Heterogeneous product and process innovations for a multi-product monopolist under finite life-cycles of production technologies.

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Abstract

Current paper analyses the influence of the length of production technologies life-cycles on the relative intensity of investments of a multi-product monopolist into different types of innovations. This monopolist is developing new versions of the basic product continuously and simultaneously invests into the production technologies of all these new products. In the paper the finite character of these products' life-cycles is assumed. These finite life-cycles are treated for the purposes of the paper as patents which are granted to the monopolist for the production and upgrading of every new product he/she introduces into the market. It is demonstrated that under the condition of finite-time life-cycles of new products the monopolist prefers to invest into process innovations for already introduced products rather than introducing the new ones in relative measure. It is argued, that this is not the case under infinite life-cycles for all these products where the monopolist invests relatively the same amounts into both types of innovations.

Keywords: Heterogeneous Innovations, Economic Dynamics, Multiproduct Monopoly, Product Life-cycle, Distributed Control

JEL codes: C02, L0, O31.

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1 Introduction

The main focus of this paper is on the optimal behaviour of a multi-product monopolist firm in the field of its innovative policy in the presence of time constraints on its activities. Namely, what are the optimal relations between investments into the generation of innovations of different types if such an innovative activity may be carried out only during some limited time.

The multi-product monopolist is modelled as a single planning agent in the industry (market). He/she decides upon investments into the development of new products (variety expansion process in the sense of (Grossman and Helpman 1993)) and into the improvement of production technology or quality for every of already introduced products (quality improvement in the sense of (Aghion and Howitt 1992)). The process of variety expansion is described as continuous in time thus yielding infinitely many new potential products. The range of such products is a continuum. The special feature of the framework is that every such a product possesses its own quality characteristic and the quality-improving process is described on two levels: as an aggregate process, which depends on variety expansion (product innovations) and as an ansamble of separate quality processes for each of the products, which does not depend on variety expansion but in turn influences this last.

Such a description of multi-product innovations follows the ideas of (Lambertini and Orsini 2001), (Lin 2004) and mainly (Bondarev 2010), where such a multi-product monopolist behaviour for limited time-horizon is introduced. In the current paper the infinite planning horizon is adopted instead, as in (Bondarev 2012). This allows for modelling long-run behaviour of the agent and for explicit solution for the dynamics of both types of innovations.

Current paper differs from this last one by the following additional assumption. Namely, the improvement of quality for every new product is limited by a finite life-cycle of the product (length of which is identical across products). This finite life-cycle is considered as to describe the effect of patenting: upon the invention of the product, the monopolist is granted the exclusive right to develop and sell this product during some fixed time, τ. After this time passes, the product development becomes common knowledge and no further economic profit may be derived from it. In the paper the question of how the investments of a monopolist would be distributed across development of qualities of different products and variety enhancing innovations is studied. It turns out, that the longer is the life-cycle (patent) of each of these new products, the more incentives an agent has to invest both into quality improvements for all the products and into the development and introduction of new products. Thus the infinite-time patent is the optimal case for the innovator which is not very surprising. It is shown that such a limiting case fully coincides with the model in (Bondarev 2012) and thus is its natural extension.

However, in relative measure investments into quality improvement as an aggregate grows faster than the investments into the introduction of new products as the length of the products life-cycles increases. This stresses the fact that it is more profitable for the monopolist to utilise already existing products through their further refinements rather then introduce new ones. It may be also noted that the logic which suggests patents to be limited instead of infinite in their length is not the same as in the classic intellectual property rights literature (see for example, (Scotchmer 1996; Menell and Scotchmer 2005) for references). As there are no any government authority or profit and competition incentives, the only thing that may be analysed in the model is how the length of the patent influences the rates of innovations. Although it turns out that the infinite-time patents would maximize the output of innovations of both types there is an argument of distribution of investments between different types of innovative activity. It is this argument that suggests the limited nature of a socially desired patent.

To stimulate the introduction of new products the patent's length should be limited by some finite number. Such an argument replicates the argument of Nordhaus in his seminal paper (Nordhaus 1967) but from completely different grounds: patents should be limited because of the interplay between quality-improving and variety-enhancing innovations and not to stimulate innovations from other agents as in this classic paper.

