Cramer–Rao lower bounds for change points in additive and multiplicative noise

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Abstract

The paper addresses the problem of determining the Cramer–Rao lower bounds (CRLBs) for noise and change-point parameters, for steplike signals corrupted by multiplicative and/or additive white noise. Closed-form expressions for the signal and noise CRLBs are first derived for an ideal step with a known change point. For an unknown change-point, the noise-free signal is modeled by a sigmoidal function parametrized by location and step rise parameters. The noise and step change CRLBs corresponding to this model are shown to be well approximated by the more tractable expressions derived for a known change-point. The paper also shows that the step location parameter is asymptotically decoupled from the other parameters, which allows us to derive simple CRLBs for the step location. These bounds are then compared with the corresponding mean square errors of the maximum likelihood estimators in the pure multiplicative case. The comparison illustrates convergence and efficiency of the ML estimator. An extension to colored multiplicative noise is also discussed.

Keywords: Cramer–Rao lower bounds; Change-points; Multiplicative noise

1. Introduction

Change-point estimation and detection is important in many signal processing applications. These applications include segmentation, fault detection or monitoring (for an overview see [2] and references therein). Most studies have been carried out for signals contaminated by additive noise. However, the observed process may also be corrupted by multiplicative noise. Some examples of multiplicative noise occur in image processing (speckle) [13] or communication signals (fading channels) [16]. This paper addresses the problem of determining a lower bound on the estimation error of change-point parameters, when the changes are corrupted by multiplicative and/or additive noise.

The covariance matrix of any unbiased estimator $\hat{\theta} = [\hat{\theta}_1, \ldots, \hat{\theta}_p]^{\mathrm{T}}$ ($^{\mathrm{T}}$ denotes transposition) of an unknown parameter vector $\theta = [\theta_1, \ldots, \theta_p]^{\mathrm{T}}$ cannot be lower than the inverse of the Fisher information matrix (FIM) of θ denoted the Cramer–Rao lower bound (CRLB) of θ (the study is restricted to unbiased estimators). In other

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words, $C_{\hat{\theta}} - I^{-1}(\theta)$ is a positive semidefinite matrix, where $C_{\hat{\theta}}$ is the covariance matrix of $\hat{\theta}$, $I(\theta) = [I(\theta)]_{ij} = [-E(\frac{\partial^2 \ln p(y;\theta)}{\partial\theta_i \partial\theta_j})]_{ij}$ is the FIM of θ and $p(y;\theta)$ is the probability density function (pdf) of the observation vector $y = [y(1), \ldots, y(N)]^{\mathrm{T}}$ given θ. The Cramer–Rao inequality requires some regularity conditions which are detailed in standard textbooks such as [3] or [14] (see also Appendix A). When such regularity conditions are satisfied, the variance of $\hat{\theta}_i$ cannot be lower that the ith diagonal element of I^{-1} (θ) denoted $[I^{-1}(\theta)]_{ii}$ known as the CRLB of θ_i. Unfortunately, when the observed signal is subjected to abrupt changes such as steps, the observation vector pdf cannot be differentiated with respect to the change location. Consequently, the FIM, its inverse and the noise and signal parameter CRLBs cannot be computed.

The problem of determining CRLBs for step-change parameters has recently received much attention in the literature [1,17,9,20–22,10]. Reza et al. [17] have modeled the sudden changes in steplike signals by sigmoidal functions. Modified CRLBs (in the sense of [24, p. 72] or [6]) have then be derived by assuming uniform or triangular prior for the change-point location. Sadler and Swami [21] have simplified the closed-form expression of the change-point location FIM which was expressed only as a summation in [17]. Their results have been generalized to step changes corrupted by additive and multiplicative noises in [9,20–22]. The presence of multiplicative noise has also been taken into account in [9], where the maximum likelihood estimator (MLE) of amplitude-modulated time series as well as closed-form expressions for the finite sample signal and noise parameter CRLBs were derived. An approximate CRLB for the change-point location in signals with step-like singularities embedded in additive white Gaussian noise has been derived in [1]. The bound was obtained by assuming that the step change can be modeled as the output of a linear filter with finite bandwidth (a cubic spline) driven by a unit step. A similar model (convolution of a unit step with a Gaussian filter) has been used successfully in [10] to model 2-D blurred step edges.

Most previous works on step change CRLBs have focused on the step location parameter, as if it were decoupled from the step amplitude and the noise means and variances. However, the computation of the FIM for the signal and noise parameters clearly shows that the estimation of the step location cannot generally be decoupled from that of the other parameters. This paper derives asymptotic signal and noise parameter CRLBs for step-like signals corrupted by additive and/or multiplicative Gaussian noises. It is important to note that these CRLBs take into account the coupling between signal and noise parameters. The asymptotic (when the number of samples tends to infinity) decoupling between the step location and the other parameters is also discussed.

The problem is formulated in Section 2. Section 3 derives closed-form expressions for the CRLB of the parameters of an ideal step change with known change location. Section 4 studies signal and noise CRLBs for an unknown change location. The step-like signals are modeled by a sigmoidal function with rise-time parameter α. The major contributions of this section are to show that 1) the asymptotic (step rise parameter $\alpha \to \infty$, and number of samples $N \to \infty$) noise parameter and step amplitude CRLBs obtained for an unknown change-point location t_0 equal the ideal step CRLBs derived for a known change-point location t_0, 2) the step location is asymptotically ($N \to \infty$) decoupled from the other parameters, which yields interesting closed-form expressions for the step location CRLBs. Section 5 compares the noise and step change parameter CRLBs to the corresponding mean square errors (MSEs) of the MLEs, in the pure multiplicative noise case. Section 6 discusses generalization to multiplicative colored noise modeled as an autoregressive (AR) process.

2. Problem formulation

The noisy steplike signal is modeled by

$$y(t) = g(t)s_\alpha(t) + v(t) = g(t)s(\alpha(t - t_0)) + v(t), \tag{1}$$

where t_0 is the actual change location, α is a rise-time parameter, $s(t)$ models the change-point shape and $g(t)$, $v(t)$ are the multiplicative and additive noises, respectively. This paper adopts the sigmoidal function to model the change-point shape, defined by [17,22]

$$s_\alpha(t) = \frac{1 + A + \mathrm{e}^{-\alpha(t-t_0)}}{1 + \mathrm{e}^{-\alpha(t-t_0)}}. \tag{2}$$

Parameter α determines how fast or slow the noise-free signal $s_\alpha(t)$ changes its amplitude from 1 to $1 + A$. In particular, the sigmoidal function approaches the ideal step when parameter α approaches infinity. The sampled noisy signal is obtained by setting $t = nT$, $n = 1, \ldots, N$, where T is the sampling period (without loss of generality, $T = 1$ in this paper). The sampled noise-free signal can then be written $s_\alpha(n) = s(\alpha(nT - t_0))$, where $t_0 = (n_0 + \tau)T$ is the actual change location, n_0 is the sample point after which there is a sudden change in the signal and τ is a deterministic parameter such that $\tau \in [0, 1]$ (note that parameter τ was alternatively modeled as a random variable or a deterministic parameter in [17,21,22]). The multiplicative noise sequence $g(n)$ is assumed to be an iid Gaussian sequence with mean μ_g and variance σ_g^2. The additive noise sequence $v(n)$ is assumed to be zero-mean iid Gaussian with variance σ_v^2. The sequences $g(n)$ and $v(n)$ are assumed statistically independent.

The signal defined in (1) can represent a line in an SAR image intensity corrupted by multiplicative speckle noise (in the case of two different terrain fields imaged by the SAR). The speckle noise is usually modeled as a stationary non-Gaussian process (distributed according to a Gamma distribution). However, in some SAR image processing systems, the speckle is reduced by incoherently averaging N_i uncorrelated images. When N_i is sufficiently large, the resulting reduced-speckle image intensities are approximately Gaussian distributed (using the central limit theorem) [4]. This paper is applicable to SAR imaging systems for which the Gaussian assumption for g_n and v_n is valid. However, it is interesting to note that the non-Gaussian case could also be considered in the purely additive and purely multiplicative noise cases (using appropriate scalar multipliers, see [20,22]).

3. CRLBs for an ideal change point with known location

Denote by $H(t)$ the Heaviside function ($H(t) = 1$ if $t > 0$ and $H(t) = 0$ else). In this section, it is assumed that $\alpha \to \infty$, and that the change-point location $t = 0$ is know; hence the noise free signal is $s_\infty(t) = 1 + AH(t - t_0)$ with amplitude $A > -1$ (the SAR image intensity is positive) and a known change-point location t_0. The parameter A is referred to as fractional step change or step amplitude. Note that a similar model $s_\infty(t) = m_2 + m_1 H(t - t_0)$ (with unknown parameters m_1 and m_2) has been considered in [17]. However, the proposed model is not restrictive since one is interested in relative changes. The unknown parameter vector is $\theta = [\mu_g, \sigma_g^2, \sigma_v^2, A]^{\mathrm{T}}$, where $\mu_g \neq 0$. Note that the condition $\mu_g \neq 0$ (which is not restrictive in SAR imagery since $\mu_g = 1$) is necessary to ensure the identifiability of the unknown parameters. Indeed, under the Gaussian assumption, the probability density function (pdf) the observations only depends on the means and variances of the two segments before and after the change, i.e., $\mu_b = \mu_g$, $\mu_a = \mu_g(1 + A)$, $\sigma_b^2 = \sigma_v^2 + \sigma_g^2$ and $\sigma_a^2 = \sigma_v^2 + \sigma_g^2(1 + A)^2$, respectively. When $\mu_g = 0$, the means before and after the change are both zero so that we have three equations and four unknowns, which results in a loss of identifiability. Even if we know that $\mu_g = 0$, we still have loss of identifiability; this is discussed further after Eq. (6). This section derives closed-form expressions for the CRLB of the unknown parameter vector θ. It is important to note that the Cramer–Rao inequality is subjected to regularity conditions which are satisfied when the change location t_0 is known (as shown in Appendix A).

