Fractional robust control of lightly damped systems

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Abstract: **The article proposes a method to design a robust controller ensuring the damping ratio of a closed-loop control. The method uses a contour parameterized by the damping ratio in the Nichols plane and the complex non-integer (or fractional) differentiation to compute a transfer function whose open-loop Nichols locus tangents this contour, thus ensuring dynamic performance. The proposed method is applied to a flexible structure (a clamped-free beam with piezoelectric ceramics). The aims of the control loop are to decrease the vibrations and to ensure the damping ratio of the controlled system.**

1. INTRODUCTION

The reduction of structural vibration has been challenging engineers for many years; innumerable applications exist where vibration control is beneficial, if not essential.
In the control of vibrations, the damping ratio is an important data since it indicates how quickly the vibrations decrease. When control of vibration is at stake, it can be useful to control this parameter.
This article deals with the control of uncertain plants. The method used to compute robust controller is the CRONE method [1]. CRONE control is a frequency-domain based methodology to design robust linear controller using complex-order differentiation [2]. As the method uses the frequency domain, it is necessary to define an element that quantifies the damping ratio in the frequency domain. The "iso-damping" contour defined by Oustaloup [3] is a contour whose graduation is the damping ratio ζ in the Nichols plane. So by computing a transfer function whose open-loop Nichols locus tangents this contour, the damping ratio of the closed-loop can be controlled.

The article falls into two parts. Section 2 introduces the transfer function of a complex non-integer integrator defining a generalized template which will be considered as part of an open-loop Nichols locus [1]. This transfer

function is used first for the construction in the Nichols plane of a network of iso-damping contours.
Section 3 describes the CRONE control based on complex-order differentiation. The interest of the fractional order is to define a transfer function with few parameters and thus to simplify the computations.
Section 4 introduces the methodology based on the CRONE control to compute a closed-loop ensuring the damping ratio. The methodology is applied to a flexible structure which is a free-clamped beam with co-localized piezoelectric ceramics used as actuator to limit the vibrations and as sensor to measure these vibrations. Different masses are fixed at the extremity of the beam and change its characteristics. Robust control can thus be tested. Two cases are studied to validate the method. The first case is a closed –loop with a damping ratio of 0.1 and the second case is a closed –loop with a damping ratio of 0.7.

II. COMPLEX NON-INTEGER INTEGRATION, ISODAMPING CONTOURS AND OPEN LOOP TRANSFER FUNCTION

A. Generalized template and non-integer integration

A "vertical template" [1] - that is to say a vertical segment in the Nichols plane - is obtained using the real fractional (or non-integer) integration [4]. Indeed, the vertical template (Fig.1) is described by the transfer function of a real non-integer integrator of order, n, which defines its phase placement at crossover frequency ω_{cg}, $-n90°$:

$$\beta(s) = \left(\frac{\omega_{cg}}{s}\right)^n \text{ for } \omega \in \left[\omega_A, \omega_B\right], \ n \in \text{R}. \tag{1}$$

From the extension of the description of the vertical template, the "generalized template" - that is to say an any-direction straight line segment in the Nichols plane - can be obtained using the complex non-integer integration of order n. The real part defines its phase placement at ω_{cg}, $-\text{Re}(n)90°$, and the imaginary part defines its angle to the vertical (Fig.1). The generalized template is thus described by the transfer function [1]:

$$\beta(s) = \left(\cosh\left(b\frac{\pi}{2} \right) \right)^{\mathrm{sign}(b)} \left(\frac{\omega_{\mathrm{cg}}}{s} \right)^a \left(\mathrm{Re}_{/i}\left[\left(\frac{\omega_{\mathrm{cg}}}{s} \right)^{ib} \right] \right)^{-\mathrm{sign}(b)} . \quad (2)$$

The imaginary unit i of the integration order n ($n = a + ib$) is independent of the imaginary unit j of the variable s ($s=\sigma+j\omega$).

