6$.

The question of the unilateral effect of microcracks is also examined through the analysis of the overall elastic properties (62 and 63) and in particular through the dependence of coefficients $\{c_p(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,10}$ with the microcracks status. Accordingly, in both cases (isotropic-like and anisotropic couplings), the macroscopic stiffness is strongly affected by the microcracks opening–closure state. Let denote:

$$\mathrm{L}(\mathbf{m}), \kappa(\mathbf{m}) = \begin{cases} L^{\mathrm{open}}(\mathbf{m}), \kappa^{\mathrm{open}}(\mathbf{m}), & \text{if } g(\mathbf{E}, \mathbf{n}, \mathbf{A}) > 0 \\ L^{\mathrm{clos}}(\mathbf{m}), \kappa^{\mathrm{clos}}(\mathbf{m}), & \text{if } g(\mathbf{E}, \mathbf{n}, \mathbf{A}) \le 0 \end{cases} \tag{70}$$

In the open state first, we note that elastic properties $L^{\mathrm{open}}(\mathbf{m})$ and $\kappa^{\mathrm{open}}(\mathbf{m})$ are deteriorated by the presence of cracks ($L^{\mathrm{open}}(\mathbf{m}) \le L^0(\mathbf{m})$, $\kappa^{\mathrm{open}}(\mathbf{m}) \le \kappa^0(\mathbf{m})$ whatever $\mathbf{m}$), mostly in:

- directions $\mathbf{m}$ corresponding exactly to the normal $\mathbf{n}$ to cracks when the crack is in one of the orthotropy axes, according to considerations on anisotropies interaction (Figs. (4-a) and (5-a)); this stands also in agreement with results obtained for isotropic media;

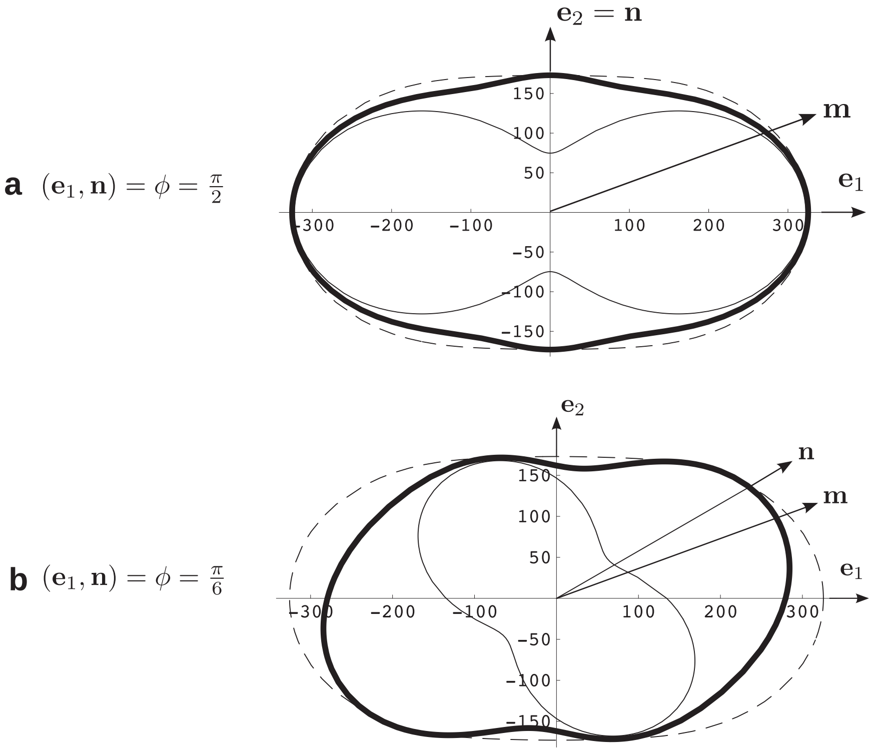

Fig. 4. Roses of the generalized elongation modulus $L(\mathbf{m})$ (in GPa) of a composite material weakened by a single array of microcracks of unit normal $\mathbf{n}$ (——: open state, ——: closed state, - - -: virgin material).

