Detection and estimation of abrupt changes contaminated by multiplicative Gaussian noise

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Abstract

The problem of abrupt change detection has received much attention in the literature. The Neyman–Pearson detector can be derived and yields the well-known CUSUM algorithm, when the abrupt change is contaminated by an additive noise. However, a multiplicative noise has been observed in many signal processing applications. These applications include radar, sonar, communication and image processing. This paper addresses the problem of abrupt change detection in presence of multiplicative noise. The optimal Neyman–Pearson detector is studied when the abrupt change and noise parameters are known. The parameters are unknown in most practical applications and have to be estimated. The maximum likelihood estimator is then derived for these parameters. The Neyman–Pearson detector combined with the maximum likelihood estimator yields the generalized likelihood ratio detector.

Zusammenfassung

Das Problem der Detektion einer abrupten änderung hat in der Literatur grosse Beachtung gefunden. Der Neyman–Pearson Detektor kann hergeleitet werden und führt, wenn die abrupte änderung mit additivem Rauschen überlagert ist, zu dem allgemein bekannten CUSUM-Algorithmus. In vielen Anwendungsgebieten, wie z.B. Radar, Sonar, Kommunikationstechnik und Bildverarbeitung, wird jedoch ein multiplikativ überlagertes Rauschen beobachtet. Dieser Beitrag behandelt das Problem der Detektion von abrupten änderungen bei der Anwesenheit von multiplikativem Rauschen. Der optimale Neyman–Pearson Detektor wird untersucht, wenn die Parameter der abrupten änderung und des Rauschens bekannt sind. Diese Parameter sind in den meisten praktischen Anwendungen unbekannt und müssen geschätzt werden. Anschliessend wird für diese Parameter der Maximum Likelihood Schätzer hergeleitet. Der Neyman–Pearson Detektor kombiniert mit dem Maximum Likelihood Schätzer führt zu dem verallgemeinerten Likelihood-Verhältnis-Detektor.

Résumé

La détection de ruptures est un problème qui a été considéré avec beaucoup d'intérêt dans la littérature. Lorsque la rupture est noyée dans un bruit additif, le test de Neyman–Pearson donne naissance à l'algorithme CUSUM. Cependant, la présence d'un bruit multiplicatif a été observée dans de nombreuses applications comme le radar, le sonar, les

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télécommunications ou le traitement d'images. Cet article étudie le problème de la détection de ruptures noyées dans un bruit multiplicatif. Lorsque les paramètres de la rupture et du bruit sont connus, le test optimal de Neyman–Pearson est étudié. Dans la plupart des applications, les paramètres sont inconnus et doivent être estimés. L'estimateur du maximum de vraisemblance pour ces paramètres est alors étudié. L'utilisation conjointe de l'estimateur du maximum de vraisemblance et du test de Neyman–Pearson permet d'étudier le test du rapport de vraisemblance généralisé.

Keywords: Abrupt changes; Multiplicative noise; Neyman–Pearson detector; Maximum likelihood detector

1. Introduction

There is an increasing interest in multiplicative noise models for many signal processing applications. These applications include image processing (speckle) [3,19], radar systems [2] and random communication models (fading channels) ([14], Chapter 7), [20]. This paper addresses the problems of detection and estimation of abrupt changes contaminated by multiplicative Gaussian noise. These problems arise in many practical applications such as segmentation or fault detection [1,17]. The example of edge detection in synthetic aperture radar (SAR) images is detailed. However, the studied problem is of great interest in many other applications.

In previous studies [5,11,15], the SPECtral ANalysis (SPECAN) algorithm (combined with parametric spectral estimation) was shown to offer spatial resolution improvement and speckle noise reduction. The SPECAN performs a line-by-line processing of the SAR image. Following the conventional SAR processor (i.e. matched filtering + SPECAN), the received signal can be modelled by

$$x(n) = b(n)s(n) = (m + y(n))s(n), \quad n \in \{1, \ldots, N\}, \quad (1)$$

$b(n) = m + y(n)$ is the multiplicative speckle noise (with mean m) and $s(n)$ is a line of the SAR image. The statistical properties of $y(n)$ and $s(n)$ can be defined as follows, when intensity images are considered.

The *speckle noise* is usually modelled as a stationary exponentially distributed process. However, in many applications including SAR image processing, the speckle is reduced by incoherently averaging N_i uncorrelated images for large values of N_i [3]. The resulting reduced-speckle image intensities are approximately Gaussian distributed (using the central limit theorem). Here, the multiplicative noise $y(n)$ is assumed to be a zero-mean Gaussian stationary AR(p) process with parameters σ^2 and $a = (a_1, \ldots, a_p)^T$. The multiplicative noise $y(n)$ can be modelled by an AR process for the following reasons:

- for any real-valued stationary process $y(n)$ with continuous spectral density $S(f)$, it is possible to find an AR process whose spectral density is arbitrarily close to $S(f)$ ([4], p. 130).
- zero-mean Gaussian processes are completely defined by their spectra.

An *ideal abrupt change* in an image line can be modelled as a step of amplitude A located at time t_0[1,3]:

$$\begin{aligned} s_{\text{ideal}}(n) &= 1 + A, \quad n > n_0, \\ s_{\text{ideal}}(n) &= 1, \quad n \leqslant n_0, \end{aligned} \quad (2)$$

where T is the sampling period and n_0 denotes the sample point after which there is a sudden change in the signal. The location of the actual change is $t_0 = n_0 T + \tau$, with $0 < \tau < T$. Eq. (2) can represent two fields with different reflectivities in piecewise constant backgrounds. To ensure some regularity conditions (differentiability of the likelihood function with respect to t_0), the abrupt change is modelled by an amplitude A sigmoidal function at time t_0 [12]:

$$s_\alpha(n) = \frac{1 + A + e^{-\alpha(nT - n_0 T - \tau)}}{1 + e^{-\alpha(nT - n_0 T - \tau)}}, \quad n \in \{1, \ldots, N\}. \quad (3)$$

Parameter α determines how fast or slow the noise-free signal $s_\alpha(n)$ changes its amplitude from 1 to $1 + A$ about t_0. Note that the sigmoidal function approaches the ideal abrupt change when parameter α approaches infinity:

$$\lim_{\alpha \to +\infty} s_\alpha(n) = 1 + A, \quad n > n_0,$$
$$\lim_{\alpha \to +\infty} s_\alpha(n) = 1, \quad n \leqslant n_0. \tag{4}$$

Abrupt change detection and estimation, in the presence of *an additive noise*, has been studied for long time (see [1] and references therein for an overview). The new contribution here is the development of several detection algorithms for *signals multiplied by the noise*. The study is restricted to off-line change detection algorithms [1]. However, most algorithms could be modified for on-line change detection problems. Note that the variance is constant (the same before and after the change) when the noise is additive. Contrary to additive noise, the multiplicative noise causes both mean and variance jumps.

