Steering vector errors and diagonal loading

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Abstract: Diagonal loading is one of the most widely used and effective methods to improve robustness of adaptive beamformers. The authors consider its application to the case of steering vector errors, i.e. when there exists a mismatch between the actual steering vector of interest and the presumed one. More precisely, the problem addressed is that of optimally selecting the loading level with a view to maximising the signal-to-interference-plus-noise ratio in the presence of steering vector errors. First, an expression is derived for the optimal loading for a given steering vector error and it is shown that this loading is negative. Next, random steering errors are considered and the optimal loading is averaged with respect to the probability density function of the steering vector errors, yielding a very simple expression for the average optimal loading. Numerical simulations attest to the validity of the analysis and show that diagonal loading with the derived optimal loading factor provides a performance close to optimum.

1 Introduction

Steering vector errors are known to be a major cause of performance degradation of adaptive beamformers, especially when the signal of interest (SOI) is present in the measurements. In this case, the latter is considered as an interference, leading to the so-called self-nulling phenomenon [1, 2]. Now, mismatches between the actual steering vector and the presumed steering vector are an unavoidable component in most applications using an array of sensors. Local scattering around the source, an inhomogeneous propagation medium, uncalibrated arrays or arrays undergoing deformations are among the many potential factors that can contribute to such errors. Hence, designing robust adaptive beamformers that can maintain good signal to interference plus noise ratio (SINR) under these conditions is of utmost importance [1, 3]. Among the many robust adaptive beamformers proposed in the literature, diagonal loading emerges as the most widely used due to its simplicity and its effectiveness in handling a wide variety of errors, including steering vector and finite-sample errors. Also, it has some very nice interpretations such as equalising the least significant eigenvalues of the covariance matrix or constraining the white noise gain [1]. Interestingly enough, it has also proved to be the solution to worst-case approaches proposed recently in [4–7], whose principle is to protect the beamformer's response for all steering vectors which lie in some ellipsoid centred around the nominal steering vector. These different interpretations plead for diagonal loading and make it a method of choice in most applications.

However, selecting the loading level remains a crucial and open question for which no theoretically sound solution exists. Indeed, the loading level enables to balance between a fully adaptive beamformer (no loading) and the conventional non-adaptive beamformer (infinite loading). Hence, its performance can vary quite significantly and finding an optimal loading level would be of major interest. It is usually admitted that a good rule of thumb is to select the loading level some 5–10 dB about the noise level (see e.g. [1], Chap. 6). In the worst-case approaches [4–7], the loading level depends on the size of the ellipsoid in which the actual steering vector is expected to lie. However, it is not clear how to choose the ellipsoid's size although some proposals are hinted at in [7]. A meaningful way of selecting the loading level is to fix the white noise gain (WNG), as suggested in [8], since diagonal loading corresponds to constraining the WNG. This is a physically appealing approach as the WNG enables to control the degree of adaptivity of the beamformer. In this paper, we attempt to provide a theoretical answer to this question. More precisely, we address the problem of finding the loading level which results in maximum SINR in the presence of steering vector errors. Towards this end, we derive an expression for the optimal loading level for any steering vector error. Since the optimal loading level, and thus the corresponding SINR, depends on the actual steering vector, we next consider random steering vector errors. The optimal loading level is then averaged with respect to (w.r.t) the probability density function (pdf) of the steering vector errors, resulting in a simple formula for the average optimal loading level.

2 Data model

We consider an array composed of m sensors and assume that the array's output can be written as

$$\boldsymbol{x}_t = \boldsymbol{a} s_t + \boldsymbol{n}_t \qquad t = 1, \ldots, N \tag{1}$$

where

- $\boldsymbol{a}$ is the actual (unknown) steering vector of the source of interest. We assume that $\boldsymbol{a}$ differs from the nominal or presumed steering vector $\bar{\boldsymbol{a}}$ due, for example, to uncertainties about the direction of arrival (DOA), unknown gains and phases of the sensors, etc.;
- s_t is the signal of interest waveform and is assumed to be a zero-mean random process with power $P = \mathcal{E}\{|s_t|^2\}$;

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- $\boldsymbol{n}_t$ is the noise contribution, including K interferences and thermal noise. Hence, the covariance matrix $\boldsymbol{C} = \mathcal{E}\{\boldsymbol{n}_t\boldsymbol{n}_t^H\} = \boldsymbol{A}_J\boldsymbol{P}_J\boldsymbol{A}_J^H + \sigma^2\boldsymbol{I}$ where $\boldsymbol{A}_J$ stands for the K-dimensional interference subspace.

In order to handle this problem, we focus here on the use of diagonal loading-based beamformers of the form

$$\boldsymbol{w} = (\boldsymbol{R} + \lambda\boldsymbol{I})^{-1}\bar{\boldsymbol{a}} \tag{2}$$

where $\boldsymbol{R}$ is the covariance matrix, $\boldsymbol{I}$ is the identity matrix of size m and λ is a real weighting factor. Observe that we do not consider here finite-sample effects, i.e. we assume that the true covariance matrix is available. In order to consider both steering vector errors and finite-sample effects, one needs to assume that the two errors are of the same order of magnitude, typically $O(1/N)$. See [9] for a detailed and comprehensive discussion on this issue. However, this assumption may seem arbitrary since the errors are not likely to depend on N. Therefore, herein we consider that N is large enough so that the steering vector errors dominate.

3 Derivation of optimal loading level

In this Section, we consider the problem of finding λ which results in a maximum SINR at the output of the beamformer. The analysis is carried out in two steps. First, we assume that the actual steering vector $\boldsymbol{a}$ is fixed and we derive the optimal value of the loading level. Since the latter will depend on $\boldsymbol{a}$, we derive in a second step its average value by assuming that the steering vector error $\boldsymbol{a} - \bar{\boldsymbol{a}}$ is random with known second-order statistics.

