Liquidity Contractions, Incomplete Financial Participation and the Prevalence of Negative Equity Non-recourse Loans

Miguel Iraola and Juan Pablo Torres-Martínez

May 2013

LIQUIDITY CONTRACTIONS, INCOMPLETE FINANCIAL PARTICIPATION AND THE PREVALENCE OF NEGATIVE EQUITY NON-RECOURSE LOANS

MIGUEL A. IRAOLA AND JUAN PABLO TORRES-MARTÍNEZ

Abstract. We address a dynamic general equilibrium model where securities are backed by collateralized loans, and borrowers face endogenous liquidity contractions and financial participation constraints. Although the only payment enforcement is the seizure of collateral guarantees, restrictions on credit access make individually optimal payment strategies—coupon payment, prepayment, and default—sensitive to idiosyncratic factors. In particular, the lack of liquidity and the presence of financial participation constraints rationalize the prevalence of negative equity loans. We prove equilibrium existence, characterize optimal payment strategies, and provide a numerical example illustrating our main results.

Keywords. Asset-Backed Securities - Liquidity Contractions - Incomplete Financial Participation
JEL Classification. D50, D52.

1. Introduction

The recent financial crisis 2007-2009 was preceded by an intense rise in the volume of securitized debt (see Figure I). This process and the complexity of securitization operations fostered the origination of subprime mortgages, increasing the fragility of the financial system. Indeed, the fall of house prices in 2006 reduced home equity and increased delinquency rates (shown in Figure II), starting the financial crisis. The devastating economic effects of these events emphasize the relevance of understanding investors' optimal strategies in asset-backed security markets.

This paper presents a general equilibrium model which captures the three most relevant risk factors underlying asset-backed security markets: credit risk, prepayment risk, and interest rate risk. Despite their importance, these risks have been partially incorporated in general equilibrium models, making it difficult to understand the interdependence of optimal decisions and asset prices in securitized debt markets. Additionally, in our model the presence of restrictions to credit access makes individually optimal payment strategies (i.e., coupon payment, prepayment, and default) sensitive to idiosyncratic factors. This property seems to be crucial in a theoretical analysis of collateralized asset markets. As emphasized by Deng, Quigley and Van Order (2000), idiosyncratic

Date: May, 2013.

The authors want to thank Manuel Santos for his valuable comments. Previous versions of this work appeared under the title *"Liquidity Contractions and Prepayment Risk on Collateralized Asset Markets"*. J.P.Torres-Martínez acknowledges financial support from Conicyt (Chilean Research Council) through Fondecyt project 1120294.

components are essential to estimate default and prepayment risks. Moreover, liquidity contractions and financial participation constraints have also been considered as fundamental ingredients to reproduce some empirical asset pricing moments (see Guvenen (2009)), and to analyze the impact of the FED's Large Scale Asset Purchases (LSAP) (see Chen, Cúrdia and Ferrero (2011) and Gertler and Karadi (2012)).

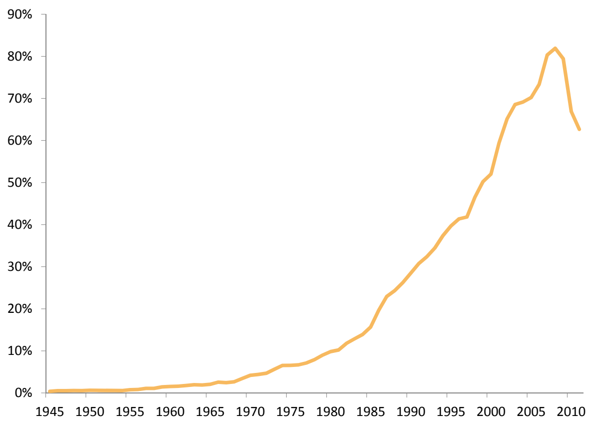


Figure I: Total Securitized Debt over GPD in the Unites States.

Source: Flow of Funds account of the United States (Tables L.122-124).[1]

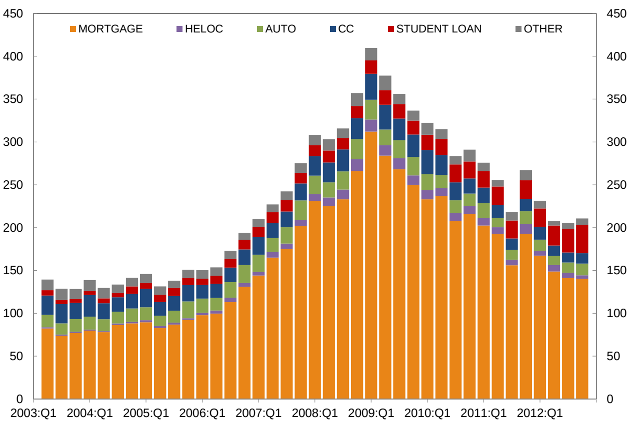


Figure II: New Delinquent Balances by Loan Type

Source: Federal Reserve Bank of New York Consumer Credit Panel .[2]

---

[1]Sum of Government Sponsored Enterprises, Agency and GSE-Backed Mortgage Pools, and ABS Issuers.

[2]MORTGAGE: Mortgage loans. HELOC: Home Equity Lines of Credit. AUTO: Auto loans. CC: Credit Card Loans. OTHER: Consumer finance and retail loans.

We consider a general equilibrium model with non-recourse collateralized loans, prepayment risk, liquidity contractions and incomplete financial participation. In particular, Geanakoplos and Zame (1997, 2002, 2007) general equilibrium model of collateralized debt is extended to a three-period setting with long-lived securities.[3] We show that, in contrast to the previous literature on incomplete financial participation, equilibrium exists without requiring financial survival assumptions or impatience conditions on preferences (cf. Aouani and Cornet (2011), Cornet and Gopalan (2010), Seghir and Torres-Martínez (2011)).

In our model, credit contracts are collateralized by durable goods, and the only payment enforcement is the seizure of these guarantees.[4] Each credit contract is characterized by its emission node, coupon payments, prepayment rule, and collateral requirements. After the emission of a credit line, borrowers have the possibility to pay the coupon or close short positions by either delivering the collateral or prepaying. Furthermore, the set of available credit instruments varies exogenously across states and depends on agents' identity, while the set of available investment opportunities varies endogenously as a consequence of individual default and prepayment decisions. Hence, financial markets could become more incomplete as a consequence of individually optimal actions.

Interest rate risk is captured by price-dependent coupon payments. Both coupon payments and prepayment rules are just required to be continuous functions of prices and hence, a wide variety of specifications are possible. We show that our model could incorporate debt contracts with exogenously specified interest rates, and forward and backward looking prepayment rules. Furthermore, liquidity contractions and exogenous financial participation constraints allow for differentiated optimal payment schemes across agents and, therefore, credit risk depends on idiosyncratic characteristics. That is, in our model the possible lack of liquidity or the existence of exogenous financial participation constraints could make debtors more willing to pay coupons. For instance, borrowers may decide to honor coupons associated with negative equity mortgages—debts with a lower collateral value than the associated prepayment cost. This behavior cannot be rationalized in a model

[3]The model of Geanakoplos and Zame (1997, 2002, 2007) has previously been extended by Araujo, Páscoa and Torres-Martínez (2005, 2011) to an infinite-horizon economy with long-lived securities and sequential trading. The authors show that exogenous debt limits or transversality conditions are not needed to prove equilibrium existence (cf., Magill and Quinzii (1994, 1996), Hernández and Santos (1996), Levine and Zame (1996), Araujo, Páscoa and Torres-Martínez (2002), Kubler and Schmedders (2003)). A major divergence of our model with respect to Araujo, Páscoa and Torres-Martínez (2005, 2011) is the incorporation of liquidity contractions and incomplete financial participation, which are crucial frictions to capture heterogeneous optimal payment strategies.

[4]Different payment enforcement mechanisms have been considered in the literature of general equilibrium addressing credit risk. The effect of utility penalties on payment behavior has been analyzed by Dubey, Geanakoplos and Shubik (1989, 2005), and Zame (1993); participation constraints have been considered by Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann (2000); bankruptcy mechanisms have been considered by Araujo and Páscoa (2002), Sabarwal (2003), and Poblete-Cazenave and Torres-Martínez (2012).

without credit tightening—as in Araujo, Páscoa and Torres-Martínez (2005, 2011)—since borrowers optimally decide to default on negative equity loans.

The prevalence of negative equity non-recourse loans is an empirically observed pattern in a significant portion of mortgage loans in the United States. Indeed, most borrowers with negative equity mortgages (84.8% at the end of the first quarter in 2012) are honoring their commitments. In the second quarter of 2012 over 22% (around 10.8 million) of mortgage loans were in negative equity,[5] a figure that hovers around 23% since the third quarter of 2009. Additionally, although there are significant differences in the share of negative equity loans across states, it is particularly relevant that in some non-recourse states, as Arizona and California, the share of mortgage loans in negative equity is substantially above the aggregate mean (see Figure III below).[6]

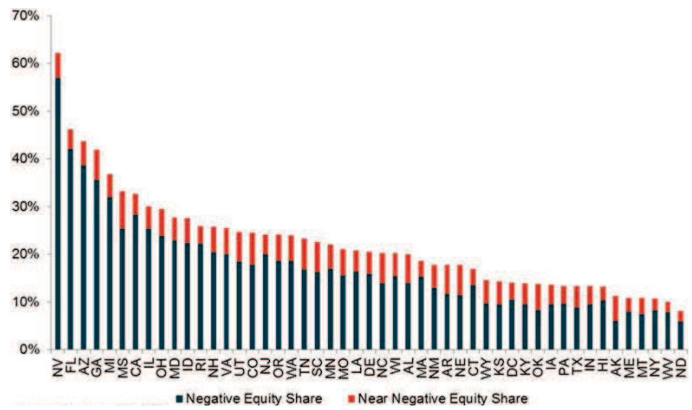


Figure III: Percentage of Mortgages in Negative Equity (2012-Q2)
Source: CoreLogic, www.corelogic.com.[7]

Our equilibrium analysis of individually optimal payment strategies reveals that, some agents who are borrowing constrained could optimally decide to continue paying negative equity loans. However, the existence of more attractive credit opportunities—in terms of downpayment and interest rates—may trigger agents' decision to close debts. Hence, agents could optimally decide to maintain underwater loans as a response to liquidity contractions. These result reveals a novel chanel which rationalizes the persistence of negative equity non-recourse mortgages in the absence of additional payment enforcements. Indeed, the existence of a high proportion of negative equity mortgages in

[5]According to CoreLogic data: http://www.corelogic.com

[6]The non-recourse states are Alaska, Arizona, California, Connecticut, Idaho, Minnesota, North Carolina, North Dakota, Oregon, Texas, Utah, and Washington. Note that, even in these states default entails additional costs such as taxes (Form 1099-A) and a negative credit rating. However, our model identifies a novel channel that rationalizes the existence of underwater mortgages in the absence of additional enforcement mechanisms.

[7]*Near Negative Equity* denotes mortgages with less than 5% of equity.

the United States coincides with a period of a significant tightening of credit standards (see Figure IV).

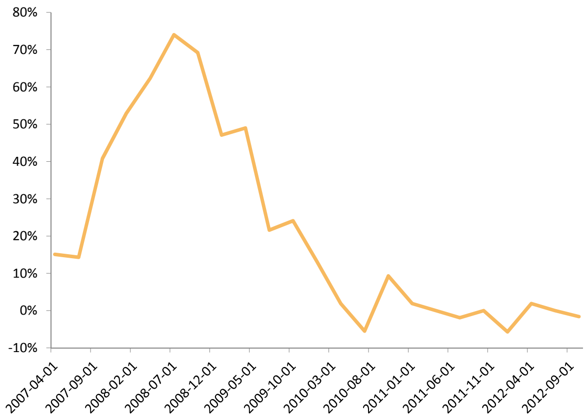


Figure IV: Net Percentage of Domestic Respondents Tightening Standards for Prime Mortgage Loans
Source: Board of Governors of the Federal Reserve System.

We provide a numerical example below illustrating all possible payment strategies in our model: payment, prepayment, and default. We show that different agents may adopt differentiated optimal payment decisions and discuss the effect of financial markets liquidity and/or financial participation constraints on these decisions. In particular, it is shown that underwater mortgages are a possible equilibrium outcome.

The rest of the paper proceeds as follows: Section 2 sets out the model, notation and equilibrium definition, Section 3 establishes equilibrium existence, Section 4 characterizes optimal payment strategies, Section 5 contains a numerical example, and Section 6 provides some concluding remarks. The proofs of our results are left to an appendix.

