Collusion in Board of Directors

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1 Introduction

There is a large literature on the composition of the boards as well as the monitoring role and the advisory role of the boards. Nevertheless, the problem of collusion between the CEO and the board has receive little attention. The aim of this paper is to study collusive aspects of the board of directors. Our paper sheds light on the problem of composition of the board of directors. We study what is the optimal composition of the board of directors in particular if it is preferable to have a insider-oriented board or a outsider-oriented board with a majority of independent directors. We consider that a board of independent directors that are all chosen directly by the CEO is a friendly board if the independent directors follow the decision of the CEO. We study the case of collusion considering a CEO facing a choice of projects. We propose a model where we have different projects each with a certain level of risk. The choice of the best project for the company is function of remuneration of the CEO as well as the private benefits of the CEO.

2 The Model

2.1 The CEO and the Projects of the Company

A firm can undertake a project which yields an uncertain payoff. The firm is run for the shareholders by a CEO, i.e. the CEO's task is to select the project that will be undertaken by the firm.

The CEO's ability to succeed in the projects may be either low, $\beta = \underline{\beta}$, with probability (γ) or high, $\beta = \overline{\beta}$ with probability $(1 - \gamma)$. As $\underline{\beta}$ corresponds to a low CEO's ability and $\overline{\beta}$ to a high ability, we have $\overline{\beta} \geq \underline{\beta}$.

We assume that the firm can undertake two kinds of projects. The implementation of those projects initially require a fixed investment I by the firm's shareholders. The characteriscs of those projects are the following:

- Project 1 either succeeds, that is, yields verifiable income $R > 0$ or fails, that is, yields no income. The probability of success is denoted by (q_1). Moreover, this project may have a low probability of success, that is, $q_1 = \underline{p}\beta_i$with probability (ν) or may have a high probability of success $q_1 = \overline{p}\beta_i$ with probability $(1 - \nu)$ where $\beta_i \in \left\{\overline{\beta}; \underline{\beta}\right\}$ is the CEO's ability to succeed in the projects.

- In the same way, Project 2 either succeeds, that is, yields verifiable income $R > 0$ or fails, that is, yields no income. The probability of success is denoted by (q_2). Moreover, this project may have a low probability of success, that is, $q_2 = (\underline{p} - \varepsilon)\beta_i$ with probability (ν) or may have a high probability of success $q_2 = (\overline{p} + \varepsilon)\beta_i$ with probability $(1 - \nu)$ where $\beta_i \in \left\{\overline{\beta}; \underline{\beta}\right\}$ is the CEO's ability to succeed in the projects.

The success and the failure of both projects are assumed to be perfectly correlated i.e. (ν) represents the probability that the economic context is bad for yhe type of projects considered by the firm while (ε) represents the relative volativity of project two compared to project one.

The CEO perfectly knows both her ability's type and the probability of success of the projects while shareholders only know their prior probability distribution.

The CEO may therefore send signals to shareholders about her type and the probability of success of the selected project:

$$\begin{cases} \sigma_{LL} = (L, L) \Leftrightarrow & \beta = \underline{\beta} \text{ and } q_2 = \underline{p} - \varepsilon \ (\Leftrightarrow q_1 = \underline{p}) \\ \sigma_{LH} = (L, H) \Leftrightarrow & \beta = \underline{\beta} \text{ and } q_2 = \overline{p} + \varepsilon \ (\Leftrightarrow q_1 = \overline{p}) \\ \sigma_{HL} = (H, L) \Leftrightarrow & \beta = \overline{\beta} \text{ and } q_2 = \underline{p} - \varepsilon \ (\Leftrightarrow q_1 = \underline{p}) \\ \sigma_{HH} = (H, H) \Leftrightarrow & \beta = \overline{\beta} \text{ and } q_2 = \overline{p} + \varepsilon \ (\Leftrightarrow q_1 = \overline{p}) \end{cases}$$

The CEO's compensation (paid by the firm's shareholders) is composed by a fixed part α_{ij} and a variable part $\mu_{ij}\Pi$ that depends on the profits from the project (Π) where $i \in \{L, H\}$ corresponds to the CEO's signal about her ability (called hereafter the CEO's type) and $j \in \{L, H\}$ corresponds to the CEO's signal about the probability of success of the project (called hereafter the project's type).

When Project 2 is selected while it has a low probability of success $q_2 = (\underline{p} - \varepsilon)$, the CEO receives a private benefit B which represents his private compensation for choosing a project that poorly performs.

The CEO's reservation wage is w.

2.2 The Board of Directors

Shareholders also have the possibility to hire a Board. The Board has both a supervising and a consulting job, i.e. he may have information about the type of the project and can communicate it to shareholders but may also monitor the information communicated by the CEO.

The structure of the Board is endogenous, in the sense that shareholders choose it. More particularly, shareholders can choose the degree of independence of the Board. Lower is this degree of independence, more the Board's information about the type of the project is precise, but also more the Board is prone to engage in collusion with the CEO, both due to his relationship with the CEO and his executive role in the firm for instance.

We model the degree of independence of the board by a variable $\tau \in [1, +\infty]$ that acts as a discount factor for the collusion's rents. When his degree of independence, τ, increases, the amount of information hold by a Board decreases while his willingness to engage in collusion decreases.

Let $\xi(\tau) = \frac{1}{\tau}$ be t-he probability that a Board with a degree of independence τ has gathered the true information about the economic context.When τ increases, Board members are more independant and less prone to collusion. However, as they have less information about the firm, their probability of knowing the truth is lower. We also assume that the CEO incurs a fine F when the Board reveals to the shareholders that she has announced that the project has a high probability of success while it is a project with a low probability of success, i.e. the case in which she gets the bonus B.

We are particularly interested in determining the value of the degree of independence τ such that the Board is Independent i.e. is completely honest and never accepts to engage in collusion with the CEO (this however means that he has a less precise information about the type of the project).

When collusion takes place, we assume that the CEO shares the collusive profits with the Board.

The Board's wage is w_0.

2.3 Multidimensional Screening Model

This model is a multidimensional screening model. Solving this kind of model is usually very complex (see Rochet and Chone, 1998). However, the structure of the model allows us to reduce this problem's complexity in a usual unidimensional screening model as the CEO's objective only depends on one parameter, θ_{ij}, that can be defined in the following way:

$$\begin{cases} \theta_{LL} = \underline{p}\underline{\beta} \\ \theta_{LH} = (\overline{p} + \varepsilon)\,\underline{\beta} \\ \theta_{HL} = \underline{p}\overline{\beta} \\ \theta_{HH} = (\overline{p} + \varepsilon)\,\overline{\beta} \end{cases}$$

In this paper we assume that $\left(\underline{p} - \varepsilon\right)\overline{\beta} \geq (\overline{p} + \varepsilon)\,\underline{\beta}$, i.e. a high ability CEO undertaking a project with a low probability of success has more chances to succeed than a low ability CEO undertaking a project with a high probability of success, this assumption put forward the positive role of the CEO in his management of projects.

The shareholders maximize their expected profits:

$$\begin{aligned} W &= \nu\gamma\left[\left(\underline{p}\underline{\beta}R-I\right)-\alpha_{LL}-\mu_{LL}\left(\underline{p}\underline{\beta}R-I\right)\right] \\ &+(1-\nu)\gamma\left[\left(\left(\overline{p}+\varepsilon\right)\underline{\beta}R-I\right)-\alpha_{LH}-\mu_{LH}\left(\left(\overline{p}+\varepsilon\right)\underline{\beta}R-I\right)\right] \\ &+\nu(1-\gamma)\left[\left(\underline{p}\overline{\beta}R-I\right)-\alpha_{HL}-\mu_{HL}\left(\underline{p}\overline{\beta}R-I\right)\right] \\ &+(1-\nu)(1-\gamma)\left[\left(\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right)-\alpha_{HH}-\mu_{HH}\left(\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right)\right] \end{aligned}$$

3 No Board

When they do not hire a Board of Directors, shareholders maximize their expected profits under the usual Participation and Incentive constraints. PC_{ij} is the Participation constraint of a CEO with ability $i \in \{H, L\}$ when the project is of type $j \in \{H, L\}$. The Participation constraints ensure that the CEO will earn at least her reservation wage w. $IC_{ij\to kl}$ is the Incentive constraint of a CEO who reveals that her ability is $k \in \{H, L\}$ and the project is of type $l \in \{H, L\}$ while her true ability is i and the true type of the project is j. The Incentives constraints ensure that the CEO earns a higher wage revealing the truth than lying to the shareholders. Through this process, shareholders induce the CEO to reveal his real type. Those constraints are stated here:

$$\alpha_{LL}+\mu_{LL}\left[\underline{p}\underline{\beta}R-I\right]\geq w \tag{PC_{LL}}$$

$$\alpha_{LH}+\mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R-I\right]\geq w \tag{PC_{LH}}$$

$$\alpha_{HL}+\mu_{HL}\left[\underline{p}\overline{\beta}R-I\right]\geq w \tag{PC_{HL}}$$

$$\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right]\geq w \tag{PC_{HH}}$$

$$\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right]\geq \alpha_{HL}+\mu_{HL}\left[\overline{p}\overline{\beta}R-I\right] \tag{$IC_{HH\to HL}$}$$

$$\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right]\geq \alpha_{LH}+\mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right] \tag{$IC_{HH\to LH}$}$$

$$\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right]\geq \alpha_{LL}+\mu_{LL}\left[\overline{p}\overline{\beta}R-I\right] \tag{$IC_{HH\to LL}$}$$

$$\alpha_{HL}+\mu_{HL}\left[\underline{p}\overline{\beta}R-I\right]\geq \alpha_{HH}+\mu_{HH}\left[\left(\underline{p}-\varepsilon\right)\overline{\beta}R-I\right]+B \tag{$IC_{HL\to HH}$}$$

$$\alpha_{HL}+\mu_{HL}\left[\underline{p}\overline{\beta}R-I\right]\geq \alpha_{LH}+\mu_{LH}\left[\left(\underline{p}-\varepsilon\right)\overline{\beta}R-I\right]+B \tag{$IC_{HL\to LH}$}$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\underline{p}\overline{\beta}R - I\right] \quad (IC_{HL\to LL})$$

$$\alpha_{LH} + \mu_{LH}\left[(\overline{p}+\varepsilon)\underline{\beta}R - I\right] \geq \alpha_{HH} + \mu_{HH}\left[(\overline{p}+\varepsilon)\underline{\beta}R - I\right] \quad (IC_{LH\to HH})$$

$$\alpha_{LH} + \mu_{LH}\left[(\overline{p}+\varepsilon)\underline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\overline{p}\underline{\beta}R - I\right] \quad (IC_{LH\to HL})$$

$$\alpha_{LH} + \mu_{LH}\left[(\overline{p}+\varepsilon)\underline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\overline{p}\underline{\beta}R - I\right] \quad (IC_{LH\to LL})$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p\beta}R - I\right] \geq \alpha_{HH} + \mu_{HH}\left[\left(\underline{p}-\varepsilon\right)\underline{\beta}R - I\right] + B \quad (IC_{LL\to HH})$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\underline{p\beta}R - I\right] \quad (IC_{LL\to HL})$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p\beta}R - I\right] \geq \alpha_{LH} + \mu_{LH}\left[\left(\underline{p}-\varepsilon\right)\underline{\beta}R - I\right] + B \quad (IC_{LL\to LH})$$

Moreover, the Spence Mirrlees condition has to be satisfied, that is:

$$\mu_{HH} \geq \mu_{HL} \geq \mu_{LH} \geq \mu_{LL}$$

By assumption, we know that the following condition is satisfied:

$$\left(\underline{p}-\varepsilon\right)\overline{\beta} - (\overline{p}+\varepsilon)\underline{\beta} \geq 0 \quad (1)$$

As usual in this kind of problem, the binding constraints are[1] :

$$\alpha_{LL} + \mu_{LL}\left[\underline{p\beta}R - I\right] = w \quad (PC_{LL})$$

$$\begin{aligned} \alpha_{LH} + \mu_{LH}\left[(\overline{p}+\varepsilon)\underline{\beta}R - I\right] &= \alpha_{LL} + \mu_{LL}\left[\overline{p}\underline{\beta}R - I\right] \quad (IC_{LH\to LL}) \\ &= \alpha_{LL} + \mu_{LL}\left[\underline{p\beta}R - I\right] + \mu_{LL}R\underline{\beta}\Delta p \\ &= w + \mu_{LL}R\underline{\beta}\Delta p \end{aligned}$$

$$\begin{aligned} \alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] &= \alpha_{LH} + \mu_{LH}\left[\left(\underline{p}-\varepsilon\right)\overline{\beta}R - I\right] + B \quad (IC_{HL\to LH}) \\ &= \alpha_{LH} + \mu_{LH}\left[(\overline{p}+\varepsilon)\underline{\beta}R - I\right] - \mu_{LH}R\left[(\overline{p}+\varepsilon)\underline{\beta} - \left(\underline{p}-\varepsilon\right)\overline{\beta}\right] + B \\ &= w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right] + B \end{aligned}$$

[1] We check that all constraints are satisfied in the Appendix.

$$
\begin{aligned}
\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right] &= \alpha_{HL}+\mu_{HL}\left[\overline{p}\overline{\beta}R-I\right] && (IC_{HH\to HL})\\
&= \alpha_{HL}+\mu_{HL}\left[\underline{p}\overline{\beta}R-I\right]+\mu_{HL}R\overline{\beta}\Delta p\\
&= w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+\mu_{HL}R\overline{\beta}\Delta p+B
\end{aligned}
$$

The optimal contract when there is no Board in the firm's organizaton is characterized in the following Proposition:

Proposition 1 *When they do not hire a Board of Directors, shareholders must conced the following informational rents to a CEO*

$$
\begin{aligned}
U_{LL} &= w\\
U_{LH} &= w\\
U_{HL} &= w+\frac{B\left(\underline{p}-\varepsilon\right)\Delta\beta}{\underline{\beta}\left[\Delta p+2\varepsilon\right]}\\
U_{HH} &= \begin{cases} w+\frac{B\left(\underline{p}-\varepsilon\right)\overline{p}(\Delta\beta)^2}{\underline{\beta}[\Delta p+2\varepsilon]\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]} \;\; if\; \varepsilon\leq\varepsilon_{nb}=\frac{\underline{\beta}\Delta p}{\Delta\beta+\frac{\underline{p}}{\overline{p}}\overline{\beta}-\underline{\beta}}\\ w+\frac{B\Delta\beta(\overline{p}+\varepsilon)}{\underline{\beta}[\Delta p+2\varepsilon]} \;\; if\; \varepsilon\geq\varepsilon_{nb}\end{cases}
\end{aligned}
$$

Moreover, the shareholders' expected profits are

$$
W_{NB}=\begin{cases} E(\pi)-w-\frac{(1-\gamma)B\left(\underline{p}-\varepsilon\right)\Delta\beta}{\underline{\beta}(\Delta p+2\varepsilon)}\left[\frac{-\nu\overline{\beta}\Delta p+\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\overline{p}\underline{\beta}}\right] \;\; if\; \varepsilon\leq\varepsilon_{nb}\\ E(\pi)-w-(1-\gamma)\left(\overline{p}+\varepsilon-\nu\Delta p-2\nu\varepsilon\right)\frac{B\Delta\beta}{\underline{\beta}(\Delta p+2\varepsilon)} \;\; if\; \varepsilon\geq\varepsilon_{nb}\end{cases}
$$

A low ability CEO does not receive any rent whatever the type of project she advises to select. However, when her signal pushes shareholders to select the project with the highest volatility (Project 2), she receives a variable wage while she only gets a fixed wage when shareholders are induced to select Project 1.

A high ability CEO receives an informational rent which is higher when her signal induces shareholders to select the project with the highest volatility (Project 2) than when shareholders are induced to select Project 1. Moreover, the variable part of her wage is higher when project 2 is finally selected than when it is Project 1. But, in all cases, the variable part of a high ability CEO is higher that the one of a low ability CEO.