3.1 Optimisation for a given steering vector error

Let us first assume that $\boldsymbol{a}$ is fixed and use the subscript $_{|\boldsymbol{a}}$ to emphasise it. Let

$$\boldsymbol{w}_{|\boldsymbol{a}} = (\boldsymbol{R}_{|\boldsymbol{a}} + \lambda\boldsymbol{I})^{-1}\bar{\boldsymbol{a}} \tag{3}$$

denote the weight vector where $\boldsymbol{R}_{|\boldsymbol{a}} = P\boldsymbol{a}\boldsymbol{a}^H + \boldsymbol{C}$ stands for the covariance matrix for a given $\boldsymbol{a}$. The conditional SINR corresponding to the weight vector in (3) is thus given by

$$\mathrm{SINR}_{|\boldsymbol{a}} = \frac{P\left|\boldsymbol{w}_{|\boldsymbol{a}}^H\boldsymbol{a}\right|^2}{\boldsymbol{w}_{|\boldsymbol{a}}^H\boldsymbol{C}\boldsymbol{w}_{|\boldsymbol{a}}} = \frac{P\left|\bar{\boldsymbol{a}}^H(\boldsymbol{R}_{|\boldsymbol{a}} + \lambda\boldsymbol{I})^{-1}\boldsymbol{a}\right|^2}{\bar{\boldsymbol{a}}^H(\boldsymbol{R}_{|\boldsymbol{a}} + \lambda\boldsymbol{I})^{-1}\boldsymbol{C}(\boldsymbol{R}_{|\boldsymbol{a}} + \lambda\boldsymbol{I})^{-1}\bar{\boldsymbol{a}}} \tag{4}$$

Using Woodbury's identity, it can be shown that

$$(\boldsymbol{R}_{|\boldsymbol{a}} + \lambda\boldsymbol{I})^{-1}\boldsymbol{a} = \frac{(\boldsymbol{C} + \lambda\boldsymbol{I})^{-1}\boldsymbol{a}}{1 + P\boldsymbol{a}^H(\boldsymbol{C} + \lambda\boldsymbol{I})^{-1}\boldsymbol{a}}$$
$$(\boldsymbol{R}_{|\boldsymbol{a}} + \lambda\boldsymbol{I})^{-1}\bar{\boldsymbol{a}} = (\boldsymbol{C} + \lambda\boldsymbol{I})^{-1}\left(\bar{\boldsymbol{a}} - \frac{P\boldsymbol{a}^H(\boldsymbol{C} + \lambda\boldsymbol{I})^{-1}\bar{\boldsymbol{a}}}{1 + P\boldsymbol{a}^H(\boldsymbol{C} + \lambda\boldsymbol{I})^{-1}\boldsymbol{a}}\boldsymbol{a}\right) \tag{5}$$

Using the previous expressions and after some straightforward algebraic manipulations yields the following expression:

$$\mathrm{SINR}_{|\boldsymbol{a}} = \frac{1}{P(\boldsymbol{a} - \gamma(\boldsymbol{a},\lambda)\bar{\boldsymbol{a}})^H\boldsymbol{Z}(\lambda)(\boldsymbol{a} - \gamma(\boldsymbol{a},\lambda)\bar{\boldsymbol{a}})} \tag{6}$$

where

$$\gamma(\boldsymbol{a},\lambda) = \frac{1 + P\boldsymbol{a}^H(\boldsymbol{C} + \lambda\boldsymbol{I})^{-1}\boldsymbol{a}}{P\boldsymbol{a}^H(\boldsymbol{C} + \lambda\boldsymbol{I})^{-1}\bar{\boldsymbol{a}}}$$
$$\boldsymbol{Z}(\lambda) = (\boldsymbol{C} + \lambda\boldsymbol{I})^{-1}\boldsymbol{C}(\boldsymbol{C} + \lambda\boldsymbol{I})^{-1} \tag{7}$$

For notational convenience, let

$$f(\lambda) = (\boldsymbol{a} - \gamma(\boldsymbol{a},\lambda)\bar{\boldsymbol{a}})^H\boldsymbol{Z}(\lambda)(\boldsymbol{a} - \gamma(\boldsymbol{a},\lambda)\bar{\boldsymbol{a}}) \tag{8}$$

be the function we wish to minimise w.r.t λ. Also, let

$$\boldsymbol{C} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H = \boldsymbol{U}_J\boldsymbol{\Lambda}_J\boldsymbol{U}_J^H + \sigma^2\boldsymbol{U}_n\boldsymbol{U}_n^H \tag{9}$$

be the eigen-decomposition of the interference plus noise covariance matrix $\boldsymbol{C}$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$, $\boldsymbol{U} = [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_m]$, $\Lambda_J = \mathrm{diag}(\lambda_1, \ldots, \lambda_K)$, $\boldsymbol{U}_J = [\boldsymbol{u}_1 \ \cdots \ \boldsymbol{u}_K]$, $\boldsymbol{U}_n = [\boldsymbol{u}_{K+1} \ \cdots \ \boldsymbol{u}_m]$, and where the eigenvalues are arranged in decreasing order.

In the sequel we assume that for $k = 1, \ldots, K$ and $\ell = K+1, \ldots, m$

$$\frac{\left|\boldsymbol{a}^H\boldsymbol{u}_k\right|^2}{\lambda_k} \ll \frac{\left|\boldsymbol{a}^H\boldsymbol{u}_\ell\right|^2}{\sigma^2}; \quad \frac{\left|\bar{\boldsymbol{a}}^H\boldsymbol{u}_k\right|^2}{\lambda_k} \ll \frac{\left|\bar{\boldsymbol{a}}^H\boldsymbol{u}_\ell\right|^2}{\sigma^2} \tag{10}$$

This approximation is valid as soon as the eigenvalues corresponding to the interferences (λ_k for $k = 1, \ldots, K$) are large compared to the noise level σ^2 (high interference to noise ratio (INR)) or as soon as the projection of the steering vector of interest $\boldsymbol{a}$ (as well as the presumed steering vector) onto the interference subspace is small which amounts to consider that the interferences are outside the main beam of the array. It should be pointed out that it is in this type of situation that the use of a robust beamformer based on diagonal loading is advisable. Indeed, as discussed in [8, 10], robust adaptive beamformers using diagonal loading are suitable when one wishes to recover a weak signal in the presence of strong interferences located outside the main beam. In contrast, in the case of main beam jamming, it would perform rather poorly (mainly due to a lack of resolution compared to the fully adaptive beamformer without loading) and therefore its use would not be recommended anyway. Consequently, the assumption in (10) is natural within the framework considered.

