Optimal control of fractional systems: a diffusive formulation

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***Abstract*— Optimal control of fractional linear systems on a finite horizon can be classically formulated using the adjoint system. But the adjoint of a causal fractional integral or derivative operator happens to be an anti-causal operator: hence, the adjoint equations are not easy to solve in the first place. Using an equivalent diffusive realization helps transform the original problem into a coupled system of PDEs, for which the adjoint system can be more easily derived and properly studied.**

I. INTRODUCTION

Fractional differential systems have become quite popular in the recent decades, giving rise to a wide literature, both on the theoretical and on the applied sides; monogaphs, and special issues of international journals are now devoted to this active research field. However, even if different scientific communities seem to have been involved in these questions, still very few papers are concerned with the question of *optimal* control of *fractional* differential systems (in e.g. [20] or [1], ad hoc finite-dimensional approximations of fractional derivatives are used in the first place, and classical optimal control methods are being applied in the second place; no proof of convergence of the process is provided).

A first reason for that could be that optimal control of infinite-dimensional systems is a quite involved and technical field, but a second one lies in the very nature of fractional operators. They are causal, but highly non-local in time (with a weakly integrable singularity at the origin); hence their adjoint becomes necessarily anti-causal and still non-local in time. Thus, one can easily imagine that the complexity of the theory for forward fractional dynamical systems becomes even more intricate when the coupled equations of the adjoint systems are derived; because we will be left with coupled forward and backward fractional dynamics in order to solve the optimal control problem: at first glance, it seems very unlikely that Riccati equations (if any) could be either analysed or even solved (not to speak of adequate numerical schemes for these) in such a complicated setting.

In order to overcome this intrinsic difficulty, we propose to use the equivalent *diffusive* representations of fractional systems, and to work on it, as for infinite dimensional systems of integer order.

The outline of the paper is as follows: in § II, we borrow from [10] a primer on diffusive representation, in a causal setting only; in § III, the adjoints are first computed in an input-output representation, then represented in state-space through anti-causal diffusive realizations; finally in § IV a family of fractional models are presented, with increasing complexity: starting from a finite-dimensional model, going through a fractional toy-model, and ending by a fully infinite-dimensional model arising in acoustics.

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II. A PRIMER ON DIFFUSIVE REPRESENTATION

In this section, we focus on the *causal* solution of a family of first-order ordinary differential equations (ODEs). Hence, the mathematical setting is the convolution algebra $\mathcal{D}'_+(\mathbb{R})$ of causal distributions.

A. An Elementary Approach

Consider the numerical identity, valid for $\delta > 1$:

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^\delta} = \frac{\frac{\pi}{\delta}}{\sin(\frac{\pi}{\delta})}\,.$$

Letting $s \in \mathbb{R}^+_*$ and substituting $x = (\frac{\xi}{s})^{\frac{1}{\delta}}$ in the above *numerical* identity, we get:

$$\int_0^\infty \frac{\sin(\frac{\pi}{\delta})}{\pi} \frac{1}{\xi^{1-\frac{1}{\delta}}} \frac{1}{s+\xi}\,\mathrm{d}\xi = \frac{1}{s^{1-\frac{1}{\delta}}}$$

Finally, performing an analytic continuation from $\mathbb{R}^{+*}$ to $\mathbb{C}\setminus\mathbb{R}^-$ for both sides of the above identity in the complex variable s, and letting $\beta := 1-\frac{1}{\delta} \in (0,1)$, we get the *functional* identity:

$$\begin{array}{rll} H_\beta : & \mathbb{C}\setminus\mathbb{R}^- & \to \mathbb{C} \\ & s & \mapsto \displaystyle\int_0^\infty \mu_\beta(\xi)\,\frac{1}{s+\xi}\,\mathrm{d}\xi = \frac{1}{s^\beta}\,, \end{array} \quad (1)$$

with density $\mu_\beta(\xi) = \frac{\sin(\beta\,\pi)}{\pi}\,\xi^{-\beta}$.

Applying an inverse Laplace transform to both sides gives:

$$\begin{array}{rll} h_\beta : & \mathbb{R}^+ & \to \mathbb{R} \\ & t & \mapsto \displaystyle\int_0^\infty \mu_\beta(\xi)\,e^{-\xi\,t}\,\mathrm{d}\xi = \frac{1}{\Gamma(\beta)} t^{\beta-1}\,. \end{array} \quad (2)$$

B. Input-output Representations

Let u and $y = I^\beta u$ be the input and output of the *causal* fractional integral of order β, defined by the Riemann-Liouville formula $y = h_\beta \star u = \int_0^t h_\beta(t-\tau)\,u(\tau)\,\mathrm{d}\tau$ in the time domain, which reads $Y(s) = H_\beta(s)\,U(s)$ in the Laplace domain.

