

Detection of Multiplicative Noise in Stationary Random Processes Using Second- and Higher Order Statistics

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Abstract—This paper addresses the problem of detecting the presence of colored multiplicative noise, when the information process can be modeled as a parametric ARMA process. For the case of zero-mean multiplicative noise, a cumulant based suboptimal detector is studied. This detector tests the nullity of a specific cumulant slice. A second detector is developed when the multiplicative noise is nonzero mean. This detector consists of filtering the data by an estimated AR filter. Cumulants of the residual data are then shown to be well suited to the detection problem. Theoretical expressions for the asymptotic probability of detection are given. Simulation-derived finite-sample ROC curves are shown for different sets of model parameters.

Index Terms—ARMA processes, detection, higher order statistics, multiplicative noise.

I. INTRODUCTION AND PROBLEM FORMULATION

ADDITIVE noise models have been intensively studied in many signal processing applications. Indeed, these models allow us to approximate a large class of physical mechanisms contaminated by measurement noise. Consequently, detection of known signals with unknown parameters, as well as detection of random signals embedded in Gaussian or non-Gaussian additive noise, has received much attention in the literature (see, for instance, [18] and references therein for an overview). However, these detectors can fail dramatically when the signal is corrupted by nonadditive noise components. This paper addresses the problem of detecting the presence of multiplicative noise in stationary random processes. Multiplicative noise has been observed in many signal processing applications, such as communication systems (e.g., fading channels, where the amplitude of the signal is modeled as a Rayleigh or Rician random variable) [32], radar (in particular, weather radar/lidar observations—see [14]), synthetic aperture radar (SAR) [28], passive and active sonar [15], [16], and mechanical vibrations [23], [25], [33], [45], where the multiplicative noise is mainly caused by nonlinearities in the observed system.

The detection of known signals embedded in multiplicative noise has been studied recently in several papers. For instance,

locally optimum detection in multiplicative noise was studied in [19] and [35] for known signals; suboptimal approaches, based on ranks, were developed in [2] and [44], and robust detectors were derived in [6] and [7].

The detection of random signals in multiplicative noise has been considered in [3] and [36], for instance. These approaches require knowledge of the noise pdf's and typically assume that the noises are i.i.d. The resulting detectors in the known signal case can be expressed as generalized correlators; in the case of random signals, the detectors are very complicated [36] and assume that at least the autocorrelation sequence of the signal is known. In [3] and [36], the observed process is of the form

$$y(n) = \theta x(n)(1 + e(n)) + u(n)$$

and the detection problem is

$$H_0: \theta = 0, \quad H_1: \theta > 0.$$

However, the model we consider [see (1)] is somewhat different and is motivated by fault detection problems in geared systems. When the signal models are more specific, it is possible to obtain more concrete results, e.g., the problem of discriminating random amplitude (i.e., multiplicative noise) from constant amplitude harmonics was studied in [29] and [42].

The parameter estimation problem (i.e., when multiplicative noise is known to be present) has been well studied, see, e.g., [1], [5], [24], [37], [39], [43], and references therein.

The detection problem of interest is the following composite hypothesis testing problem:

$$\begin{aligned} H_0: y(n) &= y_0(n) \triangleq x(n) + u_0(n) \\ H_1: y(n) &= y_1(n) \triangleq e(n)x(n) + u_1(n) \end{aligned} \quad (1)$$

where the signal $x(n)$ is modeled as an ARMA process and the noise processes as an MA process. The problem of detecting the presence of multiplicative noise that corrupts the signal of interest (SOI)—here $x(n)$ —is of great importance.

- 1) Appropriate techniques for parameter estimation in the additive noise environment can fail dramatically (or be statistically inefficient) when the observed process is contaminated by multiplicative noise. For example, if $x(n)$ is an AR process, applying the usual Yule–Walker type equations would lead to inconsistent estimates under H_1 ; the modified equations [37] yield consistent estimates under both hypotheses but are very inefficient under H_0 . In wireless communications, the multiplicative noise $e(n)$

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represents fading and is usually modeled as a zero-mean process; the optimal receiver structure for the AWGN channel differs from that for the fading channel [32], [41]; choosing the incorrect structure would lead to increased errors in detection. Consequently, it is important to determine the correct underlying model.

- 2) The presence of multiplicative noise can be informative in some applications. For instance, in mechanics, a gear fault induces multiplicative noise in the vibration signal. In this case, the detection of the multiplicative noise allows the detection of (localized or distributed) gear faults [23], [25], [33], [34], [45]. Here, the signal of interest is modeled as sums of harmonics [25] or as an ARMA process [4], [22]. In drilling applications, nonlinearities lead to random amplitude modulation, and timely detection of the multiplicative noise is important to prevent failure of the drilling system.

This paper studies a statistical test for the detection of colored multiplicative noise when the SOI can be modeled as an ARMA process. The choice of parametric ARMA modeling for the SOI is motivated by the fact that for any continuous spectral density $S(f)$, an ARMA process can be found with a spectral density arbitrarily close to $S(f)$ [10, p. 132].

We now state our modeling assumptions for the composite hypothesis testing problem in (1). The signal of interest $x(n)$ is an ARMA(p, q) process

$$x(n) = -\sum_{j=1}^p a_j x(n-j) + \sum_{j=0}^q b_j g(n-j) \quad (2)$$

in which $g(n)$ is an i.i.d. sequence. The noise processes $u_0(n)$ and $u_1(n)$ are possibly non-Gaussian with finite memories q_0 and q_1 , respectively. In particular, we assume that the sequences $u_0(n)$ and $u_1(n)$ are (Gaussian or non-Gaussian) MA(q_0) and MA(q_1) processes, respectively, i.e.,

$$u_0(n) = \sum_{j=0}^{q_0} \theta_{0,j} v_0(n-j) \quad (3)$$

$$u_1(n) = \sum_{j=0}^{q_1} \theta_{1,j} v_1(n-j) \quad (4)$$

in which $v_0(n)$ and $v_1(n)$ are i.i.d. The simple case of the i.i.d. multiplicative noise $e(n)$ case was studied in [13]. However, colored multiplicative noise occurs in many applications. For instance, it is more realistic to model the speckle noise in image processing by a bandlimited process containing only lower spatial frequencies [8]. In this case, the algorithm developed in [13] cannot be used. We assume that the multiplicative noise $e(n)$ is an MA(q_e) process driven by an i.i.d. sequence $\varepsilon(n)$, whose distribution is unknown

$$e(n) = \sum_{j=0}^{q_e} \beta_j \varepsilon(n-j). \quad (5)$$

The sequences $g(n)$, $v_0(n)$, $v_1(n)$, and $\varepsilon(n)$ are assumed to be mutually independent. Further, the processes $x(n)$, $e(n)$, $u_0(n)$,

and $u_1(n)$ are assumed to be weakly mixing (i.e., with absolutely summable cumulants).¹

Optimal detectors based on the Neyman–Pearson criterion can be derived when statistical properties concerning the signal and noise processes are available. Such detectors were studied in [38] for detecting and classifying signals corrupted by additive and multiplicative noise, although the models were rather different. However, the Neyman–Pearson detector can lead to intractable computation and, of course, the signal and noises pdf’s must be known. This paper studies suboptimal detectors based on second- and higher-order cumulants (HOCs).