Under this assumption, we first show that $\gamma(\boldsymbol{a},\lambda)$ is a linear function of λ. Indeed,

$$\begin{aligned}\boldsymbol{a}^H(\boldsymbol{C} + \lambda\boldsymbol{I})^{-1}\boldsymbol{a} &= \sum_{k=1}^{m}\frac{\left|\boldsymbol{a}^H\boldsymbol{u}_k\right|^2}{(\lambda + \lambda_k)} \\ &\simeq \frac{\sum_{k=K+1}^{m}\left|\boldsymbol{a}^H\boldsymbol{u}_k\right|^2}{\lambda + \sigma^2} \\ &= \frac{\boldsymbol{a}^H\boldsymbol{U}_n\boldsymbol{U}_n^H\boldsymbol{a}}{\lambda + \sigma^2} = \frac{\boldsymbol{a}_n^H\boldsymbol{a}_n}{\lambda + \sigma^2}\end{aligned} \tag{11}$$

where $\boldsymbol{U}_n = [\boldsymbol{u}_{K+1} \ \cdots \ \boldsymbol{u}_m]$ is the sub-dominant subspace of $\boldsymbol{C}$ and $\boldsymbol{a}_n = \boldsymbol{U}_n^H\boldsymbol{a}$. Similarly,

$$\begin{aligned}\boldsymbol{a}^H(\boldsymbol{C} + \lambda\boldsymbol{I})^{-1}\bar{\boldsymbol{a}} &= \sum_{k=1}^{m}\frac{\boldsymbol{a}^H\boldsymbol{u}_k\boldsymbol{u}_k^H\bar{\boldsymbol{a}}}{(\lambda + \lambda_k)} \\ &\simeq \frac{\sum_{k=K+1}^{m}\boldsymbol{a}^H\boldsymbol{u}_k\boldsymbol{u}_k^H\bar{\boldsymbol{a}}}{\lambda + \sigma^2} \\ &= \frac{\boldsymbol{a}^H\boldsymbol{U}_n\boldsymbol{U}_n^H\bar{\boldsymbol{a}}}{\lambda + \sigma^2} = \frac{\boldsymbol{a}_n^H\bar{\boldsymbol{a}}_n}{\lambda + \sigma^2}\end{aligned} \tag{12}$$

with $\bar{\boldsymbol{a}}_n = \boldsymbol{U}_n^H\bar{\boldsymbol{a}}$. Using (11) and (12) along with the expression (7) for $\gamma(\boldsymbol{a},\lambda)$, it follows that

$$\begin{aligned}\gamma(\boldsymbol{a},\lambda) &\simeq \frac{\sigma^2 + P\boldsymbol{a}_n^H\boldsymbol{a}_n}{P\boldsymbol{a}_n^H\bar{\boldsymbol{a}}_n} + \frac{\lambda}{P\boldsymbol{a}_n^H\bar{\boldsymbol{a}}_n} \\ &\triangleq \gamma_0 + \gamma_0'\lambda\end{aligned} \tag{13}$$

Therefore

$$
\begin{aligned}
f(\lambda) &= \sum_{k=1}^{m} \frac{\lambda_k \left| (\boldsymbol{a} - \gamma(\boldsymbol{a}, \lambda)\overline{\boldsymbol{a}})^H \boldsymbol{u}_k \right|^2}{(\lambda + \lambda_k)^2} \\
&\simeq \sum_{k=K+1}^{m} \frac{\sigma^2 \left| (\boldsymbol{a} - \gamma(\boldsymbol{a}, \lambda)\overline{\boldsymbol{a}})^H \boldsymbol{u}_k \right|^2}{(\lambda + \sigma^2)^2} \\
&= \frac{\sigma^2}{(\lambda + \sigma^2)^2} \left\| \boldsymbol{U}_n^H (\boldsymbol{a} - \gamma(\boldsymbol{a}, \lambda)\overline{\boldsymbol{a}}) \right\|^2 \\
&= \frac{\sigma^2}{(\lambda + \sigma^2)^2} \left\| \boldsymbol{\alpha} + \lambda \boldsymbol{\alpha}' \right\|^2
\end{aligned} \tag{14}
$$

with $\boldsymbol{\alpha} \triangleq \boldsymbol{U}_n^H(\boldsymbol{a} - \gamma_0 \overline{\boldsymbol{a}})$ and $\boldsymbol{\alpha}' = -\gamma_0' \boldsymbol{U}_n^H \overline{\boldsymbol{a}}$. The bottom right-hand side of (14) is only an approximation of the true $f(\lambda)$ in (8) which holds under the hypothesis (10). The expression in (14) is likely not to be very accurate for large values of λ as the ratios

$$
\frac{\lambda_k \left| \boldsymbol{a}^H \boldsymbol{u}_k \right|^2}{(\lambda + \lambda_k)^2} \quad \text{and} \quad \frac{\sigma^2 \left| \boldsymbol{a}^H \boldsymbol{u}_\ell \right|^2}{(\lambda + \sigma^2)^2}
$$

are involved whereas the hypotheses consider

$$
\frac{\left| \boldsymbol{a}^H \boldsymbol{u}_k \right|^2}{\lambda_k} \quad \text{and} \quad \frac{\left| \boldsymbol{a}^H \boldsymbol{u}_\ell \right|^2}{\sigma^2}
$$

respectively. However, choosing a large value for λ is not advisable since it would be tantamount to using a non-adaptive beamformer whose performance is likely to be very poor compared to that of an adaptive beamformer. Observe that it is sensible to choose a λ whose magnitude is a few decibels about the noise level (otherwise the interferences would be buried in the artificial noise). For λ close to the optimal value, we will show in the next Section that (14) closely matches the true $f(\lambda)$.