- directions $\mathbf{m}$ close to $\mathbf{n}$ for an arbitrary defect orientation (Figs. (4-b) and (5-b)); this latter case is attributed to the competition between structural and induced anisotropies and to the stronger influence of the second one as density d increases.

On the other hand, the closure of microcracks modifies their contribution; the resulting macroscopic stiffness is much more complex than in the open state, especially for an arbitrary orientation of microcracks. Indeed, we observe a global recovery of elastic properties, that is:

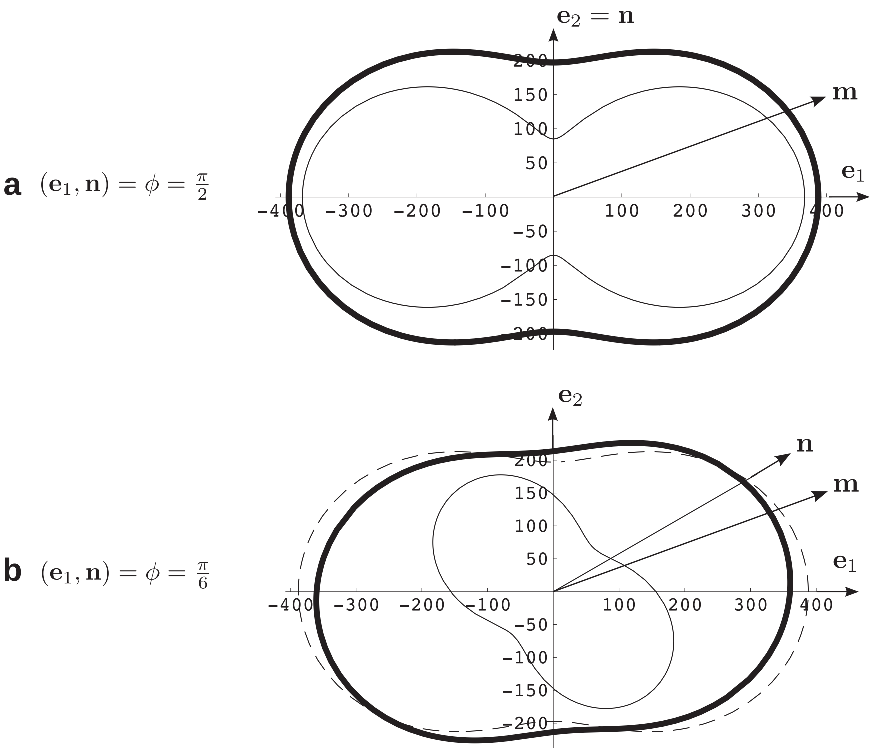

Fig. 5. Roses of the generalized bulk modulus $\kappa(\mathbf{m})$ (in GPa) of a composite material weakened by a single array of microcracks of unit normal $\mathbf{n}$ (——: open state, ——: closed state, - - -: virgin material).

$$\forall \mathbf{m}, L^{\text{open}}(\mathbf{m}) \leq L^{\text{clos}}(\mathbf{m}) \text{ and } \kappa^{\text{open}}(\mathbf{m}) \leq \kappa^{\text{clos}}(\mathbf{m}) \tag{71}$$

Microcracks closure is then associated to a partial microcrack deactivation as it exists some $\mathbf{m}$ such that $L^{\text{clos}}(\mathbf{m})$ (resp. $\kappa^{\text{clos}}(\mathbf{m})$) remains different from $L^0(\mathbf{m})$ (resp. $\kappa^0(\mathbf{m})$). Moreover, it is quite difficult to quantify in the general case microcracks closure effects since some directions $\mathbf{m}$ are such that $L^{\text{clos}}(\mathbf{m}) \leq L^0(\mathbf{m})$ (resp. $\kappa^{\text{clos}}(\mathbf{m}) \leq \kappa^0(\mathbf{m})$) and for others $L^0(\mathbf{m}) \leq L^{\text{clos}}(\mathbf{m})$ (resp. $\kappa^0(\mathbf{m}) \leq \kappa^{\text{clos}}(\mathbf{m})$). Yet, some significant unilateral effects should be highlighted:

