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On Convergence of the Auxiliary-Vector Beamformer With Rank-Deficient Covariance Matrices

Olivier Besson, *Senior Member, IEEE*, Julien Montesinos, *Student Member, IEEE*, and Cécile Larue de Tournemine

***Abstract*—The auxiliary-vector beamformer is an algorithm that generates iteratively a sequence of beamformers which, under the assumption of a positive definite covariance matrix $\boldsymbol{R}$, converges to the minimum variance distortionless response beamformer, without resorting to any matrix inversion. In the case where $\boldsymbol{R}$ is rank-deficient, e.g., when $\boldsymbol{R}$ is substituted for the sample covariance matrix and the number of snapshots is less than the number of array elements, the behavior of the AV beamformer is not known theoretically. In this letter, we derive a new convergence result and show that the AV beamformer weights converge when $\boldsymbol{R}$ is rank-deficient, and that the limit belongs to the class of reduced-rank beamformers.**

***Index Terms*—Adaptive beamforming, rank-deficient covariance matrix, reduced-rank beamformer.**

I. Introduction

The minimum variance distortionless response (MVDR) beamformer amounts to solving the following minimization problem [1]

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w} \quad \text{subject to } \boldsymbol{w}^H \boldsymbol{v} = 1 \tag{1}$$

where $\boldsymbol{v} \in \mathbb{C}^m$ is the signature of the signal of interest (SOI), with m the number of array elements, and $\boldsymbol{R}$ stands for the interference plus noise covariance matrix. The solution to (1) is known to be given by

$$\boldsymbol{w}_{\text{MVDR}} = \left(\boldsymbol{v}^H \boldsymbol{R}^{-1} \boldsymbol{v}\right)^{-1} \boldsymbol{R}^{-1} \boldsymbol{v} \tag{2}$$

and the MVDR beamformer enables one to achieve the optimal signal to interference plus noise ratio (SINR), which is equal to $\text{SINR}_{\text{opt}} = P \boldsymbol{v}^H \boldsymbol{R}^{-1} \boldsymbol{v}$ where P denotes the SOI power [1]. In practice, only K snapshots $\boldsymbol{X} = [\boldsymbol{x}(1) \ \cdots \ \boldsymbol{x}(K)]$ are available and $\boldsymbol{R}$ is usually substituted for the sample covariance matrix (SCM) $\hat{\boldsymbol{R}} = K^{-1} \boldsymbol{X} \boldsymbol{X}^H$ in (2) to yield

$$\boldsymbol{w}_{\text{MVDR}}^{\text{smi}} = \left(\boldsymbol{v}^H \hat{\boldsymbol{R}}^{-1} \boldsymbol{v}\right)^{-1} \hat{\boldsymbol{R}}^{-1} \boldsymbol{v} \tag{3}$$

where the superscript "smi" stands for sample matrix inversion. The measure of effectiveness (MOE) of (3), *viz.*, the number of snapshots required to maintain an average SINR loss, with respect to SINR_{opt}, of less than 3 dB, is known to be $2m - 3$ snapshots [2]. In applications that employ space-time adaptive processing [3], non-stationarities and the bandwidth over which frequency averaging can be done often make it impossible to obtain this number of snapshots. Moreover, with $K < m$, $\hat{\boldsymbol{R}}$ is rank-deficient and therefore (3) cannot even be implemented. Observe that, for $K \geq m$, $\hat{\boldsymbol{R}}$ is invertible with probability 1 but, in this case, one needs to invert an $m \times m$ matrix, which can be prohibitive in terms of computational cost. The need for beamformers with an enhanced MOE and algorithms that could avoid the inversion of a large matrix has raised a large amount of studies in the recent years [1]. Several strategies have been proposed to combat these two drawbacks. A first solution is diagonal loading [4], [5] which consists in adding a scaled identity matrix to $\hat{\boldsymbol{R}}$ before inversion. The resulting beamformer has an MOE that is commensurate with twice the number of interference J, and is thus very effective [4]. However, it does not alleviate the need to invert an $m \times m$ covariance matrix. A second and important class of fast-converging beamformers consists of the so-called reduced-rank (RR) beamformers which exploit the fact that the interference generally occupy a low-rank subspace, and the latter can be estimated from eigenvalue decomposition (EVD) of the sample covariance matrix. The principal component method [6], [7], the cross-spectral metric method [8], [9] and the dominant mode rejection method [10] belong to this class and were shown to converge within a very small number of snapshots. However they require the EVD of $\hat{\boldsymbol{R}}$. The multistage Wiener filter (MWF) [11], which operates in a Krylov subspace [12] and can thus be implemented through a conjugate gradient (CG) algorithm [13], constitutes one of the seldom approaches that combines the advantages of having a small MOE and of avoiding matrix inversion, at least when implemented with a CG algorithm.

