When can we do better than autarky?

Guoqiang Tian and Yuzhe Zhang

Texas A&M University at College Station, Texas A&M University at College Station

January 2013

When can we do better than autarky?

Guoqiang Tian[1] and Yuzhe Zhang[2]

Department of Economics, Texas A&M University, College Station, TX, 77843, United States

Abstract

This paper provides a necessary and sufficient condition for the existence of nonautarkic contract in a risk sharing model with two-sided lack of commitment. Verifying the condition takes just one Guassian elimination of a matrix.

Keywords: Lack of commitment, Risk sharing, Autarky
JEL: D82, D86

1. Introduction

The theory of contracting with two-sided lack of commitment has been applied to study a wide range of economic issues, including international business cycles (cf. Kehoe and Perri (2002)), consumption inequality (cf. Kocherlakota (1996) and Ligon et al. (2002)), and wage contracts (cf. Thomas and Worrall (1988)). In this theory, a commonly made assumption is that some nonautarkic risk sharing arrangement is sustainable (in the sense that no one would leave the contract). To satisfy this assumption, researchers focus on sufficiently patient economic agents, in which case a Folk-theorem argument shows that nearly any allocation is sustainable. Away from this extreme, a natural question is: Under what conditions does a nonautarkic and sustainable risk sharing arrangement exist?

To answer this question, we study agents' incentives to participate in risk sharing. We linearize their utilities around autarkic endowment, which allows us to calculate in closed form the cost and the benefit of participation. Hence the condition for participation is simply that the benefit exceeds the cost. Besides answering the above question, the analysis of the linearized model provides clear economic insights on agents' incentives that are difficult to identify in the original nonlinear model.

[1]E-mail address: gtian@tamu.edu.
[2]E-mail address: yuzhe-zhang@econmail.tamu.edu.

2. Model

The model is similar to that in Ligon et al. (2002). There are two agents at time zero, with preferences

$$E\left[\sum\nolimits_{t=0}^{\infty}\delta^t u(c_t^1)\right] \text{ and } E\left[\sum\nolimits_{t=0}^{\infty}\delta^t v(c_t^2)\right],$$

where c_t^i $(i=1,2)$ is agent i's consumption at time t, $\delta \in (0,1)$ is their common discount factor, and E is the expectation operator. Both agents are risk averse, i.e., $u'' < 0, v'' < 0$. In each period t, agent i's income y_i depends on the state of the nature s, which is drawn from a finite set $\{1,2,...,S\}$ and follows a Markov chain. Let Π be the transition matrix $[\pi_{sr}]_{s,r=1}^S$, where π_{sr} is the transition probability from state s to state r. We assume $\pi_{sr} > 0$ for all s and r to simplify the analysis.

A risk sharing contract specifies for each t and each history $h_t \equiv (s_0, s_1, ..., s_t)$ a transfer $\tau(h_t)$ to be made from agent 1 to agent 2. Transfers can be negative. Neither agent can commit; if one defaults, then both of them go to autarky (i.e., transfers are zero thereafter). Conditional on h_t, the expected surplus of agent 1 over autarky is

$$U(h_t) \equiv u(y_1(s_t) - \tau(h_t)) - u(y_1(s_t)) + E\sum_{j=t+1}^{\infty}\delta^{j-t}\left(u(y_1(s_j) - \tau(h_j)) - u(y_1(s_j))\right),$$

and the surplus of agent 2, $V(h_t)$, is defined similarly. A contract is *sustainable* if $U(h_t) \geq 0$ and $V(h_t) \geq 0$, for all h_t. All contracts to be discussed in this paper are sustainable.