2. The Model

Information structure. We consider a dynamic economy $\mathcal{E}$ with three periods. There is uncertainty about the state of nature that will be realized, which belongs to a finite set S. The common and symmetric information available at period $t \in \{0,1,2\}$ is given by a partition of S, denoted by $\mathbb{F}_t$. We assume that there is no information at $t=0$, i.e., $\mathbb{F}_0 = \{S\}$. Available information may increase through time and economic agents are perfectly informed in the last period. That is, (i) $\mathbb{F}_{t+1}$ is at least as fine as $\mathbb{F}_t$, where $t \in \{0,1\}$; and (ii) $\mathbb{F}_2 = \{\{s\} : s \in S\}$.

A node is a pair (t,σ), where $t \in \{0,1,2\}$ and $\sigma \in \mathbb{F}_t$. Let D be the set of nodes in the economy and ξ_0 be the unique initial node. We denote by $t(\xi)$ the date associated with $\xi \in D$, and by D_t the set of nodes dated t. A node $\mu = (t(\mu), \sigma_\mu)$ is a successor of $\xi = (t(\xi), \sigma_\xi)$, denoted by $\mu > \xi$, when both $t(\mu) > t(\xi)$ and $\sigma_\mu \subseteq \sigma_\xi$. Let ξ^+ be the set of immediate successor nodes of $\xi \in D$.

Physical markets. At each node in D there is a finite and ordered set of commodities, L, which are traded in spot markets and may suffer transformations through time. A bundle of commodities $v \in \mathbb{R}^L_+$ consumed at $\xi \in D$ is transformed into a bundle $Y_\mu v$ at each node $\mu \in \xi^+$, where Y_μ is a $(L \times L)$-matrix with non-negative entries. Let $p_\xi = (p_{\xi,l}\,;\, l \in L) \in \mathbb{R}^L_+$ be the vector of spot prices at $\xi \in D$ and $p = (p_\xi\,;\, \xi \in D)$ be the process of commodity prices.

Financial instruments. At each $\xi \in D \setminus D_2$ a finite and ordered set $J(\xi)$ of collateralized credit contracts can be issued. Let $q_\xi = (q_{\xi,j}\,;\, j \in J(\xi)) \in \mathbb{R}^{J(\xi)}_+$ be the vector of prices of credit contracts issued at $\xi \in D \setminus D_2$, and define $q = (q_\xi\,;\, \xi \in D \setminus D_2)$. Promises associated with $j \in J(\xi)$ are pooled into a pass-through security that distributes payments made by borrowers of credit contract j. Without loss of generality, we identify the pass-through security associated with a credit contract j with the same subindex, and we assume that the price of the pass-through security coincides with the price of the credit contract at the issue node.

Securities issued in the first period can be renegotiated. Hence, let $\pi_\mu = (\pi_{\mu,j}\,;\, j \in J(\xi_0)) \in \mathbb{R}^{J(\xi_0)}_+$ be the resale price of securities at $\mu \in D_1$, and denote by $\pi = (\pi_\mu\,;\, \mu \in D_1)$ the process of pass-through resell prices. Let $\mathcal{P} := \prod_{\xi \in D} \left(\mathbb{R}^L_+ \setminus \{0\}\right) \times \prod_{\xi \in D \setminus D_2} \mathbb{R}^{J(\xi)}_+ \times \mathbb{R}^{D_1 \times J(\xi_0)}_+$ be the space of commodity and financial prices (p, q, π).[8]

Financial trading rules. The seller of one unit of credit contract $j \in J(\xi)$ receives at ξ an amount of resources $q_{\xi,j}$, is burdened to pledge a physical collateral $C_{\xi,j} \in \mathbb{R}^L_+ \setminus \{0\}$, and promises to pay a coupon $A_{\mu,j}(p, q, \pi)$ at each node $\mu > \xi$. It is assumed that borrowers hold and consume collateral guarantees. Furthermore, each credit line incorporates a prepayment rule, which specifies the payment needed to reduce the amount of debt before terminal nodes. More precisely, borrowers of $j \in J(\xi_0)$ can reduce at $\xi \in D_1$ their short-positions in one unit by paying an amount of resources $B_{\xi,j}(p, q, \pi)$. At the end of this section we discuss the generality of our approach to credit contracts.

At intermediate nodes, heterogeneous payments across agents could be observed as a consequence of liquidity shrinkages. That is, different borrowers of a credit contract $j \in J(\xi_0)$ may adopt different decisions at $\xi \in D_1$: some may pay, while others could prepay or default on their promises.

Since the only enforcement in case of default is the seizure of collateral, at terminal nodes borrowers follow strategic default. Hence, at every $\mu \in D_2$, borrowers of one unit of a credit contract $j \in J(\xi_0)$ pay the minimum between the original promise $A_{\mu,j}(p, q, \pi)$ and the market value of the collateral guarantee $p_\mu C_{\mu,j}$, where $C_{\mu,j} := Y_\mu Y_{\mu^-} C_{\xi_0,j}$ and μ^- is the immediate predecessor node of μ. Analogously, given $\xi \in D_1$, borrowers of one unit of $j \in J(\xi)$ pay at each terminal node $\mu \in \xi^+$ the minimum between $A_{\mu,j}(p, q, \pi)$ and $p_\mu C_{\mu,j}$, where $C_{\mu,j} := Y_\mu C_{\xi,j}$. To shorten notations, let

[8] Our definition of the space of commodity prices does not allow the vector of commodity prices to be zero at any node. However, this definition entails no loss of generality as we assume later the strict monotonicity of preferences.

$R_{\mu,j}(p,q,\pi) := \min\{A_{\mu,j}(p,q,\pi), p_\mu C_{\mu,j}\}$ be the unitary payment of security $j \in J(\mu^-) \cup J(\xi_0)$ at a node $\mu \in D_2$.

Given $\xi \in D \backslash D_2$, buyers of one unit of pass-through security $j \in J(\xi)$ pay $q_{\xi,j}$, which entitles them to obtain a payment $N_{\mu,j}$ at each $\mu > \xi$. Unitary payments are endogenously determined and are such that, node by node, resources distributed to lenders of security j match borrowers' deliveries. Let $\mathcal{N} := \mathbb{R}_+^{D^+}$ be the space of security payments, where $D^+ = \{(\mu, j) : \exists \xi \in D, (\mu > \xi) \wedge (j \in J(\xi))\}$ is the set of pairs (μ, j) such that μ is a node where security j could yield deliveries.

Households. There is a finite set of agents, denoted by H. Each agent $h \in H$ is characterized by a utility function $U^h : \mathbb{R}_+^{D\times L} \to \mathbb{R}$ and a commodity endowment process $w^h = (w^h_\xi\,; \xi \in D) \in \mathbb{R}_+^{D\times L}$.

Individuals may face restricted access to credit contracts. Thus, $J^h(\xi) \subseteq J(\xi)$ is the set of credit contracts issued at ξ that agent $h \in H$ is able to trade. We assume that, for any $\xi \in D \setminus D_2$ and $j \in J(\xi)$, the set of individuals that can trade debt contract j is non-empty, i.e., $H_j(\xi) := \{h \in H : j \in J^h(\xi)\} \neq \emptyset$.

At every $\xi \in D$, each $h \in H$ chooses an autonomous consumption bundle $x^h_\xi \in \mathbb{R}^L_+$, i.e., consumption in excess of the required collateral. Also, each agent h selects at $\xi \in D \setminus D_2$ a debt portfolio $\varphi^h_\xi = (\varphi^h_{\xi,j}; j \in J^h(\xi)) \in \mathbb{R}_+^{J^h(\xi)}$. For each intermediate node $\xi \in D_1$, $\varphi^{\alpha,h}_{\xi,j} \in [0, \varphi^h_{\xi_0,j}]$ denotes the position on debt contract $j \in J^h(\xi_0)$ that h honors and maintains open. Analogously, $\varphi^{\beta,h}_{\xi,j} \in [0, \varphi^h_{\xi_0,j}]$ is the part of agent h debt that is prepaid at $\xi \in D_1$. Thus, agent h defaults on $\varphi^{\gamma,h}_{\xi,j} := (\varphi^h_{\xi_0,j} - \varphi^{\alpha,h}_{\xi,j} - \varphi^{\beta,h}_{\xi,j})$ units of contract $j \in J^h(\xi_0)$ at $\xi \in D_1$.

Since borrowers consume collateral bundles, the total consumption at $\xi \in D$ is given by

$$c^h_\xi(x^h, \varphi^h, \varphi^{\alpha,h}) := \begin{cases} x^h_\xi + \sum\limits_{j \in J^h(\xi)} C_{\xi,j}\varphi^h_{\xi,j}, & \text{when } \xi = \xi_0; \\ x^h_\xi + \sum\limits_{j \in J^h(\xi)} C_{\xi,j}\varphi^h_{\xi,j} + \sum\limits_{j \in J^h(\xi_0)} C_{\xi,j}\varphi^{\alpha,h}_{\xi,j}, & \text{when } \xi \in D_1; \\ x^h_\xi, & \text{when } \xi \in D_2. \end{cases}$$

The vector $\theta^h_\xi := (\theta^h_{\xi,j}; j \in J(\xi_0) \cup J(\xi)) \in \mathbb{R}_+^{J(\xi_0)\cup J(\xi)}$ denotes the portfolio of passthrough securities of agent $h \in H$ at node $\xi \in D \setminus D_2$.

Given prices $(p,q,\pi) \in \mathcal{P}$ and security payments $N = (N_{\xi,j})_{(\xi,j)\in D^+} \in \mathcal{N}$, the objective of each household $h \in H$ is to maximize utility by choosing a plan

$$(x^h, \theta^h, \varphi^h, \varphi^{\alpha,h}, \varphi^{\beta,h}) \in \mathcal{X}^h := \mathbb{R}_+^{D\times L} \times \prod_{\xi \in D\backslash D_2} \mathbb{R}_+^{J(\xi_0)\cup J(\xi)} \times \prod_{\xi \in D\backslash D_2} \mathbb{R}_+^{J^h(\xi)} \times \mathbb{R}_+^{D_1 \times J^h(\xi_0)} \times \mathbb{R}_+^{D_1 \times J^h(\xi_0)},$$

which satisfies the following constraints:

$$(\mathrm{B}_{\xi_0}) \qquad p_{\xi_0} c^h_{\xi_0}(x^h, \varphi^h, \varphi^{\alpha,h}) + q_{\xi_0}\theta^h_{\xi_0} \leq p_{\xi_0} w^h_{\xi_0} + \sum_{j \in J^h(\xi_0)} q_{\xi_0,j}\varphi^h_{\xi_0,j};$$

for every intermediate node $\xi \in D_1$,

$$(\mathrm{B}_\xi) \quad p_\xi c^h_\xi(x^h,\varphi^h,\varphi^{\alpha,h}) + \sum_{j\in J(\xi)} q_{\xi,j}\theta^h_{\xi,j} + \sum_{j\in J(\xi_0)} \pi_{\xi,j}\theta^h_{\xi,j}$$

$$\leq p_\xi\left(w^h_\xi + Y_\xi c^h_{\xi_0}(x^h,\varphi^h,\varphi^{\alpha,h})\right) + \sum_{j\in J^h(\xi)} q_{\xi,j}\varphi^h_{\xi,j} + \sum_{j\in J(\xi_0)} (\pi_{\xi,j}+N_{\xi,j})\,\theta^h_{\xi_0,j}$$

$$- \sum_{j\in J^h(\xi_0)} \left(A_{\xi,j}(p,q,\pi)\varphi^{\alpha,h}_{\xi,j} + B_{\xi,j}(p,q,\pi)\varphi^{\beta,h}_{\xi,j} + p_\xi C_{\xi,j}\varphi^{\gamma,h}_{\xi,j}\right);$$

$$(\mathrm{S}_\xi) \quad \varphi^{\gamma,h}_{\xi,j} := \varphi^h_{\xi_0,j} - \varphi^{\alpha,h}_{\xi,j} - \varphi^{\beta,h}_{\xi,j} \geq 0, \quad \forall j \in J^h(\xi_0);$$

and, for every terminal node $\xi \in D_2$,

$$(\mathrm{B}_\xi) \quad p_\xi c^h_\xi(x^h,\varphi^h,\varphi^{\alpha,h}) \leq p_\xi\left(w^h_\xi + Y_\xi c^h_{\xi^-}(x^h,\varphi^h,\varphi^{\alpha,h})\right) + \sum_{j\in J(\xi_0)\cup J(\xi^-)} N_{\xi,j}\theta^h_{\xi^-,j}$$

$$- \left(\sum_{j\in J^h(\xi^-)} R_{\xi,j}(p,q,\pi)\varphi^h_{\xi^-,j} + \sum_{j\in J^h(\xi_0)} R_{\xi,j}(p,q,\pi)\varphi^{\alpha,h}_{\xi^-,j}\right).$$

Given $(p,q,\pi,N) \in \mathcal{P}\times\mathcal{N}$, the choice set of $h \in H$—denoted by $\Gamma^h(p,q,\pi,N)$—is the collection of plans in $\mathcal{X}^h$ that satisfy budget constraints $(\mathrm{B}_\xi)_{\xi\in D}$ and portfolio restrictions $(\mathrm{S}_\xi)_{\xi\in D_1}$.