To demonstrate the importance of this interplay the paper uses comparison of homogeneous and heterogeneous products. For the case of homogeneous products there is no interplay between both types of innovations as the increase in patent's length unambiguously lead to the increase in revenue for the innovator. This is not the case for heterogeneous innovations. For these last there are two effects of opposite directions, called potential profit effect and compensation effect. Depending on their relative magnitude increase in patent's length would increase or decrease the incentives to innovate for the monopolist. Hence the requirement of limited time for patents is established independently from any arguments on competition and efficiency solely from individual rationality.

Main findings of the paper may be summarized as following:

- The value of innovative activity is maximized for the monopolist in the case of infinite-time patents.

- For heterogeneous products two different effects of opposite direction are present. However, one of them always dominates the other and thus the increase in patent's length leads to the increase of investments of both types.

- This increase is distributed unevenly across products: most of additional investments are being made into the development of already introduced products, while only small fraction is allocated to the introduction of new products.

- Depending on the form of social preferences upon products variety and quality different patenting policies may be optimal for the social planner.

The rest of this paper is organised as following: in the next section a brief review of the state of the arts is made. Then the basic framework of heterogeneous innovations is introduced following the lines of (Bondarev 2012). After that the model with limited life-cycles is formally described. The solution and results follows this section. The paper concludes with the analysis of distribution of investments into innovations of different types, where the argument of limited patents is discussed.

2 Related Literature

The question of how patents and patenting policy influence the dynamics of innovations and more generally, rates of technological change has a long history in economics. This question has been considered already in (Nordhaus 1967). In this paper the first formal model of the optimal patent's length has been considered. It has been argued that the patent need not to be of infinite length to stimulate innovative activity. The basic idea behind this statement was that one needs patents to protect and stimulate innovators, but these patents need not to be very long to stimulate further innovative activity from other innovators. In the current paper some progress towards establishing the similar argument for heterogeneous innovations is made. Similar to (Nordhaus 1967) paper one has a stand alone model of innovator in this framework while there are no competitors or potential entrants into the industry. On the other hand, one has the stream of two types of innovations here not the single one and patent has to be granted to every single product.

This form of innovative activity has also been considered in the patent literature under the name of sequential or cumulative innovations, where every next innovation is built up on the results of the previous one, (Denicolo 2002). There the question of optimal patent's length becomes more complicated as the sequential character of innovations rises questions not only of the length but also of the breadth of the patent. These questions have been extensively studied in (Gilbert and Shapiro 1990; Farrell and Shapiro 2004; Scotchmer 1996; Menell and Scotchmer 2005), and others.

One of the other approaches to the patenting problem is known under the name of patent races. Under this approach two or more agents are competing to be the first one to invent some product in order to obtain a prize which is the patent on this product and associated stream of profits from it, (Denicolo 1996; de Laat 1996). The suggested framework assumes a stand alone innovator and the model does not have any notion of patent races in it. Rather it is concentrated on optimal patent length for cumulative streams of innovations. Concerning this last there is also literature on variable length of patents for different products, as (Cornelli and Schankerman 1999). The suggested framework although assuming identical length of patents for all products may be modified to consider this also.

The model has an uncountable number of such cumulative streams of innovations represented by every single product's quality growth process. More then this the underlying process of variety expansion is also modelled. So the question arises what is the level (scope) of patents one would like to consider in such a framework? One variant is patenting of every level of quality of every product. Such an approach would generate non-smooth quality dynamics since the agent would not have stimulus to increase quality of a given product until the patent on the preceding level of quality will not expire. Then one has to assume patenting on the level of variety expansion process. Such a modification to the basic model has a clear interpretation, since every level of $n(t)$ is associated with invention of the new product. Every such product is then granted a patent, or, alternatively has a limited life-cycle within each the agent is free to develop its quality without external pressure from the market. Such a formulation of patenting problem is in line with works on cumulative innovations with patents, as in (Chang 1995; Gilbert and Shapiro 1990), but has some differences from it. In the suggested model one has not a single stream of innovations each of which is then patented or not, as in cumulative innovations literature, but rather one has stream of patented innovations and additionally dynamics of quality growth for each of this patented newly invented products (for each level of variety expansion process). One also accounts for the role of heterogeneity of these products, as it is done in (Hopenhayn and Mitchell 2001), but in a given model it would influence not only the rate of innovations (which is the rate of variety expansion in suggested framework) but also the growth rate of qualities of all these products. This makes the suggested approach richer then the preceding ones.

