PERFORMANCE ANALYSIS FOR A CLASS OF ROBUST ADAPTIVE BEAMFORMERS

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ABSTRACT

Robust adaptive beamforming is a key issue in array applications where there exist uncertainties about the steering vector of interest. Diagonal loading is one of the most popular techniques to improve robustness. Recently, worst-case approaches which consist of protecting the array's response in an ellipsoid centered around the nominal steering vector have been proposed. They amount to generalized (i.e. non necessarily diagonal) loading of the covariance matrix. In this paper, we present a theoretical analysis of the signal to interference plus noise ratio (SINR) for this class of robust beamformers, in the presence of random steering vector errors. A closed-form expression for the SINR is derived which is shown to accurately predict the SINR obtained in simulations. This theoretical formula is valid for any loading matrix. It provides insights into the influence of the loading matrix and can serve as a helpful guide to select it. Finally, the analysis enables us to predict the level of uncertainties up to which robust beamformers are effective and then depart from the optimal SINR.

1. INTRODUCTION

Adaptive beamforming is an essential task in most systems using an array of sensors. When the signal of interest (SOI) is present in the measurements, the optimal beamformer is the minimum power distortionless response (MPDR) beamformer [1] and is given, up to a scaling factor, by $\boldsymbol{R}^{-1}\boldsymbol{a}$. In the previous equation, $\boldsymbol{R}$ is the total covariance matrix (including the SOI and possibly interferences and noise) while $\boldsymbol{a}$ stands for the (presumed) steering vector of the signal of interest. In order to implement the optimal beamformer, $\boldsymbol{a}$ needs to be known precisely, a property that is most often not encountered in practice where there are unavoidably mismatches between the actual steering vector and the presumed steering vector. The reasons for that are numerous: local scattering, uncertainties about the direction of arrival, propagation through an inhomogeneous medium, fading, uncalibrated arrays, displaced sensors, etc. These mismatches are especially detrimental when the SOI is present in the data as the latter is considered as an interference and thus tends to be eliminated, see e.g. [2].

Therefore robust adaptive beamforming has emerged as a necessary constituent of most systems using an array of sensors. We refer the reader to [1, Chapter 6] for a comprehensive overview. The most widely used method, due to its simplicity and effectiveness, is diagonal loading [3] which consists of adding a scaled identity matrix to the covariance matrix prior to inversion. Interestingly enough, generalized loading turns out to be the solution to worst-case approaches recently proposed in [4–6]. In the latter references, the beamformer is designed so as to minimize the output power subject to the constraint that the beamformer's response be above some level for all the steering vectors which lie in an ellipsoid centered around the nominal or presumed steering vector of interest. This guarantees that the signal of interest, whose steering vector is expected to lie in the ellipsoid, will not be eliminated and hence robustness is improved. The solution to this problem is given by $(\boldsymbol{R}+\boldsymbol{Q})^{-1}\overline{\boldsymbol{a}}$ where $\overline{\boldsymbol{a}}$ denotes the nominal steering vector (in the absence of any uncertainty) and $\boldsymbol{Q}$ stands for the loading matrix. When the ellipsoid is a sphere the solution boils down to diagonal loading i.e. $\boldsymbol{Q} \propto \boldsymbol{I}$.

Through numerical simulations, this robust adaptive beamformer was shown to perform well, at least when the size of the ellipsoid does not grow large. However, no theoretical analysis was provided and assessing its performance remains an open problem. The finite-sample SINR analysis of the minimum variance distortionless beamformer (MVDR) was presented in [7] where the probability density function (pdf) of the SINR loss (compared to perfectly known interference plus noise covariance matrix) is derived. Similar finite-sample SINR's pdf analysis for the MPDR beamformer are reported in [8], see also [9, 10]. In contrast, analysis of diagonally loaded versions of the MPDR is scarce. A large-sample analysis of the weight vector and powers at the output of a diagonally loaded MPDR can be found in [11, 12] while [13] derives the pdf of the beam response when diagonal loading is used. In this paper, we present a theoretical analysis of the SINR obtained with a general loading matrix and in the presence of random steering vector uncertainties. The formulas obtained are quite simple and are valid for any loading matrix $\boldsymbol{Q}$ (including non diagonal or non invertible). Additionally, they are shown to predict well the performances obtained via numerical simulations. Finally, they can serve to provide insights into the choice of the loading matrix.