DEFINITION. *An equilibrium for $\mathcal{E}$ is given by prices and unitary payments $(\overline{p},\overline{q},\overline{\pi},\overline{N}) \in \mathcal{P}\times\mathcal{N}$ jointly with allocations $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H} \in \prod_{h\in H}\mathcal{X}^h$ such that,*

(i) For each $h \in H$, $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h}) \in argmax\,\{U^h(z),\, z \in \Gamma^h(\overline{p},\overline{q},\overline{\pi},\overline{N})\}$.

(ii) Asset markets are cleared,

$$\sum_{h\in H}\overline{\theta}^h_{\xi,j} = \sum_{h\in H_j(\xi)}\overline{\varphi}^h_{\xi,j}, \quad \forall \xi \in D\setminus D_2,\ \forall j \in J(\xi);$$

$$\sum_{h\in H}\overline{\theta}^h_{\mu,j} = \sum_{h\in H}\overline{\theta}^h_{\xi_0,j}, \quad \forall \mu \in D_1,\ \forall j \in J(\xi_0).$$

(iii) Physical markets are cleared,

$$\sum_{h\in H} c^h_{\xi_0}(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h}) = \sum_{h\in H} w^h_{\xi_0};$$

$$\sum_{h\in H} c^h_{\xi}(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h}) = \sum_{h\in H}\left(w^h_\xi + Y_\xi c^h_{\xi^-}(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})\right), \quad \forall \xi > \xi_0.$$

(iv) Security payments are compatible with deliveries,

$$\overline{N}_{\xi,j}\sum_{h\in H}\overline{\theta}^h_{\xi_0,j} = \sum_{h\in H_j(\xi_0)}\left(A_{\xi,j}\overline{\varphi}^{\alpha,h}_{\xi,j} + B_{\xi,j}\overline{\varphi}^{\beta,h}_{\xi,j} + \overline{p}_\xi C_{\xi,j}\overline{\varphi}^{\gamma,h}_{\xi,j}\right), \quad \forall (\xi,j) \in D_1\times J(\xi_0);$$

$$\overline{N}_{\xi,j}\sum_{h\in H}\overline{\theta}^h_{\xi_0,j} = R_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\sum_{h\in H_j(\xi_0)}\overline{\varphi}^{\alpha,h}_{\xi^-,j}, \quad \forall (\xi,j) \in D_2\times J(\xi_0);$$

$$\overline{N}_{\xi,j} = R_{\xi,j}(\overline{p},\overline{q},\overline{\pi}), \quad \forall \xi \in D_2,\ \forall j \in J(\xi^-).$$

Equilibrium existence could easily be proved if security prices and payments were zero at each node. Indeed, any pure spot commodity market equilibrium is an equilibrium for our financial economy. However, if credit lines involve non-zero promises and collateral do not fully depreciate through time, it is natural to expect positive deliveries for traded contracts (cf., Steinert and Torres-Martínez (2007)). Hence, we focus our attention on the existence of a *non-trivial equilibrium*, i.e., an equilibrium such that, for some $\xi \in D \setminus D_2$, there exists $j \in J(\xi)$ for which $(N_{\mu,j})_{\mu>\xi} \neq 0$.

On security payments. Let $\Big((\overline{p},\overline{q},\overline{\pi},\overline{N});(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\Big)$ be an equilibrium and assume that credit contract $j \in J(\xi_0)$ is traded. Then, at each $\xi \in D_1$, payment, prepayment, and default rates are given by

$$\tau^{\alpha}_{\xi,j} := \frac{\sum\limits_{h\in H_j(\xi_0)} \overline{\varphi}^{\alpha,h}_{\xi,j}}{\sum\limits_{h\in H_j(\xi_0)} \overline{\varphi}^{h}_{\xi_0,j}}; \qquad \tau^{\beta}_{\xi,j} := \frac{\sum\limits_{h\in H_j(\xi_0)} \overline{\varphi}^{\beta,h}_{\xi,j}}{\sum\limits_{h\in H_j(\xi_0)} \overline{\varphi}^{h}_{\xi_0,j}}; \qquad \tau^{\gamma}_{\xi,j} := \frac{\sum\limits_{h\in H_j(\xi_0)} \overline{\varphi}^{\gamma,h}_{\xi,j}}{\sum\limits_{h\in H_j(\xi_0)} \overline{\varphi}^{h}_{\xi_0,j}}.$$

Therefore, unitary security payments can be rewritten as a weighted mean of borrowers' deliveries. That is, $\overline{N}_{\xi,j} = \tau^{\alpha}_{\xi,j}A_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) + \tau^{\beta}_{\xi,j}\,B_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) + \tau^{\gamma}_{\xi,j}\,\overline{p}_{\xi}C_{\xi,j}$. Additionally, at every $\mu \in \xi^+$, we have $\overline{N}_{\mu,j} = (1-\tau^{\beta}_{\xi,j}-\tau^{\gamma}_{\xi,j})R_{\mu,j}(\overline{p},\overline{q},\overline{\pi})$. Hence, at terminal nodes, three forces could make security payments lower than coupon values: previous and current default rates, jointly with prepayment risk.

Rental Markets. Our model implicitly incorporates rental contracts. For instance, suppose that coupons of a credit contract $j \in J(\xi_0)$ are given by $A_{\mu,j}(p,q,\pi) = p_{\mu}C_{\mu,j}$ at any intermediate node $\mu \in D_1$, and are zero at terminal nodes. Then, from the borrower's perspective, credit contract j is equivalent to a rental contract on collateral bundle $C_{\xi_0,j}$ with a rental payment of $p_{\xi_0}C_{\xi_0,j} - q_{\xi_0,j}$. From the lender's perspective, contract k is a share on a real estate investment trust (REIT).

Positive margins between collateral and credit values. Let $\Big((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\Big)$ be a non-trivial equilibrium. As in Geanakoplos and Zame (1997, 2002, 2007), under strict monotonicity of preferences the following non-arbitrage condition holds: for each $\xi \in D \setminus D_2$ and each $j \in J(\xi)$, the collateral value is greater than the amount of credit, i.e., $\overline{p}_{\xi}C_{\xi,j} - \overline{q}_{\xi,j} > 0$. Indeed, if this condition were not satisfied, agents could take advantage of an unlimited arbitrage opportunity. They may increase their utility by increasing the short position on contract j issued at ξ, buying the associated collateral bundle with the borrowed resources, and defaulting at the successor nodes $\mu \in \xi^+$. The existence of this arbitrage opportunity is not consistent with the optimality of individual plans.

These positive margins between collateral costs and borrowed resources are crucial to guarantee equilibrium existence without imposing further assumptions on individuals' financial participation (see Section 3 for a detailed discussion).

Coupon specifications. Given prices $(p,q,\pi)\in\mathcal{P}$, for any asset $j\in J(\xi_0)$ the following coupon specifications are compatible with our framework:

(i) *Credit line with real promises:* Coupons are specified as the market value of a commodity bundle, i.e., $A_{\mu,j}(p,q,\pi):=p_\mu a_{\mu,j}$, where $a_{\mu,j}\in\mathbb{R}^L_+$. In this case, both real and nominal interest rates depend on commodity prices and, therefore, are endogenously determined in equilibrium. These contracts have been considered by Geanakoplos and Zame (1997, 2002, 2007) and Araujo, Páscoa and Torres-Martínez (2002, 2005, 2011).[9]

(ii) *Fixed rate loan:* Defined through a net real interest rate $r\in(-1,+\infty)$ which specifies a coupon $A_{\mu,j}(p,q,\pi)=\frac{1}{(d+d^2)}\,q_{\xi_0,j}\,\frac{p_\mu a}{p_{\xi_0}a}$ $\forall\mu>\xi_0$, where $d=\frac{1}{(1+r)}$, and a bundle $a\in\mathbb{R}^L_{++}$ determining the price index $\frac{p_\mu a}{p_\xi a}$, which is a measure of purchasing power variation between nodes ξ and $\mu\in\xi^+$.[10]

(iii) *Adjustable rate loan:* Coupons satisfy

$$
\begin{aligned}
A_{\mu,j}(p,q,\pi) &= (r_\mu+\kappa_\mu)\,q_{\xi_0,j}\,\frac{p_\mu a}{p_{\xi_0}a}, \quad \forall\mu\in D_1,\\
A_{\mu,j}(p,q,\pi) &= (1+r_\mu)\,(1-\kappa_{\mu^-})\,q_{\xi_0,j}\,\frac{p_\mu a}{p_{\xi_0}a}, \quad \forall\mu\in D_2,
\end{aligned}
$$

where $\kappa_\mu\in[0,1]$ is the fraction of the loan face value required to be paid at node $\mu\in D_1$. Hence, between nodes $\mu\in D$ and $\eta\in\mu^+$ the borrower is required to pay a real net interest rate $r_\eta\in(-1,+\infty)$. Since there are no further restrictions on $(\kappa_\mu)_{\mu\in D_1}$, the specification above implies a variable repayment of the loan face value through time.[11]

---

[9]In contrast with our model, Geanakoplos and Zame (1997, 2002, 2007) and Araujo, Páscoa and Torres-Martínez (2002, 2005, 2011) do not consider financial participation constraints. Furthermore, Araujo, Páscoa and Torres-Martínez (2005, 2011) assume that: (i) there are no liquidity contractions (i.e., for any $\xi\in D_1$, $J(\xi_0)\subseteq J(\xi)$); and (ii) since credit contracts in $J(\xi_0)$ are re-issued at any intermediate node, prepayment costs are captured through the market value of future promises (i.e., they implicitly assume that $B_{\xi,j}(p,q,\pi)=A_{\xi,j}(p,q,\pi)+q_{\xi,j}$, $\forall(\xi,j)\in D_1\times J(\xi_0)$).

[10]Note that it is also possible to consider an exogenously specified net nominal interest rate $i\in(-1,+\infty)$. It is sufficient to define $A_{\mu,j}(p,q,\pi)=\frac{q_{\xi_0,j}}{(e+e^2)}$, $\forall\mu>\xi_0$, where $e=\frac{1}{(1+i)}$.

[11]The fixed interest loan defined in (ii) is a particular case obtained by setting $(r_\mu,\kappa_\mu)=\left(r,\frac{1}{(2+r)}\right)$, $\forall\mu>\xi_0$. Furthermore, *hybrid loans* could be defined as in (iii) making interest rates independent of the realization of uncertainty between some consecutive periods.

Prepayment risk. Our model captures several types of forward and backward looking prepayment rules. To illustrate these possibilities, fix prices $(p, q, \pi) \in \mathcal{P}$. Given a debt contract $j \in J(\xi_0)$ assume that coupons are specified as in the adjustable-rate loan (iii) above. If at each $\mu \in D_1$ the prepayment rule is $B_{\mu,j}(p, q, \pi) = (1 + r_\mu) q_{\xi_0,j} \frac{p_\mu a}{p_{\xi_0} a}$, then the cost of prepaying a debt is equal to the actualized loan's face value adjusted by the price index. That is, we have a *backward-looking prepayment rule*.

Alternatively, some financial instruments protect lenders from prepayment risk specifying a *forward-looking prepayment cost*. In this case, borrowers who want to prepay debt before terminal nodes are required to deliver the present value of promises. To capture this possibility it is sufficient to specify strictly positive discount factors $(\rho(\mu); \mu \in D_2)$ such that, the prepayment cost $B_{\xi,j}(p, q, \pi)$ at node $\xi \in D_1$ is given by either $A_{\xi,j}(p, q, \pi) + \sum_{\mu \in \xi^+} \rho(\mu) A_{\mu,j}(p, q, \pi)$ or $A_{\xi,j}(p, q, \pi) + \sum_{\mu \in \xi^+} \rho(\mu) R_{\mu,j}(p, q, \pi)$. In the former case, the prepayment rule does not take into account that borrowers may default at terminal nodes and, therefore, induces relatively more costly prepayments compared with the latter. Discount factors could be exogenously determined to ensure a lower bound for investment returns, even when all borrowers prepay.

Finally, future payments could be discounted considering idiosyncratic characteristics of potential borrowers, with the aim of limiting prepayment risk. To this end, it is sufficient to ensure that agents are more impatient than the implicit inter-temporal discount induced by the financial contract (see Section 4 for a description of optimal payment strategies as functions of idiosyncratic factors).

