Amplitude Estimation of a Signal With Known Waveform in the Presence of Steering Vector Uncertainties

Olivier Besson, Gleb Varfolomeev, and François Vincent

Abstract—**In this correspondence, we address the problem of estimating the amplitude of a signal with known waveform received on an array of sensors and we consider the case where there exist uncertainties about the spatial signature of the signal of interest. Closed-form expressions for the Cramér–Rao bound are derived and the respective influence of the uncertainties and the number of snapshots is studied. The maximum likelihood estimator (MLE) of the signal of interest amplitude along with the covariance matrix of the interferences and noise is also derived and an iterative algorithm is presented to obtain the ML estimates.**

I. Problem Statement

In this correspondence, we consider the problem of estimating the amplitude of a signal with known waveform received by an array of sensors in the presence of spatially correlated noise. This problem has practical relevance in many applications, including active radar systems [1] where the reflection from a target is a scaled and delayed version of the emitted signal. It can also be encountered in mobile communications when, for instance, a training sequence is used. Finally, the same type of problem arises in quadrupole resonance techniques [2]. Briefly stated, the problem amounts to estimating the scalar β from N snapshots drawn from the following model:

$$\boldsymbol{x}_t = \beta \boldsymbol{a} s_t + \boldsymbol{n}_t. \tag{1}$$

In the previous equation, $\boldsymbol{x}_t$ is the array output collected at time t, s_t is the known signal waveform while β denotes its unknown amplitude and $\boldsymbol{a}$ corresponds to the array's response for the signal of interest (SOI). The noise $\boldsymbol{n}_t$ is assumed to be a zero-mean, complex-valued Gaussian process with unknown covariance matrix $\boldsymbol{C}$, i.e., $\boldsymbol{n}_t \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{C})$. The problem of estimating β in the model (1) has already received much attention in the literature. More precisely, different methods were proposed in [2]–[5] corresponding to various assumptions on $\boldsymbol{a}$, which are summarized in Table I.

In [2], the steering vector of interest is assumed to be known and the maximum likelihood estimator is derived and compared to the Capon estimator. Reference [2] also considers the extension to a temporally correlated (i.e., multichannel autoregressive) noise. In [3]–[5], the multiple signal version of (1) is considered (i.e., multiple signals with known waveforms impinge on the array) but the SOI's steering vector is assumed to be of the form $\boldsymbol{a} = \boldsymbol{a}(\theta)$ where θ is the direction-of-arrival (DOA) of the source. Hence, [3]–[5] address the problem of jointly estimating the amplitudes and DOAs of the signals of interest. [4] also addresses the case of a completely unstructured steering vector $\boldsymbol{a}$, i.e., $\boldsymbol{a}$ is assumed to be an unknown deterministic vector. Since, in this case, there exists an inherent scalar ambiguity between $\boldsymbol{a}$ and β, only $\tilde{\boldsymbol{a}} = \beta \boldsymbol{a}$ can be estimated, unless some constraint is set on $\boldsymbol{a}$ to resolve the ambiguity, for instance $\|\boldsymbol{a}\|$ is a known constant. Observe that the *a priori* knowledge about $\boldsymbol{a}$ is less and less pronounced as we go from the left to the right in Table I.

TABLE I
Summary of Array Processing From the Model $\boldsymbol{x}_t = \beta \boldsymbol{a} s_t + \boldsymbol{n}_t$ Where s_t is a Known Waveform

Model	$\boldsymbol{a} = \bar{\boldsymbol{a}}$	$\boldsymbol{a} = \boldsymbol{a}(\theta)$	$\boldsymbol{a} \sim \mathcal{CN}(\bar{\boldsymbol{a}}, \boldsymbol{C}_a)$	$\boldsymbol{a}$
Unknowns	β	β, θ	β	$\tilde{\boldsymbol{a}} = \beta \boldsymbol{a}$
Reference	[2]	[3], [4], [5]	this paper	[4]

Herein, we consider the case where the steering vector of interest is affected by random errors and thus there exist uncertainties about $\boldsymbol{a}$. This is typically the case when the source is surrounded by multiple closely spaced scatterers, rendering the spatial signature $\boldsymbol{a}$ random with a full-rank covariance matrix [6], [7]. Randomness in the array's response has been considered, e.g., in [8], [9]. It can be due to a non-perfectly calibrated array with random gains and phases. Accordingly, there can exist uncertainties about the DOA of the source of interest. Therefore, in this paper, we assume that the steering vector of interest $\boldsymbol{a}$ is drawn for a complex Gaussian distribution with mean $\bar{\boldsymbol{a}}$ and a known covariance matrix $\boldsymbol{C}_a$ which gathers the effects of uncertainties, i.e., $\boldsymbol{a} \sim \mathcal{CN}(\bar{\boldsymbol{a}}, \boldsymbol{C}_a)$. Since the errors in the steering vector are typically a combination of different and independent factors, the central limit theorem can be advocated to justify the Gaussian assumption. We also assume that $\boldsymbol{a}$ and and $\boldsymbol{n}_t$ are independent. Within this framework, we examine the influence of these random steering vector errors onto the estimation performance. Toward this end we derive the Cramér–Rao bound (CRB) and study its dependence toward N and $\boldsymbol{C}_a$. Then, we consider the maximum likelihood estimator of the signal amplitude and the noise covariance matrix. In contrast to the case of known or unknown but deterministic $\boldsymbol{a}$, it is shown that the MLE cannot be obtained in closed-form but requires an iterative procedure.

II. Cramér–Rao Bound

In this section, we derive the CRB for estimation of the unknown parameters in the model, namely the parameter vector

$$\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{\eta}_\beta \\ \boldsymbol{\eta}_c \end{bmatrix}$$

with $\boldsymbol{\eta}_\beta = [\,\mathrm{Re}[\beta] \quad \mathrm{Im}[\beta]\,]^T = [\,\beta_R \quad \beta_I\,]^T$ and where $\boldsymbol{\eta}_c$ is the $m^2 \times 1$ vector build from the (real-valued) diagonal elements of $\boldsymbol{C}$ and the real and imaginary parts of the elements below the main diagonal. Since we are mostly interested in estimating β, we will concentrate on deriving a closed-form expression for the CRB associated with β. Let $\boldsymbol{X} = [\,\boldsymbol{x}_1 \quad \cdots \quad \boldsymbol{x}_N\,]$ and $\boldsymbol{x} = \mathrm{vec}(\boldsymbol{X})$ be the vector obtained by stacking the columns of $\boldsymbol{X}$. Accordingly, let $\boldsymbol{s} = [\,s_1 \quad \cdots \quad s_N\,]^T$ be the vector of signal waveforms. Then, under the stated hypotheses, $\boldsymbol{x}$ is a Gaussian vector whose mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{C}_x$ are, respectively, given by