Using the integral representations above, together with Fubini's theorem, we get:

$$y(t) = \int_0^\infty \mu_\beta(\xi)\,[e_\xi \star u](t)\,\mathrm{d}\xi$$

with $e_\xi(t) := e^{-\xi\,t}$, and $[e_\xi \star u](t) = \int_0^t e^{-\xi\,(t-\tau)}\,u(\tau)\,\mathrm{d}\tau$.

Now for fractional *derivative* of order $\alpha \in (0,1)$ in the sense of distributions of Schwartz, we have $\widetilde{y} = D^\alpha u = D[I^{1-\alpha} u]$, and a careful computation shows that:

$$\widetilde{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u - \xi\, e_\xi \star u](t)\,\mathrm{d}\xi$$

C. State Space Representation

In both input-output representations above, introducing a state, say $\varphi(\xi,.)$ which realizes the classical convolution $\varphi(\xi,.) := [e_\xi \star u](t)$ leads to the following diffusive *realizations*, in the sense of systems theory:

$$\partial_t \varphi(\xi,t) = -\xi\,\varphi(\xi,t) + u(t),\ \varphi(\xi,0) = 0\,, \quad (3)$$

$$y(t) = \int_0^\infty \mu_\beta(\xi)\,\varphi(\xi,t)\,\mathrm{d}\xi\,; \quad (4)$$

and

$$\partial_t \widetilde{\varphi}(\xi,t) = -\xi\,\widetilde{\varphi}(\xi,t) + u(t),\ \widetilde{\varphi}(\xi,0) = 0\,, \quad (5)$$

$$\widetilde{y}(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u(t) - \xi\,\widetilde{\varphi}(\xi,t)]\,\mathrm{d}\xi\,. \quad (6)$$

These are first and extended diffusive realizations, respectively. The slight difference between (3)-(4) and (5)-(6), marked by the $\widetilde{\ }$ notation, lies in the underlying functional spaces in which these equations make sense: φ belongs to $\mathcal{H}_\beta := \{\varphi\ s.t.\ \int_0^\infty \mu_\beta(\xi)|\varphi|^2\,\mathrm{d}\xi < \infty\}$, whereas $\widetilde{\varphi}$ belongs to $\widetilde{\mathcal{H}}_\alpha := \{\widetilde{\varphi}\ s.t.\ \int_0^\infty \mu_{1-\alpha}(\xi)|\widetilde{\varphi}|^2\,\xi\,\mathrm{d}\xi < \infty\}$, see e.g. [6, ch. 2] or [14].

III. ADJOINTS OF FRACTIONAL SYSTEMS

A. Adjoints of Fractional Integrals

On $L^2(0,T)$, the adjoint of I^β_{0+}, the causal fractional integral of order $\beta \in (0,1)$ defined by

$$y(t) := I^\beta_{0+} u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1}\, u(\tau)\,\mathrm{d}\tau$$

is I^β_{T-}, the anti-causal fractional integral of the same order, defined by

$$v(\tau) := I^\beta_{T-} z(\tau) = \frac{1}{\Gamma(\beta)} \int_\tau^T (t-\tau)^{\beta-1}\, z(t)\,\mathrm{d}t\,.$$

In order to avoid to tackle such hereditary operators, we make use of the equivalent diffusive realizations introduced above, both for the direct and the adjoint systems. Indeed, the *forward* dynamical system defined by

$$\partial_t \varphi(\xi,t) = -\xi\,\varphi(\xi,t) + u(t), \quad \text{with} \quad \varphi(\xi,0) = 0 \quad (7)$$

as *initial* condition, and ouput $y(t) = \int_0^\infty \mu_\beta(\xi)\,\varphi(\xi,t)\,\mathrm{d}\xi$ provides a realization of $y = I^\beta_{0+} u$.

Whereas the *backward* dynamical system defined by

$$\partial_t \psi(\xi,\tau) = +\xi\,\psi(\xi,\tau) - z(\tau), \quad \text{with} \quad \psi(\xi,T) = 0 \quad (8)$$

as *final* condition, and ouput $v(\tau) = \int_0^\infty \mu_\beta(\xi)\,\psi(\xi,\tau)\,\mathrm{d}\xi$ is a realization of $v = I^\beta_{T-} z$, (we refer to [9] and [10] for a first introduction of anti-causal diffusive representations).

Then, the fundamental equality holds:

$$(I^\beta_{0+} u, z)_{L^2(0,T)} = (u, I^\beta_{T-} z)_{L^2(0,T)}\,. \quad (9)$$

It comes easily from two properties:

1) for a.e. $\xi > 0$, $(\varphi(\xi,.), z)_{L^2(0,T)} = (u, \psi(\xi,.))_{L^2(0,T)}$, which proves straightforward for first order ODEs like (7)-(8);
2) linearity, provided a *well-posedness* condition is fulfilled by measure μ_β (see e.g. [14]).

