Detection of an Unknown Rank-One Component in White Noise

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***Abstract*—We consider the detection of an unknown and arbitrary rank-one signal in a spatial sector scanned by a small number m of beams. We address the problem of finding the maximal invariant for the problem at hand and show that it consists of the ratio of the eigenvalues of a Wishart matrix to its trace. Next, we derive the generalized-likelihood ratio test (GLRT) along with expressions for its probability density function (pdf) under both hypotheses. Special attention is paid to the case $m = 2$, where the GLRT is shown to be a uniformly most powerful invariant (UMPI). Numerical simulations attest to the validity of the theoretical analysis and illustrate the detection performance of the GLRT.**

***Index Terms*—Array processing, detection, eigenvalues, maximal invariant statistic, Wishart matrices.**

I. Problem Statement

We consider an array of sensors aimed at scanning its field of view with the goal of detecting the presence of a source of interest in a background of noise. Usually, the presence of a target is tested for many potential (and thus assumed known) steering vectors (i.e., spatial signature) while the temporal signature of the target is considered as unknown with different assumptions about its statistical nature, depending on the type of target that is expected. Herein, we consider that some type of beamspace processing is first done and that we wish to detect a source within a spatial sector. Hence, the whole field of view is scanned on a spatial sector by spatial sector basis. Furthermore, within a spatial sector, both the spatial and temporal signature of the target are assumed to be unknown. In other words, we do not seek to detect a target at a specified direction of arrival; rather, we try to detect a rank-one component in the data (and as a by-product, we estimate its space–time signature). Of course, this simpler detector may not perform as well as a detector that would use all the array elements and would contain the actual steering vector of the target in the set of tested steering vectors. Hence, the detection scheme presented here may be viewed as a fast and simple preprocessing step; once it detects a target, a more sophisticated detection scheme (e.g., operating in the element space) might be used to confirm or to invalidate the decision.

The problem addressed herein can be formulated mathematically as that of deciding between the following two hypotheses:

$$\begin{cases} H_0: & \boldsymbol{X} = \boldsymbol{N} \\ H_1: & \boldsymbol{X} = \boldsymbol{a}\boldsymbol{s}^H + \boldsymbol{N} \end{cases} \tag{1}$$

where $\boldsymbol{X} = [\boldsymbol{x}(1) \cdots \boldsymbol{x}(N)] \in \mathbb{C}^{m \times N}$ is the data matrix, with m being the number of beams used to cover a spatial sector (or the number of array sensors if we operate in element space) and N denoting the number of available snapshots. $\boldsymbol{N} = [\boldsymbol{n}(1) \cdots \boldsymbol{n}(N)]$ corresponds to noise, and we assume that $\boldsymbol{n}(t)$ is drawn from a complex multivariate Gaussian distribution with zero mean and covariance matrix $\sigma^2\boldsymbol{I}$ with σ^2 unknown.[1] In (1), $\boldsymbol{a}$ and $\boldsymbol{s}$ stand for the *unknown* spatial and temporal signatures of the target. Treating $\boldsymbol{a}$ as an unknown and arbitrary quantity may also be meaningful when there exist uncertainties about the steering vector, i.e., when the actual steering vector may anyway suffer mismatch from the steering vector under test, or when it cannot be written simply as a function of a (possibly unknown) parameter, for instance the direction of arrival. This occurs when the array is not perfectly calibrated. In such a situation, it may not be advisable to test for the presence of a target with presumed signature $\boldsymbol{a}$ while, even if the target is present, its steering vector will be different from $\boldsymbol{a}$. It should be observed that the problem in (1) is also relevant when a monopulse radar is used to detect a target. In such a case, $m = 2$, and the rows of $\boldsymbol{X}$ correspond to the outputs of the sum and difference channels, respectively.

II. Detection

A. Invariances of the Problem

We first consider the natural invariances of the problem with a view to find the maximal invariant statistic for the problem at hand. The theory of invariance is well known (see, e.g., [1]–[3]). Briefly stated, the idea is that if the hypotheses testing problem is invariant to some group of transformations, then one should look for detection statistics that are also invariant to these transformations. This leads to the concept of maximal invariant: any function of the data that is invariant to the transformation depends on the data through the maximal invariant. Furthermore, if the maximal invariant is a scalar function of the data and has a monotone likelihood ratio, then it is a uniformly most powerful invariant (UMPI). We refer the reader to [4]–[7] for comprehensive presentations of maximal invariance and its application to various detection problems in array processing.

In our case, finding the invariances is rather straightforward. Since we wish to preserve Gaussianity of the data, we only consider linear transformations of $\boldsymbol{X}$. As σ^2 is unknown, the detection statistic should be invariant to scaling. Moreover, we wish to preserve a covariance matrix that is proportional to $\boldsymbol{I}$, and a rank-one mean under H_1. These considerations show that the hypothesis testing problem (1) is invariant under the group of transformations G defined by

$$G = \left\{ g : \boldsymbol{X} \rightarrow g(\boldsymbol{X}) = c\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}^H \right\} \tag{2}$$

where c is a scalar, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are unitary matrices. Moreover, the group of transformations induced on the parameter space is