3 Basic framework and Assumptions

First we consider the model without finite-time patents or, equivalently with infinite patent (life-cycle) time. Our main goal is to construct a framework which would allow to consider processes of product variety expansion and quality innovations simultaneously. The basic heterogeneous innovations framework described below follows the construction in (Bondarev 2012). Hence only basic assumptions and key mechanics of this model are described here.

3.1 *Basic structure*

Assume there is a single firm (a monopolist) in a given industry. The industry is mature and no growth of the demand is expected for existing products variety. Hence this monopolist is maximizing its profit by developing new products, which are then introduced to the market. The natural objective of the monopolist is the maximization of its profits, $\pi(t) \to \max$ for any given time period. This paper concentrates on just one part of activities of such a monopolist, namely on the process of its innovative activities. To put this in line with profit maximization behaviour we assumed that markets for all existing products are mature, yield some constant profit with stable prices and output. Production policy of the monopolist is assumed to follow standard rules of monopolistic behaviour under profit maximization: given (constant) demand, the monopolist is setting the price and production as to maximize its profit. In mature markets the process innovations reached their maximum and thus no further improvements to the production process may be made. Hence, the production costs are also constant in time. These considerations lead to the conclusion that in mature markets the monopolist's production and pricing (and hence profits) are constant.

Assumption 1 *For those products which are already in mature stage, the production, price and profit of the monopolist are constant.*

Because of this one may abstract from this part of monopolist's activities in the optimization problem. Now consider those products which are being introduced in the given time frame and which production is subject to process innovations. For these products the profit of the monopolist is proportional to the costs decrease which is the result of process innovations, denoted by $q_i(t)$. If we abstract from the pricing policy and assume constant demand for each of such products, this would result in the profit function per unit of production of a linear form: $\pi_i(t) = \delta * q_i(t)$. Then normalizing δ coefficient to one, we may have q_i as the only profit parameter for any product within the product range N.

Assumption 2 *The only source of new profit for the monopolist is the development of new products which leads to the increase in the existing range of products over time, $n(t) > 0$.*

Assume the process of development of products is continuous in time and yield new products (which are new versions of some basic for the industry product) with some rate. Let us call this rate the rate of variety expansion.

Assumption 3 *The product innovations, are continuous in time and new products appear at a continuous basis, $\dot{n}(t) \geq 0$.*

Assume that the range of these new products is limited from above. The product innovations are limited to upgrades of some basic product which defines the industry (e.g. cell phones industry produces different versions of cell phones but not computers). We do not model fundamental inventions, which introduce totally new products to the economy by this model and hence it is natural to require that there is limited capacity of the industry for the variety of products which are somewhat similar to each other.

Assumption 4 *Product innovations are limited by the maximal possible range of products, $n(t) \leq N$*

Assume these newly introduced products initially require very much resources for their production and hence the monopolist allocates part of its R&D capacity on process innovations related to these new products. Every new product is than intensively studied with respect to opportunities for its costs minimization. As there are numerous new products (continuum of) there are numerous streams of such cost-minimizing processes associated with every product.

Assumption 5 *Every product has its own dimension of process innovations or 'quality' which depends on time, $\forall i \in n(t) \exists q_i(t)$.*

Assume at each point in time, the monopolist has to choose optimally the level of investments being made into the development of new products (product innovations) and into the development of production of already existing products (process innovations). Both these investment streams cannot be negative.

Assumption 6 *Product innovations and process innovations require different types of investments, which vary over time, while process innovations for every product are also different* $u(t) \geq 0, g_i(t) \geq 0$

Assume also that the monopolist is the long-run player and does not restrict its planning to some certain length of time. Hence, the innovations of both types occur continuously up to infinite time.

Assumption 7 *There is no terminal time for both processes of innovations,* $0 < t \leq \infty$

The last point to mention is that we assume that all innovations are certain. This is rather strong limitation, but allows to concentrate on the key issues of this paper: heterogeneity and form of interdependence between different types of innovations.

