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ABSTRACT

Consider the model $Y_i = x_i^{'}\beta + g(t_i) + V_i$, $1 \le i \le n$. Here $x_i = (x_{i1}, \ldots, x_{ip})^{'}$ and t_i are known and nonrandom design points, $\beta = (\beta_1, \ldots, \beta_p)^{'}$ is an unknown parameter, $g(\cdot)$ is an unknown function over R^1, and V_i is a class of linear processes. Based on $g(\cdot)$ estimated by nonparametric kernel estimation or approximated by a finite series expansion, the asymptotic normalities and the strong consistencies of the LS estimator of β and an estimator of $\sigma_0^2 = EV_1^2$ are investigated.

1. INTRODUCTION

Consider the model given by

$$Y_i = x_i^{'}\beta + g(t_i) + V_i,\ i = 1,2,\ldots, \tag{1.1}$$

where $x_i = (x_{i1},\ldots,x_{ip})^{'}$ and $t_i \in T \subset R^1$ are known and non-random design points, $\beta = (\beta_1,\ldots,\beta_p)^{'}$ is an unknown parameter, $g(\cdot)$ is an unknown function over R^1, and V_i is a class of linear processes defined by Assumption 1 below.

The model defined in (1.1) belongs to a class of partly linear regression models, which was first discussed by Ansley & Wecker(1983) using a state space approach. Other related work is that of Heckman(1986), Rice(1986), Chen(1988), Speckman (1988), Robinson(1988), Andrews(1991), Eubank, et al(1993), and Gao, et al(1994a), which discussed asymptotic properties of some estimates under the case where V_i are i.i.d. random errors and (x_i,t_i) are i.i.d. random design points. More recently, Gao and Zhao(1993) investigated the asymptotic normality of the LS estimator of β for the case where the (x_i,t_i) are fixed, V_i are i.i.d. random disturbances with $EV_i = 0$ and $EV_i^2 = \sigma_0^2 < \infty$, and $g(\cdot)$ is estimated by a class of kernel estimates.

In this paper, we investigate the asymptotic normalities and the rates of strong convergence of the LS estimator of β and an estimator of $\sigma_0^2 = EV_i^2$ for the case where $g(\cdot)$ is estimated by a class of nonparametric kernel estimates or approximated by a finite series expansion $\sum_{i=1}^{k} f_i(\cdot)\gamma_i$, where $\{f_i(\cdot), i = 1,2,\ldots,\}$ is a prespecified family of functions from

$T \subset R^1$ to R^1 and $\gamma = (\gamma_1, \ldots, \gamma_k)'$ is an unknown parameter vector.

2. MAIN RESULTS

Consider the model (1.1). For constructing some estimates and stating the main results of this paper, we introduce the following assumptions.

Assumption 1. Suppose $\{V_i\}$ is the linear process

$$V_i = C(L)e_i = \sum_{j=0}^{\infty} c_j e_{i-j}, \quad C(L) = \sum_{j=0}^{\infty} c_j L^j \tag{2.1}$$

with $0 < |C(1)| < \infty$ and $C(L) = \sum_{j=0}^{\infty} c_j^2 < \infty$, and one of the following assumptions is satisfied

(i) e_i are i.i.d. with zero mean and $\sigma_e^2 = Ee_0^2 < \infty$;

(ii) e_i are i.i.d. with zero mean and fourth cumulant μ_4;

(iii) e_i are i.i.d. with zero mean and $E|e_0|^p < \infty$ for some $2 \le p < \infty$.

Assumption 2. (i) $\{F_{nk}\}$ is full column rank k_n for n large;

(ii) $\max_{1 \le i \le n} p_{ii} \to 0$ and $\max_{1 \le i, j \le n} |p_{ij}| (\log n)^2 \to 0$ as $n \to \infty$, where $\{p_{ij}\}$ denotes the ith row and jth rank element of the $n \times n$ projection matrix $F_{nk}(F_{nk}' F_{nk})^+ F_{nk}'$, and where $F_{nk} = (F_k(t_1), \ldots, F_k(t_n))'$, $F_k(\cdot) = (f_1(\cdot), \ldots, f_k(\cdot))'$, and $(\cdot)^+$ denotes the Moore-Penrose inverse.

Assumption 3. For $k = k_n < n - p$, $k_n \to \infty$ as $n \to \infty$ and $\{f_i(\cdot), i = 1, 2, \ldots\}$ given above, there exists an unknown parameter vector $\gamma = (\gamma_1, \ldots, \gamma_k)'$ such that

$$\sup_{t\in T}|\sum_{i=1}^{k} f_i(t)\gamma_i - g(t)| = o(n^{-1/4}). \tag{2.2}$$

Assumption 4. There exist some bounded functions $h_j(\cdot)$ over R^1 such that

$$x_{ij} = h_j(t_i) + u_{ij},\ 1 \le i \le n,\ 1 \le j \le p, \tag{2.3}$$

where $u_i = (u_{i1},\ldots,u_{ip})^\tau$ are real sequences satisfying

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} u_i u_i^{'} = B \tag{2.4}$$

and

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}\log n} \max_{1\le m\le n} \|\sum_{i=1}^{m} u_{j_i}\| < \infty \tag{2.5}$$

for any permutation $(j_1,\ldots,j_n)$ of the integers $(1,2,\ldots,n)$, where B is a positive definite matrix with the order $p\times p$ and $\|\cdot\|$ denotes the Euclidean norm.

Moreover

$$\max_{1\le i\le n} \|u_i\| \le C < \infty. \tag{2.6}$$

Assumption 5. For $k = k_n < n - p$, $k_n \to \infty$ as $n \to \infty$ and $\{f_i(\cdot),\ i = 1,2,\ldots\}$ given above, there exist unknown parameter vectors $\tilde{\gamma}_j = (\gamma_{1j},\ldots,\gamma_{kj})^{'}$ such that for $1 \le j \le p$

$$\sup_{t\in T}|\sum_{i=1}^{k} f_i(t)\gamma_{ij} - h_j(t)| = o(n^{-1/4}). \tag{2.7}$$

Assumption 6. $g(\cdot)$ and $h_j(\cdot)$ satisfy Lipschitz condition of order 1 on T (a compact subset of R^1).

