High Reynolds Channel Flows: Upstream interaction of various wall deformations

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Summary. The flow at high Reynolds number in the entrance of distorted channel is considered. We analyse the anticipated fluid responds to a downstream wall distortion, and we find that the non linear upstream length $\Delta = \mathrm{O}(R_e^{1/7})$, using either a new asymptotic approach called Successive Complementary Expansions Method (SCEM) with generalized asymptotic expansions and a modal analysis of the perturbed flow. Comparisons with Navier-Stokes solutions show that the mathematical model is well founded.

1 Introduction

This paper considers the upstream interaction of flows in a two-dimensional channel at high Reynolds number with wall deformations. An asymptotic model using the successive complementary expansion method with generalized asymptotic expansions, called GIBL for Global Interactive Boundary Layer [1, 3], is used. The aim is to analyse the non linear asymptotic length Δ of the upstream influence of an *accident* at $x = x_0$ at the walls. As Smith [2] we found that $\Delta = \mathrm{O}(R_e^{1/7})$, where R_e is the Reynolds number. The only hypothesis on the wall *accident* is that it is significant enough to perturbe the Poiseuille flow, so that the Poiseuille flow is no more a good approximation in the boundary layer.
Then by assuming an exponential variation in x of the perturbed flow, in order to obtain the Poiseuille flow as $x \to -\infty$ (i.e. far upstream the wall deformations), we perform an eigenvalue analysis. We thus found that the first mode is related to non-symmetric wall deformations. Two kind of wall deformations are considered (local and global distortions) and comparisons between GIBL, Navier-Stokes solutions and eigenmodes show that the model is well founded.

2 Geometrical configuration

Two kind of geometrical configuration have been considered for the *accident*: (i) a local wall perturbation as in figure 1, or (ii) a global wall curvature as in figure 2.

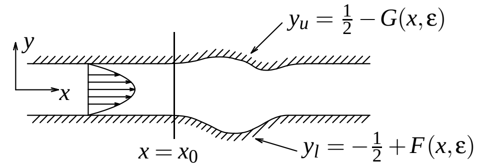

Fig. 1. (i) case: Local wall perturbation; location of the accident at $x = x_0$.

In the test case (i), the walls are deformed in a domain $x_0 \leq x \leq x_0 + L$ such as:

$$F = \frac{h_l}{2}\left(1 + \cos\frac{2\pi x}{L}\right) \; ; G = -\frac{h_u}{2}\left(1 + \cos\frac{2\pi x}{L}\right) \; . \tag{1}$$

where h_u and h_l are small parameters.
In the test case (ii), we use a generalized system of coordinates, where X and Y are distances along and perpendicular to the line $H = 0$. We call it the median line if the upper (or inner) and lower (or external) walls are respectively given by $Y = \pm\dfrac{1}{2}$.
For a point M with general coordinates X and Y, we can write $\overrightarrow{OM} = \overrightarrow{OM_0} + Y\mathbf{n}$, where $\mathbf{n}$ is the unit normal vector. Then, $\mathrm{d}\overrightarrow{M} = \mathrm{d}X\,(1 + KY)\,\tau + \mathrm{d}Y\mathbf{n}$, where τ is the unit vector tangent at M_0 to the median line in such a way that $(\tau, \mathbf{n})$ is direct; $K(X)$ is the algebraic curvature of this line. Thus, $K < 0$ in the case of figure 2. The curvature K and its variation in X are small. We thus describe the channel variable curvature for $X > 0$ by $K = \delta k(X)$, where δ is a small positive parameter. Let U and V denote the velocity components parallel and perpendicular to the line $H = 0$, then, as $\mathbf{V} = U\tau + V\mathbf{n}$, the full equations of motion written in generalized coordinates are given in [4]. These equations must be solved with boundary conditions: $U = V = 0$ for $Y = \pm\dfrac{1}{2}$.