Assumption 8 *All innovations do not have any uncertainty associated with them.*

3.2 *Objective and dynamics*

We assume the scheme of so-called 'planned' innovation: there is only one agent (social planner) who maximizes the output of innovations in any given period of time over the infinite time horizon according to some objective functional. It is defined as:

$$(1) \qquad J^{mono} \stackrel{\text{def}}{=} \int_0^\infty e^{-rt} \left(\int_0^{n(t)} q(i,t) - \frac{1}{2} g(i,t)^2 di - \frac{1}{2} u(t)^2 \right) dt \rightarrow \mathbf{max}$$

Planner is maximizing integral sum of qualities (production technologies) of all products invented until each time t minus investments being made to every invented product's quality and to the overall expansion process over the planning horizon. There is no sign of prices or profit in this formulation. We omit market clearing mechanism and all the mechanics behind the market structure. One way to motivate such a formulation of the objective functional is to assume linear in every product profit function and unitary price of each product. It is equivalent to the linearity of profit function which is a standard assumption in innovation literature (Lambertini 2009; Lambertini and Mantovani 2010). One may treat the planner here either as the central authority in some centralized planned economy either as the monopolist on some market.

Dynamics of quality growth and expansion process are governed by subsequent dynamic equations:

$$
\begin{aligned}
&\dot{n(t)} = \alpha u(t), \\
&\dot{q(i,t)} = \gamma(i)g(i,t) - \beta(i)q(i,t), \\
&\forall i \in [0,..,N] = \mathbf{I} \subset \mathbb{R}_+ \\
&\forall t \in (0,..,\infty) = \mathbf{T} \subset \mathbb{R}_+
\end{aligned} \tag{2}
$$

and static constraints:

$$
\begin{aligned}
&u(t) \geq 0; \\
&g(i,t) \geq 0; \\
&n(t) \leq N; \\
&q(i,t) \mid_{i=n(t)} = 0.
\end{aligned} \tag{3}
$$

It can be seen from above, that we assume very simple motion laws for both variety expansion and qualities growth. This is done to keep the model in th class of linear-quadratic ones and follow the specifications in IO literature, for example, in (Dawid, Greiner, and Zou 2010). To summarize these, we introduce two more assumptions:

Assumption 9 *Variety expansion process depends solely on the investments being made into the R&D by the agent and not on the already achieved level of the products variety.*

Assumption 10 *Quality growth (process innovations) depend on the investments into R&D, specific for the given product and on the deprecation of this product-specific technology over time which is the function of already achieved level of quality (technology).*

We assume zero initial quality for all products and some fixed initial range of products available. Observe, that the last constraint in (3) is equivalent to the requirement that quality of each product is zero all the time until variety expansion process, $n(t)$, reaches the position i.

Assumption 11 *Every product has zero quality (technology) level, associated with it until this product is actually invented; there are no prior knowledge on it.*

Next observation concerns $\gamma(i)$ and $\beta(i)$ functions. These are functions of efficiency of investments to every product's quality and rate of quality decay in the absence of investments depending on the product's position i respectively.

These two functions represent structural characteristics of the products space being considered as a whole, as they define relative differences between products. Here we do not consider the dependence of model results upon the form of these functions. It is sufficient to note, that heterogeneity of quality investments characteristics is essential to the model. For that we assume the simplest possible form of these functions which would linearise subsequent dynamic equations:

Assumption 12 *Efficiency of quality innovations is a decreasing function of the position of the product in the product's space* $\boldsymbol{N}$*, while rate of quality decay is constant across products;*

$$\beta(i) = \beta;$$
$$\gamma(i) = \sqrt{N-i} \times \gamma. \tag{4}$$

We employ Hamilton-Jacobi-Bellman approach to resolve the problem.

3.3 *Solution for infinite-time case*

In this infinite time horizon formulation, the straightforward application of Hamilton-Jacobi-Bellman approach yields the explicit solution for quality growth system and for variety expansion.

