Cross standard form: a solution to improve a given controller with H_2 or H_∞ specifications

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This paper introduces in cross standard form (CSF) as a solution to the inverse optimal control problem. That is, the CSF is a canonical standard problem whose unique H_∞ or H_2 optimal controller is a given controller. From the control design point of view, the general idea is to apply the CSF to a given controller in order to set up a standard problem which can be completed to handle frequency domain H_2 or H_∞ specification. The analytical formulation of the CSF proposed in this paper can be applied to reduced-, full- or augmented-order compensators or two-degree of freedom compensations. Numerical and academic examples are given.

1. Introduction

In most practical applications, the control design problem is expressed in the following terms: is it possible to improve a given controller (often, a simple low-order controller designed upon a particular know-how or good sense rules) to meet additional H_2 or H_∞ specifications? or in other terms: is it possible to take into account a given controller (which meets some closed-loop specifications) in a standard H_2 and H_∞ control problem? To address this problem, the notion of cross standard form (CSF) is introduced in this paper for a given nth-order plant and an arbitrary given stabilizing n_Kth order controller. The CSF can be seen as a solution for both inverse H_∞ and H_2 optimal control problems, that is: the CSF is a standard augmented problem whose unique H_∞ and H_2 optimal controller is an arbitrary given controller. In Alazard *et al.* (2004), the definition of the CSF was based on the possibility to determine a minimal observed-based realization of the initial controller (Alazard and Apkarian 1999). In this paper this observer-based realization is no more required and thus the CSF is directly defined by the 4 state space matrices of the plant, the 4 state space matrices of the given controller and a solution to a general non-symmetric Riccati equation. This new formulation allows the CSF to be extended to the case of low-order controller ($n_K < n$) and also to two-degree of freedom controllers.

The interest for inverse optimal control problems motivates lots of works (Kalman 1964, Molinari 1973, Fujii 1987, Fujii and Khargonekar 1988, Lenz *et al.* 1988, Sebe 2001). The practical interest of such solutions lies in the possibility to mix various approaches or take into account different kinds of specifications (Sugimoto and Yamamoto 1987, Shimomura and Fujii 1997, Sugimoto 1998). In the particular case of the H_∞ optimal control problem, the various contributions address restrictive cases: state feedback controller in Fujii and Khargonekar (1988), single-output–single-output controller and specific sensitivity problem in Lenz *et al.* (1988). However, a solution for the general case (multi-input–multi-output, dynamic output feedback or arbitrary order) has never been stated. This general case is addressed in Sebe (2001): for a given weight system $W(s)$ and a given controller $K(s)$, the problem is to find all the plants $G(s)$ such that $\|F_l(F_l(W, G),\ K)\|_\infty < \gamma$ (see figure 1). Note that the problem addressed in this paper is different: the issue is to find $W(s)$ for a given $K(s)$ and a given $G(s)$. That is, the lower right-hand transfer matrix G of the standard augmented plant $P = F_l(W,\ G)$ must be equal to the

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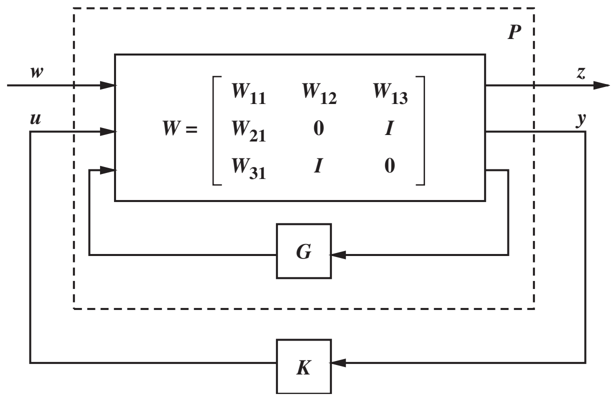

Figure 1. Block diagram of standard plant P, weight function W, model G and controller K.

model of the plant between the control input u and the measured output y.

The convex closed-loop technique Boyd and Vandenberghe (2003) seems also an attractive approach to take into account a given controller and additional H_2 and H_∞ constraints. But such an approach needs a Youla parameterization of the controller and so is limited to full-order (observer-based) controllers. Furthermore, this approach leads to very high order controllers.

The paper is organized as follows. In §2 the CSF is defined as a solution to H_2 and H_∞ inverse optimal control problems, for an nth order linear time invariant (LTI) system and a stabilizing n_kth order LTI controller. In §3 an analytical expression of the CSF is proposed for low-order controllers ($n_k \le n$) and the existence of a CSF a discussed. In §4 this result is extended for augmented-order controllers and so encompasses previous results presented in Alazard *et al.* (2004). A generalization for two-degrees of freedom controllers is also stated. Finally, an academic example is proposed to highlight the way to use CSF to take into account an initial low-order compensator and a robustness specification in an augmented standard problem.