$$\boldsymbol{\mu} = \beta \boldsymbol{s} \otimes \bar{\boldsymbol{a}} \tag{2a}$$

$$\boldsymbol{C}_x = \boldsymbol{I}_N \otimes \boldsymbol{C} + |\beta|^2 \boldsymbol{s}\boldsymbol{s}^H \otimes \boldsymbol{C}_a \tag{2b}$$

where $\otimes$ stands for the Kronecker product [10]. The (k, ℓ) element of the Fisher information matrix (FIM) can be written as [10], [11]

$$\boldsymbol{F}_{k,\ell} = 2\mathrm{Re}\left[\frac{\partial \boldsymbol{\mu}^H}{\partial \boldsymbol{\eta}_k} \boldsymbol{C}_x^{-1} \frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\eta}_\ell}\right] + \mathrm{Tr}\left\{\boldsymbol{C}_x^{-1} \frac{\partial \boldsymbol{C}_x}{\partial \boldsymbol{\eta}_k} \boldsymbol{C}_x^{-1} \frac{\partial \boldsymbol{C}_x}{\partial \boldsymbol{\eta}_\ell}\right\}. \tag{3}$$

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O. Besson and F. Vincent are with the Department of Avionics and Systems, ENSICA, 31056 Toulouse, France (e-mail: besson@ensica.fr; vincent@ensica.fr).

G. Varfolomeev is with St. Petersburg State University of Aerospace Instrumentation, St. Petersburg, Russia.

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The FIM will have the following partitioned form:

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_{\boldsymbol{\eta}_\beta\boldsymbol{\eta}_\beta} & \boldsymbol{F}_{\boldsymbol{\eta}_c\boldsymbol{\eta}_\beta}^T \\ \boldsymbol{F}_{\boldsymbol{\eta}_c\boldsymbol{\eta}_\beta} & \boldsymbol{F}_{\boldsymbol{\eta}_c\boldsymbol{\eta}_c} \end{bmatrix} \tag{4}$$

where the partitioning corresponds to that of $\boldsymbol{\eta}$. We now derive the FIM on a block-by-block basis. Prior to that, we introduce some notations and derive matrix relations that will be used repeatedly in the sequel. For any invertible matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ and any matrices $\boldsymbol{B}$ and $\boldsymbol{D}$ of conformable size, one can show that [10]

$$[\boldsymbol{A}+\boldsymbol{BCD}]^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\left[\boldsymbol{C}^{-1}+\boldsymbol{DA}^{-1}\boldsymbol{B}\right]^{-1}\boldsymbol{DA}^{-1}. \tag{5}$$

Applying (5) to (2b) and observing that $\boldsymbol{ss}^H \otimes \boldsymbol{C}_a = (\boldsymbol{s}\otimes\boldsymbol{I}_m)\boldsymbol{C}_a\left(\boldsymbol{s}^H\otimes\boldsymbol{I}_m\right)$, we obtain

$$\begin{aligned}\boldsymbol{C}_x^{-1} &= \left[(\boldsymbol{I}_N\otimes\boldsymbol{C}) + |\beta|^2(\boldsymbol{s}\otimes\boldsymbol{I}_m)\boldsymbol{C}_a\left(\boldsymbol{s}^H\otimes\boldsymbol{I}_m\right)\right]^{-1} \\ &= \boldsymbol{I}_N\otimes\boldsymbol{C}^{-1} - |\beta|^2\boldsymbol{ss}^H\otimes\widetilde{\boldsymbol{C}}^{-1}\end{aligned} \tag{6}$$

where $\widetilde{\boldsymbol{C}}^{-1} \triangleq \boldsymbol{C}^{-1}\left[|\beta|^2E_s\boldsymbol{C}^{-1}+\boldsymbol{C}_a^{-1}\right]^{-1}\boldsymbol{C}^{-1}$ and $E_s = \boldsymbol{s}^H\boldsymbol{s}$. Let us also define

$$\boldsymbol{Z}^{-1} \triangleq \boldsymbol{C}^{-1} - |\beta|^2E_s\widetilde{\boldsymbol{C}}^{-1} = \left[\boldsymbol{C}+|\beta|^2E_s\boldsymbol{C}_a\right]^{-1} \tag{7}$$

where, to obtain the second equality, we again made use of (5). Let us consider now the block $\boldsymbol{F}_{\boldsymbol{\eta}_\beta\boldsymbol{\eta}_\beta}$. Since $\boldsymbol{\mu} = \beta\boldsymbol{s}\otimes\overline{\boldsymbol{a}}$, it follows that

$$\frac{\partial\boldsymbol{\mu}}{\partial\beta_R} = \boldsymbol{s}\otimes\overline{\boldsymbol{a}}; \quad \frac{\partial\boldsymbol{\mu}}{\partial\beta_I} = i\boldsymbol{s}\otimes\overline{\boldsymbol{a}}. \tag{8}$$

Therefore

$$\begin{aligned}&\frac{\partial\boldsymbol{\mu}^H}{\partial\beta_R}\boldsymbol{C}_x^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\beta_R} \\ &\quad= \left(\boldsymbol{s}^H\otimes\overline{\boldsymbol{a}}^H\right)\left\{\boldsymbol{I}_N\otimes\boldsymbol{C}^{-1} - |\beta|^2\boldsymbol{ss}^H\otimes\widetilde{\boldsymbol{C}}^{-1}\right\}(\boldsymbol{s}\otimes\overline{\boldsymbol{a}}) \\ &\quad= E_s\overline{\boldsymbol{a}}^H\left[\boldsymbol{C}^{-1}-|\beta|^2E_s\widetilde{\boldsymbol{C}}^{-1}\right]\overline{\boldsymbol{a}} = E_s\overline{\boldsymbol{a}}^H\boldsymbol{Z}^{-1}\overline{\boldsymbol{a}} \\ &\quad= \frac{\partial\boldsymbol{\mu}^H}{\partial\beta_I}\boldsymbol{C}_x^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\beta_I}\end{aligned} \tag{9}$$

and

$$\mathrm{Re}\left[\frac{\partial\boldsymbol{\mu}^H}{\partial\beta_R}\boldsymbol{C}_x^{-1}\frac{\partial\boldsymbol{\mu}}{\partial\beta_I}\right] = 0. \tag{10}$$

Similarly, we have that

$$\frac{\partial\boldsymbol{C}_x}{\partial\beta_R} = 2\beta_R\boldsymbol{ss}^H\otimes\boldsymbol{C}_a; \quad \frac{\partial\boldsymbol{C}_x}{\partial\beta_I} = 2\beta_I\boldsymbol{ss}^H\otimes\boldsymbol{C}_a. \tag{11}$$