$$\bar{G} = \left\{ \bar{g} : [\boldsymbol{a} \quad \boldsymbol{s} \quad \sigma^2] \rightarrow [a\boldsymbol{A}\boldsymbol{a} \quad b\boldsymbol{B}\boldsymbol{s} \quad c^2\sigma^2] \right\} \tag{3}$$

with $c = ab^*$. The maximal invariant $\boldsymbol{m}(\boldsymbol{X})$ of the problem must then satisfy the following conditions:

$$\begin{aligned} \boldsymbol{m}(\boldsymbol{X}) &= \boldsymbol{m}\left(g(\boldsymbol{X})\right) \qquad \forall g \in G \\ \boldsymbol{m}(\boldsymbol{X}_1) &= \boldsymbol{m}(\boldsymbol{X}_2) \Rightarrow \exists g \in G \ \text{ such that } \ \boldsymbol{X}_2 = g(\boldsymbol{X}_1). \end{aligned}$$

The maximal invariant statistic with respect to G is given by the following proposition.

[1]This assumption corresponds to the case where the noise covariance matrix is known up to a scaling factor. If the noise covariance matrix is not proportional to $\boldsymbol{I}$ but known, a prewhitening step can be applied to obtain the model in (1). Note that assuming a completely unknown noise covariance matrix is not feasible here as we do not have any secondary data (i.e., free of the signal of interest), which would enable us to estimate the noise covariance matrix.

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Proposition 1: The maximal invariant statistic with respect to the group of transformations (2) is $\{t^{-1}\lambda_k\}_{k=1}^{m-1}$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_m$ are the m ordered eigenvalues of $\boldsymbol{X}\boldsymbol{X}^H$ and $t = \mathrm{Tr}\{\boldsymbol{X}\boldsymbol{X}\}^H$.

Proof: Obviously, $t^{-1}\lambda_k$ is invariant to scaling and to pre- and postmultiplication of $\boldsymbol{X}$ by unitary matrices. Furthermore, assume that

$$\frac{\lambda_k\left(\boldsymbol{X}_2\boldsymbol{X}_2^H\right)}{\mathrm{Tr}\left\{\boldsymbol{X}_2\boldsymbol{X}_2^H\right\}} = \frac{\lambda_k\left(\boldsymbol{X}_1\boldsymbol{X}_1^H\right)}{\mathrm{Tr}\left\{\boldsymbol{X}_1\boldsymbol{X}_1^H\right\}}, \qquad k = 1, \cdots, m-1.$$

Then, as a consequence, the equality also holds for $k = m$. Therefore, $\lambda_k(\boldsymbol{X}_2\boldsymbol{X}_2^H) = c\lambda_k(\boldsymbol{X}_1\boldsymbol{X}_1^H)$ for $k = 1, \cdots, m$. Hence, the singular value decompositions of $\boldsymbol{X}_2$ and $\boldsymbol{X}_1$ are given by $\boldsymbol{X}_2 = \boldsymbol{U}_2\boldsymbol{\Sigma}_2\boldsymbol{V}_2^H$ and $\boldsymbol{X}_1 = \boldsymbol{U}_1\boldsymbol{\Sigma}_1\boldsymbol{V}_1^H$ with $\boldsymbol{\Sigma}_2 = c\boldsymbol{\Sigma}_1$. Therefore

$$\begin{aligned}\boldsymbol{X}_2 &= \boldsymbol{U}_2\boldsymbol{\Sigma}_2\boldsymbol{V}_2^H = c\boldsymbol{U}_2\boldsymbol{\Sigma}_1\boldsymbol{V}_2^H \\ &= c\left[\boldsymbol{U}_2\boldsymbol{U}_1^H\right]\boldsymbol{X}_1\left[\boldsymbol{V}_2\boldsymbol{V}_1^H\right]^H\end{aligned}$$

which completes the proof. ■

Corollary 1: The maximal invariant statistic is a scalar function of the data $\boldsymbol{X}$ only in the case $m = 2$, where it is given by the ratio of the largest eigenvalue of the matrix $\boldsymbol{X}\boldsymbol{X}^H$ to its trace. For $m > 2$, the maximal invariant is a vector-valued function of the data, and therefore a UMPI test cannot exist.

It should also be observed that the induced maximal invariant is simply $\|\boldsymbol{a}\|^2\|\boldsymbol{s}\|^2/\sigma^2$, which roughly corresponds to N times the array signal-to-noise ratio.

B. Generalized-Likelihood Ratio Test

In this section, we derive the generalized-likelihood ratio test (GLRT) for the problem in (1). As a first step toward deriving the GLRT, maximum-likelihood estimates (MLE) of the unknown parameters must be obtained. Under the stated assumptions, the likelihood function is given by [3]

$$\ell(\boldsymbol{X}) = \frac{\exp\left\{-\frac{1}{\sigma^2}\mathrm{Tr}\left\{(\boldsymbol{X} - \mu\boldsymbol{a}\boldsymbol{s}^H)(\boldsymbol{X} - \mu\boldsymbol{a}\boldsymbol{s}^H)^H\right\}\right\}}{(\pi\sigma^2)^{mN}} \tag{4}$$

where $\mu = 0$ under H_0 and $\mu = 1$ under H_1. Under H_0, only σ^2 is unknown, and its MLE is readily obtained as

$$\widehat{\sigma}_0^2 = \frac{1}{mN}\mathrm{Tr}\{\boldsymbol{X}\boldsymbol{X}^H\}. \tag{5}$$