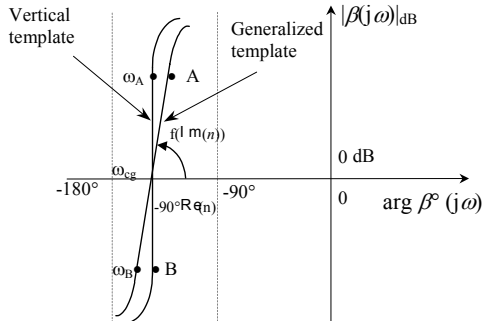

Fig.1. Representation of the vertical template and of the generalized template in the Nichols chart

B. Isodamping contour [3][5]

The easiest geometrical way to construct an isodamping contour is to use an envelope technique. The contour is then defined as the envelope tangented by a set of segments (Fig.2). In the Nichols plane, each segment of the set can be considered as the rectilinear part of an open-loop Nichols locus that ensures the closed-loop damping ratio corresponding to the contour. This rectilinear part around gain crossover frequency, ω_{cg}, is the "generalized template" defined above.
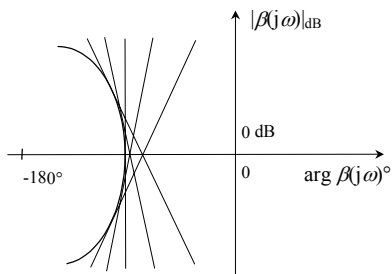

Fig.2. Envelope defining an isodamping contour in the Nichols plane

Isodamping contours can be defined analytically using a polynomial equation determined by interpolation of graphical data of each contour [3]. A contour Γ_ζ is thus defined by:

$$\Gamma_\zeta = \left\{ M(X,Y) \in P, \ X - \sum_{j=0}^{2} f_j(\zeta) Y^{2j} = 0 \right\}, \quad (3)$$

with : $f_j(\zeta) = \sum_{k=0}^{3} a_{jk}\zeta^k$, $\quad (4)$

X and Y being the coordinates expressed in degrees and in decibels and a_{jk} the coefficients given in table 1.

The equation of the tangent to Γ_ζ at point (X_i, Y_i) is deduced from relation (3) and can be written:

$$Y = \alpha_2 X + \beta_2, \quad (5),$$
with :

$$\alpha_2 = \frac{1}{2Y_i \left(\sum_{k=0}^{3} a_{1k}\zeta^k \right) + 4Y_i^3 \left(\sum_{k=0}^{3} a_{2k}\zeta^k \right)} \quad (6)$$

and

$$\beta_2 = Y_i - \frac{1}{2Y_i \left(\sum_{k=0}^{3} a_{1k}\zeta^k \right) + 4Y_i^3 \left(\sum_{k=0}^{3} a_{2k}\zeta^k \right)} X_i . \quad (7)$$

TABLE 1. VALUES OF COEFFICIENTS a_{jk}

j/k	0	1	2	3
0	-180.36	117.7	-74.316	40.376
1	-1.1538	3.8888	-5.2999	2.5417
2	-0.0057101	0.0080962	-0.0060354	0.0016158

III. CRONE CONTROL

CRONE (the French acronym of "Commande Robuste d'Ordre Non Entier") control system design [1,4] is a frequency-domain based methodology, using complex fractional differentiation. It permits the robust control of perturbed linear plants using the common unity feedback configuration. It consists on determining the nominal and optimal open-loop transfer function that guaranties the required specifications. This methodology uses fractional derivative orders (real or complex) as high level parameters that make you easy the design and optimization of the control-system. While taking into account the plant right half-plane zeros and poles, the controller is then obtained from the ratio of the open-loop frequency response to the nominal plant frequency response. Three Crone control generations have been developed, successively extending the application fields. In this paper, the third generation will be applied.

A. Open-loop transfer function

The open-loop transfer function (Fig.3) of the initial third generation Crone method is based on the generalized template described previously and takes into account:
- the accuracy specifications at low frequencies;
- the generalized template around frequency ω_{cg};
- the plant behavior at high frequencies in accordance with input sensitivity specifications for these frequencies.