- the full recovery of the elongation modulus $L(\mathbf{n})$ in the direction $\mathbf{n}$ normal to microcracks,
- the full recovery of the bulk modulus $\kappa(\mathbf{m})$ whatever the direction $\mathbf{m}$,

if and only if the crack orientation coincides with the orthotropy axes (see Figs. 4-a and 5-a). In other terms, one could write

$$\begin{cases} L^{\text{clos}}(\mathbf{n}) = L^0(\mathbf{n}) \\ \kappa^{\text{clos}}(\mathbf{m}) = \kappa^0(\mathbf{m}), \ \forall \mathbf{m} \end{cases}, \text{ if and only if } \mathbf{n} = \{\mathbf{e}_1, \mathbf{e}_2\} \left(\text{so } \phi = \left\{0, \frac{\pi}{2}\right\}\right) \tag{72}$$

Let consider the example described in Eqs. (68) and (69). Indeed, in this case, the following relations exist between coefficients $\{c_p^{\text{clos}}(\mathbb{C}^0, \mathbf{n} = \mathbf{e}_2, \mathbf{A})\}_{p=1,10}$:

$$\begin{aligned} c_1^{\text{clos}} + c_4^{\text{clos}} + c_5^{\text{clos}} + c_6^{\text{clos}} &= 0 \\ 6c_1^{\text{clos}} + c_3^{\text{clos}} + 2c_4^{\text{clos}} + 4c_5^{\text{clos}} + 2c_6^{\text{clos}} + c_8^{\text{clos}} &= 0 \\ 2c_2^{\text{clos}} + 3c_3^{\text{clos}} - 2c_4^{\text{clos}} - 3c_5^{\text{clos}} - 2c_6^{\text{clos}} &= 0 \end{aligned} \tag{73}$$

which leads to the cancellation of degradation terms in $L^{\text{clos}}(\mathbf{n})$ (for which $\psi = \phi = \pi/2$) and $\kappa^{\text{clos}}(\mathbf{m})$ whatever $\mathbf{m}$ (so whatever ψ). The result (72) makes then appear a recovery mode when microcracks stand along symmetry axes identical to the context of initially isotropic materials (Welemane and Cormery, 2002, 2003).

As a final comment, note that the assumption of dilute concentration of microdefects easily allows to extend the above considerations to an orthotropic matrix weakened by an array of N families of microcracks. Defects of the ith family have same unit normal $\mathbf{n}_i$ and mean length $2l_i$, $\mathcal{N}_i$ denotes the number of microcracks of this family per unit surface. Concerning the overall free energy, the generalized form of (54) is simply given by summation of the elementary contributions:

$$W = W_0 + \sum_{i=1}^{N} d_i \begin{bmatrix} c_1^{(i)} tr^2\mathbf{E} + c_2^{(i)} tr^2(\mathbf{E}\cdot\mathbf{A}) + c_3^{(i)} tr\mathbf{E}\, tr(\mathbf{E}\cdot\mathbf{A}) \\ + c_4^{(i)} tr^2(\mathbf{E}\cdot\mathbf{n}_i\otimes\mathbf{n}_i) + c_5^{(i)} tr\mathbf{E}\, tr(\mathbf{E}\cdot\mathbf{n}_i\otimes\mathbf{n}_i) \\ + c_6^{(i)} tr(\mathbf{E}\cdot\mathbf{E}\cdot\mathbf{n}_i\otimes\mathbf{n}_i) \\ + c_7^{(i)} tr\mathbf{E}\, tr(\mathbf{E}\cdot\mathbf{n}_i\otimes\mathbf{n}_i\cdot\mathbf{A}) + c_8^{(i)} tr(\mathbf{E}\cdot\mathbf{A})\, tr(\mathbf{E}\cdot\mathbf{n}_i\otimes\mathbf{n}_i) \\ + c_9^{(i)} tr(\mathbf{E}\cdot\mathbf{A})\, tr(\mathbf{E}\cdot\mathbf{n}_i\otimes\mathbf{n}_i\cdot\mathbf{A}) \\ + c_{10}^{(i)} tr(\mathbf{E}\cdot\mathbf{n}_i\otimes\mathbf{n}_i)\, tr(\mathbf{E}\cdot\mathbf{n}_i\otimes\mathbf{n}_i\cdot\mathbf{A}) \end{bmatrix} \tag{74}$$