A sustainable contract is *(constrained) efficient* if for any given level of agent 1's surplus it provides more surplus to agent 2 than other sustainable contracts. Ligon et al. (2002) show that, if nonautarkic contracts exist, then an efficient contract is characterized as follows. There exist $\{\bar{U}_s > 0\}_{s=1}^S$ and agent 2's surplus functions $\{V_s(\cdot) : [0, \bar{U}_s] \to \mathbb{R}\}_{s=1}^S$ such that

$$V_s(U_s) = \max_{\tau_s, \{U_r\}_{r=1}^S} \quad v(y_2(s) + \tau_s) - v(y_2(s)) + \delta\sum_{r=1}^S \pi_{sr}V_r(U_r) \tag{1}$$

$$\text{subject to} \quad u(y_1(s) - \tau_s) - u(y_1(s)) + \delta\sum_{r=1}^S \pi_{sr}U_r = U_s, \tag{2}$$

$$U_r \in [0, \bar{U}_r].$$

The surplus function $V_s(U_s)$ decreases in U_s and reaches zero at $U_s = \bar{U}_s$.

2.1. A linearized problem

Following Thomas and Worrall (1988) and Ligon et al. (2002), this subsection considers a model with utilities linearized around autarkic endowment. We show below that the linearized model not only is analytically more tractable, it also offers clear intuition about the cost and the benefit of participating in this long-term contract. Fix $\{\bar{U}_s > 0\}_{s=1}^S$ in problem (1). Suppose agents' utilities in state s are $u(y_1(s)) + u'(y_1(s))(c^1 - y_1(s))$ and $v(y_2(s)) + v'(y_2(s))(c^2 - y_2(s))$. Agent 2's problem is

$$L_s(U_s) = \max_{\tau_s, \{U_r\}_{r=1}^S} \quad v'(y_2(s))\tau_s + \delta \sum_{r=1}^S \pi_{sr} L_r(U_r)$$

$$\text{subject to} \quad -u'(y_1(s))\tau_s + \delta \sum_{r=1}^S \pi_{sr} U_r = U_s, \quad U_r \in [0, \bar{U}_r].$$

Introducing $c_s \equiv u'(y_1(s))\tau_s$, $\xi_s \equiv \frac{v'(y_2(s))}{u'(y_1(s))}$, and $A_r \equiv -U_r$, we rewrite the above as

$$L_s(A_s) = \max_{c_s, \{A_r\}_{r=1}^S} \quad \xi_s c_s + \delta \sum_{r=1}^S \pi_{sr} L_r(A_r) \tag{3}$$

$$\text{subject to} \quad c_s + \delta \sum_{r=1}^S \pi_{sr} A_r = A_s, \tag{4}$$

$$A_r \in [-\bar{U}_r, 0].$$

With a slight abuse of notation, we have used $L_s(A_s)$ to denote $L_s(U_s)$. Without loss of generality, we assume that the ratio of marginal utilities ξ_s weakly increases in s.

Problem (3) has the following interpretation. Both agent 1 and 2 have linear utilities and their consumptions are $-c_s$ and c_s, respectively. Agent 2 is subject to taste shocks $\{\xi_s\}_{s=1}^S$ while agent 1 is not. Because of the taste shocks, agent 2 prefers consumption in states with high ξ_s while agent 1 is indifferent. To facilitate trade, agent 2 opens a "bank account" with agent 1, in which agent 2's asset holding A_s represents how much agent 1 owes agent 2. Noncommitment of agent 1 requires $A_s \le 0$ (i.e., agent 2 is in debt) at all times: positive A_s would obligate agent 1 to repay and trigger his default. On the other hand, although agent 2 is in debt, he would not default as long as he can still benefit from trading with agent 1. To see the benefit, interpret (4) as agent 2's budget constraint. There are two channels through which agent 2 can move consumptions from low-taste-shock states to high-taste-shock states: (1) he can reallocate assets among future states, holding more assets in high-shock states; and (2) when the current taste shock is high, agent 2 can increase his consumption through borrowing (i.e., holding less assets in the future). Calculating these benefits is the key to understanding agent 2's default decision; the following lemma does this in closed form.