For stable minimum-phase plants, this function is written:
$$\beta(s) = \beta_l(s)\beta_{\mathrm{m}}(s)\beta_{\mathrm{h}}(s) . \quad (8)$$

• $\beta_{\mathrm{m}}(s)$, based on complex non-integer integration, is the transfer function describing the band-limited generalized template [1]:

$$\beta_{\mathrm{m}}(s) = K \left(\frac{1 + \frac{s}{\omega_{\mathrm{h}}}}{1 + \frac{s}{\omega_{\mathrm{l}}}} \right)^a \left[\mathrm{Re}_{/i}\left\{ \left(\left[\frac{1 + \left(\frac{\omega_{\mathrm{cg}}}{\omega_{\mathrm{l}}} \right)^2}{1 + \left(\frac{\omega_{\mathrm{cg}}}{\omega_{\mathrm{h}}} \right)^2} \right]^{\frac{1}{2}} \left(1 + \frac{s}{\omega_{\mathrm{h}}} \right) \right)^{ib'} \right\} \right]^{-q'\mathrm{sign}(b')} , \quad (9)$$

q' being the smallest integer such that b' verifies $|b'| < \min(|b_1|, |b_2|)$ with:

$$|b_1| = \pi / \left| \ln \left(\left(1 + \left(\frac{\omega_{cg}}{\omega_1} \right)^2 \right) / \left(1 + \left(\frac{\omega_{cg}}{\omega_h} \right)^2 \right) \right) \right| \quad (10)$$

and $|b_2| = \pi / \left| \ln \left[\left(\left(1 + \left(\frac{\omega_{cg}}{\omega_1} \right)^2 \right) / \left(1 + \left(\frac{\omega_{cg}}{\omega_h} \right)^2 \right) \right) \left(\frac{\omega_1}{\omega_h} \right)^2 \right] \right|$, (11)

and K being computed to get a gain of 0 dB at ω_{cg}.

- $\beta_l(s)$ is the transfer function of order n_1 proportional-integrator, whose corner frequency equals the low corner frequency of $\beta_m(s)$, so that joining $\beta_l(s)$ and $\beta_m(s)$ does not introduce extra parameters. $\beta_l(s)$ is defined by:

$$\beta_1(s) = \left(1 + \frac{\omega_1}{s} \right)^{n_1} . \quad (12)$$

If n_{pl} is the order of asymptotic behavior of the plant in low frequency ($\omega \ll \omega_1$), order n_1 is given by $n_1 \geq 1$ if $n_{pl} = 0$, and $n_1 \geq n_{pl}$ if $n_{pl} \geq 1$, with $n_1=1$ canceling the position error and $n_1=2$ canceling the velocity error.

- $\beta_h(s)$ is the transfer function of order n_h low-pass filter, whose corner frequency equals the high corner frequency of $\beta_m(s)$, so that joining $\beta_h(s)$ and $\beta_m(s)$ does not introduce extra parameters. $\beta_h(s)$ is defined by:

$$\beta_h(s) = 1 / \left(1 + \frac{s}{\omega_h} \right)^{n_h} . \quad (13)$$

If n_{ph} is the order of asymptotic behavior of the plant in high frequency ($\omega \gg \omega_h$), order n_h is given by $n_h \geq n_{ph}$, with $n_h = n_{ph}$ ensuring invariability of the input sensitivity function with the frequency, and $n_h > n_{ph}$ ensuring decrease.
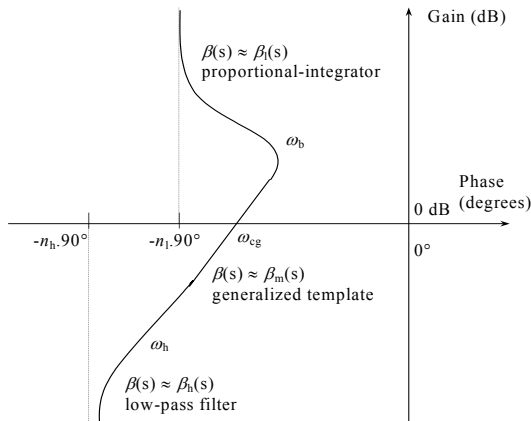