To construct the value function of the model, first observe, that problem for quality growth may be solved independently of the variety expansion process except for the time of emergence of the new product which is defined through $n(t)$ dynamics. Starting from this time (denoted by $t_i(0)$) value of each product's quality growth is independent of variety expansion process:

$$V^{mono}(q_i) = \max_{g_i(\bullet)} \int_{t_i(0)}^{\infty} e^{-r(t-t(0)_i)}(q_i(t) - \frac{1}{2}g_i(t)^2)dt \tag{5}$$

Observe also, that due to infinite time horizon, this value function does not explicitly depend on time. Thus one may construct the Hamilton-Jacobi-Bellman equation for every q_i dynamics as following:

$$rV(q_i) = \max_{g_i(\bullet)}\{q_i - \frac{1}{2}g_i^2 + \frac{\partial V(q_i)}{\partial q_i}(\gamma_i g_i - \beta_i q_i)\} \tag{6}$$

Where $\beta_i = \beta(\bullet)|_i$ and $\gamma_i = \gamma(\bullet)|_i$ with functions given by (4). Here and further on in Hamilton-Jacobi-Bellman equations we sometimes omit time argument of state and control variables to condense notation. This equation yields the system of algebraic equations on the coefficients of value function. As the given model is linear quadratic one, one may assume the value function of polynomial form, (Dockner, Jorgensen, Long, and Sorger 2000). Then equalizing powers yield polynomial of 1st degree as the only possible value function in polynomial form:

$$V^{ass}(q_i) = A_i q_i + B_i \tag{7}$$

Then one can easily solve the resulting algebraic equations on value function coefficients. The resulting optimal investments are constant and do not depend on time, but differ only across products:

$$g_i^{mono} = \frac{\gamma \times \sqrt{(N-i)}}{(r+\beta)} \tag{8}$$

Substituting this to the quality growth dynamics yields the ODE for q_i dynamics (effective only after the time of the invention of the product i, being defined from the variety expansion dynamics):

$$\dot{q_i}(t) = \frac{1}{r+\beta} \times \gamma^2(N-i) - \beta q_i(t). \tag{9}$$

which yield the piece-wise solution for q_i (taking into account the last constraint in (3)):

$$q^{mono}(t)_i = \begin{cases} \frac{\gamma^2(N-i)}{\beta(r+\beta)^2} \times (1 - e^{-\beta(t-t_i(0))}), t \geq t_i(0) : i \geq n(t); \\ \\ 0, t < t_i(0) : i < n(t). \end{cases}$$

and subsequent value function representation:

$$V^{mono}(q_i) = \begin{cases} q_i(t) + \frac{\gamma^2 \times (N-i)}{2r(r+\beta)^2}, t \geq t_i(0) : i \geq n(t); \\ \\ 0, t < t_i(0) : i < n(t). \end{cases}$$

several examples of dynamics of quality growth for different products are presented at Figure 1. Here and throughout the paper the same set of exogenous parameters of the model are used :

$$N = 100;$$
$$n_0 = 0;$$
$$r = 0.05;$$
$$\alpha = 0.5;$$
$$\gamma = 0.7;$$
$$\beta = 0.1. \quad (10)$$