Differentiating (14) and setting the result to zero, it holds that

$$
\frac{\partial f}{\partial \lambda} = 0 \Rightarrow \lambda_{|\boldsymbol{a}}^{\text{opt}} = \frac{\|\boldsymbol{\alpha}\|^2 - \sigma^2 \text{Re}\left[\boldsymbol{\alpha}^H \boldsymbol{\alpha}'\right]}{\sigma^2 \|\boldsymbol{\alpha}'\|^2 - \text{Re}\left[\boldsymbol{\alpha}^H \boldsymbol{\alpha}'\right]} \tag{15}
$$

Furthermore, using the expressions for γ_0 and γ_0' in (13), we have

$$
\begin{aligned}
\|\boldsymbol{\alpha}\|^2 &= \|\boldsymbol{a}_n\|^2 - 2\text{Re}\left[\gamma_0 \boldsymbol{a}_n^H \overline{\boldsymbol{a}}_n\right] + |\gamma_0|^2 \|\overline{\boldsymbol{a}}_n\|^2 \\
&= -\|\boldsymbol{a}_n\|^2 - 2\frac{\sigma^2}{P} + \frac{(\sigma^2 + P\|\boldsymbol{a}_n\|^2)^2 \|\overline{\boldsymbol{a}}_n\|^2}{P^2 |\boldsymbol{a}_n^H \overline{\boldsymbol{a}}_n|^2}
\end{aligned} \tag{16}
$$

$$
\left\| \boldsymbol{\alpha}' \right\|^2 = \left| \gamma_0' \right|^2 \|\overline{\boldsymbol{a}}_n\|^2 = \frac{\|\overline{\boldsymbol{a}}_n\|^2}{P^2 |\boldsymbol{a}_n^H \overline{\boldsymbol{a}}_n|^2} \tag{17}
$$

$$
\text{Re}\left[\boldsymbol{\alpha}^H \boldsymbol{\alpha}'\right] = -\frac{1}{P} + \frac{(\sigma^2 + P\|\boldsymbol{a}_n\|^2)\|\overline{\boldsymbol{a}}_n\|^2}{P^2 |\boldsymbol{a}_n^H \overline{\boldsymbol{a}}_n|^2} \tag{18}
$$

Inserting these expressions into (15) yields, after some straightforward derivations, the following simple expression for the optimal loading level for a given error on the steering vector:

$$
\lambda_{|\boldsymbol{a}}^{\text{opt}} = -(\sigma^2 + P\|\boldsymbol{a}_n\|^2) \tag{19}
$$

The following observations are in order:

- The first important thing to be noted is that $\lambda_{|\boldsymbol{a}}^{\text{opt}}$ is always negative which is quite an unexpected result as usually a positive loading level is always considered. However, this seemingly surprising should be re-examined under the following grounds. It has been noted recently that negative diagonal loading may outcome as a possible solution to the doubly constrained robust Capon beamformer of [7]. Hence, negative diagonal loading might not be such an unexpected result.
- As will be illustrated in the next Section, positive diagonal loading is also able to compensate for steering vector errors but does not manage to provide as high a SINR as negative loading with $\lambda_{|\boldsymbol{a}}^{\text{opt}}$.
- $\lambda_{|\boldsymbol{a}}^{\text{opt}}$ depends on the noise level, the source power and the squared norm of the projection of the steering vector onto the subspace orthogonal to the interference subspace. Since these parameters are not known (even if σ^2 can be accurately estimated), $\lambda_{|\boldsymbol{a}}^{\text{opt}}$ cannot be computed for any given $\boldsymbol{a}$. However, it provides a rough order of magnitude of the optimal loading level. Moreover, some further approximations can be made (see next Section) yielding an even simpler expression.
- We would like also to point out that this result holds for steering vector errors only but we do not claim that it would also be the optimal solution for finite-sample errors too. Indeed, it is our experience that a positive loading level usually provides better performance in short data samples.
- In contrast to positive diagonal loading, $\boldsymbol{R} + \lambda_{|\boldsymbol{a}}^{\text{opt}} \boldsymbol{I}$ can be rank-deficient (and hence non-invertible) if $-\lambda_{|\boldsymbol{a}}^{\text{opt}}$ coincides with an eigenvalue of the covariance matrix. Hence, care should be taken in order to avoid this potential source of problem.
- Reporting (19) in (14), it is straightforward to show that the SINR corresponding to $\lambda_{|\boldsymbol{a}}^{\text{opt}}$ is approximately

$$
\text{SINR}_{|\boldsymbol{a}}(\lambda_{|\boldsymbol{a}}^{\text{opt}}) = \frac{P\|\boldsymbol{a}_n\|^2}{\sigma^2} \tag{20}
$$

This is to be compared with the optimal performance obtained with a (hypothetical) clairvoyant beamformer which would know $\boldsymbol{a}$ and is thus given by

$$
\boldsymbol{w}_{|\boldsymbol{a}}^{\text{opt}} = \boldsymbol{C}^{-1} \boldsymbol{a} \tag{21}
$$

The SINR corresponding to the weight vector in the previous equation is

$$
\begin{aligned}
\text{SINR}_{|\boldsymbol{a}}^{\text{opt}} &= P\boldsymbol{a}^H \boldsymbol{C}^{-1} \boldsymbol{a} \\
&= \frac{P\|\boldsymbol{a}_n\|^2}{\sigma^2} + P\left\| \boldsymbol{\Lambda}_J^{-1/2} \boldsymbol{a}_J \right\|^2
\end{aligned} \tag{22}
$$

where $\boldsymbol{a}_J = \boldsymbol{U}_J^H \boldsymbol{a}$. Comparing (20) with (22), it can be conjectured that diagonal loading with the optimal loading level will have a performance very close to the optimum since the second term in the right-hand side of (22) is small under the hypothesis (10). More precisely, the difference between the two SINRs is likely to be small in the case of high INR or interferences outside the main beam. This fact will be validated in the next Section by numerical simulations.