2. DATA MODEL

We consider an array composed of m sensors and assume that the array's output can be written as

$$\boldsymbol{x}_t = \boldsymbol{a}s_t + \boldsymbol{n}_t \qquad t = 1, \cdots, N \tag{1}$$

where $\boldsymbol{a}$ is the actual steering vector of the source of interest, and s_t is the corresponding emitted signal. We assume that $\boldsymbol{a}$ is a complex-valued, circularly symmetric random vector with mean $\mathcal{E}\{\boldsymbol{a}\} = \overline{\boldsymbol{a}}$ and covariance matrix $\mathcal{E}\left\{(\boldsymbol{a}-\overline{\boldsymbol{a}})(\boldsymbol{a}-\overline{\boldsymbol{a}})^H\right\} = \boldsymbol{C}_a$. $\overline{\boldsymbol{a}}$ corresponds to the steering vector without any perturbation (e.g. for a perfectly calibrated array) while $\boldsymbol{C}_a$ captures the effects of all possible errors affecting the steering vector. s_t is a zero-mean random process with power $P = \mathcal{E}\left\{|s_t|^2\right\}$. $\boldsymbol{n}_t$ is the noise contribution, including interferences and thermal noise. $\boldsymbol{n}_t$

is assumed to be drawn from a zero-mean complex-valued Gaussian distribution with covariance matrix $\boldsymbol{C}$.

The robust adaptive beamformers of [4–6] (although their formulations are different) are obtained as the solution to the following minimization problem

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \boldsymbol{R} \boldsymbol{w} \text{ subject to } \left|\boldsymbol{w}^H \boldsymbol{a}\right| \geq 1 \quad \forall \boldsymbol{a} = \overline{\boldsymbol{a}} + \boldsymbol{B}\boldsymbol{u}; \|\boldsymbol{u}\| \leq 1 \tag{2}$$

where $\boldsymbol{R}$ is the covariance matrix and $\boldsymbol{B}$ is a $m \times r$ matrix with full column rank which defines an ellipsoid centered around $\overline{\boldsymbol{a}}$. Under mild assumptions, the solution to (2) is given by (up to a scaling factor which does not affect the SINR) $\boldsymbol{w} = (\boldsymbol{R}+\boldsymbol{Q})^{-1}\overline{\boldsymbol{a}}$ with $\boldsymbol{Q} = \lambda \boldsymbol{B}\boldsymbol{B}^H$ and where the Lagrange multiplier λ is obtained as the solution of a secular equation which involves the eigenvalue decomposition of $\boldsymbol{B}^H \boldsymbol{R}^{-1} \boldsymbol{B}$, see [5] for details. Note that when $\boldsymbol{B} = \boldsymbol{I}$ the ellipsoid becomes a sphere and the solution amounts to conventional diagonal loading.

3. SINR ANALYSIS

In this section, we provide a theoretical expression for the robust adaptive beamformer's average SINR. We proceed as follows. In a first step, we assume that $\boldsymbol{a}$ is given and derive the corresponding SINR. Then, we invoke the conditional expectation rule to compute the average SINR as

$$\overline{\text{SINR}} = \mathcal{E}_{\boldsymbol{a}}\left\{\text{SINR}_{|\boldsymbol{a}}\right\} \tag{3}$$

where $\mathcal{E}_{\boldsymbol{a}}\{.\}$ denotes the expectation with respect to (w.r.t.) the probability density function of $\boldsymbol{a}$ and $\text{SINR}_{|\boldsymbol{a}}$ corresponds to the SINR for a given $\boldsymbol{a}$. In order to obtain $\text{SINR}_{|\boldsymbol{a}}$, note that the weight vector, for a given $\boldsymbol{a}$, is given by