Eq. (1) shows that the observation vector $y = [y(1), \ldots, y(N)]^{\mathrm{T}}$ is Gaussian with mean $M_y(\theta)$ and covariance matrix $R_y(\theta)$ defined by

$$M_y(\theta) = \mu_g S_\infty 1_N, \quad R_y(\theta) = \sigma_g^2 S_\infty^2 + \sigma_v^2 I_N, \tag{3}$$

where $1_N = [1, \ldots, 1]^{\mathrm{T}}$, $S_\infty = \mathrm{diag}(s_\infty(1), \ldots, s_\infty(N))$ (diagonal matrix whose elements are $s_\infty(i)$) and I_N is the $N \times N$ identity matrix. After dropping some constants, the Gaussian log-likelihood function reduces to

$$L_y(\theta) = -\frac{1}{2}\sum_{i=1}^{N} \frac{[y(i) - \mu_g s_\infty(i)]^2}{\sigma_g^2 s_\infty^2(i) + \sigma_v^2} - \frac{1}{2}\sum_{i=1}^{N} \log(\sigma_g^2 s_\infty^2(i) + \sigma_v^2). \tag{4}$$

By differentiating $L_y(\theta)$ with respect to $\theta = [\mu_g, \sigma_g^2, \sigma_v^2, A]^{\mathrm{T}}$, the FIM of θ for a known change-point location can be expressed as follows:

$$I^*(\infty) = \begin{pmatrix} \frac{Nr}{\sigma_b^2} + \frac{N(1-r)(1+A)^2}{\sigma_a^2} & 0 & 0 & \frac{N_{\mu_g}(1-r)(1+A)}{\sigma_a^2} \\ 0 & \frac{Nr}{2\sigma_b^4} + \frac{N(1-r)(1+A)^4}{2\sigma_a^4} & \frac{Nr}{2\sigma_b^4} + \frac{N(1-r)(1+A)^2}{2\sigma_a^4} & \frac{N(1-r)\sigma_g^2(1+A)^3}{\sigma_a^4} \\ 0 & \frac{Nr}{2\sigma_b^4} + \frac{N(1-r)(1+A)^4}{2\sigma_a^4} & \frac{Nr}{2\sigma_b^4} + \frac{N(1-r)}{2\sigma_a^4} & \frac{N\sigma_g^2(1-r)(1+A)}{\sigma_a^4} \\ \frac{N_{\mu_g}(1-r)(1+A)}{\sigma_a^2} & \frac{N(1-r)\sigma_g^2(1+A)^2}{\sigma_a^4} & \frac{\sigma_g^2 N(1-r)(1+A)}{\sigma_a^4} & \frac{N_{\mu_g^2(1-r)}}{\sigma_a^2} + 2\frac{N\sigma_g^4(1-r)(1+A)^2}{\sigma_a^4} \end{pmatrix} \tag{5}$$

where $r = n_0/N$ is the fractional location of the step change. The expression of $I^*(\infty)$ shows that the parameter estimators are not automatically decoupled, even when the change-point location is known. The FIM determinant can be expressed as

$$\det I^*(\infty) = \frac{N^4 r^2 (r-1)^2 A^2 (A+2)^2 \mu_g^2}{4\sigma_b^6 \sigma_a^6}. \tag{6}$$

Eq. (6) shows that the FIM is singular for $\mu_g = 0$ which reflects the loss of identifiability discussed earlier. When $\mu_g \neq 0$, the FIM is non-singular and its inverse yields the signal and noise parameter CRLBs. Note that identifiability is obviously lost when $r = 0$ or $r = 1$, corresponding to a step change just before or after the observation window. We consider the following cases separately: (A) Pure multiplicative noise: $g(t)$ is iid Gaussian ($\sigma_v^2 = 0$), (B) Pure additive noise: $v(t)$ is iid Gaussian ($\sigma_g^2 = 0$) and (C) Additive and multiplicative noises: $g(t)$ and $v(t)$ are iid Gaussian mutually independent sequences.

3.1. Pure multiplicative noise ($\sigma_v^2 = 0$)

From the FIM expression, we obtain the CRLBs of μ_g, σ_g^2 and A:

$$\mathrm{CRLB}_m(\mu_g) = \frac{\sigma_g^2}{N}\frac{2 + \frac{\mu_g^2}{r\sigma_g^2}}{2 + \frac{\mu_g^2}{\sigma_g^2}}, \quad \mathrm{CRLB}_m(\sigma_g^2) = \frac{2\sigma_g^4}{rN}\frac{2 + \frac{r\mu_g^2}{\sigma_g^2}}{2 + \frac{\mu_g^2}{\sigma_g^2}},$$

$$\mathrm{CRLB}_m(A) = \frac{(1+A)^2}{Nr(1-r)(2 + \frac{\mu_g^2}{\sigma_g^2})}, \tag{7}$$

where we recall $r = \frac{n_0}{N}$. The following observations follow from (7):

- For $r = 1$ ($n_0 = N$), $\mathrm{CRLB}_m(\mu_g)$ and $\mathrm{CRLB}_m(\sigma_g^2)$ are identical to the well-known CRLBs obtained for white Gaussian noise with mean μ_g and variance σ_g^2, i.e., $\frac{\sigma_g^2}{N}$ and $\frac{2\sigma_g^4}{N}$ [11, p. 61].
- For $r = 0$ or $r = 1$, $\mathrm{CRLB}_m(A)$ is infinite: when the change-point occurs before the first sample or after the last sample of $y(n)$, the change parameters cannot be estimated. For a fixed vector $[\mu_g, \sigma_g^2, A]^{\mathrm{T}}$, $\mathrm{CRLB}_m(A)$ is minimum when $r(1-r)$ is maximum, i.e., $r = 1/2$. Consequently, the best estimate is obtained when the

change-point is in the middle of the observation window. Moreover, the CRLB for parameter A is very similar when $r \in [1/5, 4/5]$ as noticed in [21].

3.2. Pure additive noise ($\sigma_g^2 = 0$)

The CRLBs of μ_g, σ_v^2 and A are:

$$\mathrm{CRLB}_a(A) = \frac{[(1+A)^2 - rA(2+A)]\sigma_v^2}{r(1-r)\mu_g^2 N}, \mathrm{CRLB}_a(\mu_g) = \frac{\sigma_v^2}{rN}, \mathrm{CRLB}_a(\sigma_v^2) = \frac{2\sigma_v^4}{N}. \tag{8}$$

The following observations follow from (8):

- For $r = 0$, $\mathrm{CRLB}_a(\mu_g)$ is infinite. The observations $y(n)$ are Gaussian with mean $\mu_g(1+A)$ and variance σ_v^2. Consequently, it is impossible to estimate independently parameters μ_g and A (loss of identifiability),
- $\mathrm{CRLB}_a(\sigma_v^2)$ does not depend on the change-point position in the observation window. This is natural since there is no variance jump (the variance is the same before and after the change),
- For $r = 0$ or $r = 1$, $\mathrm{CRLB}_a(A)$ is infinite: when the change point occurs before the first sample or after the last sample of $y(n)$, the change-point parameters cannot be estimated. Moreover, for a fixed vector $[\mu_g, \sigma_v^2, A]^{\mathrm{T}}$, $\mathrm{CRLB}_a(A)$ is minimum when the integrated means before and after the step change are equal, i.e., $rN = (1-r)(1+A)$ or $r = r_{\min} = (A+1)/(A+2)$. Note that $r_{\min}$ is close 1/2 for small values of A, i.e., for weak changes.

3.3. Additive and multiplicative noises

This subsection focuses on the CRLBs for the mean of the multiplicative noise and the fractional step change, since the closed-form expresssions for the noise variance CRLBs are more difficult to interpret (see [7] for more details). Straightforward computations yield

$$\mathrm{CRLB}_{am}(A) = \frac{(\sigma_v^2 + \sigma_g^2)(1+A)^2 - rA(2+A)\sigma_v^2}{Nr(1-r)\mu_g^2}, \mathrm{CRLB}_{am}(\mu_g) = \frac{\sigma_g^2 + \sigma_v^2}{Nr}. \tag{9}$$

The following observations follow from (9):

- For $r = 0$ or $r = 1$, $\mathrm{CRLB}_{am}(A)$ is infinite. Moreover, for a fixed vector $\theta = [\mu_g, \sigma_g^2, \sigma_v^2, A]^{\mathrm{T}}$, $\mathrm{CRLB}_{am}(A)$ is minimum for $r = \frac{(A+1)\sigma_b\sigma_a - (A+1)^2\sigma_b^2}{A(A+2)\sigma_v^2} = \frac{1}{2} - \frac{A}{4} + \mathrm{o}(A)$, where $\lim_{A \to 0} \frac{\mathrm{o}(A)}{A} = 0$. In other words, the minimum fractional step change CRLB is close to 1/2 for small values of A,
- The pure additive case can be obtained from (9) by setting $\sigma_g^2 = 0$.