The paper is organized as follows. Section 2 studies the optimal Neyman–Pearson Detector (NPD) for the detection of abrupt changes contaminated by multiplicative Gaussian noise. The NPD is optimal in the sense that it minimizes the probability of non-detection (PND) for a fixed probability of false-alarm (PFA). However, the NPD requires a priori knowledge of the noise and abrupt change parameters. These parameters are unknown in practical applications and have to be estimated. Section 3 studies the maximum likelihood estimator (MLE) for these parameters. The MLE combined with the NPD yields the generalized likelihood ratio detector (GLRD), which is studied in Section 4. The conclusions are reported in Section 5.

2. Neyman–Pearson detector (NPD)

This section studies the NPD [20] for the detection of deterministic abrupt changes contaminated by multiplicative Gaussian noise.

2.1. White Gaussian noise

White Gaussian noise for $y(n)$ is a very interesting case because the NPD is especially simple. *Under hypothesis* H_0, the signal is a stationary independent and identically distributed (i.i.d.) Gaussian sequence $y(n)$ with variance σ^2 plus a constant mean m:

$$x(n) = y(n) + m. \tag{5}$$

Under hypothesis H_1, the sequence $y(n) + m$ is multiplied by a sigmoidal function with amplitude A at time t_0:

$$x(n) = (y(n) + m)s_\alpha(n). \tag{6}$$

The Neyman–Pearson test for this problem[1] is defined by [20]

$$\mathrm{H}_0 \text{ rejected if } \frac{L(X|\mathrm{H}_1)}{L(X|\mathrm{H}_0)} > k_{\mathrm{PFA}}. \tag{7}$$

In Eq. (7), $L(X|H_i)$ is a Gaussian likelihood function for the vector $X = [x(1), \ldots, x(N)]^{\mathrm{T}}$ under hypothesis H_i and k_{PFA} is a threshold depending on the fixed PFA. The log-likelihood ratio test (7) can then be written as

$$\mathrm{H}_0 \text{ rejected if } Z > S_{\mathrm{PFA}}, \tag{8}$$

with

$$Z = \frac{1}{\sigma^2} \sum_{i=1}^{N} \left(1 - \frac{1}{s_\alpha^2(i)}\right) \left[x(i) - m \frac{s_\alpha(i)}{1 + s_\alpha(i)}\right]^2. \tag{9}$$

In Eq. (8), S_{PFA} is a threshold depending on the PFA (which will be denoted S for brevity). Z can be expressed as the sum of N i.i.d. variables:

$$Z = \sum_{i=1}^{N} d_{ij}(w(i) + M_{ij})^2 \text{ under } \mathrm{H}_j, \tag{10}$$

with

$$M_{i0} = \frac{m}{\sigma} \frac{1}{1 + s_\alpha(i)}, \qquad d_{i0} = \frac{s_\alpha^2(i) - 1}{s_\alpha^2(i)},$$
$$M_{i1} = \frac{m}{\sigma} \frac{s_\alpha(i)}{1 + s_\alpha(i)}, \qquad d_{i1} = s_\alpha^2(i) - 1, \tag{11}$$

where $w(i)$ is a zero-mean unit variance Gaussian variable. The NPD for the abrupt change detection leads to a Gaussian sufficient statistic when the (white Gaussian) noise is additive. Eq. (10) shows

[1] A similar problem was studied in [18].

that the presence of a multiplicative white Gaussian noise (which causes a variance jump) leads to a non-Gaussian sufficient statistic. Note that $d_{ij} > 0 \ \forall (i,j)$ which means that Z is a positive-definite quadratic form in variables $w(i)$. The distribution of Z can then be determined under either hypothesis: the test statistic can be expressed as mixtures of central or non-central χ^2-distributions and its cumulative distribution function can be expanded in Maclaurin or Legendre series ([8], pp. 168–173). Note that the problem is simpler than the general Gaussian problem [20] for which the sufficient statistic can be an indefinite quadratic form in Gaussian variables.

For a large value of parameter α (i.e. for an *ideal abrupt change*), Eq. (10) reduces to

$$\lim_{\alpha \to +\infty} Z = d_j \sum_{i=n_0+1}^{N} (w(i) + M_j)^2 \quad \text{under } \mathrm{H}_j, \tag{12}$$

with

$$M_0 = \frac{m}{\sigma} \frac{1}{2+A}, \qquad d_0 = \frac{A(2+A)}{(1+A)^2},$$
$$M_1 = \frac{m}{\sigma} \frac{1+A}{2+A}, \qquad d_1 = A(2+A). \tag{13}$$

Under hypothesis H_j, Eq. (12) shows that the distribution of Z/d_j is a non-central χ^2-distribution with $N - n_0$ degrees of freedom and non-centrality parameter $\lambda_j = (N - n_0)M_j^2$. The PFA and PND can then be expressed as functions of the cumulative distribution function of a non-central χ^2-distribution:

$$\mathrm{PND} = \int_{-\infty}^{S/d_1} f_1(t)\,\mathrm{d}t, \qquad \mathrm{PFA} = \int_{S/d_0}^{+\infty} f_0(t)\,\mathrm{d}t. \tag{14}$$

In these equations, $f_j(t)$ denotes the probability density function of the χ^2-distribution with $N - n_0$ degrees of freedom and non-centrality parameter $\lambda_j = (N - n_0)M_j^2$. As an example, consider $N = 2048$ samples of a Gaussian distributed random sequence with $m = 1$ and $\sigma^2 = 1$. The abrupt change occurs at time $n_0 = 1024$. The variations of PFA and PND as functions of the threshold S are plotted in Fig. 1 for two different jump amplitudes $A = 0.01$ and $A = 0.05$. This test constitutes a reference to which suboptimal detectors can be compared.