Assumption 7. The probability weight functions $W_{ni}(\cdot)$ satisfy

(i) $\max_{1\le i\le n} \sum_{j=1}^{n} W_{ni}(t_j) = O(1)$,

(ii) $\max_{1\le i,j\le n} W_{ni}(t_j) = O(b_n)$,

$$\text{(iii)} \quad \max_{1\le j\le n} \sum_{i=1}^{n} W_{ni}(t_j) I(|t_j - t_i| > c_n) = O(d_n),$$

where $b_n = o(n^{-1/2}(\log n)^{-2})$, c_n satisfy $\limsup_{n\to\infty} nc_n^4 \log n < \infty$ and d_n satisfy $\limsup_{n\to\infty} nd_n^4 \log n < \infty$.

Remark 1. Assumptions 2(ii)(iii) restrict the growth rate of k_n. For example, they hold for the polynomial, trigonometric, and Fourier flexible form (FFF) functions. (See Gallant(1981)).

Remark 2. Assumptions 3 and 5 are some smoothness conditions. In almost all cases, the assumptions 3 and 5 hold if $g(\cdot)$ and $h_j(\cdot)$ are sufficiently smooth (See Schumaker(1981)). More generally, they hold whenever T is compact.

Remark 3. (i) The above u_{ij} behave like zero mean, uncorrelated random variables, and $h_j(t_i)$ are the regression of x_{ij} on t_i. Specifically, suppose that the design points (x_i, t_i) are i.i.d. random variables, and let $h_j(t_i) = E(x_{ij}|t_i)$ and $u_{ij} = x_{ij} - h_j(t_i)$ with $Eu_i u_i^{'} > 0$. Then (2.4) holds with probability 1 by applying the law of strong numbers and (2.5) holds with probability 1 by using the same reason as in the proof of Lemma 4 below.
(ii) As a matter of fact, it is easy to show that $\{h_j(\cdot)\}$ is determined uniquely under the Assumptions 4, 6 and 7. This is necessary for the explanation of the uniqueness of B in (2.4).
(iii) The above Assumption 4(2.6) is an additional assumption for constructing the law of the iterated logarithm of some estimators, which should be replaced by $E\|u_i\|^4 < \infty$ for the case where $\{u_i\}$ is i.i.d. random error.

Remark 4. As a matter of fact, there exist some weights satisfying Assumption 7. For weights

$$W_{ni}^{(1)}(t) = \frac{1}{H_n} \int_{s_{i-1}}^{s_i} K(\frac{t-s}{H_n}) ds$$

and

$$W_{ni}^{(2)}(t) = K(\frac{t-t_i}{H_n}) / \sum_{j=1}^{n} K(\frac{t-t_j}{H_n}),$$

where $s_i = 2^{-1}(t_i + t_{i+1})$, $1 \le i \le n-1$, $0 \le t_1 \le \ldots \le t_n \le 1$, $s_0 = 0$, $s_n = 1$, $K(\cdot)$ is a kernel function, H_n is a positive number sequence, where $H_n = h_n$ or r_n, h_n is bandwidth parameter and $r_n = r_n(t;t_1,\ldots,t_n)$ is the distance from t to the k_n-th nearest neighbor among the $t_i^{'}s$, and where k_n is an integer sequence. The details of justification follow by the similar reason as in Gao(1992).

Now we begin to define some estimators.

Assume that $\{x_i, t_i, Y_i; 1 \le i \le n\}$ satisfies the model

$$Y_i = x_i^{'}\beta + g(t_i) + V_i, \; i = 1,2,\ldots,n. \tag{2.8}$$

If β is known to be the true parameter, then by $EV_i = 0$ we have

$$g(t_i) = E(Y_i - x_i^{'}\beta), \; i = 1,2,\ldots,n. \tag{2.9}$$

Hence, the natural estimator of g(·) is

$$\hat{g}_n(t) = \hat{g}_n(t,\beta) = \sum_{i=1}^{n} W_{ni}(t)(Y_i - x_i^{'}\beta), \tag{2.10}$$

where $W_{ni}(t) = W_{ni}(t;t_1,\ldots,t_n)$ are defined as in Assumption 7 below.

Now, based on the model $Y_i = x_i^{'}\beta + \hat{g}_n(t_i) + V_i$, the LS estimator $\hat{\beta}$ of β can be defined by

$$\sum_{i=1}^{n}(Y_i - x_i'\hat{\beta}_n - \hat{g}_n(t_i,\hat{\beta}_n))^2 = \min!. \tag{2.11}$$

By (2.11) we obtain

$$\hat{\beta}_n = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{Y}, \tag{2.12}$$

where $\tilde{x}_i = x_i - \sum_{j=1}^{n} W_{nj}(t_i)x_j$, $\tilde{Y}_i = Y_i - \sum_{j=1}^{n} W_{nj}(t_i)Y_j$, $\tilde{X} = (\tilde{x}_1,\ldots,\tilde{x}_n)'$, and $\tilde{Y} = (\tilde{Y}_1,\ldots,\tilde{Y}_n)'$.

Next, by using Assumption 3 we define the LS estimators $\overline{\beta}_n$ and $\overline{g}_n(\cdot) = F_k(\cdot)'\overline{\theta}_n$ of β and $g(\cdot)$ by

$$\sum_{i=1}^{n}(Y_i - x_i'\overline{\beta}_n - F_k(t_i)'\overline{\theta}_n)^2 = \min!. \tag{2.13}$$

By (2.13) we get

$$\overline{\beta}_n = (\overline{X}'\overline{X})^{-1}\overline{X}'\overline{Y}, \tag{2.14}$$

where $\overline{X} = (I-P)X$, $\overline{Y} = (I-P)Y$, and $P = F(F'F)^{+}F'$.