3. Equilibrium Existence

The consideration of default and collateral requirements in our model allows to capture realistic financial contracts in a general equilibrium framework. As it is well known, the absence of debt limits may induce discontinuities on individual demands and, therefore, may compromise the existence of equilibrium (cf., Hart (1975)). However, collateral requirements impose natural endogenous debt limits as equilibrium debt is bounded by the market value of collateral guarantees. Therefore, in the absence of financial participation constraints, it is possible to establish equilibrium existence without imposing additional debt constraints or transversality conditions (cf., Geanakoplos and Zame (1997, 2002, 2007), and Araujo, Páscoa and Torres-Martínez (2002, 2005, 2011)).

To prove the existence equilibrium in a model which includes incomplete financial participation, we require that, for any prices, agents have available a positive amount of resources at any state of nature (i.e., choice sets must have a non-empty interior). Two ways to ensure this property have been proposed in the literature of classical two-period models of incomplete financial markets. First, some authors assume that, for any price, every agent has access to a positive amount of credit through all financial contracts, a property referred to as *financial survival* (cf., Angeloni and

Cornet (2006), Aouani and Cornet (2009, 2011), Cornet and Gopalan (2010)).[12] Second, Seghir and Torres-Martínez (2011) assume that preferences satisfy an impatience condition that privilege first period consumption.[13]

Our main result below establishes equilibrium existence without imposing financial survival or impatience conditions on preferences. We find an equilibrium allocation as a Cournot-Nash equilibrium of a generalized game. In this game, commodity prices are normalized to be non-zero vectors and, therefore, individuals have positive available resources at each state of nature since endowments are interior points of the consumption space. Although in the generalized game we impose upper bounds on asset prices, we do not require an impatience condition on preferences as in Seghir and Torres-Martínez (2011). The existence of a positive haircut between the collateral cost and the amount of borrowing induces natural upper bounds on financial prices.

Theorem 1. *An economy $\mathcal{E}$ that satisfies the following assumptions has a non-trivial equilibrium.*

- (A1) *For each $h \in H$, U^h is continuous, strictly increasing, and strictly quasi-concave.*
- (A2) *For each $h \in H$, $(W_\xi^h : \xi \in D) \in \mathbb{R}_{++}^{D\times L}$, with $W_{\xi_0}^h := w_{\xi_0}^h$ and $W_\xi^h := w_\xi^h + Y_\xi W_{\xi^-}^h$, $\forall \xi > \xi_0$.*
- (A3) *Given $(\xi, j) \in D^+$, $A_{\xi,j} : \mathcal{P} \to \mathbb{R}_+$ is a continuous function.*
- (A4) *Given $(\xi, j) \in D_1 \times J(\xi_0)$, $B_{\xi,j}$ is continuous and satisfies $B_{\xi,j}(\cdot) \geq A_{\xi,j}(\cdot)$.*
- (A5) *There exist $\xi \in D$ and $j \in J(\xi)$ such that, for each commodity price $p \in \mathbb{R}_{++}^{D\times L}$ there is a successor node $\mu \in \xi^+$ for which* $\min\{A_{\mu,j}(p,\cdot), \|Y_\mu C_{\mu,j}\|_\Sigma\} > 0$.

4. Characterizing Prepayment Risk and Default

Before analyzing more complex decisions, let us consider some simple characterizations of borrower's optimal payment strategies. Assuming that preferences are strictly monotone, we have that:

(1) At terminal nodes, all borrowers of a credit contract honor their commitments only if promises are lower than the collateral value, i.e., $A_{\xi,j}(p,q,\pi) < p_\xi C_{\xi,j}$.

(2) At an intermediate node $\xi \in D_1$, suppose that for some $j \in J(\xi_0)$ the collateral value is lower than the coupon value, $p_\xi C_{\xi,j} < A_{\xi,j}(p,q,\pi)$. Then, the optimal strategy of any borrower of credit contract j is to default at node ξ, because the associated collateral bundle could be consumed at a lower cost by defaulting and buying back the collateral.

[12] This assumption implies that, for any prices, agents have either a positive value of initial endowments or access to a positive amount of credit. Therefore, they have a positive amount of resources available at each state of nature.

[13] In their theoretical framework, commodity prices are normalized to ensure that at every node agents are able to obtain resources by selling their endowments. At the same time, asset prices are bounded from above to guarantee the continuity of individuals' choice set correspondences. Thus, the impatience condition is crucial to avoid frictions associated with these upper bounds on asset prices.

(3) Prepayment and default on a contract $j \in J(\xi_0)$ coexist at a node $\xi \in D_1$ only if these strategies cost the same, $B_{\xi,j}(p,q,\pi) = p_\xi C_{\xi,j}$. Indeed, both decisions finalize the contractual commitment and, thus, borrowers who want to conclude the contract before terminal nodes will always choose the less costly.

(4) Given $\xi \in D_1$ assume that $B_{\xi,j}(p,q,\pi) \neq p_\xi C_{\xi,j}$. Then, some agents could pay coupon while others could close the short position at ξ only if $A_{\xi,j}(p,q,\pi) < B_{\xi,j}(p,q,\pi) < p_\xi C_{\xi,j}$ or $A_{\xi,j}(p,q,\pi) < p_\xi C_{\xi,j} < B_{\xi,j}(p,q,\pi)$. In the first case, some borrowers of j may pay the coupon while others prepay. In the second case, a negative equity mortgage, some borrowers may default while others honor their promises maintaining the short position. We illustrate these possibilities with a numerical example in the next section.

We now provide necessary and sufficient conditions inducing borrowers to close short positions before terminal nodes, either prepaying or defaulting. These conditions depend on observable market variables and contractual characteristics. We begin with results that characterize optimal payment strategies independently of the existence of alternative credit opportunities. More precisely, at each intermediate node $\xi \in D_1$, if the cost associated with closing a debt position on $j \in J(\xi_0)$ is lower than the present value of commitments, then agents prepay or default on their j-debt. In addition, if the cost of closing a debt is higher than the present value of future commitments, then borrowers whose optimal behavior is not restricted by collateral constraints will pay the coupon and maintain open the short position.

Proposition 1. *Assume (A1)-(A2), and that for all agents $h \in H$, $U^h : \mathbb{R}^{D\times L} \to \mathbb{R}$ is continuously differentiable. Let $\Big((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\Big)$ be an equilibrium. For each $h \in H$, let $(\lambda^h(\eta))_{\eta\in D}$ be agent h's Kuhn-Tucker multipliers associated with budget constraints.*

Fix $(\xi,j) \in D_1 \times J^h(\xi_0)$ and $h \in H_j(\xi_0)$ such that $\overline{\varphi}^h_{\xi_0,j} > 0$. Define

$$\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) := \min\{B_{\xi,j}(\overline{p},\overline{q},\overline{\pi}),\overline{p}_\xi C_{\xi,j}\} - \left(A_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) + \sum_{\mu\in\xi^+}\frac{\lambda^h(\mu)}{\lambda^h(\xi)}R_{\mu,j}(\overline{p},\overline{q},\overline{\pi})\right).$$

If $\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) < 0$, then agent h closes short positions on j at ξ.
If $\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) > 0$, then agent h reduces short-positions on j at ξ only when some collateral constraints induced by credit contract j are binding at ξ, i.e., $\overline{x}^h_\xi \notin \mathbb{R}^L_{++}$.

The previous proposition shows that agents close short positions when either prepayment or default cost is sufficiently low. The evaluation of the cost level associated to the early end of a financial commitment depends of the degree of impatience of the borrower. To illustrate this idea, consider the following example, which follows the notation introduced in Section 2.

EXAMPLE. Assume that $j \in J^h(\xi_0)$ is an adjustable rate loan, that there is no default in the last period, and that the net real interest rate only depends on the time period.

Consider a backward-looking prepayment rule and let ϱ_ξ^h be agent h's equilibrium degree of impatience at node $\xi \in D_1$, defined as $\frac{1}{1+\varrho_\xi^h} = \sum_{\mu \in \xi^+} \frac{\lambda^h(\mu)}{\lambda^h(\xi)} \frac{\overline{p}_\mu a}{\overline{p}_\xi a}$. Then, it follows from Proposition 1 that agent h closes short positions on j at ξ when ϱ_ξ^h is lower than the last-period rate of interest r_2. Alternatively, consider the forward-looking prepayment rule $B_{\xi,j} = A_{\xi,j} + \sum_{\mu \in \xi^+} \rho(\mu) A_{\mu,j}$. In this case, let us define ϱ_ξ by $\frac{1}{1+\varrho_\xi} = \sum_{\mu \in \xi^+} \rho(\mu) \frac{\overline{p}_\mu a}{\overline{p}_\xi a}$. Hence, if $\varrho_\xi^h < \varrho_\xi$, then agent h closes short positions on j at ξ. Therefore, independently of the availability of alternative credit opportunities, (relatively) patient agents close short positions. $\square$

Proposition 1 ensures that, when collateral constraints are not binding, underwater loans are possible in equilibrium. Indeed, suppose that there exists a node $\xi \in D_1$ such that, for some $j \in J(\xi_0)$ and $h \in H_j(\xi_0)$ we have $\overline{\varphi}_{\xi_0,j}^h > 0$ and $B_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) > \overline{p}_\xi C_{\xi,j} > A_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) + \sum_{\mu \in \xi^+} \frac{\lambda^h(\mu)}{\lambda^h(\xi)} R_{\mu,j}(\overline{p}, \overline{q}, \overline{\pi})$. In this situation, if agent h demands autonomous consumption of all commodities used as collateral by credit contract j, then the short position on this asset is maintained at ξ.

The existence of alternative credit opportunities at a node $\xi \in D_1$ may increase borrowers' options to close a debt on a credit contract $j \in J(\xi_0)$, a possibility that is particularly relevant when the closing cost—$\min\{B_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}), \overline{p}_\xi C_{\xi,j}\}$—is higher than the present value of commitments, computed as in Proposition 1 by means of individual inter-temporal income marginal rates of substitution.

The following result shows that, if alternative credit opportunities are available, borrowers' optimal payment strategies are determined by comparing collateral margins and expected commitments across debt contracts.

PROPOSITION 2. *Assume (A1)-(A2), and that for all agents $h \in H$, $U^h : \mathbb{R}^{D \times L} \to \mathbb{R}$ is continuously differentiable. Let $\left((\overline{p}, \overline{q}, \overline{\pi}, \overline{N}), (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, \overline{\varphi}^{\alpha,h}, \overline{\varphi}^{\beta,h})_{h \in H}\right)$ be an equilibrium. Fix $(\xi, j) \in D_1 \times J^h(\xi_0)$ such that $\Psi_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) := \min\{B_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}), \overline{p}_\xi C_{\xi,j}\} - A_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) > 0$. For each $h \in H$, let $(\lambda^h(\eta))_{\eta \in D}$ be agent h's Kuhn-Tucker multipliers associated with budget constraints. Then, agent $h \in H_j(\xi_0)$ closes short positions on j if there exists an alternative credit line $k \in J^h(\xi)$ for which*

$$\frac{C_{\xi,k}}{\overline{q}_{\xi,k}} \leq \frac{C_{\xi,j}}{\Psi_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi})}, \quad \text{and} \quad \sum_{\mu \in \xi^+} \lambda^h(\mu) \frac{R_{\mu,k}(\overline{p}, \overline{q}, \overline{\pi})}{\overline{q}_{\xi,k}} < \sum_{\mu \in \xi^+} \lambda^h(\mu) \frac{R_{\mu,j}(\overline{p}, \overline{q}, \overline{\pi})}{\Psi_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi})}.$$

In this situation, agent h prepays debt j if and only if $B_{\xi,j}(\overline{p}, \overline{q}, \overline{\pi}) \leq \overline{p}_\xi C_{\xi,j}$.

Consider a rental market for bundle $C_{\xi,j}$ at node $\xi \in D_1$. That is, suppose that there exits a credit contract $k \in J(\xi)$ characterized by $C_{\xi,k} = C_{\xi,j}$ and $A_{\mu,k}(p, q, \pi) = p_\mu Y_\mu C_{\xi,j}$, at any $\mu \in \xi^+$.

Thus, given an equilibrium $\left((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\right)$, some agents can rent $C_{\xi,j}$ by paying $\overline{p}_\xi C_{\xi,j}-\overline{q}_{\xi,k}$. An agent who maintains a short position on j, in addition to the consumption of the associated collateral bundle (an action that could be implemented through the rental market) receives at terminal nodes the excess of the collateral over the promise. Hence, closing short positions on j and entering into the rental market is optimal only if the rental price of bundle $C_{\xi,j}$ is lower than the coupon value associated with j, i.e., $\overline{p}_\xi C_{\xi,j}-\overline{q}_{\xi,k}<A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})$. Additionally, it follows from Proposition 2 that, if $\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})>0$ and the rental cost of $C_{\xi,j}$ is low enough, then an agent who has access to contract k closes short positions on asset j. Indeed, it is sufficient that,

$$\overline{p}_\xi C_{\xi,j}-\overline{q}_{\xi,k}<A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})+\left(\overline{p}_\xi C_{\xi,j}-A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\right)\left(1-\frac{\sum_{\mu\in\xi^+}\lambda^h(\mu)\overline{p}_\mu C_{\mu,j}}{\sum_{\mu\in\xi^+}\lambda^h(\mu)R_{\mu,j}(\overline{p},\overline{q},\overline{\pi})}\right).$$

Notice that, this condition implies that the second inequality in Proposition 2 is satisfied. Moreover, as $\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})>0$, the right hand side of the inequality above is lower than or equal to $A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})$ implying that the first inequality in Proposition 2 is also satisfied.