B. Adjoints of Fractional Derivatives

An extension of these results to fractional derivatives can be done, but care must be taken that they are no more bounded (even compact in fact); the unboundedness of the fractional derivative operators gives rise to a specific diffusive formulation, see again [6, ch. 2] for the questions of domains. The key ingredients are:

$$\widetilde{y} = D^\alpha_{0+} u = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u - \xi\,\widetilde{\varphi}]\,\mathrm{d}\xi\,, \quad (10)$$

$$\widetilde{v} = D^\alpha_{T-} z = \int_0^\infty \mu_{1-\alpha}(\xi)\left[z - \xi\,\widetilde{\psi}\right]\mathrm{d}\xi\,. \quad (11)$$

And the fundamental equality reads:

$$(D^\alpha_{0+} u, z)_{L^2(0,T)} = (u, D^\alpha_{T-} z)_{L^2(0,T)}\,, \quad (12)$$

which now comes from the definition of the domains, for a.e. $\xi > 0$, $(u - \xi\,\widetilde{\varphi}(\xi,.), z)_{L^2(0,T)} = (u, z - \xi\,\widetilde{\psi}(\xi,.))_{L^2(0,T)}$ and linearity, provided the well-posedness holds on $\mu_{1-\alpha}$.

IV. MODELS UNDER STUDY

The objective is to minimize the energy functional $J(u_e) = \frac{1}{2}\int_0^T \|X(t)\|^2_H + u_e(t)^2\,\mathrm{d}t$ with an external input u_e on the following controlled dynamical systems: an oscillator damped by memory variables, a fractionnaly damped oscillator (both these two can be seen as toy-models), and the Webster-Lokshin wave equation.

A. An Oscillator Damped by Memory Variables

The finite-dimensional model of an oscillator damped by two types of memory variables:

$$\ddot{x} + \tilde{y} + \dot{x} + y + \omega^2\,x = u_e\,,$$

with three types of damping:

- $\dot{x} = u$, instantaneous w.r.t u;
- $y(u)$, with memory and low-pass behaviour: measure μ consists of finitely many (K) Dirac measures located at some ξ_k with positive weights μ_k;
- $\tilde{y}(u)$ with memory and high-pass behaviour: measure ν consists of finitely many (L) Dirac measures located at some ξ_l with positive weights ν_l.

B. A Fractionally Damped Osciallator

Let K and L go to infinity, in a way that is consistent with the fractional integral of order β $y = I^\beta_{0+} u$ and the fractional derivative of order α, $\tilde{y} = D^\alpha_{0+} u$, respectively; this is a toy model, a fractionally damped oscillator, studied in [13], and for which elementary propreties and numerical simulations have been presented in e.g. [2].

$$\ddot{x} + D^\alpha_{0+}[\dot{x}] + \dot{x} + I^\beta_{0+}[\dot{x}] + \omega^2\,x = u_e\,,$$

The above framework is well suited to the formulation of the optimal control problem of this system in a classical setting, with no more fractional operators and no more heredity.

C. Webster-Lokshin Wave equation

Now, the fully infinite-dimensional model of interest, is the Webster-Lokshin wave equation with a somewhat idealized boundary control operator:

$$\partial_t^2 w + \varepsilon(x) D_{0+}^{1/2}[\partial_t w] + d(x)\,\partial_t w + \eta(x)\, I_{0+}^{1/2}[\partial_t w] - \partial_x^2 w = 0\,,$$

for $0 < x < L$ and $t > 0$, with boundary control at $x = 0$, and initial conditions. Provided $\varepsilon(x) > 0$, $d(x) \geq 0$ and $\eta(x) > 0$, existence and uniqueness of solutions of this system can be proved, once the diffusive reformulation has been used, see e.g. [6, ch. 2] and [5]. The optimal control problem, formulated in the new framework presented above, will become tractable with the theory of optimal control problems for linear PDEs, because the system is now no more than the coupling of a 1D wave equation with two 1D diffusion equations.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

A general framework has been presented to reformulate optimal control problems for fractional differential systems into optimal control problems for PDE systems, thanks to a diagonal realization of fractional integrals and derivatives, known as diffusive representations.

An interesting family of examples of increasing complexity has been detailed, which proves useful in continuum mechanics and acoustics.

During the conference, numerical simulations will be provided to illustrate the practical feasability of this approach, and more precise theoretical results at hand will be given.

B. Future Works

First the examples above will have to be fully worked out.

Then, numerical simulations will be studied, using different methods which do rely on the numerical implementation of diffusive representations, such as [6, chap. 3], [5], [18], or more recently [7]; some do not, like [19].

Note that the infinite-time optimal control problem, i.e. $T \to \infty$, could prove difficult, technically speaking, because the type of asymptotic stability for fractional systems is never exponential, but algebraic; hence, admissibility conditions could be more severe in the long range $T \to \infty$.

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