4. CRLBS for an unknown location

The unknown parameter vector is $\theta = [\mu_g, \sigma_g^2, \sigma_v^2, A, t_0]^{\mathrm{T}}$. The Gaussian log-likelihood function corresponding to an ideal step only depends on the integer part of t_0, such that there is loss of identifiability. Moreover, Eq. (4) cannot be differentiated with respect to t_0, which prevents one from computing the FIM. This paper proposes to model the change-point shape by the sigmoidal function defined in (2). The first part of this section explains how to adjust the rise-time parameter which appears in the definition of the sigmoidal function. The next two parts of this section study the asymptotic step amplitude and step location CRLBs when the change point is modeled by this sigmoidal function.

4.1. Rise time parameter α

For a finite sigmoidal parameter α, the FIM of θ can be computed by differentiating the log-likelihood function defined in (4). The FIM inverse yields the noise and signal parameter CRLBs. However, this assumes that the rise-time parameter is known a priori. Several approaches have been proposed in the literature to determine an appropriate value of the rise-time parameter α. In [17], the rise time is defined as the time that it takes the signal to change its amplitude from 10% to 90%. Assuming that the rise time for the noise-free signal is approximately less than or equal to T, Reza et al. have provided the following upper bound for α:

$$\alpha \leqslant \frac{2\ln 9}{T} \simeq \frac{4.4}{T}. \tag{10}$$

However, the authors of [17] also note that the exact value of α can be obtained from specifications of the steplike signal and the characteristics of the anti-aliasing filter employed before sampling. To take into account the finite processing bandwidth, Bartov et al. [1] proposed to filter the received signal by a band-limited filter (whose impulse response is a cubic spline) and to sample the filtered signal at the Nyquist rate. In other words, the steplike signal was modeled by a filtered version of the Heaviside function. A similar approach was suggested in [10] to model edge profiles in images. In this particular case, a Gaussian filter was shown to model accurately the blurring caused by an imaging system's optics. This paper uses a similar strategy where the anti-aliasing filter has been chosen such that its output is a sigmoidal function when it is driven by the ideal step $s_\infty(t)$. Consequently, the impulse–response of the anti-aliasing filter denoted $h_\alpha(t)$ is chosen such that $s_\infty(t) * h_\alpha(t) = s_\alpha(t)$. We note that $h_\alpha(t)$ and its Fourier transform $H_\alpha(f)$ are given by:

$$h_\alpha(t) = \frac{\alpha e^{-\alpha t}}{(1+\mathrm{e}^{-\alpha t})^2}, \quad H_\alpha(f) = \int_{-\infty}^{\infty} h_\alpha(t)\mathrm{e}^{-j2\pi f t}\,\mathrm{d}t = \frac{2f\pi^2}{\sinh(2f\pi^2)}, \tag{11}$$

where sinh(.) is the hyperbolic sine function. By defining the spectral bandwidth of $s_\alpha(t)$ as the distance between the half-power points of $|H_\alpha(f)|^2$ (see for instance [15, p. 514]), we propose to optimize the value of α such that the bandwidth of $s_\alpha(t)$ is lower than $\frac{1}{T}$, according to the Nyquist criterion. This strategy provides the following relation:

$$\alpha \leqslant \frac{6.62}{T}. \tag{12}$$

Note that many different definitions of filter bandwidth may be found in the literature (see for instance [15, p. 520]), and each would provide a different value of the rise-time parameter α.

4.2. Asymptotic step amplitude CRLBs ($\alpha \to \infty, N \to \infty$)

For a finite rise-time parameter α, the FIM of the unknown parameter vector θ can be computed (see Appendix A.2 for details). The diagonal elements of the FIM inverse yield the signal and noise CRLBs. Unfortunately, the closed-form expressions are given as sums which are difficult to study. Instead, this section studies the asymptotic step amplitude CRLBs defined by setting $\alpha = \infty$. Similar results would be obtained for the asymptotic noise parameter CRLBs.

Fig. 1 displays the step amplitude CRLBs as a function of α, in the presence of additive or/and multiplicative noise (the true parameters are $N = 100$, $T = 1$, $t_0 = N/2$, i.e., $r = 1/2$, $\mu_g = 1$, $\sigma_g^2 = 1$ and $\sigma_v^2 = 1$). As can be seen, the step amplitude CRLBs converge quickly to finite values (the asymptotic step change CRLBs), as

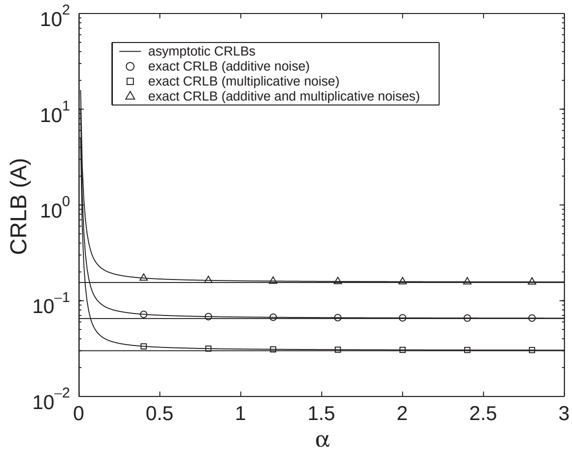

Fig. 1. CRLB(A) versus rise-time parameter (log-scale for the Y-axis).

α increases. To explain this result, we have studied the behaviour of the FIM elements (given in Appendix A.2) when $\alpha \to \infty$. Eq. (A.14) shows that the asymptotic equivalent FIM can be partitioned as follows:

$$I(\alpha) = \left(\begin{array}{cccc|c} & & & & I_{\mu_g} t_0(\alpha) \\ & I^*(\alpha) & & & I_{\sigma_g^2 t_0}(\alpha) \\ & & & & I_{\sigma_v^2 t_0}(\alpha) \\ & & & & I_{At_0}(\alpha) \\ \hline I_{\mu_g t_0}(\alpha) & I_{\sigma_g^2 t_0}(\alpha) & I_{\sigma_v^2 t_0}(\alpha) & I_{At_0}(\alpha) & I_{t_0 t_0}(\alpha) \end{array} \right). \tag{13}$$

The limit of the upper left 4×4 FIM submatrix $I^*(\alpha)$ (corresponding to known t_0) is the non-singular matrix $I^*(\infty)$ computed in (5) with $\det(I^*(\infty)) \neq 0$. In Appendix A.3 we have derived closed-form expressions for the asymptotic equivalent terms for $I_{\mu_g t_0}(\alpha)$, $I_{\sigma_g^2 t_0}(\alpha), I_{\sigma_v^2 t_0}, I_{At_0}(\alpha)$ and $I_{t_0 t_0}(\alpha)$ when $\alpha \to \infty$. These asymptotic equivalent terms lead to the following results:

- for $\tau \neq 0$,

$$\lim_{\alpha\to\infty} I_{\mu_g t_0}(\alpha) = \lim_{\alpha\to\infty} I_{\sigma_g^2 t_0} = \lim_{\alpha\to\infty} I_{\sigma_v^2 t_0}(\alpha) = \lim_{\alpha\to\infty} I_{At_0}(\alpha) = \lim_{\alpha\to\infty} I_{t_0 t_0}(\alpha) = 0, \tag{14}$$

- for $\tau = 0$,

$$\lim_{\alpha\to\infty} I_{\mu_g t_0}(\alpha) = \lim_{\alpha\to\infty} I_{\sigma_g^2 t_0} = \lim_{\alpha\to\infty} I_{\sigma_v^2 t_0}(\alpha) = \lim_{\alpha\to\infty} I_{At_0}(\alpha) = \lim_{\alpha\to\infty} I_{t_0 t_0}(\alpha) = \infty. \tag{15}$$

These equations show that, for any value of τ, the ratio between the minor associated with the step amplitude and the determinant of $I(\alpha)$ is indeterminate ($\frac{0}{0}$ or $\frac{\infty}{\infty}$) for $\alpha=\infty$. A possibility for removing the indetermination consists of inverting the asymptotic equivalent FIM (derived in Eq. (A.14) of Appendix A.2) and setting $\alpha=\infty$. Straightforward computations lead to the closed-form expressions for asymptotic step amplitude CRLBs derived in Appendix A.4. These expressions are different, depending on the value of the unknown parameter τ.