Additionally, using (6), it is straightforward to show that for any set of matrices $\{\boldsymbol{M}_k\}_{k=1}^4$ of conformable size

$$\begin{aligned}&\mathrm{Tr}\left\{\boldsymbol{C}_x^{-1}(\boldsymbol{M}_1\otimes\boldsymbol{M}_2)\boldsymbol{C}_x^{-1}(\boldsymbol{M}_3\otimes\boldsymbol{M}_4)\right\} \\ &= \mathrm{Tr}\{\boldsymbol{M}_1\boldsymbol{M}_3\}\,\mathrm{Tr}\left\{\boldsymbol{C}^{-1}\boldsymbol{M}_2\boldsymbol{C}^{-1}\boldsymbol{M}_4\right\} \\ &\quad - |\beta|^2\left(\boldsymbol{s}^H\boldsymbol{M}_3\boldsymbol{M}_1\boldsymbol{s}\right)\mathrm{Tr}\left\{\boldsymbol{C}^{-1}\boldsymbol{M}_2\widetilde{\boldsymbol{C}}^{-1}\boldsymbol{M}_4\right\} \\ &\quad - |\beta|^2\left(\boldsymbol{s}^H\boldsymbol{M}_1\boldsymbol{M}_3\boldsymbol{s}\right)\mathrm{Tr}\left\{\widetilde{\boldsymbol{C}}^{-1}\boldsymbol{M}_2\boldsymbol{C}^{-1}\boldsymbol{M}_4\right\} \\ &\quad + |\beta|^4\left(\boldsymbol{s}^H\boldsymbol{M}_1\boldsymbol{s}\right)\left(\boldsymbol{s}^H\boldsymbol{M}_3\boldsymbol{s}\right)\mathrm{Tr}\left\{\widetilde{\boldsymbol{C}}^{-1}\boldsymbol{M}_2\widetilde{\boldsymbol{C}}^{-1}\boldsymbol{M}_4\right\}.\end{aligned} \tag{12}$$

Using (12), one can write that

$$\begin{aligned}&\mathrm{Tr}\left\{\boldsymbol{C}_x^{-1}\left(\boldsymbol{ss}^H\otimes\boldsymbol{C}_a\right)\boldsymbol{C}_x^{-1}\left(\boldsymbol{ss}^H\otimes\boldsymbol{C}_a\right)\right\} \\ &= E_s^2\,\mathrm{Tr}\left\{\left(\boldsymbol{C}^{-1}-|\beta|^2E_s\widetilde{\boldsymbol{C}}^{-1}\right)\boldsymbol{C}_a\left(\boldsymbol{C}^{-1}-|\beta|^2E_s\widetilde{\boldsymbol{C}}^{-1}\right)\boldsymbol{C}_a\right\} \\ &= E_s^2\,\mathrm{Tr}\left\{\boldsymbol{Z}^{-1}\boldsymbol{C}_a\boldsymbol{Z}^{-1}\boldsymbol{C}_a\right\}.\end{aligned} \tag{13}$$

Therefore, $\boldsymbol{F}_{\boldsymbol{\eta}_\beta\boldsymbol{\eta}_\beta}$ can be compactly written as

$$\boldsymbol{F}_{\boldsymbol{\eta}_\beta\boldsymbol{\eta}_\beta} = 2E_s\left(\overline{\boldsymbol{a}}^H\boldsymbol{Z}^{-1}\overline{\boldsymbol{a}}\right)\boldsymbol{I} + 4E_s^2\,\mathrm{Tr}\left\{\boldsymbol{Z}^{-1}\boldsymbol{C}_a\boldsymbol{Z}^{-1}\boldsymbol{C}_a\right\}\boldsymbol{\eta}_\beta\boldsymbol{\eta}_\beta^T. \tag{14}$$

Let us now turn to the derivation of the other blocks of the FIM. As will be shown later, it is more convenient to work with $\boldsymbol{c} = \mathrm{vec}(\boldsymbol{C})$ rather than with $\boldsymbol{\eta}_c$: the two are linearly related by the Jacobian matrix $\boldsymbol{J}$ such that $\boldsymbol{c} = \boldsymbol{J}\boldsymbol{\eta}_c$. First, observe that

$$\frac{\partial\boldsymbol{C}_x}{\partial\boldsymbol{c}_{k+m(\ell-1)}} = \boldsymbol{I}_N\otimes\frac{\partial\boldsymbol{C}}{\partial\boldsymbol{c}_{k+m(\ell-1)}} \tag{15}$$

and $\partial\boldsymbol{C}/\partial\boldsymbol{c}_{k+m(\ell-1)}$ is a $m\times m$ matrix with all elements equal to zero except the (ℓ,k) which equals one. In particular, this implies that for any $\boldsymbol{T}\in\mathbb{C}^{m\times m}$

$$\mathrm{Tr}\left\{\boldsymbol{T}\frac{\partial\boldsymbol{C}}{\partial\boldsymbol{c}_{k+m(\ell-1)}}\right\} = \boldsymbol{T}_{k,\ell}. \tag{16}$$

Using (12), we get

$$\begin{aligned}&\mathrm{Tr}\left\{\boldsymbol{C}_x^{-1}\left(\boldsymbol{I}_N\otimes\frac{\partial\boldsymbol{C}}{\partial\boldsymbol{c}_{k+m(\ell-1)}}\right)\boldsymbol{C}_x^{-1}\left(\boldsymbol{ss}^H\otimes\boldsymbol{C}_a\right)\right\} \\ &\quad= E_s\mathrm{Tr}\left\{\boldsymbol{Z}^{-1}\frac{\partial\boldsymbol{C}}{\partial\boldsymbol{c}_{k+m(\ell-1)}}\boldsymbol{Z}^{-1}\boldsymbol{C}_a\right\} = E_s\left[\boldsymbol{Z}^{-1}\boldsymbol{C}_a\boldsymbol{Z}^{-1}\right]_{k,\ell}\end{aligned} \tag{17}$$

which, along with (11), implies that

$$\boldsymbol{F}_{\boldsymbol{c}\boldsymbol{\eta}_\beta} = 2E_s\mathrm{vec}\left(\boldsymbol{Z}^{-1}\boldsymbol{C}_a\boldsymbol{Z}^{-1}\right)\boldsymbol{\eta}_\beta^T. \tag{18}$$