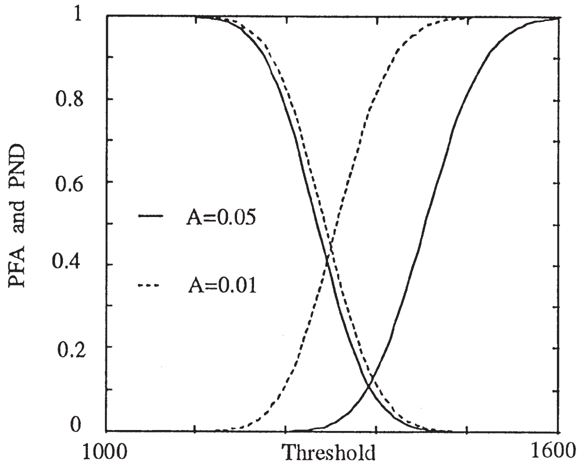

Fig. 1. PFA and PND as functions of the threshold (dashed line: $A = 0.01$, continuous line: $A = 0.05$) (white noise).

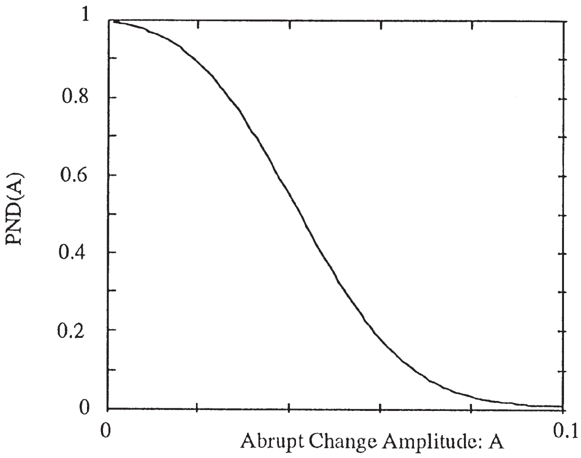

Fig. 2. PND as a function of the abrupt change amplitude for PFA = 0.01 (white noise).

The test sensitivity is then studied as a function of the jump amplitude A and the noise parameters. The PND variations as a function of A for fixed noise parameters ($m = 1$ and $\sigma^2 = 1$) and PFA (PFA = 0.01) are plotted in Fig. 2. The ROC curves, representing the variations of PD = 1-PND versus PFA, are plotted in Fig. 3, for fixed noise parameters ($m = 1$ and $\sigma^2 = 1$). Obviously, the

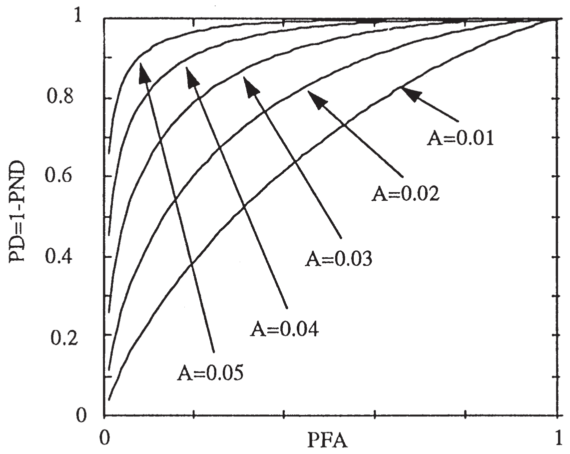

Fig. 3. ROC curves for the NPD (white noise).

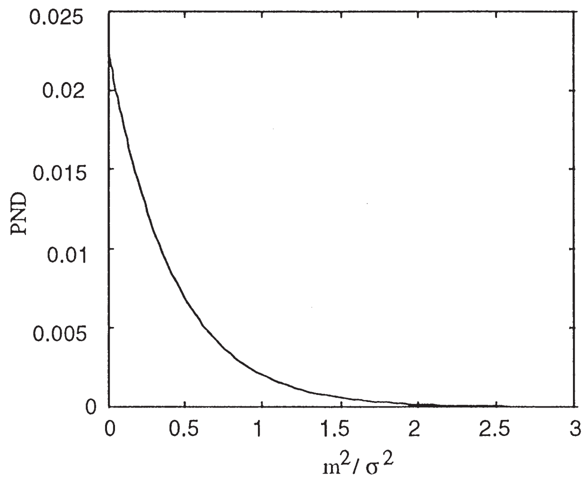

Fig. 4. PND as a function of m^2/σ^2 for $A = 0.1$ and PFA $= 0.01$ (white noise).

higher A, the better the NPD performance. Figs. 2 and 3 show that an abrupt change with amplitude $A \geqslant 0.1$ corrupted by a *multiplicative white Gaussian noise* (with $m = 1$ and $\sigma^2 = 1$) can be detected with a very good performance.

What is the noise influence on the NPD performance, for a fixed abrupt change? Eqs. (11) and (13) show that the NPD parameters are functions of the abrupt change amplitude A and the ratio m/σ. Consequently, the NPD performance is a function of m/σ, for a fixed abrupt change. The PND variations as a function of m^2/σ^2 are plotted in Fig. 4. Fig. 4 shows that an abrupt change with amplitude $A = 0.1$ corrupted by a *multiplicative white Gaussian noise* with $m^2/\sigma^2 \geqslant 1$ can be detected with a very good performance.

2.2. *AR(p) Gaussian noise*

The simple case of a multiplicative white noise is not always realistic. As it is specified in [3], it is more realistic to model the speckle by a band-limited noise process containing only lower spatial frequencies. In this case, the detection algorithm developed above cannot be used. The multiplicative colored Gaussian noise $y(n)$ is then modelled by an AR(p) process.

Under hypothesis H_0, the signal is a stationary zero-mean AR(p) Gaussian process $y(n)$ with parameters σ^2 and $a = [a_1, \ldots, a_p]^T$ plus a constant mean m:

$$x(n) = y(n) + m. \tag{15}$$

Under hypothesis H_1, the process $y(n) + m$ is multiplied by an amplitude A sigmoidal function at time t_0:

$$x(n) = (y(n) + m)s_\alpha(n). \tag{16}$$

Denote $\ln L(X|H_i)$ the Gaussian log-likelihood function for the vector $X = [x(1), \ldots, x(N)]^T$ (with mean M_i and covariance matrix Σ_i) under hypothesis H_i:

$$\ln L(X|H_i) = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\ln|\Sigma_i| - \frac{1}{2}(X - M_i)^T \Sigma_i^{-1}(X - M_i), \tag{17}$$

with $M_0 = m(1, \ldots, 1)^T$ and $M_1 = mS_\alpha = m[s_\alpha(1), \ldots, s_\alpha(N)]^T$. The inverse covariance matrix of an AR(p) process can be expressed as a function of the model parameters σ^2 and $a = [a_1, \ldots, a_p]^T$ with the Gohberg–Semencul formula (see, for instance, ([16], p. 123)):

$$\Sigma_0^{-1} = \frac{1}{\sigma^2}(FF^T - GG^T), \tag{18}$$

where $F = (f_{ij})$ and $G = (g_{ij})$ are the $N \times N$ lower triangular matrices defined by [10]

$$f_{ij} = \begin{cases} 1, & i = j, \\ a_{i-j}, & i > j, \\ 0, & i < j, \end{cases} \quad \text{and} \quad g_{ij} = \begin{cases} a_{p-i+j}, & i > j, \\ 0, & i \leqslant j. \end{cases} \tag{19}$$