However, by assuming that r and $1-r$ are bounded (i.e., the changepoint is not too close from the edges of the observation window), the asymptotic step change CRLBs for pure multiplicative noise, pure additive noise and both noises (denoted $\mathrm{CRLB}_m^\infty(A)$, $\mathrm{CRLB}_a^\infty(A)$ and $\mathrm{CRLB}_{am}^\infty(A)$), respectively satisfy the same relations

(Mathematica files are available in [7]):

$$\mathrm{CRLB}_m^{\infty}(A) = \mathrm{CRLB}_m(A)\left(1 + \mathrm{O}\left(\frac{1}{N}\right)\right), \tag{16}$$

$$\mathrm{CRLB}_a^{\infty}(A) = \mathrm{CRLB}_a(A)\left(1 + \mathrm{O}\left(\frac{1}{N}\right)\right), \tag{17}$$

$$\mathrm{CRLB}_{am}^{\infty}(A) = \mathrm{CRLB}_{am}(A)\left(1 + \mathrm{O}\left(\frac{1}{N}\right)\right), \tag{18}$$

where $N\mathrm{O}(1/N)$ remains bounded when $N \to \infty$. These equations show that the step-amplitude CRLBs for an unknown change-point location can be approximated by the step-amplitude CRLBs obtained for a known change-point location (derived in Section 3), for large values of α and N. Consequently, the simple step amplitude closed-form expressions derived in Section 3 (see (7)–(9)) can be used to approximate step amplitude CRLBs for steplike signals with unknown change-point locations. This property is illustrated in Fig. 1, which displays the exact and asymptotic step amplitude CRLBs as a function of α (the true parameters are $N = 100$, $T = 1$, $t_0 = N/2$, $A = 0.5$, $\mu_g = 1$, $\sigma_g^2 = 1$ and $\sigma_v^2 = 1$). Fig 1 also provides the convergence rate of step amplitude CRLBs to their asymptotic values when α increases. It is important to note that, for large values of α, the step-amplitude CRLBs do not depend on τ, contrary to the change-point location CRLBs.

4.3. *Asymptotic step location CRLBs* ($N \to \infty$)

For a finite value of α, the step location CRLB is defined by

$$\mathrm{CRLB}(t_0) = \frac{\det I^*(\alpha)}{\det I(\alpha)}. \tag{19}$$

When $\alpha \to \infty$, there is loss of identifiability regarding the change-point parameter t_0 (for $\alpha = \infty$, the same signal is obtained for any value of t_0 in $[n_0, n_0+1[$), which results in a singular FIM. By using (6), (14) and (15), the following results can be obtained:

- $\tau \neq 0$

$$\lim_{\alpha\to\infty} \mathrm{CRLB}_a^{\alpha}(t_0) = \lim_{\alpha\to\infty} \mathrm{CRLB}_m^{\alpha}(t_0) = \lim_{\alpha\to\infty} \mathrm{CRLB}_{am}^{\alpha}(t_0) = \infty, \tag{20}$$

- $\tau = 0$

$$\lim_{\alpha\to\infty} \mathrm{CRLB}_a^{\alpha}(t_0) = \lim_{\alpha\to\infty} \mathrm{CRLB}_m^{\alpha}(t_0) = \lim_{\alpha\to\infty} \mathrm{CRLB}_{am}^{\alpha}(t_0) = 0. \tag{21}$$

As can be seen, the step location CRLBs (obtained in the presence of additive and/or multiplicative noise) converge to ∞ or 0, depending on the value of τ (note that the results obtained for τ=0 have been emphasized in [1], in the case of a pure additive noise). Consequently, the behaviour of the step-location CRLBs when $\alpha \to \infty$ differs from that of the step-amplitude CRLBs which converge to positive constants when $\alpha \to \infty$ (see (16)–(18)).

Of course, for a finite rise-time parameter α, the step-location CRLBs can be computed from the FIM of the unknown parameter vector θ derived in Appendix A.2. However, these expressions are not simple to study. Instead this section shows that the step-location CRLBs in the presence of additive and/or multiplicative noise are approximately decoupled from the other CRLBs, for large values of N. This result leads to interesting approximations of the step location CRLBs, which are valid for large values of N.

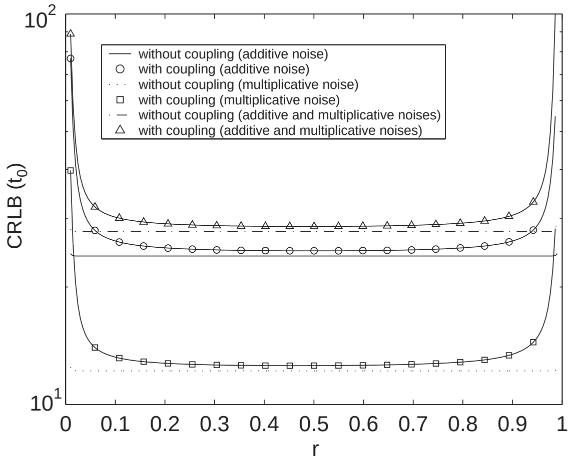

Fig. 2. Step-change CRLBs (log-scale for the Y-axis).

For this, it is important to note that $s_\alpha(i)$, $f_\alpha(i)$ and $\frac{\partial s_\alpha(i)}{\partial A}$ are bounded as follows

$$1 \leqslant s_\alpha(i) \leqslant 1 + A, \quad \sigma_v^2 + \sigma_g^2 \leqslant f_\alpha(i) \leqslant \sigma_v^2 + \sigma_g^2(1+A)^2, \quad 0 \leqslant \frac{\partial s_\alpha(i)}{\partial A} \leqslant 1. \tag{22}$$

From these inequalities, it follows that all terms of $I^*(\alpha)$ are of the form $N\mathrm{O}(1)$. Similarly, it can be shown that $I_{\mu_g t_0}(\alpha)$, $I_{\sigma_g^2 t_0}(\alpha)$, $I_{\sigma_v^2 t_0}(\alpha)$, $I_{At_0}(\alpha)$ and $I_{t_0t_0}(\alpha)$ are of the form $\mathrm{O}(1)$. By expanding $\det I(\alpha)$ with respect to its last column, the asymptotic step location CRLBs computed for a finite rise-time parameter α denoted $\mathrm{CRLB}^\alpha(t_0)$ can be shown to satisfy the following relation

$$\lim_{N\to\infty} \frac{\mathrm{CRLB}^\alpha(t_0)}{1/I_{t_0t_0}(\alpha)} = 1. \tag{23}$$

In other words, the step location is decoupled from the other parameters, when $N \to \infty$. Consequently, the step-location CRLBs, in the presence of additive and/or multiplicative noise can be approximated by the decoupled CRLBs for large values of N. This result is illustrated in Fig. 2, which shows a comparison between the step-location CRLBs obtained by assuming coupling and decoupling (in the decoupling case $\mathrm{CRLB}^\alpha(t_0) = 1/I_{t_0t_0}(\alpha)$) with the following parameters $N = 200$, $T = 1$, $t_0 = N/2$, i.e., $r = 1/2$, $A = 0.5$, $\mu_g = 1$, $\sigma_g^2 = 1$ and $\sigma_v^2 = 1$. Both bounds are clearly similar when the changepoint is not too close to the edges of observation window. When decoupling is a reasonable assumption, the change-point location CRLBs can be expressed by the following closed-form expressions [20–22]:

Additive noise

$$\mathrm{CRLB}_a^\alpha(t_0) \simeq \frac{16}{\alpha^2 A^2} \left(\frac{\mu_g^2}{\sigma_v^2} \sum_{k=1}^{N} \cosh^{-4}(\alpha(kT - t_0)/2) \right)^{-1}. \tag{24}$$

Multiplicative noise

$$\mathrm{CRLB}_m^\alpha(t_0) \simeq \frac{16}{\alpha^2 A^2} \left(\frac{2\sigma_g^2 + \mu_g^2}{\sigma_g^2} \sum_{k=1}^{N} \frac{1}{s_\alpha^2(k)} \cosh^{-4}(\alpha(kT - t_0)/2) \right)^{-1}. \tag{25}$$

Additive and multiplicative noise

$$\mathrm{CRLB}^{\alpha}_{am}(t_0) \simeq \frac{16}{\alpha^2 A^2}\left(\frac{2\sigma_g^2+\mu_g^2}{\sigma_v^2}\sum_{k=1}^{N}\left(1+\frac{\sigma_g^2}{\sigma_v^2}s_\alpha^2(k)\right)^{-1}\cosh^{-4}(\alpha(kT-t_0)/2)\right.$$

$$\left.-2\frac{\sigma_g^2}{\sigma_v^2}\sum_{k=1}^{N}\left(1+\frac{\sigma_g^2}{\sigma_v^2}s_\alpha^2(k)\right)^{-2}\cosh^{-4}(\alpha(kT-t_0)/2)\right)^{-1}. \tag{26}$$

5. Model parameter estimation

This section addresses the problem of comparing the CRLBs computed in the previous sections to the MSEs of the maximum likelihood estimates (MLEs) for the pure multiplicative noise case ($g(t)$ is iid Gaussian and $\sigma_v^2=0$). The other cases (pure additive noise or additive and multiplicative noises) could be studied similarly.