On the other hand, by Lemma 2 below we have

$$\lim_{n\to\infty} n^{-1}\tilde{X}'\tilde{X} = B \tag{2.15}$$

and

$$\lim_{n\to\infty} n^{-1}\overline{X}'\overline{X} = B. \tag{2.16}$$

So we can assume that $(\tilde{X}'\tilde{X})^{-1}$ and $(\overline{X}'\overline{X})^{-1}$ exist as n large enough.

Now, we give the main results of this paper.

Theorem 1. (i) Assume that Assumptions 4, 6, 7, and 1(i) with $\sum_{j=1}^{\infty} j^2 c_j^2 < \infty$ hold. Then as $n \to \infty$

$$\sqrt{n}(\hat{\beta}_n - \beta) \to_D N(0, \sigma^2 B^{-1}), \tag{2.17}$$

where $\sigma^2 = \sigma_e^2 C(1)^2$.

(ii) Assume that Assumptions 4, 6, 7, and 1(iii) with $\sum^{\infty} j|c_j| < \infty$

hold. Then

$$\limsup_{n\to\infty}\left(\frac{n}{2\log\log n}\right)^{1/2}|\hat{\beta}_{nj}-\beta_j| = (\sigma^2 b^{jj})^{1/2} \text{ a.s.,} \qquad (2.18)$$

where $\hat{\beta}_{nj}$ and β_j denote the jth components of $\hat{\beta}_n$ and β respectively, and $\{b^{jj}\}$ denotes the jth diagonal element of the $p \times p$ matrix B^{-1}.

Theorem 2. (i) Assume that Assumptions 4, 6, 7, and 1(ii) with $\sum_{j=1}^{\infty} jc_j^2 < \infty$ hold. Then as $n \to \infty$

$$\sqrt{n}(\hat{\sigma}_n^2 - \sigma_0^2) \to_D N(0, V_0^2), \qquad (2.19)$$

where

$\hat{\sigma}_n^2 = n^{-1}\sum_{i=1}^{n}(\tilde{Y}_i - \tilde{x}_i'\hat{\beta}_n)^2$, $\sigma_0^2 = \sigma_e^2\bar{c}_0$, $V_0^2 = \mu_4\bar{c}_0^2 + 2\sigma_e^2\sum_{j=-\infty}^{\infty}\bar{c}_j^2$, an

$\bar{c}_j = \sum_{i=0}^{\infty} c_i c_{i+j}$.

(ii) Assume that Assumptions 4, 6, 7, and 1(i) with $\sum_{j=1}^{\infty} j|c_j| < \infty$ hold. Then as $n \to \infty$

$$\lim_{n\to\infty}\hat{\sigma}_n^2 = \sigma_0^2 \text{ a.s.} \qquad (2.20)$$

Theorem 3. (i) Assume that Assumptions 2 through 5 and 1(i) with $\sum_{j=1}^{\infty} j^2 c_j^2 < \infty$ hold. Then as $n \to \infty$

$$\sqrt{n}(\bar{\beta}_n - \beta) \to_D N(0, \sigma^2 B^{-1}). \qquad (2.21)$$

(ii) Assume that Assumptions 2 through 5 and 1(iii) with $\sum_{j=1}^{\infty} j|c_j| < \infty$ hold. Then

$$\limsup_{n\to\infty}\left(\frac{n}{2\log\log n}\right)^{1/2}|\bar{\beta}_{nj}-\beta_j| = (\sigma^2 b^{jj})^{1/2} \text{ a.s.,} \qquad (2.22)$$

where $\{\bar{\beta}_{nj}\}$ denotes the jth component of $\bar{\beta}_n$.

Theorem 4. (i) Assume that Assumptions 2 through 5 and 1(ii) with $\sum_{j=1}^{\infty} jc_j^2 < \infty$ hold. Then as $n \to \infty$

$$\sqrt{n}(\overline{\sigma}_n^2 - \sigma_0^2) \to_D N(0, V_0^2), \tag{2.23}$$

where $\overline{\sigma}_n^2 = n^{-1}\overline{Y}'(I - P_{\overline{X}})\overline{Y}$ and $P_{\overline{X}} = \overline{X}(\overline{X}'\overline{X})^{-1}\overline{X}'$.

(ii) Assume that Assumptions 2 through 5 and 1(i) with $\sum_{j=1}^{\infty} j|c_j| < \infty$ hold. Then as $n \to \infty$

$$\lim_{n\to\infty} \overline{\sigma}_n^2 = \sigma_0^2 \text{ a.s.} \tag{2.24}$$

Remark 5. The above theorems only give the asymptotics for the estimators of β and σ_0^2. In fact, some results about the estimators of $g(\cdot)$ can be obtained easily.

Based on (2.10) and (2.12), we can define the estimator of $g(\cdot)$ by

$$\hat{g}_n^*(t) = \sum_{i=1}^{n} W_{ni}(t)(Y_i - x_i'\hat{\beta}_n). \tag{2.25}$$

Also, based on (2.13) the estimator of $g(\cdot)$ can be defined by

$$\overline{g}_n(t) = F_k(t)'\overline{\theta}_n. \tag{2.26}$$

Now, by Theorems 1 and 3 we can obtain the strong and weak convergence rates of $\hat{g}_n^*$ and $\overline{g}_n$. Here we omit the details for they are trivial.

Remark 6. Under the case where (x_i, t_i) are i.i.d. random variables, the above Assumption 4 should be replaced by Assumption 4°. $E\|x_i\|^2 < \infty$, $\max_{1\le j\le p} \sup_t E(x_{ij}|t)^2 < \infty$, and $B = Cov(x_1 - E(x_1|t_1))$ is positive definite.

Now, we give the corresponding results for the case where (x_i, t_i) are i.i.d. random variables.

Theorem 1^{o}. (i) Assume that Assumptions 4, 6, and 7 with probability 1 and 1(i) with $\sum_{j=1}^{\infty} j^2 c_j^2 < \infty$ hold. Let (x_i, t_i) and V_i be independent. Then (2.17) holds.

(ii) Assume that Assumptions 4, 6, and 7 with probability 1 and 1(iii) with $\sum_{j=1}^{\infty} j|c_j| < \infty$ hold. Let (x_i, t_i) and V_i be independent. Then (2.18) holds.