Fig.3. Different parts of the open-loop Nichols locus

At frequency ω_r for which the tangency will be reached, the modulus and the argument of the open-loop frequency response are expressed respectively by:

$$|\beta(j\omega_r)| dB = Y_r , \quad (14)$$

and

$$\arg \beta°(j\omega_r) = \frac{180}{\pi} \left[n_1 \left(\theta_{rl} - \frac{\pi}{2} \right) - n_h \theta_{rh} + a \left(\theta_{rh} - \theta_{rl} \right) \right]$$

(15),

with

$$\theta_{rl} = \arctan \left(\frac{\omega_r}{\omega_1} \right) \quad \text{and} \quad \theta_{rh} = \arctan \left(\frac{\omega_r}{\omega_h} \right) . \quad (16)$$

The equation of the tangent to the Nichols locus at this frequency is given by:

$$Y = \alpha_{BO} \left(X - X_{BO} \right), \quad (17)$$

with:

$$\alpha_{BO} = \left. \frac{d |\beta(j\omega)| dB}{d \arg \beta°(j\omega)} \right|_{\omega=\omega_r} \quad \text{and} \quad X_{BO} = \arg \beta°(j\omega_r). \quad (18)$$

B. CRONE methodology

The third generation CRONE methodology can be described in five points:

1 - You determine the nominal plant transfer function and the uncertainty domains. For a given frequency, an uncertainty domain (called "template" by the QFT users [6]) is the smallest hull including the possible frequency responses of the plant. The use of the edge of the domains permits to take into account the uncertainty with the smallest number of data. To construct this domain securely, the simplest way is to define it convexly.

2 - You specify some parameters of the open-loop transfer function defined for the nominal state of the plant: the number of band-limited generalized templates N^+ and N, and the rational orders n_1 and n_h.

3 - You specify the bounds of the sensibility functions that you would like to obtain. Let $M_{r_{nom}}$ be the required resonant peak of the nominal complementary sensitivity function.

4 - Using the nominal plant locus and the uncertainty domains in the Nichols chart, you optimize the parameters ω_r, a_k and b_k (for $k \neq 0$), ω_k and ω_{k+1} in order to obtain the optimal open-loop Nichols locus. An open-loop Nichols locus is defined as optimal if it tangents the $M_{r_{nom}}$ magnitude contour and if it minimizes the variations of M_r for the other parametric states. By minimizing the cost function $J = \left(M_{r \max} - M_{r nom} \right)^2$ where $M_{r_{\max}}$ is the maximal value of resonant peaks M_r, the optimal open-loop Nichols locus positions the uncertainty domains

correctly, so that they overlap the low stability margin areas as little as possible (Figure 4: case (c) is the best configuration). The minimization of J is carried out under a set of shaping constraints on the four usual sensitivity functions.

5 - The last point is the synthesis of the controller. While taking into account the plant right half-plane zeros and poles, the controller is deduced by the frequency-domain system identification of the ratio of $\beta_{\text{nom}}(j\omega)$ to the nominal plant function transfer $G_{\text{nom}}(j\omega)$. The resulting controller $K(s)$ is a rational transfer function.
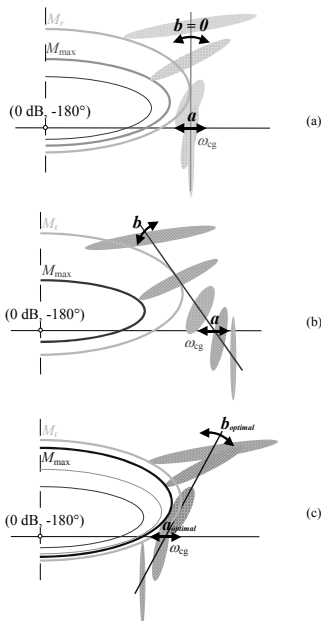
Fig.4. Optimal open-loop Nichols locus to position the uncertainty domains

IV. CRONE CONTROL APPLIED TO LIGHTLY DAMPED SYSTEMS

The methodology of the CRONE control can be adapted to plants whose damping ratio needs to be controlled. Indeed, in this article, we propose to apply the methodology of the CRONE control described in the previous section but to compute an open-loop Nichols locus that tangents an iso-damping contour instead of a magnitude contour. The objective is to control the damping ratio in a robust way.