Under H_1, σ^2, $\boldsymbol{a}$ and $\boldsymbol{s}$ are to be estimated. The MLE of σ^2 becomes

$$\widehat{\sigma}_1^2 = \frac{1}{mN}\mathrm{Tr}\left\{(\boldsymbol{X} - \boldsymbol{a}\boldsymbol{s}^H)(\boldsymbol{X} - \boldsymbol{a}\boldsymbol{s}^H)^H\right\}. \tag{6}$$

Consequently, $\boldsymbol{a}$ and $\boldsymbol{s}$ are the minimizing arguments of $\mathrm{Tr}\{(\boldsymbol{X} - \boldsymbol{a}\boldsymbol{s}^H)(\boldsymbol{X} - \boldsymbol{a}\boldsymbol{s}^H)^H\}$ and are thus given by

$$\boldsymbol{a}\boldsymbol{s}^H = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^H \tag{7}$$

where σ_1, $\boldsymbol{u}_1$, and $\boldsymbol{v}_1$ are the largest singular value, left and right singular vectors of $\boldsymbol{X}$, respectively, i.e., $\boldsymbol{X} = \sum_{k=1}^m \sigma_k\boldsymbol{u}_k\boldsymbol{v}_k^H$ with $\sigma_1 > \sigma_2 > \cdots > \sigma_m$. The mN-root generalized-likelihood ratio (GLR) takes the following form:

$$\frac{\widehat{\sigma}_0^2}{\widehat{\sigma}_1^2} = \frac{\sum_{k=1}^m \sigma_k^2}{\sum_{k=2}^m \sigma_k^2} = \frac{t}{t - \lambda_1} \tag{8}$$

with $t = \mathrm{Tr}\{\boldsymbol{X}\boldsymbol{X}^H\}$ and $\lambda_1 = \sigma_1^2$. Finally, the GLRT can be equivalently written as

$$L(\boldsymbol{X}) = \frac{\lambda_1}{t} = \frac{\lambda_{\max}(\boldsymbol{X}\boldsymbol{X}^H)}{\mathrm{Tr}\{\boldsymbol{X}\boldsymbol{X}^H\}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta. \tag{9}$$

It should be pointed out that the GLR is a function of the maximal invariant. It is exactly the maximal invariant for $m = 2$. It can also be observed that the derived detector is the ratio of $\max_{\boldsymbol{u}}(\boldsymbol{u}^H\boldsymbol{X}\boldsymbol{X}^h\boldsymbol{u}/\boldsymbol{u}^H\boldsymbol{u})$ to $\mathrm{Tr}\{\boldsymbol{X}\boldsymbol{X}^H\}$, which is kind of a normalized matched filter, with matching to the $\boldsymbol{u}$ that gives the largest output signal to noise ratio.

Remark 1: When $\boldsymbol{s}$ is assumed to be drawn from a complex multivariate Gaussian distribution with zero mean and covariance matrix $P_s\boldsymbol{I}$, rather than being an unknown parameter vector, the GLRT is still given by (9).

C. Distribution of the GLR Under the Null Hypothesis

In this section, we derive the distribution of $g \triangleq \lambda_1/t$ under H_0. As will become clear shortly, the derivation of the probability density function (pdf) of g for any m is quite complicated, if not intractable. Therefore, we will consider the general case of any m as long as possible and then focus on the special case $m = 2$, for which simple and closed-form expressions will be derived. Note that $m = 2$ corresponds to the case where two beams are used to cover a spatial sector, and we can always design the beams or define the spatial sector so that $m = 2$ is adequate. Observe also that in the case of a monopulse radar, we indeed have $m = 2$.

In the following, we assume that $N \geq m$ (to consider the inverse case, one only needs to exchange m and N in the expressions below), and we denote by $\lambda_1 > \lambda_2 > \ldots > \lambda_m \geq 0$ the first m eigenvalues of $\boldsymbol{X}\boldsymbol{X}^H$. We also define

$$z_0 = \sum_{k=1}^m \lambda_k = t; \quad z_k = z_0^{-1}\lambda_k, \qquad k = 1, \cdots, m$$

with the implicit constraint that $z_m = 1 - \sum_{k=1}^{m-1} z_k$. Let us also define $\boldsymbol{\lambda} = [\lambda_1 \cdots \lambda_m]^T$, $\boldsymbol{\Lambda} = \mathrm{diag}(\boldsymbol{\lambda})$, $\boldsymbol{z} = [z_0\ z_1 \cdots z_{m-1}]^T = [z_0\ \boldsymbol{m}^T]^T$ and $\widetilde{\boldsymbol{z}} = [z_1\ z_2 \cdots z_m]^T$. Note that $\boldsymbol{m}$ is the maximal invariant for the problem at hand. For any vector $\boldsymbol{x} = [x_1\ x_2 \cdots x_m]^T$, $\boldsymbol{V}(\boldsymbol{x})$ stands for the Vandermonde matrix whose (k, ℓ) element is x_ℓ^{k-1}. Finally, we use the following short-hand notation for the pdf's:

$$\begin{aligned} f(\boldsymbol{\lambda}) &= f_{\Lambda_1,\Lambda_2,\ldots,\Lambda_m}(\lambda_1, \lambda_2, \cdots, \lambda_m) \\ f(\boldsymbol{z}) &= f_{Z_0,Z_1,\cdots,Z_{m-1}}(z_0, z_1, \ldots, z_{m-1}) \\ f(\boldsymbol{m}) &= f_{Z_1,\cdots,Z_{m-1}}(z_1, \ldots, z_{m-1}). \end{aligned}$$