One may observe the difference in the resulting levels of products de-

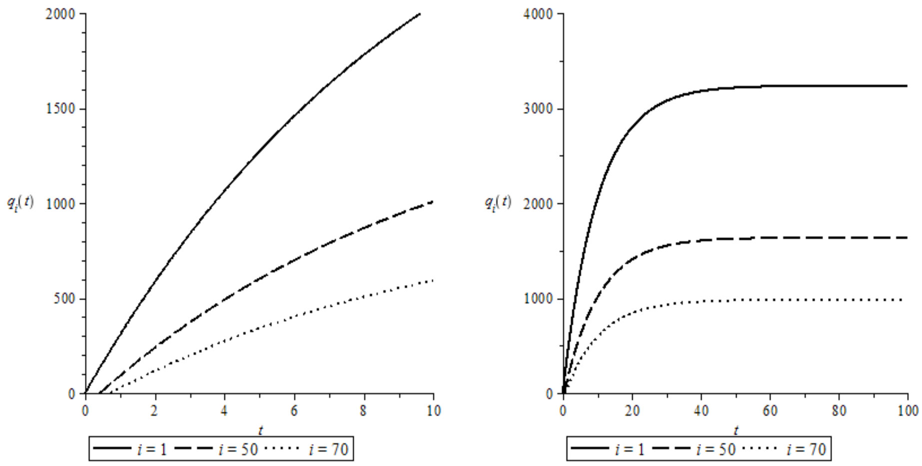

Figure 1: Difference in development of technologies for different products i

velopment: the higher is the position of the product in the product's range, the lesser is the resulting quality development level. Note, that the actual quality development processes does not start at the same zero time for all the products, but at different $t_i(0)$ which is defined from the variety expansion. This can be observed at the left on the Figure 1. This is further elaborated on in the limited life-cycle extension below, as for infinite time horizon this fact does not play a crucial role. At the same time all the products' qualities eventually reach their steady-state (maximal) levels, as it is shown at the picture to the right.

Second part of the overall value generation consists of the intensity of addition of new products at every time given the expected value of the stream of profit derived from the quality of these newly introduced products. This part may be represented by the integral over all potential stream of quality of the product over its life-cycle. At the same time the information on the value generated by the development of the quality of the product is already contained in the value function of the quality problem above, so it suffices to integrate over all potential products at initial time. Denote by $V(0)_{n(t)}$ the value of the quality growth problem above for the boundary product, $i = n(t)$. This value does not depend on the quality level, since this last is zero for the boundary product (just invented one), as it is required by (3). This value is given by:

$$V^{mono}(0)_{n(t)} = V^{mono}(q_i)|_{i=n(t),q_i=0} = \frac{\gamma^2 \times (N - n(t))}{2r(r+\beta)^2} \tag{11}$$

Variety expansion generates value through the addition of new products which are then developed through quality growth process. These yield the value function for variety expansion problem in the following form:

$$V^{mono}(n) = \max_{u(\bullet)} \int_0^{\infty} e^{-rt}\Big(\alpha u(t) \times V^{mono}(0)_{n(t)} - \frac{1}{2}u(t)^2\Big)dt \tag{12}$$

subject to dynamic and boundary constraints on variety expansion process:

$$\begin{aligned} \dot{n(t)} &= \alpha u(t); \\ n(0) &= n_0 \end{aligned} \tag{13}$$

where the value of $V^{mono}(0)_{n(t)}$ is given by equation above.
These yield the Hamilton-Jacobi-Bellman equation for the variety expansion problem:

$$rV^{mono}(n) = \max_{u(\bullet)}\Big(\alpha u \times V^{mono}(0)_{n(t)} - \frac{1}{2}u^2 + \alpha u \times \frac{\partial V^{mono}(n)}{\partial n}\Big) \tag{14}$$

Since the problem is formulated in infinite time, the coefficients of value function do not depend on time as in the case for quality dynamics. The Hamilton-Jacobi-Bellman equation then yields simple enough system of algebraic equations on coefficients of the value function (we adopt the assumption of polynomial value function form). Due to the presence of $n(t)$ variable in value function for quality growth on the right hand side, the minimal power of the polynomial here is 2, yielding quadratic value function as a conjecture:

$$V(n)^{ass} = Cn^2 + Fn + E \tag{15}$$

optimal investments to the variety expansion then are:

$$u^{mono}(t) = \frac{1}{2}\alpha\gamma^2\Big(\frac{N - n(t)}{(r+\beta)^2 r}\Big) + 2Cn(t) + F \tag{16}$$

where C, F are coefficients of the value function. Substitution of this optimal investment policy into the dynamic equation for variety expansion yields a single first-order ODE

$$\dot{n(t)} = \frac{\alpha^2\gamma^2 r(N - n(t)}{\sqrt{(r+\beta)^2 r^2(r^4 + 2r^3\beta + r^2\beta^2 + 2\alpha^2\gamma^2)} + (r+\beta)^2 r^2} \tag{17}$$

This is the simple equation with constant coefficients which may be readily solved by conventional methods, yielding optimal variety expansion path:

$$n^{mono}(t) = N + e^{-\frac{\alpha^2\gamma^2 rt}{\sqrt{(r+\beta)^2 r^2(r^4+2r^3\beta+r^2\beta^2+2\alpha^2\gamma^2)}+(r+\beta)^2 r^2}}(n_0 - N) \tag{18}$$

Figure 2 demonstrates the shape of this dynamics for different initially available ranges (all other parameters held the same as above).