Lemma 1. $L_s(A_s) = L_s(0) + \xi_s A_s$, *where*

$$L(0) = \delta(I - \delta\Pi)^{-1} B\bar{U}, \tag{5}$$

$$L(0) \equiv \begin{pmatrix} L_1(0) \\ L_2(0) \\ \vdots \\ L_S(0) \end{pmatrix}, \quad B \equiv \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \pi_{21}(\xi_2 - \xi_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{S1}(\xi_S - \xi_1) & \pi_{S2}(\xi_S - \xi_2) & \cdots & 0 \end{pmatrix}, \bar{U} \equiv \begin{pmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \vdots \\ \bar{U}_S \end{pmatrix}.$$

Proof: That $L_s(A_s)$ is linear in A_s is because agent 2's utility function is linear. To find out $L_s(0)$, note that the optimal portfolio choice in problem (3) is

$$A_r = \begin{cases} -\bar{U}_r, & \text{if } r < s; \\ 0, & \text{if } r \geq s. \end{cases}$$

Therefore, $c_s = A_s + \delta \sum_{r=1}^{s-1} \pi_{sr} \bar{U}_r$, and the Bellman equation is

$$\begin{aligned} L_s(0) &= \xi_s \left(\delta \sum_{r=1}^{s-1} \pi_{sr} \bar{U}_r \right) + \delta \sum_{r=1}^{s-1} \pi_{sr} \left(L_r(0) - \xi_r \bar{U}_r \right) + \delta \sum_{r=s}^{S} \pi_{sr} L_r(0) \\ &= \delta \sum_{r=1}^{s-1} (\xi_s - \xi_r) \pi_{sr} \bar{U}_r + \delta \sum_{r=1}^{S} \pi_{sr} L_r(0). \end{aligned}$$

Solving the above linear system of equations yields (5). ■

All elements in the matrix $(I - \delta\Pi)^{-1}$ are positive because $(I - \delta\Pi)^{-1} = \sum_{t=0}^{\infty} \delta^t \Pi^t$. This and (5) imply that $L_s(0) \geq 0$ for all s. If all ratios of marginal utilities are identical ($\xi_1 = \xi_s$ for all s), then autarky is the first best outcome. In this case, $B = 0$ and $L(0) = 0$. If there are two states with different ratios of marginal utilities, then at least one element in B is positive. Then $L_s(0) > 0$ for all s because all elements in $(I - \delta\Pi)^{-1}$ are positive.

Remark 1. $L_s(0)$ *measures agent 2's benefit from trading with agent 1. Because agent 2's initial asset holding is zero, his average consumption is zero too. Hence, the benefit is purely from shifting consumptions from low to high taste-shock states.*

3. A necessary and sufficient condition for nonautarkic contract

This section presents the main results of this paper. If nonautarkic contracts exist, then a necessary condition is as follows. Since the utility function $v(\cdot)$ is strictly concave, $v'(y_2(s))\tau_s > v(y_2(s) + \tau_s) - v(y_2(s))$ holds whenever $\tau_s \neq 0$. Therefore, if $A_s = -\bar{U}_s$, then the surplus $L_s(A_s)$ in the linearized problem (3) is higher than $V_s(\bar{U}_s)$ in (1). That is,

$$L_s(0) - \xi_s \bar{U}_s = L_s(A_s) > V_s(\bar{U}_s) = 0, \text{ for all } s, \tag{6}$$

or in matrix form $M\bar{U} < 0$ (a vector is less than zero if all the elements are less than zero), where

$$M \equiv \begin{pmatrix} \xi_1 & 0 & \cdots & 0 \\ 0 & \xi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_S \end{pmatrix} - \delta(I - \delta\Pi)^{-1}B.$$

Inequality (6) highlights the cost and the benefit of the long-term relationship for agent 2. Recall that $L_s(0)$ is agent 2's benefit when he holds zero assets. If his initial asset holding is $A_s = -\bar{U}_s$ in state s, then debt repayment costs him $\xi_s\bar{U}_s$ units of utility. This cost must be dominated by the benefit if he participates in the contract.

The above discussion shows (6) as a necessary condition for a nonautarkic contract. It turns out that (6) is also sufficient. The intuition is as follows. Under condition (6), a nonautarkic contract is sustainable with linear utilities. With nonlinear utilities $u(\cdot)$ and $v(\cdot)$, a risk sharing allocation would be sustainable if it mimics the contract with linear utilities in a small neighborhood around autarky, because $u(\cdot)$ and $v(\cdot)$ are well approximated by the linearized utilities in this neighborhood.