In order to validate this idea, it has been applied to a simple flexible structure: a clamped-free beam with two co-localised piezoelectric ceramics (figure 5). One ceramic is used as actuator and this other as sensor. This example has been chosen since it can represent different applications (ailerons, electronic boards,…). The characteristics of the structure chosen for the example are given in table 2.
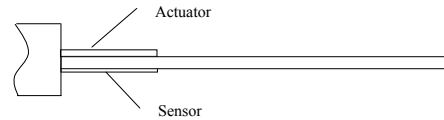
Fig.5. Clamped-free beam with piezoelectric ceramics

TABLE 2: SYSTEM PROPERTIES

	Beam	Actuator
Length (mm)	300	25
Width (mm)	20	20
Thickness (mm)	2	0.5
Density (kg/m^3)	2970	7800
Young's Modulus (GPa)	75	67
Piezoelectric Const. (pm/V)	-	-210

The free response of the flexible system to a perturbation is given in the Figure 6. It is a very lighted-damped system.
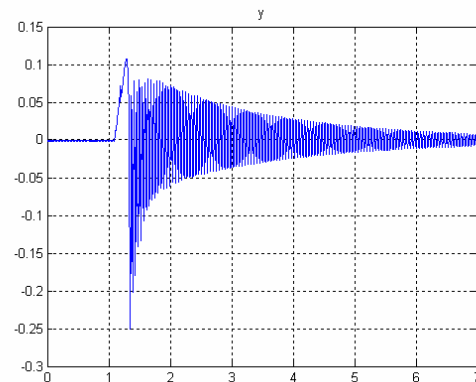
Fig.6. Free response of the flexible system

At the extremity of the beam, masses are added to modify the characteristics of the plant and to test the robustness of the controller.
The models of the plant for the different values of the mass are given in the table 3 and the figure 7 shows the transfer functions of the plant for the different cases.

TABLE 3: SYSTEM MODELS

	Mode 1	Quality factor of mode 1	Mode 2	Quality factor of mode 2
No mass	19.08	74	114.4	165
Mass 1	18.48	92	111.5	192
Mass 2	17.49	82	107.3	66
Mass 3	15.53	86	101.3	99
Mass 4	14.98	83	100.6	102
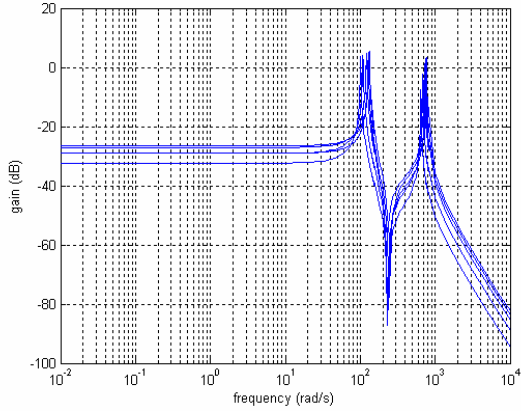
Fig.7. Transfer functions of the flexible system for the five cases of Table 3.

Fig.8 Case n°1 –Nominal open-loop Nichols locus and domains of uncertainties

Note that for damping plants, it is necessary to complete the open-loop transfer function with the resonance modes and in some cases with notch filters [7]. For the system under study, the aim is to control the vibrations for the first two modes. So the open-loop transfer function described in (8) is completed with the transfer functions of the first two resonances and of the anti-resonance of the nominal plant.

A. STUDY 1

For the first study, the aim is to ensure a damping ratio of value 0.1.
The limits on the sensitivity functions are given by the following constraints:
- the maximum plant input (1V),
- the maximum magnitude T_{max} of the complementary sensitivity function set at 3 dB,
- the maximum magnitude S_{max} of the sensitivity function set at 6dB and the minimum set at -10dB.