Also, the modifications of Theorems 2 through 4 follow by the similar reason as Theorem 1^{o}.

The proofs of Theorems 1 through 4 will be given in the following section. The proof of Theorem 1^{o} through 4^{o} will be obtained by the similar reason as those of Theorems 1 through 4.

3. PROOFS OF MAIN RESULTS

3.1 For proving the above theorems, we introduce the following lemmas.

Lemma 1. Let $C(L) = \sum_{j=0}^{\infty} c_j L^j$. Then

$$C(L) = C(1) - (1 - L)\tilde{C}(L),$$

where $\tilde{C}(L) = \sum_{j=0}^{\infty} \tilde{c}_j L^j$ and $\tilde{c}_j = \sum_{k=j+1}^{\infty} c_k$. If $p \geq 1$, then

$$\sum_{j=1}^{\infty} j^p |c_j|^p < \infty \Rightarrow \sum_{j=0}^{\infty} |\tilde{c}_j|^p < \infty \text{ and } |C(1)| < \infty.$$

If $p < 1$, then

$$\sum_{j=1}^{\infty} j^p |c_j|^p < \infty \Rightarrow \sum_{j=0}^{\infty} |\tilde{c}_j|^p < \infty.$$

Lemma 2. (i) Assume that Assumptions 6 and 7(iii) hold. Then as $n \to \infty$

$$\max_{0\le j\le p} \max_{1\le i\le n} |G_{ij}|$$
$$= \max_{0\le j\le p} \max_{1\le i\le n} |G_j(t_i) - \sum_{k=1}^{n} W_{nk}(t_i) G_j(t_k)|$$
$$= O(c_n) + O(d_n), \tag{3.1}$$

where $G_0(\cdot) = g(\cdot)$ and $G_j(\cdot) = h_j(\cdot)$ $(1 \le j \le p)$.

(ii) Assume that Assumptions 4, 6, and 7 hold. Then as $n \to \infty$

$$\max_{1\le j\le p} \max_{1\le i\le n} |\hat{h}_{nj}(t_i) - h_j(t_i)|$$
$$= O(c_n) + O(d_n) + O(a_n b_n), \tag{3.2}$$

where $\hat{h}_{nj}(t_i) = \sum_{k=1}^{n} W_{nk}(t_i) x_{kj}$ and $a_n = n^{1/2} \log n$.

Proof: The proof of (3.1) is trivial.

Here we only prove (3.2).

To prove (3.2), note that (3.1), it suffices to show that

$$\max_{1\le j\le p} \max_{1\le i\le n} |\sum_{k=1}^{n} W_{nk}(t_i) u_{kj}| = O(a_n b_n). \tag{3.3}$$

For proving (3.3), we introduce the following elementary inequality (see P_{32}, Theorem 1 of Mitrinovic(1970)).

Abel inequality: Let $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ $(b_1 \ge b_2 \ge \ldots \ge b_n \ge 0)$ be two sequences of real numbers. Let $S_k = \sum_{i=1}^{k} a_i$, $m_n = \min_{1\le k\le n} S_k$, and $M_n = \max_{1\le k\le n} S_k$. Then

$$b_1 m_n \le \sum_{i=1}^{n} a_i b_i \le b_1 M_n. \tag{3.4}$$

For applying Abel's inequality, let $a_k = u_{k1}$ and $b_k = W_{nk}(t_1)$.

Without loss of generality (W.l.o.g.), we can assume that $b_{j_1} \ge b_{j_2} \ge \ldots \ge b_{j_n}$ for some permutation $(j_1, \ldots, j_n)$ of the integers

$(1,2,\ldots,n)$. Now applying Abel's inequality,

$$|\sum_{i=1}^{n} a_i b_i| = |\sum_{i=1}^{n} a_{j_i} b_{j_i}|$$

$$\leq b_{j_1} (\min_{1\leq m\leq n} |\sum_{i=1}^{m} a_{j_i}| + \max_{1\leq m\leq n} |\sum_{i=1}^{m} a_{j_i}|)$$

$$\leq 2 \max_{1\leq i\leq n} |b_{j_i}| \cdot \max_{1\leq m\leq n} |\sum_{i=1}^{m} a_{j_i}| = O(b_n a_n). \tag{3.5}$$

Now, we have completed the proof of Lemma 2.

Lemma 3. (i) Assume that Assumptions 4, 6, and 7 hold. Then

$$\lim_{n\to\infty} \frac{1}{n} \tilde{X}' \tilde{X} = B. \tag{3.6}$$

(ii) Assume that Assumptions 2, 4, and 5 hold. Then

$$\lim_{n\to\infty} \frac{1}{n} \overline{X}' \overline{X} = B. \tag{3.7}$$

Proof: (i) By Assumption 4, we have

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} u_i u_i' = B = (b_{jk})_{1\leq j,k\leq p}. \tag{3.8}$$

On the other hand, we obtain

$$\sum_{i=1}^{n} \overline{h}_{nj}(t_i) u_{ik} = \sum_{i=1}^{n} h_{nij} u_{ik} - \sum_{i=1}^{n} \sum_{s=1}^{n} W_{ns}(t_i) u_{sj} u_{ik} = I_{jk} - J_{jk}, \tag{3.9}$$

where $\overline{h}_{nj}(t_i) = h_j(t_i) - \hat{h}_{nj}(t_i)$ and $h_{nij} = h_j(t_i) - \sum_{k=1}^{n} W_{nk}(t_i) h_j(t_k)$.

In the following, we only prove $J_{jk} = o(n)$, the other follows by the same reason.