2. Definitions

Nomenclature

A^T	A transposed
A^+	Moore-Penrose pseudo-inverse of matrix A
$A^\perp$	Orthonormal basis for the null space of A
I_n	$n \times n$ identity matrix
$\dot{x}$	time derivation ($\dot{x} = dx/dt$)
s	Laplace variable
$F_l(P, K)$	Lower linear fractional transformation of P and K
$F_u(P, \Delta)$	Upper linear-fractional transformation of P and Δ
$\lVert G(s) \rVert_2$	H_2 norm of the stable system $G(s)$
$\lVert G(s) \rVert_\infty$	H_∞ norm of the stable system $G(s)$
$G(s) := \left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	shorthand for $G(s) = C(sI - A)^{-1}B + D$

The general standard plant between exogenous input w, control input u, controlled output z and measurement output y is denoted

$$P(s) = \begin{bmatrix} P_{zw}(s) & P_{zu}(s) \\ P_{yw}(s) & P_{yu}(s) \end{bmatrix},$$

with corresponding state space realization

$$P(s) := \left[\begin{array}{c|cc} A_p & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]. \tag{1}$$

2.1. *An inverse optimal control problem*

Consider the stabilizable and detectable nth order system $G(s)$ (m inputs and p outputs) with minimal state-space realization

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \tag{2}$$

Consider also the stabilizing n_kth order controller $K_0(s)$ with minimal state-space realization

$$\begin{bmatrix} \dot{x}_k \\ u \end{bmatrix} = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} x_k \\ y \end{bmatrix}. \tag{3}$$

Definition 1: Inverse H_2 optimal problem
Find a standard plant $P(s)$ such that

- $P_{yu}(s) = G(s)$,
- K_0 stabilizes $P(s)$,
- $K_0(s) = \arg\min_{K(s)} \lVert F_l(P(s), K(s)) \rVert_2$,

(namely: $K_0(s)$ minimizes $\lVert F_l(P(s), K(s)) \rVert_2$).

Definition 2: Inverse H_∞ optimal problem
Find a standard plant $P(s)$ such that

- $P_{yu}(s) = G(s)$,
- K_0 stabilizes $P(s)$,
- $K_0(s) = \arg\min_{K(s)} \lVert F_l(P(s), K(s)) \rVert_\infty$.

2.2. *Cross standard form*

Definition 3: Cross standard form
If the standard plant $P(s)$ is such that the 4 conditions

- C1: $P_{yu}(s) = G(s)$,
- C2: K_0 stabilizes $P(s)$,

- C3: $F_l(P(s),\ K_0(s)) = 0$,
- C4: K_0 is the unique solution of the optimal H_2 or H_∞ problem $P(s)$,

are met, then $P(s)$ is called the CSF associated with the system $G(s)$ and the controller $K_0(s)$ and will be denoted $P_{CSF}(s)$ in the sequel.

By construction, the CSF solves the inverse H_2 optimal problem and the inverse H_∞ optimal problem. Note that the uniqueness condition C4 is relevant in our context since we are looking for an H_2 or H_∞ design to recopy a given controller.

3. Low-order controller case ($n_k \leq n$)

3.1. *General results*

The following proposition provides a general analytical characterization of the CSF.

Proposition 1: *For a given stabilizable and detectable nth order system $G(s)$ (2) and a given stabilizing n_kth order controller $K_0(s)$ (3) with $n_k < n$, a CSF reads*

$$P_{CSF}(s) := \left[\begin{array}{c|cc} A & T^{\#}B_k - BD_k & B \\ \hline -C_kT - D_kC & D_kDD_k - D_k & I_m - D_kD \\ C & I_p - DD_k & D \end{array}\right], \tag{4}$$

where T is a full row-rank matrix, solution of the generalized Riccati equation

$$\begin{bmatrix} -T & I \end{bmatrix} A_{cl} \begin{bmatrix} I \\ T \end{bmatrix} = 0, \tag{5}$$

where A_{cl} is the stable closed-loop dynamic matrix $F_l(G,\ K_0)$

$$A_{cl} = \begin{bmatrix} A + B(I - D_kD)^{-1}D_kC & B(I - D_kD)^{-1}C_k \\ B_k(I - DD_k)^{-1}C & A_k + B_kD(I - D_kD)^{-1}C_k \end{bmatrix}, \tag{6}$$

and where $T^{\#}$ is a right inverse of T, such that $TT^{\#} = I_{n_k}$ (see also Proposition 2).