Theorem 1. *A nonautarkic contract exists if and only if there exists* $\{\bar{U}_s > 0\}_{s=1}^S$ *to satisfy* (6).

Proof: Since necessity was already shown, this proof only shows sufficiency. If $\{\bar{U}_s > 0\}_{s=1}^S$ satisfies (6), we construct a recursive nonautarkic contract as follows. In problem (1), choose agent 1's surplus as

$$U_r = \begin{cases} \lambda\bar{U}_r, & \text{if } r < s; \\ 0, & \text{if } r \geq s, \end{cases}$$

where $\lambda > 0$ is a small number to be determined later. Choose τ_s to satisfy (2), the participation constraint of agent 1. Next we verify $V_s(\lambda\bar{U}_s) \geq 0, \forall s$, the participation constraint of agent 2. If λ is small, (2) implies $\tau_s \approx \lambda\frac{-\bar{U}_s+\sum_{r=1}^{s-1}\delta\pi_{sr}\bar{U}_r}{u'(y_1(s))}$, which further implies $v(y_2(s)+\tau_s) - v(y_2(s)) \approx \lambda\xi_s(-\bar{U}_s + \sum_{r=1}^{s-1}\delta\pi_{sr}\bar{U}_r)$. Hence for any $\epsilon > 0$, there is a sufficiently small $\lambda > 0$ such that

$$v(y_2(s)+\tau_s) - v(y_2(s)) > \lambda\left(-(\xi_s+\epsilon)\bar{U}_s + \sum_{r=1}^{s-1}\delta\pi_{sr}(\xi_s-\epsilon)\bar{U}_r\right).$$

Algebra similar to that of Lemma 1 shows that agent 2's surplus is larger than

$$\lambda\delta(I-\delta\Pi)^{-1}\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \pi_{21}(\xi_2-\xi_1-2\epsilon) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{S1}(\xi_S-\xi_1-2\epsilon) & \pi_{S2}(\xi_S-\xi_2-2\epsilon) & \cdots & 0 \end{pmatrix}\begin{pmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \vdots \\ \bar{U}_S \end{pmatrix} - \lambda\begin{pmatrix} (\xi_1+\epsilon)\bar{U}_1 \\ (\xi_2+\epsilon)\bar{U}_2 \\ \vdots \\ (\xi_S+\epsilon)\bar{U}_S \end{pmatrix}.$$

Because the above vector is positive when $\epsilon = 0$ (recall (6)), it remains positive for sufficiently small ϵ due to continuity. Therefore, agent 2's participation constraint is satisfied and the constructed nonautarkic contract is sustainable. ■

Remark 2. *Because $\delta(I - \delta\Pi)^{-1}$ increases in δ, (5) implies that the benefit $L_s(0)$ increases in δ as well. The cost $\xi_s \bar{U}_s$, however, is independent of δ. Hence it is easier to sustain a nonautarkic allocation when agents are more patient. In particular, there is a unique cutoff $\bar{\delta}$ such that nonautarkic contracts exist if and only if $\delta > \bar{\delta}$. This result reinforces the notion in the literature that there is no nonautarkic contract when δ is sufficiently small (cf. Proposition 2 (v) in Ligon et al. (2002)).*

Remark 3. *Besides patience, other factors such as large variability of agents' incomes also facilitate risk sharing. To see this, consider the two-state example in Ljungqvist and Sargent (2004, Section 20.10), where the discount factor is 0.85 and the utilities are $u(c) = v(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma = 1.1$. Agent 1's income $y_1(s)$ is i.i.d. over time and can be either $1 - \bar{y}$ or $\bar{y} \in (0.5, 1)$ with equal probability. Agent 2's income is $y_2(s) = 1 - y_1(s)$. Compute M as $\begin{pmatrix} 2.20\xi_1 - 1.20\xi_2 & 0 \\ 1.63\xi_1 - 1.63\xi_2 & \xi_2 \end{pmatrix}$ and $M\bar{U} < 0$ becomes*