By the similar reason as (3.5), and applying Abel's inequality, we can show that

$$|J_{jk}| = |\sum_{i=1}^{n} (\sum_{s=1}^{n} W_{ns}(t_i) u_{sj}) u_{ik}| \leq O(b_n a_n^2) = o(n). \tag{3.10}$$

Thus, by Lemma 2(ii) we have for all (j,k) and $n \to \infty$

$$n^{-1}\sum_{i=1}^{n}\tilde{x}_{ij}\tilde{x}_{ik} = n^{-1}\sum_{i=1}^{n}u_{ij}u_{ik} + n^{-1}\sum_{i=1}^{n}\overline{h}_{nj}(t_i)u_{ik}$$

$$+n^{-1}\sum_{i=1}^{n}\overline{h}_{nk}(t_i)u_{ij} + n^{-1}\sum_{i=1}^{n}\overline{h}_{nj}(t_i)\overline{h}_{nk}(t_i) \to b_{jk}. \qquad (3.11)$$

We now finish the proof of Lemma 3(i). The proof of Lemma 3(ii) follows from the proof of Lemma 6.1 of Gao (1994b).

Lemma 4. (i) Assume that Assumption 1(i) holds. Then for any permutation $(j_1,\ldots,j_n)$ of the integers $(1,2,\ldots,n)$

$$\max_{1\le k\le n}|\sum_{i=1}^{k}e_{j_i}| = O(n^{1/2}\log n) \text{ a.s..} \qquad (3.12)$$

(ii) Assume that Assumption 7 holds. Let $Ee_1 = 0$ and $Ee_1^{\,4} < \infty$. Then

$$\max_{1\le i\le n}|\sum_{k=1}^{n}W_{nk}(t_i)e_k| = o(n^{-1/4}(\log n)^{-1/2}) \text{ a.s..} \qquad (3.13)$$

Proof: Here we only prove (3.13), the other follows by the same reason.

Let $e_i^{'} = e_i I(|e_i| \le i^{1/4})$ and $e_i^{''} = e_i - e_i^{'}$.

Now, by Assumptions 7(i) and (ii), and applying Bennett's inequality (see Bennett(1962)), for some $C_1 > 0$ and $C_2 > 0$

$$P(\max_{1\le i\le n}|\sum_{k=1}^{n}W_{nk}(t_i)(e_k^{'} - Ee_k^{'})| \ge C_1 n^{-1/4}(\log n)^{-1/2})$$

$$\le \sum_{i=1}^{n}P(|\sum_{k=1}^{n}W_{nk}(t_i)(e_k^{'} - Ee_k^{'})| \ge C_1 n^{-1/4}(\log n)^{-1/2})$$

$$\le 2n\exp\left(-\frac{C_1^{\,2}n^{-1/2}(\log n)^{-1}}{(2\max_{1\le i\le n}\sum_{k=1}^{n}W_{nk}(t_i)^2 Ee_1^{\,2} + 2C_1 b_n(\log n)^{-1/2})}\right)$$

$$\le 2n\exp(-C_1^{\,2}C_2\log n). \qquad (3.14)$$

Thus, by choosing the proper C_1 and applying the Borel-

$$\max_{1\le i\le n}|\sum_{k=1}^{n} W_{nk}(t_i)(e_k^{'} - Ee_k^{'})| = O(n^{-1/4}(\log n)^{-1/2}) \text{ a.s.} \quad (3.15)$$

Next, by $Ee_1^4 < \infty$ we obtain

$$\sum_{i=1}^{\infty}|e_i^{''}| < \infty \text{ a.s.} \quad (3.16)$$

and

$$\max_{1\le i\le n}|\sum_{k=1}^{n} W_{nk}(t_i)e_k^{''}| \le O(b_n)\sum_{k=1}^{n}|e_k^{''}| = o(n^{-1/2}(\log n)^{-2}). \quad (3.17)$$

Also, by $Ee_1^4 < \infty$ again,

$$\max_{1\le i\le n}|\sum_{k=1}^{n} W_{nk}(t_i)Ee_k^{''}| \le O(b_n)\sum_{k=1}^{n} E|e_k| I(|e_k| \ge k^{1/4})$$

$$\le O(b_n)\sum_{k=1}^{n} k^{-3/4} Ee_1^4 = o(n^{-1/4}(\log n)^{-2}). \quad (3.18)$$

Therefore, (3.13) follows from (3.15) through (3.18).

Lemma 5. Assume that Assumption 1(ii) with $\sum_{s=1}^{\infty} sc_s^2 < \infty$ holds.

Then as $n \to \infty$

$$n^{-1/2}\sum_{i=1}^{n}(V_i^2 - \sigma_0^2) \to_D N(0, V_0^2). \quad (3.19)$$

Proof: See Theorem 3.8 of Phillips & Solo(1992).

3.2 Proofs of Theorems

(i) Now, we begin to prove (2.17).

By (2.12) we have

$$\hat{\beta}_n - \beta = (\sum_{i=1}^{n}\tilde{x}_i\tilde{x}_i^{'})^{-1}(\sum_{i=1}^{n}\tilde{x}_iV_i - \sum_{i=1}^{n}\tilde{x}_i\overline{V}_i + \sum_{i=1}^{n}\tilde{x}_i g_{ni}), \quad (3.20)$$

where $\overline{V}_i = \sum_{k=1}^{n} W_{nk}(t_i)V_k$ and $g_{ni} = g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)g(t_k)$.

First, by the similar reason as the proof of (3.5), and using Lemma 2 we have for $n \to \infty$ and all $1 \le j \le p$

$$\sum_{i=1}^{n}\tilde{x}_{ij}g_{ni} = \sum_{i=1}^{n}u_{ij}g_{ni} + \sum_{i=1}^{n}h_{nij}g_{ni} - \sum_{k=1}^{n}(\sum_{i=1}^{n} W_{nk}(t_i)g_{ni})u_{kj} = o(n^{1/2}), \quad (3.21)$$

where $h_{nij} = h_j(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)h_j(t_k)$.

Also, we obtain

$$\sum_{i=1}^{n}\tilde{x}_{ij}\overline{V}_i = \sum_{i=1}^{n}(\sum_{k=1}^{n} W_{ni}(t_k)u_{kj})V_i + \sum_{i=1}^{n}(\sum_{k=1}^{n} W_{ni}(t_k)h_{nkj})V_i$$

$$-\sum_{i=1}^{n}(\sum_{k=1}^{n}(\sum_{s=1}^{n} W_{ns}(t_k)u_{sj})W_{ni}(t_k))V_i = J_{1j} + J_{2j} + J_{3j}(say). \quad (3.22)$$

In the following, we only prove $J_{1j} = o_p(n^{1/2})$, the others follow by the same reason.

