# High Speed of Learning in Financial Markets

Laurent GERMAIN [*]

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Abstract

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We analyze the role of liquidity and collection of information in order to measure the speed of revelation of information during the preopening of order-driven markets. We extend Vives (1995) model to the case where risk averse traders receive a new private signal before each round of quotation of the preopening. We show that price discovery takes place at high speed which is consistent with the empirical studies of Biais, Hillion and Spatt (1999).

# Résumé

## Forte Vitesse d'Apprentissage sur les Marchés Financiers

Nous analysons le rôle de la liquidité et de l'arrivée d'informations dans la mesure de la vitesse de convergence des prix durant la pré-ouverture de marchés gouvernés par les ordres. Nous étendons le modèle de Vives (1995) au cas où des agents averses au risque reçoivent une nouvelle information avant chaque tour de cotation. Nous montrons que la vitesse de convergence est plus rapide et qu'elle est validée par les études empiriques de Biais, Hillion et Spatt (1999).

# 1  Introduction

How do prices aggregate dispersed information in financial markets ? This is one
of the fundamental questions in finance and in the theory of market microstruc-
ture. In his seminal paper Grossman (1976) proposes a theoretical framework to
understand the role of prices as a vector of information over one period of trad-
ing. In a dynamic setting, Kyle (1985) and Holden and Subrahmanyam (1992)
among others study the revelation of information during the trading day. Vives
(1995) focuses on the speed at which agents learn from prices during the pre-
opening of order-driven markets. Indeed, the Paris Bourse, Toronto, Tokyo or
Madrid stock exchanges have set up a preopening period in order to facilitate the
price discovery after the arrival of new information which occurs after a period
of non quotation (as overnight). In Paris, for example, during one hour and 45
minutes the participants of the market can send buy or sell orders and observe
the indicative clearing price. At any time before the opening of the market the
agents can cancel their orders. At the opening of the market a price is set and
the market is cleared.

In Vives (1995), risk averse traders are endowed with a noisy private signal $s_i$
about the liquidation value of a risky asset $v$. We extend Vives (1995) to the case
where agents receive a new signal before each round of quotation. As a matter
of fact, the flow of information received and collected by the market participants
during the preopening of financial markets is important as the agents exhange
information with each other, and try to obtain as more information as possible
in order to anticipate the opening price[1]. Then, it is likely the case that their
private information evolves over the time of the preopening. At each period,
traders observe the indicative price as well as their private signal and revise

their demand accordingly. Asymptotically, for a large number of quotations, the learning process converges to the rational expectations equilibrium and traders learn $v$, the liquidation value of the risky asset. The speed of learning is defined as the speed at which prices converge to the fully revealing rational expectations equilibrium - REE. In Vives (1995), the speed of revelation of information when the liquidity of the market is provided by market makers is $\sqrt{n}$ where $n$ is the number of round of quotations. But Biais, Hillion and Spatt (1999) in their empirical financial study on the Paris Bourse reject this speed and find a speed of revelation of information which is much higher and is equal to $\sqrt{n}^3$. In this paper, we show that such high speed of revelation of information can be attributed to the collection of new private information[2].

As in Vives (1995), to characterize the learning process we focus on two parameters :

- the reaction of the traders to their private information,

- and the speed at which prices reveal private information.

With market makers supplying the liquidity in the financial market (or alternatively competitive limit orders) - endogenous price function - our results are the following :

- the speed of learning is $n^{\frac{3}{2}}$,

- and the reaction of the traders to their private information goes to infinity.

In fact, the liquidity effect described by Vives (1995) and the informational effect, that we highlight in this model play the same symmetric role. It is why the

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revelation of information happens very quickly when traders keep on collecting information and the liquidity is endogenously provided by market makers.

In the next section, we characterize the equilibrium conditions. In section 3 we study the asymptotic properties of the market. Section 4 states some concluding remarks.

# 2    The model: financial market with competitive market makers or limit orders

At each round $n$ there is a probability $\delta_n$ that $v$ be revealed to the traders and a probability $1 - \delta_n$ that the tatonement process continues, those probabilities are independent over time. As in Vives (1995) we assume that the sequence $(\delta_n)$ is non decreasing. The probability that the tatonement process lasts until stage $n$ is $(1 - \delta_n)...(1 - \delta_1)$ which goes to 0 as the sequence $(\delta_n)$ is non-decreasing. As Biais, Hillion and Spatt (1999) notice this is different from the Paris Bourse where everybody knows the time of the opening. Nevertheless, this corresponds to certain financial markets as the Frankfurt Bourse where the opening time is random[3].

