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Overview on a selection of recent works in asymptotic analysis for wave propagation problems

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In recent years, the study of wave propagation has induced a lot of research in relation with asymptotic analysis. While many types of problems may possibly be linked both to asymptotic analysis and scattering theory, the present overview focuses on recent advances concerned with wave propagation problems involving a small perturbation of geometrical nature.

General issues Consider a well posed wave propagation problem (P_δ) with solution u_δ (called "exact") that depends on a parameter δ satisfying $k\delta << 1$ where k is the wave number. The problem (P_δ) is assumed to be a perturbed version of a limit problem (P_0) from which it differs only in a localised region that is as small as δ in one (or more) direction of space.

Our purpose is to design a numerical method for (P_δ) that would keep track of the influence of the perturbation at a reasonnable computational price. One idea is to find another well posed problem $(\widetilde{\mathrm{P}}_\delta)$ that satisfies two features:

- its dependency with respect to δ is easier to handle numerically,
- its unique solution $\widetilde{u}_\delta$ is a sufficiently sharp approximation of u_δ.

Deriving such an approximate model for (P_δ) is usually not a trivial issue. Suppose that we want to build $(\widetilde{\mathrm{P}}_\delta)$ such that $\|u_\delta - \widetilde{u}_\delta\| = O(\,\eta(\delta)\,)$ with $\eta(\delta) \to 0$. Consider the expansion

$$u_\delta(\mathbf{x}) = u_0(\mathbf{x}) + \lambda_1(\delta)u_1(\mathbf{x}) + \cdots + \lambda_N(\delta)u_N(\mathbf{x}) + O(\,\eta(\delta)\,)$$

On possible approach consists in identifying a (well posed) problem $(\widetilde{\mathrm{P}}_\delta)$ that would be satisfied by $\sum_{n=0}^{N}\lambda_n(\delta)u_n(\mathbf{x})$ up to a residual in $O(\eta(\delta))$, and proving that $\widetilde{u}_\delta$ satisfies a relevant error estimate noting that $\widetilde{u}_\delta = \sum_{n=0}^{N}\lambda_n(\delta)u_n + O(\,\eta(\delta)\,)$.

Although approximate models can be easier to handle, they may contain exotic features which would require specific numerical treatments. Once a numerical method has been proposed for $\widetilde{\mathrm{P}}_\delta$, a typical issue consists in proving that there exists $C > 0$ (independent of δ) and $\mu(h) \to 0$ such that $\|\widetilde{u}_\delta - \widetilde{u}_{\delta,h}\| \leq C\,\mu(h)$ for all $h, \delta \in (0, 1)$ where $\widetilde{u}_{\delta,h}$ is the discrete solution.

Thin layer problems In the first category of problem that we examine, the perturbation has the structure of a thin layer of thickness δ located along smooth boundaries or smooth interfaces of the domain of propagation. In such a case the exact solution admits an expansion of the form: $u_\delta = u_0 + \delta u_1 + \delta^2 u_2 + \ldots$. A possible approximate model then consists in posing the wave equation in the limit geometry, as if there were no perturbation, and taking into account the thin layer by means of modified boundary conditions called Generalized Impedance Boundary Conditions (GIBC).

Scattering by objects with thin dielectric coating is a first exemple of such a case. On this subject, Engquist and Nédélec in [15] proposed a GIBC that was much sharper than any of the modified impedance boundary conditions that had been previously introduced. Since then, a complete theoretical framework for thin coatings has been developped by Bendali, Lemrabet, Joly, Haddar and co-workers in [2, 3, 4, 14, 16], including derivation of full expansions and rigorous justifications of GIBCs of high order. Besides Poignard in [29] studied the case a high contrasted thin coating, and Chun and Hestaven in [7] validated the good computational efficiency of GIBCs in a discontinuous Galerkin context.

Scattering by a highly absorbing obstacle is another case of thin layer problem. Assume that such an object admits an absorption coefficient $\sigma = k/\delta^2$ where $\delta \to 0$. A wave penetrating such an object keeps a significant amplitude only in a region of thickness δ concentrated along the exterior boundary of the obstacle: this is the skin effect. Whereas thin coating problems had been studied for long, there existed very few works on high absorption problems until recently, except [1, 32]. Haddar, Joly and Nguyen in [17, 18] proposed results comparable to what had been established in the case of thin coatings. Péron, Dauge and co-workers in [6, 27] also proposed a derivation of full expansions for obstacles with Lipschitz boundary (not just smooth) and studied precisely the influence of the geometry on the skin thickness. Finally Haddar and Lechleiter in [19] proposed a similar analysis in the context of scattering by an unbounded obstacle.

The case of a domain of propagation containing a dielectric layer of thickness δ is a third example that caught attention only recently. In [31] Schmidt and Tordeux proposed a full expansion and approximate trasmission conditions precise in $O(\delta^2)$ for a scalar problem. Poignard and Péron in [28, 30] derived an expansion in $O(\delta^3)$ of the exact solution to an electromagnetic scattering problem. Chun, Haddar and Hestaven in [8] formally derived an expansion for a time domain electromagnetic problem and studied computational efficiency of high order GIBCs in a discontinuous Galerkin framework.

Geometric singular perturbation problems In a second part of this presentation we focus on asymptotic problems for which the solutions admit an expansion with terms that may have a singular behaviour related to a singularity appearing in the geometry. In this situation, the exact solution may admit an expansion with more complex structure than for thin layer problems. On this type of asymptotic analysis for elliptic problems, the reference book is [26]. For such cases different asymptotic approaches are possible, we comment on two: multiscale expansion method and matched asymptotics. Clear presentation and comparison of those two approaches was the subject of [13]. Concerning matched asymptotics, a reference book is [20].

Multiscale expansion method and matched asymptotics share several common features. Both techniques involve far field functions expressed in standard variables, and near field functions that depend on scaled variables, and in both cases the expansion is obtained as an interpolation between the near and the far field terms. These methods differ on the cutt-off functions used to interpolate, and on the exact procedure used to construct the far field and near field terms. The multiscale expansion method provides sharper approximations but looks less intrinsic than matched asymptotics. The construction procedure associated to matched asymptotic requires to enforce an algebraic identity called "matching principle".

Several recent works dedicated to problems of this category deserve attention. Joly, Tordeux and co-workers in [12, 21, 22, 23, 24] applied matched asymptotics and provided a full expansion and approximate models for the propagation of waves in thin slots. This work also brought deep insight on the matching principle for a whole class of problems. Dauge, Vial, Costabel and Caloz in [5, 33] studied a thin coating problem in the case of a domain whose boundary contains an angle. An important conclusion of this work is that the ansatz is directly related to the opening of the angle: this case is much more involved than the case of a smooth boundary. Besides in this situation, whether it is possible to derive high order GIBC remains an open question. The works of Tordeux and Vial have inspired further advances in many other situations including patch antennas, see [25], or diffraction by thin wires, see [9, 10, 11].

Open questions Concerning remaining open questions that we present, we would like to formulate two observations. First, asymptotic problems in a context of time domain scattering have received only few attention so far. Second, theoretical numerical analysis for approximate model in an asymptotic context still remains to be developped for most of the problems already studied from a purely analytical point of view.

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