Since $\boldsymbol{X}\boldsymbol{X}^H$ has a complex Wishart distribution $CW_m(N, \sigma^2\boldsymbol{I})$, the joint pdf of its eigenvalues is given by [8], [9]

$$f(\boldsymbol{\lambda}) = c\frac{e^{-\mathrm{Tr}\{\sigma^{-2}\boldsymbol{\Lambda}\}}}{\sigma^{2mN}}|\boldsymbol{\Lambda}|^{N-m}|\boldsymbol{V}(\boldsymbol{\lambda})|^2 \tag{10}$$

where $c^{-1} = \prod_{k=1}^m \Gamma(m-k+1)\Gamma(N-k+1)$, and $|.|$ stands for the determinant of a matrix. The Jacobian of the transformation from $\boldsymbol{\lambda}$ to $\boldsymbol{z}$ is easily seen to be z_0^{m-1}. Therefore, the joint density function of $\boldsymbol{z}$ is given by

$$f(\boldsymbol{z}) = f_{Z_0}(z_0)f(\boldsymbol{m})$$

with

$$f_{Z_0}(z_0) = \frac{z_0^{mN-1} e^{-z_0/\sigma^2}}{\Gamma(mN)\sigma^{2mN}} \tag{11a}$$

$$f(\boldsymbol{m}) = c\Gamma(mN)|\widetilde{\boldsymbol{Z}}|^{N-m} \, |\boldsymbol{V}(\widetilde{\boldsymbol{z}})|^2 \tag{11b}$$

where $\widetilde{\boldsymbol{Z}} = \mathrm{diag}(\widetilde{\boldsymbol{z}})$. It should be pointed out that z_0 is independent of $(z_1, \cdots, z_{m-1})$. Equation (11a) is the pdf of a (scaled) chi-squared random variable, and (11b) is the pdf of the maximal invariant under the null hypothesis. The pdf of $g = z_1$ can be obtained as

$$f_{Z_1}(z_1) = \int \cdots \int f(\boldsymbol{m}) dz_2 \cdots dz_{m-1} \tag{12}$$

where the integration is over the domain $0 \leq z_m < z_{m-1} < \ldots < z_1 \leq 1$ and $z_m + z_{m-1} + \ldots + z_1 = 1$. However, it appears quite complicated to obtain a closed-form expression for this integral for any m. Hence, *we now focus on the case* $m = 2$.

When $m = 2$, there is no integral in (12), and the pdf of g simply becomes

$$f_G(g|H_0) = c\Gamma(2N) g^{N-2} (1-g)^{N-2} (2g-1)^2; \quad \frac{1}{2} \leq g \leq 1. \tag{13}$$

From this simple expression, the probability of false alarm, for a given threshold η, can be written as

$$\begin{aligned} P_{\mathrm{FA}} &= \int_\eta^1 f_G(g|H_0) dg \\ &= \int_\eta^1 c\Gamma(2N) g^{N-2} (1-g)^{N-2} (2g-1)^2 dg \\ &= \frac{c\Gamma(2N)}{2^{2N-2}} \int_{(2\eta-1)^2}^1 x^{1/2} (1-x)^{N-2} dx \\ &= \frac{c\Gamma(2N)}{2^{2N-2}} \left[B\left(\frac{3}{2}, N-1\right) - B_{(2\eta-1)^2}\left(\frac{3}{2}, N-1\right) \right] \end{aligned} \tag{14}$$

where $B(a,b)$ and $B_z(a,b)$ are the Beta and incomplete Beta functions, respectively [10]. Therefore, the probability of false alarm is obtained in closed-form and the threshold η can be set from (14).

D. Distribution of the GLR Under the Alternative Hypothesis When $m = 2$

In this section, we derive an expression for the pdf of g under H_1, when $m = 2$. An expression for the pdf of $g = z_1$ for any m is derived in the Appendix, but it involves an integral, similar to what was found under H_0 (see (12)). Hence, we focus on the case $m = 2$. Under H_1, $\boldsymbol{X}$ is multivariate Gaussian with mean $\boldsymbol{M} = \boldsymbol{a}\boldsymbol{s}^H$. Therefore, the matrix $\boldsymbol{M}^H \boldsymbol{\Sigma}^{-1} \boldsymbol{M}$ has a single eigenvalue, namely $\omega_1 = \|\boldsymbol{a}\|^2 \|\boldsymbol{s}\|^2 / \sigma^2$, which is simply the induced maximal invariant. Using the results of the Appendix, which hold for any m, it can be shown that