$$\boldsymbol{w}_{|\boldsymbol{a}} = \left(\boldsymbol{R}_{|\boldsymbol{a}} + \boldsymbol{Q}\right)^{-1} \overline{\boldsymbol{a}} \tag{4}$$

where $\boldsymbol{R}_{|\boldsymbol{a}}$ denotes the covariance matrix for a given $\boldsymbol{a}$. Herein we do not consider finite-sample effects, i.e. we assume that the true covariance matrix $\boldsymbol{R}_{|\boldsymbol{a}}$ is available. In order to introduce finite-sample effects (and thereby to combine them with steering vector errors), one needs to assume that the two errors are of the same order of magnitude. In our case, this would amount to assuming that $\boldsymbol{C}_a = \overline{\boldsymbol{C}}_a/N$ where $\overline{\boldsymbol{C}}_a$ is fixed and N denotes the number of snapshots. We refer the reader to [14] for a detailed and comprehensive discussion on this issue. However, this assumption may seem arbitrary since the errors are not likely to depend on N. Therefore, herein we consider that N is large enough so that the steering vector errors dominate. We will however illustrate the finite-sample behavior of (4) in the numerical simulations.

Using (4), the conditional SINR is

$$\text{SINR}_{|\boldsymbol{a}} = \frac{P\left|\boldsymbol{w}_{|\boldsymbol{a}}^H \boldsymbol{a}\right|^2}{\boldsymbol{w}_{|\boldsymbol{a}}^H \boldsymbol{C} \boldsymbol{w}_{|\boldsymbol{a}}} = \frac{P\left|\overline{\boldsymbol{a}}^H\left(\boldsymbol{R}_{|\boldsymbol{a}}+\boldsymbol{Q}\right)^{-1}\boldsymbol{a}\right|^2}{\overline{\boldsymbol{a}}^H\left(\boldsymbol{R}_{|\boldsymbol{a}}+\boldsymbol{Q}\right)^{-1}\boldsymbol{C}\left(\boldsymbol{R}_{|\boldsymbol{a}}+\boldsymbol{Q}\right)^{-1}\overline{\boldsymbol{a}}} \tag{5}$$

However, using (1) along with the assumptions made, one has

$$\boldsymbol{R}_{|\boldsymbol{a}} = P\boldsymbol{a}\boldsymbol{a}^H + \boldsymbol{C} \tag{6}$$

Inserting (6) in (5) and after some straightforward algebraic manipulations, it can be shown that

$$\text{SINR}_{|\boldsymbol{a}} = \frac{1}{P\left(\boldsymbol{a}-\gamma(\boldsymbol{a})\overline{\boldsymbol{a}}\right)^H \boldsymbol{Z}\left(\boldsymbol{a}-\gamma(\boldsymbol{a})\overline{\boldsymbol{a}}\right)} \tag{7}$$

with

$$\gamma(\boldsymbol{a}) = \frac{1+P\boldsymbol{a}^H\widetilde{\boldsymbol{Q}}^{-1}\boldsymbol{a}}{P\boldsymbol{a}^H\widetilde{\boldsymbol{Q}}^{-1}\overline{\boldsymbol{a}}} \tag{8}$$

and where $\widetilde{\boldsymbol{Q}} = \boldsymbol{Q}+\boldsymbol{C}$ and $\boldsymbol{Z} = \widetilde{\boldsymbol{Q}}^{-1}\boldsymbol{C}\widetilde{\boldsymbol{Q}}^{-1}$. The average SINR is thus given by

$$\overline{\text{SINR}} = \int \frac{p(\boldsymbol{a})}{P\left(\boldsymbol{a}-\gamma(\boldsymbol{a})\overline{\boldsymbol{a}}\right)^H \boldsymbol{Z}\left(\boldsymbol{a}-\gamma(\boldsymbol{a})\overline{\boldsymbol{a}}\right)}\, d\boldsymbol{a} \tag{9}$$