with $d_i = \mathcal{N}_i l_i^2$ the density of the ith set of parallel microcracks and associated coefficients $\{c_p^{(i)}(\mathbb{C}^0, \mathbf{n}_i, \mathbf{A})\}_{p=1,10}$ depending on the microcracks status (open if $g(\mathbf{E}, \mathbf{n}_i, \mathbf{A}) > 0$, closed if $g(\mathbf{E}, \mathbf{n}_i, \mathbf{A}) \leq 0$). Accordingly, the recovery condition can be extended in this way: microcracks whose normal $\mathbf{n}_i$ corresponds to orthotropy axes no longer contribute in their closed state to the degradation of the elongation modulus $L(\mathbf{n}_i)$ neither to the bulk moduli $\kappa(\mathbf{m})$ whatever $\mathbf{m}$.

6. Conclusion and perspectives

This study has focused on the micromechanical determination of overall anisotropic multilinear response of 2D initially orthotropic materials weakened by microcracks under the dilute concentration assumption. These results rigorously extend the energy-based homogenization approach initially proposed by Andrieux et al. (1986) for isotropic media to the context of orthotropic materials. The micromechanical procedure leads to the derivation of original closed-form expression of the overall free energy of the microcracked material with account of closure effects for arbitrarily oriented microcracks. The consideration of such unilateral behavior constitutes the main contribution of the study. The explicit expressions obtained provide then a complete quantification of interaction effects both between primary and microcracks-induced anisotropies and between opening and closure states of cracks. Especially, the overall response highlights the recovery phenomena induced by the microcracks closure: the recovery mode is found to be identical to isotropic case when microcracks stand along symmetry axes. Illustrations of these various effects are given on the case of a SiC–SiC composite.

From these basic results, further work will now be conducted in order to develop a fully-constitutive damage model for 2D initially orthotropic materials in the framework of Continuum Damage Mechanics. The overall free energy expression derived in the present work will allow to introduce both oriented and closure effects due to microcracks in the material response and damage evolution. By this way, one can account for the dependence of such materials behavior on the loading, both on its orientation (on/off axes loadings) and on its nature (tension/compression load).

Appendix A. Expression of coefficients

This section provides detailed expressions of coefficients appearing in Eqs. (53) and (54) for a single set of microcracks with unit normal $\mathbf{n}$.

Coefficients $\{\eta_p(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,4}$ entering the opening–closure characteristic function (53) are given by:

$$\begin{aligned} \eta_1 &= a_2(k_1 + k_2) \\ \eta_2 &= a_1[k_1 + k_2 - 2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})] + a_4(k_1 - k_2)(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}) \\ \eta_3 &= a_4[k_1 + k_2 - 2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})] + a_3(k_1 - k_2)(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}) \\ \eta_4 &= -2a_2k_2 \end{aligned} \tag{A.1}$$

Coefficients $\{c_p(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,10}$ entering the free energy (54) take the following expressions according to the microcracks status:

- if cracks are open $(g(\mathbf{E}, \mathbf{n}, \mathbf{A}) > 0)$, $\{c_p(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,10} = \{c_p^{\text{open}}(\mathbb{C}^0, \mathbf{n}, \mathbf{A})\}_{p=1,10}$

$$
\begin{aligned}
c_1^{\text{open}} &= -(2a_1+a_4)a_4(k_1-k_2)(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}) - a_1^2[k_1+k_2-2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]\\
c_2^{\text{open}} &= -a_3(a_3+2a_4)(k_1-k_2)(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}) - a_4^2[k_1+k_2-2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]\\
c_3^{\text{open}} &= -2\left(a_1a_3+a_3a_4+a_4^2\right)(k_1-k_2)(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n}) - 2a_1a_4[k_1+k_2-2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]\\
c_4^{\text{open}} &= -2a_2^2k_2\\
c_5^{\text{open}} &= -2a_1a_2(k_1+k_2)\\
c_6^{\text{open}} &= -a_2^2[k_1-k_2+2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]\\
c_7^{\text{open}} &= -2a_2a_4(k_1-k_2)+4a_1a_2k_2\\
c_8^{\text{open}} &= -2a_2a_4(k_1+k_2)\\
c_9^{\text{open}} &= -2a_2a_3(k_1-k_2)+4a_2a_4k_2\\
c_{10}^{\text{open}} &= 4a_2^2k_2
\end{aligned}
\tag{A.2}
$$