The vector of unknown parameters is $\theta=[\mu_g,\sigma_g^2,A,t_0]^{\mathrm{T}}$. The MLE of θ has been derived in [9] and [23]. The maximization of the Gaussian likelihood function of $y=[y(1),\ldots,y(N)]^{\mathrm{T}}$ with respect to θ reduces to the maximization of the following criterion with respect to the signal parameters (A,t_0):

$$U(A,t_0;y)=-\sum_{i=1}^{N}\ln s_\alpha(i)-\frac{N}{2}\ln\left(\sum_{i=1}^{N}\left(\frac{y(i)}{s_\alpha(i)}-\frac{1}{N}\sum_{i=1}^{N}\frac{y(i)}{s_\alpha(i)}\right)^2\right). \tag{27}$$

Moreover, the MLEs of the multiplicative noise parameters are the conventional mean and variance estimators for the vector $[\frac{y(1)}{s_\alpha(1)},\ldots,\frac{y(N)}{s_\alpha(N)}]^{\mathrm{T}}$:

$$\hat{\mu}_g=\frac{1}{N}\sum_{i=1}^{N}\frac{y(i)}{s_\alpha(i)},\quad \hat{\sigma}_g^2=\frac{1}{N}\sum_{i=1}^{N}\left(\frac{y(i)}{s_\alpha(i)}-\hat{\mu}_g\right)^2. \tag{28}$$

We propose to optimize the cost function $U(A,t_0;y)$ with respect to A and t_0/N (as in [9]) by using the BFGS quasi-Newton algorithm with the following constraints $0<t_0/N<1$ and $0<A<A_m$, where A_m is an upper bound on the step-change parameter A ($A_m=10$ in our simulations). Once the contrast function has been optimized, the ML estimates of the noise parameters are computed as in (28). Figs. 3–6 display the MSEs of the ML estimates for the signal and noise parameters and the corresponding exact and asymptotic CRLBs (see (7) and (26)) as a function of the number of samples. The MSEs have been computed from 1000 Monte Carlo runs with the following signal and noise parameters: $\mu_g=1$, $\sigma_g^2=1$, $A=1$ and $t_0/N=\frac{1}{2}$. The shape parameter for the sigmoidal function is $\alpha=1$. These simulations illustrate the convergence of the ML estimator in the context of abrupt changes corrupted by multiplicative noise (which has been studied for instance in [12] and [5]) and its asymptotic efficiency.

6. Extension to correlated multiplicative noise

An interesting question is the following: "Are the previous results regarding asymptotic step amplitude and location CRLBs still valid when the multiplicative noise is correlated?" To answer this question, assume that the multiplicative noise can be modeled as $g(n)=\mu_g+\tilde{g}(n)$, where $\tilde{g}(n)$ is a zero-mean Gaussian stationary AR(p) process with parameters σ_g^2 and $a=[a_1,\ldots,a_p]^{\mathrm{T}}$ ($a_0=1$). Since the observation vector

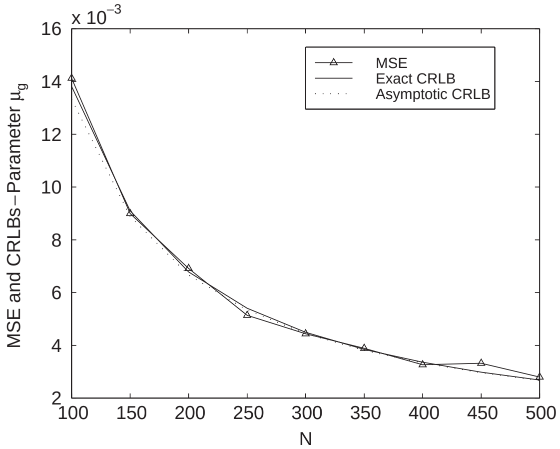

Fig. 3. CRLBs and MSE for parameter μ_g (pure multiplicative noise).

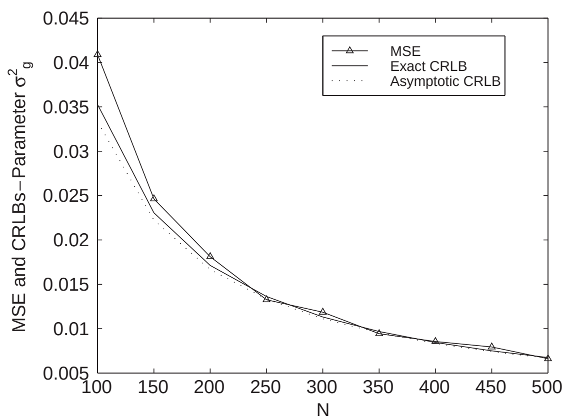

Fig. 4. CRLBs and MSE for parameter σ_g^2 (pure multiplicative noise).

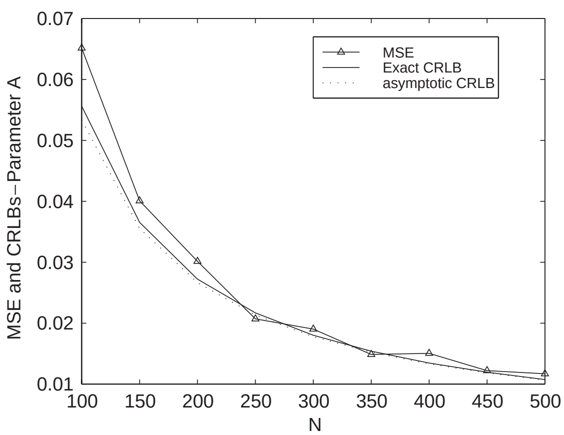

Fig. 5. CRLBs and MSE for parameter A (pure multiplicative noise).

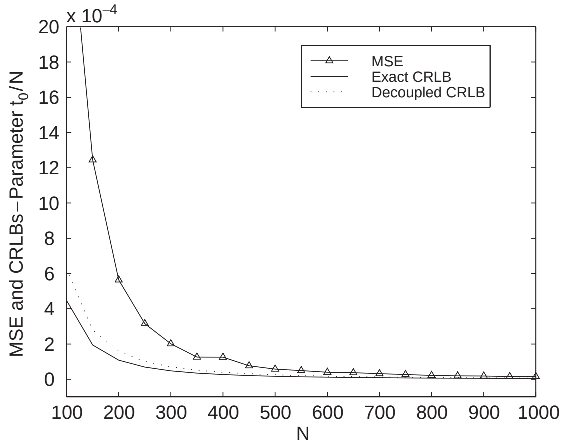

Fig. 6. CRLB and MSE for parameter t_0/N (pure multiplicative noise).

$y = [y(1), \ldots, y(N)]^{\mathrm{T}}$ is Gaussian, many equivalent expressions for the FIM of θ have been proposed in the literature. We use the following expression [18, p. 289]:

$$[I(\alpha)]_{kl} = \frac{1}{2}\,\mathrm{tr}\left(R_y^{-1}(\theta)\frac{\partial R_y(\theta)}{\partial\theta_k}R_y^{-1}(\theta)\frac{\partial R_y(\theta)}{\partial\theta_l}\right) + \left(\frac{\partial M_y(\theta)}{\partial\theta_k}\right)^{\mathrm{T}} R_y^{-1}(\theta)\frac{\partial M_y(\theta)}{\partial\theta_l}, \tag{29}$$

where $\mathrm{tr}\,(.)$ denotes the trace operator and $M_y(\theta)$, $R_y(\theta)$ are the mean and the $N \times N$ covariance matrix of the observation vector y. As can be seen in (29), the FIM of θ can be determined, provided the partial derivatives of $M_y(\theta)$ and $R_y(\theta)$ with respect to the unknown parameters can be computed. We can readily express $M_y(\theta)$

and $R_y(\theta)$ as a function of the unknown signal and noise parameters. The mean of the observation vector is

$$M_y(\theta) = \mu_g S_\alpha = \mu_g [s_\alpha(1), \ldots, s_\alpha(N)]^{\mathrm{T}}. \tag{30}$$

The inverse covariance matrix of the multiplicative noise can be expressed as a function of the model parameters σ_g^2 and a with the Gohberg–Semencul formula [18, p. 125]:

$$R_g^{-1}(\theta) = \frac{1}{\sigma_g^2}(FF^{\mathrm{T}} - GG^{\mathrm{T}}), \tag{31}$$

where $F = (f_{ij})$ and $G = (g_{ij})$ are the $N \times N$ lower triangular matrices defined by

$$f_{ij} = \begin{cases} 1 & \text{if } i = j \\ a_{i-j} & \text{if } i > j \\ 0 & \text{if } i < j \end{cases} \quad \text{and} \quad g_{ij} = \begin{cases} a_{p-i+j} & \text{if } i \geqslant j, \\ 0 & \text{if } i < j, \end{cases} \tag{32}$$

$a_0 = 1$ and $a_i = 0$ for $i > p$ and $i < 0$. Consequently, $R_y(\theta)$ is expressed as follows:

$$R_y(\theta) = \sigma_g^2 D_\alpha (FF^{\mathrm{T}} - GG^{\mathrm{T}})^{-1} D_\alpha + \sigma_v^2 I_N, \tag{33}$$

where I_N is the $N \times N$ identity matrix and $D_\alpha = \mathrm{diag}(s_\alpha(1), \ldots, s_\alpha(N))$ is the diagonal matrix whose elements are $s_\alpha(i)$, $i = 1, \ldots, N$. The derivatives of $M_y(\theta)$ and $R_y(\theta)$ with respect to the unknown parameters (computed from (30) and (33)) are summarized in Appendix A.5. From these derivatives and (29), one can compute the unknown parameter FIMs and CRLBs.