It follows from Proposition 2 above that the existence of attractive credit opportunities avoids underwater loans in equilibrium. In this direction, and under the appropriate specification of credit contracts, the finite horizon version of Araujo, Páscoa, and Torres-Martínez (2011) is a particular case of our framework.[14]

Our previous propositions are based on individual income shadow values. However, under some circumstances optimal payment strategies can be specified in terms of observable variables.

Corollary. *Assume (A1)-(A2), and that for all agents $h\in H$, $U^h:\mathbb{R}^{D\times L}\to\mathbb{R}$ is continuously differentiable. Let $\left((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\right)$ be an equilibrium. Fix $(\xi,j)\in D_1\times J(\xi_0)$ for which $\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})>0$. If the following conditions are satisfied,*

$$\frac{C_{\xi,k}}{\overline{q}_{\xi,k}}\leq\frac{C_{\xi,j}}{\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})},\quad \textit{and}\quad \left(\frac{R_{\mu,k}(\overline{p},\overline{q},\overline{\pi})}{\overline{q}_{\xi,k}}\right)_{\mu\in\xi^+}<\left(\frac{R_{\mu,j}(\overline{p},\overline{q},\overline{\pi})}{\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})}\right)_{\mu\in\xi^+},$$

then all agents $h\in H_j(\xi_0)$ close their short positions on credit contract j at ξ.

[14] Indeed, given $(\xi,j)\in D_1\times J(\xi_0)$ suppose that there exists a debt contract $k\in J(\xi)$ with the same collateral requirements and future promises as j, i.e., $C_{\xi,k}=C_{\xi,j}$ and $R_{\mu,k}(p,q,q)=R_{\mu,j}(p,q,\pi)$, $\forall(p,q,\pi)\in\mathcal{P}$, $\forall\mu\in\xi^+$. Then, j-borrowers close their debt at ξ when $\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})<\overline{q}_{\xi,k}$ and they would be indifferent if both magnitudes were equal. Since Araujo, Páscoa and Torres-Martínez (2011) do not allow for liquidity contractions, the prepayment cost is implicitly given by $A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})+\overline{q}_{\xi,k}$. Therefore, borrowers optimally conclude debt contracts at intermediate nodes, defaulting if the collateral value $\overline{p}_\xi C_{\xi,j}$ is lower than the prepayment cost $A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})+\overline{q}_{\xi,k}$. Thus, their model does not capture underwater mortgages.

5. Negative Equity in Equilibrium

The objective of this section is to illustrate the existence of differentiated optimal payment strategies when agents face credit liquidity constraints, with a particular attention to the presence of negative equity loans.

Thus, by means of an example, we show that in equilibrium we can observe: (i) prepayment of debts in presence of cheaper credit options; (ii) prepayment without alternative access to credit; (iii) payment of coupons, even for negative equity loans; and (iv) default on the original promises.

Example. Assume that there is uncertainty only between $t = 0$ and $t = 1$. In $t = 1$ there are three states of nature $\{u, m, d\}$. Thus, let $D = \{0, u, m, d, u^+, m^+, d^+\}$ be the event-tree. There is only one commodity which is perfectly durable between periods $t = 1$ and $t = 2$, and satisfies $Y_u = Y_m = 0.5$ and $Y_d = 9/22$. At each node, the commodity price is normalized to one.

Credit contracts are issued at nodes $\{0, m\}$ and are securitized into pass-through securities. One unit of credit contract j_0 issued at $\xi = 0$ delivers q_{0,j_0} to the borrower, which is burdened to constitute a collateral of $C_{0,j_0} = 11/4$ and has the commitment to pay coupons $A_{\xi,j_0} = 1$ at nodes $\xi \neq \{0, m^+\}$ and $A_{\xi,j_0} = 2$ at node $\xi = m^+$. The constituted collateral must be maintained through the duration of the contract. Borrowers may prepay their debt at nodes $\xi \in \{u, m, d\}$ delivering B_{ξ,j_0} units of the commodity, where $(B_{u,j_0}, B_{m,j_0}, B_{d,j_0}) = (5/4,\ 3/2,\ 5/4)$. Additionally, one unit of debt contract j_m issued at node $\xi = m$ delivers q_{m,j_m} to borrowers, who must constitute a collateral $C_{m,j_m} = 33/8$ and commit to pay a coupon $A_{\xi,j_m} = 1$ at node $\xi = m^+$.

Individuals can invest on securities associated with the pooling of credit contracts. The security associated with credit contract j_0 is negotiated at every node in periods $t \in \{0, 1\}$ and distributes payments made by borrowers. The unitary payment of security j_0 at node $\xi > 0$ is denoted by N_{ξ,j_0}. The security associated with credit contract j_m is negotiated only at node m and delivers N_{m^+,j_m} at node $\xi = m^+$.

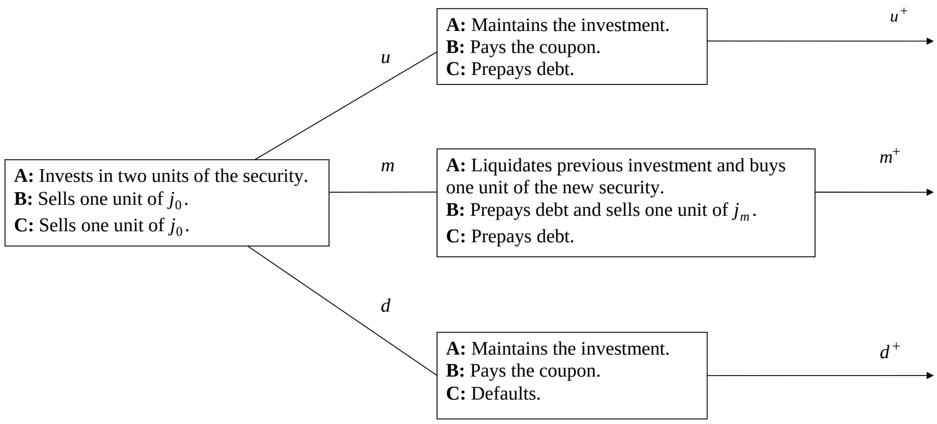
Agents $h \in \{A, B, C\}$ are characterized by the following utility functions and endowments,

$$\begin{aligned}
U^A(x_0, x_u, x_m, x_d, x_{u^+}, x_{m^+}, x_{d^+}) &= x_0 + \frac{3}{24}x_u + \frac{3}{24}x_m + \frac{12}{24}x_d + \frac{12}{96}x_{u^+} + \frac{12}{96}x_{m^+} + \frac{48}{96}x_{d^+};\\
(w_0^A, w_u^A, w_m^A, w_d^A, w_{u^+}^A, w_{m^+}^A, w_{d^+}^A) &= (3/2, 0, 0, 0, 0, 0, 0);\\
U^B(x_0, x_u, x_m, x_d, x_{u^+}, x_{m^+}, x_{d^+}) &= x_0 + \frac{2}{24}x_u + \frac{2}{24}x_m + \frac{8}{24}x_d + \frac{1}{96}x_{u^+} + \frac{1}{96}x_{m^+} + \frac{4}{96}x_{d^+};\\
(w_0^B, w_u^B, w_m^B, w_d^B, w_{u^+}^B, w_{m^+}^B, w_{d^+}^B) &= (2, 1, 1, 1, 0, 0, 1);\\
U^C(x_0, x_u, x_m, x_d, x_{u^+}, x_{m^+}, x_{d^+}) &= x_0 + \frac{1}{24}x_u + \frac{1}{24}x_m + \frac{4}{24}x_d + \frac{4}{96}x_{u^+} + \frac{4}{96}x_{m^+} + \frac{16}{96}x_{d^+};\\
(w_0^C, w_u^C, w_m^C, w_d^C, w_{u^+}^C, w_{m^+}^C, w_{d^+}^C) &= (2, 1, 1, 1, 0, 0, 1).
\end{aligned}$$

Allocations of consumption and financial positions are chosen so as to maximize utility subject to budget constraints and portfolio restrictions defined in Section 2.

An equilibrium for this economy is given by[15]

$$\begin{aligned}
((q_{0,j_0}, q_{m,j_m}); (\pi_{u,j_0}, \pi_{m,j_0}, \pi_{d,j_0})) &= \left(\left(\frac{3}{4}, \frac{1}{2}\right); \left(\frac{1}{4}, 0, \frac{1}{4}\right)\right); \\
\big((N_{u,j_0}, N_{m,j_0}, N_{d,j_0}, N_{u^+,j_0}, N_{m^+,j_0}, N_{d^+,j_0}); N_{m^+,j_m}\big) &= \left(\left(\frac{9}{8}, \frac{3}{2}, \frac{17}{16}, \frac{1}{2}, 0, \frac{1}{2}\right); 1\right); \\
(x_0^A, x_u^A, x_m^A, x_d^A, x_{u^+}^A, x_{m^+}^A, x_{d^+}^A) &= \left(0, \frac{9}{4}, \frac{10}{4}, \frac{17}{8}, \frac{13}{4}, \frac{7}{2}, \frac{25}{8}\right); \\
(x_0^B, x_u^B, x_m^B, x_d^B, x_{u^+}^B, x_{m^+}^B, x_{d^+}^B) &= \left(\frac{11}{4}, \frac{33}{8}, \frac{33}{8}, \frac{9}{8}, \frac{25}{8}, \frac{25}{8}, \frac{9}{8}\right); \\
(x_0^C, x_u^C, x_m^C, x_d^C, x_{u^+}^C, x_{m^+}^C, x_{d^+}^C) &= \left(\frac{11}{4}, \frac{31}{8}, \frac{29}{8}, 1, \frac{31}{8}, \frac{29}{8}, 2\right).
\end{aligned}$$



Note that in Araujo, Páscoa and Torres-Martínez (2011) the optimal actions of agents B and C at node d could be equilibrium outcomes only if the prepayment and collateral costs were the same. However, in our model, although the prepayment cost is $10/8$ and the depreciated collateral value is $9/8$, liquidity constraints make these heterogeneous payment strategies compatible with equilibrium.

Since marginal rates of substitution between two immediate successor nodes are measures of individual impatience, agent A is relatively patient. Moreover, as agent $A's$ endowment is concentrated

[15]The individual optimality of these allocations has been verified through a simplex algorithm.

at $t = 0$, A decides to invest in the first period. Agent B, who is more impatient than A between $t = 0$ and $t = 1$, and the most impatient consumer between periods $t = 1$ and $t = 2$, prefers to borrow at $t = 0$. Agent C, who is as patient as A between the last two periods, is the most impatient agent between periods $t = 0$ and $t = 1$ and, therefore, borrows resources at $t = 0$ to anticipate consumption.

However, between periods $t = 1$ and $t = 2$, B is more impatient than C. Therefore, at node u, where both borrowers could prepay their debts, B decides to pay the coupon and C prepays. Hence, even though agents have enough resources to prepay their debts, this decision depends on preferences and endowments. Furthermore, if there are more convenient borrowing options, the most impatient agents may prepay their debts and make use of these alternative credit instruments. For instance, at node m, agent B prepays and issues the new credit contract.

We would like to highlight that agents do not necessarily default on their debt when the collateral value is lower than the prepayment value (underwater mortgage). This decision depends on financial markets' liquidity and debtors' wealth. For instance, since B is impatient, prefers to pay coupons at d and d^+ rather than close the contract by delivering the collateral guarantee at d. Notice that agent B maintains a negative equity as a consequence of the liquidity shrinkage, although incomplete financial participation would have the same impact. □

6. Concluding Remarks

We propose a model of long-term loans backed by physical collateral, in which borrowers may prepay their debts before terminal nodes. This model extends Geanakoplos and Zame (1997, 2002, 2007) theoretical framework to allow for long-term loans, liquidity contractions, and incomplete financial participation. Without requiring financial survival assumptions or impatience conditions of preferences, we prove existence of equilibrium and provide a theoretical characterization of optimal payment strategies.