Proof: From (4), it is obvious that conditions C1 and C2 are met. In the sequel of the demonstration, without loss of generality, the feed-through matrix D of system G is assumed to be null. Then, equations (4) and (5) become

$$P_{CSF}(s) := \left[\begin{array}{c|cc} A & T^{\#}B_k - BD_k & B \\ \hline -C_kT - D_kC & -D_k & I_m \\ C & I_p & 0 \end{array}\right] \tag{7}$$

and

$$T(A + BD_kC) + TBC_kT - B_kC - A_kT = 0. \tag{8}$$

A state space realization of $F_l(P_{CSF}, K_0)$ associated with state vector $\begin{bmatrix} x^T, & x_k^T \end{bmatrix}^T$ reads

$$F_l(P_{CSF}, K_0) := \left[\begin{array}{cc|c} A + BD_kC & BC_k & T^{\#}B_k \\ B_kC & A_K & B_k \\ \hline -C_kT & C_k & 0 \end{array}\right].$$

Let us consider the change of state coordinates

$$M = M^{-1} = \begin{bmatrix} I_n & 0 \\ T & -I_{n_k} \end{bmatrix}, \tag{9}$$

where T is a solution of (5) and $TT^{\#} = I_{n_k}$. The new state space realization of $F_l(P_{CSF}, K_0)$ reads

$$F_l(P_{CSF}, K_0) := \left[\begin{array}{cc|c} A + B(D_kC + C_kT) & -BC_k & T^{\#}B_k \\ 0 & A_K - TBC_k & 0 \\ \hline 0 & -C_k & 0 \end{array}\right]. \tag{10}$$

Therefore then $n + n_k$ stable close-loop eigenvalues are composed of

- n eigenvalues of $A + B(D_kC + C_kT)$ which are unobservable by the controlled output z of P_{CSF},
- n_k eigenvalues of $A_k - TBC_k$ which are uncontrollable by the exogenous input w of P_{CSF}.

Thus, condition C3 is met

$$F_l(P_{CSF}(s), K_0(s)) = 0.$$

In the next section it is shown that it is always possible de find a right inverse $T^{\#}$ of T such that the uniqueness condition C4 is met and that ends the proof. □

The general block-diagram associated with P_{CSF} is depicted in figure 2. One can notice that the CSF is a one block problem and can be seen as a combination of well-known output estimation (OE) problem and

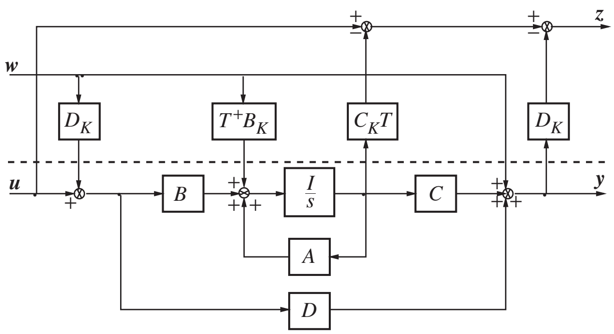

Figure 2. Block-diagram of cross standard form.

disturbance feed-forward (DF) problem (Zhou *et al.* 1996). So, if both cross transfers, $P_{zu}(s)$ and $P_{yw}(s)$, are minimum phase (no zero in the closed right half plane), then both H_2 and H_∞ syntheses converge towards the same H_∞performance index (γ) (Zhou 1992). But for the standard problem P_{CSF}, one can state that $\gamma = 0$ and that both syntheses are exactly equal.

3.2. *Uniqueness condition*

The uniqueness condition (C4) can be proven considering the H_2-optimal controller of P_{CSF}. First of all, in order to make the direct feed-trough between exogenous inputs and controlled outputs in P_{CSF} vanish, a simple change of variable ($u \leftarrow u - D_k y$) is performed to transform P_{CSF} into the problem (remember the assumption $D = 0$)

$$\overline{P_{CSF}}(s) := \left[\begin{array}{c|cc} A + BD_kC & T^\# B_k & B \\ \hline -C_kT & 0 & I_m \\ C & I_p & 0 \end{array}\right] \quad (11)$$

and thus

$$F_l(P_{CSF}, K) = F_l(\overline{P_{CSF}}, K - D_k),$$
$$\arg\min_K ||F_l(P_{CSF}, K)|| = \arg\min_K ||F_l(\overline{P_{CSF}}, K)|| + D_k.$$