3.2 Optimisation for random steering vector errors

Since the optimal loading level $\lambda_{|\boldsymbol{a}}^{\text{opt}}$ depends on the actual steering vector which is unknown, we propose to characterise it 'on average'. Towards this end, we assume that the steering vector $\boldsymbol{a}$ is random with correlation matrix $\boldsymbol{R}_{\boldsymbol{a}} = \mathcal{E}_{\boldsymbol{a}}\{\boldsymbol{a}\boldsymbol{a}^H\}$ where $\mathcal{E}_{\boldsymbol{a}}\{\cdot\}$ stands for the statistical expectation with respect to the pdf of $\boldsymbol{a}$. Under the stated assumptions, it is straightforward to see that

$$\overline{\lambda}^{\text{opt}} \triangleq \mathcal{E}_{\boldsymbol{a}}\left\{\lambda^{\text{opt}}_{|\boldsymbol{a}}\right\} = -\left(\sigma^2 + P\text{Tr}\left\{\boldsymbol{U}_n\boldsymbol{U}_n^H\mathcal{E}\{\boldsymbol{a}\boldsymbol{a}^H\}\right\}\right)$$
$$= -\left(\sigma^2 + P\text{Tr}\left\{\boldsymbol{U}_n\boldsymbol{U}_n^H\boldsymbol{R}_{\boldsymbol{a}}\right\}\right) \qquad (23)$$

A further simplification can be made by noting that, under the stated hypotheses, the projection of the steering vector onto the interference subspace is small so that $\|\boldsymbol{a}_n\|^2$ can be replaced by $\|\boldsymbol{a}\|^2$ in (19). Taking the expectation with this modification, we end up with the following very simple expression

$$\overline{\lambda}^{\text{opt}}_{\text{approx}} = -\left(\sigma^2 + P\text{Tr}\{\boldsymbol{R}_{\boldsymbol{a}}\}\right) \qquad (24)$$

We stress the fact that this is a very simple expression which depends in a simple way on the noise level, the source power and the steering vector correlation matrix $\boldsymbol{R}_{\boldsymbol{a}}$. Observe that (24) is still simpler in the case of DOA uncertainties. Indeed, assume that $\boldsymbol{a} = \boldsymbol{a}(\widetilde{\theta})$ where $\widetilde{\theta}$ is a random variable with mean $\overline{\theta}$ and some *a priori* pdf $p(\widetilde{\theta})$. Whatever $p(\widetilde{\theta})$, since $\|\boldsymbol{a}\|^2 = m$, we necessarily have $\text{Tr}\{\boldsymbol{R}_{\boldsymbol{a}}\} = m$ and thus

$$\overline{\lambda}^{\text{opt}}_{\text{approx}} = -(\sigma^2 + Pm)$$

in the case of DOA uncertainties or pointing errors. As will be illustrated next, the use of $\overline{\lambda}^{\text{opt}}$ or $\overline{\lambda}^{\text{opt}}_{\text{approx}}$ enables us to obtain a SINR comparable with that of the clairvoyant beamformer in most situations.

4 Numerical illustrations

In this Section, we assess the validity of the analysis presented above for both fixed and random steering vector errors. In all simulations, we consider a uniform linear array of $m = 10$ sensors spaced a half-wavelength apart.

In a first series of simulations, we consider a fixed steering vector error and validate (19). Towards this end, we consider the case of pointing errors, i.e. the source of interest impinges from $\Delta\theta$ while its DOA is assumed to be $0°$. In addition to the signal of interest, two interferences are present whose DOAs are $-20°$, $30°$ and whose powers are 20 dB and 30 dB above the white noise level, respectively. Figure 1 displays the exact SINR, given by (4), and the approximated SINR, computed from (14), against the loading level λ, for various values of $\Delta\theta$, ranging from one fiftieth to one half the null-to-null beamwidth (BW_{NN}) of the array. The vertical line corresponds to $\lambda^{\text{opt}}_{|\boldsymbol{a}}$. The horizontal (upper) line corresponds to the clairvoyant beamformer and thus to the optimal SINR. From inspection of this Figure, the following conclusions can be drawn:

- The approximation in (14) is very accurate, at least for not too large values of λ.
- The optimal SINR is always obtained for a negative value of λ and this value is very close to $\lambda^{\text{opt}}_{|\boldsymbol{a}}$. This assesses the validity of our analysis.
- The SINR obtained with $\lambda^{\text{opt}}_{|\boldsymbol{a}}$ is very close to the optimum SINR.
- A positive loading level also enables to compensate for the pointing errors but it does not provide a SINR as large as that obtained with the optimal negative loading level. The difference is more pronounced when $\Delta\theta$ increases.
- In contrast to positive diagonal loading where a large range of values for λ roughly provide the same SINR, the SINR varies more significantly around the maximum for negative loading levels. Hence, selecting a negative λ may be more delicate.