- When the ARMA process $x(n)$ and the additive noise $u_0(n)$ are Gaussian, the observed signal $y(n)$ is Gaussian under hypothesis H_0 and generally non-Gaussian under hypothesis H_1 . Any Gaussianity test such as the frequency-domain tests of Hinich or Rao–Gabr [31, p. 42] or the lag-domain tests in [11] and [26] can then be used for multiplicative noise detection; however, in the colored noise case, these generic tests may involve hundreds of thousands of samples [11], and the power (probability of detection) has been evaluated analytically only in the i.i.d. case.
- When $x(n)$, or $u_0(n)$, or both are non-Gaussian, Gaussianity tests are no longer useful for problem (1) since the observed signal $y(n)$ is generally non-Gaussian under both hypotheses. It should be noted that the process is a noisy linear process under hypothesis H_0 , whereas it is a nonlinear process under hypothesis H_1 . Consequently, linearity tests for noisy signals such as that in [40] could be used for detection. However, those tests are quite complicated since they use the bispectrum (or possibly trispectrum) and require multiple grids on which a modified bi-coherence (or tricoherence) function should be computed. Moreover, they are applicable to arbitrary linear/nonlinear processes and do not make use of the particular structure of the processes $y_0(n)$ and $y_1(n)$. Therefore, we derive new detectors that are based on the specific structure of the processes involved in the multiplicative noise detection problem. The paper focuses on suboptimal multiplicative noise detectors based on cumulants. The main advantage of these detectors is that they only require mild assumptions regarding the distributions of $x(n)$ and $e(n)$. The only *a priori* information required is the mean of the multiplicative noise $e(n)$.
- We can interpret $e(n)$ as the signal of interest and $x(n)$ (and the other processes) as noise; in this context, the problem is similar to that studied in [38]. However, the scenario of “signal” modulating only one component of an ever-present two component noise is somewhat restrictive.

We develop different detectors for the zero-mean and nonzero mean cases. The zero-mean case is studied in Section II; we show that an appropriate vector of cumulants is zero under H_1 and nonzero under H_0 . The nonzero mean case is considered in

¹The processes $x(n)$, $e(n)$, $u_0(n)$, and $u_1(n)$ are modeled as stable ARMA and MA processes; consequently, these processes are weakly mixing if the moments/cumulants of their innovation sequences are finite [9, p. 26].

Section III; we first develop an algorithm to estimate the AR parameters of the ARMA process $x(n)$ and inverse filter the data (i.e., the applied filter is MA); we show that an appropriate vector of cumulants of this residual process is zero under H_0 and nonzero under H_1 . Sample estimates of the cumulants of $y(n)$ are asymptotically Gaussian; this enables us to show that the test statistic is central F distributed under hypothesis H_0 and noncentral F distributed under hypothesis H_1 . Closed-form expressions for the ROC are given in Section IV. Simulation examples are reported in Section V, and some discussion may be found in Section VI.

II. ZERO-MEAN MULTIPLICATIVE NOISE (ZM DETECTOR)

In this section, we assume that the multiplicative noise $e(n)$ is zero mean. Let $\mu_s = E[s(n)]$ denote the mean of the stationary random process $s(n)$, $M_k^s(\underline{\rho})$ and $C_k^s(\underline{\rho})$ its k th-order moment and cumulant computed at the $(k-1)$ -D lag $\underline{\rho} = (\rho_1, \rho_2, \dots, \rho_{k-1})$. Let $\mathcal{N}_k(r)$ define the region of support of the k th-order cumulants of an MA(r) process. Let

$$\mathcal{N}_k^+(r) = \{\underline{\rho} | 0 \leq \rho_{k-1} \leq \dots \leq \rho_1 \leq r\}.$$

Then

$$\mathcal{N}_k(r) = \mathcal{N}_k^+(r) \cup \text{Symmetric extensions of } \mathcal{N}_k^+(r). \quad (6)$$

Since the sequences $u_0(n)$ and $u_1(n)$ are independent of the processes $e(n)$ and $x(n)$, it follows that

$$\begin{aligned} H_0: C_k^y(\underline{\rho}) &= C_k^x(\underline{\rho}), & \underline{\rho} \notin \mathcal{N}_k(q_0) \\ H_1: C_k^y(\underline{\rho}) &= C_k^{ex}(\underline{\rho}), & \underline{\rho} \notin \mathcal{N}_k(q_1) \end{aligned} \quad (7)$$

where ex denotes the product process $e(n)x(n)$. Let us define the set of lags

$$\begin{aligned} S_k(q_e) &= \{\underline{\rho} | |\rho_i| > q_e \forall i \in \{1, \dots, k-1\} \\ &\text{and } |\rho_i - \rho_j| > q_e \text{ for } i \neq j\}. \end{aligned} \quad (8)$$

In Appendix A, we show that choosing $\underline{\rho} \in S_k(q_e)$ yields the following detection problem:

$$\begin{aligned} H_0: C_k^y(\underline{\rho}) &= C_k^x(\underline{\rho}), & \underline{\rho} \in S_k(q_e) \setminus \mathcal{N}_k(q_0) \\ H_1: C_k^y(\underline{\rho}) &= C_k^{ex}(\underline{\rho}) = 0, & \underline{\rho} \in S_k(q_e) \setminus \mathcal{N}_k(q_1) \end{aligned} \quad (9)$$

where $a \in A \setminus B$ means that $a \in A$ and $a \notin B$. Suppose that $C_k^x(\underline{\rho}) \neq 0 \forall \underline{\rho} \in S_k(q_e)$ [which is true if the k th-order cumulant of $g(n)$, which is the innovations of the ARMA process, is nonzero and either $p \geq 1$ and/or $q > q_e$]. Then, the detector reduces to testing the nullity of an appropriate vector \mathbf{C}_k of the k th-order cumulants of $y(n)$ computed at lags $\underline{\rho} \in S_k(q_e) \setminus (\mathcal{N}_k(q_0) \cup \mathcal{N}_k(q_1))$ [which is nonempty if $q_e > \max(q_0, q_1)$]

$$H_0: \mathbf{C}_k = \mathbf{C}_k^0 \neq \mathbf{0} \quad H_1: \mathbf{C}_k = \mathbf{C}_k^1 = \mathbf{0}. \quad (10)$$

It is important to note that this procedure can be used for other linear processes, as well as for nonlinear processes, since (9) depends only on the fact that the multiplicative noise $e(n)$ is zero mean. The only requirements are the following:

- 1) The SOI $x(n)$ must have memory strictly greater than $\max(q_0, q_1)$.
- 2) $q_e > \max(q_0, q_1)$.
- 3) Multiplicative noise $e(n)$ must be zero mean.