Another alternative is the auxiliary vector (AV) beamformer which was introduced in [14]. Starting with the conventional beamformer, the AV beamformer adds non-orthogonal auxiliary vectors and generates an infinite sequence of beamformers. For the sake of readability, and because it will be used in the next section, we reproduce in Algorithm 1 the successive steps of this algorithm. Therein, $\boldsymbol{P}_{\boldsymbol{v}}^{\perp}$ denotes the projection matrix on the subspace orthogonal to $\boldsymbol{v}$. The input covariance matrix $\boldsymbol{R}$ of Algorithm 1 should be viewed as a "generic" input matrix even if, in practice, Algorithm 1 will generally be fed with $\hat{\boldsymbol{R}}$. However, for the sake of generality, we keep the notation $\boldsymbol{R}$ to denote the input matrix of Algorithm 1. In [14], it was proved that, with $\boldsymbol{R}$ a positive definite covariance matrix, the AV weight vector converges to $\boldsymbol{w}_{\text{MVDR}}$ in (2) as n goes to infinity. Moreover, in finite samples, its performance was shown to be commensurate

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O. Besson and J. Montesinos are with the University of Toulouse, ISAE/TéSA, 14-16 Port Saint Etienne, 31000 Toulouse (e-mail: olivier.besson@isae.fr; julien.montesinos@tesa.prd.fr).

C. Larue de Tournemine is with Thales Alenia Space, 31037 Toulouse, France (e-mail: Cecile.Larue-de-Tournemine@thalesaleniaspace.com).

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with that of the MWF [15], [16]. Therefore, it is a very interesting solution to consider. However, in "sample-starved" scenarios where $K < m$, $\hat{\boldsymbol{R}}$—which, in practice, will replace $\boldsymbol{R}$ as an input to Algorithm 1—becomes rank-deficient and therefore it is of utmost importance to study the behavior of the AV beamformer in such a case. Note also that in some applications, the true covariance matrix may also be rank-deficient. Interestingly enough, there is no result of convergence for the AV filter when the input matrix is rank-deficient while this may occur frequently in practice. In this letter, we fill this gap. We provide a new convergence result and derive, *for any rank-deficient input covariance matrix* $\boldsymbol{R}$, the limit of the AV beamformer $\boldsymbol{w}_n$ of Algorithm 1.

Algorithm 1 Auxiliary-vector beamformer

Input: $\boldsymbol{R}, \boldsymbol{v}$
1: $\boldsymbol{w}_0 = (\boldsymbol{v}^H\boldsymbol{v})^{-1}\boldsymbol{v}$
2: **for** $n = 1, \ldots$ **do**
3: $\boldsymbol{g}_n = \boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{R}\boldsymbol{w}_{n-1}$
4: **if** $\boldsymbol{g}_n = \boldsymbol{0}$ **then**
5: exit
6: **else**
7: $\mu_n = (\boldsymbol{g}_n^H\boldsymbol{R}\boldsymbol{w}_{n-1})/(\boldsymbol{g}_n^H\boldsymbol{R}\boldsymbol{g}_n)$
8: $\boldsymbol{w}_n = \boldsymbol{w}_{n-1} - \mu_n\boldsymbol{g}_n$
9: **end if**
10: **end for**

Output: sequence of beamformers $\boldsymbol{w}_n$

II. Main Result

The main result of this letter is summarized in the next proposition. In the sequel, $\mathcal{R}\{\cdot\}$ stands for the range space of the matrix between braces.