In this market there are three types of agents - it is a competitive version of Kyle (1985):

- Market makers who set the price conditionally on past volumes $\omega^n$,

$$p_n = E(v|\omega^n), \tag{1}$$

where $\omega^n = (\omega_1, ..., \omega_{n-1}, \omega_n)$[4].

- A continuum of informed agents in the interval $[0, 1]$ maximizing CARA utility $U(x) = -e^{-\rho x}$ where $\rho$ is the risk aversion coefficient[5].

  Each agent $i$ at round $n$ is endowed with a set of signals $(s_i^n = s_{i1}, ..., s_{in})$ with

  $$s_{in} = v + \varepsilon_{in}$$

  where $v$ is the liquidation value of the normally distributed risky asset $\tilde{v} \rightsquigarrow N(\bar{v}, \sigma_v^2)$ and $(\varepsilon_i)_i \rightsquigarrow N(0, \sigma_{\varepsilon_i}^2)$ are independent normally distributed variables and independent to $v$ too[6].

  Even if we consider that those private signals are independent over time, we will see that a correlation does not change any of the results and that one could consider the case of an AR(1) where $\varepsilon_{in} = \gamma \, \varepsilon_{i(n-1)} + h_{in}$ and $(h_{in})_{i,n}$ variables are independent with $v$ and $h_{in} \rightsquigarrow N(0, \sigma_{h_i}^2)$.

- At each round, noise traders submit inelastic demand,

  $$u_t \rightsquigarrow \mathcal{N}(0, \sigma_u^2),$$

  where $u_t$ are independent with $(\varepsilon_i)_i$ and with $v$.

Informed agents maximize their expected utility at each period conditional to $I_{in}$ their information both private and public. Past prices $(p_\tau, \tau < n)$ and past volumes $(\omega_\tau, \tau < n)$ are public information. Therefore $I_{in}$ is equal to $(s_i^n, p^{n-1})$ which are the past signals received from $t = 1$ to $t = n$ and past prices until $n - 1$ [7].

We define $X_{in}$ as the quantity demanded by the $i^{th}$ agent and $\pi_{in} = X_{in}(v - p_n)$ his profit. Each agent solves the following program :

$$\max_{X_{in}} \; \mathrm{E}(U(X_{in}(v - p_n))|I_{in}) \text{ with } U(x) = -\exp{-(\rho x)}$$

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Therefore, the optimal quantity is:

$$X^*_{in} = \frac{E(v - p_n | I_{in})}{\rho Var(v - p_n | I_{in})}$$

(2)

We focus on linear equilibria:

$$X_{in} = a_{in} f_{in} + F_n(p^{n-1})$$

where $f_{in}$ is the OLS estimator of v over $s^n_i$ (the mean of the private signals) and $F_n$ is a deterministic function of past prices.

By the SLLN we have $\int_0^1 f_{in} di = v$ and therefore the aggregate volume is:

$$\omega_n = a_n v + F_n(p^{n-1}) + u_n$$

(3)

The following proposition characterizes the linear equilibrium.

**Proposition 1** *There is a unique linear equilibrium where:*

$$X_n(s^n, p^{n-1}) = a_n(f_{in} - p_{n-1})$$

(4)

*and $\omega_n$ the total volume is equal to:*

$$\int_0^1 X_n di + u_n = \int_0^1 a_n(f_{in} - p_{n-1}) di + u_n = a_n(v - p_{n-1}) + u_n$$

*The equilibrium price function is:*

$$p_n = p_{n-1} + \lambda_n \omega_n$$

*where $a_n$ and $\lambda_n$ are defined in the proof in the Appendix.*

# 3 Asymptotic properties: financial market with competitive market makers or limit orders

The speed of revelation of information is captured by the asymptotic variance $\tau_n = Var(v|p^n)^{-1}$ which is the informativeness of prices[8] . When $\tau_n = +\infty$, $v$ is revealed.

We compute $\nu$ such that $n^{-\nu}\tau_n$ goes to a constant $A\tau_\infty$ as n goes to infinity. $A\tau_\infty$ is the asymptotic precision. By Amemya (1985) theorem :

$$\sqrt{n^{-\nu}}\tau_n(p_n - v) \rightsquigarrow \mathcal{N}(0, (A\tau_\infty)^{-1})$$

Vives (1995) shows that the presence of competitive market makers allow the traders to react increasingly to their private information. In this case the speed of convergence is $\sqrt{n}$, and $\sqrt{n}(p_n - v) \rightsquigarrow N(0, \sigma_u^2\rho^2\sigma_\varepsilon^4)$. The following proposition characterizes the reaction of the traders to their signals and the speed of the price discovery process in our general set up.