III. NONZERO MEAN MULTIPLICATIVE NOISE (NZM DETECTOR)

In this section, we assume that the multiplicative noise $e(n)$ has nonzero mean, i.e., $\mu_e \neq 0$. In this case, the detector derived in the previous section is no longer valid. The proposed algorithm is a three-step process. First, we derive an algorithm to estimate the AR parameters of the ARMA process $x(n)$; we show that the estimator is consistent under both hypotheses. We next filter the observed data $y(n)$ by the estimated AR filter to form the residual process $z(n)$, which is noisy MA under H_0 , but does not have the MA structure under H_1 . We then show that appropriate cumulants of the process $z(n)$ are useful for detection.

A. AR Parameter Estimation

In this section, we derive an estimator for the AR parameters of the ARMA process $x(n)$; we show that the estimator is consistent under both hypotheses. This algorithm exploits the fact that appropriate cumulants of $y(n)$ satisfy the same higher order Yule-Walker equations under both hypotheses. The additive noises $u_0(n)$ and $u_1(n)$ are defined in (3) and (4), respectively. For brevity, the study is conducted with covariances, making the implicit assumption that there are no inherent all-pass factors in the ARMA model generating $x(n)$. However, it can be generalized to higher order cumulants if $x(n)$ is non-Gaussian, which would allow us to relax the allpass assumption; further, the additive noises $u_0(n)$ and $u_1(n)$ could be either non-Gaussian MA processes or arbitrarily colored (i.e., perhaps MA) Gaussian processes.

Since second-order cumulants are symmetric, only positive lags ρ will be considered.

- i) Under hypothesis H_0 , $C_2^y(\rho) = C_2^x(\rho) + C_2^{u_0}(\rho)$, with $C_2^{u_0}(\rho) = \sigma_{u_0}^2 \sum_{j=0}^{q_0} \theta_{0,j} \theta_{0,j+\rho} = 0$, for $\rho > q_0$ since $u_0(n)$ is an MA(q_0) process. Consequently, the covariances of $y(n)$ satisfy the well-known Yule-Walker equations

$$\sum_{j=0}^p a_j C_2^y(\rho - j) = 0 \quad (11)$$

(with $a_0 = 1$), provided that $\rho > q$ and $\rho - j > q_0$, $\forall j = 0, \dots, p$, which will be denoted condition \mathbf{C}_1 :

$$\text{condition } \mathbf{C}_1: \rho > \max(p + q_0, q). \quad (12)$$

- ii) Under hypothesis H_1 , $C_2^y(\rho) = C_2^{ex}(\rho) + C_2^{u_1}(\rho)$, with $C_2^{u_1}(\rho) = 0$ for $\rho > q_1$. Moreover

$$\begin{aligned} C_2^{ex}(\rho) &= M_2^{ex}(\rho) - \mu_e^2 \\ &= M_2^e(\rho) M_2^x(\rho) - \mu_e^2 \mu_x^2 \\ &= M_2^e(\rho) (C_2^x(\rho) + \mu_x^2) - \mu_e^2 \mu_x^2. \end{aligned}$$

Since $c(n)$ is an $\text{MA}(q_e)$ process, $M_2^c(\rho) = C_2^c(\rho) + \mu_e^2 = \mu_e^2$ for $\rho > q_e$. Thus, under H_1

$$C_2^y(\rho) = \mu_e^2 C_2^x(\rho), \quad \text{for } \rho > \max(q_1, q_e).$$

It follows that

$$\sum_{j=0}^p a_j C_2^y(\rho - j) = \mu_e^2 \sum_{j=0}^p a_j C_2^x(\rho - j) = 0 \quad (13)$$

if $\rho > q$ and $|\rho - j| > \max(q_1, q_e)$ for any $j \in \{0, \dots, p\}$. Thus, the validity condition is

$$\text{condition } \mathbf{C}_2: \rho > \max(p + q_1, p + q_e, q). \quad (14)$$

Equations (11)–(14) show that the AR parameter vector $\underline{a} = [a_1, \dots, a_p]^t$ satisfies the same Yule–Walker equations under both hypotheses

$$\underline{\mathcal{C}}_2(m, \rho) \underline{a} = -\underline{\mathcal{C}}_2(m, \rho) \quad (15)$$

for some $m \geq p$, provided that

$$\text{condition } \mathbf{C}_3: \rho > \rho_0 \triangleq \max(p + q_0, p + q_1, p + q_e, q) \quad (16)$$

is satisfied. In (15), $\underline{\mathcal{C}}_2(m, \rho)$ is the $m \times p$ full-rank Toeplitz matrix with (i, j) entry, $C_2^y(\rho - j + i)$, $j = 1, \dots, p$, $i = 0, \dots, m - 1$, and

$$\underline{\mathcal{C}}_2(m, \rho) = [C_2^y(\rho), \dots, C_2^y(\rho + m - 1)]^t.$$

Several remarks are now appropriate.

- 1) Equation (13) shows that the condition $\mu_e \neq 0$ is required. Indeed, when $\mu_e = 0$, $\underline{\mathcal{C}}_2(m, \rho_0) = \mathbf{0}$, and $\underline{\mathcal{C}}_2(m, \rho_0) = \mathbf{0}$.
- 2) Equation (15) can be obtained for orders $k > 2$, leading to

$$\underline{\mathcal{C}}_k(m, \rho_1, \dots, \rho_{k-1}) \underline{a} = -\underline{\mathcal{C}}_k(m, \rho_1, \dots, \rho_{k-1})$$

provided that appropriate relations between $\rho_1, \rho_2, \dots, \rho_{k-1}, p, q, q_e, q_0, q_1$ are satisfied. For instance, for third-order cumulants, we must have $\rho_1 > q$ and $(\rho_1 - j, \rho_2) \in S_3 \setminus (\mathcal{N}_3(q_0) \cup \mathcal{N}_3(q_1))$, $j = 0, \dots, p$, which implies $(\rho_1 > \rho_0$ and $\rho_2 > \max(q_0, q_1))$ or $(\rho_1 > \max(q_0 + p, q_1 + p, q)$ and $\rho_2 > \max(q_e, q_0, q_1))$.

- 3) In practice, sample covariances or cumulants will replace theoretical covariances or cumulants for AR parameter estimation; sample estimates of the cumulants, and hence the AR parameter estimates, are strongly consistent under the assumed mixing conditions.
- 4) Equation (15) was derived assuming that model orders p, q, q_e, q_0 , and q_1 were known. However, it can be easily modified to yield consistent AR parameter estimates when only upper bounds $\bar{p}, \bar{q}, \bar{q}_e, \bar{q}_0$, and \bar{q}_1 are available.

B. MA Detector

Denote by $z_i(n)$ the output of the FIR filter with Z -transform $A(z) = \sum_{k=0}^p a_k z^{-k}$ driven by $y_i(n)$. The detection problem can be rewritten as

$$\begin{aligned} H_0: z(n) &= z_0(n) = \sum_{j=0}^q b_j g(n-j) + \sum_{j=0}^p a_j u_0(n-j) \\ H_1: z(n) &= z_1(n) = \sum_{j=0}^p a_j c(n-j)x(n-j) \\ &\quad + \sum_{j=0}^p a_j u_1(n-j). \end{aligned} \quad (17)$$

Equation (17) shows that 1) $z_0(n)$ is the sum of an $\text{MA}(q)$ sequence and an $\text{MA}(p+q_0)$ sequence [since $u_0(n)$ is an $\text{MA}(q_0)$ process], and 2) $z_1(n)$ is the sum of an $\text{MA}(p+q_1)$ process and a non-Gaussian and nonlinear sequence; the latter sequence cannot be modeled as an MA process.