We again focus on the pure multiplicative colored noise case ($\tilde{g}(n)$ is a zero-mean AR(p) Gaussian process and $\sigma_v^2 = 0$). The unknown parameter vector is $\theta = [a, \mu_g, \sigma_g^2, A, t_0]^{\mathrm{T}}$. The estimation of deterministic signals corrupted by pure multiplicative autoregressive noise (referred to as amplitude-modulated signals) has been intensively studied in [9]. In particular, by expressing the quadratic form in the observations, appearing in the likelihood function, as a quadratic form in the AR parameters, Ghogho and Garel derived an interesting expression for the FIM (see proposition 1 of [9]). In this expression, many FIM entries involve matrices of size $(p+1) \times (p+1)$ instead of matrices of size $N \times N$ (as those given in [8]), which results in better conditioned FIMs. Unfortunately, closed-form expressions of the FIM inverses are difficult to obtain, which prevents any theoretical analysis. This section shows via a typical example that the asymptotic step amplitude ($\alpha \to \infty, N \to \infty$) and asymptotic decoupled step location CRLBs ($N \to \infty$) provide good approximations for the exact-step amplitude and location CRLBs.

6.1. Asymptotic step amplitude CRLBs ($\alpha \to \infty, N \to \infty$)

Fig. 7 displays the step amplitude CRLB as a function of α, in the presence of pure multiplicative colored noise (the parameters are $N = 200$, $T = 1$, $t_0 = N/2$, $A = 0.5$, $\mu_g = 1$, $\sigma_g^2 = 1$, $a = [-0.2, 0.153]^{\mathrm{T}}$ and $\sigma_v^2 = 1$). As can be seen, the step amplitude CRLB converges quickly to its asymptotic values, when α increases.

6.2. Asymptotic step location CRLBs ($N \to \infty$)

Looking carefully at proposition 1 of [9], we can see that some terms related to AR parameters such as $I_{a_i \sigma_v^2}$ are independent of N. Consequently, the step location parameter is not asymptotically decoupled from the other parameters, as in the case of white multiplicative noise. Fig. 8 displays the step location CRLBs in the presence of pure colored multiplicative noise, by assuming coupling and decoupling (the parameters are $N = 200$, $T = 1$, $t_0 = N/2$, $A = 0.5$, $\mu_g = 1$, $\sigma_g^2 = 1$, $a = [-0.2, 0.153]^{\mathrm{T}}$ and $\sigma_v^2 = 1$). This figure shows that (despite coupling between the step location and the other parameters) the asymptotic decoupled step-location CRLBs provide a good approximation for the exact step-location CRLBs.

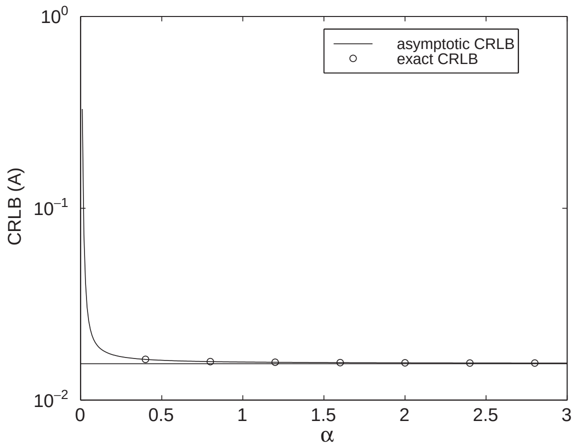

Fig. 7. Step-amplitude CRLBs (log-scale for the Y-axis, multiplicative colored noise).

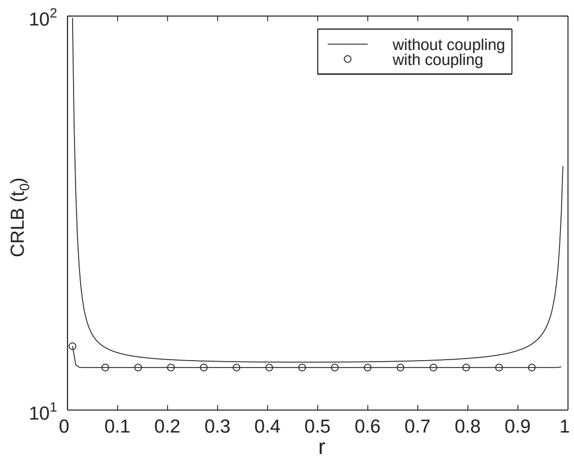

Fig. 8. Step-amplitude CRLBs (log-scale for the Y-axis, multiplicative colored noise).

7. Conclusions

This paper has derived and studied interesting CRLB closed-form expressions for the parameters of change points corrupted by additive and multiplicative white Gaussian noises:

- *The step change* and *noise parameter* CRLBs for an unknown change location can be approximated accurately with the asymptotic ($\alpha \to \infty, N \to \infty$) bounds computed for a known-step location,
- *The step location* parameter is asymptotically ($N \to \infty$) decoupled from the step change and noise parameters. Consequently, the CRLBs for this parameter can be computed as if the other parameters were known, yielding simpler closed-form expressions.

The CRLBs for the change-point and noise parameters have been compared with the MSEs of the corresponding maximum likelihood estimators. This comparison has illustrated the well-known convergence and asymptotic efficiency of this estimator. Most results have been obtained by assuming that the multiplicative noise is white and Gaussian. However, extensions to colored multiplicative noise were also discussed.

Appendix A.

A.1. Regularity conditions for Cramér–Rao inequality

Suppose that $\hat{\theta}(y)$ is an unbiased estimate of the parameter vector $\theta \in \Lambda$, where Λ is an open interval referred to as the parameter space. Denote as $p_y(\theta)$ the probability density function of the observation vector $y = [y(1), \ldots, y(N)]^{\mathrm{T}} \in \Gamma$, where Γ is the observation space. The regularity conditions for the Cramér–Rao inequality can be defined as follows [14, p. 169]:

1. $\frac{\partial p_y(\theta)}{\partial \theta}$ exists and is finite for all $\theta \in \Lambda$ and all y in the support of $p_y(\theta)$.
2. $\int_\Gamma \frac{\partial p_y(\theta)}{\partial \theta} \,\mathrm{d}y = \frac{\partial}{\partial \theta} \int_\Gamma p_y(\theta)\,\mathrm{d}y = 0$, for all $\theta \in \Lambda$.
3. $\frac{\partial^2 p_y(\theta)}{\partial \theta^2}$ exists and is finite for all $\theta \in \Lambda$ and all y in the support of $p_y(\theta)$.
4. $\int_\Gamma \frac{\partial^2 p_y(\theta)}{\partial \theta^2} \,\mathrm{d}y = \frac{\partial^2}{\partial \theta^2} \int_\Gamma p_y(\theta)\,\mathrm{d}y = 0$, for all $\theta \in \Lambda$.

When these conditions are satisfied, the variance of any unbiased estimate $\hat{\theta}(y)$ such that $\int_\Gamma \hat{\theta}(y) \frac{\partial p_y(\theta)}{\partial \theta}\,\mathrm{d}y = \frac{\partial}{\partial \theta} \int_\Gamma \hat{\theta}(y) p_y(\theta)\,\mathrm{d}y$ satisfies the Cramér–Rao inequality:

$$Var[\hat{\theta}(y)] \geqslant I_\theta^{-1}, \tag{A.1}$$

where I_θ is the Fisher information matrix of θ. This appendix explains how the previous regularity conditions can be verified for model (1) in the case of a known change location t_0. In this case, the unknown parameter vector is $\theta = [\mu_g, \sigma_g^2, \sigma_v^2, A]^{\mathrm{T}}$. Conditions (1) and (3) are obviously satisfied since $p_y(\theta)$ is the pdf of the Gaussian distribution with mean $M_y(\theta)$ and covariance matrix $R_y(\theta)$. To verify Conditions (2) and (4) for $\theta_i = \theta_j = \mu_g$, we have to differentiate $p_y(\theta) = \exp(L_y(\theta))$ with respect to μ_g:

$$\frac{\partial p_y(\theta)}{\partial \mu_g} = p_y(\theta) \sum_{i=1}^{N} \frac{s_\infty(i)[y(i) - \mu_g s_\infty(i)]}{\sigma_g^2 s_\infty^2(i) + \sigma_v^2}, \tag{A.2}$$

$$\frac{\partial^2 p_y(\theta)}{\partial \mu_g^2} = p_y(\theta) \left(\sum_{i=1}^{N} \frac{s_\infty(i)[y(i) - \mu_g s_\infty(i)]}{\sigma_g^2 s_\infty^2(i) + \sigma_v^2} \right)^2 - p_y(\theta) \sum_{i=1}^{N} \frac{s_\infty^2(i)}{\sigma_g^2 s_\infty^2(i) + \sigma_v^2}. \tag{A.3}$$

Since $E[y(i)] = \mu_g s_\infty(i)$ and $\operatorname{cov}(y(i), y(j)) = E[(y(i) - \mu_g s_\infty(i))(y(j) - \mu_g s_\infty(j))] = 0$ for $i \neq j$, the following results are obtained:

$$\int_\Gamma \frac{\partial p_y(\theta)}{\partial \mu_g}\,\mathrm{d}y = 0, \tag{A.4}$$

$$\int_\Gamma \frac{\partial^2 p_y(\theta)}{\partial \mu_g^2}\,\mathrm{d}y = 0. \tag{A.5}$$

Similar results can be easily obtained for any $(\theta_i, \theta_j) \in \{\mu_g, \sigma_g^2, \sigma_v^2, A\}^2$ which allow to show that all the regularity conditions are verified for model (1) in the case of a known change location t_0.