- if cracks are closed $(g(\mathbf{E},\mathbf{n},\mathbf{A})=0), \{c_p(\mathbb{C}^0,\mathbf{n},\mathbf{A})\}_{p=1,10} = \{c_p^{clos}(\mathbb{C}^0,\mathbf{n},\mathbf{A})\}_{p=1,10}$

$$
\begin{aligned}
c_1^{\text{clos}} &= -\frac{(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})[1-(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]}{k_1^2-k_2^2}[a_4(k_1-k_2)-2a_1k_2] \times [2a_1k_2[k_1+k_2-2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]+a_4(k_1-k_2)[k_1+k_2+2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]]\\
c_2^{\text{clos}} &= -\frac{(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})[1-(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]}{k_1^2-k_2^2}[a_3(k_1-k_2)-2a_4k_2] \times [2a_4k_2[k_1+k_2-2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]+a_3(k_1-k_2)[k_1+k_2+2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]]\\
c_3^{\text{clos}} &= \frac{2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})[1-(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]}{k_1^2-k_2^2}\times\left[4a_4k_2^2[a_4(k_1-k_2)(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})+a_1[k_1+k_2-2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]]+a_3(k_1-k_2)\right.\\
&\quad \left.\times\left[4a_1k_2^2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})-a_4(k_1-k_2)[k_1+k_2+2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]\right]\right]\\
c_4^{\text{clos}} &= \frac{a_2^2}{k_1^2-k_2^2}\left[k_1^3-k_1k_2[k_1+k_2-2k_1(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]+k_2^3\left[1+6(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})-8(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})^2\right]\right]\\
c_5^{\text{clos}} &= \frac{2a_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})}{k_1-k_2}\left[4a_1k_2^2[1-(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]+a_4(k_1-k_2)[k_1-k_2+2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]\right]\\
c_6^{\text{clos}} &= -\frac{a_2^2}{k_1^2-k_2^2}[k_1-k_2+2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})] \times \left[k_1^2-k_2^2\left[1+4(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})-4(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})^2\right]\right]\\
c_7^{\text{clos}} &= -\frac{2a_2}{k_1^2-k_2^2}\left[8a_1k_2^3(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})[1-(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]+a_4(k_1-k_2)\left[k_1^2+2k_1k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})-k_2^2\left[1+2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})-4(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})^2\right]\right]\right]\\
c_8^{\text{clos}} &= \frac{2a_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})}{k_1-k_2}\left[4a_4k_2^2[1-(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]+a_3(k_1-k_2)[k_1-k_2+2k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]\right]\\
c_9^{\text{clos}} &= -\frac{2a_2}{k_1^2-k_2^2}\left[8a_4k_2^3(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})[1-(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]+a_3(k_1-k_2)\left[k_1^2+2k_1k_2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})-k_2^2\left[1+2(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})-4(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})^2\right]\right]\right]\\
c_{10}^{\text{clos}} &= -\frac{16k_2^3a_2^2}{k_1^2-k_2^2}(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})[1-(\mathbf{n}\cdot\mathbf{A}\cdot\mathbf{n})]
\end{aligned}
\tag{A.3}
$$

References

Allix, O., Cluzel, C., Gasser, A., Ladevèze, P., 1993. Modélisation des composites céramique-céramique à différentes échelles. Rev. Comp. Mater. Avancés 3, 277–297.

Andrieux, S., Bamberger, Y., Marigo, J., 1986. Un modèle de matériau microfissuré pour les bétons et les roches. J. Méc. Theo. Appl. 5, 471–513.

Aubard, X., 1995. Modelling of the mechanical behaviour of 2-D SiC–SiC composite at a mesoscale. Comp. Sci. Tech. 54, 371–378.

Baste, S., Aristégui, C., 1998. Induced anisotropy and crack systems orientations of a ceramic matrix composite under off-principal axis loading. Mech. Mater. 29, 19–41.