where $p(\boldsymbol{a})$ is the probability density function of $\boldsymbol{a}$. Obtaining a closed-form expression for the previous integral appears to be an intractable task. Hence, we prefer to approximate this integral. Towards this end, it can be shown that for any scalar function $f(\boldsymbol{a})$ and assuming that $\boldsymbol{a}$ is circularly symmetric

$$\int f(\boldsymbol{a})p(\boldsymbol{a})\, d\boldsymbol{a} \simeq f(\overline{\boldsymbol{a}}) + \text{Tr}\left\{\left.\frac{\partial^2 f}{\partial\boldsymbol{a}\partial\boldsymbol{a}^H}\right|_{\overline{\boldsymbol{a}}} \boldsymbol{C}_a\right\} \tag{10}$$

where $\text{Tr}\{.\}$ stands for the trace of the matrix between braces. It should be pointed out that the previous approximation does not require complete knowledge of the pdf of $\boldsymbol{a}$ but only of its mean and covariance matrix, which is an appealing feature. We now apply the result (10) to $\text{SINR}_{|\boldsymbol{a}}$. However, in the purpose of simplifying subsequent derivations, we first simplify the expression of $\text{SINR}_{|\boldsymbol{a}}$ by approximating $\gamma(\boldsymbol{a})$. More precisely, we propose to approximate $\gamma(\boldsymbol{a})$ by its statistical mean. Observing that

$$\frac{\partial\gamma}{\partial\boldsymbol{a}} = \frac{\widetilde{\boldsymbol{Q}}^{-1}\left(\boldsymbol{a}-\gamma\overline{\boldsymbol{a}}\right)}{\boldsymbol{a}^H\widetilde{\boldsymbol{Q}}^{-1}\overline{\boldsymbol{a}}} \tag{11a}$$

$$\frac{\partial^2\gamma}{\partial\boldsymbol{a}\partial\boldsymbol{a}^H} = \frac{\widetilde{\boldsymbol{Q}}^{-1}}{\boldsymbol{a}^H\widetilde{\boldsymbol{Q}}^{-1}\overline{\boldsymbol{a}}}\left(\boldsymbol{I} - \frac{\overline{\boldsymbol{a}}\boldsymbol{a}^H\widetilde{\boldsymbol{Q}}^{-1}}{\boldsymbol{a}^H\widetilde{\boldsymbol{Q}}^{-1}\overline{\boldsymbol{a}}}\right) \tag{11b}$$

it follows, using (10) that

$$\mathcal{E}_{\boldsymbol{a}}\{\gamma(\boldsymbol{a})\} \simeq \frac{1+P\overline{\boldsymbol{a}}^H\widetilde{\boldsymbol{Q}}^{-1}\overline{\boldsymbol{a}}}{P\overline{\boldsymbol{a}}^H\widetilde{\boldsymbol{Q}}^{-1}\overline{\boldsymbol{a}}} + \frac{\text{Tr}\left\{\widetilde{\boldsymbol{Q}}^{-1}\boldsymbol{C}_a\right\}}{\overline{\boldsymbol{a}}^H\widetilde{\boldsymbol{Q}}^{-1}\overline{\boldsymbol{a}}} - \frac{\overline{\boldsymbol{a}}^H\widetilde{\boldsymbol{Q}}^{-1}\boldsymbol{C}_a\widetilde{\boldsymbol{Q}}^{-1}\overline{\boldsymbol{a}}}{\left(\overline{\boldsymbol{a}}^H\widetilde{\boldsymbol{Q}}^{-1}\overline{\boldsymbol{a}}\right)^2} \triangleq \gamma_0 \tag{12}$$

We propose to replace $\gamma(\boldsymbol{a})$ by γ_0 in (7) so that

$$\text{SINR}_{|\boldsymbol{a}} \simeq \frac{1}{P\left(\boldsymbol{a}-\gamma_0\overline{\boldsymbol{a}}\right)^H \boldsymbol{Z}\left(\boldsymbol{a}-\gamma_0\overline{\boldsymbol{a}}\right)} \tag{13}$$