4 No Collusion

In this section, we assume that collusion is not possible between the Board of Directors and the CEO[2].When shareholders hire a Board, the CEO may incur a loss F when the Board has found that she has announced that the Project has a high probability of success while it is a low probability of success project, i.e. the case in which she has the bonus B. The Participation and Incentive constraints are now:

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq w \tag{PC_{LL}}$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p} + \varepsilon\right)\underline{\beta}R - I\right] \geq w \tag{PC_{LH}}$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq w \tag{PC_{HL}}$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p} + \varepsilon\right)\overline{\beta}R - I\right] \geq w \tag{PC_{HH}}$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p} + \varepsilon\right)\overline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\overline{p}\overline{\beta}R - I\right] \tag{$IC_{HH\to HL}$}$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p} + \varepsilon\right)\overline{\beta}R - I\right] \geq \alpha_{LH} + \mu_{LH}\left[\left(\overline{p} + \varepsilon\right)\overline{\beta}R - I\right] \tag{$IC_{HH\to LH}$}$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p} + \varepsilon\right)\overline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\overline{p}\overline{\beta}R - I\right] \tag{$IC_{HH\to LL}$}$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \left(1 - \xi(\tau)\right)\left\{\alpha_{HH} + \mu_{HH}\left[\left(\underline{p} - \varepsilon\right)\overline{\beta}R - I\right] + B\right\} + \xi(\tau)\left(w - F\right) \tag{$IC_{HL\to HH}$}$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \left(1 - \xi(\tau)\right)\left\{\alpha_{LH} + \mu_{LH}\left[\left(\underline{p} - \varepsilon\right)\overline{\beta}R - I\right] + B\right\} + \xi(\tau)\left(w - F\right) \tag{$IC_{HL\to LH}$}$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\underline{p}\overline{\beta}R - I\right] \tag{$IC_{HL\to LL}$}$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p} + \varepsilon\right)\underline{\beta}R - I\right] \geq \alpha_{HH} + \mu_{HH}\left[\left(\overline{p} + \varepsilon\right)\underline{\beta}R - I\right] \tag{$IC_{LH\to HH}$}$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p} + \varepsilon\right)\underline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\overline{p}\underline{\beta}R - I\right] \tag{$IC_{LH\to HL}$}$$

[2] We examine the case of collusion in the next section.

$$\alpha_{LH} + \mu_{LH} \left[\left(\overline{p} + \varepsilon \right) \underline{\beta} R - I \right] \geq \alpha_{LL} + \mu_{LL} \left[\overline{p} \underline{\beta} R - I \right] \tag{$IC_{LH \to LL}$}$$

$$\alpha_{LL} + \mu_{LL} \left[\underline{p} \underline{\beta} R - I \right] \geq \left(1 - \xi(\tau) \right) \left\{ \alpha_{HH} + \mu_{HH} \left[\left(\underline{p} - \varepsilon \right) \underline{\beta} R - I \right] + B \right\} + \xi(\tau) \left(w - F \right) \tag{$IC_{LL \to HH}$}$$

$$\alpha_{LL} + \mu_{LL} \left[\underline{p} \underline{\beta} R - I \right] \geq \alpha_{HL} + \mu_{HL} \left[\underline{p} \underline{\beta} R - I \right] \tag{$IC_{LL \to HL}$}$$

$$\alpha_{LL} + \mu_{LL} \left[\underline{p} \underline{\beta} R - I \right] \geq \left(1 - \xi(\tau) \right) \left\{ \alpha_{LH} + \mu_{LH} \left[\left(\underline{p} - \varepsilon \right) \underline{\beta} R - I \right] + B \right\} + \xi(\tau) \left(w - F \right) \tag{$IC_{LL \to LH}$}$$

We also assume that the CEO faces a limited liability constraint, i.e., even if the Board found that the CEO has sent the wrong signal, she cannot get less than her reservation wage. This gives:

$$\begin{aligned} \left(1 - \xi(\tau) \right) \left\{ w + B \right\} + \xi(\tau) \left(w - F \right) \geq w \\ \Leftrightarrow B \geq \frac{\xi(\tau)}{\left(1 - \xi(\tau) \right)} F \end{aligned} \tag{LL}$$

The binding constraints are:

$$\alpha_{LL} + \mu_{LL} \left[\underline{p} \underline{\beta} R - I \right] = w \tag{PC_{LL}}$$

$$\begin{aligned} \alpha_{LH} + \mu_{LH} \left[\left(\overline{p} + \varepsilon \right) \underline{\beta} R - I \right] &= \alpha_{LL} + \mu_{LL} \left[\overline{p} \underline{\beta} R - I \right] \\ &= \alpha_{LL} + \mu_{LL} \left[\underline{p} \underline{\beta} R - I \right] + \mu_{LL} R \underline{\beta} \Delta p \\ &= w + \mu_{LL} R \underline{\beta} \Delta p \end{aligned} \tag{$IC_{LH \to LL}$}$$

$$\begin{aligned} \alpha_{HL} + \mu_{HL} \left[\underline{p} \overline{\beta} R - I \right] &= \left(1 - \xi(\tau) \right) \left\{ \alpha_{LH} + \mu_{LH} \left[\left(\underline{p} - \varepsilon \right) \overline{\beta} R - I \right] + B \right\} + \xi(\tau) \left(w - F \right) \\ &= \left(1 - \xi(\tau) \right) \left\{ \begin{array}{c} \alpha_{LH} + \mu_{LH} \left[\left(\overline{p} + \varepsilon \right) \underline{\beta} R - I \right] \\ -\mu_{LH} R \left[\left(\overline{p} + \varepsilon \right) \underline{\beta} - \left(\underline{p} - \varepsilon \right) \overline{\beta} \right] + B \end{array} \right\} + \xi(\tau) \left(w - F \right) \\ &= \left(1 - \xi(\tau) \right) \left\{ w + \mu_{LL} R \underline{\beta} \Delta p + \mu_{LH} R \left[\underline{p} \overline{\beta} - \overline{p} \underline{\beta} - \varepsilon \left(\underline{\beta} + \overline{\beta} \right) \right] + B \right\} + \xi(\tau) \left(w - F \right) \end{aligned} \tag{$IC_{HL \to LH}$}$$

$$\begin{aligned} \alpha_{HH} + \mu_{HH} \left[\left(\overline{p} + \varepsilon \right) \overline{\beta} R - I \right] &= \alpha_{HL} + \mu_{HL} \left[\overline{p} \overline{\beta} R - I \right] \\ &= \alpha_{HL} + \mu_{HL} \left[\underline{p} \overline{\beta} R - I \right] + \mu_{HL} R \overline{\beta} \Delta p \\ &= \left(1 - \xi(\tau) \right) \left\{ w + \mu_{LL} R \underline{\beta} \Delta p + \mu_{LH} R \left[\underline{p} \overline{\beta} - \overline{p} \underline{\beta} - \varepsilon \left(\underline{\beta} + \overline{\beta} \right) \right] + B \right\} \\ &\quad + \xi(\tau) \left(w - F \right) + \mu_{HL} R \overline{\beta} \Delta p \end{aligned} \tag{$IC_{HH \to HL}$}$$

The optimal contract when there is a Board of Director and when collusion is not achievable is characterized in the following Proposition:

Proposition 2 *When they hire a Board of Directors and when collusion is not possible, shareholders must concede the following informational rents to a CEO*

$$\begin{aligned}
U_{LL} &= w \\
U_{LH} &= w \\
U_{HL} &= w + (1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\Delta\beta\left(\frac{\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]}\right) \\
U_{HH} &= \begin{cases} w+\frac{\left(\underline{p}-\varepsilon\right)\overline{p}(\Delta\beta)^2[(1-\xi(\tau))B-\xi(\tau)F]}{\underline{\beta}[\Delta p+2\varepsilon]\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]} & \textit{if } \varepsilon\leq\varepsilon_{ib}=\frac{\underline{\beta}\Delta p-\xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta+\frac{\underline{p}}{\overline{p}}\overline{\beta}-\underline{\beta}} \\ w+\frac{(\overline{p}+\varepsilon)\Delta\beta\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]} & \textit{if } \varepsilon\geq\varepsilon_{ib} \end{cases}
\end{aligned}$$

Moreover, the shareholders' expected profits are

$$W_{IB}=\begin{cases} E(\pi)-w-w_0-(1-\gamma)(1-\xi(\tau))(\underline{p}-\varepsilon)\Delta\beta\frac{\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}(\Delta p+2\varepsilon)}\left[\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right] & \textit{if } \varepsilon\leq\varepsilon_{ib} \\ E(\pi)-w-w_0-(1-\gamma)\Delta\beta\frac{\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}(\Delta p+2\varepsilon)}\left[\nu(1-\xi(\tau))(\underline{p}-\varepsilon)+(1-\nu)(\overline{p}+\varepsilon)\right] & \textit{if } \varepsilon\geq\varepsilon_{ib} \end{cases}$$

In this case, the optimal contract has the same form than without Board, i.e. no rent for a low ability CEO and a positive rent for a high ability CEO which is higher when Project 2 is selected following her advice. However, we can note that the informational rents extracted by a CEO when there is an a Board of Directors having no possibility to collude are lower than when there is no Board whatever the CEO's type.

We can therefore immediately conclude that there exists a Board's wage $\widetilde{w_0}$ such that for all $w_0 \leq \widetilde{w_0}$, hiring an a Board is always beneficial for the shareholders when collusion is not possible, i.e. $W_{IB} \geq W_{NB}$ for all $w_0 \leq \widetilde{w_0}$.

5 Collusive Board

We now examine a framework in which the CEO and the Board of Directors may collude when this is profitable for them.

In the following inequalities, w_L is the income of a board that announces that the project has a low probability of success, w_H is the income of a board that announces that the project has a high probability of success, $w_\emptyset$ is the income of a board that announces that it has no information regarding the project probability of success, w_0 is the income of a board that says the truth, i.e. an Independent Board.

The following constraints reveal that the CEO-Board coalition had better tell the truth than collude with each other.

$$\gamma\left[U_{LL}-w+w_L\right]+(1-\gamma)\left[U_{HL}-w+w_L\right]\geq\gamma\left[\frac{U_{LH}-w}{\tau}+w_\emptyset\right]+(1-\gamma)\left[\frac{U_{HH}-w}{\tau}+w_\emptyset\right]$$

$$\Leftrightarrow w_L\geq\gamma\left[\frac{U_{LH}-w}{\tau}-(U_{LL}-w)\right]+(1-\gamma)\left[\frac{U_{HH}-w}{\tau}-(U_{HL}-w)\right]+w_\emptyset$$

We set $A=\gamma\left[\frac{U_{LH}-w}{\tau}-(U_{LL}-w)\right]+(1-\gamma)\left[\frac{U_{HH}-w}{\tau}-(U_{HL}-w)\right]$

$$\gamma\left[U_{LH}-w+w_H\right]+(1-\gamma)\left[U_{HH}-w+w_H\right]\geq\gamma\left[\frac{U_{LL}-w}{\tau}+w_\emptyset\right]+(1-\gamma)\left[\frac{U_{HL}-w}{\tau}+w_\emptyset\right]$$

$$\Leftrightarrow w_H\geq\gamma\left[\frac{U_{LL}-w}{\tau}-(U_{LH}-w)\right]+(1-\gamma)\left[\frac{U_{HL}-w}{\tau}-(U_{HH}-w)\right]+w_\emptyset$$

We set $B=\gamma\left[\frac{U_{LL}-w}{\tau}-(U_{LH}-w)\right]+(1-\gamma)\left[\frac{U_{HL}-w}{\tau}-(U_{HH}-w)\right]$

Since we have $U_{LL}\leq U_{LH}\leq U_{HL}\leq U_{HH}$ and $\tau\geq 1$, necessarily $B\leq 0$.

We then have 4 constraints to satify:

$$\begin{aligned} w_L &\geq A+w_\emptyset \ (1)\\ w_H &\geq B+w_\emptyset \ (2)\\ w_L &\geq w_0 \ (3)\\ w_H &\geq w_0 \ (4)\end{aligned}$$

5.1 Collusion-Proof contract

In this situation, shareholders want to avoid collusion in the Board. The only case they have to take into account is when the Board says there is a low probablity of success (the Board is more likely to lie when the project is of a low probability of success ; there is no point in lying when it is of a high probability of success): therefore we will always have $w_L\geq w_H$. They can try to use w_L to pay the Board into revealing the truth: if they set w_L high enough, collusion might be avoided. The shareholders' expected profits have the following form:

$$\begin{aligned} W_{CP} &= E(\pi)-\gamma\nu U_{LL}-\gamma(1-\nu)U_{LH}-(1-\gamma)\nu U_{HL}-(1-\gamma)(1-\nu)U_{HH}\\ &\quad -\nu\xi(\tau)w_L-(1-\nu)\xi(\tau)w_H-(1-\xi(\tau))w_0\end{aligned}$$

In that case, the constraint on w_L is binding. Since they want to maximize their income, shareholders set $w_H=w_\emptyset=w_0$ (because w_0 is the lowest wage of the board).

$$
\begin{aligned}
w_L &= \gamma\left[\frac{U_{LH}-w}{\tau}-(U_{LL}-w)\right]+(1-\gamma)\left[\frac{U_{HH}-w}{\tau}-(U_{HL}-w)\right]+w_\emptyset \\
&= \gamma\left[\frac{U_{LH}-U_{LL}}{\tau}+\frac{1-\tau}{\tau}(U_{LL}-w)\right]+(1-\gamma)\left[\frac{U_{HH}-U_{HL}}{\tau}+\frac{1-\tau}{\tau}(U_{HL}-w)\right]+w_\emptyset \\
w_H &= w_\emptyset = w_0
\end{aligned}
$$

We can remark that there exists τ_0 such that $w_L \geq w_0 \Longleftrightarrow \tau \leq \tau_0$. This means that for $\tau \geq \tau_0$, engaging in collusion is not beneficial for the coalition Board-CEO and the optimal contract is the same as with an Independent Board. Actually, when $\tau \geq \tau_0$, the Board will not collude whatsoever happens. Shareholders don't need to induce the Board to say the truth because he will do it anyway. So, we have in this case

$$
w_L = w_H = w_0
$$

We are now characterizing τ_0

$$
\begin{aligned}
w_L \geq w_0 &\Longleftrightarrow \tau\left[\gamma(U_{LL}-w)+(1-\gamma)(U_{HL}-w)\right] \leq \gamma(U_{LH}-w)+(1-\gamma)(U_{HH}-w) \\
&\Longleftrightarrow \tau \leq \frac{U_{HH}-w}{U_{HL}-w}
\end{aligned}
$$

$$
\tau_0 = \begin{cases} \frac{1}{1-\xi(\tau)}\frac{\overline{p}+\varepsilon}{\underline{p}-\varepsilon} \text{ if } \varepsilon \geq \varepsilon_{ib} \\ \frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \text{ if } \varepsilon \leq \varepsilon_{ib} \end{cases}
$$

However, on the interval $[1;\tau_0]$, since shareholders have paid enough to avoid collusion, the CEO's rents are those of an Independent Board. For those degree of independence, since shareholders have paid enough to avoid collusion, the CEO's rents are those of an Independent Board:

$$
\begin{aligned}
U_{LL} &= w \\
U_{LH} &= w \\
U_{HL} &= w+(1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\Delta\beta\left(\frac{\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]}\right) \\
U_{HH} &= \begin{cases} w+\frac{\left(\underline{p}-\varepsilon\right)\overline{p}(\Delta\beta)^2\left[(1-\xi(\tau))B-\xi(\tau)F\right]}{\underline{\beta}\left[\Delta p+2\varepsilon\right]\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]} \text{ if } \varepsilon \leq \varepsilon_{ib} \\ w+\frac{(\overline{p}+\varepsilon)\Delta\beta\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]} \text{ if } \varepsilon \geq \varepsilon_{ib} \end{cases}
\end{aligned}
$$

This is stated in the following Proposition.

Proposition 3 *Assume that collusion between the Board of Directors and the CEO is possible. In the optimal collusion proof contract, when they hire a Board of Directors, shareholders must concede the same rents to a CEO than in the presence of an Independent Board.*

In this case, the shareholders' expected profits are

$$W_{CP} = \begin{cases} E(\pi) - w - w_0 - (1-\gamma)(1-\xi(\tau))(\underline{p}-\varepsilon)\Delta\beta \frac{\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}(\Delta p + 2\varepsilon)} \left[\nu(1-\xi(\tau)) + (1-\nu+\frac{\xi(\tau)\nu}{\tau})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right] & \textit{if } \varepsilon \leq \\ E(\pi) - w - w_0 - (1-\gamma)\Delta\beta \frac{\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}(\Delta p + 2\varepsilon)} \left[\nu(1-\xi(\tau))^2(\underline{p}-\varepsilon) + (1-\nu+\frac{\xi(\tau)\nu}{\tau})(\overline{p}+\varepsilon)\right] & \textit{if } \varepsilon \geq \varepsilon_{it} \end{cases}$$

Moreover, there exists τ_0 such that for Boards of Directors with a degree of independence $\tau \geq \tau_0$, it is not beneficial to engage in collusion.

5.2 Collusion Free contract

We now characterize the optimal collusion free contract. In this case, shareholders would have to pay too much to avoid collusion, they therefore decide to let it happen because they would throw money out of the window if they paid the board. The shareholders' expected profits have the following form:

$$\begin{aligned} W_{CF} &= E(\pi) - \gamma\nu U_{LL} - \gamma(1-\nu)U_{LH} - (1-\gamma)\nu U_{HL} - (1-\gamma)(1-\nu)U_{HH} \\ &\quad -\nu\xi(\tau)w_L - (1-\nu)\xi(\tau)w_H - (1-\xi(\tau))w_\emptyset \end{aligned}$$

This is optimal to set $w_L = w_0$. Inequalities (1) and (2) do not need to be satisfied. Subsequently, we have:

$$w_L = w_H = w_\emptyset = w_0$$

Since the Board is collusive, shareholders should not trust what it says for their own good. Therefore, the CEO's rents are those of a No Board case.

$$\begin{aligned} U_{LL} &= w \\ U_{LH} &= w \\ U_{HL} &= w + \frac{B(\underline{p}-\varepsilon)\Delta\beta}{\underline{\beta}[\Delta p + 2\varepsilon]} \\ U_{HH} &= \begin{cases} w + \frac{B(\underline{p}-\varepsilon)\overline{p}(\Delta\beta)^2}{\underline{\beta}[\Delta p + 2\varepsilon][\underline{p}\overline{\beta}-\overline{p}\underline{\beta}]} & \text{if } \varepsilon \leq \varepsilon_{nb} \\ w + \frac{B\Delta\beta(\overline{p}+\varepsilon)}{\underline{\beta}[\Delta p + 2\varepsilon]} & \text{if } \varepsilon \geq \varepsilon_{nb} \end{cases} \end{aligned}$$

Proposition 4 *Assume that collusion between the Board of Directors and the CEO is possible. In the optimal collusion free contract, when they hire a Board of Directors, shareholders must concede the same rents to a CEO than without any Board.*

In this case, the shareholders' expected profits are

$$W_{CF} = \begin{cases} E(\pi) - w_0 - w - (1-\gamma)\left[\nu + (1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right]\frac{B\Delta\beta(\underline{p}-\varepsilon)}{\underline{\beta}(\Delta p + 2\varepsilon)} & \textit{if } \varepsilon \leq \varepsilon_{nb} \\ E(\pi) - w_0 - w - (1-\gamma)\left[(\overline{p}+\varepsilon) - \nu(\Delta p + 2\varepsilon)\right]\frac{B\Delta\beta}{\underline{\beta}(\Delta p + 2\varepsilon)} & \textit{if } \varepsilon \geq \varepsilon_{nb} \end{cases}$$

5.3 Optimal Contract with collusion

Let's remind that $\xi(\tau) = \frac{1}{\tau}$.