Finally, using (12) and after some straightforward calculations, it comes

$$\begin{aligned}&\mathrm{Tr}\left\{\boldsymbol{C}_x^{-1}\left(\boldsymbol{I}_N\otimes\frac{\partial\boldsymbol{C}}{\partial\boldsymbol{c}_{k+m(\ell-1)}}\right)\boldsymbol{C}_x^{-1}\left(\boldsymbol{I}_N\otimes\frac{\partial\boldsymbol{C}}{\partial\boldsymbol{c}^*_{p+m(q-1)}}\right)\right\} \\ &= \mathrm{Tr}\left\{\boldsymbol{Z}^{-1}\frac{\partial\boldsymbol{C}}{\partial\boldsymbol{c}_{k+m(\ell-1)}}\boldsymbol{Z}^{-1}\frac{\partial\boldsymbol{C}}{\partial\boldsymbol{c}^*_{p+m(q-1)}}\right\} \\ &\quad + (N-1)\mathrm{Tr}\left\{\boldsymbol{C}^{-1}\frac{\partial\boldsymbol{C}}{\partial\boldsymbol{c}_{k+m(\ell-1)}}\boldsymbol{C}^{-1}\frac{\partial\boldsymbol{C}}{\partial\boldsymbol{c}^*_{p+m(q-1)}}\right\} \\ &= \left[\boldsymbol{Z}^{-1}\right]_{q,\ell}\left[\boldsymbol{Z}^{-1}\right]_{k,p} + (N-1)\left[\boldsymbol{C}^{-1}\right]_{q,\ell}\left[\boldsymbol{C}^{-1}\right]_{k,p} \\ &= \left[\boldsymbol{Z}^{-T}\otimes\boldsymbol{Z}^{-1} + (N-1)\boldsymbol{C}^{-T}\otimes\boldsymbol{C}^{-1}\right]_{k+m(\ell-1),p+m(q-1)}\end{aligned} \tag{19}$$

and hence

$$\boldsymbol{F}_{\boldsymbol{cc}} = \boldsymbol{Z}^{-T}\otimes\boldsymbol{Z}^{-1} + (N-1)\boldsymbol{C}^{-T}\otimes\boldsymbol{C}^{-1}. \tag{20}$$

Equations (14), (18), and (20) provide all necessary material to compute the FIM and hence the CRB. Since we are mainly interested in estimating β, we now derive a closed-form expression for CRB(β). Using the fact that $\boldsymbol{F}_{\boldsymbol{\eta}_c\boldsymbol{\eta}_\beta} = \boldsymbol{J}^H\boldsymbol{F}_{\boldsymbol{c}\boldsymbol{\eta}_\beta}$, $\boldsymbol{F}_{\boldsymbol{\eta}_c\boldsymbol{\eta}_c} = \boldsymbol{J}^H\boldsymbol{F}_{\boldsymbol{cc}}\boldsymbol{J}$ along with a lemma for the inverse of partitioned matrices [10], we obtain

$$\begin{aligned}\mathrm{CRB}(\boldsymbol{\eta}_\beta) &= \left(\boldsymbol{F}_{\boldsymbol{\eta}_\beta\boldsymbol{\eta}_\beta} - \boldsymbol{F}_{\boldsymbol{c}\boldsymbol{\eta}_\beta}^H\boldsymbol{J}\left(\boldsymbol{J}^H\boldsymbol{F}_{\boldsymbol{cc}}\boldsymbol{J}\right)^{-1}\boldsymbol{J}^H\boldsymbol{F}_{\boldsymbol{c}\boldsymbol{\eta}_\beta}\right)^{-1} \\ &= \left(\boldsymbol{F}_{\boldsymbol{\eta}_\beta\boldsymbol{\eta}_\beta} - \boldsymbol{F}_{\boldsymbol{c}\boldsymbol{\eta}_\beta}^H\boldsymbol{F}_{\boldsymbol{cc}}^{-1}\boldsymbol{F}_{\boldsymbol{c}\boldsymbol{\eta}_\beta}\right)^{-1} = \left(\nu\boldsymbol{I} + \upsilon\boldsymbol{\eta}_\beta\boldsymbol{\eta}_\beta^T\right)^{-1}\end{aligned} \tag{21}$$

with

$$\nu = 2E_s\left(\overline{\boldsymbol{a}}^H\boldsymbol{Z}^{-1}\overline{\boldsymbol{a}}\right) \tag{22a}$$

$$\upsilon = 4E_s^2\mathrm{vec}(\boldsymbol{C}_a)^H\left[\boldsymbol{Z}^T\otimes\boldsymbol{Z} + \frac{\boldsymbol{C}^T\otimes\boldsymbol{C}}{N-1}\right]^{-1}\mathrm{vec}(\boldsymbol{C}_a) \tag{22b}$$

and where, to derive υ, we made use of (5), (7), and (20) to obtain

$$\boldsymbol{F}_{\boldsymbol{cc}}^{-1} = \left(\boldsymbol{Z}^T \otimes \boldsymbol{Z}\right) - \left(\boldsymbol{Z}^T \otimes \boldsymbol{Z}\right)\left[\boldsymbol{Z}^T \otimes \boldsymbol{Z} + \frac{\boldsymbol{C}^T \otimes \boldsymbol{C}}{N-1}\right]^{-1}\left(\boldsymbol{Z}^T \otimes \boldsymbol{Z}\right). \quad (23)$$

Note that CRB$(\boldsymbol{\eta}_\beta)$ is not diagonal and that its diagonal elements are not identical. For any unbiased estimate $\widehat{\beta} = \widehat{\beta}_R + i\widehat{\beta}_I$ of β

$$\mathcal{E}\left\{\left|\widehat{\beta} - \beta\right|^2\right\} = \mathcal{E}\left\{\left(\widehat{\beta}_R - \beta_R\right)^2\right\} + \mathcal{E}\left\{\left(\widehat{\beta}_I - \beta_I\right)^2\right\} \geq \text{Tr}\left\{\text{CRB}(\boldsymbol{\eta}_\beta)\right\} \triangleq \text{CRB}(\beta). \quad (24)$$

Since the eigenvalues of $\nu\boldsymbol{I} + \upsilon\boldsymbol{\eta}_\beta\boldsymbol{\eta}_\beta^T$ are ν and $\nu + |\beta|^2\,\upsilon$, it follows that

$$\text{CRB}(\beta) = \frac{1}{\nu} + \frac{1}{\nu + |\beta|^2\,\upsilon}. \quad (25)$$

Equations (22) and (25) provide closed-form expressions for the CRB. The following remarks are in order.