We show that, agents decide to close their debts before terminal dates—either prepaying or defaulting—if closing a short position is less costly than the expected present value of commitments. However, this condition is not homogeneous across agents and, hence, optimal payment strategies depend on individual characteristics. Moreover, this decision also depends on financial markets liquidity and on participation constraints. The absence of better credit opportunities makes some individuals more prone to honor original commitments in order to maintain the consumption of collateralized durable goods. That is, borrowers can react to liquidity shrinkages or to limited access to credit by paying coupons of debt instead of closing short positions. Therefore, the lack of liquidity or the presence of participation constraints could make it optimal for borrowers to honor their commitments even when the collateral value is lower than the prepayment value (underwater mortgage). We also provide a numerical example illustrating that optimal payment strategies—payment, prepayment, and default—depend on individual characteristics and financial markets liquidity.

It is well known that, without liquidity contractions, collateral avoids Ponzi schemes and equilibrium exists in infinite horizon collateralized asset markets. Furthermore, the absence of asset pricing bubbles on durable commodity prices avoids bubbles on collateralized securities (see Araujo, Páscoa, and Torres-Martínez (2002, 2005, 2011)). As a matter of future research, we plan to extend these results to our model with liquidity contractions, incomplete financial participation, and prepayment.

Appendix

Proof of Theorem 1. We construct a non-trivial equilibrium for our economy $\mathcal{E}$ as a Nash equilibrium of a generalized game $\mathcal{G}(Q)$ where abstract players choose prices and security payments, and agents maximize objective functions by choosing allocations in truncated budget sets. The parameter Q is an upper bound for financial prices, which is non-binding in equilibrium.

Spaces of strategies

In $\mathcal{G}(Q)$ prices are restricted to belong to $\Delta(Q) := \left(\prod_{\xi\in D\setminus D_2}\Delta_\xi(Q)\right)\times\left(\prod_{\xi\in D_2}\Delta_\xi\right)\subset\mathcal{P}$, where

$$\begin{aligned}
\Delta_{\xi_0}(Q) &:= \left\{(p_{\xi_0},q_{\xi_0})\in\mathbb{R}^L_+\times\mathbb{R}^{J(\xi_0)}_+ : \|p_{\xi_0}\|_\Sigma = 1 \ \wedge\ q_{\xi_0}\in[0,Q]^{J(\xi_0)}\right\};\\
\Delta_{\xi}(Q) &:= \left\{(p_{\xi},q_{\xi},\pi_\xi)\in\mathbb{R}^L_+\times\mathbb{R}^{J(\xi)}_+\times\mathbb{R}^{J(\xi_0)}_+ : \|p_{\xi}\|_\Sigma = 1 \ \wedge\ (q_{\xi},\pi_\xi)\in[0,Q]^{J(\xi)\cup J(\xi_0)}\right\},\ \ \forall\xi\in D_1;\\
\Delta_\xi &:= \left\{p_\xi\in\mathbb{R}^L_+ : \|p_\xi\|_\Sigma = 1\right\},\ \ \forall\xi\in D_2;
\end{aligned}$$

and the upper bound Q is exogenously fixed and satisfies

$$Q > \max_{\xi\in D\setminus D_2}\ \max_{j\in J(\xi)}\|C_{\xi,j}\|_\Sigma.$$

Our equilibrium definition guarantees that there exists $\Omega > 0$ such that, each collection of allocations $(x^h,\theta^h,\varphi^h,\varphi^{\alpha,h},\varphi^{\beta,h})_{h\in H}\in\prod_{h\in H}\mathcal{X}^h$ satisfying $(\mathrm{S}_\xi)_{\xi\in D_1}$ and market clearing conditions (ii)-(iii) is bounded from above by $\Omega(1,\ldots,1)\in\prod_{h\in H}\mathcal{X}^h$. Therefore, we assume that

$$\Omega > 2\max_{\xi\in D\setminus D_2}\frac{\sum\limits_{h\in H}\|W^h_\xi\|_\Sigma}{\min\limits_{k\in J(\xi)}\|C_{\xi,k}\|_\Sigma}.$$

Let $\mathcal{X}^h(\Omega)$ be the set of allocations in $\mathcal{X}^h$ such that its coordinates are lower than or equal to 2Ω. Since $\Delta(Q)$ is compact, Assumptions (A3)-(A4) and condition (iv) in the equilibrium definition guarantee that the unitary security payments associated with traded debt contracts are bounded. Thus, there exists $\Phi > 0$ such that, for each traded debt contract j we have $N_{\mu,j} < \Phi$, $\forall\mu\in D : (\mu,j)\in D^+$. Hence, we restrict security payments $N_{\mu,j}$ to $\mathcal{N}_{\mu,j}(\Phi) := [0,\Phi]$, $\forall j\in J(\xi)$, $\forall\mu>\xi$. Let $\mathcal{N}(\Phi) := \prod_{(\mu,j)\in D^+}\mathcal{N}_{\mu,j}(\Phi)$.

Players

The generalized game $\mathcal{G}(Q)$ has a finite number of players whose objectives are:

(i) Given a vector of prices and security payments $((p,q,\pi),N)\in\Delta(Q)\times\mathcal{N}(\Phi)$, each agent $h\in H$ maximizes the objective function U^h choosing allocations in $\Gamma^h(p,q,\pi,N)\cap\mathcal{X}^h(\Omega)$.

(ii) Given allocations $(x^h,\theta^h,\varphi^h,\varphi^{\alpha,h},\varphi^{\beta,h})_{h\in H}\in\prod_{h\in H}\mathcal{X}^h(\Omega)$,

- A player chooses prices $(p_{\xi_0},q_{\xi_0})\in\Delta_{\xi_0}(Q)$ to maximize

$$p_{\xi_0}\sum_{h\in H}\left(c^h_{\xi_0}(x^h,\varphi^h,\varphi^{\alpha,h}) - w^h_{\xi_0}\right) + \sum_{j\in J(\xi_0)} q_{\xi_0,j}\left(\sum_{h\in H}\theta^h_{\xi_0,j} - \sum_{h\in H_j(\xi_0)}\varphi^h_{\xi_0,j}\right).$$

- For each $\xi\in D_1$, a player chooses prices $(p_\xi,q_\xi,\pi_\xi)\in\Delta_\xi(Q)$ to maximize

$$p_\xi\sum_{h\in H}\left(c^h_\xi(x^h,\varphi^h,\varphi^{\alpha,h}) - w^h_\xi - Y_\xi c^h_{\xi_0}(x^h,\varphi^h,\varphi^{\alpha,h})\right)$$

$$+\sum_{j\in J(\xi)} q_{\xi,j}\left(\sum_{h\in H}\theta^h_{\xi,j}-\sum_{h\in H_j(\xi)}\varphi^h_{\xi,j}\right)+\sum_{j\in J(\xi_0)}\pi_{\xi,j}\sum_{h\in H}\left(\theta^h_{\xi,j}-\theta^h_{\xi_0,j}\right).$$

- For each $\xi \in D_2$, a player chooses $p_\xi \in \Delta_\xi$ to maximize

$$p_\xi\sum_{h\in H}\left(c^h_\xi(x^h,\varphi^h,\varphi^{\alpha,h})-w^h_\xi-Y_\xi c^h_{\xi^-}(x^h,\varphi^h,\varphi^{\alpha,h})\right).$$

(iii) Given $\left((p,q,\pi),(x^h,\theta^h,\varphi^h,\varphi^{\alpha,h},\varphi^{\beta,h})_{h\in H}\right)\in\Delta(Q)\times\prod_{h\in H}\mathcal{X}^h(\Omega)$,

- For each $(\mu,j)\in D_1\times J(\xi_0)$, a player chooses $N_{\mu,j}\in[R_{\mu,j}(p,q,\pi),\Phi]$ to maximize

$$-\left(N_{\mu,j}\sum_{h\in H_j(\xi_0)}\varphi^h_{\xi_0,j}-\sum_{h\in H_j(\xi_0)}\left(A_{\mu,j}(p,q,\pi)\varphi^{\alpha,h}_{\mu,j}+B_{\mu,j}(p,q,\pi)\varphi^{\beta,h}_{\mu,j}+p_\mu C_{\mu,j}\varphi^{\gamma,h}_{\xi,j}\right)\right)^2.$$

- For each $(\mu,j)\in D_2\times J(\xi_0)$, a player chooses $N_{\mu,j}\in\mathcal{N}_{\mu,j}(\Phi)$ to maximize

$$-\left(N_{\mu,j}\sum_{h\in H_j(\xi_0)}\varphi^h_{\xi_0,j}-R_{\mu,j}(p,q,\pi)\sum_{h\in H_j(\xi_0)}\varphi^{h,\alpha}_{\mu^-,j}\right)^2.$$

- For each $(\mu,j)\in D_2\times J(\xi)$, a player chooses $N_{\mu,j}\in\mathcal{N}_{\mu,j}(\Phi)$ to maximize $-\left(N_{\mu,j}-R_{\mu,j}(p,q,\pi)\right)^2$.

A vector $\left((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\right)\in\Delta(Q)\times\mathcal{N}(\Phi)\times\prod_{h\in H}\mathcal{X}^h(\Omega)$ is a *Cournot-Nash equilibrium* for the generalized game $\mathcal{G}(Q)$ if it solves all the problems above.

Existence of Cournot-Nash equilibria for $\mathcal{G}(Q)$

Under Assumptions (A1)-(A4), each player in the generalized game $\mathcal{G}(Q)$ has a continuous correspondence of admissible strategies, with non-empty, compact, and convex values. Also, players' objective functions are continuous and quasi-concave on their own strategy. Since $\Delta(Q)\times\mathcal{N}(\Phi)\times\prod_{h\in H}\mathcal{X}^h(\Omega)$ is non-empty, convex, and compact, Berge's Maximum Theorem guarantees that best-reply correspondences are upper hemicontinuous and have non-empty, compact and convex values.

Applying Kakutani Fixed Point Theorem to the set-value mapping which associates to each $z\in\Delta(Q)\times\mathcal{N}(\Phi)\times\prod_{h\in H}\mathcal{X}^h(\Omega)$ the cartesian product of players' best-reply strategies to z, we obtain an equilibrium for $\mathcal{G}(Q)$.

From Cournot-Nash equilibria of $\mathcal{G}(Q)$ to non-trivial equilibria of $\mathcal{E}$

Let $\left((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\right)$ be a Cournot-Nash equilibrium for $\mathcal{G}(Q)$.

We have that,

(1) $$\sum_{h\in H}c^h_\xi(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})\le\sum_{h\in H}W^h_\xi\quad\Longrightarrow\quad\overline{q}_{\xi,j}<\overline{p}_\xi C_{\xi,j},\quad\forall\xi\in D\backslash D_2,\ \forall j\in J(\xi);$$ [16]

[16]Fix $\xi\in D\backslash D_2$ such that $\sum_{h\in H}c^h_\xi(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})\le\sum_{h\in H}W^h_\xi$ and assume that, for some $j\in J(\xi)$, $\overline{p}_\xi C_{\xi,j}\le\overline{q}_{\xi,j}$. The strict monotonicity of preferences (Assumption (A1)) implies that, for every player $h\in H_j(\xi)$, $\overline{\varphi}^h_{\xi,j}=\Omega$. Otherwise, player h could increase U^h without any additional cost, by increasing the short-position on j at node ξ, consuming the associated collateral, and defaulting on this additional short-position at the successor nodes of ξ. Therefore, it follows that $\Omega\left\|C_{\xi,j}\right\|_\Sigma\le\|C_{\xi,j}\|_\Sigma\sum_{h\in H_j(\xi)}\overline{\varphi}^h_{\xi,j}\le\left\|\sum_{h\in H}c^h_\xi(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})\right\|_\Sigma\le\sum_{h\in H}\|W^h_\xi\|_\Sigma$, which contradicts the fact that $\Omega>2\max_{\xi\in D\backslash D_2}\frac{\sum_{h\in H}\|W^h_\xi\|_\Sigma}{\min_{k\in J(\xi)}\|C_{\xi,k}\|_\Sigma}$.

(2) $$\sum_{h\in H}\overline{\theta}^h_{\xi_0,j}<\sum_{h\in H}\overline{\theta}^h_{\xi,j}\quad\Longrightarrow\quad\overline{\pi}_{\xi,j}\leq\overline{p}_{\xi}C_{\xi,j},\quad\forall\xi\in D_1,\,\forall j\in J(\xi_0).$$ [17]

Therefore, if there is no excess of demand in physical market at a node $\xi\in D\setminus D_2$, then upper bounds on credit contract prices are non-binding at this node, i.e., for any $j\in J(\xi)$ we have that $\overline{q}_{\xi,j}<Q$. Analogously, if there is excess of demand for security $j\in J(\xi_0)$ at $\xi\in D_1$, then $\overline{\pi}_{\xi,j}<Q$.

