Decision making under risk with continuous states of nature

Mazurek Jiri

Silesian University - School of Business Administration

25. November 2012

Decision Making under Risk with Continuous States of Nature

Jiří Mazurek

Abstract: Many real-world decision making situation are associated with uncertainty regarding future state of the World. Traditionally, in such situation different (and discrete) *scenarios* – future states of nature – are considered. This domain of decision making is denoted as decision making under risk. However, limitation to some set of discrete scenarios is somewhat unnatural as future reality might not choose one of considered scenarios, but some other scenario or a scenario in between. The aim of this paper is to propose a more natural approach with continuum states of nature, where all scenarios expressed by their probability density function from some reasonable interval are taken into consideration. The approach is illustrated by a numerical example and is compared with the corresponding decision making under risk with discrete states of nature.

Keywords: continuous states of nature, decision making under risk, scenario, utility function.

JEL: D81

1. Introduction

Many real-world decision making situation in economics, politics, environmental protection, etc. are associated with uncertainty regarding future state of the World. When the probabilities of future states of the World (so called 'scenarios') are known, then the problem is referred as *decision making under risk*. When the probabilities of scenarios are not known, then the problem is referred as *decision making under uncertainty*. To deal with the both situations the *expected utility function approach* was proposed already by von Neumann and Morgenstern (1944): the best alternative is the alternative with the highest expected value of a utility function over all scenarios. Later, this approach was enriched by a notion of a *risk aversion*, see Arrow (1971) or Kahnemann and Tversky (1979), and today decision making under risk or uncertainty has numerous applications in many areas of human action, see for example Johnson and Busemayer (2010) or Abdellaoui et. al. (2007).

Now, let's consider economic situation at the beginning of the year 2012 from a point of view of a small enterprise. Will the world experience another recession or not? Should they hire new employees and expand, or should they rather turn into cost-saving regime?

Traditionally, in such situation different (and discrete) *scenarios* – future states of nature – are considered. Imagine that one possible scenario would be small economic growth (say 1 % of GDP), the second stagnation (0 % of GDP), and the third small decline (–1 % of GDP). Each scenario can be assigned its probability (e.g. by some financial expert). The best alternative for firm's behaviour can be evaluated from the utility function for all alternatives under all scenarios, with the best alternative achieving the highest value. This approach is called the *decision making under risk with discrete states of nature*.

However, one needn't limitate his thoughts to just three scenarios (cases), which are often somewhat artificial (why we use exactly 1 %, and not, let's say, 0.87 %?). More natural approach would be to evaluate all scenarios between some reasonable limits, say between –1 % to +1 % of GDP growth in the next year. This approach can be called the *decision making under risk with continuous states of nature*.

The aim of this article is to formulate the decision making under risk with continuum states of nature, illustrate its use on a numerical example, and to compare it with decision under risk with discrete states of nature.

The paper is organized as follows: Section 2 decision making under risk with discrete states of nature is briefly described, in Section 3 a continuum of states of nature is introduced and a numerical example is provided in Section 4. Conclusions close the article.

2. Decision making under risk with discrete states of nature

Let S_1 to S_n be future states of nature with probabilities p_1 to p_n, such that $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$; let A_1 to A_k be alternatives and v_{ij} pay-offs for *i*-th alternative under *j*-th state of nature. All this information can be described in a *decision matrix* (see Table 1). Furthermore, it is supposed that higher pay-offs are considered 'better'.

Table 1. A general form of a decision matrix in a decision making under risk.

	S_1	S_2	...	S_n
A_1	v_{11}	...	...	v_{1n}
A_2	...	...	...	...
...	...	...	...	...
A_k	v_{k1}	...	...	v_{kn}
	p_1	p_2	...	p_n

A utility function of an alternative *j* under all states of nature (all scenarios) is given as (Ramík and Perzina, 2008):

$$E_j = \sum_{i=1}^{n} p_i \cdot v_{ij} \tag{1}$$

Hence, each alternative *j* is assigned the number (expected value) E_j, and the best alternative is that with the maximum expected value:

$$\max_k E_k = \sum_{i=1}^{n} p_i \cdot v_{ij} \tag{2}$$

A numerical example of this approach is provided in Section 4.

3. Decision making under risk with continuous states of nature

Let $S(x)$ be a continuum of states of nature, $x \in [a,b]$, let $\upsilon_i(x)$ be a pay-off function for the *i*-th alternative and let $p(x)$ be a probability density function of states $S(x)$, satisfying the following conditions:

$$p(x) \geq 0 \text{ for } x \in [a,b] \tag{3a}$$

$$\int_a^b p(x)dx = 1 \tag{3b}$$

Then, by analogy with the utilitily function (1), each alternative *j* is assigned the number (expected value of a utility function) over all scenarios:

$$E_j = \int_a^b p(x) v_j(x) dx \tag{4}$$

Again, the best alternative attains the maximum expected value (4).

The venue of this approach is that we don't have to break our considerations into small number of (somewhat artificially and deliberately chosen) cases, as we are considering all possibilities (future scenarios) bounded by some limits *a* and *b*. Secondary, we can focus on arbitrary subinterval of interest between limits *a* and *b* (if there is a good reason to narrow the limits), which is not possible with discrete scenarios.

The weakness of this approach rests in the fact that we have to describe $p(x)$ and $v_1(x)$ as functions, which might be difficult in practice especially for the probability density function due to constraints (3a) and (3b). However, complicated functions can be approaximated by simpler ones, such as linear, quadratic, exponential or logarithmic functions.