**Proposition 2** *The reaction $(a_n)$ to private information is increasing and goes to infinity.*

*The speed of convergence is $n^{\frac{3}{2}}$ :*

$$n^{\frac{3}{2}}(P_n - v) \rightsquigarrow \mathcal{N}(0, \sigma_u^2\rho^2\sigma_\varepsilon^4)$$

*Proof: see Appendix*

The speed of revelation of information is therefore the cube of that in Vives (1995). In fact, in our model informed agents have an unbiaised estimator of $v$ $f_{in}$ on top of the equilibrium price which is too an unbiaised estimator of $v$. Informed agents can compute a new unbiaised estimator the precision of which

is a weighted average of the two others $f_{in}$ et $y_n$. As a consequence, the precision of their information is higher than in the case where they had only one private signal over all the preopening period which accelerates the learning process.

Whereas in Vives (1995), the reaction to the private information $(a_n)$ is increasing, and goes to a limit $(\rho\sigma_\varepsilon^2)^{-1}$, in this model there is no limit to the reaction of the agents to their private information. $a_n$ is not bounded because the depth of the market $\lambda_n = \dfrac{a_n\tau_u}{\tau_n}$ is of the order of $n^{-3}$ whereas the order is $n^{-1}$ with only one private signal. This shows the role of competitive market makers or limit orders in inducing revelation of information in a financial market where there is an important collection of information. $(a_n)$ goes to infinity does not mean that the demand of the informed goes to infinity. This demand is finite and $E|X|$ goes to 0 [9]. Indeed, each agent is competitive and the reaction to their private information is infinite because at the limit they know $v$ perfectly.

# 4   Conclusion

We have shown that if the traders collect private information at each round of quotation during the preopening period of an order-driven market then more information is incorporated into prices and the speed of convergence towards the fully revealing REE is consistent with empirical studies. Indeed, in the case where the liquidity is endogenous the speed of learning predicted by our model is the same as the speed estimated by Biais, Hillion and Spatt (1999) that is to say $n^{\frac{3}{2}}$.

# APPENDIX

PROOF OF PROPOSITION 1

Using (3) the price function is given by:

$$p_n = E(v|\omega^n) = E(v|a_1 v + u_1 + F_1(p_0), ..., a_n v + u_n + F_n(p^{n-1})).$$

Let define $z_t = a_t v + u_t$ and $y_t$ the OLS estimator of v over $(z^t)$. Vives (1995) shows that $p_t = E(v|z^t)$ (consequently to the fact that the functions $F(.)$ are deterministic) and so that $p_t = E(v|y_t)$ by definition of $y_t$.

Note that the main difference between the model presented in this article and the one described in Vives (1995) is that the trader receives sequential signals. We know that the information collected to evaluate $v$ can be summarized by a single variable, which is the OLS estimator of v over the $(s_i)$.

We note $f_{in}$ this variable [10].

Therefore, $I_{in} = (s_i^n, p^{n-1})$ can be summarized by $f_{in}$, the OLS estimator of $s_i^n$, and $y_{n-1}$, the OLS estimator of $p^{n-1}$.

As in Vives (1995), we assume that the demand of a client is a linear function of his private information (summarized by $f_{in}$) and the public information (summarized by $y_n$), and we find that:

$$X_{in} = a_n(f_{in} - p_{n-1})$$

As in Vives (1995) we can write $p_n = p_{n-1} + \lambda_n \omega_n$ where the aggregate volume $\omega_n = a_n(v - p_{n-1}) + u_n$ and the depth of the market is $\lambda_n = \dfrac{a_n \tau_u}{\tau_n}$ with $\tau_n =$

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$\tau_v + \tau_u A_n$ and $A_n = \sum_{t=1}^{n} a_t^2$ and we show that $a_n$ is defined as the solution of the following cubic equation :

$$F(a_n) = (\rho \sigma_\varepsilon^2 n^{-1} a_n - 1)\tau_{n-1} + \rho a_n^3 \frac{\tau_u}{\tau_n} = 0$$

Morever, we will show in the next proof that this maximum is unique.

<div align="right">QED</div>

PROOF OF PROPOSITION 2

From the proposition above, $a_n$ and $\tau_n$ can be seen as the roots of a cubic equation, more precisely :

$$F(a_n, \tau_n) = 0$$

Where $F(a_n, \tau_n) = (\rho \sigma_\varepsilon^2 n^{-1} a_n - 1)\tau_{n-1} + \rho a_n^3 \frac{\tau_u}{\tau_n}$.

On the intervall $\Re_{+*} \times \Re_{+*}$, the function F is $C^1$ and $\frac{\partial F}{\partial a}$ is stricly positive. Applying the theorem of implicit functions, at each point $(a_n^*, \tau_n^*)$ solution of the equation above, there exists a function $\psi$ defined on an open intervall $I_{\tau_n^*}$ (containing $\tau_n^*$) which goes to an open intervall $I_{a_n^*}$ (containing $a_n^*$) such that :

- For every $\tau$ included in $I_{\tau_n^*}$, if $a$ is solution of $F(\tau, a) = 0$, then $a = \psi(\tau)$.