$$\begin{aligned} &f_{\Lambda_1,\Lambda_2}(\lambda_1, \lambda_2) \\ &\quad = \frac{e^{-\omega_1}}{\omega_1 [\Gamma(N-1)]^2} e^{-(\lambda_1+\lambda_2)} (\lambda_1\lambda_2)^{N-2} (\lambda_1 - \lambda_2) \\ &\quad\quad \times [{}_0F_1(N-1; \lambda_1\omega_1) - {}_0F_1(N-1; \lambda_2\omega_1)] \end{aligned} \tag{15}$$

$$\begin{aligned} &f_{Z_0,Z_1}(z_0, z_1) \\ &\quad = \frac{e^{-\omega_1} e^{-z_0} z_0^{2(N-1)}}{\omega_1 [\Gamma(N-1)]^2} z_1^{N-2} (1-z_1)^{N-2} (2z_1 - 1) \\ &\quad\quad \times [{}_0F_1(N-1; z_0 z_1 \omega_1) - {}_0F_1(N-1; z_0(1-z_1)\omega_1)] \end{aligned} \tag{16}$$

where ${}_0F_1(.;.)$ is the Bessel type of hypergeometric function [10]. Note that under H_1, z_0 and z_1 are not independent, in contrast to what happens under H_0. The pdf of $g = z_1$ can be obtained by integrating (16) over z_0, i.e.,

$$\begin{aligned} &f_G(g|H_1) \\ &\quad = \frac{e^{-\omega_1}}{\omega_1 [\Gamma(N-1)]^2} g^{N-2} (1-g)^{N-2} (2g-1) \\ &\quad\quad \times \int_0^\infty e^{-z_0} z_0^{2(N-1)} [{}_0F_1(N-1; z_0 g\omega_1) \\ &\quad\quad\quad\quad - {}_0F_1(N-1; z_0(1-g)\omega_1)] dz_0 \\ &\quad = \frac{\Gamma(2N-1)}{[\Gamma(N-1)]^2} \frac{e^{-\omega_1}}{\omega_1} g^{N-2} (1-g)^{N-2} (2g-1) \\ &\quad\quad \times [{}_1F_1(2N-1; N-1; g\omega_1) \\ &\quad\quad\quad - {}_1F_1(2N-1; N-1; (1-g)\omega_1)]. \end{aligned} \tag{17}$$

It is instructive to note the similarities and differences between (13) and (17), which give the pdf of the GLR under both hypotheses. Note that when $\omega_1 \to 0$, the pdf in (17) converges to that in (13) as, for small x, ${}_1F_1(a;b;x) \simeq 1 + (a/b)x$. We also point out that the distribution of the GLR depends on the unknown parameters of the problem only through ω_1, which is the induced maximal invariant. The expression in (17) enables us to compute the probability of detection for a given threshold.

E. Optimality of the GLRT When $m = 2$

Equipped with the pdf's of the GLR under both hypotheses, we now show that the GLRT is indeed UMPI when $m = 2$. Toward this end, let us observe that

$$\begin{aligned} C^{-1}(2g-1) \frac{f_G(g|H_1)}{f_G(g|H_0)} &= {}_1F_1(2N-1; N-1; g\omega_1) \\ &\quad - {}_1F_1(2N-1; N-1; (1-g)\omega_1). \end{aligned} \tag{18}$$

Let us define $x = 2g - 1$, $\alpha = \omega_1/2$ and consider

$$\begin{aligned} h(x) &= \frac{1}{x} [{}_1F_1(2N-1; N-1; \alpha(1+x)) \\ &\quad\quad - {}_1F_1(2N-1; N-1; \alpha(1-x))] \\ &= \frac{1}{x} \sum_{k=0}^\infty \frac{(2N-1)_k}{(N-1)_k} \frac{\alpha^k}{k!} \left[(1+x)^k - (1-x)^k \right]. \end{aligned} \tag{19}$$

It is straightforward to show that $x^{-1}[(1+x)^k - (1-x)^k]$ is a monotone function of x. Therefore, $f_G(g|H_1)/f_G(g|H_0) = Ch(2g-1)$ is a monotone function of g, which proves that the GLRT is UMPI when $m = 2$.

III. Numerical Examples

In this section, we illustrate the performance of the GLRT. Through the simulations, we consider an array of 16 elements with half-wavelength separation. A beamspace transformation is designed so as to detect sources within the spatial sector $[10^\circ, 20^\circ]$. Fig. 1 displays the beampatterns corresponding to each of the two beams. The data matrix $\boldsymbol{X}$ consists of the outputs of the beamspace matrix.

We assume that a unit power source is impinging on the array with a direction of arr1ival equal to 13°. We define the signal-to-noise ratio (SNR) as

$$\mathrm{SNR} = 10 \log_{10} \frac{\|\boldsymbol{a}\|^2}{\sigma^2}. \tag{20}$$

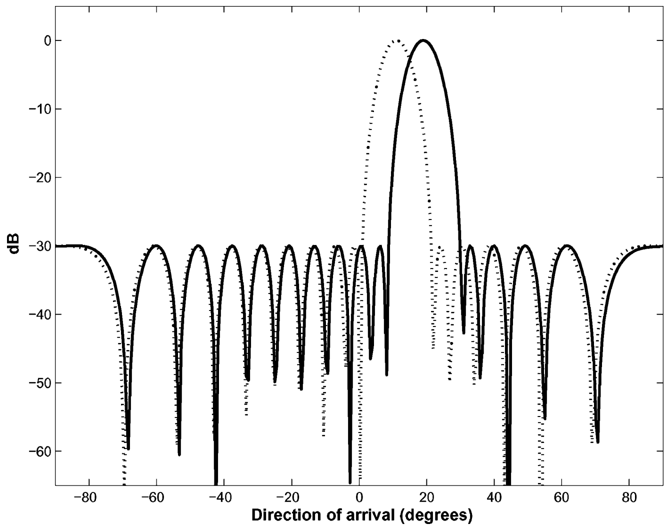

Fig. 1. Coverage of the $[10^\circ, 20^\circ]$ sector with two beams.