Notice that, for each agent $h\in H$ the allocation $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})$ belongs to $\Gamma^h(\overline{p},\overline{q},\overline{\pi},\overline{N})\cap\mathcal{X}^h(\Omega)$ and, therefore, it satisfies inequalities $(\mathrm{B}_\xi)_{\xi\in D}$ and $(\mathrm{S}_\xi)_{\xi\in D_1}$. Hence, adding restrictions (B_{ξ_0}) across agents, we conclude that the objective function of the player who chooses $(\overline{p}_{\xi_0},\overline{q}_{\xi_0})$ has an optimal value less than or equal to zero. Since $(\overline{p}_{\xi_0},\overline{q}_{\xi_0})\in\Delta_{\xi_0}(Q)$, this implies that

$$\sum_{h\in H}\left(c^h_{\xi_0}(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})-w^h_{\xi_0}\right)\leq 0,\qquad\sum_{h\in H}\overline{\theta}^h_{\xi_0}\leq\sum_{h\in H_j(\xi_0)}\overline{\varphi}^h_{\xi_0}.$$ [18]

Hence, for each agent h, $(\overline{x}^h_{\xi_0},\overline{\theta}^h_{\xi_0},\overline{\varphi}^h_{\xi_0})\leq\Omega(1,\ldots,1)$. That is, upper bounds on individual allocations chosen at ξ_0 are non-binding. For this reason, monotonicity of preferences implies that both $\overline{p}_{\xi_0}\gg 0$ and budget constraints at ξ_0 are binding. We conclude that the equilibrium value of the objective function of the player who chooses $(\overline{p}_{\xi_0},\overline{q}_{\xi_0})$ is zero, which in turn implies that commodity markets feasibility holds at ξ_0 and, for each $j\in J(\xi_0)$, $\sum_{h\in H}\overline{\theta}^h_{\xi_0,j}\leq\sum_{h\in H_j(\xi_0)}\overline{\varphi}^h_{\xi_0,j}$, $\overline{q}_{\xi_0,j}\left(\sum_{h\in H}\overline{\theta}^h_{\xi_0,j}-\sum_{h\in H_j(\xi_0)}\overline{\varphi}^h_{\xi_0,j}\right)=0$.

Fix an intermediate node $\xi\in D_1$. The definition of Φ guarantees that, for each $j\in J(\xi_0)$,

$$\overline{N}_{\xi,j}\sum_{h\in H_j(\xi_0)}\overline{\varphi}^h_{\xi_0,j}=\sum_{h\in H_j(\xi_0)}\left(A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j}+B_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\beta,h}_{\xi,j}+\overline{p}_\xi C_{\xi,j}(\overline{\varphi}^h_{\xi_0,j}-\overline{\varphi}^{\alpha,h}_{\xi,j}-\overline{\varphi}^{\beta,h}_{\xi,j})\right).$$

From this identity, adding (B_ξ) across agents and given that $\sum_{h\in H}\overline{\theta}^h_{\xi_0}\leq\sum_{h\in H_j(\xi_0)}\overline{\varphi}^h_{\xi_0}$, we get

$$\overline{p}_\xi\sum_{h\in H}\left(c^h_\xi(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})-w^h_\xi-Y_\xi c^h_{\xi_0}(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})\right)$$
$$+\sum_{j\in J(\xi)}\overline{q}_{\xi,j}\left(\sum_{h\in H}\overline{\theta}^h_{\xi,j}-\sum_{h\in H_j(\xi)}\overline{\varphi}^h_{\xi,j}\right)+\sum_{j\in J(\xi_0)}\overline{\pi}_{\xi,j}\sum_{h\in H}\left(\overline{\theta}^h_{\xi,j}-\overline{\theta}^h_{\xi_0,j}\right)\leq 0.$$

Therefore, as in ξ_0, upper bounds on individual consumption allocations are non-binding at ξ. Hence, the monotonicity of preferences ensures that $\overline{p}_\xi\gg 0$ and that budget constraints at ξ are satisfied with equality. Furthermore, for any $j\in J(\xi)$ we have that $\sum_{h\in H}\overline{\theta}^h_{\xi,j}\leq\sum_{h\in H_j(\xi)}\overline{\varphi}^h_{\xi,j}$, otherwise $\overline{q}_{\xi,j}=Q$ which contradicts (1). Analogously, it follows from condition (2) that, for every $j\in J(\xi_0)$ we have that $\sum_{h\in H}\left(\overline{\theta}^h_{\xi,j}-\overline{\theta}^h_{\xi_0,j}\right)\leq 0$.

---

[17]Fix $\xi\in D_1$ and $j\in J(\xi_0)$ such that $\sum_{h\in H}\overline{\theta}^h_{\xi_0,j}<\sum_{h\in H}\overline{\theta}^h_{\xi,j}$. Then, there is at least one lender of security j at node ξ. On the other hand, if $\overline{\pi}_{\xi,j}>\overline{p}_\xi C_{\xi,j}$, agents prefer to buy bundle $C_{\xi,j}$ instead of investing on security j. Indeed, buying $C_{\xi,j}$ provides future payments greater than or equal to security j's deliveries and, at the same time, allows to consume $C_{\xi,j}$. Therefore, the existence of long positions on j at node ξ implies that $\overline{\pi}_{\xi,j}\leq\overline{p}_\xi C_{\xi,j}$.

[18]If there exists excess of demand in a commodity market l, then the player who determines prices at ξ_0 can make his objective function positive by fixing asset prices to zero and concentrating commodity prices on l, i.e., setting $p_l=1$. Thus, there is no excess of demand in commodity markets. Hence, if there is some credit contract $j\in J(\xi_0)$ with excess of investment, the player who chooses prices will make $\overline{q}_{\xi_0,j}=Q$, which is a contradiction.

Thus, at node ξ commodity markets feasibility conditions hold and

$$\sum_{h\in H}\overline{\theta}^h_{\xi,j}\le\sum_{h\in H_j(\xi)}\overline{\varphi}^h_{\xi,j},\quad \overline{q}_{\xi,j}\left(\sum_{h\in H}\overline{\theta}^h_{\xi,j}-\sum_{h\in H_j(\xi)}\overline{\varphi}^h_{\xi,j}\right)=0,\quad \forall j\in J(\xi),$$

$$\sum_{h\in H}\left(\overline{\theta}^h_{\xi,j}-\overline{\theta}^h_{\xi_0,j}\right)\le 0,\quad \overline{\pi}_{\xi,j}\sum_{h\in H}\left(\overline{\theta}^h_{\xi,j}-\overline{\theta}^h_{\xi_0,j}\right)=0,\quad \forall j\in J(\xi_0).$$

Fix a terminal node $\xi\in D_2$. Analogous arguments to those made above guarantee that, for each $j\in J(\xi^-)$, $\overline{N}_{\xi,j}=R_{\xi,j}(\overline{p},\overline{q},\overline{\pi})$. Also, for each $j\in J(\xi_0)$,

$$\overline{N}_{\xi,j}\sum_{h\in H_j(\xi_0)}\overline{\varphi}^h_{\xi_0,j}=R_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\sum_{h\in H_j(\xi_0)}\overline{\varphi}^{\alpha,h}_{\xi^-,j}.$$

Hence, inequalities $\sum_{h\in H}\overline{\theta}^h_{\xi^-,k}\le\sum_{h\in H}\overline{\theta}^h_{\xi_0,k}\le\sum_{h\in H_j(\xi_0)}\overline{\varphi}^h_{\xi_0,k}$ and $\sum_{h\in H}\overline{\theta}^h_{\xi^-,j}\le\sum_{h\in H_j(\xi^-)}\overline{\varphi}^h_{\xi^-,j}$, which hold for any $(k,j)\in J(\xi_0)\times J(\xi^-)$, guarantee that after adding (B_ξ) across agents we get that,

$$\overline{p}_\xi\sum_{h\in H}\left(c^h_\xi(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})-w^h_\xi-Y_\xi c^h_{\xi_0}(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})\right)\le 0.$$

Since $\overline{p}_\xi\in\Delta_\xi$, there is no excess of demand in commodity markets at ξ and, hence, upper bounds on individual allocations chosen at ξ are non-binding. We conclude that commodity markets feasibility conditions hold at ξ and that $\overline{p}_\xi\gg 0$.

Therefore, $\overline{p}\gg 0$ and commodity market clearing conditions hold at D.

Given $\xi\in D\setminus D_2$ and $j\in J(\xi)$, $\overline{N}_{\mu,j}\ge R_{\mu,j}(\overline{p},\overline{q},\overline{\pi})$, $\forall\mu\in\xi^+$. Since $\overline{p}\gg 0$, Assumption (A5) guarantees that there exists at least one security with non-trivial payments. Moreover, for each security with non-trivial payments, the market clearing condition holds at the emission node. Otherwise $\overline{q}_{\xi,j}=0$, a contradiction with the strict monotonicity of preferences and the fact that upper bounds on optimal individual allocations are non-binding. Therefore, for each $\xi\in D\setminus D_2$ and $j\in J(\xi)$ such that $(N_{\mu,j})_{\mu>\xi}\neq 0$, we obtain $\sum_{h\in H}\overline{\theta}^h_{\xi,j}=\sum_{h\in H_j(\xi)}\overline{\varphi}^h_{\xi,j}$.

If for some $\xi\in D$ and $j\in J(\xi)$, $\sum_{h\in H}\overline{\theta}^h_{\xi,j}<\sum_{h\in H_j(\xi)}\overline{\varphi}^h_{\xi,j}$, then $\overline{q}_{\xi,j}=0$ and $(\overline{N}_{\mu,j})_{\mu>\xi}=0$. Therefore, maintaining optimality, we can substitute $\overline{\theta}^h_{\xi,j}$ with

$$\widehat{\theta}^h_{\xi,j}:=\begin{cases}\overline{\varphi}^h_{\xi,j}, & \text{for any } h\in H_j(\xi_0);\\ 0, & \text{otherwise.}\end{cases}$$

Also, if there exist $(\mu,j)\in D_1\times J(\xi_0)$ for which $\sum_{h\in H}\left(\overline{\theta}^h_{\mu,j}-\overline{\theta}^h_{\xi_0,j}\right)<0$, then $\overline{\pi}_{\mu,j}=0$ and $(\overline{N}_{\xi,j})_{\xi\in D_2}=0$.[19] Therefore, we can substitute $\overline{\theta}^h_{\mu,j}$ with $\widehat{\theta}^h_{\mu,j}:=\overline{\theta}^h_{\xi_0,j}$ maintaining optimality. After these modifications, financial market clearing conditions hold.

Furthermore, these substitutions guarantee that, for each $\xi\in D_1$ and $j\in J(\xi_0)$,

$$\overline{N}_{\xi,j}\sum_{h\in H}\overline{\theta}^h_{\xi_0,j}=\sum_{h\in H_j(\xi_0)}\left(A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j}+B_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\beta,h}_{\xi,j}+\overline{p}_\xi C_{\xi,j}(\overline{\varphi}^h_{\xi_0,j}-\overline{\varphi}^{\alpha,h}_{\xi,j}-\overline{\varphi}^{\beta,h}_{\xi,j})\right).$$

[19] This may be a consequence of debt prepayment.

Also, for each $\xi \in D_2$ and $j \in J(\xi_0)$, we have

$$\overline{N}_{\xi,j} \sum_{h\in H} \overline{\theta}^h_{\xi_0,j} - R_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) \sum_{h\in H_j(\xi_0)} \overline{\varphi}^{\alpha,h}_{\xi^-,j} = 0.$$

It follows that $\left((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\widehat{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\right)$ satisfies conditions (ii)-(iv) in our equilibrium definition, with at least one security with non-trivial payments.

Therefore, to ensure that $\left((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\widehat{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\right)$ is a non-trivial equilibrium for $\mathcal{E}$ it is sufficient to show that, for each $h \in H$ the allocation $\overline{z}^h := (\overline{x}^h,\widehat{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})$ is an optimal choice in $\Gamma^h(\overline{p},\overline{q},\overline{\pi},\overline{N})$. Suppose by contradiction that for some $h \in H$ there exists another allocation $\widetilde{z}^h := (\widetilde{x}^h,\widetilde{\theta}^h,\widetilde{\varphi}^h,\widetilde{\varphi}^{\alpha,h},\widetilde{\varphi}^{\beta,h}) \in \Gamma^h(\overline{p},\overline{q},\overline{\pi},\overline{N})$ such that,

$$U^h\left(\left(c^h_\xi(\widetilde{x}^h,\widetilde{\varphi}^h,\widetilde{\varphi}^{\alpha,h})\right)_{\xi\in D}\right) > U^h\left(\left(c^h_\xi(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})\right)_{\xi\in D}\right).$$

Since $\overline{z}^h$ is in the interior (relative to $\mathcal{X}$) of $\Gamma^h(\overline{p},\overline{q},\overline{\pi},\overline{N}) \cap \mathcal{X}^h(\Omega)$ and U^h is strictly quasi-concave, there exists $\lambda \in (0,1)$ such that, $(x^h_\lambda,\theta^h_\lambda,\varphi^h_\lambda,\varphi^{\alpha,h}_\lambda,\varphi^{\beta,h}_\lambda) := \lambda\overline{z}^h + (1-\lambda)\widetilde{z}^h \in \Gamma^h(\overline{p},\overline{q},\overline{\pi},\overline{N}) \cap \mathcal{X}^h(\Omega)$ and $U^h\left(\left(c^h_\xi(x^h_\lambda,\varphi^h_\lambda,\varphi^{\alpha,h}_\lambda)\right)_{\xi\in D}\right) > U^h\left(\left(c^h_\xi(\overline{x}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h})\right)_{\xi\in D}\right)$, a contradiction.