- Through numerical evaluation, it was observed that in most cases, one can accurately approximate the CRB as follows. Observe that $\boldsymbol{C} = O(1)$ while $\boldsymbol{Z} = \max\left\{O(1), O\left(N\,\|\boldsymbol{C}_a\|\right)\right\}$ as $E_s = \boldsymbol{s}^H\boldsymbol{s} = O(N)$. Hence, for not too small N, $(1/(N-1))\boldsymbol{C}^T \otimes \boldsymbol{C}$ is small compared to $\boldsymbol{Z}^T \otimes \boldsymbol{Z}$. This in turn implies that $\upsilon \simeq 4E_s^2\text{Tr}\left\{\boldsymbol{Z}^{-1}\boldsymbol{C}_a\boldsymbol{Z}^{-1}\boldsymbol{C}_a\right\}$. With this approximation, the CRB becomes

$$\text{CRB}(\beta) \simeq \text{Tr}\left\{\boldsymbol{F}_{\boldsymbol{\eta}_\beta\boldsymbol{\eta}_\beta}^{-1}\right\} = \frac{1}{E_s\left(\overline{\boldsymbol{a}}^H\boldsymbol{Z}^{-1}\overline{\boldsymbol{a}}\right)} \times\left\{1 - \frac{|\beta|^2\,E_s\text{Tr}\left\{\boldsymbol{Z}^{-1}\boldsymbol{C}_a\boldsymbol{Z}^{-1}\boldsymbol{C}_a\right\}}{\overline{\boldsymbol{a}}^H\boldsymbol{Z}^{-1}\overline{\boldsymbol{a}} + 2\,|\beta|^2\,E_s\text{Tr}\left\{\boldsymbol{Z}^{-1}\boldsymbol{C}_a\boldsymbol{Z}^{-1}\boldsymbol{C}_a\right\}}\right\} \quad (26)$$

  which provides a rather simple expression for the CRB. It will be shown through numerical examples that the approximated CRB (26) is very close to the exact CRB (25).
- Where there are no uncertainties, then $\boldsymbol{C}_a = \boldsymbol{0}$ and the steering vector is known to be $\overline{\boldsymbol{a}}$. In this case, $\nu = 2E_s\left(\overline{\boldsymbol{a}}^H\boldsymbol{C}^{-1}\overline{\boldsymbol{a}}\right)$, $\upsilon = 0$ and the CRB reduces to

$$\text{CRB}(\beta)|_{\boldsymbol{C}_a=\boldsymbol{0}} = \frac{1}{E_s\left(\overline{\boldsymbol{a}}^H\boldsymbol{C}^{-1}\overline{\boldsymbol{a}}\right)} \quad (27)$$

  which coincides with the expression derived in [2]. Hence, the CRB's expression in (25) generalizes the result obtained in [2] to the case of random steering vector errors.
- When the steering vector is known the CRB decreases as $1/N$ and thus goes to zero as N goes to infinity. In contrast, when N tends to infinity, and assuming that $\|\boldsymbol{C}_a\|$ is constant

$$\nu \underset{N\to\infty}{\simeq} \frac{2\left(\overline{\boldsymbol{a}}^H\boldsymbol{C}_a^{-1}\overline{\boldsymbol{a}}\right)}{|\beta|^2}; \quad \upsilon \underset{N\to\infty}{\simeq} \frac{4m}{|\beta|^4} \quad (28)$$

  and hence the CRB does not go to zero. This is a logical since the steering vector $\boldsymbol{a}$ is random [11] and even when $N \to \infty$ the performance is limited by the steering vector errors covariance matrix $\boldsymbol{C}_a$. For instance, when $\boldsymbol{C}_a = \sigma_a^2\boldsymbol{I}$ and $\overline{\boldsymbol{a}}^H\overline{\boldsymbol{a}} = m$, the asymptotic (in N) CRB is approximately $|\beta|^2\,\sigma_a^2/m$ and is therefore proportional to the variance of the steering vector errors.

III. Maximum Likelihood Estimation

We now turn to the derivation of the MLE of β and $\boldsymbol{C}$. The probability density function (pdf) of the observations can be written as

$$p(\boldsymbol{x}; \beta, \boldsymbol{C}) = \int p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})p(\boldsymbol{a})\,d\boldsymbol{a}$$

where $p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})$ is the conditional pdf, given $\boldsymbol{a}$, $p(\boldsymbol{a})$ is the a priori pdf of the steering vector of interest and where the semicolon indicates that the pdf depends on β and $\boldsymbol{C}$. For a given $\boldsymbol{a}$, the snapshots are Gaussian so that the conditional pdf is given by

$$p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C}) = \frac{\exp\left\{-\sum_{t=1}^{N}\left(\boldsymbol{x}_t - \beta\boldsymbol{a}s_t\right)^H\boldsymbol{C}^{-1}\left(\boldsymbol{x}_t - \beta\boldsymbol{a}s_t\right)\right\}}{\pi^{mN}\,|\boldsymbol{C}|^N} \quad (29)$$

where $|\boldsymbol{C}|$ stands for the determinant of matrix $\boldsymbol{C}$. Since $p(\boldsymbol{a})$ does not depend on β and $\boldsymbol{C}$, we consider the derivatives of $p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})$ with respect to β and $\boldsymbol{C}$. For notational convenience let us define $\Lambda(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C}) \triangleq \ln p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})$. First note that

$$\begin{aligned}\frac{\partial p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})}{\partial\beta} &= p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})\frac{\partial\Lambda(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})}{\partial\beta}\\ &= p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})\frac{\partial}{\partial\beta}\cdot\left\{-\sum_{t=1}^{N}\left(\boldsymbol{x}_t - \beta\boldsymbol{a}s_t\right)^H\boldsymbol{C}^{-1}\left(\boldsymbol{x}_t - \beta\boldsymbol{a}s_t\right)\right\}\\ &= p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})\left\{\boldsymbol{a}^H\boldsymbol{C}^{-1}\boldsymbol{X}\boldsymbol{s}^* - \beta\left(\boldsymbol{a}^H\boldsymbol{C}^{-1}\boldsymbol{a}\right)\boldsymbol{s}^H\boldsymbol{s}\right\}\end{aligned} \quad (30)$$

and consequently

$$\begin{aligned}\frac{\partial p(\boldsymbol{x}; \beta, \boldsymbol{C})}{\partial\beta} &= \int\frac{\partial p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})}{\partial\beta}p(\boldsymbol{a})d\boldsymbol{a}\\ &\propto \int\left\{\boldsymbol{a}^H\boldsymbol{C}^{-1}\boldsymbol{X}\boldsymbol{s}^* - \beta\left(\boldsymbol{a}^H\boldsymbol{C}^{-1}\boldsymbol{a}\right)E_s\right\}p(\boldsymbol{a}|\boldsymbol{x}; \beta, C)d\boldsymbol{a}\\ &= \boldsymbol{a}_{\text{post}}^H\boldsymbol{C}^{-1}\boldsymbol{X}\boldsymbol{s}^*\\ &\quad - \beta E_s\left[\boldsymbol{a}_{\text{post}}^H\boldsymbol{C}^{-1}a_{\text{post}} + \text{Tr}\left\{\boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}\boldsymbol{C}^{-1}\right\}\right]\end{aligned} \quad (31)$$

where $\boldsymbol{a}_{\text{post}} \triangleq \mathcal{E}\left\{\boldsymbol{a}|\boldsymbol{x}; \beta, \boldsymbol{C}\right\} = \int \boldsymbol{a}p(\boldsymbol{a}|\boldsymbol{x}; \beta, C)\,d\boldsymbol{a}$ is the *a posteriori* mean and $\boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}$ is the *a posteriori* covariance matrix. In order to derive the last equality in (31), we used the fact that

$$\int\left(\boldsymbol{a}^H\boldsymbol{\zeta}\right)p(\boldsymbol{a}|\boldsymbol{x}; \beta, \boldsymbol{C})d\boldsymbol{a} = \boldsymbol{a}_{\text{post}}^H\boldsymbol{\zeta} \quad (32a)$$

$$\int\left(\boldsymbol{a}^H\boldsymbol{\Gamma}\boldsymbol{a}\right)p(\boldsymbol{a}|\boldsymbol{x}; \beta, \boldsymbol{C})d\boldsymbol{a} = \boldsymbol{a}_{\text{post}}^H\boldsymbol{\Gamma}\boldsymbol{a}_{\text{post}} + \text{Tr}\left\{\boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}\boldsymbol{\Gamma}\right\}. \quad (32b)$$