On the other hand, applying Lemma to Assumption 1(2.1) we get

$$V_i = C(1)e_i + \tilde{e}_{i-1} - \tilde{e}_i, \quad (3.23)$$

where $\tilde{e}_i = \tilde{C}(L)e_i = \sum_{j=0}^{\infty}\tilde{c}_j e_{i-j}$ and $\tilde{c}_j = \sum_{k=j+1}^{\infty} c_k$.

Now sum (3.23) to find

$$\sum_{i=1}^{m} V_{j_i} = C(1)\sum_{i=1}^{m} e_{j_i} + \tilde{e}_0 - \tilde{e}_{j_m}, \ 1 \le m \le n. \quad (3.24)$$

For simplicity and without loss of generality, it can be assumed that $(j_1, j_2, \ldots, j_n) = (1, 2, \ldots, n)$ in Assumption 4(2.5).

Thus, by the similar reason as the proof of (3.5) we obtain for some $C > 0$

$$|J_{1j}| \le C\max_{1\le m\le n}|\sum_{i=1}^{m} e_i|\cdot \max_{1\le s\le n}|\sum_{i=1}^{s} u_{ij}|\cdot \max_{1\le i,k\le n} W_{ni}(t_k)$$

$$+C\max_{1\le s\le n}|\sum_{i=1}^{s} u_{ij}|\cdot \max_{1\le i,k\le n} W_{ni}(t_k)\cdot(|\tilde{e}_0| + \max_{1\le m\le n}|\tilde{e}_m|). \quad (3.25)$$

Now note that Assumption 4(2.5), Assumption 7(ii), and Lemma 4, in order to prove

$$J_{1j} = o_p(n^{1/2}), \quad (3.26)$$

it suffices to show that

$$n^{-1}\tilde{e}_0^{\,2} \to_p 0 \text{ and } n^{-1}\max_{1\le m\le n}|\tilde{e}_m|^2 \to_p 0. \quad (3.27)$$

The former holds if $E|\tilde{e}_0|^2 < \infty$, which holds if $\sum_{j=1}^{\infty} j^2 c_j^{\,2} < \infty$.

The latter is equivalent to

$$n^{-1}\sum_{m=1}^{n} \tilde{e}_m^{\,2} I(\tilde{e}_m^{\,2} > nc) \to_p 0 \text{ for any } c > 0 \tag{3.28}$$

[cf. Hall & Heyde(1980), page 53, and (36) below].

But (3.28) holds because

$$E[\tilde{e}_0^{\,2} I(\tilde{e}_0^{\,2} > nc)] \to 0 \tag{3.29}$$

by dominated convergence since $\sum_{j=1}^{\infty} j^2 c_j^{\,2} < \infty$ ensures that $E|\tilde{e}_0|^2 < \infty$.

Secondly, by Assumption 4(2.3) we have

$$\sum_{i=1}^{n} \tilde{x}_i V_i = \sum_{i=1}^{n} u_i V_i + \sum_{i=1}^{n} H_{ni} V_i - \sum_{i=1}^{n} (\sum_{k=1}^{n} W_{nk}(t_i) u_k) V_i$$
$$= M_{1n} + M_{2n} + M_{3n} (say), \tag{3.30}$$

where $H_{ni} = (h_{ni1}, \ldots, h_{nip})'$.

By the same reason as (3.23) through (3.29), and using Lemmas 2 and 4, and Assumptions 4(2.5) and 7(ii), we obtain for $i = 2,3$

$$M_{in} = o_p(n^{1/2}). \tag{3.31}$$

On the other hand, by (3.23) we get

$$M_{1n} = \sum_{i=1}^{n} u_i V_i = C(1)\sum_{i=1}^{n} u_i e_i + \sum_{i=1}^{n} u_i(\tilde{e}_{i-1} - \tilde{e}_i)$$
$$= M_{4n} + M_{5n}. \tag{3.32}$$

And by the similar reason as (3.25), and using Assumption 4(2.6), for all $1 \le j \le p$

$$M_{5n} = \sum_{i=1}^{n} u_i(\tilde{e}_{i-1} - \tilde{e}_i) = o_p(n^{1/2}). \tag{3.33}$$

Now, note that (3.20) through (3.33), in order to prove (2.17), it suffices to show that

$$n^{1/2}(\sum_{i=1}^{n} u_i u_i^{'})^{-1} M_{4n} \to_D N(0, C(1)^2 \sigma_e^{\,2} B^{-1}), \tag{3.34}$$

which holds because the Lindeberg condition is met by Assumption 4(2.4) and Lemma 3(i). (Here Lemma 3 of Wu(1981) and Proposition 2.2 of Huber(1973) may be used for checking the Lindeberg condition).

(ii) In the following, we will prove Theorem 1(ii).

In order to prove Theorem 1(ii), note that (3.20) through (3.25) and (3.30) through (3.33), it suffices to show that for $i = 1,2,3$ and $k = 2,3$

$$J_{ij} = o((n \log\log n)^{1/2}) \text{ a.s.,} \tag{3.35}$$

$$M_{kn} = o((n \log\log n)^{1/2}) \text{ a.s.,} \tag{3.36}$$

and

$$\limsup_{n\to\infty} |\sum_{i=1}^{n} (\sum_{k=1}^{p} b^{jk} u_{ik}) V_i| / (2n \log\log n)^{1/2} = (\sigma^2 b^{jj})^{1/2} \text{ a.s..} \tag{3.37}$$

In the following, we only prove (3.35) with $i = 1$, the others follow by the similar reason.