4. A numerical example

In this section an illustrative example on both approaches is provided. We turn back to the example in Introduction: Let there be scenarios with the different economic growth (from 0 % to 4 % GDP) and three alternative strategies of some enterprise with the corresponding pay-offs. The goal is to find the best strategy under given assumptions.

At first we consider the discrete states of nature: Let S_1 to S_5 be (distinct) future states of nature with probabilities p_1 to p_5, such that $p_i \geq 0$ and $\sum_{i=1}^{5} p_i = 1$; let A_1 to A_3 be alternative strategies and let v_{ij} be pay-offs for *i*-th alternative under *j*-th state of nature, see Table 2.

Table 2. A decision matrix of an illustrative example.

	S_1 (0 %)	S_2 (1 %)	S_3 (2 %)	S_4 (3 %)	S_5 (4 %)
A_1	4	5	6	7	8
A_2	1	2	4	8	16
A_3	3	5	7	9	11
p_i	0	0.25	0.5	0.25	0

Using relation (1) we obtain the expected values of a utility function for all three alternative strategies:

$$E_1 = \sum_{i=1}^{5} p_i \cdot v_{ij} = 0 \cdot 4 + 0.25 \cdot 5 + 0.5 \cdot 6 + 0.25 \cdot 7 + 0 \cdot 8 = 6,$$

$$E_2 = 4.5,$$

$$E_3 = 7$$

Hence, the best strategy is A_3.

When continuous states of nature are considered, $p(x)$ and $v_i(x)$ for each alternative *i* must be expressed as functions. In this case we might construct these functions from the discrete case in Table 2, as row values of Table 2 can be regarded as an output of linear or exponential functions. The function $p(x)$ for all scenarios is shown in Figure 1.

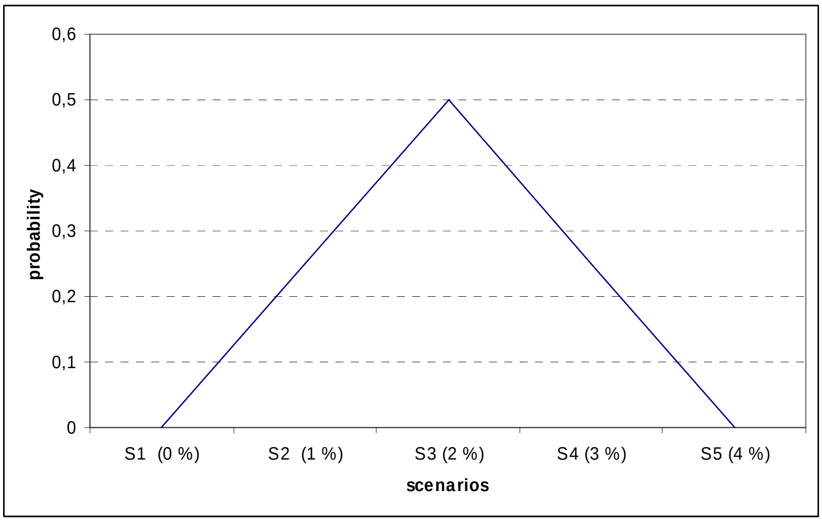

Figure 1. The probability density function of all scenarios.

Let $p(x) = 0.25x$ for $x \in [0, 2]$ and $p(x) = 1 - 0.25x$ for $x \in [2, 4]$, (it is easy to verify that constraints (3a) and (3b) for the probability density function are satisfied),

$v_1(x) = x + 4$,

$v_2(x) = 2^{x-1}$,

$v_3(x) = 2x + 3$,

$a = 0, b = 4$.

Using relation (4) we obtain the expected values of a utility function for all three alternative strategies:

$$E_1 = \int_0^4 p(x)v_1(x)dx = \int_0^2 0.25x \cdot (x+4)\,dx + \int_2^4 (1-0.25x) \cdot (x+4)\,dx = 6,$$

$$E_2 = 4.683,$$

$$E_3 = 7.$$

The highest expected value is attained by the strategy A_3. This result is the same as for the discrete case (because pay-off functions were mostly linear).

Nevertheless, the continuous approach is more general in nature, as $p(x)$ and $v_i(x)$ are functions, not only given numbers, which opens more space for modelling scenario probabilities as well as pay-offs of alternatives. Also, this approach can be easily extended to group decision making under risk with the use of suitable aggregation operators. A risk aversion could be included into a continuous model as well.

5. Conclusions

The aim of this article was to demonstrate the use of a decision making under risk with continuum states of nature, which can be considered more natural and also more general approach than a decision making under risk with discrete states of nature. Future research might focus for example on a decision making under uncertainty or a decision making with fuzzy scenarios.

References

Abdellaoui, M., Luce, R. D., Machina, M. J., Munier, B. (Eds.). (2007). Uncertainty and Risk: Mental, Formal, Experimental Representations, series: *Theory and Decision Library* C, Vol. 41.

Arrow, K. J. (1971). The theory of risk aversion. *Essays in the Theory of Risk Bearing*, Markham Publ. Co., Chicago, pp. 90-109.

Johnson, J. G., Busemeyer, J. R. (2010). Decision making under risk and uncertainty. *Wiley Interdisciplinary Reviews: Cognitive Science*, Vol. 1, No. 5, pp. 736–749.

Kahneman, D., Tversky, A. (1979). Prospect Theory: An Analysis of Decision under Risk. *Econometrica*, Vol. 47, No. 2, pp. 263-292.

Neumann von, J., Morgenstern, O. (1944). *Theory of Games and Economic Behavior*, Princeton, NJ, Princeton University Press.

Ramík, J., Perzina, R. (2008). *Modern methods of evaluation and decision making* (*Moderní metody hodnocení a rozhodování*). OPF SU, Karviná.