Under hypothesis H_1, the inverse covariance matrix of the vector X can be expressed as

$$\Sigma_1^{-1} = \frac{1}{\sigma^2} D(FF^T - GG^T)D, \tag{20}$$

D being a diagonal matrix defined by $D = \mathrm{diag}(1/s_\alpha(1), \ldots, 1/s_\alpha(N))$. The Neyman–Pearson test is then defined by

$$H_0 \text{ rejected if } \quad Q_0(X) - Q_1(X) > S. \tag{21}$$

In Eq. (21), $Q_0(X)$ and $Q_1(X)$ are the two positive-definite quadratic forms:

$$Q_0(X) = (X - M_0)^T \Sigma_0^{-1}(X - M_0), \tag{22}$$

$$Q_1(X) = (X./S_\alpha - M_0)^T \Sigma_0^{-1}(X./S_\alpha - M_0), \tag{23}$$

where (./) denotes the element-by-element division. In the general case, the quadratic form $Q(X) = Q_0(X) - Q_1(X)$ is indefinite. Relatively little attention has been devoted to the problem of obtaining the distribution of indefinite quadratic forms of Gaussian vectors. $Q(X)$ can be represented as a mixture of differences between pairs of independent central χ^2-distributions [7], ([8], p. 174). Unfortunately, these expansions are difficult to study. Instead, the distribution of $Q(X)$ can be approximated leading to a simple test statistic. For example, Fig. 5(a) and (b) show a comparison between the histograms of $Q(X)$ and the fitted Gaussian pdf's (under both hypotheses) with 95% confidence intervals (computed as in [13], p. 251). These figures show that the Gaussian probability density function (PDF) is a good approximation for the PDF of $Q(X)$ under hypotheses H_0 and H_1. The approximate ROC curves (derived from the fitted Gaussian PDF for $Q(X)$) are shown in Fig. 6 as a function of the jump amplitude A for a low-pass

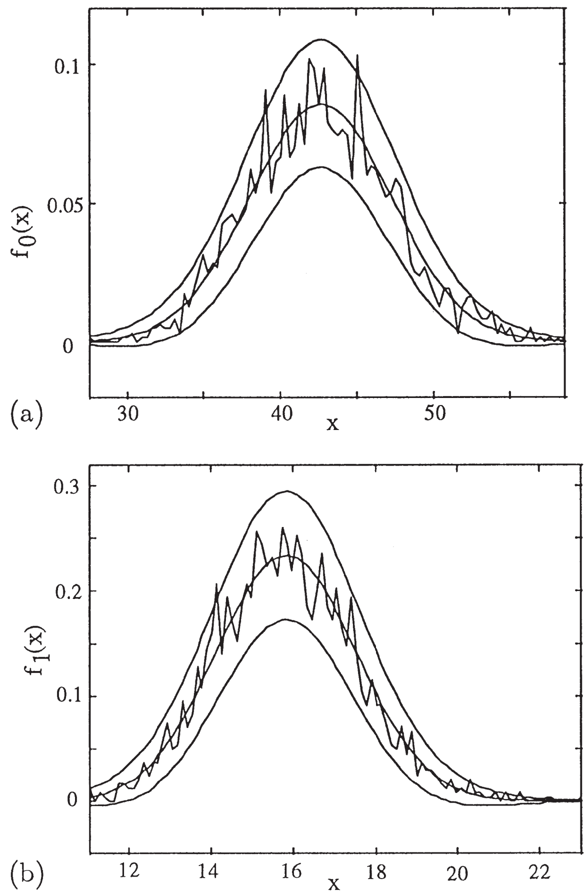

Fig. 5. Histograms and PDF of $Q(X)$ with 95% confidence intervals: (a) under H_0, (b) under H_1.

spectrum ($a = [-0.2, 0.153]^T$). The NPD performance for the detection of abrupt changes contaminated by multiplicative *colored* noise is very similar to the performance obtained with a multiplicative *white* Gaussian noise. Consequently, as previously, an abrupt change with amplitude $A \geqslant 0.1$ corrupted by a *multiplicative colored Gaussian noise* with $m^2/\sigma^2 \geqslant 1$ can be detected with a very good performance.

The optimal NPD gives a reference to which suboptimal detectors can be compared. However, the NPD requires a priori knowledge of the noise and abrupt change parameters. These parameters are unknown in practical applications and have to

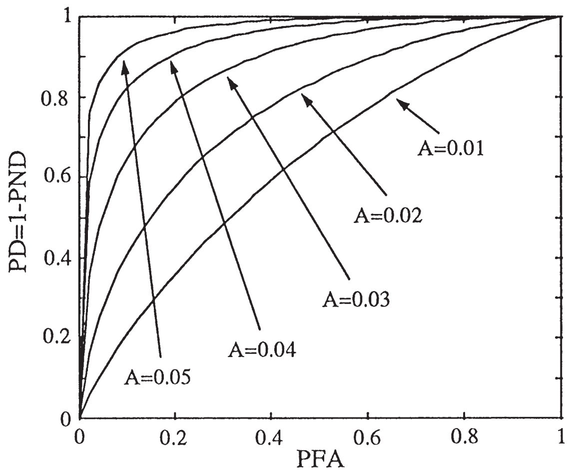

Fig. 6. ROC curves for the NPD (colored noise).

be estimated. The next part of the paper derives the Maximum likelihood estimator (MLE) for these parameters.

3. Maximum likelihood estimator (MLE)

The maximum likelihood principle [20] provides a method to estimate a parameter vector θ from a finite length data record $X = [x(1), \ldots, x(N)]^T$. Under hypothesis H_0, X is a Gaussian white or AR(p) process plus a constant mean whose parameters can be estimated with the conventional autocorrelation or covariance methods [9]. This section focuses on estimating the noise and sigmoid parameters under hypothesis H_1. Note that parameter α (which characterizes the abrupt change shape) is assumed to be known. All simulations have been performed with $\alpha = 10$, which corresponds to a very sharp change (very close to the ideal abrupt change).