In Doyle *et al.* (1989) and Zhou *et al.* (1996), it is demonstrated that a standard problem P has a unique H_2-optimal controller if and only if P is a regular problem. That is, in our case, if cross transfers

$$P_{zu}(s) := \left[\begin{array}{c|c} A + BD_kC & B \\ \hline -C_kT & I_m \end{array}\right]$$

and

$$P_{yw}(s) := \left[\begin{array}{c|c} A + BD_kC & T^\# B_k \\ \hline C & I_p \end{array}\right]$$

have no invariant zeros on the $j\omega$ axis. It is clear that the n zeros of $P_{zu}(s)$ are the n eigenvalues of $\phi_{zu} = A + B(D_kC + C_kT)$ and, considering (10), belong to the set of $n + n_k$ closed-loop eigenvalues and thus, are stable by assumption. So, $P_{zu}(s)$ has no zeros on the $j\omega$ axis.

The problem of the zeros of $P_{yw}(s)$ is more complex, the n zeros of $P_{yw}(s)$ are the n eigenvalues of $\phi_{yw} = A + BD_kC - T^\# B_kC$. Then, pre-multiplying ϕ_{yw} by $N = [\,T^+ \quad T^\perp\,]$, post-multiplying by $N^{-1} = [\,T^T \quad T^\perp\,]^T$ and using (8) we have

$$N^{-1}\phi_{yw}N = \begin{bmatrix} A_k - TBC_k & 0 \\ * & T^{\perp^T}(A + BD_kC - T^\# B_kC)T^\perp \end{bmatrix}.$$

So, the n zeros of $P_{yw}(s)$ are composed of

- n_k eigenvalues of $A_k - TBC_k$. Considering (10), these eigenvalues belong to the set of $n + n_k$ closed-loop eigenvalues and thus, are stable by assumption,
- $n - n_k$ eigenvalues of $\varphi(T^\#) = T^{\perp^T}(A + BD_kC - T^\# B_kC)T^\perp$ whose location in the complex plane is discussed in the following proposition.

Proposition 2: It is always possible to find a right inverse $T^\#$ of T such that all the $n - n_k$ eigenvalues of $\phi(T^\#)$ (and thus all the n zeros of the cross transfer P_{yw}) are not on the $j\omega$ axis.

Proof: The set of right-inverse matrices of T can be parameterized in the following way (Brookes 2005):

$$T^\# = T^+ + T^\perp X,$$

where X is a $(n - n_k) \times n_k$ arbitrary matrix. Then

$$\varphi(T^\#) = \varphi(X) = T^{\perp^T}(A + BD_kC)T^\perp - XB_kCT^\perp. \quad (12)$$

X allows the $n - n_k$ eigenvalues of ϕ to be assigned in the s plane (see numerical example in remark 3). The computation of X is in fact an eigenvalue assignment problem by a state feedback X^T on the pair $(T^{\perp^T}(A + BD_kC)^T T^\perp, (B_kCT^\perp)^T)$. □

Therefore, Proposition 2 allows to state that $P_{zu}(s)$ has no zeros on the $j\omega$ axis. Thus $\overline{P_{CSF}}(s)$ is regular and $K_0(s)$ is the unique solution of the H_2-optimal problem P_{CSF}.

As $F_l(P_{CSF}, K_0) = 0$, all controllers solution of the H_∞-optimal problem are also solutions of the H_2-optimal problem. Thus $K_0(s)$ is also the unique solution of the H_∞-optimal problem P_{CSF}.

3.3. *Existence of a CSF for a given {G(s), K₀(s)}*

The CSF setup is based on the existence of a full row rank matrix T (solution of the generalized Riccati equation (5)). This Riccati equation can be solved using the technique of invariant subspaces.

(i) Find an invariant subspace ($S = \mathrm{Im}(U)$) of dimension n of A_{cl}

$$A_{cl}U - U\Lambda. \tag{13}$$

This subspace is associated with a set of n eigenvalues, spec(Λ), among $n + n_k$ eigenvalues of A_{cl}. Such subspaces can be calculated with a Schur decomposition of A_{cl} (see Golub and van Loan (1996) for more details).

(ii) Split the vectors of U.

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad U_1 \in \mathbb{R}^{n \times n}$$

(iii) Calculate the solution $T = U_2 U_1^{-1}$.

The existence of such a matrix T, when A_{cl} has distinct eigenvalues, is demonstrated in Narasimhamurthi and Wu (1977). However in general case, T is not unique. Furthermore T must be right invertible (i.e. U_2 must be full row rank).