Only one generalized template is used in the definition of the open-loop. Two notch filters are also added and theirs expressions are:

$$N_{f1} = \frac{\frac{p^2}{50^2} - 3\frac{p}{50} + 1}{\frac{p^2}{130^2} + \frac{p}{130} + 1} \text{ and } N_{f2} = \frac{\frac{p^2}{220^2} - 0.16\frac{p}{220} + 1}{\frac{p^2}{386^2} + \frac{p}{386} + 1} .$$

The parameters obtained after optimization are:
$a = 2.06$, $b = 0.278$; $w_r = 12$ rad/s ; $Y_r = $ -5.28dB; $w_0 = 8.42$ rad/s ; $w_1 = 1000$ rad/s .

The Nichols locus of the nominal open-loop and the domains of uncertainties are given in Figure 8. The Nichols locus tangents the iso-damping contour of value 0.1 and the domains uncertainties do not penetrate into the contours.
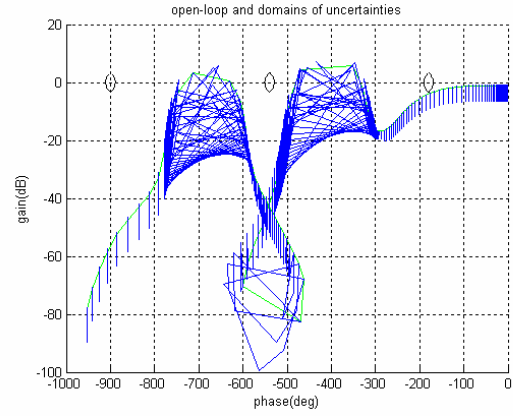
The transfer function of the controller is synthesized by a frequency-domain identification and its expression is:

$$C(p) = \frac{-2.89e^{-11}p^7 + 2.24e^{-9}p^6 - 1.165e^{-6}p^5 - 1.11e^{-4}p^4 - 6.89e^{-3}p^3 - 0.17p^2 - 0.86p + 9.66}{3.65e^{-14}p^8 + 2.59e^{-11}p^7 + 8.79e^{-9}p^6 + 1.48e^{-6}p^5 + 1.175e^{-4}p^4 + 3.7e^{-3}p^3 + 5.4e^{-2}p^2 + 0.376p + 1}$$

Tests are achieved on the plant. The flexible structure is deviated from its equilibrium position and released. Figure 9 shows the plant input u and the plant output y in the case where no mass is added. The damping ratio measured on the output is 0.12. It is close to the expected value of 0.1.
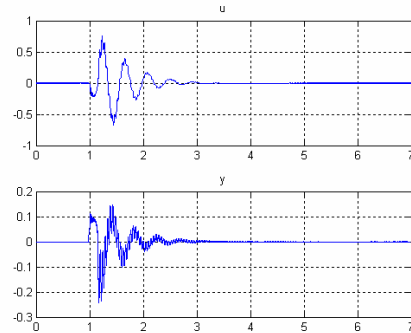

Fig.9. Case n°1 - Plant input and output if no mass added

Figure 10 shows the plant output for the four different values of added mass and the robustness of the controller can be noticed.
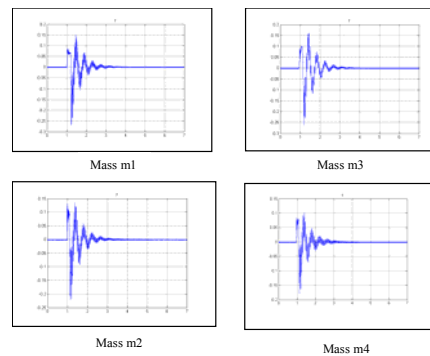

Fig.10. Case n°1 - Plant outputs in four different cases

B. STUDY 2

For the second study, the aim is to ensure a damping ratio of value 0.7.

The constraints on the sensitivity functions are the same as in the case of study 1.