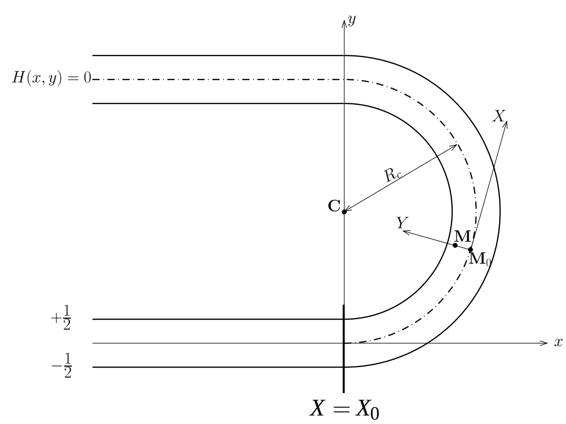

Fig. 2. (ii) case: Global wall curvature; location of the accident at $X = X_0$.

3 Fully established flow in a curved channel

For a channel of constant curvature δ, the fully established flow U_0 is solution of

$$(1+\delta Y)\frac{d^2U_0}{dY^2}+\delta\frac{dU_0}{dY}-\frac{\delta^2}{1+K_0Y}U_0=-GR_e \tag{2}$$

where $G=-\dfrac{\partial P}{\partial X}$ is constant, and with $U_0=0$ for $Y=\pm\dfrac{1}{2}$. Notice that for $\delta=0$ we retrieve the equation for the Poiseuille flow: $\dfrac{d^2U_0}{dY^2}=-2$.
The exact solution is given by:

$$U_0(Y)=\frac{1}{64}GR_e\frac{f(\delta,Y)}{(\delta^2(1+\delta Y))} \tag{3}$$

where

$$\begin{aligned} f(\delta,Y) = {} & \left[\delta^3(1-4Y^2)+8\delta^2Y(2Y-1)+4\delta(-4Y^2+8Y-3)+16(1-2Y)\right]\ln\left(\frac{2-\delta}{2\delta}\right)+ \\ & \left[-\delta^3(1-4Y^2)+8\delta^2Y(2Y+1)+4\delta(4Y^2+8Y+3)+16(1+2Y)\right]\ln\left(\frac{2+\delta}{2\delta}\right)- \\ & 32\left(2\delta Y+\delta^2Y^2+1\right)\ln\left(\frac{1+\delta Y}{\delta}\right) \end{aligned}$$

As shown in figure 3, the corresponding exact solution $U_0(y)$ bends towards the internal wall of the bend. Notice that, for a small constant curvature δ and for a flow

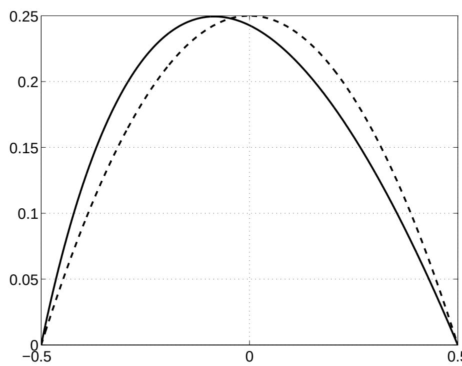

Fig. 3. Velocity profile $U_0(Y)$; Poiseuille flow (dashed line); profile for $\delta=1$ (straight line).

rate of $1/6$, an approximate solution $O(\delta)$ is $U_0=\left(\dfrac{1}{4}-Y^2\right)\left(1-\dfrac{2\delta}{3}Y\right)$, which implies a skin friction of $C_f\dfrac{R_e}{2}=1\mp\dfrac{\delta}{3}$.