We would like to point out that using γ_0 in lieu of $\gamma(\boldsymbol{a})$ in (10) does not result in a less accurate expression for the average SINR, as will be illustrated below. The average SINR is thus approximated by

$$\overline{\text{SINR}} \simeq \int \frac{p(\boldsymbol{a})}{P\left(\boldsymbol{a}-\gamma_0\overline{\boldsymbol{a}}\right)^H \boldsymbol{Z}\left(\boldsymbol{a}-\gamma_0\overline{\boldsymbol{a}}\right)}\, d\boldsymbol{a} \tag{14}$$

Let $f(\boldsymbol{a}) = \left(\boldsymbol{a}-\gamma_0\overline{\boldsymbol{a}}\right)^H \boldsymbol{Z}\left(\boldsymbol{a}-\gamma_0\overline{\boldsymbol{a}}\right)$ so that $\text{SINR}_{|\boldsymbol{a}} \simeq [Pf(\boldsymbol{a})]^{-1}$. First, observe that

$$\frac{\partial 1/f}{\partial\boldsymbol{a}} = \frac{-1}{f^2}\frac{\partial f}{\partial\boldsymbol{a}}$$

$$\frac{\partial^2 1/f}{\partial\boldsymbol{a}\partial\boldsymbol{a}^H} = \frac{-1}{f^2}\frac{\partial^2 f}{\partial\boldsymbol{a}\partial\boldsymbol{a}^H} + \frac{2}{f^3}\frac{\partial f}{\partial\boldsymbol{a}}\frac{\partial f}{\partial\boldsymbol{a}^H} \tag{15}$$

Next, using

$$\frac{\partial f}{\partial \boldsymbol{a}} = \boldsymbol{Z}\left(\boldsymbol{a} - \gamma_0 \overline{\boldsymbol{a}}\right); \quad \frac{\partial^2 f}{\partial \boldsymbol{a} \partial \boldsymbol{a}^H} = \boldsymbol{Z} \tag{16}$$

along with (10), (14) and (15) yields

$$\overline{\mathrm{SINR}} \simeq \frac{1}{Pf(\overline{\boldsymbol{a}})} - \frac{\mathrm{Tr}\left\{\boldsymbol{Z}\boldsymbol{C}_a\right\}}{Pf^2(\overline{\boldsymbol{a}})} + \frac{2\left|1-\gamma_0\right|^2 \overline{\boldsymbol{a}}^H \boldsymbol{Z}\boldsymbol{C}_a\boldsymbol{Z}\overline{\boldsymbol{a}}}{Pf^3(\overline{\boldsymbol{a}})}$$
$$f(\overline{\boldsymbol{a}}) = \left|1-\gamma_0\right|^2 \overline{\boldsymbol{a}}^H \boldsymbol{Z}\overline{\boldsymbol{a}} \tag{17}$$

The previous equation provides a closed-form and compact expression for the average SINR. We stress the fact that it holds for a large class of robust adaptive beamformers as it holds for any loading matrix $\boldsymbol{Q}$ and any steering vector error covariance matrix $\boldsymbol{C}_a$. As will be illustrated in the next section, it predicts very accurately the average SINR obtained through Monte-Carlo simulations. Hence, it can serve as a useful tool to obtain rapid insights into the choice of the loading matrix $\boldsymbol{Q}$ without resorting to extensive simulations.