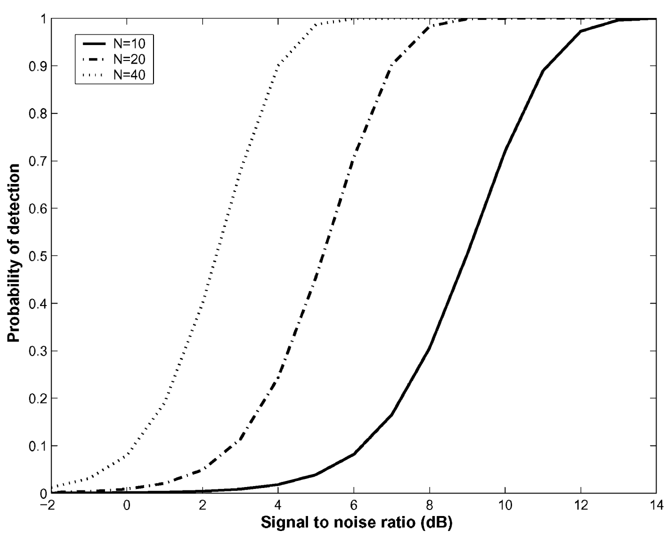

Fig. 2. Probability of detection of the GLRT versus SNR. $P_{\mathrm{FA}} = 10^{-4}$ and varying N.

In Figs. 2 and 3, we display the probability of detection of the GLRT as a function of the SNR, for different values of N and P_{FA}. As the distribution of the GLR under H_1 depends on ω_1 only, and since the latter is roughly proportional to N, we can observe a 3-dB gain when the number of snapshots is doubled. Note also that, when N increases, the slope of the curve in Fig. 2 increases significantly, indicating that a sufficient number of snapshots is required for the GLRT to perform well (note that $N = 40$ corresponds approximately to twice the number of elements in the initial full array).

IV. Conclusion

We considered the problem of detecting an arbitrary rank-one component in noise, operating in the beamspace domain. The GLRT was shown to be the ratio of the largest eigenvalue of a Wishart matrix to its trace. When $m = 2$ beams are used to cover a spatial sector, the GLRT was proved to be UMPI. Moreover, new results concerning the distribution of the (normalized) eigenvalues of a Wishart matrix were presented, which enabled us to provide closed-form expressions for the distribution of the GLRT under the null and the alternative hypotheses.

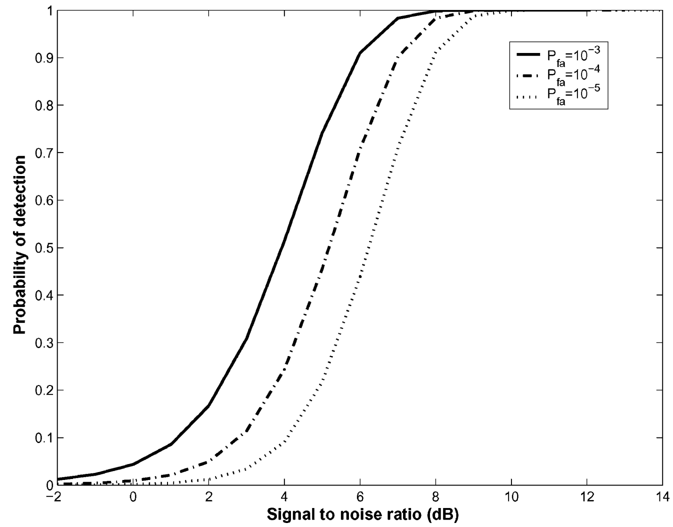

Fig. 3. Probability of detection of the GLRT versus SNR. $N = 20$ and varying P_{FA}.

Appendix

In this appendix, we provide expressions for the joint pdf of the maximal invariant $\boldsymbol{m}$ under H_1 in the case where the matrix $\boldsymbol{M}^H\boldsymbol{\Sigma}^{-1}\boldsymbol{M}$ has a single eigenvalue ω_1. Toward this end, we first derive an expression for the joint pdf of $(z_0, z_1, \cdots, z_{m-1})$ and then integrate over z_0. The joint distributions are derived in two steps. First, we consider the case where all eigenvalues of $\boldsymbol{M}^H\boldsymbol{\Sigma}^{-1}\boldsymbol{M}$, $\omega_1 > \omega_2 > \cdots > \omega_m$ are distinct and nonzero. Then, we take the limit of the distributions as $\omega_2, \cdots, \omega_m$ tend to zero. In the following, we denote $\boldsymbol{\omega} = [\omega_1 \cdots \omega_m]$ and $\boldsymbol{\Omega} = \mathrm{diag}(\boldsymbol{\omega})$.