4 Global Interactive Boundary Layer (GIBL) model

According to the SCEM, a Uniformaly Valid Approximation (UVA) for the velocity and pressure fields (U,V,P) is obtained by complementing the core approximation $(U_1 = u_0 + \delta u_1, V_1 = \delta v_1, P_1 = p_0 + \delta p_1)$ such as:

$$\begin{aligned} U &= u_0(Y) + \delta\left[u_1(X,Y,\delta) + U_{BL}(X,\eta,\delta)\right] \\ V &= \delta\left[v_1(X,Y,\delta) + \varepsilon V_{BL}(X,\eta,\delta)\right] \\ P &= p_0(X) + \delta\left[p_1(X,Y,\delta,\varepsilon) + \Delta(\varepsilon) P_{BL}(X,\eta,\delta,\varepsilon)\right] \end{aligned} \tag{4}$$

where $\lim_{\eta\to\infty} U_{BL} = 0$, $\lim_{\eta\to\infty} V_{BL} = 0$ and $\lim_{\eta\to\infty} P_{BL} = 0$ (see [4] for more details).
Thus, we obtain Uniformaly Valid Approximation (UVA) equations:

$$\begin{aligned} \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} &= 0 \\ U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} &= -\frac{\partial P_1}{\partial X} + \frac{1}{R_e}\frac{\partial}{\partial Y}\left[(1+KY)\frac{\partial U}{\partial Y}\right] \end{aligned}$$

with the following boundary conditions, $U = V = 0$, for $Y = \pm\frac{1}{2}$. The core equations being:

$$\begin{aligned} u_0\frac{\partial V_1}{\partial X} - K u_0^2 &= -\frac{\partial P_1}{\partial Y} \\ -u_0\frac{\partial V_1}{\partial Y} + V_1\frac{du_0}{dY} &= -\frac{\partial (P_1 - p_0)}{\partial X} \end{aligned}$$

A simplified model for the pressure gives $\frac{\partial P_1}{\partial X} = \frac{\mathrm{d}p_0}{\mathrm{d}X} + \delta\left(A''' + k'\right)\int_{\eta_c}^{\eta} u_0^2(\eta')\,\mathrm{d}\eta' + \delta B'(X)$. At the medline, i.e. for $\eta = \eta_c$, since the UVA V should match the core approximation V_1, we impose the coupling condition $V = V_1 = -A'(X)u_0$.
For more details about GIBL, see the companion paper [5].

5 Upstream interaction

5.1 Upstream length

In a straight channel, upstream of the wall *accident*, for $x < 0$, the GIBL and core equations become:

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = -\frac{\partial P_1}{\partial x} + \frac{1}{Re}\frac{\partial^2 U}{\partial y^2} \tag{5}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{6}$$

$$-U_0\frac{\partial V_1}{\partial y} + V_1\frac{\mathrm{d}U_0}{\mathrm{d}y} = -\frac{\partial (P_1 - P_0)}{\partial x} \tag{7}$$

$$U_0\frac{\partial V_1}{\partial x} = -\frac{\partial (P_1 - P_0)}{\partial y} \tag{8}$$

We now consider perturbations of the following form: $U = U_0 + \varepsilon u$, $V = \varepsilon v$ and $P_1 = P_0 + \lambda p_1$.
If the critical unknown streamwise length scale is Δ, then, with $\overline{x} = \dfrac{x}{\Delta}$ and thus $\overline{V} = \Delta V$, we obtain from (5,6,7,8) the following perturbation equations:

$$\frac{\partial u}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial y} = 0 \tag{9}$$

$$U_0 \frac{\partial u}{\partial \overline{x}} + \overline{v}\frac{\mathrm{d}U_0}{\mathrm{d}y} + \varepsilon\left(u\frac{\partial u}{\partial \overline{x}} + \overline{v}\frac{\partial u}{\partial y}\right) = -\frac{\lambda}{\varepsilon}\frac{\partial p_1}{\partial \overline{x}} + \frac{\Delta}{Re}\frac{\partial^2 u}{\partial y^2} \tag{10}$$

$$-U_0 \frac{\partial \overline{v_1}}{\partial y} + \overline{v_1}\frac{\mathrm{d}U_0}{\mathrm{d}y} = -\frac{\lambda}{\varepsilon}\frac{\partial p_1}{\partial \overline{x}} \tag{11}$$

$$U_0 \frac{\partial \overline{v_1}}{\partial \overline{x}} = -\frac{\lambda \Delta^2}{\varepsilon}\frac{\partial p_1}{\partial y} \tag{12}$$