$$(2.20\xi_1 - 1.20\xi_2)\bar{U}_1 < 0, \quad (7)$$

$$(1.63\xi_1 - 1.63\xi_2)\bar{U}_1 + \xi_2\bar{U}_2 < 0. \quad (8)$$

If $\bar{U}_2 = \frac{1.63(\xi_2 - \xi_1)\bar{U}_1}{2\xi_2}$, then (8) is always satisfied. Inequality (7) and $\bar{U}_1 > 0$ require $2.20\xi_1 - 1.20\xi_2 < 0$, or $\bar{y} > 0.57$. In other words, (nonautarkic) risk sharing exists in this example if and only if agents' incomes exhibit enough variability. This result is consistent with the finding in Krueger and Perri (2011) that public income insurance through progressive income taxation reduces private risk sharing.

Although condition (6) is intuitive, it is not easily verifiable as $\{\bar{U}_s\}_{s=1}^S$ is unknown a priori. Below we relate condition (6) (i.e., $M\bar{U} < 0$ for some $\bar{U} > 0$) to the determinants of the principal minors of M. We begin with an illustrative example.

Example 1. *Suppose $S = 3$ and $\xi_1 < \xi_2 < \xi_3$. Denote the matrix $\delta(I - \delta\Pi)^{-1}B$ as $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 0 \end{pmatrix}$, where elements (a to f) are positive. The third column is zero because the third column of B is zero. Then $M = \begin{pmatrix} \xi_1 - a & -b & 0 \\ -c & \xi_2 - d & 0 \\ -e & -f & \xi_3 \end{pmatrix}$. Since the third inequality, $-e\bar{U}_1 - f\bar{U}_2 + \xi_3\bar{U}_3 < 0$, is always satisfied if $\bar{U}_3$ equals $\frac{e\bar{U}_1 + f\bar{U}_2}{2\xi_3}$, we focus on finding $\bar{U}_1 > 0$, $\bar{U}_2 > 0$ to satisfy the first two inequalities:*

$$(\xi_1 - a)\bar{U}_1 - b\bar{U}_2 < 0, \quad (9)$$

$$-c\bar{U}_1 + (\xi_2 - d)\bar{U}_2 < 0. \quad (10)$$

If $\xi_1 - a \le 0$, then $(\bar{U}_1, \bar{U}_2) = (1, \epsilon)$ satisfies both inequalities if $\epsilon > 0$ is sufficiently small. Similarly, if $\xi_2 - d \le 0$, then $(\bar{U}_1, \bar{U}_2) = (\epsilon, 1)$ satisfies both inequalities. What happens if both $\xi_1 - a$ and $\xi_2 - d$ are positive? In this case, (9) *and* (10) *can be rewritten as*

$$\frac{\xi_1 - a}{b} < \frac{\bar{U}_2}{\bar{U}_1}, \quad \frac{\bar{U}_2}{\bar{U}_1} < \frac{c}{\xi_2 - d}.$$

A solution exists if and only if $\frac{\xi_1 - a}{b} < \frac{c}{\xi_2 - d}$, i.e., the determinant of $\begin{pmatrix} \xi_1 - a & -b \\ -c & \xi_2 - d \end{pmatrix}$ is negative.

The following theorem generalizes the above conditions on determinants. Let T be the largest s such that $\xi_s < \xi_S$ (i.e., $\xi_r = \xi_S$ for all $r > T$) and $M(1:s, 1:s)$ be the principal minor of M containing the first s rows and s columns.

Theorem 2. *There exists $\{\bar{U}_s > 0\}_{s=1}^S$ to satisfy* (6) *if and only if either* $\det(M(1:T, 1:T)) < 0$ *or* $\det(M(1:s, 1:s)) \le 0$ *for some $s < T$.*[3]