Accordingly

$$\begin{aligned}\frac{\partial p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})}{\partial\boldsymbol{C}} &= p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})\frac{\partial\Lambda(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})}{\partial\boldsymbol{C}}\\ &= p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})\frac{\partial}{\partial\boldsymbol{C}}\left\{-N\ln|\boldsymbol{C}|\right\} - p(\boldsymbol{x}|\boldsymbol{a}; \beta, C)\frac{\partial}{\partial\boldsymbol{C}}\cdot\left\{\sum_{t=1}^{N}\left(\boldsymbol{x}_t - \beta\boldsymbol{a}s_t\right)^H\boldsymbol{C}^{-1}\left(\boldsymbol{x}_t - \beta\boldsymbol{a}s_t\right)\right\}\\ &= -Np(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})\boldsymbol{C}^{-1}\\ &\quad + p(\boldsymbol{x}|\boldsymbol{a}; \beta, \boldsymbol{C})\boldsymbol{C}^{-1}\left(\boldsymbol{X} - \beta\boldsymbol{a}\boldsymbol{s}^T\right)\left(\boldsymbol{X} - \beta\boldsymbol{a}\boldsymbol{s}^T\right)^H\boldsymbol{C}^{-1}\end{aligned} \quad (33)$$

which yields (34), shown at the bottom of the page. Both $\boldsymbol{a}_{\mathrm{post}}$ and $\boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}$ can be expressed as a function of the data matrix $\boldsymbol{X}$. More precisely, the *a posteriori* covariance matrix $\boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}$ is given by [11, p. 532]

$$\begin{aligned}\boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}} &= \boldsymbol{C}_a - |\beta|^2 \left(\boldsymbol{s}^H \otimes \boldsymbol{C}_a\right) \boldsymbol{C}_x^{-1} \left(\boldsymbol{s} \otimes \boldsymbol{C}_a\right) \\ &= \boldsymbol{C}_a - |\beta|^2 \boldsymbol{C}_a \left(\boldsymbol{s}^H \otimes \boldsymbol{I}_m\right) \boldsymbol{C}_x^{-1} \left(\boldsymbol{s} \otimes \boldsymbol{I}_m\right) \boldsymbol{C}_a \\ &= \left[\boldsymbol{C}_a^{-1} + |\beta|^2 E_s \boldsymbol{C}^{-1}\right]^{-1} \end{aligned} \tag{35}$$

where we used (5) and the expression of $\boldsymbol{C}_x^{-1}$ in (6). Similarly, the *a posteriori* mean is given by

$$\begin{aligned}\boldsymbol{a}_{\mathrm{post}} &= \overline{\boldsymbol{a}} + \beta^* \left(\boldsymbol{s}^H \otimes \boldsymbol{C}_a\right) \boldsymbol{C}_x^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \\ &= \overline{\boldsymbol{a}} + \beta^* \boldsymbol{C}_a \left(\boldsymbol{s}^H \otimes \boldsymbol{I}_m\right) \boldsymbol{C}_x^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \\ &= \boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}} \left(\boldsymbol{C}_a^{-1} \overline{\boldsymbol{a}} + \beta^* \boldsymbol{C}^{-1} \boldsymbol{X} \boldsymbol{s}^*\right). \end{aligned} \tag{36}$$

Setting the derivatives in (31) and (34) to zero, it follows that

$$\beta = \frac{\boldsymbol{a}_{\mathrm{post}}^H \boldsymbol{C}^{-1} \boldsymbol{X} \boldsymbol{s}^*}{E_s \left[\boldsymbol{a}_{\mathrm{post}}^H \boldsymbol{C}^{-1} \boldsymbol{a}_{\mathrm{post}} + \mathrm{Tr}\left\{\boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}} \boldsymbol{C}^{-1}\right\}\right]} \tag{37a}$$

$$\boldsymbol{C} = \frac{1}{N} \left(\boldsymbol{X} - \beta \boldsymbol{a}_{\mathrm{post}} \boldsymbol{s}^T\right) \left(\boldsymbol{X} - \beta \boldsymbol{a}_{\mathrm{post}} \boldsymbol{s}^T\right)^H + |\beta|^2 \frac{E_s}{N} \boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}. \tag{37b}$$

Before pursuing the derivation of the ML estimator, the following remarks are in order. The ML estimate of $\boldsymbol{C}$ in (37b) is the sum of two terms. The first corresponds to the MLE of $\boldsymbol{C}$ would the steering vector be known and equal to $\boldsymbol{a}_{\mathrm{post}}$ while the second accounts for the information brought by the observations. Note also that (37a) corresponds to the MLE of β, would $\boldsymbol{a} = \boldsymbol{a}_{\mathrm{post}}$ and $\boldsymbol{C}$ be known.

Equations (35)–(37) form the basis for computing the ML estimates. However, it can be observed that β depends on $\boldsymbol{C}$ and $\boldsymbol{a}_{\mathrm{post}}$, which itself depends on β and $\boldsymbol{C}$. Similarly, $\boldsymbol{C}$ is a function of β and $\boldsymbol{a}_{\mathrm{post}}$. Therefore, there does not exist any closed-form solution for the problem at hand and we have to resort to an iterative procedure. Equations (35)–(37) suggest the iterative scheme of Table II to estimate β and $\boldsymbol{C}$.

In the simulations presented below, $\boldsymbol{a}_{\mathrm{post}}^{(1)} = \overline{\boldsymbol{a}}$ and the initial guess of the noise covariance matrix $\boldsymbol{C}$ was the unstructured MLE derived in [4], or equivalently the matrix $\boldsymbol{T}$ in [2, eq. (10)]. Note that this is not the true MLE $\widehat{\boldsymbol{C}}_{ML}$ of $\boldsymbol{C}$ when $\boldsymbol{a}$ is known. It is interesting to note that, for the data model of [2], a better estimate of the covariance matrix, e.g., replacing $\boldsymbol{T}$ by $\widehat{\boldsymbol{C}}_{ML}$ does not yield a better estimate of β, in contrast to the MLE in the present problem. Although it appears very difficult to prove that the scheme of Table II converges, we did not encounter any convergence problem in practice. The algorithm typically converges within 10 to 20 iterations.

