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Assessment of the Spectral Volume Method on Inviscid and Viscous Flows

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Abstract The compact high-order *'Spectral Volume Method'* (SVM, Wang (2002)) designed for conservation laws on unstructured grids is presented. Its spectral reconstruction is exposed briefly and its applications to the Euler equations are presented through several test cases to assess its accuracy and stability. Comparisons with classical methods such as MUSCL show the superiority of SVM. The SVM method arises as a high-order accurate scheme, geometrically flexible and computationally efficient.

1 INTRODUCTION

Despite the constant improvements in computational and data processing resources, the continuously growing requirements of computational fluid dynamics still remain unsatisfied. In the last decade, the CFD community showed a growing interest in high-order approximations to address these issues (WENO, Discontinuous Galerkin, ...). An attractive choice is the *'Spectral Volume Method'* (SVM) proposed and developed by Wang et al. [6, 5, 3] which achieves high-order accuracy on unstructured grids through polynomial reconstruction.

To assess the performance of the SVM method, different test cases were computed with *Typhon*, an unstructured open-source code. The numerical experiments were chosen to cover a large set of flow configurations from continuous quasi-incompressible problems to shock wave propagations and mixing flows.

The results presented here are up to expectations with a significant increase in accuracy and a reduction in CPU time compared to a usual second order method.

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2 Spectral Volume Method for the 2D Euler equations

The SVM method achieves high-order accuracy on unstructured grids through polynomial reconstruction within initial grid cells (spectral volumes, SVs) with a subdivision of the each SV into polygonal control volumes (CVs, Fig. (1)). The spectral splitting of the SV is designed to minimize internal reconstruction oscillations [1]. The flux computations are finally achieved with Gaussian quadrature directly inferred from CV states ponderation with a constant and unique set of coefficients for each SV in the whole domain.

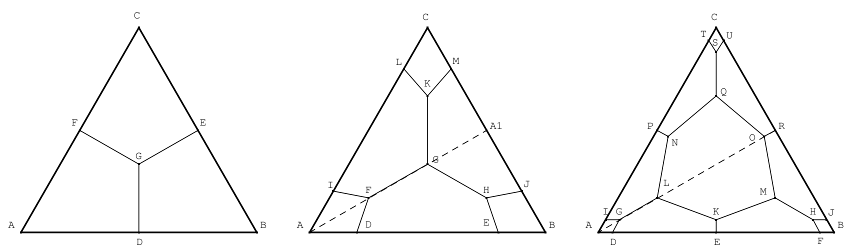

Fig. 1 Geometric splitting definitions for the second, third and fourth order partitions

While high-order reconstructions are often based on projections and gradient evaluations on a quite large region of neighboring cells, the SVM compact stencil makes it computationally efficient and attractive. The polynomial reconstruction used remains exact (at a given order) on arbitrarily shaped triangles.

2.1 Spectral volumes partitions

The subdivision of a spectral volume S_i in different controle volumes C_{ij} is an essential step for the SVM scheme. For example, some geometric partitions can lead to degenerated systems and thus should be excluded. The splitting procedure must conserve the existing symmetries in a triangle, use straight edges and consider convex CVs. Fig. 1 shows different geometric partitions of a spectral volume for different desired orders. While the second order splitting definition is unique, the higher order partitions introduce some geometric parameters. The third order partitions is defined by two parameters: $\alpha = \frac{AD}{AB}$ and $\beta = \frac{AF}{AA1}$. The fourth order partitions have four degrees of freedom: $\alpha = \frac{AD}{AB}$, $\beta = \frac{AG}{AR}$, $\gamma = \frac{OR}{AR}$ and $\delta = \frac{AL}{AR}$. The accuracy and stability of a given spectral volume scheme depends on the choice of these geometric parameters [1, 2].

Table 1 sums up the different implemented and tested partitions with their geometric parameters.

Partition	Order	α	β	δ	γ
SVM2	2	-	-	-	-
SVM3W	3	1/4	1/3	-	-
SVM3K	3	91/1000	18/100	-	-
SVM3K2	3	0.1093621117	0.1730022492	-	-
SVM4W	4	1/15	2/15	2/15	1/15
SVM4K	4	78/1000	104/1000	52/1000	351/1000
SVM4K2	4	0.0326228301	0.042508082	0.0504398911	0.1562524902

Table 1 Geometric parameters for the different SVM partitions

The SVMW splittings were proposed by Wang et al. and designed by minimizing the Lebesgue constant over the SV. Whereas the minimization of this constant provides a good assessment of the quality of the SVM splitting, different observations show that it is not a sufficient condition for the stability of the scheme. Thus, Abeele proposed other geometric splittings noted as SVMK and SVMK2 [1, 2] .