3.1. White Gaussian noise

The noise and sigmoid parameters are (m,σ^2) and (A,t_0) such that $\theta = (m,\sigma^2,A,t_0)^T$. The MLE of θ denoted $\hat{\theta}_{ML}$ maximizes the Gaussian log-likelihood function:

$$\ln L(X;\theta) = -\frac{N}{2}\ln(2\pi\sigma^2) - \sum_{i=1}^{N} \ln s_\alpha(i) - \frac{1}{2\sigma^2}\left\{\sum_{i=1}^{N}\left(\frac{x(i)}{s_\alpha(i)} - m\right)^2\right\} \tag{24}$$

over a subset Θ of $\mathbb{R}^3 \times]T,NT[$. When $(A,t_0)^T$ is known, the MLE of $(m,\sigma^2)^T$ is obtained by setting the partial derivatives of $\ln L(X;\theta)$ with respect to m and σ^2 to zero:

$$\hat{m}_{ML} = \frac{1}{N}\sum_{i=1}^{N}\frac{x(i)}{s_\alpha(i)}, \tag{25}$$

$$\hat{\sigma}^2_{ML} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{x(i)}{s_\alpha(i)} - \hat{m}_{ML}\right)^2. \tag{26}$$

These estimators are the conventional mean and variance estimators for the observation vector $(x(1)/s_\alpha(1), \ldots, x(N)/s_\alpha(N))^T$. When $(A,t_0)^T$ is unknown, the expression of $\hat{m}_{ML}$ and $\hat{\sigma}^2_{ML}$ are substituted in Eq. (24). After dropping the constant terms, the following criterion has to be maximized with respect to A and t_0:

$$J_1(X;A,t_0) = -\sum_{i=1}^{N}\ln s_\alpha(i) - \frac{N}{2}\ln\left\{\sum_{i=1}^{N}\left(\frac{x(i)}{s_\alpha(i)} - f(X;A,t_0)\right)^2\right\} \tag{27}$$

with

$$f(X;A,t_0) = \frac{1}{N}\sum_{i=1}^{N}\frac{x(i)}{s_\alpha(i)}. \tag{28}$$

Setting the partial derivatives of $J_1(X;A,t_0)$ with respect to A and t_0 to zero generally yields nonlinear equations in A and t_0. These equations do not yield closed-form expressions for the estimates. A numerical approach then has to be used. However, for large values of parameter α (i.e. for an *ideal abrupt change*), Eq. (27) reduces to

$$J_1^{ideal}(X;A,n_0) = -(N-n_0)\ln(1+A) - \frac{N}{2}\ln\left\{\sum_{i=1}^{N}\left(\frac{x(i)}{s_{ideal}(i)} - f_{ideal}(X;A,n_0)\right)^2\right\} \tag{29}$$

with

$$f_{\text{ideal}}(X;A,n_0) = \frac{1}{N}\left[\sum_{i=1}^{n_0} x(i) + \frac{1}{1+A}\sum_{i=n_0+1}^{N} x(i)\right], \tag{30}$$

$$s_{\text{ideal}}(i) = \begin{cases} 1, & i \leqslant n_0, \\ 1+A, & i > n_0. \end{cases} \tag{31}$$

$[\partial J_1^{\text{ideal}}(X;A,n_0)]/\partial A = 0$ leads to a quadratic equation in A. This equation gives a closed-form expression for A, denoted as $g(X;n_0)$, as a function of the jump time n_0 and the observation vector X. The MLEs for A and t_0 are then

$$\begin{aligned} \hat{t}_{0\text{ML}} &= \arg \max_{k\in\{1,\ldots,N\}} J_3(X;k), \\ \hat{A}_{\text{ML}} &= g(X;\hat{t}_{0\text{ML}}), \end{aligned} \tag{32}$$

with $J_3(X;k) = J_1(X;g(X;k),k)$. The MLEs for m and σ^2 can be deduced from Eqs. (25) and (26) by replacing $s_\alpha(i)$ by

$$\hat{s}_\alpha(i) = \begin{cases} 1, & i \leqslant \hat{n}_{0\text{ML}}, \\ 1+\hat{A}_{\text{ML}}, & i > \hat{n}_{0\text{ML}}. \end{cases} \tag{33}$$

The following remarks apply for an ideal abrupt change:

- The maximization of $L(X;\theta)$ over the whole parameter vector θ is equivalent to the maximization of $J_3(X;n_0)$ with respect to n_0 only [6].
- The maximization of $J_3(X;n_0)$ with respect to n_0 is discrete and is very simple to implement. However, the usual MLE properties are not guaranteed, since the regularity conditions on $J_3(X;n_0)$ with respect to n_0 are not satisfied.

The mean and standard deviation of the abrupt change parameter MLEs, computed with 500 Monte Carlo runs, are shown in Fig. 7(a) and (b) for different numbers of samples N. The true parameters are $m = 1$, $\sigma^2 = 1$, $A = 0.5$ and $n_0 = N/2$. The comparison between the true parameters and the estimates show the ML algorithm efficiency.

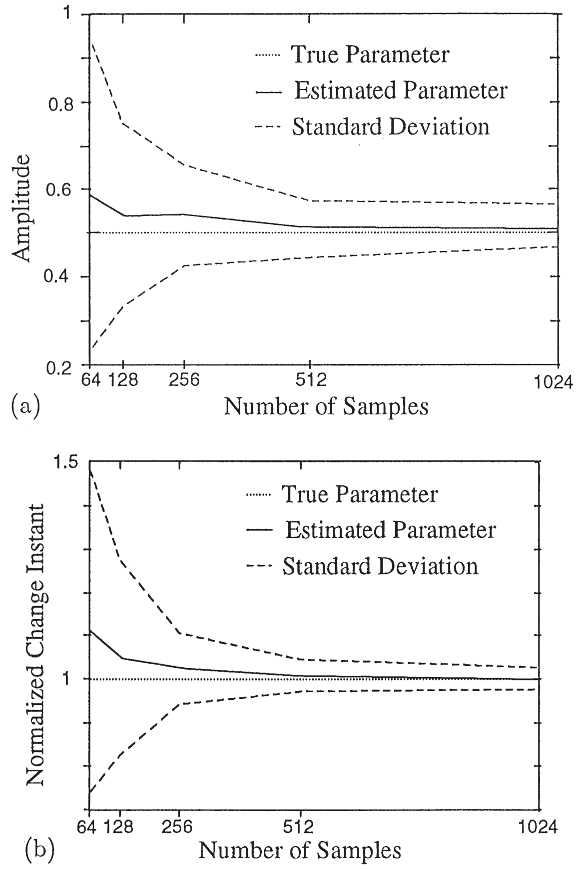

Fig. 7. Mean and standard deviation of estimated parameters for different numbers of samples N (white noise): (a) $\hat{A}_{\text{ML}}$, (b) $\hat{n}_{0\text{ML}}/n_0$.