TABLE II
ITERATIVE SCHEME FOR COMPUTING THE MLE

Initialisation: $\boldsymbol{C}^{(1)}$; $\quad \boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}^{(1)} = \boldsymbol{C}_a$; $\quad \boldsymbol{a}_{\mathrm{post}}^{(1)}$

Recursion: for $n = 1, \cdots,$

$$\beta^{(n+1)} = \frac{\boldsymbol{a}_{\mathrm{post}}^{(n)H} \boldsymbol{C}^{(n)-1} \boldsymbol{X} \boldsymbol{s}^*}{E_s \left[\boldsymbol{a}_{\mathrm{post}}^{(n)H} \boldsymbol{C}^{(n)-1} \boldsymbol{a}_{\mathrm{post}}^{(n)} + \mathrm{Tr}\left\{\boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}^{(n)} \boldsymbol{C}^{(n)-1}\right\}\right]}$$

$$\boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}^{(n+1)} = \left(|\beta|^{(n+1)2} E_s \boldsymbol{C}^{(n)-1} + \boldsymbol{C}_a^{-1}\right)^{-1}$$

$$\boldsymbol{a}_{\mathrm{post}}^{(n+1)} = \boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}^{(n+1)} \left(\boldsymbol{C}_a^{-1} \overline{\boldsymbol{a}} + \beta^{(n+1)*} \boldsymbol{C}^{(n)-1} \boldsymbol{X} \boldsymbol{s}^*\right)$$

$$\boldsymbol{C}^{(n+1)} = N^{-1} \left(\boldsymbol{X} - \beta^{(n+1)} \boldsymbol{a}_{\mathrm{post}}^{(n+1)} \boldsymbol{s}^T\right) \left(\boldsymbol{X} - \beta^{(n+1)} \boldsymbol{a}_{\mathrm{post}}^{(n)} \boldsymbol{s}^T\right)^H + |\beta|^{(n+1)2} N^{-1} E_s \boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}^{(n+1)}$$

IV. Numerical Examples

In this section, we provide numerical illustrations of the results derived in the previous sections. We consider an uniform linear array with $m = 6$ elements separated a half-wavelength. We consider a scenario where the SOI impinges from the broadside of the array and undergoes local scattering so that its spatial signature is [6]

$$\boldsymbol{a} = \overline{\boldsymbol{a}} + \frac{1}{\sqrt{L}} \sum_{k=1}^{L} g_k \boldsymbol{a}(\widetilde{\theta}_k) \tag{38}$$

where g_k are zero-mean, independent and identically distributed random variables with power σ_g^2 and $\widetilde{\theta}_k$ are independent uniformly distributed random variables with standard deviation (i.e., angular spread) $\sigma_\theta = 15^\circ$. $\overline{\boldsymbol{a}} = \boldsymbol{a}(0^\circ)$ corresponds to the spatial signature of the line of sight component. The model in (38) is typical of coherent local scattering for which there exists a line of sight component along with multiple scatterers randomly distributed around the user of interest. In the simulations, we use $L = 10$. We define the uncertainty ratio (UR) as $UR = 10\log_{10}\left(\mathrm{Tr}\{\boldsymbol{C}_a\}/\overline{\boldsymbol{a}}^H \overline{\boldsymbol{a}}\right)$. *UR* is a measure of the degree of uncertainty in $\boldsymbol{a}$. The noise component comprises a white noise contribution with power σ^2 and two interferences with respective DOAs -30° and 20°, and powers 20 dB above the white noise level. The signal to noise ratio is defined as $\mathrm{SNR} = -10\log_{10}\left(\sigma^2\right)$. The SNR is set to SNR = 3 dB in all simulations.

First, we study the influence of $\boldsymbol{C}_a$, N and SNR onto the Cramér–Rao bound and we examine the validity of the approximated formula (26). Toward this end, we display the exact and approximated CRBs versus *UR* for different values of N (see Fig. 1) and different values of the SNR; see Fig. 2. From inspection of these figures, it can be observed that (26) provides a very accurate approximation of the exact CRB. Note also that when *UR* is very small the CRB is roughly proportional to $(N \times \mathrm{SNR})^{-1}$, see e.g., the 3 dB improvement of the CRB when N goes from $N = 10$ to $N = 20$ with $UR = -30$ dB, or the 6 dB improvement when SNR goes from

$$\begin{aligned}\frac{\partial p(\boldsymbol{x};\beta,\boldsymbol{C})}{\partial \boldsymbol{C}} &= \int \frac{\partial p(\boldsymbol{x}|\boldsymbol{a};\beta,\boldsymbol{C})}{\partial \boldsymbol{C}} p(\boldsymbol{a})\, d\boldsymbol{a} \\ &\propto -\boldsymbol{C}^{-1} \int \left\{\boldsymbol{I} - N^{-1}\left(\boldsymbol{X} - \beta \boldsymbol{a}\boldsymbol{s}^T\right)\left(\boldsymbol{X} - \beta \boldsymbol{a}\boldsymbol{s}^T\right)^H \boldsymbol{C}^{-1}\right\} p(a|\boldsymbol{x};\beta,\boldsymbol{C})\, d\boldsymbol{a} \\ &= -\boldsymbol{C}^{-1}\left\{\boldsymbol{I} - \left[N^{-1}\left(\boldsymbol{X} - \beta \boldsymbol{a}_{\mathrm{post}}\boldsymbol{s}^T\right)\left(\boldsymbol{X} - \beta \boldsymbol{a}_{\mathrm{post}}\boldsymbol{s}^T\right)^H + |\beta|^2 \frac{E_s}{N} \boldsymbol{C}_{\boldsymbol{a}|\boldsymbol{x}}\right] \boldsymbol{C}^{-1}\right\}. \end{aligned} \tag{34}$$

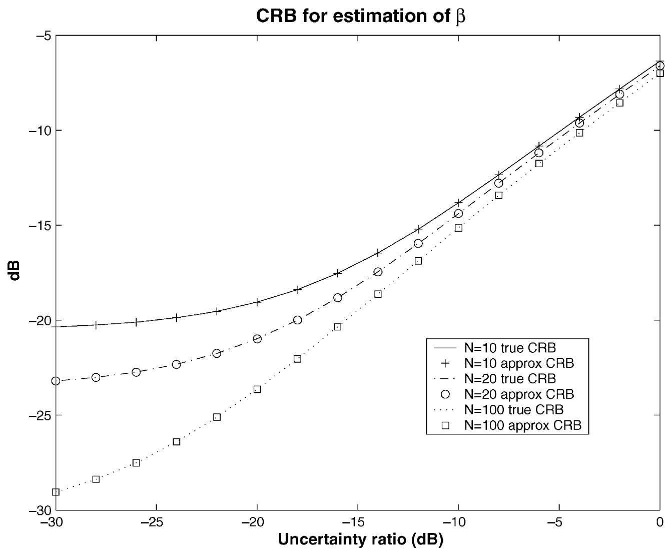

Fig. 1. Exact and approximated Cramér–Rao bounds versus the uncertainty ratio. $\beta = 1$ and SNR $= 3$ dB.