For completeness, we now vary $\Delta\theta$ and, for each value of $\Delta\theta$, we look for the loading level that results in the largest

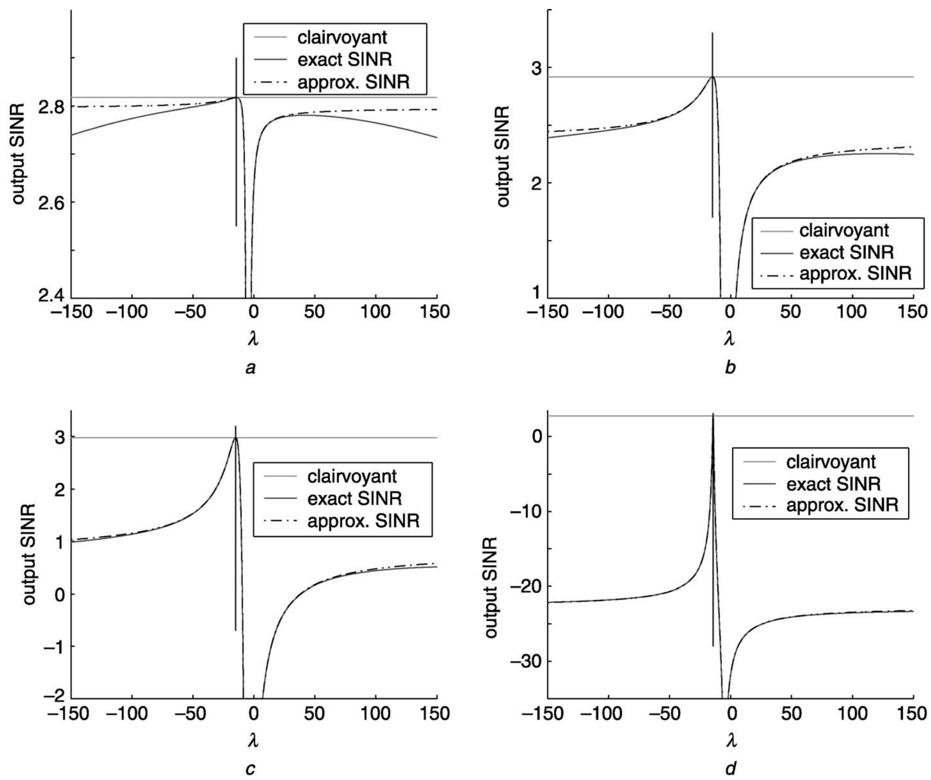

Fig. 1 *Fixed pointing errors. Exact and approximated SINR against loading level* λ

a Pointing error: $0.02\,BW_{NN}$
b Pointing error: $0.1\,BW_{NN}$
c Pointing error: $0.2\,BW_{NN}$
d Pointing error: $0.5\,BW_{NN}$

SINR. In Fig. 2, this optimal loading level is compared with $\lambda^{\text{opt}}_{|a}$ and with $-(\sigma^2 + Pm)$. As can be seen, $\lambda^{\text{opt}}_{|a}$ really provides the optimal level, except for very small pointing errors. However, in the latter case, despite the fact that $\lambda^{\text{opt}}_{|a}$ is not exactly the optimal value, the SINR loss is negligible. In addition, observe that for small pointing errors diagonal loading is not really useful. Also, notice that $\lambda^{\text{opt}}_{|a}$ is rather close to $-(\sigma^2 + Pm)$ as $\|\boldsymbol{a}_n\|^2 \simeq \|\boldsymbol{a}\|^2$; hence this latter value may be used as a further approximation without too much penalising performance.

In a second series of simulations, we consider random steering vector errors and $\boldsymbol{a}$ is varied randomly in each Monte Carlo run. We consider here three cases, namely DOA uncertainties, local scattering and uncalibrated arrays:

DOA uncertainties: in this case, the true DOA of the source of interest is uniformly distributed on $[-\Delta\theta, \Delta\theta]$ while the assumed DOA is 0° and $\bar{\boldsymbol{a}} = \boldsymbol{a}(0^\circ)$. The SINR will be plotted versus $\Delta\theta$ and the latter is normalised to the array null-to-null beamwidth.

Local scattering: in this case, the steering vector can be written as [11]

$$\boldsymbol{a} = \bar{\boldsymbol{a}} + \frac{1}{\sqrt{L}} \sum_{k=1}^{L} g_k \boldsymbol{a}(\widetilde{\theta}_k) \tag{25}$$

where g_k are zero-mean, independent and identically distributed random variables with power σ_g^2 and $\widetilde{\theta}_k$ are independent random variables with pdf $p(\widetilde{\theta})$. $\bar{\boldsymbol{a}}$ corresponds to the mean steering vector, i.e. the spatial signature of the line of sight component. The covariance matrix of the errors is given by [11]

$$\boldsymbol{C_a} = \sigma_g^2 \int \boldsymbol{a}(\widetilde{\theta})\boldsymbol{a}^H(\widetilde{\theta})p(\widetilde{\theta})\, d\widetilde{\theta} \tag{26}$$

In the numerical simulations below, we assume a Gaussian distribution for the scatterers with standard deviation (referred to as angular spread in the literature) $\sigma_\theta = 15^\circ$.

Uncalibrated arrays: in this case, the gain and phases of the elements are not known exactly, and the steering vector can be written as $\boldsymbol{a} = \bar{\boldsymbol{a}} \odot (\boldsymbol{1} + \boldsymbol{\Delta_a})$ where $\bar{\boldsymbol{a}}$ is the response of the perfectly calibrated array and $\boldsymbol{\Delta_a}$ is a zero-mean Gaussian vector with covariance matrix $\sigma_a^2\boldsymbol{I}$. Under these assumptions, $\boldsymbol{a}$ is random, with mean $\bar{\boldsymbol{a}}$ and covariance matrix $\boldsymbol{C_a} = \bar{\boldsymbol{a}}\bar{\boldsymbol{a}}^H \odot \sigma_a^2\boldsymbol{I}$.