2.2 Spectral volume method assets

The spectral volume method uses a compact stencil which is a great advantage compared to other high order reconstructions. The SVM geometric splitting is designed and optimised in a spectral way to minimize internal reconstruction oscillations known as Runge phenomena. The reconstructed field is continuous over the entire SV, therefore internal faces do not constitute Riemann problems, which reduces the flux computation cost and contains the problem of data limitation to the SV faces. The other interesting aspect of the SVM is the homothetic nature of the splitting reconstruction: no new metric terms need be kept in memory. The interpolation on Gauss points for fluxes computation is directly inferred from a weighting of CV states with constant and unique coefficients for the whole domain.

Lastly, while the usual finite-volume and finite-difference methods depend strongly on the grid quality and density, the SVM reconstruction remains exact (at a given order) on arbitrarily shaped triangles.

3 Numerical experiments

To assess the performance of the SVM method, different test cases were computed with *Typhon*, an open-source unstructured solver [4]. The numerical experiments were chosen to cover a large set of flow configurations. In the next sections, we focus on the results of two chosen cases: the evolution of a convected vortex and a simple Mach reflection on a wedge. The HLLC solver is used for the Riemann problem at faces. Time integration is performed with a third order TVD Runge-Kutta scheme and SVM results are compared to results provided by a usual MUSCL method.

3.1 Convected Vortex

We consider a vortex evolution problem governed by: $\frac{\partial p}{\partial t} = \rho \frac{{v_\theta}^2}{r}$ and convected with a speed of $V_{conv} = 20$ in the x direction on a regular domain of $[-5:5] \times [-5:5]$. Periodic boundary conditions are set in the x and y directions. Thus, the vortex crosses the whole domain twice from left to right between t=0 and t=1. No limiters were employed for the SVM simulations.

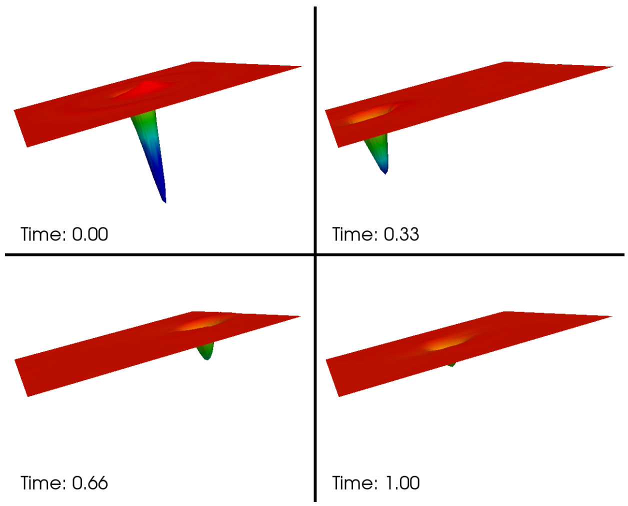

Fig. 2 Overview of numerical damping of a convected vortex using MUSCL scheme

The overview of the vortex damping is presented on figure 2 for the MUSCL simulation using the Van Albada limiter. While the theoretical maximum transverse velocity is $V_{ymax} = 30$, $V_{ymax} \sim 10$ is obtained for this MUSCL simulation, thereby showing that this test case is very sensitive to numerical dissipation.

For comparison purposes, the velocity profiles are considered on figure 3 at t=1 along a horizontal line passing through the vortex center. With the SVM schemes, the velocity profile is better preserved when the order is increased. Thus, $V_{ymax} \sim 15$, $V_{ymax} \sim 25$ and $V_{ymax} \sim 30$ are obtained for the second, third and fourth order respectively.

Another interesting aspect of SVM simulations is the reduction in CPU time per time step compared to a classic MUSCL method using the same number of control volumes: 50% for 2^{nd} order, 32% to 35% for 3^{rd} order and 22% to 25% for 4^{th} order. The CPU time savings are due to the absence of gradient evaluation for inviscid fluxes and the continuity of state variables through internal faces. This latter fact reduces the limitation problems and several cases can be computed without any limitation procedure.