3.2. AR(p) Gaussian noise

The noise and sigmoid parameters are (m,σ^2,a) and (A,t_0) such that $\theta = (m,\sigma^2,a^{\text{T}},A,t_0)^{\text{T}}$ with $a = [a(1),\ldots,a(p)]^{\text{T}}$. The exact maximization of the Gaussian likelihood function (17) produces a set of highly non-linear equations even in the pure AR case ($A = 0$) [10]. However, the likelihood function maximization can be approximated by maximizing the conditional likelihood function $L(x(p+1),\ldots,x(N)\,|\,x(1),\ldots,x(p);\theta)$ for large data records ([9], p. 186). The driving AR(p) process $u(n)$ is assumed an i.i.d. sequence with zero mean and variance σ^2. The Jacobian matrix determinant of the transformation from $\tilde{U} = [u(p+1),\ldots,u(N)]^{\text{T}}$ to $\tilde{X} = [x(p+1),\ldots,x(N)]^{\text{T}}$ is

$$\det(J) = \left[\prod_{i=p+1}^{N} s_\alpha(i)\right]^{-1}.$$

Consequently, the PDF for $\tilde{X}$ conditioned on the p first values $x(1),\ldots,x(p)$ can be determined:

$$L(\tilde{X}\,|\,x(1),\ldots,x(p);\theta) = \frac{\det(J)}{(2\pi\sigma^2)^{(N-p)/2}}\exp\left(-\frac{1}{2\sigma^2}f(x;m,\sigma^2,a,S_\alpha)\right), \quad (34)$$

with

$$f(x;m,\sigma^2,a,S_\alpha) = \sum_{i=p+1}^{N}\left[\frac{x(i)}{s_\alpha(i)} + \sum_{k=1}^{p} a(k)\frac{x(i-k)}{s_\alpha(i-k)} - m\left(1+\sum_{k=1}^{p} a(k)\right)\right]^2. \quad (35)$$

Setting the partial derivatives of $\ln L(\tilde{X}\,|\,x(1),\ldots,x(p);\theta)$ with respect to m and σ^2 to zero yields

$$\hat{m}_{\mathrm{ML}} = \frac{1}{(N-p)(1+\sum_{k=1}^{p}a(k))} \times \sum_{i=p+1}^{N}\left(\frac{x(i)}{s_\alpha(i)} + \sum_{k=1}^{p} a(k)\frac{x(i-k)}{s_\alpha(i-k)}\right), \quad (36)$$

$$\hat{\sigma}^2_{\mathrm{ML}} = \frac{1}{N-p}\sum_{i=p+1}^{N}\left(\frac{x(i)}{s_\alpha(i)} + \sum_{k=1}^{p} a(k)\frac{x(i-k)}{s_\alpha(i-k)} - \hat{m}_{\mathrm{ML}}\left[1+\sum_{k=1}^{p} a(k)\right]\right)^2. \quad (37)$$

Note that Eqs. (36) and (37) reduce to Eqs. (25) and (26) when $a=0$. Replacing m and σ^2 in $\ln L(\tilde{X}\,|\,x(1),\ldots,x(p);\theta)$ by their estimates from Eqs. (36) and (37), the criterion J_1^{AR} has to be maximized with respect to a, A and t_0 with

$$J_1^{\mathrm{AR}}(\tilde{X};a,A,t_0) = -\sum_{i=p+1}^{N}\ln s_\alpha(i) - \frac{N-p}{2}\ln Q(a),$$

$$Q(a) = \sum_{i=p+1}^{N}\left(v(i,0) + \sum_{k=1}^{p} a(k)v(i,k)\right)^2,$$

$$v(i,k) = \frac{x(i-k)}{s_\alpha(i-k)} - \frac{1}{N-p}\sum_{j=p+1}^{N}\frac{x(j-k)}{s_\alpha(j-k)}. \quad (38)$$

J_1^{AR} is maximized over a by minimizing $Q(a)$. Note that $Q(a)$ is a quadratic form in a. As a result, its differentiation yields a global minimum (which may not be unique) defined by the matrix equation $W\hat{a}_{\mathrm{ML}} = -w$ with

$$W = \begin{bmatrix} w(1,1) & w(1,2) & \cdots & w(1,p) \\ w(2,1) & w(2,2) & \cdots & w(2,p) \\ \vdots & \vdots & \ddots & \vdots \\ w(p,1) & w(p,2) & \cdots & w(p,p) \end{bmatrix},$$

$$w = \begin{bmatrix} w(1,0) \\ w(2,0) \\ \vdots \\ w(p,0) \end{bmatrix} \quad (39)$$

and

$$w(j,k) = \frac{1}{N-p}\sum_{i=p+1}^{N} v(i,j)v(i,k). \quad (40)$$

When $(A,t_0)^{\mathrm{T}}$ is unknown, $\hat{a}_{\mathrm{ML}}$ is substituted in J_1^{AR}. The maximization of $L(\tilde{X}\,|\,x(1),\ldots,x(p);\theta)$ over the whole parameter vector θ is equivalent to the maximization of $J_2^{\mathrm{AR}}(\tilde{X};A,t_0) = J_1^{\mathrm{AR}}(\tilde{X};\hat{a}_{MV},A,t_0)$ with respect to $(A,t_0)^{\mathrm{T}}$ only [6]. The MLE for the parameter vector $(A,t_0)^{\mathrm{T}}$ is

$$\hat{t}_0 = \arg\max_{k\in\{1m\ldots,N\}}\left\{\sup_A J_2^{\mathrm{AR}}(\tilde{X};A,k)\right\}, \quad (41)$$

$$\hat{A} = \arg\sup_A J_2^{\mathrm{AR}}(\tilde{X};A,\hat{t}_0). \quad (42)$$

This case is significantly more complicated than the white Gaussian multiplicative noise case. The differentiation of $J_2^{\mathrm{AR}}(\tilde{X};A,t_0)$ with respect to A yields a set of non-linear equations which cannot be easily solved. An analytical closed-form expression of $\hat{A}$ cannot be found even in the case of an *ideal abrupt change*. A numerical method then has to be used for the estimation of $\sup_A J_2^{\mathrm{AR}}(\tilde{X};A,k)$. This paper proposes to use the conventional iterative quasi-Newton BFGS algorithm (available in the Matlab optimization toolbox). Partial derivatives are computed using a numerical differentiation method via finite differences (although they can be analytically derived with high computational cost). In general, the cost function $J_2^{\mathrm{AR}}(\tilde{X};A,k)$

has several local maxima. Thus, the optimization procedure has to be initialized sufficiently close to the global maximum.