Some necessary conditions for U_1 to be invertible and U_2 to be full row rank can be given.

- All uncontrollable eigenvalues in the (A, B) pair are also eigenvalues of A_{cl} and must be in spec(Λ) for U_1 to be invertible (see Alazard and Apkarian (1999) for the proof).
- All uncontrollable eigenvalues in the (A_k, B_k) pair are also eigenvalues of A_{cl} and must be in spec(Λ) for U_2 to be full row rank.

Proof: Let λ an uncontrollable eigenvalue in the (A_k, B_k) pair associated with left eigen-vector u. That is,

$$u^T[A_k - \lambda I_{n_k}] = 0 \quad \text{and} \quad u^T B_k = 0, \tag{14}$$

then pre-multiplying (13) by $[0_{1\times n} \quad u^T]$, it follows that ($A_{cl}$ is defined in equation (6) with assumption $D = 0$)

$$u^T B_k C U_1 + u^T A_k U_2 = u^T U_2 \Lambda,$$

or (using (14))

$$u^T U_2(\Lambda - \lambda I_n) = 0.$$

So, if $\lambda \notin \text{spec}(\Lambda)$ then $u^T U_2 = 0$, that is, U_2 is not right invertible. □

Remark 1: From a practical point of view, conditions for the non-existence of a right invertible matrix T are very restrictive. Note also that the set of solutions can be reduced so that T is real. A necessary condition for T to be real is: U must be auto-conjugate and spec(A_{cl}) must contain at least on real eigenvalue if n is odd.

Remark 2: If $K_0(s)$ is not a stabilizing controller, it is still possible to solve in T the generalized equation (5). The standard problem P_{CSF} given by (4) is still regular if there are no closed-loop eigenvalues on the imaginary axis. Then, H_2 or H_∞ design on the standard plant P_{CSF} will provide a stabilizing controller $\hat{K}(s) \neq K_0(s)$. It can be shown (Alazard *et al.* 1999) that the closed-loop eigenvalues of $F_l(P_{CSF}, \hat{K})$ are then assigned on the stable eigenvalues and on the opposites of the unstable eigenvalues of $F_l(P_{CSF}, K_0)$. This property will be used in the example given in § 5.

Proposition 3 (Existence of a CSF)**:** *The non-existence of a full row rank matrix T solution of the generalized non-symmetric Riccati equation* (5) *implies the non-existence of a CSF for $G(s)$ and $K_0(s)$.*

Proof (by contraction)**:** Let us assume that a regular CSF exists for the strictly proper stabilizing controller $K_0(s) - D_k$ and for the stabilizable and detectable modified system $\bar{G}(s)$ (such a change of variable is not restrictive)

$$\bar{G}(s) := \left[\begin{array}{c|c} A + BD_kC & B \\ \hline C & 0 \end{array}\right].$$

Then it is shown in Doyle *et al.* (1989) that the unique solution $\widehat{K_{H_2}}$ of the corresponding H_2 optimal problem involves a state feedback gain K and a state estimator gain G. The nth order state space realization of such a controller associated with the state vector $\hat{x}$ reads

$$\widehat{K_{H_2}} := \left[\begin{array}{c|c} A + BD_kC + KB + GC & -G \\ \hline K & 0 \end{array}\right]. \tag{15}$$

As the solution is unique: $\widehat{K_{H_2}}(s) = K_0(s) - D_k$. Thus the state space realization (15) in non-minimal if $n_k < n$. Thus a projection matrix $S_{n_k \times n}$ (full-row rank) exist such that $x_k = S\hat{x}$ and

$$\begin{aligned} S(A + BD_kC + BK + GC) &= A_kS \\ -SG &= B_k \\ K &= C_kS. \end{aligned}$$

So S solves the following equation

$$S(A + BD_kC) + SBC_kS - B_kC - A_kS = 0. \tag{16}$$

This equation is exactly the same as the Riccati equation (8) in T. Thus, if T (or S) does not exist, then the CSF for given $\bar{G}(s)$ and $K_0(s) - D_k$ (or $G(s)$ and $K_0(s)$) does not exist.

Remark 3: This last proposition highlights that the unique controller $\hat{K}(s)$ provided by H_2 or H_∞ design on P_{CSF} is non-minimal. It can be shown that the $n - n_k$ non-minimal dynamics in $\hat{K}(s)$ are assigned to the eigenvalues of $\phi(X)$ (equation (12)) and thus can be assigned by a suitable choice of X.