In order to find the optimal contract in presence of collusion, W_{CB}, we have to compare W_{CP} and W_{CF} and to find which one is the highest depending on τ. Indeed, the shareholders will choose to design the contract (Collusion Proof or Collusion Free) in order to maximize their objective. As $\varepsilon_{ib} \leq \varepsilon_{nb}$ we only have three cases:

1. $\varepsilon \leq \varepsilon_{ib}$
2. $\varepsilon_{ib} \leq \varepsilon \leq \varepsilon_{nb}$
3. $\varepsilon_{nb} \leq \varepsilon$

The following Proposition characterizes the optimal contract when collusion is achievable.

Proposition 5 *For all $\tau \in [1, \tau_0]$, the optimal contract is the collusion proof contract for all ε.*

This allows us to state that the shareholders's welfare, W_{CB} that depends on τ is, for all $\tau \in [1, \tau_0]$:

$$W_{CB}(\tau) = max(W_{CP}; W_{CF}) = W_{CP}(\tau)$$

This is an important result as it means that when collusion is achievable and is profitable fot the coalition Board/CEO, it is beneficial for the shareholders to offer a contract preventing collusion to emerge. However, this is costly in terms of informational rents.

This result and those of the previopus sections allow us to characterize what is the optimal structure of teh Board of Directors from the shareholders' perspective.

6 Optimal Structure of the Board

We are now able to find what is the optimal Board's degree of independence τ^* maximizing the piecewise continuous shareholders's welfare $W_{CB}(\tau)$.

We have to take care about corner solutions as $\tau \in [1; \tau_0]$.

In order to simplify the computations, we rewrite the intervals of discontinuity of $W_{CB}(\tau)$ in order to build them with repect to τ. This gives

$$\varepsilon \geq \varepsilon_{ib} = \frac{\overline{p}\underline{\beta}\Delta p - \frac{1}{\tau}\overline{p}\underline{p}\Delta\beta}{\frac{(\tau-1)}{\tau}\Delta\beta\overline{p} + \underline{p}\overline{\beta} - \overline{p}\underline{\beta}} \Leftrightarrow \tau \leq \frac{\frac{\Delta\beta\overline{p}}{\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]}}{\frac{\Delta\beta\overline{p}}{\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}} = \widehat{\tau}$$

Hence, when $\left[\frac{\Delta\beta\overline{p}}{\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right] \leq 0, \Longleftrightarrow \varepsilon \ \geq \frac{\overline{p}\underline{\beta}\Delta p}{\Delta\beta\overline{p} + \left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]} = \widehat{\varepsilon},$

$$\varepsilon \geq \varepsilon_{ib} \text{ for all } \tau$$

and when $\left[\frac{\Delta\beta\overline{p}}{\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]}-\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right]\geq 0,\Longleftrightarrow \varepsilon\leq\frac{\overline{p}\underline{\beta}\Delta p}{\Delta\beta\overline{p}+\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]}=\widehat{\varepsilon},$

$$\begin{aligned}\varepsilon &\geq \varepsilon_{ib} \text{ for } \tau\leq\widehat{\tau}, \text{ and}\\ \varepsilon &\leq \varepsilon_{ib} \text{ for } \tau\geq\widehat{\tau}\end{aligned}$$

The shareholders thus have the following objective function[3]:

When $\varepsilon\leq\widehat{\varepsilon}$,

$$W_{CB}(\tau)=\begin{cases} \begin{array}{c} E(\pi)-w-w_0 \\ -(1-\gamma)\Delta\beta\frac{\left[B-\frac{1}{\tau-1}F\right]}{\underline{\beta}(\Delta p+2\varepsilon)}\left[\begin{array}{c}\nu(\frac{\tau-1}{\tau})^2(\underline{p}-\varepsilon)\\ +(1-\nu+\frac{\nu}{\tau^2})(\overline{p}+\varepsilon)\end{array}\right]\end{array} & \text{if } \tau\leq\widehat{\tau}\\ \begin{array}{c} E(\pi)-w-w_0 \\ -(1-\gamma)(\frac{\tau-1}{\tau})(\underline{p}-\varepsilon)\Delta\beta\frac{\left[B-\frac{1}{\tau-1}F\right]}{\underline{\beta}(\Delta p+2\varepsilon)}\left[\begin{array}{c}\nu(\frac{\tau-1}{\tau})\\ +(1-\nu+\frac{\nu}{\tau^2})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\end{array}\right]\end{array} & \text{if } \widehat{\tau}\leq\tau\leq\tau_0\\ \begin{array}{c} E(\pi)-w-w_0 \\ -(1-\gamma)(\frac{\tau-1}{\tau})(\underline{p}-\varepsilon)\Delta\beta\frac{\left[B-\frac{1}{\tau-1}F\right]}{\underline{\beta}(\Delta p+2\varepsilon)}\left[\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right]\end{array} & \text{if } \tau\geq\tau_0\end{cases}$$

When $\varepsilon\geq\widehat{\varepsilon}$,

$$W_{CB}(\tau)=\begin{cases} \begin{array}{c} E(\pi)-w-w_0 \\ -(1-\gamma)\Delta\beta\frac{\left[B-\frac{1}{\tau-1}F\right]}{\underline{\beta}(\Delta p+2\varepsilon)}\left[\begin{array}{c}\nu(\frac{\tau-1}{\tau})^2(\underline{p}-\varepsilon)\\ +(1-\nu+\frac{\nu}{\tau^2})(\overline{p}+\varepsilon)\end{array}\right]\end{array} & \text{if } \tau\leq\tau_0\\ \begin{array}{c} E(\pi)-w-w_0 \\ -(1-\gamma)\Delta\beta\frac{\left[B-\frac{1}{\tau-1}F\right]}{\underline{\beta}(\Delta p+2\varepsilon)}\left[\nu(\frac{\tau-1}{\tau})(\underline{p}-\varepsilon)+(1-\nu)(\overline{p}+\varepsilon)\right]\end{array} & \text{if } \tau\geq\tau_0\end{cases}$$

Lemma 6 *$W_{CB}(\tau)$ is non increasing on each interval of discontinuity.*

This allows us to compute the Board's optimal degree of independence, τ^*.

As $W_{CB}(\tau)$ is non increasing on each interval, we have to compare its value for the lower bound of each interval[4].

The following proposition summarizes our results:

Proposition 7 *When $\varepsilon\leq\widehat{\varepsilon}$, $\tau_{\min}\geq\tau^*_{\min}$ and $\tau_0\geq\overline{\tau_0}$, it is optimal for the shareholders to select a Board of Directors with a low degree of independence, i.e. $\tau^*=\tau_{\min}$ and to offer contracts avoiding collusion between the Board and the CEO.*

In all other cases, it is optimal for the shareholders to select a Board of Directors with a high degree of independence, i.e. $\tau^*=\tau_0$. In this case, the shareholders should not care about collusion because collusion is not profitable for such Boards.

[3] As $\widehat{\tau}\leq\tau_0\ \forall\varepsilon$

[4] Assuming that $\tau_{\min}$ is such that $\left[B-\frac{1}{\tau_{\min}-1}F\right]=K>0$.

Contrary to the usual idea that the optimal Board should be independant, we find that shareholders may prefer to select a Board of Directors with a low degree of independence. However, the result is not due, as in Adams and Ferreira (2007), to the fact that the CEO is more prone to reveal information to a "friendly" Board. Here, there is a trade-off between the information that shareholders may extract from the Board and the costs from both extracting it and avoiding collusion. When the risk of both projects are close, when there are constraints (ethical or technical) on the level of collusion that can be sustained and when the degree of independence necessary to have a perfectly honest Board is too high, the optimal structure is a Board with a low degree of independence. However, when project 2 is too risky compared to project 1 or when collusion is not too costly for the coalition or when he degree of independence necessary to have a perfectly honest Board is low enough, the optimal structure is a Board with a high degree of independence and the shareholders should not care about collusion because collusion is not profitable for such Boards.

6.1 Camparative statics

We now have to find what are the effects of $B, F, \nu, \overline{p}, \underline{p}, \overline{\beta}, \underline{\beta}, \varepsilon$ on the optimal τ^*. In particular, it would be interesting to know how the region of parameters such that a Board with a low degree of independence is optimal varies when those parameters vary.

When $\frac{B}{F}$ is high (i.e. B high or F low), the optimal τ seems to be $\tau^* = \tau_{\min}$ (conversely, when $\frac{B}{F}$ is low , $\tau^* = \tau_0$)

When ε is high, the optimal τ is $\tau^* = \tau_0$ (conversely, when ε is low , $\tau^* = \tau_{\min}$)

7 Conclusion

8 Appendix

Proof of Proposition 1. When they do not hire a Board of Directors, shareholders maximize their expected profits under the usual Participation and Incentive constraints. PC_{ij} is the Participation constraint of a CEO with ability $i \in \{H, L\}$ when the project is of type $j \in \{H, L\}$. The Participation constraints ensure that the CEO will earn at least her reservation wage w. $IC_{ij \to kl}$ is the Incentive constraint of a CEO who reveals that her ability is $k \in \{H, L\}$ and the project is of type $l \in \{H, L\}$ while her true ability is i and the true type of the project is j. The Incentives constraints ensure that the CEO earns a higher wage revealing the truth than lying to the shareholders. Through this process, shareholders induce the CEO to reveal his real type. Those constraints are stated here:

$$\alpha_{LL} + \mu_{LL} \left[\underline{p}\underline{\beta} R - I \right] \geq w \qquad (PC_{LL})$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] \geq w \tag{PC_{LH}}$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq w \tag{PC_{HL}}$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R - I\right] \geq w \tag{PC_{HH}}$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\overline{p}\overline{\beta}R - I\right] \tag{$IC_{HH\to HL}$}$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R - I\right] \geq \alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R - I\right] \tag{$IC_{HH\to LH}$}$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\overline{p}\overline{\beta}R - I\right] \tag{$IC_{HH\to LL}$}$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \alpha_{HH} + \mu_{HH}\left[\left(\underline{p}-\varepsilon\right)\overline{\beta}R - I\right] + B \tag{$IC_{HL\to HH}$}$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \alpha_{LH} + \mu_{LH}\left[\left(\underline{p}-\varepsilon\right)\overline{\beta}R - I\right] + B \tag{$IC_{HL\to LH}$}$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\underline{p}\overline{\beta}R - I\right] \tag{$IC_{HL\to LL}$}$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] \geq \alpha_{HH} + \mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] \tag{$IC_{LH\to HH}$}$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\overline{p}\underline{\beta}R - I\right] \tag{$IC_{LH\to HL}$}$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\overline{p}\underline{\beta}R - I\right] \tag{$IC_{LH\to LL}$}$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq \alpha_{HH} + \mu_{HH}\left[\left(\underline{p}-\varepsilon\right)\underline{\beta}R - I\right] + B \tag{$IC_{LL\to HH}$}$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\underline{p}\underline{\beta}R - I\right] \tag{$IC_{LL\to HL}$}$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq \alpha_{LH} + \mu_{LH}\left[\left(\underline{p}-\varepsilon\right)\underline{\beta}R - I\right] + B \tag{$IC_{LL\to LH}$}$$

Moreover, the Spence Mirrlees condition has to be satisfied, that is:

$$\mu_{HH} \geq \mu_{HL} \geq \mu_{LH} \geq \mu_{LL}$$

By assumption, we know that (1) is satisfied:

$$\left(\underline{p}-\varepsilon\right)\overline{\beta} - \left(\overline{p}+\varepsilon\right)\underline{\beta} \geq 0$$

As usual in this kind of problem, the binding constraints are :

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] = w \tag{PC_{LL}}$$

$$\begin{aligned} \alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] &= \alpha_{LL} + \mu_{LL}\left[\overline{p}\underline{\beta}R - I\right] \\ &= \alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] + \mu_{LL}R\underline{\beta}\Delta p \\ &= w + \mu_{LL}R\underline{\beta}\Delta p \end{aligned} \tag{$IC_{LH\to LL}$}$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] = \alpha_{LH} + \mu_{LH}\left[\left(\underline{p} - \varepsilon\right)\overline{\beta}R - I\right] + B \qquad (IC_{HL\to LH})$$

$$= \alpha_{LH} + \mu_{LH}\left[\left(\overline{p} + \varepsilon\right)\underline{\beta}R - I\right] - \mu_{LH}R\left[\left(\overline{p} + \varepsilon\right)\underline{\beta} - \left(\underline{p} - \varepsilon\right)\overline{\beta}\right] + B$$

$$= w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + B$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p} + \varepsilon\right)\overline{\beta}R - I\right] = \alpha_{HL} + \mu_{HL}\left[\overline{p}\overline{\beta}R - I\right] \qquad (IC_{HH\to HL})$$

$$= \alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] + \mu_{HL}R\overline{\beta}\Delta p$$

$$= w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + \mu_{HL}R\overline{\beta}\Delta p + B$$

In order to minimize the CEO's informational rents, shareholders set μ_{LL}, μ_{LH} and μ_{HL} as low as possible while satisfying the other Incentive constraints.

We now check what are the conditions due to the other Incentive constraints (and will check later that Participation constraints are satisfied). There is no constraint on μ_{LL}, we can therefore set:

$$\mu_{LL} = 0$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq \alpha_{LH} + \mu_{LH}\left[\left(\underline{p} - \varepsilon\right)\underline{\beta}R - I\right] + B = \qquad (IC_{LL\to LH})$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p} + \varepsilon\right)\underline{\beta}R - I\right] - \mu_{LH}R\underline{\beta}\left[\left(\overline{p} + \varepsilon\right) - \left(\underline{p} - \varepsilon\right)\right] + B$$

$$\Leftrightarrow w \geq w + \mu_{LL}R\underline{\beta}\Delta p - \mu_{LH}R\underline{\beta}\left[\Delta p + 2\varepsilon\right] + B$$

$$\Leftrightarrow \mu_{LH}R\underline{\beta}\left[\Delta p + 2\varepsilon\right] \geq B$$

$$\Leftrightarrow \mu_{LH} \geq \frac{B}{R\underline{\beta}\left[\Delta p + 2\varepsilon\right]}$$

and then

$$\mu_{LH} = \frac{B}{R\underline{\beta}\left[\Delta p + 2\varepsilon\right]}$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p} + \varepsilon\right)\overline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\overline{p}\overline{\beta}R - I\right] = w + \mu_{LL}R\left[\overline{p}\overline{\beta} - \underline{p}\underline{\beta}\right] \qquad (IC_{HH\to LL})$$

$$\Leftrightarrow w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + \mu_{HL}R\overline{\beta}\Delta p + B \geq w + \mu_{LL}R\left[\overline{p}\overline{\beta} - \underline{p}\underline{\beta}\right]$$

$$\Leftrightarrow -\mu_{LL}R\left[\overline{p}\Delta\beta\right] + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + \mu_{HL}R\overline{\beta}\Delta p + B \geq 0$$

$$\Leftrightarrow \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + \mu_{HL}R\overline{\beta}\Delta p + B \geq 0$$

which is satisfied, as $\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] \geq 0$.