Therefore $\left((\overline{p},\overline{q},\overline{\pi},\overline{N}),(\overline{x}^h,\widehat{\theta}^h,\overline{\varphi}^h,\overline{\varphi}^{\alpha,h},\overline{\varphi}^{\beta,h})_{h\in H}\right)$ is a non-trivial equilibrium for $\mathcal{E}$. □

Proof of Proposition 1. From Arrow and Enthoven (1961), the usual Kuhn-Tucker conditions are necessary for optimality. From the partial derivatives of agent h's Lagrangian function with respect to $\overline{x}^h_\xi$ and $\overline{\varphi}^{\alpha,h}_{\xi,j}$ we obtain,

$$\overline{p}_\xi C_{\xi,j} = A_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) + \sum_{\mu\in\xi^+} \frac{\lambda^h(\mu)}{\lambda^h(\xi)} R_{\mu,j}(\overline{p},\overline{q},\overline{\pi}) + \frac{\kappa^h_{\xi,j} + \nu^h_\xi C_{\xi,j} - \eta^{\alpha,h}_{\xi,j}}{\lambda^h(\xi)},$$

where $\kappa^h_{\xi,j} \geq 0$ is the Kuhn-Tucker multiplier of constraint $\overline{\varphi}^{\alpha,h}_{\xi,j} + \overline{\varphi}^{\beta,h}_{\xi,j} \leq \overline{\varphi}^h_{\xi_0,j}$, $\nu^h_\xi \in \mathbb{R}^L_+$ is the vector of multipliers associated with $\overline{x}^h_\xi \geq 0$, and $\eta^{\alpha,h}_{\xi,j} \geq 0$ is the multiplier of the non-negativity constraint of $\overline{\varphi}^{\alpha,h}_{\xi,j}$. From this condition, and using the partial derivative of agent h's Lagrangian function with respect to $\overline{\varphi}^{\beta,h}_{\xi,j}$ we have,

$$B_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) = A_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) + \sum_{\mu\in\xi^+} \frac{\lambda^h(\mu)}{\lambda^h(\xi)} R_{\mu,j}(\overline{p},\overline{q},\overline{\pi}) + \frac{\eta^{\beta,h}_{\xi,j} + \nu^h_\xi C_{\xi,j} - \eta^{\alpha,h}_{\xi,j}}{\lambda^h(\xi)},$$

where $\eta^{\beta,h}_{\xi,j} \geq 0$ is the multiplier of the non-negativity constraint of $\overline{\varphi}^{\beta,h}_{\xi,j}$. Thus, we obtain that,

$$\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) = \frac{\nu^h_\xi C_{\xi,j} - \eta^{\alpha,h}_{\xi,j}}{\lambda^h(\xi)} + \frac{\min\{\eta^{\beta,h}_{\xi,j},\kappa^h_{\xi,j}\}}{\lambda^h(\xi)}.$$

Therefore, $\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) < 0$ implies that $\eta^{\alpha,h}_{\xi,j} > 0$. Thus, when $\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) < 0$ agent h closes short positions on j at ξ. On the other hand, suppose that $\Phi^h_{\xi,j}(\overline{p},\overline{q},\overline{\pi}) > 0$ and that collateral constraints of credit contract j do not generate frictions on agent h's optimal decisions at ξ, i.e., $\nu^h_\xi = 0$. Then, $\min\{\eta^{\beta,h}_{\xi,j},\kappa^h_{\xi,j}\} > 0$. Hence, both $\overline{\varphi}^{\beta,h}_{\xi,j} = 0$ and $\overline{\varphi}^{\alpha,h}_{\xi,j} + \overline{\varphi}^{\beta,h}_{\xi,j} = \overline{\varphi}^h_{\xi_0,j}$, implying $\overline{\varphi}^{\alpha,h}_{\xi,j} = \overline{\varphi}^h_{\xi_0,j}$. □

Proof of Proposition 2. Assume that for some debt contract $k \in J(\xi)$ conditions above hold. Suppose that, after issuing $\overline{\varphi}^h_{\xi_0,j}$ units of j at ξ_0, j-borrower h maintains a position $\overline{\varphi}^{\alpha,h}_{\xi,j} \in (0,\overline{\varphi}^h_{\xi_0,j}]$ at node ξ.

Therefore, h should pay $A_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j}$, consume the collateral bundle $C_{\xi,j}\overline{\varphi}^{\alpha,h}_{\xi,j}$, and deliver a payment $R_{\mu,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j}$ at each terminal node $\mu \in \xi^+$. It can be shown that this strategy is not optimal.

Indeed, consider the following alternative: agent h closes the short position $\overline{\varphi}^{\alpha,h}_{\xi,j}$ and trades $\widetilde{\varphi}_k\overline{\varphi}^{\alpha,h}_{\xi,j}$ units of debt contract k, where $\widetilde{\varphi}_k = \frac{\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})}{\overline{q}_{\xi,k}}$. There is no additional cost at ξ, i.e., $\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})\overline{\varphi}^{\alpha,h}_{\xi,j} - \overline{q}_{\xi,k}\widetilde{\varphi}_k\overline{\varphi}^{\alpha,h}_{\xi,j} = 0$. Since $\frac{C_{\xi,k}}{\overline{q}_{\xi,k}} \leq \frac{C_{\xi,j}}{\Psi_{\xi,j}(\overline{p},\overline{q},\overline{\pi})}$, the original consumption bundle at ξ satisfies agent h's new collateral requirements. Finally, the new payments at terminal nodes imply that the Lagrangian function increases, as $\sum_{\mu\in\xi^+} \lambda^h(\mu)\left(R_{\mu,j}(\overline{p},\overline{q},\overline{\pi}) - R_{\mu,k}(\overline{p},\overline{q},\overline{\pi})\widetilde{\varphi}_k\right)\overline{\varphi}^{\alpha,h}_{\xi,j} > 0$.

Hence, any strategy that maintains open a short position on j at ξ is not optimal. $\square$

References

[1] Alvarez, F., and U. J. Jermann (2000): "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica,* 68, 4, 775-797.

[2] Angeloni, L. and B. Cornet (2006): "Existence of Financial Equilibria in a Multi-Period Stochastic Economy," *Advances in Mathematical Economics*, 8, 933-955.

[3] Aouani, Z. and B. Cornet (2009): "Existence of Financial Equilibria with Restricted Participation," *Journal of Mathematical Economics*, 45, 772-786.

[4] Aouani, Z. and B. Cornet (2011): "Reduced Equivalent Form of a Financial Structure," *Journal of Mathematical Economics*, doi:10.1016/j.jmateco.2010.12.015

[5] Araujo, A., Páscoa, M.R. (2002): "Bankruptcy in a Model of Unsecured Claims," *Economic Theory,* 20, 455-481.

[6] Araujo, A., M.R. Páscoa, and J.P. Torres-Martínez (2002): "Collateral Avoids Ponzi Schemes in Incomplete Markets," *Econometrica,* 70, 1613-1638.

[7] Araujo, A., M.R. Páscoa, and J.P. Torres-Martínez (2005): "Bubbles, Collateral and Monetary Equilibrium," working paper, Department of Economics, Pontifical Catholic University of Rio de Janeiro. Available at http://www.econ.puc-rio.br/pdf/td513.pdf

[8] Araujo, A., M.R. Páscoa, and J.P. Torres-Martínez (2011): "Long-lived Collateralized Assets and Bubbles," *Journal of Mathematical Economics*, 47, 3, 260-271.

[9] Arrow, K., and A. Enthoven (1961): "Quasi-Concave Programming," *Econometrica*, 29, 779-800.

[10] Chen, H., V. Cúrdia, and A. Ferrero (2011): "The Macroeconomic Effects of Large-Scale Asset Purchase Programs," Federal Reserve Bank of New York Staff Report no. 527.

[11] Cornet, B., and R. Gopalan (2010): "Arbitrage and Equilibrium with Portfolio Constraints ," *Economic Theory*, doi: 10.1007/s00199-009-0506-5.

[12] Deng, Y., J.M. Quigley, and R. Van Order (2000): "Mortgage Terminations, Heterogeneity and the Exercise of Mortgage Options," *Econometrica*, vol. 68, 2, 275-307.

[13] Dubey, P., J. Geanakoplos, and M. Shubik (1989): "Default and Efficiency in a General Equilibrium Model with Incomplete Markets," Cowles Foundation Discussion Paper No. 879R.

[14] Dubey, P., J. Geanakoplos, and M. Shubik (2005): "Default and Punishment in General Equilibrium," *Econometrica*, 73, 1-37.

[15] Garriga, C., and D. Schlagenhauf (2009): "Home Equity, Foreclosures, and Bailouts," Working Paper.

[16] Geanakoplos, J., and W.R. Zame (1997): "Collateral, Default and Market Crashes," Cowles Foundation Discussion Paper.

[17] Geanakoplos, J., and W.R. Zame (2002): "Collateral and the Enforcement of Intertemporal Contracts," Cowles Foundation Discussion Paper.

[18] Geanakoplos, J., and W.R. Zame (2007): "Collateralized Asset Markets," UCLA Working Paper.

[19] Gertler, M., and P. Karadi (2012): "QE123: A Framework for Analyzing Large Scale Asset Purchases as a Monetary Policy Tool," Working Paper.

[20] Guvenen, F. (2009): "A Parsimonious Macroeconomic Model for Asset Pricing," *Econometrica*, 2009, 77, 6, 1711-1740.

[21] Hart, O. (1975): "On the Optimality of Equilibrium when the Market Structure is Incomplete", *Journal of Economic Theory*, 11, 418-443.

[22] Hernández, A., and M. Santos (1996): "Competitive Equilibria for Infinite-Horizon Economies with Incomplete Markets," *Journal of Economic Theory*, 71, 102-130.

[23] Kehoe, T., and D.K. Levine (1993): "Debt Constrained Asset Markets," *Review of Economic Studies*, 60, 865-888.

[24] Kocherlakota, N. (1996): "Implications of Efficient Risk Sharing without Committment," *Review of Economic Studies*, 63, 4, 595-609.

[25] Kubler, F., and K. Schmedders (2003): "Stationary equilibria in asset pricing models with incomplete markets," *Econometrica* 71, 1767-1793.

[26] Levine, D.K., and W.R. Zame (1996): "Debt Constraints and Equilibrium in Infinite Horizon Economies with Incomplete Markets," *Journal of Mathematical Economics*, 26, 103-131.

[27] Magill, M., and M. Quinzii (1994): "Infinite Horizon Incomplete Markets," *Econometrica*, vol. 62, 4, 853-880.

[28] Magill, M., and M. Quinzii (1996): "Incomplete Markets over an Infinite Horizon: Long-lived Securities and Speculative Bubbles," *Journal of Mathematical Economics*, 26, 133-170.

[29] Poblete-Cazenave, R. and Torres-Martínez, J.P. (2012): "Equilibrium with Limited-Recourse Collateralized Loans," *Economic Theory,* DOI:10.1007/s00199-011-0685-8.

[30] Sabarwal, T. (2003): "Competitive Equilibria with Incomplete Markets and Endogenous Bankruptcy," *Contributions to Theoretical Economics,* 3(1).

[31] Seghir, A., and J.P.Torres-Martínez (2011): "On Equilibrium Existence with Endogenous Restricted Financial Participation," *Journal of Mathematical Economics*, volume 47, issue 1, pages 37-42.

[32] Steinert, M., and J.P. Torres-Martinez (2007): "General Equilibrium in CLO Markets," *Journal of Mathematical Economics*, 43, 6, 709-734.

[33] Zame, W. (1993): "Efficiency and the Role of Default when Security Markets are Incomplete," *American Economic Review*, 83, 1142-1164.

MIGUEL A. IRAOLA
DEPARTMENT OF ECONOMICS, UNIVERSITY OF MIAMI
CENTER FOR ECONOMIC RESEARCH, CIE-ITAM
E-mail address: `miraola@bus.miami.edu; miguel.iraola@itam.mx`

JUAN PABLO TORRES-MARTÍNEZ
DEPARTMENT OF ECONOMICS, UNIVERSITY OF CHILE
E-mail address: `juan.torres@fen.uchile.cl`