4. NUMERICAL ILLUSTRATIONS AND CONCLUSIONS

The aim of this section is threefold. Firstly, we assess the validity of the theoretical formula (17) by comparing it with the actual SINR obtained through Monte-Carlo simulations. Secondly, we provide illustrations of up to which level of uncertainty generalized loading can compensate for steering vector errors and still provide a performance close to optimum. Thirdly, we provide rules of thumb for selecting the shape and the size of the loading matrix. In all simulations, we consider a uniform linear array of $m = 10$ sensors spaced a half-wavelength apart. The signal of interest impinges from broadside and thus $\overline{\boldsymbol{a}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$. The noise component consists of a white noise contribution with power σ_n^2 and two interferences whose DOAs are $-20°$, $30°$ and whose powers are 20dB and 30dB above the white noise level, respectively. We define the *uncertainty ratio* (UR) and the signal to noise ratio (SNR) as $UR = 10\log_{10}\left(\frac{\mathrm{Tr}\{\boldsymbol{C}_a\}}{\overline{\boldsymbol{a}}^H\overline{\boldsymbol{a}}}\right)$ and $SNR = 10\log_{10}\left(\frac{P\left(\overline{\boldsymbol{a}}^H\overline{\boldsymbol{a}}+\mathrm{Tr}\{\boldsymbol{C}_a\}\right)}{\sigma_n^2}\right)$, respectively. In all simulations, the SINR is evaluated as follows. $N_r = 500$ Monte-Carlo simulations are run with a different random $\boldsymbol{a}$ and, for a given weight vector $\boldsymbol{w}$, the *average SINR* is computed as

$$\overline{\mathrm{SINR}}(\boldsymbol{w}) = \frac{1}{N_r}\sum_{n=1}^{N_r} \frac{P\left|\boldsymbol{w}^H\boldsymbol{a}(n)\right|^2}{\boldsymbol{w}^H\boldsymbol{C}\boldsymbol{w}} \tag{18}$$

The robust adaptive beamformer (4) -which is referred to as RB in the figures-, will be compared to the following beamformers:

- the MVDR beamformer which is given by

$$\boldsymbol{w}_{\mathrm{MVDR}} = \mathcal{P}\left\{\boldsymbol{C}^{-1}\left(\overline{\boldsymbol{a}}\overline{\boldsymbol{a}}^H + \boldsymbol{C}_a\right)\right\} \tag{19}$$

  where $\mathcal{P}\left\{.\right\}$ stands for the principal eigenvector of the matrix between braces.

- a (hypothetical) clairvoyant optimum beamformer which maximizes the SINR for any given $\boldsymbol{a}$ and is thus given by

$$\boldsymbol{w}_{|\boldsymbol{a}}^{\mathrm{opt}} = \boldsymbol{C}^{-1}\boldsymbol{a} \tag{20}$$

- the sample covariance matrix (SCM) version of (4), i.e.

$$\boldsymbol{w}_{\mathrm{SCM}} = \left(\widehat{\boldsymbol{R}} + \boldsymbol{Q}\right)^{-1}\overline{\boldsymbol{a}}; \quad \widehat{\boldsymbol{R}} = \frac{1}{N}\sum_{t=1}^{N}\boldsymbol{x}_t\boldsymbol{x}_t^H \tag{21}$$

  The performance of the SCM robust beamformer will be evaluated by (18). It will enable us to take into account finite-sample effects.

In a first series of simulation, we consider the case where the steering vector errors are drawn from a zero-mean complex-valued Gaussian distribution with covariance matrix $\boldsymbol{C}_a = \sigma_a^2\boldsymbol{I}$. We study the performance of the beamformers versus the uncertainty ratio. The loading matrix $\boldsymbol{Q} = \lambda\boldsymbol{I}$ and we define the loading level (LL) as $LL = 10\log_{10}\left(\frac{\lambda}{\sigma_n^2}\right)$. Note that it corresponds to a loading level relative to the white noise power. In the simulations, LL is chosen as $LL = 5$dB. The results are displayed in Figure 1. The following observations are in order. *The theoretical formula (17) is seen to predict very accurately* the SINR obtained in simulations, to within 0.2dB for $UR \leq -2$dB. Hence (17) provides a very good picture of the robust adaptive beamformer's performance in most situations. Moreover, the finite-sample behavior of the robust beamformer is also close to the theoretical formula. The robust beamformer has a performance rather close to that of the MVDR, at least for UR's below -6dB. For higher UR, the robust beamformer can no longer compensate for the uncertainties; hence one must turn to other solutions. It should be pointed out that, for $UR \geq -6$dB, the MVDR also becomes less performant than the clairvoyant beamformer, which makes use of the actual steering vector. This suggests that *for high uncertainties the remedy would be to obtain additional information about the actual steering vector*, for instance by estimating it, rather than to protect the array's response over a larger and larger ellipsoid.