Proposition 1: Let $\boldsymbol{R}$ be a rank-deficient covariance matrix whose EVD is $\boldsymbol{R} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H$ with $\boldsymbol{U} = [\boldsymbol{u}_1\ \boldsymbol{u}_2\ \cdots\ \boldsymbol{u}_p] \in \mathbb{C}^{m\times p}$ a semi-unitary matrix. Then, assuming that $\boldsymbol{v} \notin \mathcal{R}\{\boldsymbol{U}\}$, the limit of the weight vector $\boldsymbol{w}_n$ of Algorithm 1 is

$$\lim_{n\to\infty} \boldsymbol{w}_n = \frac{\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}}{\boldsymbol{v}^H\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}} \triangleq \boldsymbol{w}_{\text{av}-\infty} \tag{4}$$

where $\boldsymbol{U}_\perp$ is an orthonormal basis for $\mathcal{R}\{\boldsymbol{U}\}^\perp$.

Proof: The proof begins along the same lines as in [14] but additional results are rapidly needed to handle the case of a rank-deficient $\boldsymbol{R}$. Similarly to [14], we observe that

$$\begin{aligned}\boldsymbol{g}_n^H\boldsymbol{g}_{n+1} &= \boldsymbol{g}_n^H\boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{R}\boldsymbol{w}_n = \boldsymbol{g}_n^H\boldsymbol{R}\boldsymbol{w}_n \\ &= \boldsymbol{g}_n^H\boldsymbol{R}\boldsymbol{w}_{n-1} - \mu_n\boldsymbol{g}_n^H\boldsymbol{R}\boldsymbol{g}_n = 0\end{aligned} \tag{5}$$

where we used the fact that $\boldsymbol{g}_n = \boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{g}_n$—see line 3 of Algorithm 1—and the definition of μ_n. Next, we show that, despite the fact that $\boldsymbol{R}$ is rank-deficient, μ_n remains bounded. Towards this end, one can write

$$\boldsymbol{g}_n^H\boldsymbol{R}\boldsymbol{g}_n = \boldsymbol{g}_n^H\boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{R}\boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{g}_n = \boldsymbol{g}_n^H\boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H\boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{g}_n. \tag{6}$$

Therefore $\boldsymbol{g}_n^H\boldsymbol{R}\boldsymbol{g}_n = 0$ if and only if $\boldsymbol{U}^H\boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{g}_n = \boldsymbol{0}$ which implies that $\boldsymbol{g}_n \in \mathcal{R}\{\boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{U}\}^\perp$. However this is impossible—unless $\boldsymbol{g}_n = \boldsymbol{0}$—since, from line 3 of Algorithm 1, $\boldsymbol{g}_n \in \mathcal{R}\{\boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{U}\}$. It follows that

$$\lambda_p \le \frac{\boldsymbol{g}_n^H\boldsymbol{R}\boldsymbol{g}_n}{\boldsymbol{g}_n^H\boldsymbol{g}_n} \le \lambda_1 \tag{7}$$

where λ_1 and λ_p are, respectively, the maximal and minimal non-zero eigenvalues of $\boldsymbol{R}$. Consequently μ_n is bounded. From the previous observations, one has

$$\begin{aligned}\boldsymbol{w}_n^H\boldsymbol{R}\boldsymbol{w}_n &= \boldsymbol{w}_n^H\boldsymbol{R}(\boldsymbol{w}_{n-1} - \mu_n\boldsymbol{g}_n) = \boldsymbol{w}_n^H\boldsymbol{R}\boldsymbol{w}_{n-1} \\ &= \boldsymbol{w}_{n-1}^H\boldsymbol{R}\boldsymbol{w}_{n-1} - \mu_n\boldsymbol{g}_n^H\boldsymbol{R}\boldsymbol{w}_{n-1} \\ &= \boldsymbol{w}_{n-1}^H\boldsymbol{R}\boldsymbol{w}_{n-1} - \mu_n\boldsymbol{g}_n^H\boldsymbol{g}_n\end{aligned} \tag{8}$$

where, to obtain the last line, we used the fact that $\boldsymbol{g}_n^H\boldsymbol{R}\boldsymbol{w}_{n-1} = \boldsymbol{g}_n^H\boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{R}\boldsymbol{w}_{n-1} = \boldsymbol{g}_n^H\boldsymbol{g}_n$. The sequence $\boldsymbol{w}_n^H\boldsymbol{R}\boldsymbol{w}_n$ is a monotonically decreasing sequence of non-negative numbers. Since μ_n is bounded, we conclude as in [14] that $\boldsymbol{g}_n$ converges to $\boldsymbol{0}$ as n tends to infinity. In particular it implies that

$$\lim_{n\to\infty} \boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{R}\boldsymbol{w}_n = \boldsymbol{0}. \tag{9}$$