If ε is the boundary layer thickness, the first significant perturbation is such as $U_0 = \mathrm{O}(\varepsilon)$, $\overline{v} = \mathrm{O}(\varepsilon)$ in the boundary layers, which implies from (10) that ε, $\dfrac{\lambda}{\varepsilon}$ and $\dfrac{\Delta}{\varepsilon^2 R_e}$ are of same order.
An upstream interaction takes place if we have a generation of a significant transverse pressure gradient in the core flow, which implies from (12) that $\dfrac{\lambda\Delta^2}{\varepsilon} = \mathrm{O}(1)$. Thus, we easily obtain (as did Smith [2] by regular asymptotic expansions) the following crucial orders:

$$\Delta = \mathrm{O}(R_e^{1/7}), \quad \varepsilon = \mathrm{O}(R_e^{-2/7}) \quad \text{and} \quad \lambda = \mathrm{O}(R_e^{-4/7}). \tag{13}$$

5.2 Eigenmode analysis

For $x < 0$, the linearized UVA system of equations may be written as :

$$\begin{cases} U_0 \dfrac{\partial u}{\partial x} + U_0' v = -\dfrac{\partial p_1}{\partial x} + \dfrac{1}{Re}\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ U_0 \dfrac{\partial v_1}{\partial x} = -\dfrac{\partial p_1}{\partial y} \end{cases} \tag{14}$$

By replacing v_1 by v in the transverse core momemtum equation, and by assuming the following form for u, v and p_1 :

$$u(x,y) = \hat{u}(y)\mathrm{e}^{\theta x}, \quad v(x,y) = \hat{v}(y)\mathrm{e}^{\theta x}, \quad p_1(x,y) = \hat{p}_1(y)\mathrm{e}^{\theta x} \tag{15}$$

we obtain for the perturbations:

$$\theta \begin{pmatrix} U_0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & U_0 & 0 \end{pmatrix} \hat{q} = \begin{pmatrix} \frac{D^2}{Re} & -U_0' & 0 \\ 0 & -D^1 & 0 \\ 0 & 0 & -D^1 \end{pmatrix} \hat{q} \tag{16}$$

where $\hat{q} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{p}_1 \end{pmatrix}$, $D^1 = \frac{\partial}{\partial y}$ and $D^2 = \frac{\partial^2}{\partial y^2}$.

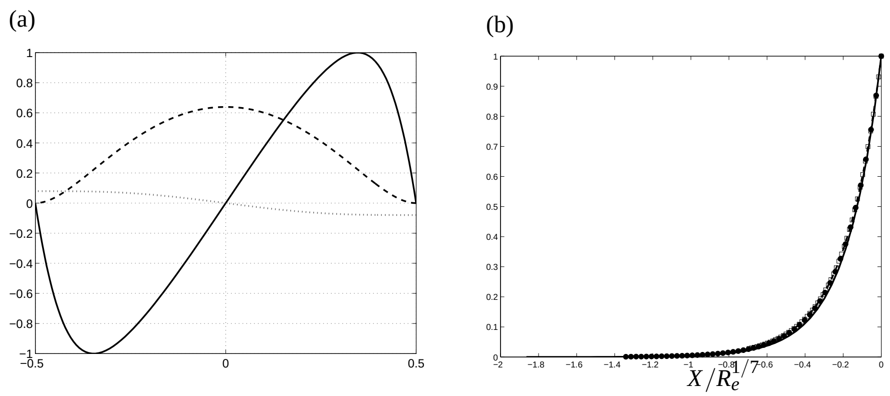

Fig. 4. (a) Profiles of the first mode eigenfunctions $\hat{u}$ (straight line), $\hat{v}$ (dashed line) and $\hat{p}_1$ (dotted line) for $Re = 1000$; (b) Upstream influence of the first mode for $Re = 10^3$ (straight line), 10^4 (black circle), 10^5 (dashed line), 10^6 (white square).