Numerical example: The results if this section are illustrated on a very simple example. Let us consider the system

$$G(s) = \frac{1}{s^2 - 2s + 1} := \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right],$$

and the initial controller

$$K_0(s) = \frac{-30s}{s+10} := \left[\begin{array}{c|c} -10 & 10 \\ \hline 30 & -30 \end{array}\right].$$

$$P_{CSF}(s) := \left[\begin{array}{cc|cc} A & T^+(A_k - TBC_k)V & T^+B_k - BD_k & B \\ 0 & V^+A_kV & V^+B_k & 0 \\ \hline -(C_kT + D_kC) & -C_kV & -D_k & I_m \\ C & 0 & I_p & 0 \end{array}\right]. \quad (17)$$

The only real solution T of (8) reads

$$T = \begin{bmatrix} -0.11485 & 0.98248 \end{bmatrix}.$$

Let us choose $T^\# = T^+$, then the CSF reads

$$P_{CSF} := \left[\begin{array}{cc|cc} 2 & -1 & 28.826 & 1 \\ 1 & 0 & 10.0410 & 0 \\ \hline 3.4455 & 0.52566 & 30 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

It is easy to check that the optimal H_∞ controller reads

$$K_\infty(s) = \frac{-30s(s+1.487)}{(s+10)(s+1.487)}.$$

Furthermore, equation (12) reads

$$\varphi(X) = -1.4866 - 1.1611X \quad \text{and} \quad \varphi(1.3035) = -3.$$

Then the choice

$$T^\# = T^+ + 1.3035T^\perp = \begin{bmatrix} 1.1773 & 1.1555 \end{bmatrix}^T$$

leads to a new P_{CSF} and a new optimal H_∞ controller

$$K_\infty(s) = \frac{-30s(s+3)}{(s+10)(s+3)}.$$

In both designs, K_∞ is not minimal and $K_\infty = K_0$.

4. Generalization

For briefness, the proof of following results are omitted.

4.1. *Augmented-order controller case* ($n_k > n$)

It is supposed that $D = 0$, $K_0(s)$ is minimal and $n_k > n$. This implies that $P_{CSF}(s)$ must be at least of order n_k. Then the CSF can be dressed in the following way.

- solve in T (now, a column matrix because $n_k > n$) the generalized Riccati equation (5),
- compute a matrix $V_{n_k \times (n_k - n)}$ such that $[T \quad V]$ is invertible ($V = T^{T^\perp}$),
- compute T^+ and V^+ such that $T^+T = I$, $V^+V = I$ and $TT^+ + VV^+ = I$.

Then, a CSF satisfying C1 to C4 reads (17).

4.2. *2 degrees of freedom controllers*

The general connection of a standard plant $P(s)$ with a 2 degrees of freedom controller is depicted in figure 3.

A state space realization of this controller between both inputs (measurement y and input reference e) and both outputs (control signal u and monitoring signal z_c) is

$$K_0(s) : \begin{bmatrix} \dot{x}_k \\ \hline u \\ z_c \end{bmatrix} = \left[\begin{array}{cc|c} A_k & B_{k_1} & B_{k_2} \\ \hline C_{k_1} & D_{k_{11}} & D_{k_{12}} \\ C_{k_2} & D_{k_{21}} & D_{k_{22}} \end{array}\right] \begin{bmatrix} x_k \\ \hline y \\ e \end{bmatrix}$$

$$= \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \begin{bmatrix} \dot{x}_k \\ \hline y \\ e \end{bmatrix}.$$

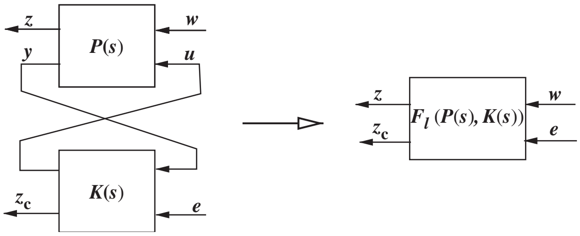

Figure 3. 2 degrees of freedom controller.

Case $n_k \le n$: The general solution P_{CSF} presented in equation (4) and the block diagram in figure 2 are still valid but involve augmented parameters

$$B \leftarrow \begin{bmatrix} B & 0_{n\times q} \end{bmatrix}, \quad C \leftarrow \begin{bmatrix} C \\ 0_{r\times n} \end{bmatrix}, \quad D \leftarrow \begin{bmatrix} D & 0_{p\times q} \\ 0_{r\times m} & 0_{r\times q} \end{bmatrix},$$

where r and q are length of vector e and z_c respectively. This CSF works with an augmented control input $[u^T \quad z_c^T]^T$ and an augmented measurement output $[y^T \quad e^T]^T$.