In contrast to [14] we now show that $\boldsymbol{R}\boldsymbol{w}_n$ also converges to $\boldsymbol{0}$. From the previous equation, one can write

$$\boldsymbol{R}\boldsymbol{w}_n - \frac{\boldsymbol{v}^H\boldsymbol{R}\boldsymbol{w}_n}{\boldsymbol{v}^H\boldsymbol{v}}\boldsymbol{v} \underset{n\to\infty}{\longrightarrow} \boldsymbol{0}. \tag{10}$$

Premultiplying by $\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H$ and observing that, by assumption, $\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v} \neq \boldsymbol{0}$ and that $\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{R} = \boldsymbol{0}$, we conclude that $\boldsymbol{v}^H\boldsymbol{R}\boldsymbol{w}_n$ converges to zero as n tends to infinity. This implies that $\boldsymbol{P}_{\boldsymbol{v}}\boldsymbol{R}\boldsymbol{w}_n$ also converges to $\boldsymbol{0}$ and finally

$$\lim_{n\to\infty} \boldsymbol{R}\boldsymbol{w}_n = \boldsymbol{0}. \tag{11}$$

Therefore, the component of $\boldsymbol{w}_n$ in $\mathcal{R}\{\boldsymbol{U}\}$ converges to $\boldsymbol{0}$, or equivalently

$$\lim_{n\to\infty} \boldsymbol{U}\boldsymbol{U}^H\boldsymbol{w}_n = \boldsymbol{0}. \tag{12}$$

Let us now consider the component of $\boldsymbol{w}_n$ in $\mathcal{R}\{\boldsymbol{U}_\perp\}$. From line 8 of Algorithm 1, one can write

$$\begin{aligned}\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{w}_n &= \boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{w}_{n-1} - \mu_n\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{P}_{\boldsymbol{v}}^{\perp}\boldsymbol{R}\boldsymbol{w}_{n-1} \\ &= \boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{w}_{n-1} - \mu_n\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{R}\boldsymbol{w}_{n-1} \\ &\quad + \mu_n\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{P}_{\boldsymbol{v}}\boldsymbol{R}\boldsymbol{w}_{n-1} \\ &= \boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{w}_{n-1} + \mu_n\frac{\boldsymbol{v}^H\boldsymbol{R}\boldsymbol{w}_{n-1}}{\boldsymbol{v}^H\boldsymbol{v}}\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}.\end{aligned} \tag{13}$$

Hence, since $\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{w}_0 = (\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v})/(\boldsymbol{v}^H\boldsymbol{v})$, it follows from (13) that $\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{w}_n$ is aligned with $\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}$ *for every* n. Otherwise stated

$$\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{w}_n = \alpha_n\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v} \quad \forall n. \tag{14}$$

Furthermore, since $\boldsymbol{v}^H\boldsymbol{w}_n = 1$ by construction,

$$\boldsymbol{v}^H\boldsymbol{U}\boldsymbol{U}^H\boldsymbol{w}_n + \alpha_n\boldsymbol{v}^H\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v} = 1. \quad (15)$$

Since $\boldsymbol{U}\boldsymbol{U}^H\boldsymbol{w}_n$ converges to $\mathbf{0}$, it follows that

$$\lim_{n\to\infty}\alpha_n = (\boldsymbol{v}^H\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v})^{-1} \quad (16)$$

from which we conclude that

$$\lim_{n\to\infty}\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{w}_n = \frac{\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}}{\boldsymbol{v}^H\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}}. \quad (17)$$

Finally, (12) and (17) imply that

$$\lim_{n\to\infty}\boldsymbol{w}_n = \frac{\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}}{\boldsymbol{v}^H\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}}. \quad (18)$$

which terminates our proof. ∎

Some remarks concerning Proposition 1 are provided as follows.