We just have now to find the eigenvalues and eigenfunctions of the matrix $B^{-1}A$, where :

$$A = \begin{pmatrix} \frac{D^2}{Re} & -U_0' & 0 \\ 0 & -D^1 & 0 \\ 0 & 0 & -D^1 \end{pmatrix} \text{ and } B = \begin{pmatrix} U_0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & U_0 & 0 \end{pmatrix} \tag{17}$$

For $R_e = 1000$, the first positive eigenvalue found is $\theta_1 \simeq 2.0441$. The figure 4(a) represents the eigenfunctions of this mode. As shown in figure 4(b), by computing this first positive eigenvalue for different Reynolds number ranging from 10^3 to 10^6, we obtain that the corresponding upstream influence $\Delta = O(R_e^{1/7})$ as in the analysis of the section 5.1.

6 Results

Both the order analysis of section 5.1 and the eigenmode analysis of section 5.2 show that $\Delta = O(R_e^{1/7})$. We now compute the flow field using the GIBL model described in section 4 for different accident types at $x = 0$.

First, we have considered a straight channel connected at $x = 0$ to a curved channel of constant curvature. The figures 5 (a) and (b) represent the median curved length evolution of $V(X, \eta_c)$ for, respectively, a fixed $\delta = 0.2$ at different Reynolds numbers, and a fixed $R_e = 1000$ at different wall curvature. These two results confirm that $\Delta = \mathrm{O}(R_e^{1/7})$.

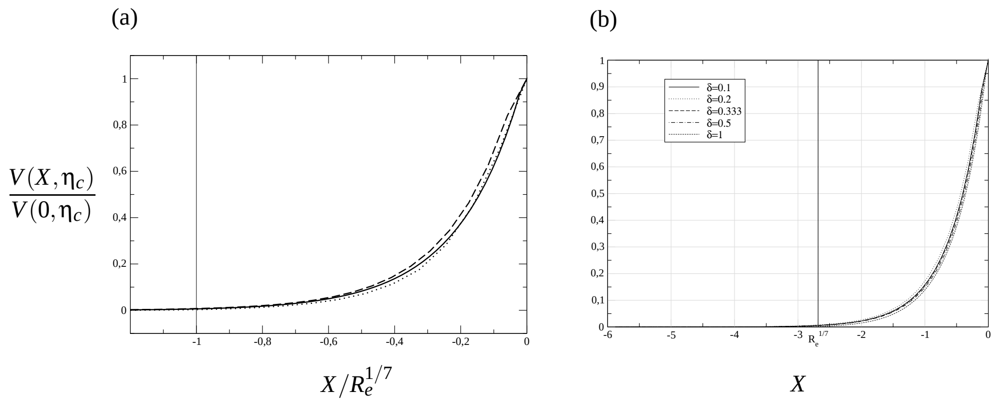

Fig. 5. (ii) case: straight channel connected at $x = 0$ to a curved channel of constant curvature; (a) $\delta = 0.2$, R_e from 100 to 10000; (b) $R_e = 1000$, δ from 0.1 to 1

Then, we have considered an asymmetrically perturbed straight channel at $x = 0$ with $L = 4H$ and $h_u = h_l = 0.3$. The figure 6 represents the streamwise evolution of $V(x, \eta_c)$, where we recover as previously $\Delta = \mathrm{O}(R_e^{1/7})$.