Baste, S., El Bouazzaoui, R., 1996. Cracking orientation and induced anisotropy of a ceramic matrix composite under off-axis loading. J. Mater. Sci. 31, 1575–1584.

Bristow, J., 1960. Microcracks, and the static and dynamic elastic constants of annealed heavily cold-worked metals. Br. J. Appl. Phys. 11, 81–85.

Budiansky, B., O'Connel, J., 1976. Elastic moduli of a cracked solid. Int. J. Solids Structures 12, 81–97.

Carol, I., Willam, K., 1996. Spurious energy dissipation/generation in stiffness recovery models for elastic degradation and damage. Int. J. Solids Structures 33, 2939–2957.

Chaboche, J., 1992. On the difficulties associated with the active/passive unilateral condition. Int. J. Damage Mech. 1, 148–171.

Chaboche, J., 1993. Development of CDM for elastic solids sustaining anisotropic and unilateral damage. Int. J. Damage Mech. 2, 311–329.

Chaboche, J., Lesne, P., Maire, J., 1996. Un modèle de comportement et d'endommagement conjuguant des variables de dommages tensorielle et scalaires. C. R. Acad. Sci. Paris Série IIb 322, 187–193.

Chaboche, J., Maire, J., 2001. New progress in micromechanics-based CDM models and their application to cmc's. Comp. Sci. Tech. 61, 2239–2246.

Chaboche, J., Maire, J., 2002. A new micromechanics based CDM model and its application to CMC's. Aero. Sci. Tech. 6, 131–145.

Challamel, N., Halm, D., Dragon, A., 2006. On the non-conservativeness of a class of anisotropic damage models with unilateral effects. C. R. Mécanique 334, 414–418.

Christensen, R., 1979. Mechanics of Composite Materials. Wiley and Sons, New-York.

Cormery, F., Welemane, H., 2002. A critical review of some damage models with unilateral effect. Mech. Res. Comm. 29, 391–395.

Cormery, F., Welemane, H., 2010. A stress-based macroscopic approach for microcracks unilateral effect. Comput. Mater. Sci. 47, 727–738.

Costanzo, F., Boyd, J., Allen, D., 1996. Micromechanics and homogenization of inelastic composite materials with growing cracks. J. Mech. Phys. Solids 44, 333–370.

Curnier, A., He, Q., Zysset, P., 1995. Conewise linear elastic materials. J. Elasticity 37, 1–38.

Deudé, V., Dormieux, L., Kondo, D., Pensée, V., 2002. Propriétés élastiques non linéaires d'un milieu mésofissuré. C. R. Mécanique 330, 587–592.

Dormieux, L., Kondo, D., 2009. Stress-based estimates and bounds of effective elastic properties: the case of cracked media with unilateral effects. Comput. Mater. Sci. 46, 173–179.

Gruescu, C., Monchiet, V., Kondo, D., 2005. Eshelby tensor for a crack in an orthotropic elastic medium. C. R. Mécanique 333, 467–473.

Halm, D., Dragon, A., 1996. A model of anisotropic damage by mesocrack growth; unilateral effect. Int. J. Damage Mech. 5, 384–402.

Halm, D., Dragon, A., Charles, Y., 2002. A modular damage model for quasi-brittle solids – interaction between initial and induced anisotropy. Arch. Appl. Mech. 72, 498–510.

Hashin, Z., 1983. Analysis of composite materials – survey. J. Appl. Mech. 50, 481–505.

Hashin, Z., 1988. The differential scheme and its application to cracked materials. J. Mech. Phys. Solids 36, 719–734.

Hazanov, S., 1998. Hill condition and overall properties of composites. Arch. Appl. Mech. 68, 385–394.

He, Q., Curnier, A., 1995. A more fundamental approach to damaged elastic stress-strain relations. Int. J. Solids Struct. 32, 1433–1457.

Hill, R., 1965. A self consistent mechanics of composite materials. J. Mech. Phys. Solids 13, 213–222.

Horii, H., Nemat-Nasser, S., 1983. Overall moduli of solids with microcracks: load induced anisotropy. J. Mech. Phys. Solids 31, 155–171.