Remark 1: A less rigorous but intuitively appealing convergence study can be obtained as follows. For notational convenience, let us temporarily note $\boldsymbol{w}_n(\boldsymbol{R})$ the weight vector sequence of Algorithm 1 corresponding to the input matrix $\boldsymbol{R}$. In order to obtain the limit of $\boldsymbol{w}_n(\boldsymbol{R})$, we consider the limit of $\boldsymbol{w}_n(\boldsymbol{R}+\sigma^2\boldsymbol{I})$ and then let σ^2 tend to zero. For any $\sigma^2 > 0$, the matrix $\boldsymbol{R}+\sigma^2\boldsymbol{I}$ is positive definite, and the convergence theorem of [14] applies, i.e.,

$$\lim_{n\to\infty}\boldsymbol{w}_n(\boldsymbol{R}+\sigma^2\boldsymbol{I}) = \frac{(\boldsymbol{R}+\sigma^2\boldsymbol{I})^{-1}\boldsymbol{v}}{\boldsymbol{v}^H(\boldsymbol{R}+\sigma^2\boldsymbol{I})^{-1}\boldsymbol{v}} \triangleq \boldsymbol{w}_\infty(\boldsymbol{R}+\sigma^2\boldsymbol{I}). \quad (19)$$

Therefore, all "trajectories" of $\boldsymbol{w}_n(\boldsymbol{R}+\sigma^2\boldsymbol{I})$ converge to $\boldsymbol{w}_\infty(\boldsymbol{R}+\sigma^2\boldsymbol{I})$. In order to infer the limit of $\boldsymbol{w}_n(\boldsymbol{R})$, let us now examine the trajectory of $\boldsymbol{w}_\infty(\boldsymbol{R}+\sigma^2\boldsymbol{I})$ when σ^2 goes to zero. For any σ^2, one has

$$\begin{aligned}(\boldsymbol{R}+\sigma^2\boldsymbol{I})^{-1} &= (\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H+\sigma^2\boldsymbol{I})^{-1}\\ &= [\boldsymbol{U}(\boldsymbol{\Lambda}+\sigma^2\boldsymbol{I})\boldsymbol{U}^H+\sigma^2\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H]^{-1}\\ &= \boldsymbol{U}(\boldsymbol{\Lambda}+\sigma^2\boldsymbol{I})^{-1}\boldsymbol{U}^H+\sigma^{-2}\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\\ &= \sigma^{-2}[\boldsymbol{U}(\boldsymbol{I}+\sigma^{-2}\boldsymbol{\Lambda})^{-1}\boldsymbol{U}^H+\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H].\end{aligned} \quad (20)$$

Consequently

$$\begin{aligned}\boldsymbol{w}_\infty(\boldsymbol{R}+\sigma^2\boldsymbol{I}) &= \frac{\boldsymbol{U}(\boldsymbol{I}+\sigma^{-2}\boldsymbol{\Lambda})^{-1}\boldsymbol{U}^H\boldsymbol{v}+\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}}{\boldsymbol{v}^H\boldsymbol{U}(\boldsymbol{I}+\sigma^{-2}\boldsymbol{\Lambda})^{-1}\boldsymbol{U}^H\boldsymbol{v}+\boldsymbol{v}^H\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}}\\ &\underset{\sigma^2\to 0}{\longrightarrow}\frac{\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}}{\boldsymbol{v}^H\boldsymbol{U}_\perp\boldsymbol{U}_\perp^H\boldsymbol{v}} = \boldsymbol{w}_{\text{av}-\infty}.\end{aligned} \quad (21)$$

Although the previous derivations do not constitute a proof, they provide an intuitive explanation to the fact that $\boldsymbol{w}_n(\boldsymbol{R})$ converges to $\boldsymbol{w}_{\text{av}-\infty}$.