A.2. FIM for $\theta=[\mu_g,\sigma_g^2,\sigma_v^2,A,t_0]^{\mathrm{T}}$ in the presence of additive and multiplicative iid Gaussian noises (unknown change-point location)

$$I(\alpha)=\begin{pmatrix} I_{\mu_g\mu_g} & 0 & 0 & I_{\mu_g A} & I_{\mu_g t_0} \\ 0 & I_{\sigma_g^2\sigma_g^2} & I_{\sigma_g^2\sigma_v^2} & I_{\sigma_g^2 A} & I_{\sigma_g^2 t_0} \\ 0 & I_{\sigma_v^2\sigma_g^2} & I_{\sigma_v^2\sigma_v^2} & I_{\sigma_v^2 A} & I_{\sigma_v^2 t_0} \\ I_{A\mu_g} & I_{A\sigma_g^2} & I_{A\sigma_v^2} & I_{AA} & I_{At_0} \\ I_{t_0\mu_g} & I_{t_0\sigma_g^2} & I_{t_0\sigma_v^2} & I_{t_0 A} & I_{t_0 t_0} \end{pmatrix}, \tag{A.6}$$

where $f_\alpha(i)=\sigma_v^2+\sigma_g^2 s_\alpha^2(i)$ and

$$I_{\mu_g\mu_g}=\sum_{i=1}^{N}\frac{s_\alpha^2(i)}{f_\alpha(i)}, \quad I_{\mu_g A}=I_{A\mu_g}=\mu_g\sum_{i=1}^{N}\frac{s_\alpha(i)}{f_\alpha(i)}\frac{\partial s_\alpha(i)}{\partial A}, \quad I_{\sigma_g^2\sigma_g^2}=\frac{1}{2}\sum_{i=1}^{N}\frac{s_\alpha^4(i)}{f_\alpha^2(i)}, \tag{A.7}$$

$$I_{\sigma_g^2\sigma_v^2}=I_{\sigma_v^2\sigma_g^2}=\frac{1}{2}\sum_{i=1}^{N}\frac{s_\alpha^2(i)}{f_\alpha^2(i)}, \quad I_{\sigma_g^2 A}=I_{A\sigma_g^2}=\sigma_g^2\sum_{i=1}^{N}\frac{s_\alpha^3(i)\frac{\partial s_\alpha(i)}{\partial A}}{f_\alpha^2(i)}, \tag{A.8}$$

$$I_{\sigma_v^2 A}=I_{A\sigma_v^2}=\sigma_g^2\sum_{i=1}^{N}\frac{s_\alpha(i)\frac{\partial s_\alpha(i)}{\partial A}}{f_\alpha^2(i)}, \quad I_{\sigma_v^2 t_0}=I_{t_0\sigma_v^2}=\sigma_g^2\sum_{i=1}^{N}\frac{s_\alpha(i)\frac{\partial s_\alpha(i)}{\partial t_0}}{f_\alpha^2(i)}, \tag{A.9}$$

$$I_{\sigma_v^2\sigma_v^2}=\frac{1}{2}\sum_{i=1}^{N}\frac{1}{f_\alpha^2(i)}, \quad I_{AA}=2\sigma_g^4\sum_{i=1}^{N}\frac{s_\alpha^2(i)(\frac{\partial s_\alpha(i)}{\partial A})^2}{f_\alpha^2(i)}+\mu_g^2\sum_{i=1}^{N}\frac{(\frac{\partial s_\alpha(i)}{\partial A})^2}{f_\alpha(i)}, \tag{A.10}$$

$$I_{t_0\mu_g}=I_{\mu_g t_0}=\mu_g\sum_{i=1}^{N}\frac{s_\alpha(i)}{f_\alpha(i)}\frac{\partial s_\alpha(i)}{\partial t_0}, \quad I_{t_0\sigma_g^2}=I_{\sigma_g^2 t_0}=\sigma_g^2\sum_{i=1}^{N}\frac{s_\alpha^3(i)\frac{\partial s_\alpha(i)}{\partial t_0}}{f_\alpha^2(i)}, \tag{A.11}$$

$$I_{t_0 A}=I_{At_0}=2\sigma_g^4\sum_{i=1}^{N}\frac{s_\alpha^2(i)\frac{\partial s_\alpha(i)}{\partial A}\frac{\partial s_\alpha(i)}{\partial t_0}}{f_\alpha^2(i)}+\mu_g^2\sum_{i=1}^{N}\frac{\frac{\partial s_\alpha(i)}{\partial A}\frac{\partial s_\alpha(i)}{\partial t_0}}{f_\alpha(i)}, \tag{A.12}$$

$$I_{t_0 t_0}=2\sigma_g^4\sum_{i=1}^{N}\frac{s_\alpha^2(i)(\frac{\partial s_\alpha(i)}{\partial t_0})^2}{f_\alpha^2(i)}+\mu_g^2\sum_{i=1}^{N}\frac{(\frac{\partial s_\alpha(i)}{\partial t_0})^2}{f_\alpha(i)}. \tag{A.13}$$

As explained in [19], I_{AA}, I_{At_0} and $I_{t_0t_0}$ consist of two terms: the first term disappears if the multiplicative noise is zero mean, and the second term disappears if the multiplicative noise has zero variance.

A.3. Asymptotic equivalent FIM for $\theta=[\mu_g,\sigma_g^2,\sigma_v^2,A]^{\mathrm{T}}$ when $\alpha\to\infty$ (unknown change-point location)

$$\tilde{I}(\alpha)=\begin{pmatrix} \tilde{I}_{\mu_g\mu_g} & 0 & 0 & \tilde{I}_{\mu_g A} & \tilde{I}_{\mu_g t_0} \\ 0 & \tilde{I}_{\sigma_g^2\sigma_g^2} & \tilde{I}_{\sigma_g^2\sigma_v^2} & \tilde{I}_{\sigma_g^2 A} & \tilde{I}_{\sigma_g^2 t_0} \\ 0 & \tilde{I}_{\sigma_v^2\sigma_g^2} & \tilde{I}_{\sigma_v^2\sigma_v^2} & \tilde{I}_{\sigma_v^2 A} & \tilde{I}_{\sigma_v^2 t_0} \\ \tilde{I}_{A\mu_g} & \tilde{I}_{A\sigma_g^2} & \tilde{I}_{A\sigma_v^2} & \tilde{I}_{AA} & \tilde{I}_{At_0} \\ \tilde{I}_{t_0\mu_g} & \tilde{I}_{t_0\sigma_g^2} & \tilde{I}_{t_0\sigma_v^2} & \tilde{I}_{t_0 A} & I_{t_0 t_0} \end{pmatrix}, \tag{A.14}$$

where

$$\tilde{I}_{\mu_g\mu_g} = \frac{Nr}{\sigma_b^2} + \frac{N(1-r)(1+A)^2}{\sigma_a^2}, \quad \tilde{I}_{\mu_g A} = \frac{N\mu_g(1-r)(1+A)}{\sigma_a^2}, \tag{A.15}$$

$$\tilde{I}_{\sigma_g^2\sigma_g^2} = \frac{Nr}{2\sigma_b^4} + \frac{N(1-r)(1+A)^4}{2\sigma_a^4}, \quad \tilde{I}_{\sigma_g^2\sigma_v^2} = \frac{Nr}{2\sigma_b^4} + \frac{N(1-r)(1+A)^2}{2\sigma_a^4}, \tag{A.16}$$

$$\tilde{I}_{\sigma_g^2 A} = \frac{N(1-r)\sigma_g^2(1+A)^3}{\sigma_a^4}, \quad \tilde{I}_{\sigma_v^2\sigma_v^2} = \frac{Nr}{2\sigma_b^4} + \frac{N(1-r)}{2\sigma_a^4}, \tag{A.17}$$

$$\tilde{I}_{\sigma_g^2 A} = \frac{N(1-r)\sigma_g^2(1+A)}{\sigma_a^4}, \quad \tilde{I}_{AA} = \frac{N\mu_g^2(1-r)}{\sigma_a^4} + 2\frac{N\sigma_g^4(1-r)(1+A)^2}{\sigma_a^4}, \tag{A.18}$$

$$\tilde{I}_{\mu_g t_0} = -\mu_g\left(\frac{A\alpha}{\sigma_b^2}\,\mathrm{e}^{-\alpha\tau} + \frac{\alpha A(1+A)}{\sigma_a^2}\,\mathrm{e}^{\alpha(\tau-1)}\right), \tag{A.19}$$

$$\tilde{I}_{\sigma_g^2 t_0} = -\sigma_g^2\left(\frac{A\alpha}{\sigma_b^4}\,\mathrm{e}^{-\alpha\tau} + \frac{\alpha A(1+A)^3}{\sigma_a^4}\,\mathrm{e}^{\alpha(\tau-1)}\right), \tag{A.20}$$