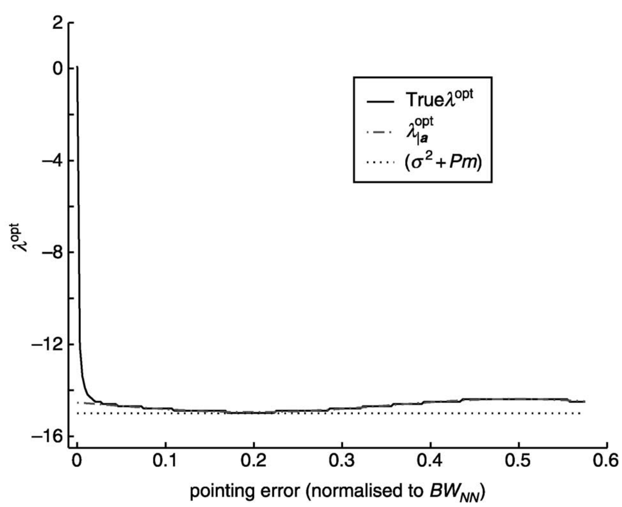

Fig. 2 *Fixed pointing errors*

Comparison between the exact optimal value of λ and $\lambda^{\text{opt}}_{|a}$ against pointing error

For the two latter cases, local scattering and uncalibrated arrays, we define the uncertainty ratio (UR) as

$$UR = 10\log_{10}\left(\frac{\text{Tr}\{\boldsymbol{C_a}\}}{\bar{\boldsymbol{a}}^H\bar{\boldsymbol{a}}}\right)$$

In all cases, the signal to noise ratio (SNR) is defined as

$$SNR = 10\log_{10}\left(\frac{P\text{Tr}\{\boldsymbol{R_a}\}}{\sigma^2}\right)$$

and corresponds to the array SNR. Similarly to the fixed error case, two interferences are present with the same characteristics as previously. In all simulations, the SINR is evaluated as follows. $N_r = 500$ Monte Carlo simulations are run with a different random $\boldsymbol{a}$ and, for a given weight vector $\boldsymbol{w}$, the average SINR is computed as

$$\overline{\text{SINR}}(\boldsymbol{w}) = \frac{1}{N_r}\sum_{n=1}^{N_r} \frac{P|\boldsymbol{w}^H\boldsymbol{a}(n)|^2}{\boldsymbol{w}^H\boldsymbol{C}\boldsymbol{w}} \tag{27}$$

The average SINR obtained with the clairvoyant beamformer (21) is, cf. (22)

$$\overline{\text{SINR}}^{\text{opt}} = \mathcal{E}_{\boldsymbol{a}}\left\{\text{SINR}^{\text{opt}}_{|a}\right\} = P\text{Tr}\{\boldsymbol{C}^{-1}\boldsymbol{R_a}\} \tag{28}$$

For comparison purposes, we also display the performance of the minimum variance distortionless response (MVDR) beamformer, which is given by

$$\begin{aligned}\boldsymbol{w}_{\text{MVDR}} &= \arg\max_{\boldsymbol{w}} \frac{\mathcal{E}\left\{\left|\boldsymbol{w}^H\boldsymbol{a}s_t\right|^2\right\}}{\mathcal{E}\left\{\left|\boldsymbol{w}^H\boldsymbol{n}_t\right|^2\right\}} \\ &= \arg\max_{\boldsymbol{w}} \frac{\boldsymbol{w}^H\boldsymbol{R_a}\boldsymbol{w}}{\boldsymbol{w}^H\boldsymbol{C}\boldsymbol{w}} \\ &= \mathcal{P}\{\boldsymbol{C}^{-1}\boldsymbol{R_a}\}\end{aligned} \tag{29}$$

where $\mathcal{P}\{\cdot\}$ is the principal eigenvector of the matrix between braces. The average SINR associated with the MVDR beamformer is readily obtained as

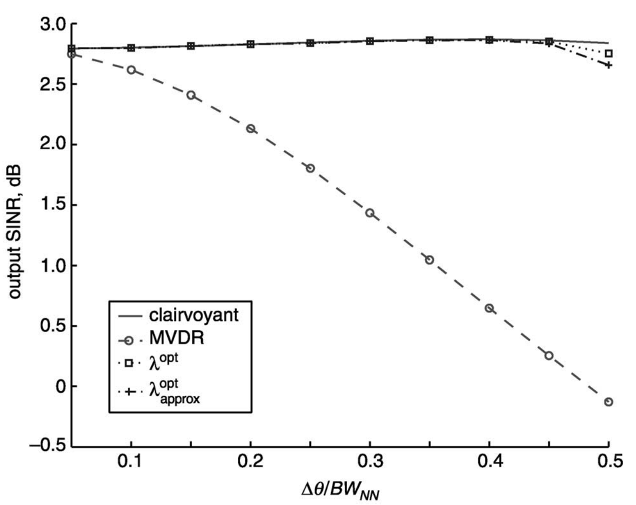

Fig. 3 *DOA uncertainties: output SINR for the clairvoyant, MVDR and diagonally loaded beamformers against Δθ*

$SNR = 3$ dB

$$\overline{\mathrm{SINR}}_{\mathrm{MVDR}} = \mathcal{E}_{\boldsymbol{a}}\left\{\frac{P\left|\boldsymbol{w}_{\mathrm{MVDR}}^{H}\boldsymbol{a}\right|^{2}}{\boldsymbol{w}_{\mathrm{MVDR}}^{H}\boldsymbol{C}\boldsymbol{w}_{\mathrm{MVDR}}}\right\}$$
$$= P\lambda_{\max}\{\boldsymbol{C}^{-1}\boldsymbol{R}_{\boldsymbol{a}}\} \qquad (30)$$

where $\lambda_{\max}\{\cdot\}$ corresponds to the maximum eigenvalue.

Figures 3, 4 and 5 compare the performances of the clairvoyant beamformer, the MVDR beamformer and the diagonally loaded beamformers with $\lambda = \bar{\lambda}^{\mathrm{opt}}$ and $\lambda = \bar{\lambda}^{\mathrm{opt}}_{\mathrm{approx}}$. These Figures plot the average SINR versus $\Delta\theta$ or the uncertainty ratio depending on the case considered. The following observations can be made:

- In all cases, the diagonally loaded beamformer, either with $\lambda = \bar{\lambda}^{\mathrm{opt}}$ or $\lambda = \bar{\lambda}^{\mathrm{opt}}_{\mathrm{approx}}$, performs as well as the clairvoyant beamformer, at least up to $UR = -2$ dB where it slightly departs from the optimum. Note that in the case of DOA uncertainties, diagonal loading is able to provide the optimal SINR even for uncertainties up to half the null-to-null beamwidth.
- Using $\lambda = \bar{\lambda}^{\mathrm{opt}}_{\mathrm{approx}}$ instead of $\lambda = \bar{\lambda}^{\mathrm{opt}}$ results in a marginal degradation.
- Both diagonally loaded beamformers outperform the MVDR beamformer, especially at high UR or large $\Delta\theta$, where the MVDR's performance drops abruptly.