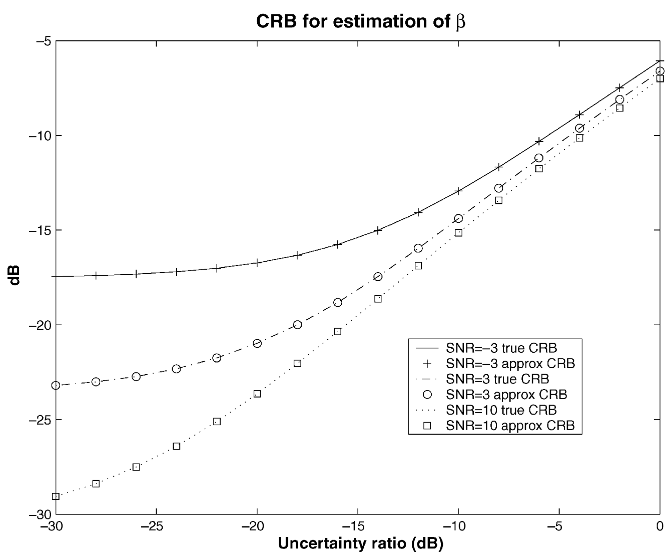

Fig. 2. Exact and approximated Cramér–Rao bounds versus the uncertainty ratio. $\beta = 1$ and $N = 20$.

SNR $= -3$ dB to SNR $= 3$ dB with $UR = -30$ dB. However, when *UR* increases, this improvement decreases. Indeed for *UR* above some threshold (say -10 dB), the CRB still depends on N (or SNR) but more and more on $\boldsymbol{C}_a$. Increasing N or having a better SNR does not result in a significant improvement. For moderate to large *UR* the CRB is roughly proportional to *UR* (i.e., nearly independent of N and SNR), indicating that the parameter with largest influence onto the performance is $\boldsymbol{C}_a$.

Next, we study the performance of the ML estimator and compare it with the MLE which assumes that $\boldsymbol{a}$ is known and equal to $\overline{\boldsymbol{a}}$ (we refer to it as the $\mathrm{MLE}_{a\ \mathrm{priori}}$ in the figure). Fig. 3 displays the mean-square errors of the two estimators along with the CRB versus *UR*. It can be observed that, for very small *UR*, there is hardly no difference between the two methods. In contrast, when *UR* increases the difference becomes significant indicating that it is really worth taking into account the steering vector's uncertainties. Finally, note that the MLE has a performance very close to the CRB for all values of *UR*.

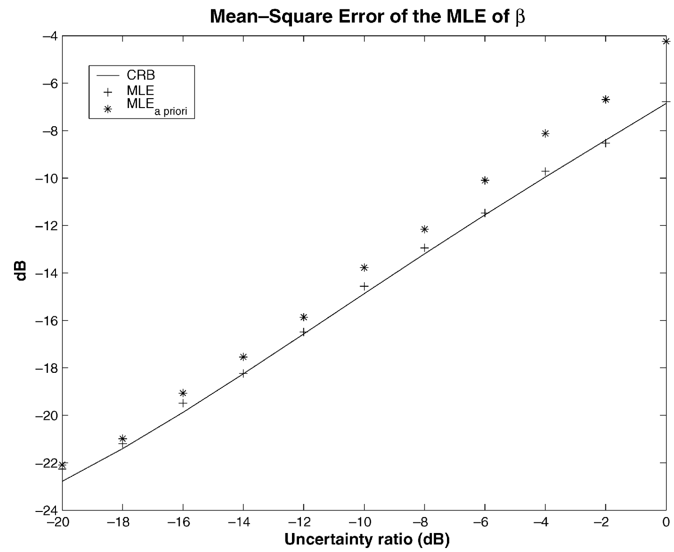

Fig. 3. Mean-square error of the maximum likelihood estimators versus *UR*. $\beta = 1, N = 50$ and SNR $= 3$ dB.

V. Conclusions

In this paper, we considered the problem of estimating the amplitude of a signal with known waveform received on an array of sensors with uncertainties about the spatial signature. We examined the influence of these random errors onto the estimation performance by deriving closed-form expressions for the CRB. Furthermore, we derived the MLE for the problem at hand and showed that the uncertainties should be taken into account in the ML procedure, even though it results in a more complicated algorithm.

References

[1] A. Farina, *Antenna Based Signal Processing Techniques for Radar Systems*. Boston, MA: Artech House, 1992.
[2] Y. Jiang, P. Stoica, and J. Li, "Array signal processing in the known waveform and steering vector case," *IEEE Trans. Signal Process.*, vol. 52, no. 1, pp. 23–35, Jan. 2004.
[3] J. Li and R. Compton, "Maximum likelihood angle estimation for signals with known waveforms," *IEEE Trans. Signal Process.*, vol. 41, no. 9, pp. 2850–2862, Sep. 1993.
[4] J. Li, B. Hadler, P. Stoica, and M. Viberg, "Computationally efficient angle estimation for signals with known waveforms," *IEEE Trans. Signal Process.*, vol. 43, no. 9, pp. 2154–2163, Sep. 1995.
[5] M. Viberg, P. Stoica, and B. Ottersten, "Maximum likelihood array processing in spatially correlated noise fields using parameterized signals," *IEEE Trans. Signal Process.*, vol. 45, no. 4, pp. 996–1004, Apr. 1997.
[6] S. Shahbazpanahi, A. Gershman, Z.-Q. Luo, and K. Wong, "Robust adaptive beamforming for general-rank signal models," *IEEE Trans. Signal Process.*, vol. 51, no. 9, pp. 2257–2269, Sep. 2003.
[7] S. Valaee, B. Champagne, and P. Kabal, "Parametric localization of distributed sources," *IEEE Trans. Signal Process.*, vol. 43, no. 9, pp. 2144–2153, Sep. 1995.
[8] K. Bell, Y. Ephraim, and H. V. Trees, "A Bayesian approach to robust adaptive beamforming," *IEEE Trans. Signal Process.*, vol. 48, no. 2, pp. 386–398, Feb. 2000.
[9] M. Viberg and A. Swindlehurst, "Analysis of the combined effects of finite samples and model errors on array processing performance," *IEEE Trans. Signal Process.*, vol. 42, no. 11, pp. 3073–3083, Nov. 1994.
[10] T. Söderström and P. Stoica, *System Identification*. London, U.K.: Prentice-Hall International, 1989.
[11] S. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1993.