Now, note that (3.25), Lemma 4, and Assumptions 4(2.5) and 7(ii), in order to prove

$$J_{1j} = o((n \log\log n)^{1/2}) \text{ a.s.,} \tag{3.38}$$

it suffices to show that

$$\tilde{e}_0 = o((n \log\log n)^{1/2}) \text{ a.s.} \tag{3.39}$$

and

$$\max_{1\le m\le n} |\tilde{e}_m| = o((n \log\log n)^{1/2}) \text{ a.s..} \tag{3.40}$$

But (3.39) holds by the Borel-Cantelli lemma if

$$E|\tilde{e}_n|^q < \infty \text{ for some } q > 2.$$

The latter condition holds under $\sum_{j=1}^{\infty}\tilde{c}_j^{\,2} < \infty$ (which by Lemma 1 holds if $\sum_{j=1}^{\infty} j^2 c_j^{\,2} < \infty$ holds) and Assumption 1(iii) with $q < p$.

On the other hand, observe that

$$|\tilde{e}_m| \le \sum_{i=0}^{m} |\tilde{c}_i||e_{m-i}| + \sum_{i=1}^{m} \tilde{c}_{m+i}|e_{-i}|$$

$$\le \max_{0\le i\le m}|e_i|(\sum_{i=0}^{m}|\tilde{c}_i|) + \sum_{i=1}^{\infty}\hat{c}_i|e_{-i}| \tag{3.41}$$

with $\hat{c}_i = \sum_{j=i+1}^{\infty}|c_j|$ and

$$\sum_{j=1}^{\infty}|\tilde{c}_j| \le \sum_{i=1}^{\infty}\hat{c}_i = \sum_{j=1}^{\infty}\sum_{i=j+1}^{\infty}|c_j| \le \sum_{j=1}^{\infty} j|c_j| < \infty. \tag{3.42}$$

Thus, note that (3.41) and $\sum_{i=1}^{\infty}\hat{c}_i|e_{-i}| < \infty$ a.s. (since its expectation is finite), in order to prove (3.40), it suffices to show that as $n \to \infty$

$$(n \log\log n)^{-1/2} \max_{1\le i\le n}|e_i| \to 0 \tag{3.43}$$

which follows from $Ee_1^{\,2} < \infty$.

The next thing is to prove (3.37).

Note that (3.23) we find

$$\sum_{i=1}^{n}(\sum_{k=1}^{p} b^{jk}u_{ik})V_i = C(1)\sum_{i=1}^{n}(\sum_{k=1}^{p} b^{jk}u_{ik})e_i + \sum_{i=1}^{n}(\sum_{k=1}^{p} b^{jk}u_{ik})(\tilde{e}_{i-1} - \tilde{e}_i)$$

$$= L_{1n} + L_{2n}. \tag{3.44}$$

Now, by the similar reason as (3.33), and using (3.39) and (3.40),

$$L_{2n} = \sum_{i=1}^{n}(\sum_{k=1}^{p} b^{jk}u_{ik})(\tilde{e}_{i-1} - \tilde{e}_i) = o((n\log\log n)^{1/2}) \text{ a.s.}. \tag{3.45}$$

Before completing the proof of (3.37), we introduce the next proposition.

Proposition: Let $Z_1, Z_2, \ldots,$ be independent random variables with $EZ_i = 0$ and $\max_{i\geq 1} E|Z_i|^{2+c} < \infty$ for some $c > 0$. Let again $\liminf_{n\to\infty} n^{-1}\sum_{i=1}^{n} Var(Z_i) > 0$. Then

$$\limsup_{n\to\infty} |S_n|/(2s_n^2 \log\log s_n^2)^{1/2} = 1 \text{ a.s.}, \tag{3.46}$$

where $S_n = \sum_{i=1}^{n} Z_i$ and $s_n^2 = \sum_{i=1}^{n} EZ_i^2$. (See Corollary 5.2.3 of Stout (1974)).

Let $Z_{ij} = \sum_{k=1}^{p} b^{jk} u_{ik} e_i$, then we obtain $EZ_{ij} = 0$ and $E|Z_{ij}|^{2+c} = |\sum_{k=1}^{p} b^{jk} u_{ik}|^{2+c} E|e_i|^{2+c} \leq C_j(p) \max_{i\geq 1} \max_{k\geq 1} |u_{ik}|^{2+c} \cdot E|e_i|^{2+c} < \infty$.

Now, applying Assumption 4(2.4), we have

$$\liminf_{n\to\infty} n^{-1}\sum_{i=1}^{n} EZ_{ij}^2 = \liminf_{n\to\infty} n^{-1}\sum_{i=1}^{n}(\sum_{k=1}^{p} b^{jk} u_{ik})^2 Ee_i^2$$

$$= \sigma^2 (b^j)' (\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} u_i u_i') b^j = \sigma^2 (b^j)' B(b^j) = \sigma^2 b^{jj} > 0, \tag{3.47}$$

where $b^j = (b^{j1}, \ldots, b^{jj}, \ldots, b^{jp})'$.

Thus, applying the above Proposition we complete the proof of (3.37).

Up to now, we finish the proof of Theorem 1(ii).

(iii) Thirdly, we begin to prove Theorem 2(i).

Observe that

$$\hat{\sigma}_n^2 = n^{-1}\tilde{Y}'(I - \tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}')\tilde{Y}$$

$$= n^{-1}V'V - n^{-1}V'\tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}'V + n^{-1}\hat{G}'(I - \tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}')\hat{G}$$

$$-2n^{-1}\hat{G}'\tilde{X}(\tilde{X}'\tilde{X})^{-1}\tilde{X}'V + 2\hat{G}'V$$

(3.48)

where $\hat{G} = (g(t_1) - \hat{g}_n(t_1), \ldots, g(t_n) - \hat{g}_n(t_n))'$, $V = (V_1, \ldots, V_n)'$, and $\hat{g}_n(\cdot)$ is defined as in (2.10).