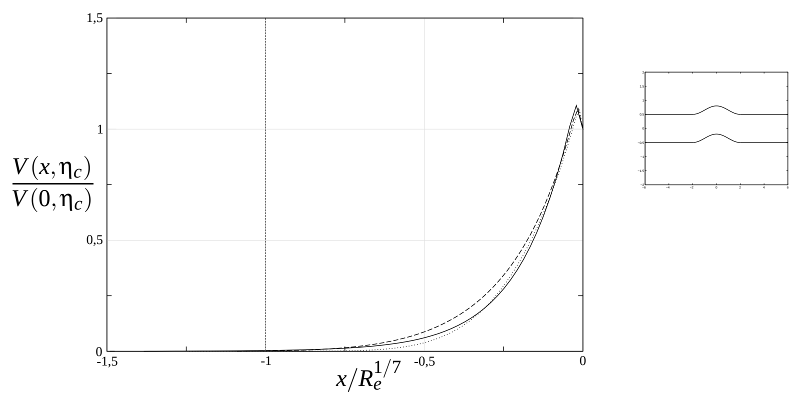

Fig. 6. (i) case: straight channel perturbed at $x = 0$ with $L = 4H$, $h_u = h_l = 0.3$; x-evolution of the adimensionnalized $V(x, \eta_c)$ for different values of R_e (from 100 to 10000).

Finally, we have compared the Navier-Stokes, GIBL and eigenmode analysis results. As shown in figure 7, all the results are very similar.

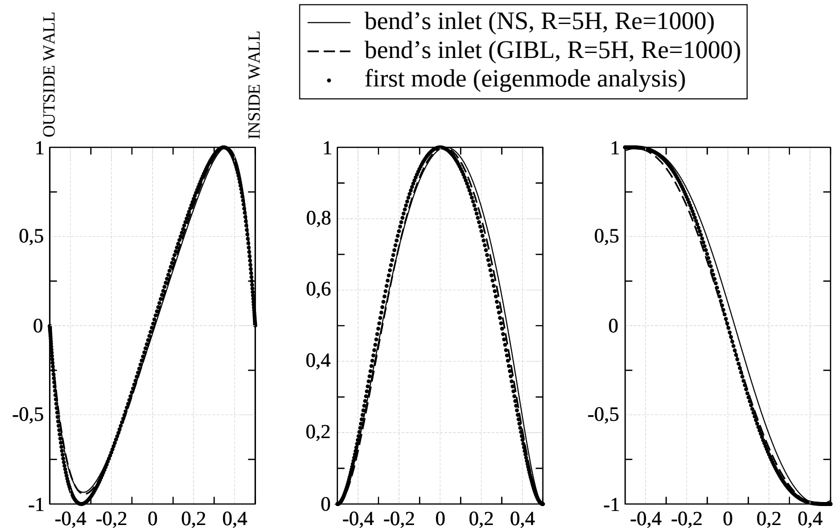

Fig. 7. NS, GIBL, first eigenmode comparaison; $R_e = 1000$, $\delta = 0.2$; left: u profile; middle: v profile; right: p_1 profile.

7 Conclusion

The non linear upstream effect on a channel flow submitted to asymmetric disturbance has been studied. By using three differents tools, a new asymptotic approach called Successive Complementary Expansions Method (SCEM) with generalized asymptotic expansions, a modal analysis and direct Navier-Stokes computations, we found that the upstream influence length $\Delta = O(R_e^{1/7})$.

References

1. J. Cousteix and J. Mauss,*Asymptotic Analysis and Boundary Layers*. Scientific Computation, vol. XVIII, Springer, Berlin, Heidelberg, 2007.
2. F.T. Smith, Upstream interactions in channel flows. *Journal of Fluid Mechanics* **79** (4), 631–655, (1977).
3. P. Cathalifaud, J. Mauss and J. Cousteix, Non linear aspects of high Reynolds number channel flows. *European Journal of Mechanics B/Fluids* **29** (4), 295–304 (2010).
4. M. Zagzoule, P. Cathalifaud, J. Mauss and J. Cousteix, Uniformaly Valid Approximation Flow analysis in Curved Channels. *submitted to Physics of Fluids* (2010).
5. M. Zagzoule, P. Cathalifaud, J. Mauss and J. Cousteix, High Reynolds Channel Flows: Variable curvature. *submitted to Lecture Notes in Computational Science and Engineering* (2010).