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\underline{p}\overline{\beta}R - I\right] = w + \mu_{LL}R\underline{p}\Delta\beta \qquad (IC_{HL\to LL})$$

$$\Leftrightarrow w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + B \geq w + \mu_{LL}R\underline{p}\Delta\beta$$

$$\Leftrightarrow \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] - \mu_{LL}R\left[\underline{p}\overline{\beta} - \underline{\beta}\overline{p}\right] + B \geq 0$$

$$\Leftrightarrow \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + B \geq 0$$

As $\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]\geq 0$, $(IC_{HL\to LL})$ is not binding.

$$\alpha_{LH}+\mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R-I\right]\geq\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R-I\right] \qquad (IC_{LH\to HH})$$

$$=\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right]-\mu_{HH}R\left(\overline{p}+\varepsilon\right)\Delta\beta$$

$$\Leftrightarrow w+\mu_{LL}R\underline{\beta}\Delta p\geq w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]$$

$$+\mu_{HL}R\overline{\beta}\Delta p+B-\mu_{HH}R\left(\overline{p}+\varepsilon\right)\Delta\beta$$

$$\Leftrightarrow\mu_{HH}R\left(\overline{p}+\varepsilon\right)\Delta\beta\geq\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+\mu_{HL}R\overline{\beta}\Delta p+B$$

$$\Leftrightarrow\mu_{HH}R\left(\overline{p}+\varepsilon\right)\Delta\beta-\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]-\mu_{HL}R\overline{\beta}\Delta p-B\geq 0$$

$$\Leftrightarrow\mu_{HH}R\left(\overline{p}+\varepsilon\right)\Delta\beta\geq\frac{B}{\underline{\beta}\left[\Delta p+2\varepsilon\right]}\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+\frac{B\left[\Delta p+2\varepsilon\right]\overline{\beta}}{\underline{\beta}\left[\Delta p+2\varepsilon\right]}$$

$$\Leftrightarrow\mu_{HH}R\left(\overline{p}+\varepsilon\right)\Delta\beta\geq\frac{B}{\underline{\beta}\left[\Delta p+2\varepsilon\right]}\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)+\overline{p}\overline{\beta}-\underline{p}\overline{\beta}+2\varepsilon\overline{\beta}\right]$$

$$\Leftrightarrow\mu_{HH}R\left(\overline{p}+\varepsilon\right)\Delta\beta\geq\frac{B\left(\overline{p}+\varepsilon\right)\Delta\beta}{\underline{\beta}\left[\Delta p+2\varepsilon\right]}$$

$$\Leftrightarrow\mu_{HH}\geq\frac{B}{R\underline{\beta}\left[\Delta p+2\varepsilon\right]}=\mu_{LH}.$$

This is satisfied from the Spence Mirlees condition.

$$\alpha_{LH}+\mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R-I\right]\geq\alpha_{HL}+\mu_{HL}\left[\overline{p}\underline{\beta}R-I\right]=\alpha_{HL}+\mu_{HL}\left[\underline{p}\overline{\beta}R-I\right]-\mu_{HL}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right] \qquad (IC_{LH\to HL})$$

$$\Leftrightarrow w+\mu_{LL}R\underline{\beta}\Delta p\geq w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]-\mu_{HL}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]+B$$

$$\Leftrightarrow\mu_{HL}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]\geq\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B$$

$$\Leftrightarrow\mu_{HL}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]\geq\frac{B\Delta\beta\left[\underline{p}-\varepsilon\right]}{\underline{\beta}\left[\Delta p+2\varepsilon\right]}$$

$$\Leftrightarrow\mu_{HL}\geq\frac{B\left[\underline{p}-\varepsilon\right]\Delta\beta}{R\underline{\beta}\left[\Delta p+2\varepsilon\right]\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]}=\mu_{HL}^{1}$$

$$\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right]\geq\alpha_{LH}+\mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right]=w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left(\overline{p}+\varepsilon\right)\Delta\beta \qquad (IC_{HH\to LH})$$

$$\Leftrightarrow w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+\mu_{HL}R\overline{\beta}\Delta p+B\geq w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left(\overline{p}+\varepsilon\right)\Delta\beta$$

$$\Leftrightarrow\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\overline{\beta}\overline{p}+\overline{p}\underline{\beta}-\varepsilon\overline{\beta}+\varepsilon\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+\mu_{HL}R\overline{\beta}\Delta p+B\geq 0$$

$$\Leftrightarrow\mu_{HL}R\overline{\beta}\Delta p-\mu_{LH}R\overline{\beta}\left[\Delta p+2\varepsilon\right]+B\geq 0$$

$$\Leftrightarrow\mu_{HL}\geq\frac{B\Delta\beta}{R\Delta p\overline{\beta}\underline{\beta}}=\mu_{HL}^{2}$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\underline{p}\underline{\beta}R - I\right] = \alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] - \mu_{HL}R\underline{p}\Delta\beta \qquad (IC_{LL\to HL})$$

$$\Leftrightarrow w \geq w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] - \mu_{HL}R\underline{p}\Delta\beta + B$$

$$\Leftrightarrow 0 \geq \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] - \mu_{HL}R\underline{p}\Delta\beta + B$$

$$\mu_{HL}R\underline{p}\Delta\beta \geq \frac{BR\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right]}{R\underline{\beta}\left[\Delta p + 2\varepsilon\right]} + B$$

$$\mu_{HL}R\underline{p}\Delta\beta \geq \frac{B\left(\underline{p} - \varepsilon\right)\Delta\beta}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]}$$

$$\mu_{HL} \geq \frac{B\left(\underline{p} - \varepsilon\right)}{R\underline{p}\underline{\beta}\left[\Delta p + 2\varepsilon\right]}$$

This is always verified as $\frac{B\left(\underline{p}-\varepsilon\right)}{R\underline{\beta}[\Delta p+2\varepsilon]\underline{p}} \leq \mu_{LH}$ and due to the Spence Mirlees condition.

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq \alpha_{LH} + \mu_{LH}\left[\left(\underline{p} - \varepsilon\right)\underline{\beta}R - I\right] + B = \qquad (IC_{LL\to LH})$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p} + \varepsilon\right)\underline{\beta}R - I\right] - \mu_{LH}R\underline{\beta}\left[\left(\overline{p} + \varepsilon\right) - \left(\underline{p} - \varepsilon\right)\right] + B$$

$$\Leftrightarrow w \geq w + \mu_{LL}R\underline{\beta}\Delta p - \mu_{LH}R\underline{\beta}\left[\Delta p + 2\varepsilon\right] + B$$

$(IC_{LL\to LH})$ is thus not binding

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \alpha_{HH} + \mu_{HH}\left[\left(\underline{p} - \varepsilon\right)\overline{\beta}R - I\right] + B \qquad (IC_{HL\to HH})$$

$$= w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + \mu_{HL}R\overline{\beta}\Delta p - \mu_{HH}R\overline{\beta}\left[\Delta p + 2\varepsilon\right] + 2B$$

$$\Leftrightarrow w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + B \geq \begin{array}{c} w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] \\ +\mu_{HL}R\overline{\beta}\Delta p - \mu_{HH}R\overline{\beta}\left[\Delta p + 2\varepsilon\right] + 2B \end{array}$$

$$\Leftrightarrow \mu_{HH}R\overline{\beta}\left[\Delta p + 2\varepsilon\right] \geq \mu_{HL}R\overline{\beta}\Delta p + B$$

$$\Leftrightarrow \mu_{HH} \geq \frac{B\left[\underline{p} - \varepsilon\right]\Delta\beta}{R\underline{\beta}\left[\Delta p + 2\varepsilon\right]^2\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]}\Delta p + \frac{B}{R\overline{\beta}\left[\Delta p + 2\varepsilon\right]} = \mu^1_{HH}$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq \alpha_{HH} + \mu_{HH}\left[\left(\underline{p} - \varepsilon\right)\underline{\beta}R - I\right] + B = \qquad (IC_{LL\to HH})$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p} + \varepsilon\right)\overline{\beta}R - I\right] - \mu_{HH}R\left[\left(\overline{p} + \varepsilon\right)\overline{\beta} - \left(\underline{p} - \varepsilon\right)\underline{\beta}\right] + B$$

$$\Leftrightarrow w \geq w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + \mu_{HL}R\overline{\beta}\Delta p - \mu_{HH}R\left[\left(\overline{p}\overline{\beta} - \underline{p}\underline{\beta}\right) + \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + 2B$$

$$\Leftrightarrow \mu_{HH} \geq \begin{array}{c} \mu_{LH}\frac{\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]}{\left[\left(\overline{p}\overline{\beta}-\underline{p}\underline{\beta}\right)+\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]} \\ +\mu_{HL}\frac{\overline{\beta}\Delta p}{\left[\left(\overline{p}\overline{\beta}-\underline{p}\underline{\beta}\right)+\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]} \\ +\frac{2B}{R\left[\left(\overline{p}\overline{\beta}-\underline{p}\underline{\beta}\right)+\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]} \end{array} = \mu^2_{HH}$$

We therefore have:

$$\begin{aligned}
\mu_{LL} &= 0 \\
\mu_{LH} &= \frac{B}{R\underline{\beta}\left[\Delta p + 2\varepsilon\right]} \\
\mu_{HL} &= \max\left\{\mu_{LH}; \mu_{HL}^{1}; \mu_{HL}^{2}\right\} \\
\mu_{HH} &\geq \max\left\{\mu_{HL}; \mu_{HH}^{1}; \mu_{HH}^{2}\right\}
\end{aligned}$$

We now have to show that $\mu_{HL} = \begin{cases} \mu_{HL}^{1} \text{ if } \leq \frac{\underline{\beta}\Delta p}{\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{nb} \\ \mu_{HL}^{2} \text{ if } \varepsilon \geq \frac{\underline{\beta}\Delta p}{\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{nb} \end{cases}$

We only have six cases:

1. $\mu_{LH} \leq \mu_{HL}^{1} \leq \mu_{HL}^{2} \Longleftrightarrow \mu_{HL} = \mu_{HL}^{2}$ if $\varepsilon\Delta\beta \leq \Delta p\underline{\beta} \leq \varepsilon\left[\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}\right]$. Indeed, we have :

$$\begin{aligned}
\mu_{LH} \leq \mu_{HL}^{1} \leq \mu_{HL}^{2} &\Longleftrightarrow \frac{1}{\Delta p + 2\varepsilon} \leq \frac{\left(\underline{p} - \varepsilon\right)\Delta\beta}{\left(\Delta p + 2\varepsilon\right)\left(\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right)} \leq \frac{\Delta\beta}{\Delta p\overline{\beta}} \\
&\Longleftrightarrow \begin{cases} \underline{p}\overline{\beta} - \overline{p}\underline{\beta} \leq \left(\underline{p} - \varepsilon\right)\Delta\beta \\ \left(\underline{p} - \varepsilon\right)\Delta p\overline{\beta} \leq \left(\Delta p + 2\varepsilon\right)\left(\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right) \end{cases} \\
&\Longleftrightarrow \begin{cases} \varepsilon\Delta\beta \leq \Delta p\underline{\beta} \\ -\varepsilon\Delta p\overline{\beta} \leq -\Delta p\overline{p}\underline{\beta} - 2\varepsilon\overline{p}\underline{\beta} + 2\varepsilon\underline{p}\overline{\beta} \end{cases} \\
&\Longleftrightarrow \begin{cases} \varepsilon\Delta\beta \leq \Delta p\underline{\beta} \\ \Delta p\overline{p}\underline{\beta} \leq \varepsilon\left(\underline{p}\overline{\beta} - 2\overline{p}\underline{\beta} + \overline{p}\overline{\beta}\right) \end{cases} \\
&\Longleftrightarrow \begin{cases} \varepsilon\Delta\beta \leq \Delta p\underline{\beta} \\ \Delta p\underline{\beta} \leq \varepsilon\left(\Delta\beta + \overline{\beta}\frac{\underline{p}}{\overline{p}} - \underline{\beta}\right) \end{cases}
\end{aligned}$$

For the following cases (2, 3 and 4), we use the same inequalities to obtain.

2. $\mu_{HL}^{1} \leq \mu_{LH} \leq \mu_{HL}^{2} \Longleftrightarrow \mu_{HL} = \mu_{HL}^{2}$ if $\Delta p\underline{\beta} \leq \varepsilon\Delta\beta$

3. $\mu_{LH} \leq \mu_{HL}^{2} \leq \mu_{HL}^{1} \Longleftrightarrow \mu_{HL} = \mu_{HL}^{1}$ if $\varepsilon\left[\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}\right] \leq \Delta p\underline{\beta} \leq 2\varepsilon\Delta\beta$

4. $\mu_{HL}^{2} \leq \mu_{LH} \leq \mu_{HL}^{1} \Longleftrightarrow \mu_{HL} = \mu_{HL}^{1}$ if $\Delta p\underline{\beta} \geq 2\varepsilon\Delta\beta$

5. $\mu_{HL}^{2} \leq \mu_{HL}^{1} \leq \mu_{LH} \Longleftrightarrow$ impossible. Indeed, we would eventually obtain

$$\varepsilon\Delta\beta \geq \Delta p\underline{\beta} \geq \varepsilon\left(\Delta\beta + \overline{\beta}\frac{\underline{p}}{\overline{p}} - \underline{\beta}\right)$$

which is not possible because the last term is strictly superior to the first one.

6. $\mu_{HL}^{1} \leq \mu_{HL}^{2} \leq \mu_{LH} \Longleftrightarrow$ impossible

We therefore have the result of the lemma.

And then :

$$U_{LL} = \alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] = w$$

$$U_{LH} = \alpha_{LH} + \mu_{LH}\left[\left(\overline{p} + \varepsilon\right)\underline{\beta}R - I\right] = w + \mu_{LL}R\underline{\beta}\Delta p = w$$

$$U_{HL} = \alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] = w + \frac{B\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right]}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]} + B$$
$$\Longleftrightarrow U_{HL} = w + \frac{B\left(\underline{p} - \varepsilon\right)\Delta\beta}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]}$$

Moreover, when $\varepsilon \leq \frac{\underline{\beta}\Delta p}{\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{nb}$

$$U_{HH} = \alpha_{HH} + \mu_{HH}\left[\left(\overline{p} + \varepsilon\right)\overline{\beta}R - I\right] = w + \frac{B\left(\underline{p} - \varepsilon\right)\Delta\beta}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]} + \frac{B\left[\underline{p} - \varepsilon\right]\Delta\beta\overline{\beta}\Delta p}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]}$$
$$= w + \frac{B\left(\underline{p} - \varepsilon\right)\Delta\beta\left[\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right] + \overline{\beta}\Delta p\right]}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]}$$
$$\Longleftrightarrow U_{HH} = w + \frac{B\left(\underline{p} - \varepsilon\right)\overline{p}\left(\Delta\beta\right)^2}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]}$$

Moreover, when $\varepsilon \geq \frac{\underline{\beta}\Delta p}{\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{nb}$

$$U_{HH} = \alpha_{HH} + \mu_{HH}\left[\left(\overline{p} + \varepsilon\right)\overline{\beta}R - I\right] = w + \frac{B\left(\underline{p} - \varepsilon\right)\Delta\beta}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]} + \frac{B\Delta\beta}{\underline{\beta}}$$
$$= w + \frac{B\Delta\beta\left[\left(\underline{p} - \varepsilon\right) + \left(\Delta p + 2\varepsilon\right)\right]}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]}$$
$$\Longleftrightarrow U_{HH} = w + \frac{B\Delta\beta\left(\overline{p} + \varepsilon\right)}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]}$$

To sum up, here are the CEO' informational rents when there is No Board:

$$\begin{aligned}
U_{LL} &= w \\
U_{LH} &= w \\
U_{HL} &= w + \frac{B\left(\underline{p} - \varepsilon\right)\Delta\beta}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]} \\
U_{HH} &= \begin{cases} w + \frac{B\left(\underline{p} - \varepsilon\right)\overline{p}\left(\Delta\beta\right)^2}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]} \text{ if } \varepsilon \leq \varepsilon_{nb} \\ w + \frac{B\Delta\beta\left(\overline{p} + \varepsilon\right)}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]} \text{ if } \varepsilon \geq \varepsilon_{nb} \end{cases}
\end{aligned}$$

We can verify now that we have

$$U_{LL} \leq U_{LH} \leq U_{HL} \leq U_{HH}$$

.

When $\varepsilon \leq \varepsilon_{nb}$, we need to see if $\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \overline{p}\underline{\beta}} \geq 1$, which is true since $\underline{p}\overline{\beta} - \overline{p}\underline{\beta} = \overline{p}\Delta\beta - \overline{\beta}\Delta p$.
Subsequently, we have $U_{HL} \leq U_{HH}$.
When $\varepsilon \geq \varepsilon_{nb}$, since $\underline{p} - \varepsilon \leq \overline{p} + \varepsilon$, we necessarily have $U_{HL} \leq U_{HH}$.

Rewriting the shareholders' expected profits depending on those infomational rents, when there is no board, we have:

$$W_{NB} = E(\pi) - \gamma\nu U_{LL} - \gamma\left(1-\nu\right)U_{LH} - \left(1-\gamma\right)\nu U_{HL} - \left(1-\gamma\right)\left(1-\nu\right)U_{HH}$$

This gives, for $\varepsilon \leq \frac{\underline{\beta}\Delta p}{\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{nb}$

$$W_{NB} = E(\pi) - w - \frac{\left(1-\gamma\right)B\left(\underline{p}-\varepsilon\right)\Delta\beta}{\underline{\beta}\left(\Delta p + 2\varepsilon\right)}\left[\frac{-\nu\overline{\beta}\Delta p + \overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \overline{p}\underline{\beta}}\right]$$

For $\varepsilon \geq \frac{\underline{\beta}\Delta p}{\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{nb}$

$$W_{NB} = E(\pi) - w - \left(1-\gamma\right)\left(\overline{p} + \varepsilon - \nu\Delta p - 2\nu\varepsilon\right)\frac{B\Delta\beta}{\underline{\beta}\left(\Delta p + 2\varepsilon\right)}$$

■

Proof of Proposition 2. The Participation and Incentive constraints are now:

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq w \qquad (PC_{LL})$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] \geq w \qquad (PC_{LH})$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq w \qquad (PC_{HL})$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R - I\right] \geq w \qquad (PC_{HH})$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\overline{p}\overline{\beta}R - I\right] \qquad (IC_{HH\to HL})$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R - I\right] \geq \alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R - I\right] \qquad (IC_{HH\to LH})$$

$$\alpha_{HH} + \mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\overline{p}\overline{\beta}R - I\right] \qquad (IC_{HH\to LL})$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \left(1-\xi(\tau)\right)\left\{\alpha_{HH} + \mu_{HH}\left[\left(\underline{p}-\varepsilon\right)\overline{\beta}R - I\right] + B\right\} + \xi(\tau)\left(w-F\right) \qquad (IC_{HL\to HH})$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \left(1-\xi(\tau)\right)\left\{\alpha_{LH} + \mu_{LH}\left[\left(\underline{p}-\varepsilon\right)\overline{\beta}R - I\right] + B\right\} + \xi(\tau)\left(w-F\right) \qquad (IC_{HL\to LH})$$

$$\alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\underline{p}\overline{\beta}R - I\right] \qquad (IC_{HL\to LL})$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] \geq \alpha_{HH} + \mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] \qquad (IC_{LH\to HH})$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\overline{p}\underline{\beta}R - I\right] \qquad (IC_{LH\to HL})$$

$$\alpha_{LH} + \mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R - I\right] \geq \alpha_{LL} + \mu_{LL}\left[\overline{p}\underline{\beta}R - I\right] \qquad (IC_{LH\to LL})$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq \left(1-\xi(\tau)\right)\left\{\alpha_{HH} + \mu_{HH}\left[\left(\underline{p}-\varepsilon\right)\underline{\beta}R - I\right] + B\right\} + \xi(\tau)\left(w-F\right) \qquad (IC_{LL\to HH})$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\underline{p}\underline{\beta}R - I\right] \qquad (IC_{LL\to HL})$$

$$\alpha_{LL}+\mu_{LL}\left[\underline{p}\beta R-I\right] \geq (1-\xi(\tau))\left\{\alpha_{LH}+\mu_{LH}\left[\left(\underline{p}-\varepsilon\right)\underline{\beta}R-I\right]+B\right\}+\xi(\tau)\left(w-F\right) \quad (IC_{LL\to LH})$$