This expansion dynamics is bounded from above by the maximal range

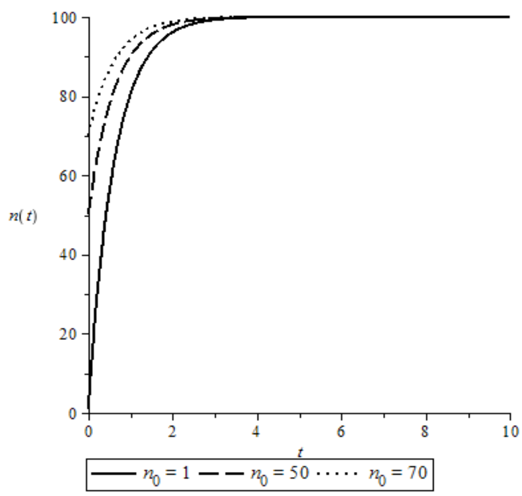

Figure 2: Convergence of different variety expansion paths

of potential products' space, N, and is approaching it at the decreasing rate. N is the steady-state level of the range of products. Irrespectively of the chosen particular range, it is achieved by the variety expansion process only in infinite time. This model with infinite-time horizon just replicates the model in (Bondarev 2012), where further details may be found concerning the solution concept. In what follows we use this model as a benchmark for the model with finite life-cycles of products being introduced into the market.

Putting then both innovations streams together one may schematically reconstruct the shape of the overall evolution of the products space as it is

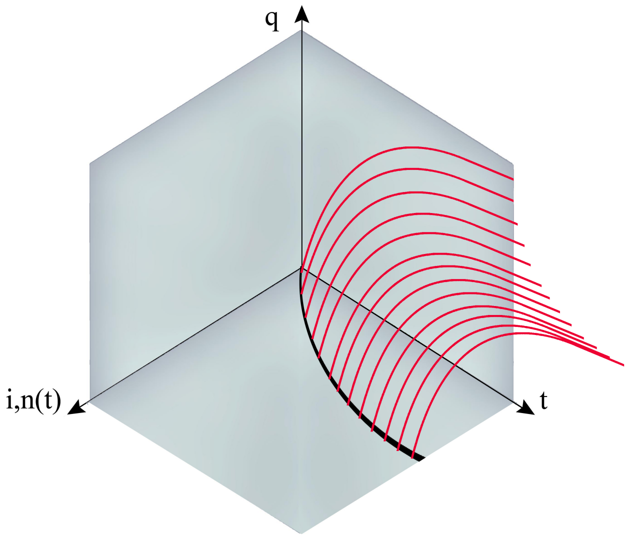

Figure 3: Reconstruction of product and process innovations

done at Figure 3. Here the underlying process of product innovations at each point of its curve generates the associated flow of quality growth innovation, which in turn, form the surface. This surface demonstrates the overall qi, t process dynamics as a function of products variety expansion.

4 Finite life-cycles model

4.1 *Objective function*

In this section the modification of the basic model presented above is made that allows for finite life-cycles of all the introduced products. Here we abstract from any competition as well as from patent races type of models. The only question of interest in current setup with one agent model should be: whether the introduction of limited time patents would stimulate or depress innovative activity of both types. For that purpose one may handle patents as products' life-cycles: after the expiration of the patent's time the agent cannot use his/her achieved quality level of the given product for profitable activities (e.g. sell this product). This of course is not true in real economies, but one can imagine the high density of competition on the product market which approaches the perfect one. As soon as the patent expires, all quality development of this product becomes the common knowledge to all the competitors and hence the agent in the model is no longer able to derive non-zero economic profit from it and thus he is no longer interested in quality investments in this product. In terms of the model this means that every product from $n(t)$ range has a limited time life-cycle (determined by the patent's length) during which its quality is developed by the agent. After this time development stops. At the same time such a setup would make sense only if at any given time the agent may switch his activities to the variety expansion investments thus earning himself another portion of patents. For this we use the infinite time horizon model with respect to variety expansion process from the previous section. Now put all these considerations in formal terms. The objective functional of the agent is almost the same as in the basic model:

$$J^{patent} \stackrel{\text{def}}{=}$$

$$\stackrel{\text{def}}{=} \max_{u(\bullet),g(\bullet)} \int_0^{\infty} e^{-rt} \left(\alpha u(t) \int_{t_i(0)}^{t_i(0)+\tau} \{q(n,s+t) - \frac{1}{2} g(n,s+t)^2\} ds - \frac{1}{2} u(t)^2 \right) dt \tag{19}$$

Such an objective functional is derived from the (1) while allowing for finite time of development of quality q_i of each new product. To see this, consider the decomposition of the problem into quality growth and variety expansion problems as above.