3.2.1. Initialization of $\hat{t}_0$

There is a simultaneous mean value and variance jump after the abrupt change instant when the multiplicative noise is non-zero mean ($m \neq 0$). The off-line estimation procedure described in ([1], p. 66) for mean changes then can be used for the initialization of $\hat{t}_0$:

$$\hat{t}_0 = \arg \max_{1 \leqslant k \leqslant N} \{ -(k-1)(N-k+1) \times [m_0(k)^2 - m_1(k)^2]\}. \quad (43)$$

$m_0(k)$ and $m_1(k)$ are the usual mean estimates before and after k, respectively,

$$m_0(k) = \frac{1}{k-1}\sum_{i=1}^{k-1} x(i), \qquad m_1(k) = \frac{1}{N-k+1}\sum_{i=k}^{N} x(i). \quad (44)$$

A mean value jump occurs in the signal $x^2(n)$ when the multiplicative noise is zero-mean ($m = 0$). This jump can be used for the initialization of $\hat{t}_0$.

3.2.2. Initialization of $\hat{A}$

Once the abrupt change instant has been estimated, the amplitude can be estimated as

$$\hat{A} = \frac{m_1(\hat{t}_0)}{m_0(\hat{t}_0)} - 1. \quad (45)$$

The mean and standard deviation of the abrupt change parameter MLEs are shown in Fig. 8(a) and (b) for different numbers of samples N. The true parameters are $m = 1$, $\sigma^2 = 1$, $a = [-0.2, 0.153]^T$ (low-pass spectrum), $A = 0.5$ and $n_0 = N/2$. The comparison between the true parameters and the estimates shows the ML algorithm good performance.

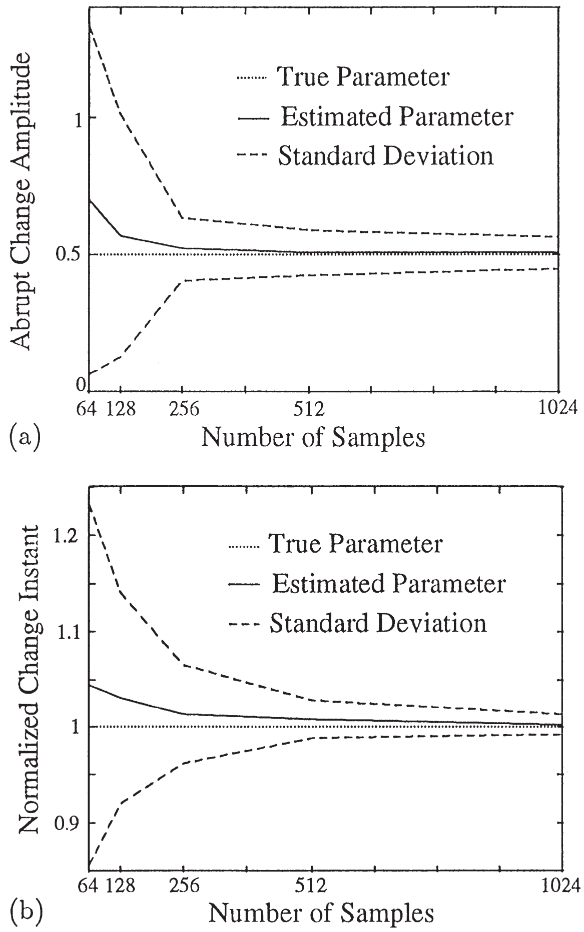

Fig. 8. Mean and standard deviation of estimated parameters for different numbers of samples N (colored noise): (a) $\hat{A}_{\mathrm{ML}}$, (b) $\hat{n}_{0\mathrm{ML}}/n_0$.

4. Generalized likelihood ratio detector (GLRD)

The generalized likelihood ratio detector (GLRD) estimates the unknown parameters under hypotheses H_0 and H_1 using the maximum likelihood procedure and uses these estimates in the Neyman–Pearson test defined in Eq. (7). The GLR test for our problem is [20]

$$H_0 \text{ rejected if } \quad \frac{L(X|\hat{\theta}_1)}{L(X|\hat{\theta}_0)} > k_{\mathrm{PFA}}, \quad (46)$$

where $\hat{\theta}_i$ denotes the MLE of θ under hypothesis H_i. According to the first section, the GLRD is defined by

$$H_0 \text{ rejected if } \quad \hat{Q}_0(X) - \hat{Q}_1(X) > S, \quad (47)$$

with

$$\hat{Q}_0(X) = (X - \hat{M}_0)^{\mathrm{T}} \hat{\Sigma}_0^{-1} (X - \hat{M}_0), \tag{48}$$

$$\hat{Q}_1(X) = (X./\hat{S}_\alpha - \hat{M}_0)^{\mathrm{T}} \hat{\Sigma}_0^{-1} (X./\hat{S}_\alpha - \hat{M}_0). \tag{49}$$

Eqs. (48) and (49) show that the test statistics can be computed from noise and abrupt change parameter estimates. The threshold S corresponding to a fixed PFA is determined as in the first section (i.e. using the non-central χ^2-distribution for the white noise case or the fitted Gaussian PDF for the colored noise case), after replacing noise and abrupt change parameter by their ML estimates. The GLRD ROC curves are depicted in Fig. 9(a) and (b) as a function of the abrupt change amplitude, for white and colored noise (same parameters than in the first section). Fig. 9(a) and (b) show that

- the GLRD performance is significantly reduced compared to NPD performance. This is due to the ML estimation procedure.
- the GLRD performance for a colored noise can be significantly reduced with respect to the GLRD performance obtained with a white noise. This is again due to the ML procedure which requires a numerical optimization algorithm when the noise is colored.
- an abrupt change with amplitude $A \geqslant 0.1$ (respectively $A \geqslant 0.4$) corrupted by a *multiplicative white* (*respectively colored*) Gaussian noise (with $m^2/\sigma^2 \geqslant 1$) can be detected with a very good performance.