Proof: Necessity: Suppose there exists $\{\bar{U}_s > 0\}_{s=1}^S$ to satisfy (6). If $M(1,1) \le 0$, then $\det(M(1:1, 1:1)) \le 0$ and the proof is finished. Otherwise, if $M(1,1) > 0$, then use Guassian elimination to eliminate $M(s,1)$ for all $s > 1$. Denote the new matrix as $\tilde{M}$. If $\tilde{M}(2,2) \le 0$, then $\det(M(1:2, 1:2)) = \det(\tilde{M}(1:2, 1:2)) \le 0$. (Note that the elementary row operations in Guassian elimination do not change the determinant of $M(1:2, 1:2)$). Otherwise, if $\tilde{M}(2,2) > 0$, then use Guassian elimination again. We can repeat this procedure as long as $\tilde{M}(s,s) > 0$ and finally reach the elimination of $\bar{U}_T$. We show that $\tilde{M}(T,T) < 0$ if $\tilde{M}(s,s) > 0$ for all $s < T$. Because all off-diagonal elements in $M(1:T, 1:T)$ are negative, the elementary row operations in Guassian elimination always add one row to another and hence preserve the signs of the inequalities in (6). The inequality in row T becomes $\tilde{M}(T,T)\bar{U}_T < 0$, which implies $\tilde{M}(T,T) < 0$ because $\bar{U}_T > 0$. Therefore, $\det(M(1:T, 1:T)) = \det(\tilde{M}(1:T, 1:T)) < 0$.

Sufficiency: Construct a solution to (6) as follows. Carry out Guassian elimination until $\det(M(1:s, 1:s)) \le 0$ for the first time. This means $\tilde{M}(r,r) > 0$ for all $r < s$ and $\tilde{M}(s,s) \le 0$. Let $\bar{U}_s \equiv 1$ and then recursively define $\bar{U}_r \equiv \frac{-\sum_{k=r+1}^s \tilde{M}(r,k)\bar{U}_k}{\tilde{M}(r,r)} > 0$ for $r = s-1, s-2, ..., 1$. That $\sum_{k=1}^s \tilde{M}(r,k)\bar{U}_k \begin{cases} = 0, & \text{if } r < s; \\ \le 0, & \text{if } r = s, \end{cases}$ implies that $\sum_{k=1}^s M(r,k)\bar{U}_k \begin{cases} = 0, & \text{if } r < s; \\ \le 0, & \text{if } r = s. \end{cases}$ Define $\bar{U}_r \equiv \epsilon > 0$ for $r = s+1, ..., T$. Finally, we verify $\sum_{k=1}^T M(r,k)\bar{U}_k < 0$ for all $r = 1, ..., T$. If $r \le s$, then the inequality follows from $M(r,k) < 0$ and $\bar{U}_k > 0$ for $k > s$. If $r > s$, then $\sum_{k=1}^T M(r,k)\bar{U}_k < 0$

[3]Equivalently, autarky is the only sustainable contract if and only if $\det(M(1:T, 1:T)) \ge 0$ and $\det(M(1:s, 1:s)) > 0$ for all $s < T$.

holds when $\epsilon > 0$ is sufficiently small, because $M(r, s) < 0$ and $\bar{U}_s = 1$. ■

Remark 4. *As shown in the proof of Theorem 2, it takes only one Guassian elimination of M to verify the condition of this paper. This can be done easily in most numerical software.*

4. Conclusion

This paper establishes a necessary and sufficient condition for the existence of nonautarkic contracts in a model where two risk-averse agents face stochastic income and cannot commit. This condition is easy to verify as it boils down to computing determinants of matrices. The analysis of linearized utilities in the paper helps us understand the tradeoffs in an agent's participation decision.

References

Kehoe, P., Perri, F., 2002. International business cycles with endogenous incomplete markets. Econometrica 70 (3), 907–928.

Kocherlakota, N., 1996. Implications of efficient risk sharing without commmitment. Review of Economic Studies 63 (4), 595–609.

Krueger, D., Perri, F., 2011. Public versus private risk sharing. Journal of Economic Theory 146, 920–956.

Ligon, E., Thomas, J., Worrall, T., 2002. Informal insurance arrangements with limited commitment: Theory and evidence from village economies. Review of Economic Studies 69, 209–44.

Ljungqvist, L., Sargent, T., 2004. Recursive Macroeconomic Theory, Second Edition. The MIT Press, Cambridge, Massachusetts, USA.

Thomas, J., Worrall, T., 1988. Self-enforcing wage contracts. Review of Economic Studies 55, 541–554.