It follows from these two remarks that 1) the k th-order cumulants of $z_0(n)$ are zero except on the finite set of lags $\Lambda_k^0 \triangleq \mathcal{N}_k(q) \cup \mathcal{N}_k(p+q_0)$, and the k th-order cumulants of $z_1(n)$ are nonzero for a specific set of lags Λ_k^1 . It is shown in Appendix B that $C_2^{z_1}(\rho) = 0$ if $\rho \notin \Lambda_2^1 \triangleq \mathcal{N}_2(\max(q, p+q_1, p+q_e))$ and $C_3^{z_1}(\rho_1, \rho_2) \neq 0$ in general so that $\Lambda_3^1 \triangleq \mathbb{Z}^2$.

The detection procedure consists, then, of testing a cumulant vector \mathbf{C}_k^z whose lags belong to the set $\Lambda_k \triangleq \Lambda_k^1 \setminus \Lambda_k^0$

$$H_0: \mathbf{C}_k^z = \mathbf{0} \quad H_1: \mathbf{C}_k^z \neq \mathbf{0} \quad (18)$$

provided, of course, that $\Lambda_k \neq \emptyset$. Indeed, if $\max(p+q_0, q) = \max(p+q_1, p+q_e, q)$ [which occurs, for instance, if the noises $u_0(n)$, $u_1(n)$ and $c(n)$ are i.i.d., or if $q \geq \max(p+q_0, p+q_1, p+q_e)$], $\Lambda_2 = \emptyset$ so that second-order cumulants cannot be used for detection in this case. Thus, in case of uncertainty concerning model orders (in particular, if they are overestimated) third- or higher order cumulants must be considered.

Moreover, it is interesting to note that (17) was derived assuming that the AR parameter vector \underline{a} is known. In practice, this vector is unknown and must be estimated, perhaps using the procedure described in Section III-A. The k th-order cumulant vector \mathbf{C}_k^z is then estimated via the sample cumulants of the output of the FIR filter with Z -transform $\hat{A}(z) = \sum_{k=0}^p \hat{a}_k z^{-k}$ driven by $y_i(n)$. Strong consistency of the sample cumulant estimators guarantees that \hat{a}_k is a strongly consistent estimator of a_k .

IV. STUDY OF THE TEST STATISTICS

In this section, we give the distribution of the test statistics for our detection problem and obtain closed-form expressions for the ROC's.

Equations (10) and (18) show that in both the zero mean and nonzero mean cases, the detection problem reduces to testing the nullity of an appropriate k th-order cumulant vector

$$H_0: \mathbf{C} = \mathbf{C}_0 = \mathbf{0} \quad H_1: \mathbf{C} = \mathbf{C}_1 \neq \mathbf{0}. \quad (19)$$

Note that hypotheses H_0 and H_1 in (10) have to be interchanged to agree with (19). Denote by $\hat{\mathbf{C}}_N$ the vector obtained by replacing the true cumulants in \mathbf{C} by their usual estimates computed from N samples (from the procedure given in [27]). The cumulant estimates of an ARMA process driven by a weakly mixing input (i.e., with absolutely summable cumulants) are asymptotically Gaussian with [20, Sec. 10.5]:

$$\lim_{N \rightarrow +\infty} E[\hat{\mathbf{C}}_N] = \mathbf{C}$$

$$\lim_{N \rightarrow +\infty} NE \left[(\hat{\mathbf{C}}_N - \mathbf{C})(\hat{\mathbf{C}}_N - \mathbf{C})^t \right] = \Sigma.$$

The process $e(n)x(n)$ is weakly mixing, provided that $e(n)$ and $x(n)$ are both weakly mixing. Consequently, the asymptotic statistical behavior of the HOC vector estimate $\hat{\mathbf{C}}_N$ is

$$H_0: \sqrt{N}\hat{\mathbf{C}}_N \sim \mathcal{N}(\mathbf{0}, \Sigma_0)$$

$$H_1: \sqrt{N}(\hat{\mathbf{C}}_N - \mathbf{C}_1) \sim \mathcal{N}(\mathbf{0}, \Sigma_1) \quad (20)$$

where the matrices Σ_0 and Σ_1 are independent of N . If Σ_0 , Σ_1 , and \mathbf{C}_1 are known, the asymptotic statistics of $\hat{\mathbf{C}}_N$ can be used to derive a likelihood ratio test based on the k th-order cumulants (rather than the data whose distribution is unknown). However, we focus on the composite hypotheses test (19) in which matrices Σ_0 , Σ_1 , and \mathbf{C}_1 are all unknown. Consider a segmentation of an N -sample signal into M segments of K samples. The segment size K is assumed to be large enough to obtain approximately normally distributed cumulant estimates from each segment. Moreover, any two adjacent slices must be sufficiently separated to yield approximately independent slices (see [17] for a segmentation procedure description). Finally, the segmentation gives M independent estimates of $\hat{\mathbf{C}}_K$ (denoted $\hat{\mathbf{C}}_{K,j}$ for $j = 1, \dots, M$) (Note, however, that M is not supposed to tend to infinity; it is only assumed that $M > d$, where d denotes the dimension of the vector $\hat{\mathbf{C}}_N$). Define $\bar{\mathbf{C}}$ and $\hat{\mathbf{S}}$ as the sample mean and covariance matrix of the sequence $(\hat{\mathbf{C}}_{K,j})_{j=1, \dots, M}$

$$\bar{\mathbf{C}} = \frac{1}{M} \sum_{j=1}^M \hat{\mathbf{C}}_{K,j}$$

$$\hat{\mathbf{S}} = \frac{1}{M-1} \sum_{j=1}^M (\hat{\mathbf{C}}_{K,j} - \bar{\mathbf{C}})(\hat{\mathbf{C}}_{K,j} - \bar{\mathbf{C}})^t.$$

Using the asymptotic normality of vector $(\hat{\mathbf{C}}_{K,1}^t, \dots, \hat{\mathbf{C}}_{K,M}^t)^t$, the generalized likelihood ratio detector for the detection problem (20) is given by [21]

$$H_0 \text{ rejected if } T^2 \triangleq M\bar{\mathbf{C}}^t \hat{\mathbf{S}}^{-1} \bar{\mathbf{C}} > \lambda \quad (21)$$

where λ is a threshold that can be determined from the distribution of T^2 under the null hypothesis and a fixed probability of false alarm (PFA). Giri [21] showed that the statistic $((M-d)/((M-1)d))T^2$ has an F -distribution with $(d, M-d)$ degrees of freedom under the null hypothesis. Further, under hypothesis H_1 , the distribution of $((M-d)/((M-1)d))T^2$ is

noncentral F with $(d, M-d)$ degrees of freedom and noncentrality parameter

$$\nu = MK\mathbf{C}_1^t \Sigma_1^{-1} \mathbf{C}_1.$$

Let $f(X; d_1, d_2, \xi)$ denote the cumulative distribution function of a noncentral F -distribution with d_1 and d_2 degrees of freedom and noncentrality parameter ξ , and let $f^{-1}(x; d_1, d_2, \xi)$ denote its inverse. The probability of detection (PD) can then be obtained from the PFA as follows:

$$\lambda = \frac{(M-1)d}{M-d} f^{-1}(1 - PFA; d, M-d, 0)$$

$$PD = 1 - f\left(\frac{M-d}{(M-1)d} \lambda; d, M-d, \nu\right). \quad (22)$$

Analytically, the ROC can be written as

$$PD = 1 - f(f^{-1}(1 - PFA; d, M-d, 0); d, M-d, \nu)$$

where we recall that

- d dimension of the statistic \mathbf{C} ;
- K number of samples per segment;
- M number of segments (so that $MK \leq N$, where N is the total number of samples);
- ν noncentrality parameter.