We also assume that the CEO faces a limited liability constraint, i.e., even if the Board found that the CEO has sent the wrong signal, she cannot get less than her reservation wage. This gives:

$$(1-\xi(\tau))\left\{w+B\right\}+\xi(\tau)\left(w-F\right) \geq w \qquad (LL)$$
$$\Leftrightarrow B \geq \frac{\xi(\tau)}{(1-\xi(\tau))}F$$

The binding constraints are:

$$\alpha_{LL}+\mu_{LL}\left[\underline{p}\beta R-I\right]=w \qquad (PC_{LL})$$

$$\begin{aligned} \alpha_{LH}+\mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R-I\right] &= \alpha_{LL}+\mu_{LL}\left[\overline{p}\underline{\beta}R-I\right] && (IC_{LH\to LL}) \\ &= \alpha_{LL}+\mu_{LL}\left[\underline{p}\beta R-I\right]+\mu_{LL}R\underline{\beta}\Delta p \\ &= w+\mu_{LL}R\underline{\beta}\Delta p \end{aligned}$$

$$\begin{aligned} \alpha_{HL}+\mu_{HL}\left[\underline{p}\overline{\beta}R-I\right] &= (1-\xi(\tau))\left\{\alpha_{LH}+\mu_{LH}\left[\left(\underline{p}-\varepsilon\right)\overline{\beta}R-I\right]+B\right\}+\xi(\tau)\left(w-F\right) && (IC_{HL\to LH}) \\ &= (1-\xi(\tau))\left\{\begin{array}{c} \alpha_{LH}+\mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R-I\right] \\ -\mu_{LH}R\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}-\left(\underline{p}-\varepsilon\right)\overline{\beta}\right]+B \end{array}\right\}+\xi(\tau)\left(w-F\right) \\ &= (1-\xi(\tau))\left\{w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B\right\}+\xi(\tau)\left(w-F\right) \end{aligned}$$

$$\begin{aligned} \alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right] &= \alpha_{HL}+\mu_{HL}\left[\overline{p}\overline{\beta}R-I\right] && (IC_{HH\to HL}) \\ &= \alpha_{HL}+\mu_{HL}\left[\underline{p}\overline{\beta}R-I\right]+\mu_{HL}R\overline{\beta}\Delta p \\ &= (1-\xi(\tau))\left\{w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B\right\} \\ &\quad +\xi(\tau)\left(w-F\right)+\mu_{HL}R\overline{\beta}\Delta p \end{aligned}$$

Again, in order to minimize the informational rents, shareholders will set μ_{LL}, μ_{LH} and μ_{HL} as low as possible while satisfying the other incentive constraints.

We now check what are the conditions due to the other Incentive constraints (and will check later that Participation constraints are satisfied).

$$\mu_{LL}=0$$

$$\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right] \geq \alpha_{LL}+\mu_{LL}\left[\overline{p}\overline{\beta}R-I\right] \qquad (IC_{HH\to LL})$$
$$= w+\mu_{LL}R\left[\overline{p}\overline{\beta}-\underline{p}\beta\right]=w$$

$$\alpha_{LL}+\mu_{LL}\left[\underline{p}\beta R-I\right]\geq(1-\xi(\tau))\left\{\alpha_{LH}+\mu_{LH}\left[\left(\underline{p}-\varepsilon\right)\underline{\beta}R-I\right]+B\right\}+\xi(\tau)\left(w-F\right)= \qquad (IC_{LL\to LH})$$

$$(1-\xi(\tau))\left\{\begin{array}{c} w+\mu_{LL}R\underline{\beta}\Delta p \\ -\mu_{LH}R\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]+B \end{array}\right\}+\xi(\tau)\left(w-F\right)$$

$$\Leftrightarrow\mu_{LH}R\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]\geq B-\frac{\xi(\tau)}{(1-\xi(\tau))}F$$

$$\Leftrightarrow\mu_{LH}\geq\frac{B-\frac{\xi(\tau)}{(1-\xi(\tau))}F}{R\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]}$$

As $\frac{\xi(\tau)}{(1-\xi(\tau))}F-B\leq 0$, we have

$$\mu_{LH}=\frac{B-\frac{\xi(\tau)}{(1-\xi(\tau))}F}{R\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]}$$

$$\alpha_{HL}+\mu_{HL}\left[\underline{p}\overline{\beta}R-I\right]\geq\alpha_{LL}+\mu_{LL}\left[\underline{p}\overline{\beta}R-I\right]=w+\mu_{LL}R\underline{p}\Delta\beta \qquad (IC_{HL\to LL})$$

$$\Leftrightarrow(1-\xi(\tau))\left\{\begin{array}{c} \mu_{LL}R\underline{\beta}\Delta p \\ +\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B \end{array}\right\}-\xi(\tau)F\geq\mu_{LL}R\underline{p}\Delta\beta$$

$$\Leftrightarrow(1-\xi(\tau))\left\{\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B\right\}-\xi(\tau)F\geq 0$$

$$\Leftrightarrow\mu_{LH}\geq\frac{\frac{\xi(\tau)}{(1-\xi(\tau))}F-B}{R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]}$$

As $\frac{\xi(\tau)}{(1-\xi(\tau))}F-B\leq 0$, this is satisfied

$$\alpha_{HH}+\mu_{HH}\left[(\overline{p}+\varepsilon)\overline{\beta}R-I\right]\geq\alpha_{LH}+\mu_{LH}\left[(\overline{p}+\varepsilon)\overline{\beta}R-I\right]=w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left(\overline{p}+\varepsilon\right)\Delta\beta \qquad (IC_{HH\to LH})$$

$$\Leftrightarrow\left\{\begin{array}{c} (1-\xi(\tau))\left\{w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B\right\} \\ +\xi(\tau)\left(w-F\right)+\mu_{HL}R\overline{\beta}\Delta p \end{array}\right\}\geq w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left(\overline{p}+\varepsilon\right)\Delta\beta$$

$$\Leftrightarrow(1-\xi(\tau))\left\{\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B\right\}+\mu_{HL}R\overline{\beta}\Delta p-\xi(\tau)F\geq\mu_{LH}R\left(\overline{p}+\varepsilon\right)\Delta\beta$$

$$\mu_{HL}R\overline{\beta}\Delta p\geq\left(\frac{-(1-\xi(\tau))\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+(\overline{p}+\varepsilon)\Delta\beta}{\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]}-(1-\xi(\tau))\right)\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]$$

$$\mu_{HL}\geq\frac{(\overline{p}+\varepsilon)\Delta\beta-(1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\Delta\beta}{\overline{\beta}\Delta p}\left(\frac{\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{R\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]}\right)$$

$$\Leftrightarrow\mu_{HL}\geq\mu_{LH}\frac{\left[(\overline{p}+\varepsilon)-(1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\right]\Delta\beta}{\overline{\beta}\Delta p}=\mu_{HL}^{1}$$

$$\alpha_{LH} + \mu_{LH}\left[(\overline{p}+\varepsilon)\,\underline{\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\overline{p}\underline{\beta}R - I\right] \qquad (IC_{LH\to HL})$$

$$= \alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] - \mu_{HL}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]$$

$$\Leftrightarrow w + \mu_{LL}R\underline{\beta}\Delta p \geq \left\{ \begin{array}{c} (1-\xi(\tau))\left\{ \begin{array}{c} w + \mu_{LL}R\underline{\beta}\Delta p \\ +\mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right] + B \end{array} \right\} \\ +\xi(\tau)\,(w-F) - \mu_{HL}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right] \end{array} \right\}$$

$$\Leftrightarrow \mu_{HL}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right] \geq (1-\xi(\tau))\left\{\mu_{LH}R\left[\left(\underline{p}-\varepsilon\right)\overline{\beta} - (\overline{p}+\varepsilon)\,\underline{\beta}\right] + B\right\} - \xi(\tau)F$$

$$\mu_{HL} \geq \frac{(1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\Delta\beta}{\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]}\left(\frac{\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{R\underline{\beta}\left[(\overline{p}+\varepsilon) - \left(\underline{p}-\varepsilon\right)\right]}\right)$$

$$\mu_{HL} \geq \frac{(1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\Delta\beta}{\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]}\mu_{LH} = \mu^2_{HL}$$

We can verify that

$$\mu_{HL} = \left\{ \begin{array}{l} \mu^1_{HL} \text{ if } \varepsilon \geq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{ib} \\ \mu^2_{HL} \text{ if } \varepsilon \leq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{ib} \end{array} \right.$$

Indeed, we have :

$$\begin{aligned} \mu_{HL} = \mu^1_{HL} &\iff \mu^1_{HL} \geq \mu^2_{HL} \\ &\iff \frac{\left[\Delta p + 2\varepsilon + \xi(\tau)\left(\underline{p}-\varepsilon\right)\right]\Delta\beta}{\overline{\beta}\Delta p} \geq \frac{(1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\Delta\beta}{\underline{p}\overline{\beta} - \overline{p}\underline{\beta}} \\ &\iff \varepsilon\left[(2-\xi(\tau))\left(\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right) + (1-\xi(\tau))\,\overline{\beta}\Delta p\right] \geq (1-\xi(\tau))\,\underline{p}\overline{\beta}\Delta p - \left(\Delta p + \xi(\tau)\underline{p}\right)\left(\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right) \\ &\iff \varepsilon\left[\underline{p}\overline{\beta}\,(2-\xi(\tau)-1+\xi(\tau)) + \overline{p}\left((1-\xi(\tau))\,\overline{\beta} - (2-\xi(\tau))\,\underline{\beta}\right)\right] \geq \overline{p}\underline{\beta}\Delta p + \xi(\tau)\left(-\underline{p}\overline{\beta}\left(\overline{p}-\underline{p}\right) - \right. \\ &\iff \varepsilon\left[\underline{p}\overline{\beta} + \overline{p}\left((1-\xi(\tau))\,\Delta\beta - \underline{\beta}\right)\right] \geq \overline{p}\underline{\beta}\Delta p + \xi(\tau)\left(-\underline{p}\overline{p}\Delta\beta\right) \\ &\iff \varepsilon \geq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\,\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} \end{aligned}$$

Since we have $\mu_{HL} = \mu^1_{HL}$ we necessarily have $\varepsilon \geq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}}$, that is to say

$$\begin{aligned} \left[(1-\xi(\tau))\,\Delta\beta\right]\varepsilon + \varepsilon\left[\frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}\right] - \underline{\beta}\Delta p + \xi(\tau)\underline{p}\Delta\beta &\geq 0 \\ \xi(\tau)\left(\underline{p}-\varepsilon\right)\Delta\beta - \underline{\beta}\Delta p &\geq -\varepsilon\Delta\beta - \varepsilon\left[\frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}\right] \end{aligned}$$

We therefore have

$$\begin{aligned} \left[(\overline{p}+\varepsilon) - (1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\right]\Delta\beta - \overline{\beta}\Delta p &= -\overline{\beta}\Delta p + \xi(\tau)\left(\underline{p}-\varepsilon\right)\Delta\beta + (\Delta p + 2\varepsilon)\,\Delta\beta \\ &\geq \varepsilon\Delta\beta - \varepsilon\left[\frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}\right] \\ &= \varepsilon\overline{\beta}\frac{\Delta p}{\overline{p}} \geq 0 \end{aligned}$$

And

$$\mu_{HL}^1 \geq \mu_{LH}$$

In the same way, when $\mu_{HL} = \mu_{HL}^2$ we have $\varepsilon \leq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}}$, that is to say

$$\begin{aligned} \left[(1-\xi(\tau))\,\Delta\beta\right]\varepsilon + \varepsilon\left[\frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}\right] - \underline{\beta}\Delta p + \xi(\tau)\underline{p}\Delta\beta &\leq 0 \\ -\xi(\tau)\left(\underline{p} - \varepsilon\right)\Delta\beta - \varepsilon\Delta\beta + \underline{\beta}\Delta p &\geq \varepsilon\left[\frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}\right] \end{aligned}$$

We then have

$$\begin{aligned} (1-\xi(\tau))\left(\underline{p} - \varepsilon\right)\Delta\beta - \left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right] &= \underline{\beta}\Delta p - \varepsilon\Delta\beta - \xi(\tau)\left(\underline{p} - \varepsilon\right)\Delta\beta \\ &\geq \varepsilon\left[\frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}\right] \geq 0 \end{aligned}$$

And

$$\mu_{HL}^2 \geq \mu_{LH}$$

$$\alpha_{LL} + \mu_{LL}\left[\underline{p\beta}R - I\right] \geq \alpha_{HL} + \mu_{HL}\left[\underline{p\beta}R - I\right] = \alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] - \mu_{HL}R\underline{p}\Delta\beta \qquad (IC_{LL\to HL})$$

$$\begin{aligned} &\Leftrightarrow 0 \geq (1-\xi(\tau))\left\{\mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + B\right\} - \xi(\tau)F - \mu_{HL}R\underline{p}\Delta\beta \\ &\Leftrightarrow \mu_{HL}R\underline{p}\Delta\beta \geq (1-\xi(\tau))\left\{\mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta} + \overline{\beta}\right)\right] + B\right\} - \xi(\tau)F \\ &\Leftrightarrow \mu_{HL} \geq \frac{(1-\xi(\tau))\left(\underline{p} - \varepsilon\right)\Delta\beta}{\underline{p}\Delta\beta}\left(\frac{\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{R\underline{\beta}\left[(\overline{p} + \varepsilon) - \left(\underline{p} - \varepsilon\right)\right]}\right) \\ &\Leftrightarrow \mu_{HL} \geq \frac{(1-\xi(\tau))\left(\underline{p} - \varepsilon\right)\Delta\beta}{\underline{p}\Delta\beta}\mu_{LH} \end{aligned}$$

Since $(1-\xi(\tau))\,\underline{p}\Delta\beta \leq \underline{p}\Delta\beta$ and since $\mu_{HH} \geq \mu_{LH}$, $IC_{LL\to HL}$ is also satisfied.

Finally, we get

$$\begin{aligned} \mu_{HL} &= \max\left\{\mu_{HL}^1; \mu_{HL}^2\right\} \\ &= \begin{cases} \mu_{HL}^1 \text{ if } \varepsilon \geq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{ib} \\ \mu_{HL}^2 \text{ if } \varepsilon \leq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{ib} \end{cases} \end{aligned}$$

$$\alpha_{LL}+\mu_{LL}\left[\underline{p}\underline{\beta}R-I\right]\geq(1-\xi(\tau))\left\{\alpha_{HH}+\mu_{HH}\left[\left(\underline{p}-\varepsilon\right)\underline{\beta}R-I\right]+B\right\}+\xi(\tau)\left(w-F\right)= \tag{$IC_{LL\to HH}$}$$

$$(1-\xi(\tau))\left\{\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right]-\mu_{HH}R\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}-\left(\underline{p}-\varepsilon\right)\underline{\beta}\right]+B\right\}+\xi(\tau)\left(w-F\right)$$

$$\Leftrightarrow w\geq(1-\xi(\tau))\left\{\begin{array}{c}(1-\xi(\tau))\left\{w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B\right\}\\ +\xi(\tau)\left(w-F\right)+\mu_{HL}R\overline{\beta}\Delta p-\mu_{HH}R\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}-\left(\underline{p}-\varepsilon\right)\underline{\beta}\right]+B\end{array}\right\}+\xi(\tau)\left(w-F\right)$$

$$\Leftrightarrow 0\geq(1-\xi(\tau))\left\{\begin{array}{c}(1-\xi(\tau))\left\{\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B\right\}-\xi(\tau)F+\mu_{HL}R\overline{\beta}\Delta p\\ -\mu_{HH}R\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}-\left(\underline{p}-\varepsilon\right)\underline{\beta}\right]+B\end{array}\right\}-\xi(\tau)F$$

$$\Leftrightarrow \mu_{HH}\geq\left\{\begin{array}{c}(1-\xi(\tau))\left\{\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B\right\}\\ -\xi(\tau)F+\mu_{HL}R\overline{\beta}\Delta p\end{array}\right\}\left(\frac{\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{R\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}-\left(\underline{p}-\varepsilon\right)\underline{\beta}\right]}\right)=\mu_{HH}^{1}$$

$$\alpha_{HL}+\mu_{HL}\left[\underline{p}\overline{\beta}R-I\right]\geq(1-\xi(\tau))\left\{\alpha_{HH}+\mu_{HH}\left[\left(\underline{p}-\varepsilon\right)\overline{\beta}R-I\right]+B\right\}+\xi(\tau)\left(w-F\right) \tag{$IC_{HL\to HH}$}$$

$$=(1-\xi(\tau))\left\{\begin{array}{c}(1-\xi(\tau))\left\{w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B\right\}\\ +\xi(\tau)\left(w-F\right)+\mu_{HL}R\overline{\beta}\Delta p-\mu_{HH}R\overline{\beta}\left[\Delta p+2\varepsilon\right]+B\end{array}\right\}+\xi(\tau)\left(w-F\right)$$

$$\Leftrightarrow(1-\xi(\tau))\left\{w+\mu_{LL}R\underline{\beta}\Delta p+\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+B\right\}+\xi(\tau)\left(w-F\right)\geq$$

$$(1-\xi(\tau))\left\{\begin{array}{c}(1-\xi(\tau))\left\{\begin{array}{c}w+\mu_{LL}R\underline{\beta}\Delta p+\\ \mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]\\ +\mu_{HL}R\overline{\beta}\Delta p+B\end{array}\right\}\\ +\xi(\tau)\left(w-F\right)-\mu_{HH}R\overline{\beta}\left[\Delta p+2\varepsilon\right]+B\end{array}\right\}+\xi(\tau)\left(w-F\right)$$

$$\Leftrightarrow \mu_{HH}\geq\frac{(1-\xi(\tau))\,\mu_{HL}R\overline{\beta}\Delta p-\xi(\tau)\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]+(1-\xi(\tau))\,B-\xi(\tau)F}{R\overline{\beta}\left[\Delta p+2\varepsilon\right]}=\mu_{HH}^{2}$$

$$\alpha_{LH}+\mu_{LH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R-I\right]\geq\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\underline{\beta}R-I\right] \tag{$IC_{LH\to HH}$}$$

$$=\alpha_{HH}+\mu_{HH}\left[\left(\overline{p}+\varepsilon\right)\overline{\beta}R-I\right]-\mu_{HH}R\left(\overline{p}+\varepsilon\right)\Delta\beta$$

$$\Leftrightarrow \mu_{HH}R\left(\overline{p}+\varepsilon\right)\Delta\beta\geq\begin{array}{c}(1-\xi(\tau))\,\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]\\ +\mu_{HL}R\overline{\beta}\Delta p\end{array}+(1-\xi(\tau))\,B-\xi(\tau)F$$

$$\Leftrightarrow \mu_{HH}\geq\frac{\begin{array}{c}(1-\xi(\tau))\,\mu_{LH}R\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}-\varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right]\\ +\mu_{HL}R\overline{\beta}\Delta p\end{array}+(1-\xi(\tau))\,B-\xi(\tau)F}{R\left(\overline{p}+\varepsilon\right)\Delta\beta}=\mu_{HH}^{3}$$