Now, using (3.23) and Lemma 4(ii), we can obtain that

$$E[\sum_{i=1}^{n}(\sum_{j=1}^{n} W_{nj}(t_i)(\tilde{e}_{j-1} - \tilde{e}_j))^2] = o(n) \tag{3.49}$$

and

$$\sum_{i=1}^{n}(\sum_{j=1}^{n} W_{nj}(t_i)V_j)^2 \leq 2nC(1))^2 \max_{1\leq i\leq n}|\sum_{j=1}^{n} W_{nj}(t_i)e_j|^2$$

$$+2\sum_{i=1}^{n}(\sum_{j=1}^{n} W_{nj}(t_i)(\tilde{e}_{j-1} - \tilde{e}_j))^2 = o_p(n^{1/2}). \tag{3.50}$$

Thus, by Lemma 2(i) we get

$$n^{1/2}|I_{3n}| \leq n^{-1/2}\sum_{i=1}^{n}(g(t_i) - \hat{g}_n(t_i))^2$$

$$\leq 2n^{1/2}\max_{1\leq i\leq n}|g(t_i) - \sum_{k=1}^{n} W_{nk}(t_i)g(t_k)|^2 + 2n^{-1/2}\sum_{i=1}^{n}(\sum_{j=1}^{n} W_{nj}(t_i)V_j)^2$$

$$= o_p(1). \tag{3.51}$$

Next, by (3.21) through (3.34), and using Lemma 3(i), we have for $k = 2,4$

$$I_{kn} = o_p(n^{-1/2}). \tag{3.52}$$

On the other hand, by the straightforward computation, and applying Markov's inequality, we can obtain that

$$nI_{5n} = \sum_{i=1}^{n}(g(t_i) - \hat{g}_n(t_i))V_i$$

$$= \sum_{i=1}^{n} g_{ni}V_i - \sum_{i=1}^{n} W_{ni}(t_i)V_i^2 - \sum_{i=1}^{n}(\sum_{k=1,\neq i}^{n} W_{nk}(t_i)V_k)V_i$$

$$= o_p(n^{1/2}). \tag{3.53}$$

Therefore, note that (3.48) through (3.53), in order to

$$\sqrt{n}(I_{1n} - \sigma_0^2) \to_D N(0, V_0^2), \tag{3.54}$$

which follows from Lemma 5.

(iv) In order to prove Theorem 2(ii), note that (3.48), it suffices to show that as $n \to \infty$

$$I_{1n} \to \sigma_0^2 \text{ a.s.} \tag{3.55}$$

and for $k = 2,3,4,5$

$$I_{kn} \to 0 \text{ a.s..} \tag{3.56}$$

The proof of (3.55) follows from Theorem 3.7 of Phillips & Solo(1992).

Next, by Lemma 3(i), (3.21), and (3.35) through (3.37), we have for $k = 2,4$

$$I_{kn} \to 0 \text{ a.s..} \tag{3.57}$$

For proving $I_{3n} \to 0$ a.s., note that (3.50), it suffices to show that

$$\max_{1\le i\le n} |\sum_{k=1}^{n} W_{nk}(t_i)(\tilde{e}_{k-1} - \tilde{e}_k)| = o(1) \text{ a.s.,} \tag{3.58}$$

which follows from (3.39), (3.40), and the following (using the similar reason as (3.33))

$$\begin{aligned} &\max_{1\le i\le n} |\sum_{k=1}^{n} W_{nk}(t_i)(\tilde{e}_{k-1} - \tilde{e}_k)| \\ &\le C \max_{1\le m\le n} |\tilde{e}_0 - \tilde{e}_m| \cdot \max_{i,k\ge 1} W_{nk}(t_i) \\ &= o((n\log n \log n)^{1/2}) \cdot o(n^{-1/2}(\log n)^{-2}) = o(1) \text{ a.s..} \end{aligned} \tag{3.59}$$

The remainder thing is to show that

$$I_{5n} \to 0 \text{ a.s.,} \tag{3.60}$$

which follows from the Borel-Cantelli lemma and the following

$$E[\sum_{i=1}^{n}(\sum_{k=1\ne i}^{n} W_{nk}(t_i)V_k)V_i]^2 = o(n(\log n)^{-2}). \tag{3.61}$$

The computation of (3.61) is omitted here for it is lengthy.

Hence, we finish the proofs of Theorems 1 and 2.

(v) The proof of Theorem 3(i)

By (2.14) we have

$$\bar{\beta}_n - \beta = (\bar{X}'\bar{X})^{-1}(\bar{X}'\bar{G} + \bar{X}'\bar{V})$$
$$= (\bar{X}'\bar{X})^{-1}(\bar{X}'\bar{G} + \bar{X}'V), \tag{3.62}$$

where $\bar{G} = (I-P)G$ and $\bar{V} = (I-P)V$.

Now, by the similar reason as the proof of Lemma 3(ii), and using Assumption 3, we obtain

$$\bar{X}'\bar{G} = o(n^{1/2}). \tag{3.63}$$

Next, by the similar reason as (3.30) through (3.34), and using Lemma 3(ii), we get

$$\sqrt{n}(\bar{X}'\bar{X})^{-1}\bar{X}'V \to_D N(0, C(1)^2\sigma_e{}^2B^{-1}). \tag{3.64}$$

Thus, the proof of Theorem 3(i) follows from Lemma 3(ii) and (3.62) through (3.64).

(vi) For proving Theorem 3(ii), observe that

$$\sum_{i=1}^{n}\bar{x}_i V_i = \sum_{i=1}^{n} u_i V_i + \sum_{i=1}^{n}\bar{h}_i V_i - \sum_{i=1}^{n} u_{in} V_i$$
$$= K_{1n} + K_{2n} + K_{3n} \text{ (say)}, \tag{3.65}$$

where $\{\bar{h}_i\}$ denotes the ith component of $\bar{H} = (I-P)H = (\bar{h}_1, \ldots, \bar{h}_i, \ldots, \bar{h}_n)'$ and $\{u_{in}\}$ denotes the ith component of $PU = (u_{1n}, \ldots, u_{in}, \ldots, u_{nn})'$.

Note that (3.37), (3.62) through (3.63), and (3.65), in order to prove Theorem 3(ii), it suffices to show that for $i = 2,3$

$$K_{in} = o((n \log\log n)^{1/2}) \text{ a.s.}. \tag{3.66}$$

By using Assumptions 3(i)(ii) and 5, the proof of (3.66) follows by the similar reason as (3.36).

The proof of Theorem 4 follows by the similar reason as

Theorem 2, which can also be obtained by the similar reason as the proof of Theorem 3.3 of Gao(1994b). #

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