Note that for a fixed PFA , and a fixed number of slices M , PD is an increasing function of the noncentrality parameter ν such that $\lim_{\nu \rightarrow +\infty} PD(\nu) = 1$. Moreover, for given model parameters, \mathbf{C}_1 and Σ_1 are fixed and independent of the number of samples N . Thus, $\lim_{N \rightarrow +\infty} \nu(N) = +\infty$, and $\lim_{N \rightarrow +\infty} PD(N) = 1$ for all model parameters.

V. SIMULATION RESULTS

Many simulations have been performed to validate the theoretical results; we report a few representative examples here.

We consider the zero-mean case first, with interchanged hypotheses, i.e.,

$$H'_0: y(n) = y_1(n) \triangleq e(n)x(n) + u_1(n)$$

$$H'_1: y(n) = y_0(n) \triangleq x(n) + u_0(n). \quad (23)$$

PFA and PD are defined for the problem in (23) by

$$PFA = P[\text{reject } H'_0/H'_0 \text{ is true}]$$

$$= P[\text{multiplicative noise is not detected} \\ / \text{multiplicative noise is present}].$$

and

$$PD = P[\text{accept } H'_1/H'_1 \text{ is true}]$$

$$= P[\text{multiplicative noise is not detected} \\ / \text{multiplicative noise is absent}].$$

The signal $x(n)$ is an ARMA(2,2) process with poles $\rho = 0.7e^{\pm j2\pi \times 0.1}$, and the MA parameters are [1; 0.4; 0.8]. The innovations sequence $g(n)$ in (2) is an exponentially distributed i.i.d. sequence such that $\mu_x = 5$ and $\sigma_x^2 = 1$. Additive noise processes $u_0(n)$ and $u_1(n)$ are MA(1) Gaussian processes with $\text{SNR}_{x, u_i} \triangleq \sigma_x^2 / \sigma_{u_i}^2 = 10$, $i = 0, 1$. Multiplicative noise

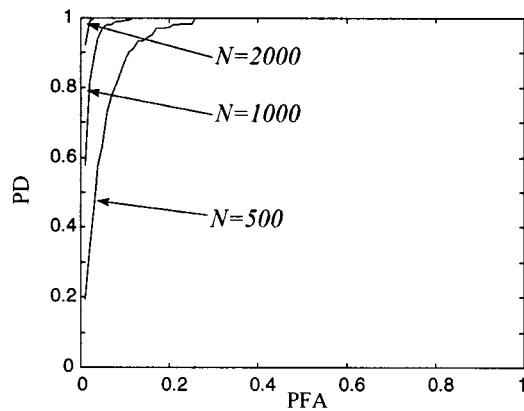
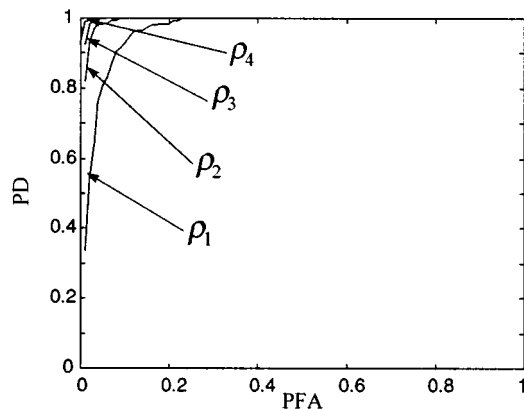
Fig. 1. ZM detector ROC's as a function of the number of samples N .

Fig. 2. ZM detector ROC's as a function of the poles.

$e(n)$ is an MA(2) process with parameters $[1; 0.3; -0.2]$; the innovations process $\varepsilon(n)$ in (5) is a zero-mean exponentially distributed i.i.d. sequence. The number of segments was fixed at $M = 7$, and the cumulant order is $k = 3$. Two-hundred Monte Carlo runs were used to study the performance of the detector.

In Fig. 1, we show the performance of the ZM detector as a function of the number of samples N . Obviously, performance improves as the amount of data increases. However, it should be noted that performance is quite satisfactory even for $N = 1000$.

We next fixed the number of samples at $N = 2000$ and varied the pole locations. Fig. 2 shows the ROC's obtained for different set of poles:

- $\rho_1 = 0.2e^{\pm j2\pi \times 0.1}$;
- $\rho_2 = 0.4e^{\pm j2\pi \times 0.1}$;
- $\rho_3 = 0.6e^{\pm j2\pi \times 0.1}$;
- $\rho_4 = 0.7e^{\pm j2\pi \times 0.1}$;
- $\rho_5 = 0.9e^{\pm j2\pi \times 0.1}$;
- $\rho_6 = 0.99e^{\pm j2\pi \times 0.1}$.

As the poles move closer to the unit circle, the process becomes increasingly more narrowband; therefore, the effective memory of the process increases. In the problem formulation, it helps if the memory of the SOI is much greater than that of the noise processes. Consequently, the performance of the detector improves.

We then fixed the number of samples at $N = 2000$ and the AR poles at $\rho = 0.7e^{\pm j2\pi \times 0.1}$ and varied the signal-to-

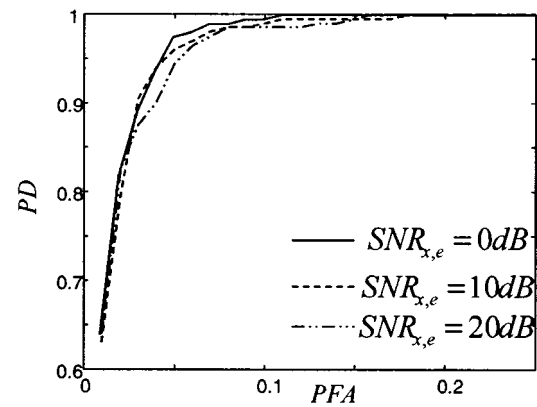
Fig. 3. ZM detector ROC's as a function of $\text{SNR}_{x,e}$.