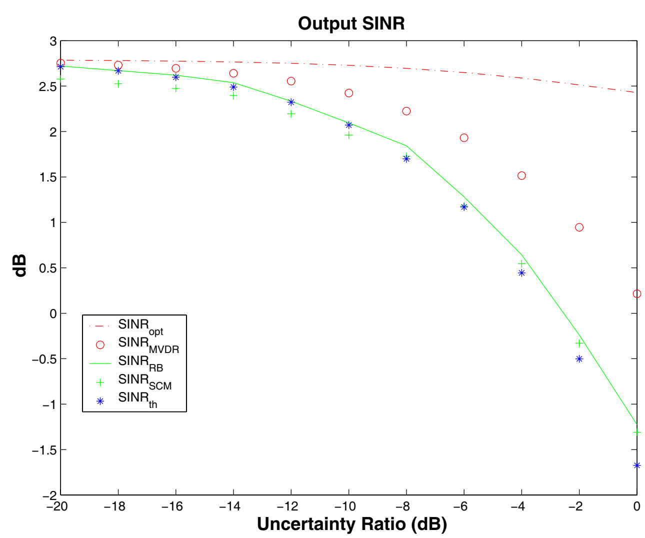

Fig. 1. Output SINR versus uncertainty ratio. $SNR = 3$dB, $LL = 5$dB and $N = 200$. $\boldsymbol{C}_a = \sigma_a^2\boldsymbol{I}, \boldsymbol{Q} \propto \boldsymbol{I}$.

Next, the influence of the loading level is studied in Figure 2. In this figure, the loading matrix is still proportional to the identity matrix and $UR = -6$dB. Again, it can be seen that the theoretical SINR is very close to the practical SINR. Also, it can be observed

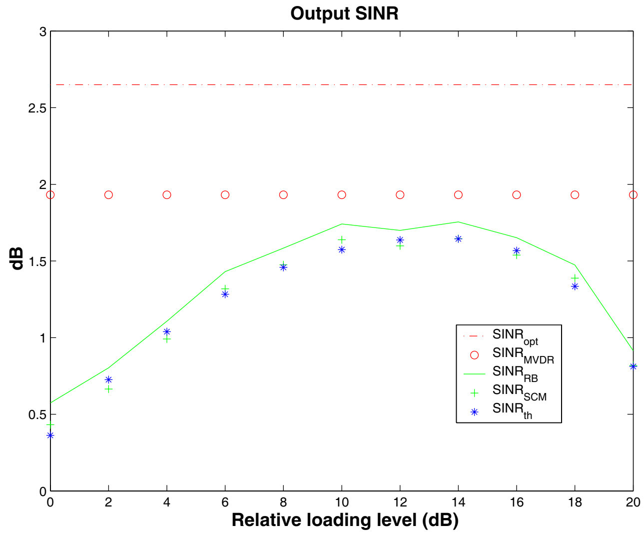

Fig. 2. Output SINR versus loading level. $SNR = 3$dB, $UR = -6$dB and $N = 200$. $\boldsymbol{C}_a = \sigma_a^2 \boldsymbol{I}, \boldsymbol{Q} \propto \boldsymbol{I}$.