Kachanov, M., 1993. Advances in Applied Mechanics. Academic Press, New-York, Ch, Elastic solids with many cracks and related problems, pp. 259–445.

Krajcinovic, D., 1996. Damage Mechanics. In: Applied Mathematics and Mechanics, vol. 41. Elsevier Science, Amsterdam.

Laws, N., 1977. A note on interaction energies associated with cracks in anisotropic solids. Phil. Mag 36 (2), 367–372.

Laws, N., 1983. Stiffness changes in unidirectional composites caused by crack systems. Mech. Mater. 2, 123–137.

Laws, N., Brockenbrough, E., 1987. The effect of microcrack systems on the loss of stiffness of brittle solids. Int. J. Solids Struct. 23, 1247–1268.

Leguillon, D., Sanchez-Palencia, E., 1982. On the behaviour of a cracked elastic body with (or without) friction. J. Méc. Théo. Appli 1, 195–209.

Lekhnitskii, S., 1963. Theory of Elasticity of Anisotropic Elastic Body. Holden-Day, San Francisco.

Maire, J., Chaboche, J., 1997. A new formulation of continuum damage mechanics (CDM) for composite materials. Aero. Sci. Tech. 4, 247–257.

Maire, J., Lesne, P., 1998. An explicit damage model for the design of composites structures. Comp. Sci. Tech. 58, 773–778.

Mauge, C., Kachanov, M., 1994. Effective elastic properties of anisotropic materials with arbitrarily oriented cracks. J. Mech. Phys. Solids 42, 1–24.

Morvan, J., Baste, S., 1998. Effect of the opening/closure of microcracks on the nonlinear behavior of a 2D C–SiC composite under cyclic loading. Int. J. Damage Mech. 7, 381–402.

Murakami, S., Kamiya, K., 1997. Constitutive and damage evolution equations of elastic–brittle materials based on irreversible thermodynamics. Int. J. Mech. Sci. 39, 473–486.

Nemat-Nasser, S., Hori, M., 1993. Micromechanics: Overall Properties of Heterogeneous Materials. In: Applied Mathematics and Mechanics, vol. 37. Elsevier Science, Amsterdam.

Pensée, V., Kondo, D., Dormieux, L., 2002. Micromechanical analysis of anisotropic damage in brittle materials. J. Eng. Mater. 128, 889–897.

Ponte Castañeda, P., Willis, J., 1995. The effect of spatial distribution on the effective behavior of composite materials and cracked media. J. Mech. Phys. Solids 43 (12), 1919–1951.

Santare, M., Crocombe, A., Anlas, G., 1995. Anisotropic effective moduli of materials with microcracks. Engng Fract. Mech. 52, 833–842.

Suquet, P., 1982. Plasticité et homogénéisation. Ph.D. thesis, Univ. Paris 6, France.

Telega, J., 1990. Homogenization of fissured elastic solids in the presence of unilateral conditions and friction. Comput. Mech. 6, 109–127.

Tsukrov, I., Kachanov, M., 2000. Effective moduli of an anisotropic material with elliptical holes of arbitrary orientational distribution. Int. J. Solids Struct. 37, 5919–5941.

Wang, X., Gazonas, G., Santare, M.H., 2009. On the effective electroelastic properties of microcracked generally anisotropic solids. Int. J. Fract 158, 27–40.

Welemane, H., Cormery, F., 2002. Some remarks on the damage unilateral effect modelling for microcracked materials. Int. J. Damage Mech. 11, 65–86.

Welemane, H., Cormery, F., 2003. An alternative 3D model for damage induced anisotropy and unilateral effect in microcracked materials. J. Phys. IV 105, 329–336.

Welemane, H., Goidescu, C., 2010. Isotropic brittle damage and unilateral effect. C. R. Mécanique 338, 271–276.

Zaoui, A., 2002. Continuum micromechanics: survey. J. Eng. Mech. 128, 808–816.

Zhu, Q., Kondo, D., Shao, J., Pensée, V., 2008. Micromechanical modelling of anisotropic damage in brittle rocks and application. Int. J. Rock Mech. Mining Sci. 45, 467–477.