TABLE I
ESTIMATED PFA AND PD FOR THE ZM DETECTOR—DESIGN PFA WAS 0.05. $x(n)$ WAS AN ARMA(2,2) PROCESS WITH EXPONENTIALLY DISTRIBUTED INPUT PROCESS, AND $u_0(n)$ AND $u_1(n)$ WERE GAUSSIAN MA(1) PROCESSES

N	estimated PFA	estimated PD
1000	0.045	0.975
3000	0.050	1.000
5000	0.045	1.000
8000	0.046	1.000
10000	0.054	1.000

noise ratio $\text{SNR}_{x,e} \triangleq \sigma_x^2 / \sigma_e^2$; results are shown in Fig. 3. It can be seen that the ZM detector's performance is insensitive to $\text{SNR}_{x,e}$ (note that for legibility, ROC's are zoomed in on $PFA \in [0; 0.25]$). Indeed, the hypotheses of the binary testing problem (10) have been interchanged in order to agree with the procedure developed in the previous section. Recall, moreover, that the probability of detection PD derived in (22) only depends on the process parameters of the alternative hypothesis via the noncentrality parameter ν . Now, these parameters are those of the process $y_0(n) = x(n) + u_0(n)$ because of the hypothesis interchange. Therefore, the multiplicative noise parameters do not appear in the noncentrality parameter. This explains why the performance of the ZM detector is blind to $\text{SNR}_{x,e}$.

Table I shows the actual PFA for a design $PFA = 0.05$. This table proves that the ZM detector yields good performance since it maintains the prescribed PFA .

We next consider the nonzero mean case for which we use the NZM detector:

$$\begin{aligned} H_0: y(n) &= y_0(n) \triangleq x(n) + u_0(n) \\ H_1: y(n) &= y_1(n) \triangleq e(n)x(n) + u_1(n). \end{aligned} \quad (24)$$

PFA and PD are defined for the problem in (24) by

$$\begin{aligned} PFA &= P[\text{reject } H_0 / H_0 \text{ is true}] \\ &= P[\text{multiplicative noise is detected} \\ &\quad / \text{multiplicative noise is absent}], \end{aligned}$$

and

$$\begin{aligned} PD &= P[\text{accept } H_1 / H_1 \text{ is true}] \\ &= P[\text{multiplicative noise is detected} \\ &\quad / \text{multiplicative noise is present.}] \end{aligned}$$

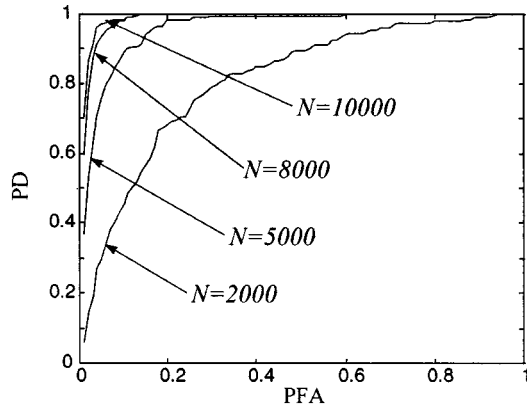
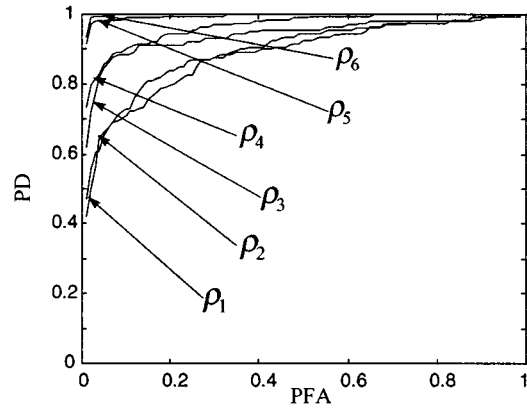
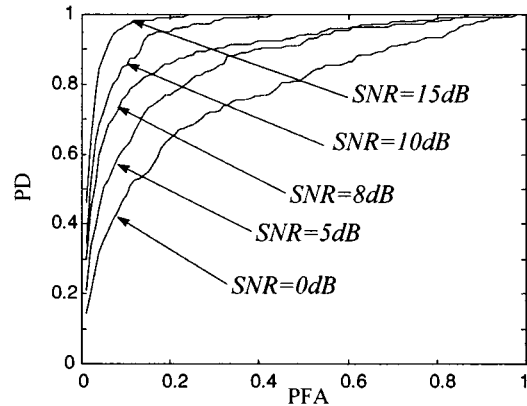
Fig. 4. NZM detector ROC's as a function of the number of samples N 

Fig. 5. NZM detector ROC's as a function of poles.

Fig. 6. NZM detector ROC's as a function of $\text{SNR}_{x,e}$.

Figs. 4–6 are the counterparts of Figs. 1–3; all the parameters in the simulations were kept the same, except that the mean of the multiplicative process was nonzero ($\mu_e = 1$).

Fig. 4 shows the ROC's for a different number of samples; as may be expected, the performance is not as good as that of the ZM detector for the same number of samples, and the detector requires many more samples to give a satisfactory PD . This can be explained by the greater complexity of the NZM detector. Indeed, it is based on AR parameter estimation, which needs the estimation of $p + m + 1$ cumulants (for $m \geq p$). In our simulations, $p + m + 1 = 5$, which implies that N must be large to ensure the convergence of the parameter estimates. Note also

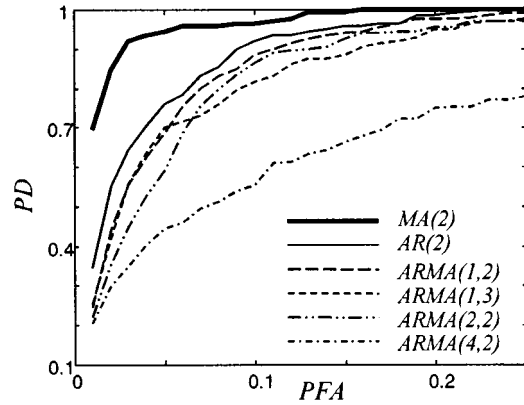


Fig. 7. NZM detector ROC's for different multiplicative noise models.

TABLE II
ESTIMATED PFA AND PD FOR THE NZM DETECTOR—DESIGN PFA WAS 0.05. $x(n)$ WAS A GAUSSIAN ARMA(2,2) PROCESS, AND $u_0(n)$ AND $u_1(n)$ WERE GAUSSIAN MA(1) PROCESSES

N	estimated PFA	estimated PD
1000	0.047	0.171
3000	0.061	0.692
5000	0.052	0.8980
8000	0.060	0.9880
10000	0.059	0.9960

that performance improves as $\text{SNR}_{x,e}$ decreases. Indeed, when the variance of the multiplicative noise is low, it is close to an (unknown) scale constant. The AR process $x(n)$ is then weakly disturbed, and detection is difficult.