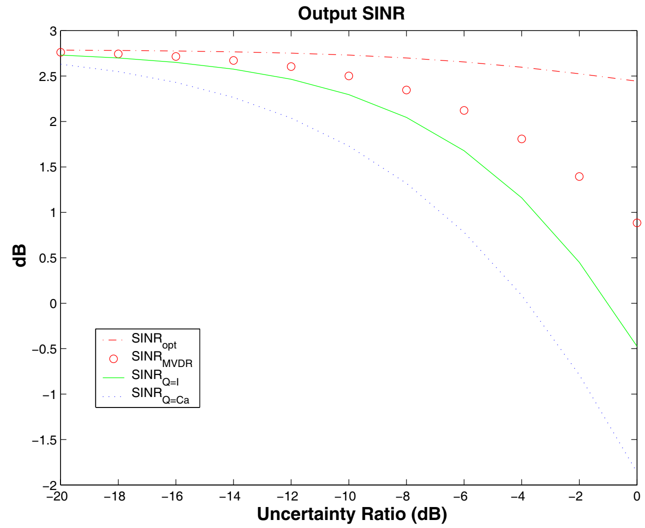

Fig. 3. Coherent local scattering. Optimum SINR obtained with the robust beamformer versus uncertainty level. $SNR = 3$dB.

that, despite LL has an influence onto the final SINR, there exist a large range of values for LL which provide a similar performance.

Finally, we study the influence of the *shape of the loading matrix* which is directly related to the form of the ellipsoid in (2). Intuitively, since $\mathcal{E}\left\{(\boldsymbol{a}-\overline{\boldsymbol{a}})(\boldsymbol{a}-\overline{\boldsymbol{a}})^H\right\} = \boldsymbol{C}_a$, it follows that $\boldsymbol{a}$ can be written as $\boldsymbol{a} = \overline{\boldsymbol{a}} + \boldsymbol{C}_a^{1/2}\boldsymbol{u}$ with $\boldsymbol{C}_a^{1/2}$ a square-root of $\boldsymbol{C}_a$. This suggests that $\boldsymbol{B}$ should be related to $\boldsymbol{C}_a^{1/2}$, or equivalently that $\boldsymbol{Q} \propto \boldsymbol{C}_a$. Hence, we check whether this intuitive hypothesis results in a better performance than using diagonal loading only. Towards this end, we consider the case of local scattering for which the steering vector cas be written as $\boldsymbol{a} = \overline{\boldsymbol{a}} + \frac{1}{\sqrt{L}}\sum_{k=1}^{L} g_k \boldsymbol{a}(\widetilde{\theta}_k)$ where g_k are zero-mean, independent and identically distributed random variables with power σ_g^2 and $\widetilde{\theta}_k$ are independent random variables with pdf $p(\widetilde{\theta})$. The covariance matrix of the errors is given by

$$\boldsymbol{C}_a = \sigma_g^2 \int \boldsymbol{a}(\widetilde{\theta})\boldsymbol{a}^H(\widetilde{\theta})p(\widetilde{\theta})\, d\widetilde{\theta} = \sigma_g^2 \breve{\boldsymbol{C}}_a \tag{22}$$

In the simulations presented below, we assume a Gaussian distribution for the scatterers with standard deviation (referred to as angular spread in the literature) $\sigma_\theta = 15°$. For each value of UR, we consider both $\boldsymbol{Q} = \lambda \boldsymbol{I}$ and $\boldsymbol{Q} = \lambda \breve{\boldsymbol{C}}_a$ (note that $\mathrm{Tr}\left\{\breve{\boldsymbol{C}}_a\right\} = \mathrm{Tr}\left\{\boldsymbol{I}\right\}$) and we look for the value of λ that results in the optimal average SINR in (17). We plot in Figure 3 the obtained optimum SINR. Examination of this figure reveals that the performance obtained with $\boldsymbol{Q} \propto \boldsymbol{C}_a$ is always inferior to that obtained with $\boldsymbol{Q} \propto \boldsymbol{I}$. So, even if the steering vector covariance matrix is not a scaled identity matrix, there is no gain in using $\boldsymbol{Q} \propto \boldsymbol{C}_a$ instead of diagonal loading. Note also, as discussed in [5], that choosing $\boldsymbol{Q} \propto \boldsymbol{C}_a$ implies that one has a strong a priori on the shape of the errors, which is seldom the case. Hence, robustness may be endangered if $\boldsymbol{Q}$ is chosen as $\boldsymbol{Q} \propto \boldsymbol{C}_a$ whereas the true covariance matrix of the steering vector errors is not $\boldsymbol{C}_a$. This provides an additional argument in favor of diagonal loading.

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