We thus have:

$$\begin{aligned}
\mu_{LL} &= 0 \\
\mu_{LH} &= \frac{B - \frac{\xi(\tau)}{(1-\xi(\tau))}F}{R\underline{\beta}\left[(\overline{p}+\varepsilon) - \left(\underline{p}-\varepsilon\right)\right]} \\
\mu_{HL} &= \begin{cases} \mu^1_{HL} \text{ if } \varepsilon \geq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{ib} \\ \mu^2_{HL} \text{ if } \varepsilon \leq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{ib} \end{cases} \\
\mu_{HH} &\geq \max\left\{\mu_{HL}; \mu^1_{HH}; \mu^2_{HH}; \mu^3_{HH}\right\}
\end{aligned}$$

$$U_{LL} = \alpha_{LL} + \mu_{LL}\left[\underline{p}\underline{\beta}R - I\right] = w$$

$$U_{LH} = \alpha_{LH} + \mu_{LH}\left[(\overline{p}+\varepsilon)\underline{\beta}R - I\right] = w + \mu_{LL}R\underline{\beta}\Delta p = w$$

$$\begin{aligned}
U_{HL} &= \alpha_{HL} + \mu_{HL}\left[\underline{p}\overline{\beta}R - I\right] = (1-\xi(\tau))\left\{w + \mu_{LL}R\underline{\beta}\Delta p + \mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right] + B\right\} + \xi(\tau)(w - \\
&= w + (1-\xi(\tau))\left\{\mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right] + B\right\} - \xi(\tau)F \\
&= w + (1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\Delta\beta\left(\frac{\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon) - \left(\underline{p}-\varepsilon\right)\right]}\right)
\end{aligned}$$

$$\begin{aligned}
U_{HH} &= \alpha_{HH} + \mu_{HH}\left[(\overline{p}+\varepsilon)\overline{\beta}R - I\right] = (1-\xi(\tau))\left\{\begin{array}{c} w + \mu_{LL}R\underline{\beta}\Delta p \\ +\mu_{LH}R\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta} - \varepsilon\left(\underline{\beta}+\overline{\beta}\right)\right] + B \end{array}\right\} \\
&\quad +\xi(\tau)(w-F) + \mu_{HL}R\overline{\beta}\Delta p \\
&= w + (1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\Delta\beta\left(\frac{\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon) - \left(\underline{p}-\varepsilon\right)\right]}\right) + \mu_{HL}R\overline{\beta}\Delta p \\
&= \begin{cases} w + \frac{\left(\underline{p}-\varepsilon\right)\overline{p}(\Delta\beta)^2[(1-\xi(\tau))B - \xi(\tau)F]}{\underline{\beta}[\Delta p + 2\varepsilon]\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]} \text{ if } \varepsilon \leq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{ib} \\ w + \frac{(\overline{p}+\varepsilon)\Delta\beta\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon) - \left(\underline{p}-\varepsilon\right)\right]} \text{ if } \varepsilon \geq \frac{\underline{\beta}\Delta p - \xi(\tau)\underline{p}\Delta\beta}{(1-\xi(\tau))\Delta\beta + \frac{\underline{p}}{\overline{p}}\overline{\beta} - \underline{\beta}} = \varepsilon_{ib} \end{cases}
\end{aligned}$$

To sum up, here are the CEO utilities when there is an Independant Board:

$$\begin{aligned}
U_{LL} &= w \\
U_{LH} &= w \\
U_{HL} &= w + (1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\Delta\beta\left(\frac{\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon) - \left(\underline{p}-\varepsilon\right)\right]}\right) \\
U_{HH} &= \begin{cases} w + \frac{\left(\underline{p}-\varepsilon\right)\overline{p}(\Delta\beta)^2[(1-\xi(\tau))B - \xi(\tau)F]}{\underline{\beta}[\Delta p + 2\varepsilon]\left[\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right]} \text{ if } \varepsilon \leq \varepsilon_{ib} \\ w + \frac{(\overline{p}+\varepsilon)\Delta\beta\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon) - \left(\underline{p}-\varepsilon\right)\right]} \text{ if } \varepsilon \geq \varepsilon_{ib} \end{cases}
\end{aligned}$$

We can verify now that we have

$$U_{LL} \leq U_{LH} \leq U_{HL} \leq U_{HH}$$

When $\varepsilon \leq \varepsilon_{ib}$, we need to see if $\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\overline{p}\underline{\beta}} \geq 1$, which is true since $\underline{p}\overline{\beta} - \overline{p}\underline{\beta} = \overline{p}\Delta\beta - \overline{\beta}\Delta p$. Subsequently, we have $U_{HL} \leq U_{HH}$.

When $\varepsilon \geq \varepsilon_{ib}$, since $(1-\xi(\tau))\left(\underline{p}-\varepsilon\right) \leq \overline{p}+\varepsilon$, we necessarily have $U_{HL} \leq U_{HH}$.

One can remark that types (HL) and (HH) informational rents are lower with an Independent Board than without Board.

$$\begin{aligned}
U_{HLib} \leq U_{HLnb} &\iff (1-\xi(\tau))\left(\underline{p}-\varepsilon\right)\Delta\beta\left(\frac{\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]}\right) \leq \frac{B\left(\underline{p}-\varepsilon\right)\Delta\beta}{\underline{\beta}\left[\Delta p+2\varepsilon\right]} \\
&\iff (1-\xi(\tau))\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right] \leq B \text{ , which is true}
\end{aligned}$$

Since we have $\varepsilon_{ib} \leq \varepsilon_{nb}$ we only have three possible cases to consider for U_{HH}.

$$\begin{aligned}
\varepsilon_{ib}-\varepsilon_{nb} &= \frac{\underline{\beta}\Delta p-\xi\underline{p}\Delta\beta}{(1-\xi)\Delta\beta+\frac{\underline{p}}{\overline{p}}\overline{\beta}-\underline{\beta}} - \frac{\underline{\beta}\Delta p}{\Delta\beta+\frac{\underline{p}}{\overline{p}}\overline{\beta}-\underline{\beta}} \\
\varepsilon_{ib}-\varepsilon_{nb} \leq 0 &\iff \left(\underline{\beta}\Delta p-\xi\underline{p}\Delta\beta\right)\left(\Delta\beta+\frac{\underline{p}}{\overline{p}}\overline{\beta}-\underline{\beta}\right) - \left(\underline{\beta}\Delta p\right)\left((1-\xi)\Delta\beta+\frac{\underline{p}}{\overline{p}}\overline{\beta}-\underline{\beta}\right) \leq 0 \\
&\iff \Delta\beta\left[\underline{\beta}\Delta p-\xi\underline{p}\Delta\beta-(1-\xi)\underline{\beta}\Delta p\right] - \left(\frac{\underline{p}}{\overline{p}}\overline{\beta}-\underline{\beta}\right)\left(\xi\underline{p}\Delta\beta+\underline{\beta}\Delta p-\underline{\beta}\Delta p\right) \leq 0 \\
&\iff \Delta\beta\xi\left(\underline{\beta}\Delta p-\underline{p}\Delta\beta\right) - \xi\underline{p}\Delta\beta\left(\frac{\underline{p}}{\overline{p}}\overline{\beta}-\underline{\beta}\right) \leq 0 \\
&\iff \frac{\Delta\beta\xi}{\overline{p}}\left(\overline{p}\overline{p}\underline{\beta}-\overline{p}\underline{p}\underline{\beta}-\overline{p}\underline{p}\overline{\beta}+\overline{p}\underline{p}\underline{\beta}-\underline{p}\underline{p}\overline{\beta}+\overline{p}\underline{p}\underline{\beta}\right) \leq 0 \\
&\iff \frac{\Delta\beta\xi}{\overline{p}}\left(\overline{p}+\underline{p}\right)\left(\underline{\beta}\overline{p}-\underline{p}\overline{\beta}\right) \leq 0
\end{aligned}$$

which is true since we have $\underline{\beta}\overline{p}-\underline{p}\overline{\beta} \leq 0$

1. When $\varepsilon \leq \varepsilon_{ib}$

$$\begin{aligned}
U_{HHib}-U_{HHnb} &= \frac{\left(\underline{p}-\varepsilon\right)\overline{p}\left(\Delta\beta\right)^2\left[(1-\xi(\tau))B-\xi(\tau)F\right]}{\underline{\beta}\left[\Delta p+2\varepsilon\right]\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]} - \frac{B\left(\underline{p}-\varepsilon\right)\overline{p}\left(\Delta\beta\right)^2}{\underline{\beta}\left[\Delta p+2\varepsilon\right]\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]} \\
sgn(U_{HHib}-U_{HHnb}) &= sgn(-\xi(\tau)(B+F)\left(\underline{p}-\varepsilon\right)(\Delta\beta)^2) \leq 0
\end{aligned}$$

2. When $\varepsilon_{ib} \leq \varepsilon \leq \varepsilon_{nb}$

$$\begin{aligned}
U_{HHib}-U_{HHnb} &= \frac{(\overline{p}+\varepsilon)\Delta\beta\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[(\overline{p}+\varepsilon)-\left(\underline{p}-\varepsilon\right)\right]} - \frac{B\left(\underline{p}-\varepsilon\right)\overline{p}\left(\Delta\beta\right)^2}{\underline{\beta}\left[\Delta p+2\varepsilon\right]\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right]} \\
sgn(U_{HHib}-U_{HHnb}) &= sgn\left[(\overline{p}+\varepsilon)\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]\left[\underline{p}\overline{\beta}-\overline{p}\underline{\beta}\right] - B\left(\underline{p}-\varepsilon\right)\overline{p}\Delta\beta\right]
\end{aligned}$$

Since $B \geq \left[B - \frac{\xi(\tau)}{(1-\xi(\tau))} F\right]$ we need to prove that $\left(\underline{p} - \varepsilon\right) \overline{p} \Delta\beta \geq \left(\overline{p} + \varepsilon\right) \left(\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right)$

$$\begin{aligned}
\left(\underline{p} - \varepsilon\right) \overline{p}\Delta\beta - \left(\overline{p} + \varepsilon\right)\left(\underline{p}\overline{\beta} - \overline{p}\underline{\beta}\right) &= \underline{p}\overline{p}\Delta\beta - \overline{p}\underline{p}\overline{\beta} + \overline{p}\overline{p}\underline{\beta} - \varepsilon\left(\overline{p}\Delta\beta - \overline{\beta}\underline{p} + \overline{p}\underline{\beta}\right) \\
&= \overline{p}\left[\underline{\beta}\Delta p - \varepsilon\left(\Delta\beta - \frac{\underline{p}}{\overline{p}}\overline{\beta} + \underline{\beta}\right)\right]
\end{aligned}$$

Since $\varepsilon \leq \varepsilon_{nb}$, we have $\underline{\beta}\Delta p - \varepsilon\left(\Delta\beta - \frac{\underline{p}}{\overline{p}}\overline{\beta} + \underline{\beta}\right) \geq 0$

3. When $\varepsilon_{ib} \leq \varepsilon_{nb} \leq \varepsilon$

$$U_{HHib} - U_{HHnb} = \frac{\left(\overline{p} + \varepsilon\right)\Delta\beta\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}\left[\left(\overline{p} + \varepsilon\right) - \left(\underline{p} - \varepsilon\right)\right]} - \frac{B\Delta\beta\left(\overline{p} + \varepsilon\right)}{\underline{\beta}\left[\Delta p + 2\varepsilon\right]}$$

Since $B \geq \left[B - \frac{\xi(\tau)}{(1-\xi(\tau))} F\right]$ we have $U_{HHib} \leq U_{HHnb}$.

We can now calculate the income of the shareholders. There are two cases to consider.

When $\varepsilon \leq \varepsilon_{ib}$,

$$W_{IB} = E(\pi) - w - w_0 - (1-\gamma)(1-\xi(\tau))(\underline{p} - \varepsilon)\Delta\beta\frac{\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}(\Delta p + 2\varepsilon)}\left[\nu + (1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \underline{\beta}\overline{p}}\right]$$

When $\varepsilon \geq \varepsilon_{ib}$,

$$W_{IB} = E(\pi) - w - w_0 - (1-\gamma)\Delta\beta\frac{\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}(\Delta p + 2\varepsilon)}\left[\nu(1-\xi(\tau))(\underline{p} - \varepsilon) + (1-\nu)(\overline{p} + \varepsilon)\right]$$

■

Proof. We have to find for which values of τ, the contract is collusion proof and also prove that this bound is between 1 and τ_0.

1. $\varepsilon \leq \varepsilon_{ib}$

$$\begin{aligned}
W_{CP} - W_{CF} &= -(1-\gamma)(1-\xi(\tau))(\underline{p} - \varepsilon)\Delta\beta\frac{\left[B - \frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}(\Delta p + 2\varepsilon)}\left[\nu(1-\xi(\tau)) + (1 - \nu + \frac{\xi(\tau)\nu}{\tau})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \underline{\beta}\overline{p}}\right] \\
&\quad + (1-\gamma)\left[\nu + (1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \underline{\beta}\overline{p}}\right]\frac{B\Delta\beta(\underline{p} - \varepsilon)}{\underline{\beta}(\Delta p + 2\varepsilon)} \geq 0 \text{ for all } \tau \in [1, \tau_0]
\end{aligned}$$

with $\tau_0 = \frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \underline{\beta}\overline{p}}$ for $\varepsilon \leq \varepsilon_{ib}$.

Indeed,

$$W_{CP} - W_{CF} = \frac{(1-\gamma)(\underline{p} - \varepsilon)\Delta\beta}{\underline{\beta}(\Delta p + 2\varepsilon)}\left(\left[\nu + (1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \underline{\beta}\overline{p}}\right]B - \left[(1-\xi(\tau))B - \xi(\tau)F\right]\left[\nu(1-\xi(\tau)) + (1 - \nu + \frac{\xi(\tau}{\tau}\right.\right.$$

As, we have $\xi(\tau)^{ED}(\tau) = \frac{1}{\tau}$, this gives

$$W_{CP} - W_{CF} = \frac{(1-\gamma)(\underline{p} - \varepsilon)\Delta\beta}{\underline{\beta}(\Delta p + 2\varepsilon)}\left(\begin{array}{c} \left[\left(\nu + (1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \underline{\beta}\overline{p}}\right)(B + F) + \nu B\right]\tau^2 \\ -\nu\left[B + F + B\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \underline{\beta}\overline{p}}\right]\tau \\ +\nu\left(B + F\right)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \underline{\beta}\overline{p}} \end{array}\right)$$

This polynomial in τ with a positive second degree term has 2 positive roots. If those roots are both lower than τ_0, then, $W_{CP} - W_{CF} \geq 0$ for all $\tau \in [1, \tau_0]$.

The lowest root is

$$\tau_1 = \frac{\nu\left[B+F+B\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right] - \sqrt{\nu^2\left[B+F+B\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right]^2 - 4\nu\left(B+F\right)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\left[\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)(B+F)+\nu B\right.}}{2\left[\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)(B+F)+\nu B\right]}$$

we have

$$\begin{aligned}
&\tau_1 \geq \tau_0\\
&\iff 4\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)^2\left[\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)(B+F)+\nu B\right]^2\\
&-4\nu\left(B+F+B\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\left[\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)(B+F)+\nu B\right]\\
&-4\nu\left(B+F\right)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\left[\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)(B+F)+\nu B\right] \geq 0\\
&\iff 4\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)^2\left[\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)(B+F)+\nu B\right]\left[\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)(B+F)\right] \geq 0
\end{aligned}$$

which is true. $W_{CP} - W_{CF}$ is therefore positive for all $\tau \in [1, \tau_0]$.

The optimal contract is the collusion proof contract for $\varepsilon \leq \varepsilon_{ib}$.