The case $n_k > n$ is not given here, but could be easily found with previous cases.

5. Academic example

This example is given to illustrate the way to use the CSF to take into account an initial low-order controller and a parametric robustness specification in an augmented standard problem. The initial controller $K_0(s)$ is a well-known proportional derivative controller designed on a simplified model $G_0(s)$ to assign the dominant dynamics. Although $K_0(s)$ does not stabilize the full-order model $G(s)$, the CSF is used to find a stabilizing controller $K(s)$ allowing the dominant dynamics to be assigned on the desired value in spite of parametric uncertainties.

Let us consider the simplified model of a positioning device between the force u (input) applied on the load m (mass) and its position x (output)

$$G_0(s) = \frac{1}{ms^2}.$$

A simple proportional-derivative controller is designed to assign the poles of $G_0(s)$ to a second order dynamics (pulsation ω, damping ratio ξ). A fast pole (-10ω) is included for $K_0(s)$ to be proper

$$K_0(s) = -m\frac{\omega^2 + 2\xi\omega s}{1 + (s/10\omega)}.$$

A state space realization of $K_0(s)$ reads

$$\left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] = \left[\begin{array}{c|c} -10\omega & 20\omega - 1 \\ \hline 10\omega^3 m & 20\xi\omega^2 m \end{array}\right] \qquad (18)$$

N.A.: $m = 1\,Kg$, $\omega = 1\ rd/s$, $\xi = \sqrt{2}/2$.

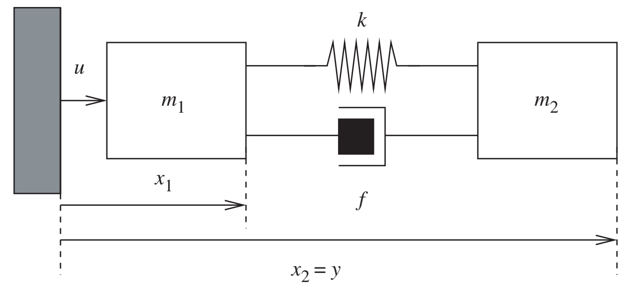

Figure 4. Spring-mass model.

The closed loop dynamics assignment is highlighted on the root locus of the loop gain $-K_0(s)G_0(s)$ (figure 5: grey curves).

The full order model $G(s)$ takes into account the stiffness k and the low damping factor f of the transmission between the mass m_1 of the positioning device and the mass m_2 of the payload (see figure 4) whose position x_2 is the only measurement. Furthermore, the stiffness k is uncertain,

$$k \in \left[(\bar{k}(1-\delta),\ \bar{k}(1+\delta)\right].$$

N.A.: $m_1 = m_2 = 0.5\,Kg$; $\bar{k} = 1N/m$; $\delta = 0.3$ (30%); $f = 0.0025\ Ns/m$.

One can show Alazard *et al.* (1999) that the uncertain model can be represented by the following $M(s) - \Delta$ form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_1 \\ \ddot{x}_2 \\ \hline z_{\Delta_k} \\ y \end{bmatrix} = \underbrace{\left[\begin{array}{c|cc} A & B_\Delta & B \\ \hline C_\Delta & D_\Delta & D_\Delta \\ C & D_\Delta & D \end{array}\right]}_{:=M(s)} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \\ \hline w_{\Delta_k} \\ u \end{bmatrix}$$

$$w_{\Delta_k} = \Delta z_{\Delta_k} \quad \forall \Delta \in [-1, +1]$$

with

$$M(s) := \left[\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\bar{k}}{m_1} & \frac{\bar{k}}{m_1} & -\frac{f}{m_1} & \frac{f}{m_1} & -\frac{\sqrt{\delta}}{m_1} & \frac{1}{m_1} \\ \frac{\bar{k}}{m_2} & -\frac{\bar{k}}{m_2} & \frac{f}{m_2} & -\frac{f}{m_2} & \frac{\sqrt{\delta}}{m_2} & 0 \\ \hline \sqrt{\delta\bar{k}} & -\sqrt{\delta\bar{k}} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

and $G(s) = F_u(M(s), 0) = D + C(sI - A)^{-1}B$.