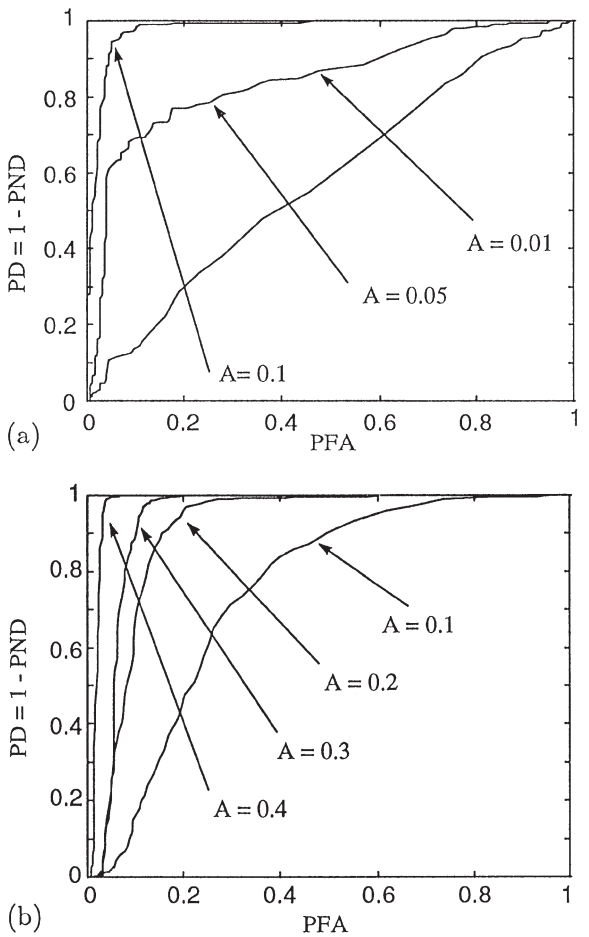

Fig. 9. ROC curves for the GLRD: (a) white noise, (b) colored noise.

5. Summary and conclusions

This paper studied the detection of abrupt changes contaminated by multiplicative Gaussian noise. The optimal Neyman–Pearson detector (NPD) was derived for white and colored noise. The NPD performance was shown to be similar in both cases. The NPD provides a reference to which suboptimal detectors can be compared. However, the NPD requires knowledge of the abrupt change and noise parameters. The abrupt change and noise parameters are unknown in practical applications and have to be estimated.

The maximum likelihood estimator (MLE) was then derived for these parameters. When the noise is white, the MLE algorithm reduced to a discrete maximization (easy to implement). When the noise is colored, a numerical algorithm was used for the maximization with respect to the abrupt change parameters.

The MLE was then combined with the NPD to yield the generalized likelihood ratio detector (GLRD). The GLRD performance was shown to be significantly reduced compared to the NPD performance, because of the estimation procedure. Moreover, the GLRD performance was shown to be different for white and colored noise: the numerical optimization algorithm, required when the noise is colored, leaded to smaller performance with respect to the white noise case.

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References

[1] M. Basseville, I.V. Nikiforov, Detection of Abrupt Changes: Theory and Application, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[2] O. Besson, P. Stoica, Sinusoidal signals with random amplitude: Least-squares estimators and their statistical analysis, IEEE Trans. Acoust. Speech Signal Process. 43 (11) (November 1995) 2733–2744.

[3] A.C. Bovik, On detecting edges in speckle imagery, IEEE Trans. Acoust. Speech Signal Process. 36 (10) (October 1988) 1618–1627.

[4] P.J. Brockwell, R.A. Davis, Time Series: Theory and Methods, 2nd ed., Springer, Berlin, 1990.

[5] J.C. Curlander, R.N. McDonough, Synthetic Aperture Radar, Wiley-Interscience, New York, 1991.

[6] M. Ghogho, Maximum likelihood estimation of AR modulated signals, Proc. EUSIPCO'96, September 1996.

[7] J.P. Imhof, Computing the distribution of quadratic forms in normal variables, Biometrika 48 (1961) 443–446.

[8] N.L. Johnson, S. Kotz, Continuous Univariate Distributions-2, 1st ed., Wiley, New York, 1970.

[9] S.M. Kay, Modern Spectral Estimation: Theory and Applications, Prentice-Hall, Englewood Cliffs, NJ, 1988.

[10] L.T. McWhorter, L.L. Scharf, Nonlinear maximum likelihood estimation of autoregressive time series, IEEE Trans. Signal Process. 43 (12) (December 1995) 2909–2919.

[11] G. Mesnager, C. Lambert-Nebout, F. Castanié, Spatial resolution improvement using a SPECAN approach in SAR data processing, Proc. IGARSS'95, July 1995.

[12] A. Moghaddamjoo, M. Doroodchi, Cramer-Rao lower bound on locations of sudden changes in a steplike signal, IEEE Trans. Signal Process. 44 (10) (October 1996) 2551–2556.

[13] A. Papoulis, Probability Random Variables and Stochastic Processes, 3rd ed., McGraw-Hill, New York, 1991.

[14] J.G. Proakis, Digital Communications, McGraw-Hill, New York, 1989.

[15] M. Sack et al., Application of efficient linear FM matching filtering algorithms to SAR processing, Proc. IEE 132 (1985).

[16] P. Stoica, R. Moses, Introduction to Spectral Analysis, Prentice Hall, Englewood Cliffs, NJ, 1997.

[17] A. Swami, Multiplicative noise models: Parameter estimation using cumulants, Signal Processing 36 (1994) 355–373.

[18] A. Swami, G.B. Giannakis, Signal detection and classification in multiplicative and additive noise, in: Proc. IEEE Workshop on Non-linear Signal and Image Processing, Halkidiki, Greece, 20–22 June 1995, pp. 839–842.

[19] M. Tur, K.C. Chin, J.W. Goodman, When is speckle noise multiplicative?, Appl. Opt. 21 (1982) 1157–1159.

[20] H.L. Van Trees, Detection, Estimation and Modulation Theory. Part 1, Wiley, New York, 1968.