2. $\varepsilon_{ib} \leq \varepsilon \leq \varepsilon_{nb}$

$$\begin{aligned}
W_{CP} - W_{CF} &= -(1-\gamma)\Delta\beta\frac{\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}(\Delta p+2\varepsilon)}\left[\nu(1-\xi(\tau))^2(\underline{p}-\varepsilon)+(1-\nu+\frac{\xi(\tau)\nu}{\tau})(\overline{p}+\varepsilon)\right]\\
&\quad+(1-\gamma)\left[\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right]\frac{B\Delta\beta(\underline{p}-\varepsilon)}{\underline{\beta}(\Delta p+2\varepsilon)} \geq 0 \text{ for all } \tau \in [1, \tau_0]
\end{aligned}$$

with $\tau_0 = \frac{\overline{p}+\varepsilon}{\underline{p}-\varepsilon} + 1$ if $\varepsilon \geq \varepsilon_{ib}$.

Indeed,

$$W_{CP}-W_{CF} = \frac{(1-\gamma)\Delta\beta}{\underline{\beta}(\Delta p+2\varepsilon)}\left(\left[\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right]B(\underline{p}-\varepsilon)+\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]\left[\nu(1-\xi(\tau))^2(\underline{p}-\varepsilon)+(1-\nu\right.\right.$$

As, we have $\xi^{ED}(\tau) = \frac{1}{\tau}$, this gives

$$W_{CP}-W_{CF} = \frac{(1-\gamma)\Delta\beta(\underline{p}-\varepsilon)}{\underline{\beta}(\Delta p+2\varepsilon)(\tau-1)}\left(\begin{array}{c}
\left[(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}-(1-\nu)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right]B\tau^3\\
\left[-\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)B+2\nu B+\left(\nu+(1-\nu)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)(B+F)\right]\tau^2\\
-\nu\left[B\left(1+\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)+2\nu\left(B+F\right)\right]\tau\\
+\nu\left(B+F\right)\left(1+\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)
\end{array}\right)$$

We are now able to show that this degree 3 polynomial, denote it $P(\tau)$, is negative for all $\tau \in [1, \tau_0]$. Indeed

$$\frac{\partial P(\tau)}{\partial \tau} = \left(\begin{array}{c} 3\tau^2 B(1-\nu)\left[\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right] \\ +2\tau\left[-\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)B + 2\nu B + \left(\nu+(1-\nu)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)(B+F)\right] \\ -\left[\nu B\left(1+\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right) + 2\nu(B+F)\right] \end{array} \right)$$

Moreover, as

$$\varepsilon \geq \varepsilon_{ib} \Longleftrightarrow \frac{\tau\left(\underline{\beta}\overline{p}(\tau_0-2)\right) - \varepsilon\overline{\beta}\tau_0}{\overline{p}\Delta\beta} \leq 1$$

we have

$$\begin{aligned} & \tau^2 B(1-\nu)\left[\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right] \\ = \; & \tau B(1-\nu)\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)\left(\frac{\tau\left(\underline{\beta}\overline{p}(\tau_0-2)\right) - \varepsilon\overline{\beta}\tau_0}{\overline{p}\Delta\beta}\right) \leq \tau B(1-\nu)\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right) \end{aligned}$$

and thus

$$\begin{aligned} \frac{\partial P(\tau)}{\partial \tau} \; & \leq \; \left(\begin{array}{c} 3\tau B(1-\nu)\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right) \\ +2\tau\left[-\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)B + 2\nu B + (1-\nu)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}(\tau_0-1)(B+F)\right] \\ -\left[\nu B\tau_0 + 2\nu(B+F)\right] \end{array} \right) \leq 0 \\ \Longleftrightarrow \; & \tau \leq \frac{\nu B\tau_0 + 2\nu(B+F)}{\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)B + \nu B + \left(\nu+(1-\nu)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)(B+F)} \end{aligned}$$

Moreover,

$$\frac{\nu B\tau_0 + 2\nu(B+F)}{\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)B + \nu B + \left(\nu+(1-\nu)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)(B+F)} \leq \tau_0$$

Hence, $\frac{\partial P(\tau)}{\partial \tau}$ is negative for all $\tau \in [1, \tau_0]$.

Finally, we will show that $\left(W_{CP}-W_{CF}\right)(\tau_0)\geq 0$

$$\left(W_{CP}-W_{CF}\right)(\tau_0)\geq 0 \Longleftrightarrow$$

$$\frac{(1-\gamma)\Delta\beta(\underline{p}-\varepsilon)}{\underline{\beta}(\Delta p+2\varepsilon)\tau_0}\begin{pmatrix} \left[(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}-(1-\nu)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right]B\tau_0^3 \\ \left[-\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)B+2\nu B+\left(\nu+(1-\nu)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)(B+F)\right]\tau_0^2 \\ -\left[\nu B\tau_0+2\nu\left(B+F\right)\right]\tau_0+\nu\left(B+F\right)\tau_0 \end{pmatrix}\geq 0$$

$$\Longleftrightarrow \begin{pmatrix} \left[(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}-(1-\nu)\left(\tau_0-1\right)\right]B\tau_0^2 \\ \left[-\left(\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)B+2\nu B+\left(\nu+(1-\nu)\left(\tau_0-1\right)\right)(B+F)\right]\tau_0 \\ -\nu B\tau_0-\nu\left(B+F\right) \end{pmatrix}\geq 0$$

$$\Longleftrightarrow \begin{pmatrix} B(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\tau_0\left(\tau_0-1\right)\left[\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}-\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right] \\ +(1-\nu)\tau_0\left(\tau_0-1\right)F+\nu\left(\tau_0-1\right)\left(B+F\right) \end{pmatrix}\geq 0$$

However, as

$$\begin{aligned} \varepsilon &\leq \varepsilon_{nb}\Longleftrightarrow \\ \frac{\overline{\beta}\Delta p}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}(\underline{p}-\varepsilon) &\geq \frac{\overline{p}\Delta\beta\underline{p}+\left[\underline{p}\overline{\beta}-\underline{\beta}\overline{p}\right]\underline{p}-\underline{p}\overline{\beta}\Delta p}{\left[\underline{p}\overline{\beta}-\underline{\beta}\overline{p}\right]}-(\underline{p}-\varepsilon) \end{aligned}$$

we have, together with $\overline{p}\Delta\beta\geq\overline{\beta}\Delta p$

$$\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}-\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\geq\frac{\overline{\beta}\Delta p}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}-\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\geq\frac{\overline{p}\Delta\beta\underline{p}+\left[\underline{p}\overline{\beta}-\underline{\beta}\overline{p}\right]\underline{p}-\underline{p}\overline{\beta}\Delta p}{\left[\underline{p}\overline{\beta}-\underline{\beta}\overline{p}\right](\underline{p}-\varepsilon)}-1-\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\geq 0$$

As $\left[\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}-\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right]\geq 0$, $\left(W_{CP}-W_{CF}\right)(\tau_0)$ is thus positive and as $\frac{\partial P(\tau)}{\partial\tau}$ is negative for all $\tau\in[1,\tau_0]$, $W_{CP}-W_{CF}$ is therefore positive for all $\tau\in[1,\tau_0]$.

The optimal contract is the collusion proof contract for $\varepsilon_{ib}\leq\varepsilon\leq\varepsilon_{nb}$.

3. $\varepsilon_{ib}\leq\varepsilon_{nb}\leq\varepsilon$

$$\begin{aligned} W_{CP}-W_{CF} &= -(1-\gamma)\Delta\beta\frac{\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]}{\underline{\beta}(\Delta p+2\varepsilon)}\left[\nu(1-\xi(\tau))^2(\underline{p}-\varepsilon)+(1-\nu+\frac{\xi(\tau)\nu}{\tau})(\overline{p}+\varepsilon)\right] \\ &\quad +(1-\gamma)\left[(\overline{p}+\varepsilon)-\nu(\Delta p+2\varepsilon)\right]\frac{B\Delta\beta}{\underline{\beta}(\Delta p+2\varepsilon)}\geq 0 \text{ for all } \tau\in[1,\tau_0] \end{aligned}$$

with $\tau_0=\frac{\overline{p}+\varepsilon}{\underline{p}-\varepsilon}+1$ if $\varepsilon\geq\varepsilon_{ib}$.

Indeed,

$$W_{CP}-W_{CF}=\frac{(1-\gamma)(\underline{p}-\varepsilon)\Delta\beta}{\underline{\beta}(\Delta p+2\varepsilon)}\left(\left[\nu+(1-\nu)\left(\tau_0-1\right)\right]B-\left[B-\frac{\xi(\tau)}{(1-\xi(\tau))}F\right]\left[\nu(1-\xi(\tau))^2+(1-\nu+\frac{\xi(\tau)\nu}{\tau}\right.\right.$$

As, we have $\xi^{ED}(\tau)=\frac{1}{\tau}$, this gives

$$W_{CP}-W_{CF}=\frac{(1-\gamma)(\underline{p}-\varepsilon)\Delta\beta}{\underline{\beta}(\Delta p+2\varepsilon)(\tau-1)}\begin{pmatrix}\left[\nu\left(2B+F\right)+\left(1-\nu\right)\left(\tau_0-1\right)F\right]\tau^2\\-\nu\left[B\tau_0+2\left(B+F\right)\right]\tau\\+\nu\left(B+F\right)\tau_0\end{pmatrix}$$

This polynomial in τ with a positive second degree term has 2 positive roots. If those roots are both lower than τ_0, then, $W_{CP}-W_{CF}\geq 0$ for all $\tau\in[1,\tau_0]$.

The lowest root is

$$\tau_1=\frac{\nu\left[B\tau_0+2\left(B+F\right)\right]-\sqrt{\nu^2\left[B\tau_0+2\left(B+F\right)\right]^2-4\nu\left(B+F\right)\tau_0\left[\nu\left(2B+F\right)+\left(1-\nu\right)\left(\tau_0-1\right)F\right]}}{2\left[\nu\left(2B+F\right)+\left(1-\nu\right)\left(\tau_0-1\right)F\right]}$$

we have

$$\begin{aligned}&\tau_1\geq\tau_0\\ \Longleftrightarrow\ &4\tau_0^2\left[\nu\left(2B+F\right)+\left(1-\nu\right)\left(\tau_0-1\right)F\right]^2-4\nu\tau_0\left[B\tau_0+\left(B+F\right)\right]\left[\nu\left(2B+F\right)+\left(1-\nu\right)\left(\tau_0-1\right)F\right]\geq 0\\ \Longleftrightarrow\ &4\tau_0\left(\tau_0-1\right)\left[\nu\left(2B+F\right)+\left(1-\nu\right)\left(\tau_0-1\right)F\right]\left[\nu\left(B+F\right)+(1-\nu)\tau_0F\right]\geq 0\end{aligned}$$

which is true. $W_{CP}-W_{CF}$ is therefore positive for all $\tau\in[1,\tau_0]$.

The optimal contract is the collusion proof contract for $\varepsilon_{ib}\leq\varepsilon_{nb}\leq\varepsilon$. ■

Proof of Lemma 1. Maximizing $W_{CB}(\tau)$ is equivalent to

When $\varepsilon\leq\widehat{\varepsilon}$,

$$W_{CB}(\tau)=\begin{cases}-\frac{1}{(\tau-1)\tau^2}\left[(\tau-1)B-F\right]\begin{bmatrix}\nu(\tau-1)^2(\underline{p}-\varepsilon)\\+((1-\nu)\tau^2+\nu)(\overline{p}+\varepsilon)\end{bmatrix} & \text{if } \tau\leq\widehat{\tau}\\ -\frac{1}{\tau^3}\left[(\tau-1)B-F\right]\begin{bmatrix}\nu(\tau-1)\tau\\+((1-\nu)\tau^2+\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\end{bmatrix} & \text{if } \widehat{\tau}\leq\tau\leq\tau_0\\ -\frac{1}{\tau}\left[(\tau-1)B-F\right]\left[\nu+(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right] & \text{if } \tau\geq\tau_0\end{cases}$$

When $\varepsilon\geq\widehat{\varepsilon}$,

$$W_{CB}(\tau)=\begin{cases}-\frac{1}{(\tau-1)\tau^2}\left[(\tau-1)B-F\right]\begin{bmatrix}\nu\left(\tau-1\right)^2\left(\underline{p}-\varepsilon\right)\\+((1-\nu)\tau^2+\nu)(\overline{p}+\varepsilon)\end{bmatrix} & \text{if } \tau\leq\tau_0\\ -\frac{1}{\tau}\left[(\tau-1)B-F\right]\left[\nu(\tau-1)(\underline{p}-\varepsilon)+(1-\nu)(\overline{p}+\varepsilon)\right] & \text{if } \tau\geq\tau_0\end{cases}$$

When $\varepsilon\leq\widehat{\varepsilon}$, Iif $\tau \leq \widehat{\tau}$

$$
\begin{aligned}
\frac{\partial W_{CB}(\tau)}{\partial \tau} &= -\frac{\left[\begin{array}{c} \left(B\left[\begin{array}{c} \nu(\tau-1)^2(\underline{p}-\varepsilon) \\ +((1-\nu)\,\tau^2+\nu)(\overline{p}+\varepsilon) \end{array}\right] + \left[(\tau-1)\,B-F\right]\left[\begin{array}{c} 2\nu(\tau-1)(\underline{p}-\varepsilon) \\ +2\,(1-\nu)\,\tau(\overline{p}+\varepsilon) \end{array}\right]\right)(\tau-1)\,\tau^2 \\ -\left[(\tau-1)\,B-F\right]\left[\begin{array}{c} \nu(\tau-1)^2(\underline{p}-\varepsilon) \\ +((1-\nu)\,\tau^2+\nu)(\overline{p}+\varepsilon) \end{array}\right]\left(2\,(\tau-1)\,\tau+\tau^2\right) \end{array}\right]}{(\tau-1)^2\,\tau^4} \\
&= -\frac{\left[\begin{array}{c} \left(B\left[\begin{array}{c} \nu(\tau-1)^2(\underline{p}-\varepsilon) \\ +((1-\nu)\,\tau^2+\nu)(\overline{p}+\varepsilon) \end{array}\right](\tau-1)\,\tau^2 + \left[(\tau-1)\,B-F\right]\left[(1-\nu)\,\tau^4(\overline{p}+\varepsilon)\right]\right) \\ -\left[(\tau-1)\,B-F\right](\overline{p}+\varepsilon)\nu\tau\,(\tau-1) \end{array}\right]}{(\tau-1)^2\,\tau^4} \\
&= -\frac{\left[\left(B\left[\begin{array}{c} \nu(\tau-1)^2(\underline{p}-\varepsilon) \\ +((1-\nu)\,\tau^2+\nu)(\overline{p}+\varepsilon) \end{array}\right](\tau-1)\,\tau + \left[(\tau-1)\,B-F\right](\overline{p}+\varepsilon)\left[(1-\nu)\,\tau^3-\nu\,(\tau-1)\right]\right)\right.}{(\tau-1)^2\,\tau^4} \\
&= -\frac{\left(B\left[\begin{array}{c} \nu(\tau-1)^3\tau(\underline{p}-\varepsilon) \\ +\,(1-\nu)\,\tau^3\,(2\tau-1)\,(\overline{p}+\varepsilon) \end{array}\right] - \left[B+F\right](\overline{p}+\varepsilon)\left[(1-\nu)\,\tau^3-\nu\,(\tau-1)\right]\right)}{(\tau-1)^2\,\tau^4} \\
&= -\frac{\left(\begin{array}{c} B\left[\nu(\tau-1)^3\tau(\underline{p}-\varepsilon)+2\,(1-\nu)\,\tau^3\,(\tau-1)\,(\overline{p}+\varepsilon)\right] \\ +\left[B+F\right](\overline{p}+\varepsilon)\nu\,(\tau-1)-F(\overline{p}+\varepsilon)\,(1-\nu)\,\tau^3 \end{array}\right)}{(\tau-1)^2\,\tau^4} \\
&= -\frac{\left(\begin{array}{c} B\left[\nu(\tau-1)^3\tau(\underline{p}-\varepsilon)+(1-\nu)\,\tau^3\,(\tau-1)\,(\overline{p}+\varepsilon)\right] \\ +\left[B+F\right](\overline{p}+\varepsilon)\nu\,(\tau-1)+(\overline{p}+\varepsilon)\,(1-\nu)\,\tau^3\left[(\tau-1)\,B-F\right] \end{array}\right)}{(\tau-1)^2\,\tau^4}
\end{aligned}
$$

if $\widehat{\tau} \leq \tau \leq \tau_0$

$$
\begin{aligned}
\frac{\partial W_{CB}(\tau)}{\partial \tau} &= -\frac{1}{\tau^3}\left[(\tau-1)\,B-F\right]\left[\begin{array}{c} \nu(\tau-1)\tau \\ +((1-\nu)\,\tau^2+\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right] \\
&= -\frac{B\tau^3\left[\begin{array}{c} \nu(\tau-1) \\ +((1-\nu)\,\tau^2+\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right] + \left[(\tau-1)\,B-F\right]\left[\nu\tau^3-\nu\,(\tau-1)\,\tau^3-\left[(1-\nu)\,\tau^4+\nu\right]\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right.}{\tau^6} \\
&= -\frac{\begin{array}{c} B\left[\nu\tau^4+\nu\,(\tau-1)^2\,\tau^3+\nu\tau\left(\tau^2-1\right)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right] \\ +\left[B+F\right]\left[\nu\,(\tau-1)\,\tau^3+\left[(1-\nu)\,\tau^4+\nu\right]\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right]-\nu\tau^3\left[B+F\right] \end{array}}{\tau^6} \\
&= -\frac{\begin{array}{c} B\left[\nu\,(\tau-1)^2\,\tau^3+\nu\tau\left(\tau^2-1\right)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right] \\ +\left[B+F\right]\left[\nu\,(\tau-1)\,\tau^3+\left[(1-\nu)\,\tau^4+\nu\right]\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right]+\nu\tau^3\left[(\tau-1)\,B-F\right] \end{array}}{\tau^6} \leq 0
\end{aligned}
$$

If $\tau \geq \tau_0$

$$\begin{aligned} \frac{\partial W_{CB}(\tau)}{\partial \tau} &= -\frac{[B\tau - (\tau-1)B + F]}{\tau^2}\left[\nu + (1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \underline{\beta}\overline{p}}\right] \\ &= -\frac{[B+F]}{\tau^2}\left[\nu + (1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta} - \underline{\beta}\overline{p}}\right] \leq 0 \text{ if } \tau \geq \tau_0 \end{aligned}$$

When $\varepsilon \geq \widehat{\varepsilon}$,

if $\tau \leq \tau_0$

$$\frac{\partial W_{CB}(\tau)}{\partial \tau} = -\frac{1}{(\tau-1)\tau^2}[(\tau-1)B - F]\left[\begin{array}{c} \nu(\tau-1)^2(\underline{p}-\varepsilon) \\ +((1-\nu)\tau^2 + \nu)(\overline{p}+\varepsilon) \end{array}\right]$$

if $\tau \geq \tau_0$

$$\begin{aligned} \frac{\partial W_{CB}(\tau)}{\partial \tau} &= -\frac{\left[\begin{array}{c} \tau\left(B\left[\nu(\tau-1)(\underline{p}-\varepsilon) + (1-\nu)(\overline{p}+\varepsilon)\right] + [(\tau-1)B - F]\nu(\underline{p}-\varepsilon)\right) \\ -[(\tau-1)B - F]\left[\nu(\tau-1)(\underline{p}-\varepsilon) + (1-\nu)(\overline{p}+\varepsilon)\right] \end{array}\right]}{\tau} \\ &= -\frac{\left[\begin{array}{c} \tau[(\tau-1)B - F]\nu(\underline{p}-\varepsilon) \\ +[B+F]\left[\nu(\tau-1)(\underline{p}-\varepsilon) + (1-\nu)(\overline{p}+\varepsilon)\right] \end{array}\right]}{\tau} \leq 0 \end{aligned}$$

■

Proof. As $W_{CB}(\tau)$ is non increasing on each interval, we have to compare its value for the lower bound of each interval[5].