The joint density of $(\lambda_1, \cdots, \lambda_m)$ can be written as [9, eq. (45)]

$$f(\boldsymbol{\lambda}) = c' \left[\frac{|\boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\omega})|}{|\boldsymbol{V}(\boldsymbol{\omega})|}\right] \times \left[|\boldsymbol{V}(\boldsymbol{\lambda})|\, |\boldsymbol{\Lambda}|^{N-m} e^{-\mathrm{Tr}\{\boldsymbol{\Lambda}\}}\right] \tag{21}$$

with

$$c' = \frac{e^{-\mathrm{Tr}\{\boldsymbol{\Omega}\}}}{[\Gamma(N-m+1)]^m}$$

and where the (k, ℓ) element of $\boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\omega})$ is ${}_0F_1(N-m+1; \lambda_k\omega_\ell)$. Note that only the first bracketed term in (21) depends on $\boldsymbol{\omega}$. For the sake of conciseness, let $b = N - m + 1$ and use $|{}_0F_1(b; \lambda_k\omega_\ell)|$ for $|\boldsymbol{F}(\boldsymbol{\lambda}, \boldsymbol{\omega})|$. Observing that the Jacobian of the transformation from $\boldsymbol{\lambda}$ to $\boldsymbol{z}$ is z_0^{m-1}, it follows from (21) that

$$f(\boldsymbol{z}) = c' \left[\frac{|{}_0F_1(b; z_0 z_k\omega_\ell)|}{|\boldsymbol{V}(\boldsymbol{\omega})|}\right] \times \left[z_0^{m\left[N - \frac{m-1}{2}\right]-1} e^{-z_0}\right] \times \left[|\boldsymbol{V}(\tilde{\boldsymbol{z}})|\, |\widetilde{\boldsymbol{Z}}|^{N-m}\right]. \tag{22}$$

In the previous equation, only the first bracketed term depends on $\boldsymbol{\omega}$. The second term depends on z_0 only, while the third term depends on $\tilde{\boldsymbol{z}}$. Also, observe that z_0 is not independent of $\tilde{\boldsymbol{z}}$ due to the first term, which depends on both z_0 and $\tilde{\boldsymbol{z}}$.

Let us consider us now the case where $\omega_2 = \cdots = \omega_m = 0$. In order to obtain the pdf of $\boldsymbol{\lambda}$ or $\boldsymbol{z}$, we need to make $\omega_2, \cdots, \omega_m$ converge to zero in (21) and (22). Apart from $e^{-\mathrm{Tr}\{\boldsymbol{\Omega}\}}$—which converges to $e^{-\omega_1}$—only the first bracketed term in these equations depend on $\boldsymbol{\omega}$, and we are left with the problem of finding

$$\lim_{\omega_2, \cdots, \omega_m \to 0} \frac{|{}_0F_1(b; x_k\omega_\ell)|}{|\boldsymbol{V}(\boldsymbol{\omega})|} \tag{23}$$

where x_k is either λ_k or $z_0 z_k$.

When $m = 2$, this limit can be obtained directly as

$$\lim_{\omega_2 \to 0} \frac{|{}_0F_1(b; x_k \omega_\ell)|}{|\boldsymbol{V}(\boldsymbol{\omega})|} = \frac{{}_0F_1(b; x_1\omega_1) - {}_0F_1(b; x_2\omega_1)}{-\omega_1} \tag{24}$$

where we used the fact that ${}_0F_1(b; 0) = 1$.

When $m > 2$, we have a 0/0 type of limit, and we apply l'Hôpital rule by taking the $(m - \ell)$ derivative of the columns of $\boldsymbol{V}(\boldsymbol{\omega})$ and $[{}_0F_1(b; x_k\omega_\ell]$ containing ω_ℓ with respect to ω_ℓ, and then setting $\omega_\ell = 0$, for $\ell = m, m-1, \cdots, 2$ (see [9, Appendix B] for a similar approach). Doing so, it is possible to show that

$$\lim_{\omega_2 \to 0} |\boldsymbol{V}(\boldsymbol{\omega})| = (-1)^{\frac{m(m-1)}{2}} \omega_1^{m-1} \prod_{k=1}^{m-1} \Gamma(m-k). \tag{25}$$

Next, using the fact that

$$\frac{\partial^p {}_0F_1(b; x_k\omega_\ell)}{\partial \omega_\ell^p} = \frac{x_k^p}{(b)_p} {}_0F_1(b+p; x_k\omega_\ell) \tag{26}$$

it is possible to show that

$$\lim_{\omega_2, \cdots, \omega_m \to 0} |{}_0F_1(b; x_k\omega_\ell)| = |\boldsymbol{G}(\boldsymbol{x}, \omega_1)| \tag{27}$$

with

$$\begin{aligned} [\boldsymbol{G}(\boldsymbol{x}, \omega_1)]_{(k,1)} &= {}_0F_1(b; x_k\omega_1) \\ [\boldsymbol{G}(\boldsymbol{x}, \omega_1)]_{(k,\ell)} &= \frac{x_k^{m-\ell}}{(b)_{m-\ell}}; \qquad \ell = 2, \cdots, m. \end{aligned} \tag{28}$$