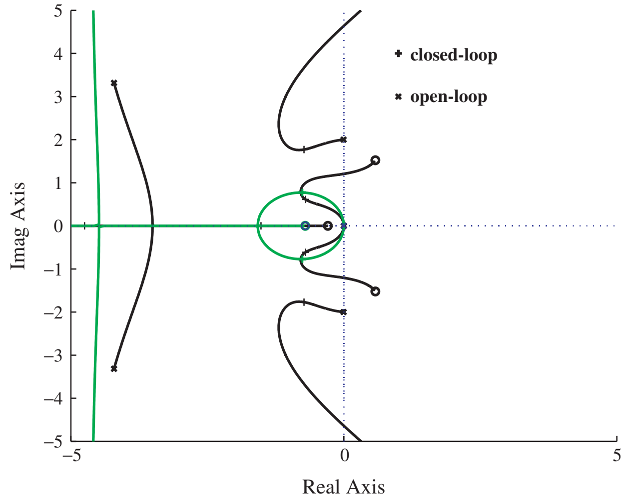

Figure 5. Root loci: $-K_0G_0$ (grey), $-\hat{K}G$ (black).

It is well known that a proportional-derivative controller cannot stabilize such a non-collocated spring-mass system. Indeed, the closed-loop dynamics of $F_l(M, K_0)$ is

$$\{0.68 \pm 2.02i, \quad -0.71 \pm 0.61i, \quad -9.95\}.$$

To solve the non-symmetric Riccati equation (5) using the invariant subspace technique, we choose a subspace associated with the 4 complex eigenvalues. Thus, the solution is

$$T = [0.0311, \quad 1.5683, \quad -0.0030, \quad -0.1577].$$

Then, one can build a two-channel standard problem

$$P(s) := \left[\begin{array}{c|ccc} A & B_\Delta & T^+B_k - BD_k & B \\ \hline C_\Delta & D_\Delta & 0 & D_\Delta \\ -C_kT - D_kC & 0 & -D_k + D_kDD_k & I_m - D_kD \\ C & D_\Delta & I_p - DD_k & D \end{array}\right]. \tag{19}$$

The first channel of this problem corresponds to the robustness problem on the uncertain parameter k. A sufficient condition for robust stability is: the H_∞ norm of this channel must be less than 1. The second channel corresponds to the CSF (equation (4)). Thus, the minimization of both channels will provide a stabilizing controller and the resulting closed-loop rigid dynamics will be inflected towards the initial stable assignment (see remark 2). Indeed the third order controller $\hat{K}(s)$ thus obtained, using the

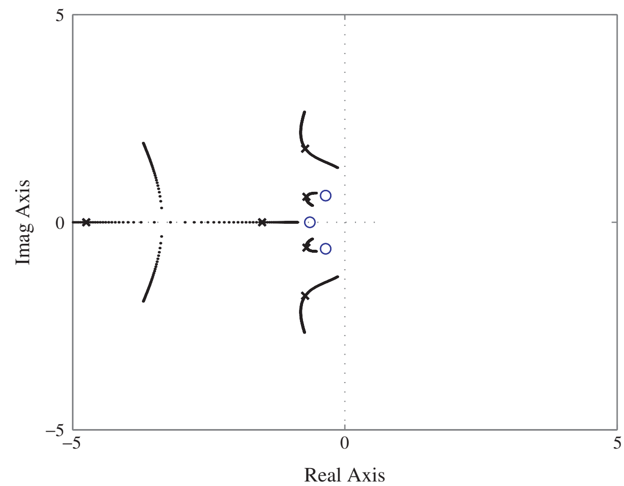

Figure 6. Root locus of $F_u(F_l(M,\hat{K}),\Delta)$ for $\Delta \in [-1,1]$.

macro-function `hinflmi` from the Matlab™ LMI control toolbox, reads

$$\hat{K}(s) = \arg\min_{K(s)} \|F_lP(s), K(s))\|_\infty = \frac{-73.62s^3 + 63.66s^2 - 170.4s - 57.16}{s^3 + 19.07s^2 + 118.4s + 305.8}$$

and the H_∞ closed-loop performance on the first channel is 1.02. One can check on the root locus of $-\hat{K}(s)G(s)$ (figure 5: black curves) that the rigid mode is correctly assigned close to the location prescribed by the initial controller $K_0(s)$. Figure 6 highlights the closed-loop stability for all values of the uncertain parameter. It appears also that this uncertainty has a weak influence on the rigid mode assignment.

6. Conclusions

The definition of the CSF and links with inverse H_2 and H_∞ optimal control problems were established. A general analytical solution was derived for any given system and any given stabilizing controller of arbitrary order. The interest of the CSF to take into account an *a priori* given controller and additional frequency-domain or robustness specifications in a general augmented standard problem was highlighted on an academic example. The reader will find in Voinot *et al.* (2003) a quite realistic application of the CSF in the context of discrete-time attitude control of a launcher, which demonstrates the flexibility and practical value of this methodology.

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