Only one generalized template is used in the definition of the open-loop. Two notch filters are also added and theirs expressions are:

$$N_{f1} = \frac{\dfrac{p^2}{65^2} - 3\dfrac{p}{65} + 1}{\dfrac{p^2}{115^2} + \dfrac{p}{115} + 1} \text{ and } N_{f2} = \frac{\dfrac{p^2}{225^2} - 0.2\dfrac{p}{225} + 1}{\dfrac{p^2}{350^2} + \dfrac{p}{350} + 1} \ .$$

The parameters obtained after optimization are:
$a = 1.18$, $b = 0.46$; $w_r = 12$ rad/s ; Y_r = -8.2 dB; $w_0 = 5.4$ rad/s ; $w_1 = 148$ rad/s .

The Nichols locus of the nominal open-loop and the domains of uncertainties are given in Figure 11. It is more difficult in this case to ensure that the domains of uncertainties do not penetrate into the iso-damping contour.

The transfer function of the controller is synthesized by a frequency-domain identification and its expression is:

$$C(p) = \frac{1.987e^{-10}p^6 - 8.439e^{-8}p^5 + 1.279e^{-6}p^4 - 3.089e^{-3}p^3 - 0.166p^2 - 0.659p + 31.62}{3e^{-14}p^7 + 4.66e^{-11}p^6 + 2.657e^{-8}p^5 + 6.9e^{-6}p^4 + 8.23e^{-4}p^3 + 3.96e^{-2}p^2 + 4.17p + 1}$$
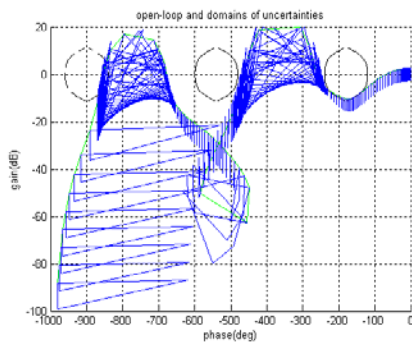

Fig.11. Case n°2 –Nominal open-loop Nichols locus and domains of uncertainties

Tests are achieved on the plant. Figure 12 shows the plant input *u* and the plant output *y* in the case where no mass is added. The plant input nearly does not exceed its maximum value and the output comes back to the equilibrium position very quickly, that is expected with a damping ratio of 0.7.
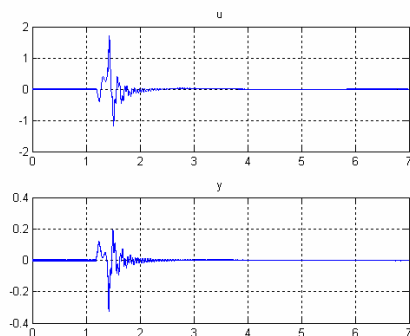

Fig.12. Case n°2 - Plant input and output if no mass added

As in the other case, figure 13 shows the output for the four different values of added mass and the robustness of the controller can be checked.
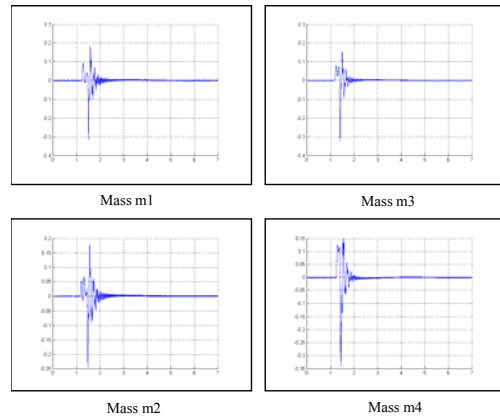

Fig.13. Case n°2 - Plant outputs in four different cases

V. CONCLUSION

This article has introduced a method to design a robust controller ensuring the damping ratio of a closed-loop control. The first part of this article (section 2) introduces the generalized template based on complex non-integer integration and recalls the method for construction of iso-damping contours by the envelope technique. This technique uses segments obtained using complex non-integer integration. Section 3 describes the CRONE methodology. Section 4 concerns the proposed methodology and its application to a flexible structure in two cases of study. The results show the efficiency of the method.

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