Finally, in a last simulation, we illustrate the fact that diagonal loading is more suitable in the case of high INR or interferences outside the main beam. We consider the simple scenario of a single interference whose DOA is varied and whose INR is 20 dB. The case of pointing errors is considered here and we assume that the DOA uncertainties are uniformly distributed in $[-0.3\,BW_{NN}\,,\,0.3\,BW_{NN}]$. Figure 6 displays the performance of the four adaptive beamformers described previously versus the interference's DOA. The null-to-null beamwidth of the array is shown in dash-dotted vertical lines. Figure 6 shows that when the interference is outside the main beam the diagonally loaded beamformers provide the same SINR as the clairvoyant beamformer and about 1.5 dB above the MVDR beamformer. When the interference enters the main beam, the performance of the diagonally loaded beamformers drops significantly, with $\bar{\lambda}^{\mathrm{opt}}_{\mathrm{approx}}$ being worse than $\bar{\lambda}^{\mathrm{opt}}$, and falls below that of the MVDR beamformer with no loading. This suggests, as was discussed in Section 2 that diagonal loading is not helpful for main-beam jamming but effective in handling strong interferences located outside the main-beam.

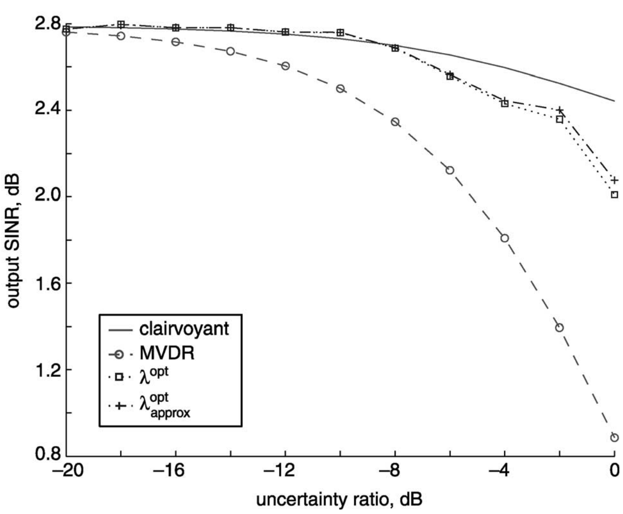

Fig. 4 *Local scattering: output SINR for the clairvoyant, MVDR and diagonally loaded beamformers against uncertainty ratio*

$SNR = 3$ dB

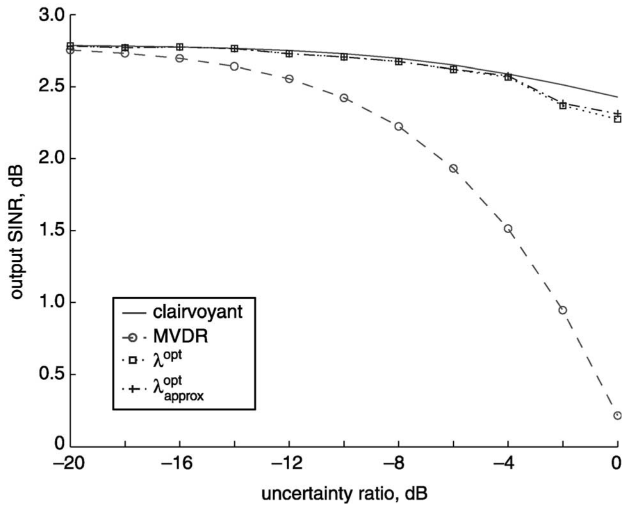

Fig. 5 *Uncalibrated array, output SINR for the clairvoyant, MVDR and diagonally loaded beamformers against uncertainty ratio*

$SNR = 3$ dB

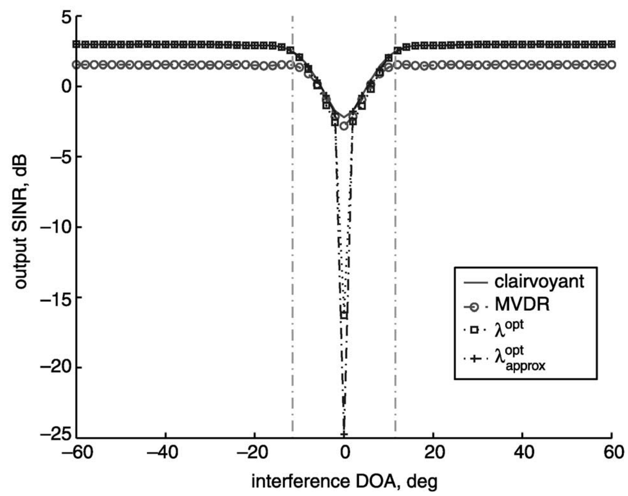

Fig. 6 *DOA uncertainties: output SINR for the clairvoyant, MVDR and diagonally loaded beamformers against the DOA of the interference*

$SNR = 3$ dB

5 Conclusions

This paper has considered the use of diagonal loading to compensate for random steering vector errors and dealt with the problem of optimally selecting the loading level. We considered the case of steering vectors errors only (i.e. no finite-sample effects) and the case of weak signal detection in the presence of strong interferences. Within this framework, it was shown that there exists a value of the loading level which results in maximal SINR, and that this optimal level is negative. A simple and closed-form expression for the average, with respect to the pdf of the steering vector, optimal loading level was also derived. Numerical simulations attested to the validity of the analysis and showed that diagonal loading with optimal selection of the loading level can provide a performance very close to that of a clairvoyant beamformer.

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