Using (25) and (27), we obtain the following expressions for the joint density of $\boldsymbol{\lambda}$ and $\boldsymbol{z}$, in the noncentral Wishart case and when $\boldsymbol{M}^H \boldsymbol{\Sigma}^{-1} \boldsymbol{M}$ has a single eigenvalue ω_1:

$$f(\boldsymbol{\lambda}) = C \frac{e^{-\omega_1}}{\omega_1^{m-1}} |\boldsymbol{G}(\boldsymbol{\lambda}, \omega_1)| \, |\boldsymbol{V}(\boldsymbol{\lambda})| \, |\boldsymbol{\Lambda}|^{N-m} e^{-\mathrm{Tr}\{\boldsymbol{\Lambda}\}} \tag{29}$$

$$\begin{aligned} f(\boldsymbol{z}) = C \frac{e^{-\omega_1}}{\omega_1^{m-1}} z_0^{m\left[N - \frac{m-1}{2}\right] - 1} e^{-z_0} \\ \times |\boldsymbol{G}(z_0 \widetilde{\boldsymbol{z}}, \omega_1)| \, |\boldsymbol{V}(\widetilde{\boldsymbol{z}})| \, |\widetilde{\boldsymbol{Z}}|^{N-m} \end{aligned} \tag{30}$$

with

$$C^{-1} = (-1)^{\frac{m(m-1)}{2}} \left[\Gamma(N-m+1)\right]^m \prod_{k=1}^{m-1} \Gamma(m-k).$$

Setting $m = 2$ in (29) and (30) yields (15) and (16), respectively.

We now consider the joint distribution of the maximal invariant $\boldsymbol{m} = [z_1 \ \cdots \ z_{m-1}]^T$. In order to obtain the latter, we need to integrate (30) with respect to z_0. To do so, let us rewrite (30) as

$$f(\boldsymbol{z}) = C_1 \, |\boldsymbol{G}(z_0 \widetilde{\boldsymbol{z}}, \omega_1)| \, z_0^{m\left[N - \frac{m-1}{2}\right] - 1} e^{-z_0} \tag{31}$$

where C_1 does not depend on z_0. Let us define $a = m[N - ((m-1)/2)]$. Using a lemma for the determinant of partitioned matrices, it follows that (with obvious definitions of vectors and matrices)

$$\begin{aligned} |\boldsymbol{G}(z_0 \widetilde{\boldsymbol{z}}, \omega_1)| &= \left| \begin{bmatrix} {}_0F_1(b; z_0 z_1 \omega_1) & \boldsymbol{\beta}^T \\ \widetilde{\boldsymbol{f}} & \boldsymbol{W} \end{bmatrix} \right| \\ &= |\boldsymbol{W}| \left({}_0F_1(b; z_0 z_1 \omega_1) - \boldsymbol{\beta}^T \boldsymbol{W}^{-1} \widetilde{\boldsymbol{f}} \right) \\ &= |\boldsymbol{W}| \sum_{k=1}^{m} \gamma_k \, {}_0F_1(b; z_0 z_k \omega_1). \end{aligned} \tag{32}$$

Now, using the fact that [11, eq. (7.522-5)]

$$\int_0^\infty e^{-z_0} z_0^{a-1} \, {}_0F_1(b; z_0 z_k \omega_1) dz_0 = \Gamma(a) \, {}_1F_1(a; b; z_k \omega_1) \tag{33}$$

it follows that

$$\begin{aligned} f(\boldsymbol{m}) &= \int_0^\infty f(\boldsymbol{z}) dz_0 \\ &= C_1 \Gamma(a) |\boldsymbol{W}| \sum_{k=1}^{m} \gamma_k \, {}_1F_1(a; b; z_k \omega_1) \\ &= C_1 \Gamma(a) \, |\boldsymbol{H}(\widetilde{\boldsymbol{z}}, \omega_1)| \end{aligned} \tag{34}$$

with

$$\begin{aligned} [\boldsymbol{H}(\widetilde{\boldsymbol{z}}, \omega_1)]_{(k,1)} &= {}_1F_1(a; b; z_k \omega_1) \\ [\boldsymbol{H}(\widetilde{\boldsymbol{z}}, \omega_1)]_{(k,\ell)} &= \frac{z_k^{m-\ell}}{(b)_{m-\ell}}; \qquad \ell = 2, \cdots, m. \end{aligned} \tag{35}$$

Finally, the distribution of the maximal invariant is given by

$$f(\boldsymbol{m}) = C' \frac{e^{-\omega_1}}{\omega_1^{m-1}} \, |\boldsymbol{H}(\widetilde{\boldsymbol{z}}, \omega_1)| \, |\boldsymbol{V}(\widetilde{\boldsymbol{z}})| \, |\widetilde{\boldsymbol{Z}}|^{N-m} \tag{36}$$

where $C' = C\Gamma(a)$. The distribution of the GLR is, in theory, given by

$$f_{Z_1}(z_1) = \int \cdots \int f(\boldsymbol{m}) dz_2 \cdots dz_{m-1} \tag{37}$$

where the integration is over the domain $0 \leq z_m < z_{m-1} < \ldots < z_1 \leq 1$ and $z_m + z_{m-1} + \ldots + z_1 = 1$. When $m = 2$, there is no integral, and setting $m = 2$ in (36) yields (17).

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