$$\tilde{I}_{\sigma_v^2 t_0} = -\sigma_g^2\left(\frac{A\alpha}{\sigma_b^4}\,\mathrm{e}^{-\alpha\tau} + \frac{\alpha A(1+A)}{\sigma_a^4}\,\mathrm{e}^{\alpha(\tau-1)}\right), \tag{A.21}$$

$$\tilde{I}_{At_0} = -2\sigma_g^4\left(\frac{A\alpha}{\sigma_b^4}\,\mathrm{e}^{-2\alpha\tau} + \frac{\alpha A(1+A)^2}{\sigma_a^4}\,\mathrm{e}^{\alpha(\tau-1)}\right) - \mu_g^2\left(\frac{A\alpha}{\sigma_b^2}\,\mathrm{e}^{-2\alpha\tau} + \frac{\alpha A}{\sigma_a^2}\,\mathrm{e}^{\alpha(\tau-1)}\right), \tag{A.22}$$

$$\tilde{I}_{t_0t_0} = 2\sigma_g^4\left(\frac{A^2\alpha^2}{\sigma_b^4}\,\mathrm{e}^{-2\alpha\tau} + \frac{\alpha^2A^2(1+A)^4}{\sigma_a^4}\,\mathrm{e}^{2\alpha(\tau-1)}\right) + \mu_g^2\left(\frac{A^2\alpha^2}{\sigma_b^2}\,\mathrm{e}^{-2\alpha\tau} + \frac{\alpha^2A^2}{\sigma_a^2}\,\mathrm{e}^{2\alpha(\tau-1)}\right). \tag{A.23}$$

A.4. Asymptotic step-amplitude CRLBs (unknown change-point location)

Straightforward computations yield the following asymptotic step-amplitude CRLBs:

Additive noise

$\tau > 0.5$

$$\mathrm{CRLB}_a^\infty(A) = \mathrm{CRLB}_a(A)\,\frac{1-\frac{1}{N}\,\frac{1}{(1+A)^2-rA(2+A)}}{1-\frac{1}{Nr}}, \tag{A.24}$$

$\tau < 0.5$

$$\mathrm{CRLB}_a^\infty(A) = \mathrm{CRLB}_a(A)\,\frac{1-\frac{1}{N}\,\frac{(A+1)^2}{(1+A)^2-rA(2+A)}}{1+\frac{1}{N(r-1)}}. \tag{A.25}$$

Multiplicative noise

$\tau > 0.5$

$$\mathrm{CRLB}_m^\infty(A) = \mathrm{CRLB}_m(A)\,\frac{N-1}{N}\,\frac{1-r}{1-r-1/N}, \tag{A.26}$$

$\tau < 0.5$

$$\mathrm{CRLB}_m^\infty(A) = \mathrm{CRLB}_m(A)\,\frac{N-1}{N}\,\frac{r}{r-1/N}. \tag{A.27}$$

Additive and multiplicative noise
$\tau > 0.5$

$$\mathrm{CRLB}^{\infty}_{am}(A) = \mathrm{CRLB}^{\infty}_{am}(A)\left(1 + \frac{1}{1+N(r-1)}\frac{r\mu_g^2\sigma_a^4}{(2(1+A)^2\sigma_g^4 + \mu_g^2\sigma_a^2)((1+A)^2\sigma_b^2(r-1) - \sigma_a^2 r)}\right), \tag{A.28}$$

$\tau < 0.5$

$$\mathrm{CRLB}^{\infty}_{am}(A) = \mathrm{CRLB}^{\infty}_{am}(A)\left(1 + \frac{1}{Nr-1}\frac{(r-1)\mu_g^2\sigma_b^4(1+A)^2}{((1+A)^2\sigma_b^2(r-1) - r\sigma_a^2)(2\sigma_g^4 + \sigma_b^2\mu_g^2)}\right). \tag{A.29}$$

A.5. FIM of $\theta = [a, \mu_g, \sigma_g^2, \sigma_v^2, A, t_0]^{\mathrm{T}}$ (additive iid Gaussian noise, multiplicative correlated Gaussian noise and unknown change-point location)

Recall that $S_\alpha = [s_\alpha(1), \ldots, s_\alpha(N)]^{\mathrm{T}}$, $D_\alpha = \mathrm{diag}(s_\alpha(1), \ldots, s_\alpha(N))$ and that $M_y(\theta)$, $R_y(\theta)$ denote the mean and the covariance matrix of the observation vector $y = [y(1), \ldots, y(N)]^{\mathrm{T}}$. The FIM of $\theta = [a, \mu_g, \sigma_g^2, \sigma_v^2, A, t_0]^{\mathrm{T}}$, in the presence of additive white Gaussian noise and multiplicative correlated Gaussian noise is

$$I(\alpha) = \begin{pmatrix} I_{aa} & 0 & I_{a\sigma_g^2} & I_{a\sigma_v^2} & I_{aA} & I_{at_0} \\ 0 & I_{\mu_g\mu_g} & 0 & 0 & I_{\mu_g A} & I_{\mu_g t_0} \\ I_{\sigma_g^2 a} & 0 & I_{\sigma_g^2\sigma_g^2} & I_{\sigma_g^2\sigma_v^2} & I_{\sigma_g^2 A} & I_{\sigma_g^2 t_0} \\ I_{\sigma_v^2 a} & 0 & I_{\sigma_v^2\sigma_g^2} & I_{\sigma_v^2\sigma_v^2} & I_{\sigma_v^2 A} & I_{\sigma_v^2 t_0} \\ I_{Aa} & I_{A\mu_g} & I_{A\sigma_g^2} & I_{A\sigma_v^2} & I_{AA} & I_{At_0} \\ I_{t_0 a} & I_{t_0\mu_g} & I_{t_0\sigma_g^2} & I_{t_0\sigma_v^2} & I_{t_0 A} & I_{t_0 t_0} \end{pmatrix}, \tag{A.30}$$

where

$$[I(\alpha)]_{k,l} = \frac{1}{2}\, tr\left(R_y^{-1}(\theta)\frac{\partial R_y(\theta)}{\partial\theta_k}R_y^{-1}(\theta)\frac{\partial R_y(\theta)}{\partial\theta_l}\right) + \left(\frac{\partial M_y(\theta)}{\partial\theta_k}\right)^{\mathrm{T}} R_y^{-1}(\theta)\frac{\partial M_y(\theta)}{\partial\theta_l}. \tag{A.31}$$

The non-zero partial derivatives of $M_y(\theta)$ and $R_y(\theta)$ with respect to the unknown signal and noise parameters are:

$$\frac{\partial M(\theta)}{\partial\mu_g} = S_\alpha, \quad \frac{\partial M(\theta)}{\partial A} = \mu_g\frac{\partial S_\alpha}{\partial A}, \quad \frac{\partial M(\theta)}{\partial t_0} = \mu_g\frac{\partial S_\alpha}{\partial t_0}, \tag{A.32}$$

$$\frac{\partial R_y(\theta)}{\partial\mu_g} = S_\alpha^{\mathrm{T}}R_y^{-1}(\theta)S_\alpha, \quad \frac{\partial R_y(\theta)}{\partial\sigma_g^2} = D_\alpha(FF^{\mathrm{T}} - GG^{\mathrm{T}})^{-1}D_\alpha, \quad \frac{\partial R_y(\theta)}{\partial\sigma_v^2} = I, \tag{A.33}$$

$$\frac{\partial R_y(\theta)}{\partial A} = 2\sigma_g^2 D_\alpha(FF^{\mathrm{T}} - GG^{\mathrm{T}})^{-1}\frac{\partial D_\alpha}{\partial A}, \quad \frac{\partial R_y(\theta)}{\partial t_0} = 2\sigma_g^2 D_\alpha(FF^{\mathrm{T}} - GG^{\mathrm{T}})^{-1}\frac{\partial D_\alpha}{\partial t_0}, \tag{A.34}$$

$$\begin{aligned}\frac{\partial R_y(\theta)}{\partial a} &= \sigma_g^2 D_\alpha\frac{\partial(FF^{\mathrm{T}} - GG^{\mathrm{T}})^{-1}}{\partial a}D_\alpha, \\ &= -\sigma_g^2 D_\alpha(FF^{\mathrm{T}} - GG^{\mathrm{T}})^{-1}\frac{\partial(FF^{\mathrm{T}} - GG^{\mathrm{T}})}{\partial a}(FF^{\mathrm{T}} - GG^{\mathrm{T}})^{-1}D_\alpha.\end{aligned} \tag{A.35}$$

The partial derivatives of $FF^{\mathrm{T}} - GG^{\mathrm{T}}$ with respect to a_k have been derived in [8]:

$$\frac{\partial(FF^{\mathrm{T}} - GG^{\mathrm{T}})}{\partial a_k} = Z_k F^{\mathrm{T}} + FZ_k^{\mathrm{T}} - Z_{N-k}G^{\mathrm{T}} - GZ_{N-k}^{\mathrm{T}}, \tag{A.36}$$

where Z_k is the down-shift matrix

$$Z_k(i,j) = \begin{cases} 1 & \text{if } i-j=k, \\ 0 & \text{otherwise.} \end{cases} \tag{A.37}$$

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