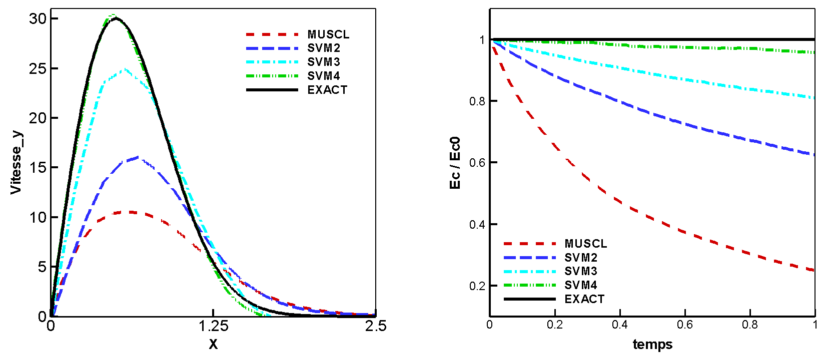

Fig. 3 Comparison of velocity profiles along a line through the vortex center at $t = 1$ (left) and kinetic energy time evolution (right)

3.2 Simple Mach reflection

This case deals with a classic problem of shock reflection with a Mach number $M_s = 1.7$ and a wedge angle of $\theta = 25°$. The numerical results were obtained on a domain of $[25 \times 16.5]$ on the x, y plane with the apex of the wedge placed at $x, y = 4.69, 0$. The upstream shock conditions are ambient conditions, with $\rho_a = 1.225$, $p_a = 1.01325 \cdot 10^5$ and $u = 0$.

Different SVM simulations were carried out with different orders. Observations show that the contours around the incident and reflected shocks and the slip surface are better captured when the order increases. To prove that the capture of these features and the better resolution of the flow are due to the increasing order, other simulations have been performed using the MUSCL scheme. The results obtained

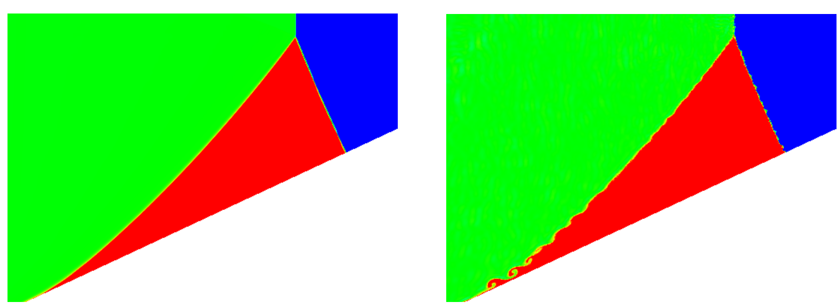

Fig. 4 Entropy contours at slip surface: MUSCL, 800.000 CV (left) and SVM4, 422.080 CV (right)

with the SVM4 scheme were compared to those provided by the MUSCL method on a finer structured cartesian grid. While the unstructured grid for SVM4 contains 422.080 CVs, the structured grid used for MUSCL simulation contains almost twice the number of CVs, i.e. 800.000 CVs. Yet, the shocks thicknesses obtained are large and no improvement of the slip surface resolution is observed when increasing the number of CVs with the MUSCL method. Figure 4 clearly shows that the SVM scheme better captures the slip surface: Kelvin-Helmholtz instabilities can be observed on the shear line. These observations suggest that the better resolution of the problem is achieved by the order increase rather than the mesh refinement.

4 Conclusion

In this study, analysis and applications of the Spectral Volume Method through applications were presented. This high-order reconstruction is particularly interesting owing to its compact support. The use of flux evaluation on split control volumes makes this method very similar to Finite-Volume method which makes the SVM implementation in existing codes relatively easy.

The results are up to expectations with a significant increase in occuracy and a reduction in CPU time compared to a MUSCL method with the same number of elements. The CPU gains are due to the absence of gradient evaluation for inviscid fluxes and the continuity of status variables through internal faces. This property reduces the limitation problems and several cases can be computed with no limitation procedure.

For all previously reported assets, the Spectral Volume Method arises as a promising high-order reconstruction mode for both academic and industrial studies especially for complex applications which require great accuracy with computational resource savings.

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