- $\psi$ is $C^1$ and
$$\psi' = -\frac{\frac{\partial F}{\partial \tau}}{\frac{\partial F}{\partial a}}$$
It is straightforward to show that on $\Re_{+*} \times \Re_{+*}$ : $\frac{\partial F}{\partial a} > 0$ and $\frac{\partial F}{\partial \tau} < 0$. Hence, at every point $(a_n^*, \tau_n^*)$ the differential of the function $\psi$ locally defined is always positive.

Therefore $a_n$ is an increasing function of $\tau_n$ and $a_n$ is unique. Moreover, $\tau_n$ is obviously increasing with n, therefore $(a_n)$ is increasing with n too. Hence, as $\tau_n > na_1$, the informativeness of prices, goes to $\infty$. Moreover we have :

$$a_n \rho(\sigma_\varepsilon^2 n^{-1} + a_n \lambda_n (\tau_{n-1})^{-1}) = 1$$

If $a_n$ goes to a finite limit, the equation above is not verified when n goes to $\infty$ as the first term goes to 0. Therefore, as $(a_n)$ increases, it goes to $\infty$.

Let f be a function of n such that $a_n = n(\rho\sigma_\varepsilon^2)^{-1} + f(n)$. By substituing $a_n$ to this new expression in the equation above, it can easely be shown that

$$lim_{n\to\infty} f(n)/n = 0$$

Therefore,

$$a_n \sim n(\rho\sigma_\varepsilon^2)^{-1}$$

Now let determine the speed of convergence.

It is straightforward that the standard deviation of the estimator $y_n$ is equal to $\sigma_u \frac{\sum_{t=1}^{t=n} a_t}{\sum_{t=1}^{t=n} a_t^2}$.

After computing the equivalent of the numerator and the denominator of this expression when n goes to $\infty$ we find that the standard deviation of $y_n$ is:

$$\sigma_{y_n} \sim (\sqrt{3}\sigma_\varepsilon^2)$$

To apply the general central limit theorem (see Amemiya (1985)) we have to check that the following condition is fullfilled:

$$lim_{n\to\infty} \frac{Max\{a_k|k=1...n\}}{A_n} = 0$$

We know that $a_n \sim n(\rho\sigma_\varepsilon^2)^{-1}$ and $A_n = \sum_{t=1}^{t=n} a_t^2 \sim n^3 \frac{1}{3}(\frac{1}{\rho\sigma_\varepsilon^2})^2$. Because $a_n$ is increasing

$$\frac{Max\{a_k|k=1...n\}}{A_n} = \frac{a_n}{A_n}$$

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which goes to zero as n goes to $\infty$

Hence, we use the theorem Amemiya (1985) and we have :

$$\sigma_{y_n}^{-1}(\sum_{t=1}^{t=n} a_t^2)^{\frac{1}{2}}(y_n - f_{in}) \rightsquigarrow N(0,1)$$

and that $\sqrt{n^3}(y_n - f_{in}) \rightsquigarrow N(0, 3\rho^2 \sigma_u^2 \sigma_\varepsilon^4)$

QED

# Footnotes

[1] This has been related to us by market practitioneers.

[2] Biais and Martinez (1999) study the preopening in Frankfurt and in Paris.

[3] It would be of interest to measure the speed of learning in Frankfurt as well in order to see if this difference could change the result.

[4] In this paper we consider competitive market makers or alternatively competition between traders who place limit orders. As Biais, Hillion, Spatt (1995) notice "the conditional probability that investors place limit orders (rather than hitting the quotes) is larger when the bid-ask spread is larger or the order book is thin."

[5] Alternatively, Medrano and Vives (2001) consider a strategic agent.

[6] One could have considered that at each round of quotation the traders receive a new signal with a positive probability. This would have diminished the speed of convergence. Nevertheless we also consider the case where the private signals follow an AR(1) and show that any correlation would have changed our results.

[7] As prices and volumes are observationally equivalent it is not necessary to include the past volumes in the information set.

[8] $\tau_i$ is the inverse of the variance $\sigma_i^2$.

[9] It is straightforward that $E|X_n| = (2/\pi)^{\frac{1}{2}} a_n \tau_{n-1}^{-1}$ where $\dfrac{a_n}{\tau_v + \tau_u A_{n-1}}$ goes to 0.

[10] Consequently, there is is a very simple isomporphism between the two models. In fact, in this proposition, it is sufficient to substitute $f_{in}$ to $s_i$ in Vives (1995) to derive the new results of this proof.

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