Remark 2: When Algorithm 1 is used with $\hat{\boldsymbol{R}}$ and $K \leq m$, the AV beamformer weights converge to a vector that operates in a low-rank subspace. Hence, the AV beamformer asymptotically (in n) belongs to the class of RR beamformers. Therefore, it should inherit their enjoyable properties in terms of MOE. Moreover, it enables one to achieve effective interference cancelation, as illustrated now. In the absence of noise, the array output is given by $\boldsymbol{X} = \boldsymbol{A}_i\boldsymbol{S}_i^H$ where the columns of $\boldsymbol{A}_i \in \mathbb{C}^{m\times J}$ are the interference steering vectors and the columns of $\boldsymbol{S}_i$ correspond to the interference waveforms. In this case, for $K \geq J$ $\text{rank}(\hat{\boldsymbol{R}}) = J$ and the principal subspace $\boldsymbol{U}$ of $\hat{\boldsymbol{R}}$ includes $\mathcal{R}\{\boldsymbol{A}_i\}$, along with $K-J$ vectors orthogonal to $\boldsymbol{A}_i$. In any case, all vectors in $\mathcal{R}\{\boldsymbol{U}_\perp\}$ are orthogonal to $\boldsymbol{A}_i$ and hence the interferences will be nulled. However, the asymptotic vector $\boldsymbol{w}_{\text{av}-\infty}$ will not use all degrees of freedom in the space orthogonal to the interferences, but only $m-K$ out of $m-J$. When noise is present, $\boldsymbol{X} = \boldsymbol{A}_i\boldsymbol{S}_i^H + \boldsymbol{N}$, $\hat{\boldsymbol{R}}$ is of rank K with probability 1 and $\mathcal{R}\{\boldsymbol{A}_i\}$ will be not be contained entirely in $\mathcal{R}\{\boldsymbol{U}\}$, but will also have components in $\mathcal{R}\{\boldsymbol{U}_\perp\}$. However, for interference dominating scenarios, i.e., for high interference to noise ratio (INR), this leakage will be small, and the columns of $\boldsymbol{U}_\perp$ will be quasi-orthogonal to $\boldsymbol{A}_i$. In other words, for high interference to noise ratio, Algorithm 1 converges to a weight vector that mostly lies in the null space of the interferences. A last but important observation is the following. In the special case when the sample support K is exactly equal to the signal subspace dimension J, $\boldsymbol{U}_\perp$ will contain the $m-J$ eigenvectors of $\hat{\boldsymbol{R}}$ corresponding to the $m-J$ smallest eigenvalues. Hence, in that case, the AV sequence converges to the minimum norm eigencanceler (MNE) of [6].

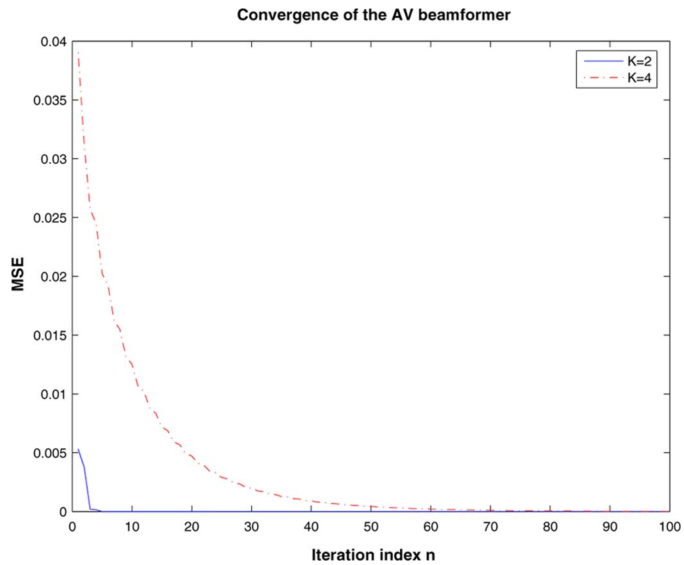

Fig. 1. Mean-square error versus iteration index.

III. Numerical Illustrations

In this section, we illustrate, through a simple scenario, the main theoretical result of this letter—namely the convergence of the AV beamformer to $\boldsymbol{w}_{\text{av}-\infty}$—as well as the transient behavior of Algorithm 1. We consider a uniform linear array with $m = 10$ elements spaced a half-wavelength apart. The array is steered at broadside (0°) and the interferences consist of 2 signals with directions of arrival $[-10°\ \ 15°]$ and interference to noise ratio [10 dB 5 dB], respectively. In Fig. 1, we plot the mean-square error (MSE)

$$\text{MSE}(n) = \|\boldsymbol{w}_n - \boldsymbol{w}_{\text{av}-\infty}\|^2 \quad (22)$$