Next, we studied the robustness of the NZM detector with respect to the multiplicative noise model. Results are given in Fig. 7. It presents ROC's obtained when the multiplicative noise is supposed to be an MA(2) process, whereas it actually corresponds to another model (the true ARMA model orders are shown in the figure). The AR parameters are [1; 0.5] [for ARMA(1,2) and ARMA(1,3) processes], [1; 0.5; 0.25] [for AR(2) and ARMA(2, 2) processes], and [1; 0.5; 0.25; -0.125; -0.06] [for the ARMA(4, 2) process]. The MA parameters are [1; 0.3; -0.2] [for ARMA(1, 2), ARMA(2, 2) and ARMA(4, 2) processes], and [1; 0.3; -0.2; 0.8] [for the ARMA(1, 3) process]. The input sequences are identical to those of Fig. 4. Fig. 7 shows that the algorithm is quite robust with respect to the multiplicative noise structure as long as the actual model does not deviate too much from the expected model (note that ROC's are zoomed in on $PFA \in [0; 0.25]$).

Finally, we simulated the case with the Gaussian "carrier" ($x(n)$) and Gaussian additive noise. In this particular case, the observed process is Gaussian under hypothesis H_0 and non-Gaussian under hypothesis H_1 . Consequently, any Gaussianity test can be applied to solve the detection problem. A comparison between the NZM detector and the Moulines–Choukri detector [26] is shown in Tables II and III. In these simulations, the noise and signal parameters are identical to those leading to Fig. 4. We fixed the theoretical PFA at 0.05 and computed the estimated PFA and PD for different number of samples N . These tables show the following: 1) The NZM detector yields estimated PFA 's close to the theoretical PFA and PD close

TABLE III
ESTIMATED PFA AND PD FOR THE MOULINES–CHOUKRI
DETECTOR—DESIGN PFA WAS 0.05. $x_0(n)$ WAS A GAUSSIAN ARMA (2,2)
PROCESS, AND $u_0(n)$ AND $u_1(n)$ WERE GAUSSIAN MA(1) PROCESSES

N	estimated PFA	estimated PD
1000	0.212	0.995
3000	0.179	0.998
5000	0.189	1.00
8000	0.209	1.00
10000	0.197	1.00

to 1, provided that $N \geq 5000$; 2) the Moulines–Choukri detector requires far fewer samples than the NZM detector to obtain PD close to 1. However, the empirical PFA is less accurate. These results allow us to conclude that the NZM detector and the Moulines–Choukri detector perform very similarly in the Gaussian context.

VI. CONCLUSION

We studied the detection of colored multiplicative noise in a stationary setting. The proposed detectors modeled the signal of interest as a parametric ARMA process and the noises as MA processes. Suboptimal detectors that do not require any knowledge of the distributions of the processes were derived. The detectors only need to know whether or not the multiplicative noise has zero mean and some bounds on the model orders. The detection of zero-mean multiplicative noise was achieved using appropriate cumulants of the observed signal. The detection of nonzero mean multiplicative noise is a little bit more complicated. It involves

- 1) AR parameter estimation;
- 2) filtering by the estimated AR filter;
- 3) computation of the cumulants of the filtered data.

For both detectors, the problem finally reduces to comparing a vector of cumulants to the null vector, leading to a test statistic with central and noncentral F distributions under the null and alternative hypotheses, respectively. Closed-form expressions for the ROC's were given.

The choice of detector to be used depends on whether or not the multiplicative noise is zero mean; in this paper, we assume that we have this *a priori* knowledge; it would be of interest to develop detectors/estimators for this. In addition, in the context of gear-fault detection, quantifying the strength of the multiplicative noise is also of interest.

APPENDIX A

REGION OF SUPPORT OF $C_k^{ex}(\rho)$

Here, we prove that $C_k^{ex}(\rho) = 0$ for $\rho \in S_k(q_e)$ if $\mu_e = 0$, where

$$S_k(q_e) = \{\rho \mid |\rho_i| > q_e \forall i \in \{1, \dots, k-1\} \text{ and } |\rho_i - \rho_j| > q_e \text{ for } i \neq j\}.$$

Recall that k th-order cumulants are related to k th-order moments via

$$C_k^{ex}(\rho) = \sum_{P(k) \in \mathcal{P}(k)} (-1)^{r-1} (r-1)! \prod_{(i_1, \dots, i_m) \in P(k)} M_m(ex(i_1), ex(i_2), \dots, ex(i_m))$$

where

$P(k)$ partition of $\{1, 2, \dots, k\}$;

r number of sets in $P(k)$;

$\mathcal{P}(k)$ set of all partitions of $\{1, 2, \dots, k\}$;

and $i_l \in \{n, n + \rho_1, \dots, n + \rho_{k-1}\} \forall l \in \{1, \dots, m\}$. Since $e(n)$ and $x(n)$ are independent processes, it follows that

$$\begin{aligned} M_m(ex(i_1), ex(i_2), \dots, ex(i_m)) \\ = M_m(e(i_1), e(i_2), \dots, e(i_m)) \\ \times M_m(x(i_1), x(i_2), \dots, x(i_m)). \end{aligned}$$

Moreover

$$\begin{aligned} M_m(e(i_1), e(i_2), \dots, e(i_m)) \\ = M_m(e(0), e(i_2 - i_1), \dots, e(i_m - i_1)). \end{aligned}$$

Since $\rho \in S_k(q_e)$, we have $|i_l - i_1| > q_e \forall l \in \{2, \dots, m\}$. Thus, $e(0)$ and $e(i_m - i_1)$ are independent for all $l \in \{2, \dots, m\}$ [since $e(n)$ is a MA(q_e) process], which implies

$$\begin{aligned} M_m(e(0), e(i_2 - i_1), \dots, e(i_m - i_1)) \\ = M_1(e(0)) M_{m-1}(e(i_2 - i_1), \dots, e(i_m - i_1)) \\ = \mu_e M_{m-1}(e(i_2 - i_1), \dots, e(i_m - i_1)) = 0 \end{aligned}$$

which establishes the result.

Note, however, that $\rho \in S_k(q_e)$ is a sufficient but not necessary condition to ensure $C_k^{ex}(\rho) = 0$. Indeed, we have the following.

- For $k = 2$, we have, since $\mu_{ex} = \mu_e \mu_x = 0$
 $C_2^{ex}(\rho) = M_2^{ex}(\rho) = M_2^e(\rho) M_2^x(\rho) = C_2^e(\rho) M_2^x(\rho)$
and $C_2^e(\rho) = 0$ for $|\rho| > q_e$. Thus, we can take
 $S_2 = \{\rho \mid |\rho| > q_e\}$.