When $\varepsilon \leq \widehat{\varepsilon}$,

$$W_{CB}(\tau_{\min}) - W_{CB}(\widehat{\tau}) =$$
$$-\left[B - \frac{1}{\tau_{\min}-1}F\right]\left[\begin{array}{c} \nu(\frac{\tau_{\min}-1}{\tau_{\min}})^2 \\ +(1-\nu+\frac{\nu}{\tau_{\min}^2})\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)} \end{array}\right] + \left[B - \frac{1}{\widehat{\tau}-1}F\right]\left[\begin{array}{c} \nu(\frac{\widehat{\tau}-1}{\widehat{\tau}})^2 \\ +(1-\nu+\frac{\nu}{\widehat{\tau}^2})(\frac{\widehat{\tau}-1}{\widehat{\tau}})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right]$$

$$W_{CB}(\tau_{\min}) - W_{CB}(\tau_0) =$$
$$-\left[B - \frac{1}{\tau_{\min}-1}F\right]\left[\begin{array}{c} \nu(\frac{\tau_{\min}-1}{\tau_{\min}})^2 \\ +(1-\nu+\frac{\nu}{\tau_{\min}^2})\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)} \end{array}\right] + \left[B - \frac{1}{\tau_0-1}F\right]\left[\begin{array}{c} \nu(\frac{\tau_0-1}{\tau_0})^2 \\ +(1-\nu)(\frac{\tau_0-1}{\tau_0})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right]$$

$$W_{CB}(\widehat{\tau}) - W_{CB}(\tau_0) =$$
$$-\left[B - \frac{1}{\widehat{\tau}-1}F\right]\left[\begin{array}{c} \nu(\frac{\widehat{\tau}-1}{\widehat{\tau}})^2 \\ +(1-\nu+\frac{\nu}{\widehat{\tau}^2})(\frac{\widehat{\tau}-1}{\widehat{\tau}})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right] + \left[B - \frac{1}{\tau_0-1}F\right]\left[\begin{array}{c} \nu(\frac{\tau_0-1}{\tau_0})^2 \\ +(1-\nu)(\frac{\tau_0-1}{\tau_0})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right]$$

[5] Assuming that $\tau_{\min}$ is such that $\left[B - \frac{1}{\tau_{\min}-1}F\right] = K > 0$.

When $\varepsilon \geq \widehat{\varepsilon}$,

$$W_{CB}(\tau_{\min}) - W_{CB}(\tau_0) =$$

$$-\left[B - \frac{1}{\tau_{\min}-1}F\right]\left[\begin{array}{c}\nu(\frac{\tau_{\min}-1}{\tau_{\min}})^2 \\ +(1-\nu+\frac{\nu}{\tau_{\min}^2})\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\end{array}\right] + \left[B - \frac{1}{\tau_0-1}F\right]\left[\nu(\frac{\tau_0-1}{\tau_0}) + (1-\nu)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right]$$

For each case, replacing $\tau_{\min}$, τ_0, $\widehat{\tau}$ by τ, the value of those differences is negative. This means that the first term is lower than the second one[6]. However, when the difference between the $\tau' s$ increases, the first term becomes higher than the second one. Thus, depending on the length of this interval, we can find the optimal τ.

Let's check what happens when the shareholders set F optimally in order to maximize their profits, i.e. such that $\left[B - \frac{1}{\tau-1}F\right] = K$.

When $\varepsilon \leq \widehat{\varepsilon}$,

$$W_{CB}(\tau_{\min}) - W_{CB}(\widehat{\tau}) =$$

$$-\left[\begin{array}{c}\nu(\frac{\tau_{\min}-1}{\tau_{\min}})^2 \\ +(1-\nu+\frac{\nu}{\tau_{\min}^2})\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\end{array}\right] + \left[\begin{array}{c}\nu(\frac{\widehat{\tau}-1}{\widehat{\tau}})^2 \\ +(1-\nu+\frac{\nu}{\widehat{\tau}^2})(\frac{\widehat{\tau}-1}{\widehat{\tau}})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\end{array}\right] \leq 0$$

$$\iff -\left[\begin{array}{c}\nu\widehat{\tau}^2(\tau_{\min}^2 - 2\tau_{\min}+1) \\ +((1-\nu)\,\tau_{\min}^2\widehat{\tau}^2 + \nu\widehat{\tau}^2)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\end{array}\right] + \left[\begin{array}{c}\nu\tau_{\min}^2(\widehat{\tau}^2 - 2\widehat{\tau}+1) \\ +((1-\nu)\,\widehat{\tau}^2+\nu)\tau_{\min}^2(\frac{\widehat{\tau}-1}{\widehat{\tau}})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\end{array}\right] \leq 0$$

$$\iff -\left[\begin{array}{c}\nu\widehat{\tau}^2(-2\tau_{\min}+1) \\ +\nu\widehat{\tau}^2\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\end{array}\right] + \tau_{\min}^2\left[\begin{array}{c}(1-\nu)\,\widehat{\tau}^2\left((\frac{\widehat{\tau}-1}{\widehat{\tau}})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right) \\ +\nu(\frac{\widehat{\tau}-1}{\widehat{\tau}})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - 2\nu\widehat{\tau}+\nu\end{array}\right] \leq 0$$

Moreover as

$$\widehat{\tau} = \frac{\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}}{\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}} \iff (\frac{\widehat{\tau}-1}{\widehat{\tau}})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)} = 0$$

We have

$$2\tau_{\min}\widehat{\tau}^2 - \widehat{\tau}^2\left(\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}+1\right) + \tau_{\min}^2\left[(\frac{\widehat{\tau}-1}{\widehat{\tau}})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - 2\widehat{\tau}+1\right] \leq 0$$

Using the fact that for $\tau \leq \widehat{\tau}$, we have $\tau_0 = \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}+1$, this gives

$$2\tau_{\min}\widehat{\tau}^2 - \widehat{\tau}^2\tau_0 + \tau_{\min}^2\left[\tau_0 - 2\widehat{\tau}\right] \leq 0$$

[6] For instance, for the case $W_{CB}(\tau_{\min}) - W_{CB}(\tau_0)$ we have:

$$\begin{aligned} & (1-\nu)(\tau-1)\overline{p}\Delta\beta(\underline{p}-\varepsilon) - ((1-\nu)\,\tau + \frac{\nu}{\tau})(\overline{p}+\varepsilon)\left[\underline{p}\overline{\beta}-\underline{\beta}\overline{p}\right] \\ = \; & (1-\nu)\left[\overline{p}\Delta\beta(\underline{p}-\varepsilon) - (\overline{p}+\varepsilon)\left[\underline{p}\overline{\beta}-\underline{\beta}\overline{p}\right]\right]\tau^2 - (1-\nu)\overline{p}\Delta\beta(\underline{p}-\varepsilon)\tau - \nu(\overline{p}+\varepsilon)\left[\underline{p}\overline{\beta}-\underline{\beta}\overline{p}\right] \\ = \; & (1-\nu)\left[\frac{\overline{p}\Delta\beta}{\left[\underline{p}\overline{\beta}-\underline{\beta}\overline{p}\right]} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right]\tau^2 - (1-\nu)\frac{\overline{p}\Delta\beta}{\left[\underline{p}\overline{\beta}-\underline{\beta}\overline{p}\right]}\tau - \nu\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)} \end{aligned}$$

This is a polynomial of degree with only one positive root which is lower than τ_0 (indeed, it is negative in τ_0)

This polynomial in $\tau_{\min}$ has 2 roots : $\tau_{\min} = \widehat{\tau}$ and $\tau_{\min} = \frac{-\widehat{\tau}}{[\tau_0 - 2\widehat{\tau}]}$. Using this, we can conclude that it is therefore negative for all $\tau_{\min} \leq \widehat{\tau}$.

$$
\begin{aligned}
& W_{CB}(\tau_{\min}) - W_{CB}(\tau_0) \\
&= -\left[\begin{array}{c} \nu(\frac{\tau_{\min}-1}{\tau_{\min}})^2 \\ +(1-\nu+\frac{\nu}{\tau_{\min}^2})\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)} \end{array}\right] + \left[\begin{array}{c} \nu(\frac{\tau_0-1}{\tau_0})^2 \\ +(1-\nu)(\frac{\tau_0-1}{\tau_0})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right] \\
&= -\left[\begin{array}{c} \nu\tau_0^2(\tau_{\min}^2 - 2\tau_{\min}+1) \\ +((1-\nu)\,\tau_{\min}^2\tau_0^2 + \nu\tau_0^2)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)} \end{array}\right] + \left[\begin{array}{c} \nu\tau_{\min}^2(\tau_0^2 - 2\tau_0 + 1) \\ +(1-\nu)\,\tau_{\min}^2(\tau_0^2-\tau_0)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right]
\end{aligned}
$$

$$
= \tau_{\min}^2\left(\begin{array}{c} \nu(\tau_0^2 - 2\tau_0+1) \\ +(1-\nu)\,(\tau_0^2-\tau_0)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array} - \nu\tau_0^2 - (1-\nu)\,\tau_0^2\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right) + 2\nu\tau_0^2\tau_{\min} - \nu\tau_0^2\left(\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}+1\right) \leq 0
$$

$$
\tau_{\min}^2\left(\nu - \tau_0\left[(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} + 2\nu\right] + (1-\nu)\,\tau_0^2\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)\right) + 2\nu\tau_0^2\tau_{\min} - \nu\tau_0^2\left(\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}+1\right) \leq 0
$$

$$
\tau_0^2\left((1-\nu)\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)\tau_{\min}^2 + 2\nu\tau_{\min} - \nu\left(\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}+1\right)\right) - \tau_{\min}^2\left[(1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} + 2\nu\right]\tau_0 + \nu \leq 0
$$

The coefficient of the degree 2 term is negative for low values of $\tau_{\min}$ and positive for high values of $\tau_{\min}$. Indeed, it is a degree 2 polynomial in $\tau_{\min}$, with only a positive root, $\tau_{\min}^*$ which is lower than τ_0 :

$$
\frac{-2\nu + \sqrt{4\nu^2 + 4\nu\,(1-\nu)\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)\left(\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}+1\right)}}{2\,(1-\nu)\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)} \leq \tau_0 = \left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)
$$

$$
4\nu\,(1-\nu)\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)\left(2\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)} - 1\right) + 4\,(1-\nu)^2\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\right)^2\left(\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right)^2 \geq
$$

When $\tau_{\min} \leq \tau_{\min}^*$, $W_{CB}(\tau_{\min}) - W_{CB}(\tau_0)$ is a concave second degree polynomial in τ_0 with only one positive root. As it is negative for $\tau_0 = \tau_{\min},$ $W_{CB}(\tau_{\min}) - W_{CB}(\tau_0) \leq 0$ for all $\tau_0 \geq \tau_{\min}$.

When $\tau_{\min} \geq \tau_{\min}^*$, $W_{CB}(\tau_{\min}) - W_{CB}(\tau_0)$ is a convex second degree polynomial in τ_0 with two positive roots (denote teh highest root $\overline{\tau_0}$). As it is negative for $\tau_0 = \tau_{\min},$ $W_{CB}(\tau_{\min}) - W_{CB}(\tau_0) \leq$

0 for all $\tau_{\min} \leq \tau_0 \leq \overline{\tau_0}$ and $W_{CB}(\tau_{\min}) - W_{CB}(\tau_0) \geq 0$ for all $\tau_0 \geq \overline{\tau_0}$.

$$\begin{aligned} & W_{CB}(\widehat{\tau}) - W_{CB}(\tau_0) \\ & -\left[\begin{array}{c} \nu(\frac{\widehat{\tau}-1}{\widehat{\tau}})^2 \\ +(1-\nu+\frac{\nu}{\widehat{\tau}^2})(\frac{\widehat{\tau}-1}{\widehat{\tau}})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right] + \left[\begin{array}{c} \nu(\frac{\tau_0-1}{\tau_0})^2 \\ +(1-\nu)(\frac{\tau_0-1}{\tau_0})\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right] \\ = & -\left[\begin{array}{c} \nu\tau_0^2(\widehat{\tau}^2 - 2\widehat{\tau}+1) \\ +\tau_0^2((1-\nu)\widehat{\tau}^2+\nu)\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)} \end{array}\right] + \left[\begin{array}{c} \nu\widehat{\tau}^2(\tau_0^2 - 2\tau_0+1) \\ +(1-\nu)\widehat{\tau}^2(\tau_0^2-\tau_0)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} \end{array}\right] \\ = & \tau_0^2\left[((1-\nu)\widehat{\tau}^2\left[\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right] + \nu(2\widehat{\tau} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)} - 1)\right] - \left[(1-\nu)\widehat{\tau}^2\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} + 2\nu\widehat{\tau}^2\right]\tau_0 + \nu\widehat{\tau}^2 \\ = & \widehat{\tau}^2\left[((1-\nu)\tau_0^2\left[\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}} - \frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right] - (1-\nu)\frac{\overline{p}\Delta\beta}{\underline{p}\overline{\beta}-\underline{\beta}\overline{p}}\tau_0 - 2\nu\tau_0 + \nu\right] + 2\nu\tau_0^2\widehat{\tau} - \nu\tau_0^3 \end{aligned}$$

When the degree 2 coefficient of this polynomial is positive, there is only one positive root and as it is negative for $\widehat{\tau} = \tau_0$, $W_{CB}(\widehat{\tau}) - W_{CB}(\tau_0) \leq 0$ for all $\widehat{\tau} \leq \tau_0$.

However, when this degree 2 coefficient is negative, there is two positive roots. As $W_{CB}(\widehat{\tau}) - W_{CB}(\tau_0) \leq 0$ for $\widehat{\tau} = \tau_0$, and $\frac{\partial(W_{CB}(\widehat{\tau})-W_{CB}(\tau_0))}{\partial\widehat{\tau}}(\tau_0) \geq 0$, we also have $W_{CB}(\widehat{\tau}) - W_{CB}(\tau_0) \leq 0$ for all $\widehat{\tau} \leq \tau_0$.

When $\varepsilon \geq \widehat{\varepsilon}$,

$$\begin{aligned} & W_{CB}(\tau_{\min}) - W_{CB}(\tau_0) \leq 0 \\ \Longleftrightarrow & -\left[\nu(\frac{\tau_{\min}-1}{\tau_{\min}})^2 + \frac{\nu}{\tau_{\min}^2}\frac{(\overline{p}+\varepsilon)}{(\underline{p}-\varepsilon)}\right] + \nu(\frac{\tau_0-1}{\tau_0}) \leq 0 \\ \Longleftrightarrow & -\tau_0\left(\tau_{\min}^2 - 2\tau_{\min}+1\right) - \tau_0^2 + \tau_0 + \tau_{\min}^2(\tau_0-1) \leq 0 \\ \Longleftrightarrow & 2\tau_0\tau_{\min} - \tau_0^2 - \tau_{\min}^2 \leq 0 \\ \Longleftrightarrow & \left[\frac{\tau_0}{\tau_{\min}}\right]^2 - 2\frac{\tau_0}{\tau_{\min}} + 1 \geq 0 \end{aligned}$$

We therefore have $W_{CB}(\tau_{\min}) \leq W_{CB}(\tau_0)$ when $\varepsilon \geq \widehat{\varepsilon}$. ■

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