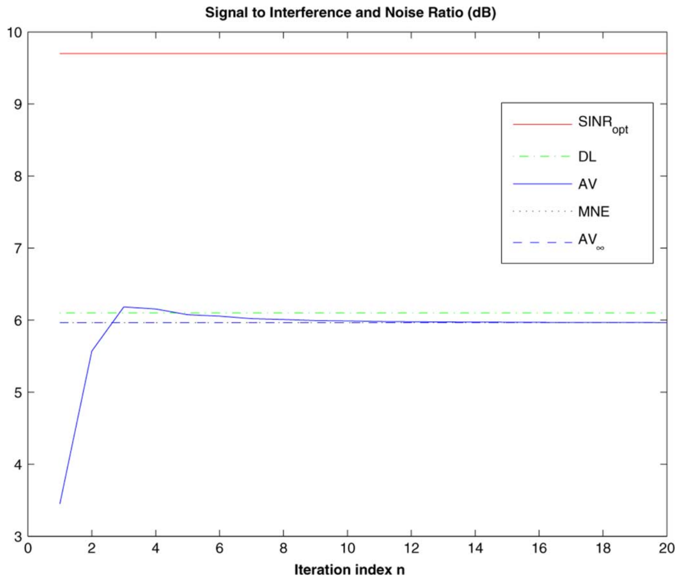

Fig. 2. Output SINR versus iteration index. $K = 2$.

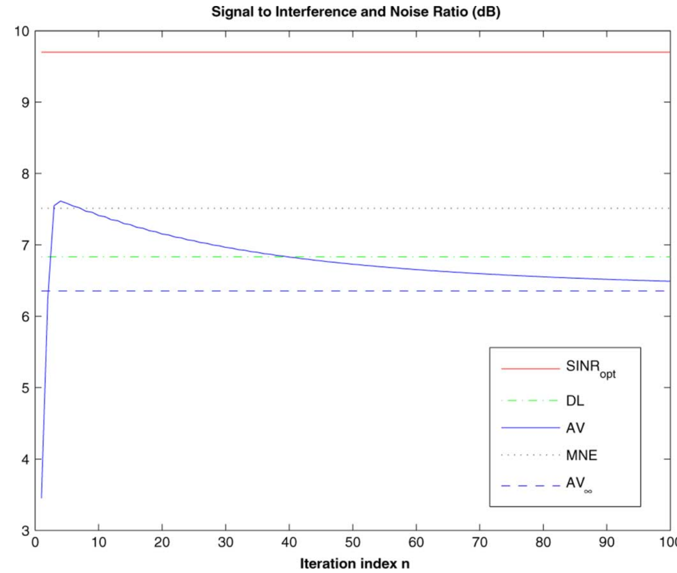

Fig. 3. Output SINR versus iteration index. $K = 4$.

versus the iteration index n, for two different levels of snasphot support, namely $K = 2$ and $K = 4$. The results are given for a particular realization of $\hat{\boldsymbol{R}}$. This figure confirms the validity of the theoretical analysis. It also shows that the convergence is slower with $K = 4$ than with $K = 2$. However, as illustrated next, the SINR obtained after convergence will be higher in the former case.

We consider now the transient behavior of the AV beamformer and compare its performance with diagonal loading (with a loading level 15 dB above the thermal noise power) and the minimum norm eigencanceler. The results are shown in Figs. 2 and 3, which correspond to $K = 2$ and $K = 4$, respectively. We also display the optimal SINR, for comparison purposes. As can be observed, when $K = J = 2$, the asymptotic vector $\boldsymbol{w}_{\mathrm{av}-\infty}$ is exactly the MNE vector while they are different when $K = 4$. It can be observed that the SINR obtained for small n can be larger than that obtained with $\boldsymbol{w}_{\mathrm{av}-\infty}$, and that the AV beamformer can outperform the MNE for some values of n. Of course, a delicate issue is to select the value at which the iterations are stopped, although a solution to this issue has been proposed in [17].

IV. Conclusion

In this letter, we presented a new convergence result for the auxiliary-vector beamformer in case of a rank-deficient input covariance matrix. This case is of utmost practical importance in low sample support where the number of snapshots is less than the number of array elements, and a sample covariance matrix is used. We proved that the limit of the AV algorithm belongs to a low-rank subspace, actually the null space of the covariance matrix, and is thus quasi-orthogonal to the interferences subspace. This result shades a new light on the behavior of the AV beamformer.

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