- For $k = 3$, we have

$$\begin{aligned} C_3^{ex}(\rho_1, \rho_2) &= M_3^{ex}(\rho_1, \rho_2) \\ &= M_3^e(\rho_1, \rho_2) M_3^x(\rho_1, \rho_2) \\ &= C_3^e(\rho_1, \rho_2) M_3^x(\rho_1, \rho_2) \end{aligned}$$

and $C_3^e(\rho_1, \rho_2) = 0$ if $|\rho_1| > q_e$ or $|\rho_2| > q_e$. Finally, we can define

$$S_3 = \{\rho \mid |\rho_1| > q_e \text{ or } |\rho_2| > q_e\}.$$

APPENDIX B

COMPUTATION OF $C_2^{z1}(\rho)$ AND $C_3^{z1}(\rho_1, \rho_2)$

We have

$$\begin{aligned} z_1(n) &= \sum_{j=0}^p a_j y_1(n-j) \\ &= \sum_{j=0}^p a_j e(n-j) x(n-j) + \sum_{j=0}^p a_j u_1(n-j). \end{aligned} \quad (25)$$

Denote by $\bar{z}(n)$ and $\bar{u}_1(n)$ the two terms in the right-hand side of (25). Since those two processes are independent, it follows that

$$C_2^{z1}(\rho) = C_2^{\bar{z}}(\rho) + C_2^{\bar{u}_1}(\rho).$$

Let $\tilde{e}(n) = e(n) - \mu_e$, $\tilde{\varepsilon}(n) = \varepsilon(n) - \mu_\varepsilon$ and $\tilde{g}(n) = g(n) - \mu_g$ so that $\tilde{e}(n)$, $\tilde{\varepsilon}(n)$, and $\tilde{g}(n)$ are zero-mean processes. Denote

by $(h_j)_{j=0, \dots, +\infty}$ the impulse response of the ARMA process, and define $H(0) = \sum_{j=0}^{+\infty} h_j$. We have

$$\begin{aligned}\bar{z}(n) &= \mu_e \sum_{j=0}^p a_j x(n-j) + \sum_{j=0}^p a_j \tilde{e}(n-j) x(n-j) \\ &= \mu_e \sum_{j=0}^q b_j g(n-j) + \mu_g H(0) \sum_{j=0}^p a_j \tilde{e}(n-j) \\ &\quad + \sum_{j=0}^p \sum_{l=0}^{+\infty} a_j h_l \tilde{e}(n-j) \tilde{g}(n-j-l).\end{aligned}\quad (26)$$

Denote $\bar{g}(n)$, $\bar{e}(n)$, and $\bar{w}(n)$ as the three terms on the right-hand side. We have

$$\begin{aligned}C_{\bar{z}}(\rho) &= Cum(\bar{g}(n) + \bar{e}(n) + \bar{w}(n), \\ &\quad \bar{g}(n+\rho) + \bar{e}(n+\rho) + \bar{w}(n+\rho)).\end{aligned}$$

Since $\tilde{e}(n)$ and $\tilde{g}(n)$ are independent and zero-mean, $\bar{g}(n)$, $\bar{e}(n)$, and $\bar{w}(n)$ are zero-mean, and the cross terms in the development of $C_{\bar{z}}(\rho)$ are zero. The nonzero mean terms are

$$\begin{aligned}Cum(\bar{g}(n), \bar{g}(n+\rho)) &= \mu_e^2 \sigma_g^2 \sum_{j=0}^q b_j b_{j+\rho}, \\ Cum(\bar{e}(n), \bar{e}(n+\rho)) &= \mu_g^2 H^2(0) \sum_{j_1=0}^p \sum_{j_2=0}^p a_{j_1} a_{j_2} \sum_{l_1=0}^{q_e} \sum_{l_2=0}^{q_e} \beta_{l_1} \beta_{l_2} \\ &\quad Cum(\tilde{e}(n-j_1-l_1), \tilde{e}(n+\rho-j_2-k_2)) \\ &= \mu_g^2 \sigma_e^2 H^2(0) \left(\sum_{j_1=0}^p \sum_{j_2=0}^p \sum_{l=0}^{q_e} a_{j_1} a_{j_2} \beta_l \beta_{\rho+l+j_1-j_2} \right) \\ Cum(\bar{w}(n), \bar{w}(n+\rho)) &= \sum_{j_1=0}^p \sum_{j_2=0}^p a_{j_1} a_{j_2} \sum_{k_1=0}^{q_e} \sum_{k_2=0}^{q_e} \beta_{k_1} \beta_{k_2} \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} h_{l_1} h_{l_2} \\ &\quad Cum(\tilde{e}(n-j_1-k_1) \tilde{g}(n-j_1-l_1), \\ &\quad \tilde{e}(n+\rho-j_2-k_2) \tilde{g}(n+\rho-j_1-l_2)) \\ &= \sigma_g^2 \sigma_e^2 \left(\sum_{j_1=0}^p \sum_{j_2=0}^p \sum_{k=0}^{q_e} \sum_{l=0}^{+\infty} a_{j_1} a_{j_2} \beta_k \beta_{\rho+k+j_1-j_2} \right. \\ &\quad \left. \times h_l h_{\rho+l+j_1-j_2} \right).\end{aligned}$$

Moreover, $u_1(n)$ is an MA(q_1) process with innovations $v_1(n)$ and parameters $(\theta_{1,j})_{j=0, \dots, q_1}$. Thus, $\bar{u}_1(n)$ is an MA($p+q_1$) process with innovations $v_1(n)$ and parameters

$$\xi_l = \sum_{j=0}^l a_j \theta_{1,l-j}, \quad l = 0, \dots, p+q_1$$

with $a_j = 0$ if $j > p$, and $\theta_{1,j} = 0$ if $j > q_1$ [note that $\xi_l = a_l$ if $u_1(n)$ is MA(0), i.e., $u_1(n)$ is i.i.d.]. Thus, we obtain

$$C_{\bar{u}_1}(\rho) = \sigma_{v_1}^2 \sum_{l=0}^{p+q_1} \xi_l \xi_{\rho+l}.$$

Finally, the second-order cumulant of $z_1(n)$ is

$$\begin{aligned}C_2^{z_1}(\rho) &= \mu_e^2 \sigma_g^2 \sum_{j=0}^q b_j b_{j+\rho} + \mu_g^2 \sigma_e^2 H^2(0) \\ &\quad \times \left(\sum_{j_1=0}^p \sum_{j_2=0}^p \sum_{l=0}^{q_e} a_{j_1} a_{j_2} \beta_l \beta_{\rho+l+j_1-j_2} \right) \\ &\quad + \sigma_g^2 \sigma_e^2 \left(\sum_{j_1=0}^p \sum_{j_2=0}^p \sum_{k=0}^{q_e} \sum_{l=0}^{+\infty} a_{j_1} a_{j_2} \beta_k \beta_{\rho+k+j_1-j_2} \right. \\ &\quad \left. \times h_l h_{\rho+l+j_1-j_2} \right) + \sigma_{v_1}^2 \sum_{l=0}^{p+q_1} \xi_l \xi_{\rho+l}.\end{aligned}$$

Obviously, the first and the last terms vanish if $|\rho| > q$ and $|\rho| > p+q_1$, respectively. Moreover, it is easy to prove that the second and the third terms are zero if $|\rho| > p+q_e$. Consequently

$$C_2^{z_1}(\rho) = 0 \quad \text{if } |\rho| > \max(q, p+q_e, p+q_1).$$

Now, the computation of $C_3^{z_1}(\rho_1, \rho_2)$ is long and tedious. Thus, it is not detailed in this paper. However, it should be noted that $C_3^{z_1}(\rho_1, \rho_2)$ is the sum of 15 terms, of which 13 are zero, except on a finite set. Now, it can be proved that the two other terms are, in general, nonzero for all $(\rho_1, \rho_2) \in \mathbb{Z}^2$ [12].

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