Start with the quality growth part. In the basic model every product is developed in infinite time and hence the value function for the quality growth of each product i is defined as (5), from 0 to infinity. Under finite life-cycles conjecture the time of development of every new product i is limited by τ. Hence the value function of quality growth is time-limited and time dependent:

$$V^{pat}(q_i, t) = \max_{g_i(\bullet)} \int_{t_i(0)}^{t_i(0)+\tau} e^{-rs} \{q_i(s) - \frac{1}{2} g_i(s)^2\} ds \tag{20}$$

Note, that value function now depends not only on the number of the product (which implicitly defines dependence on the time of emergence $t_i(0)$ also as the inverse function of i) but also on the length of the life-cycle, which is assumed to be the same for all products.

Now consider the variety expansion part. This remains unchanged, since the process of product innovations is not limited by finite life-cycles, but only by the overall planning horizon of the model which is assumed to be infinite. Hence the process of variety expansion has exactly the same shape of the value function as in the basic model, (12). Bringing together these two parts of the overall value generation in the model yields the value function as of (19). Observe that the overall value function is maximized with respect to all the quality growth investments, $g(i,t)$, while the value function (20) contains the optimal investments only for the given product i.

The difference is in the bounded integration term over the length of the patent τ for quality development only. All dynamic constraints (2) remain the same as in the original model with subsequent change of the area of definition: q_i is now defined on the time domain $[t_i(0); t_i(0)+\tau]$, while $n(t)$ is defined on $[0;\infty)$. To solve such a model we make use of Hamilton-Jacobi-Bellman approach in the same manner as in the previous section. Again we decompose the initial problem into quality dynamics and variety expansion dynamics. Note, however, that this time quality growth is time-dependent and time horizon for this problem should be finite, thus yielding time-varying coefficients of value function for the quality growth problem.

4.2 *Solution for quality growth*

The solution of the problem of quality growth for the finite life-cycles case follow the same steps as for the basic infinite time model. First we derive the Hamilton-Jacobi-Bellman equation for the development of every product i, then assume the polynomial form of the associated value function and derive optimal investments.These are then used to solve for the optimal dynamics of quality of product i *within the duration of its life-cycle*, τ. This defines the difference in dynamics with the basic model above.
The Hamilton-Jacobi-Bellman equation for the development of every product i now depends on t and not only on the quality level itself:

$$rV^{pat}(q_i,t) + \frac{\partial V^{pat}(q_i,t)}{\partial t} = \max_{g_i(\bullet)}\Big\{ q_i - \frac{1}{2}g_i^2 + \frac{\partial V^{pat}(q_i,t)}{\partial q_i} \times (\gamma\sqrt{(N-i)}g_i - \beta q_i)\Big\}, \quad (21)$$

$t \in [t_i(0),..,t_i(0)+\tau]$

One may assume the same linear form of value function for this problem as in the basic model, but with time-varying coefficients:

$$V^{ass}(q_i,t) = A_i(t)q_i + B_i(t). \quad (22)$$

Then the first-order condition for every product's quality growth is:

$$-g_i + \frac{\partial V(q_i,t)}{\partial q_i} \times (\gamma\sqrt{(N-i)}) = 0;$$

$$g_i^{pat} = A_i(t) \times (\gamma\sqrt{(N-i)}). \tag{23}$$

However, due to the limited life-cycle, one have a system of 2 differential equations on value function coefficients rather then algebraic equations:

$$\begin{aligned}
&\dot{A_i}(t) = (r+\beta)A_i(t) - 1;\\
&\dot{B_i}(t) = rB_i(t) - \frac{1}{2}\gamma^2(N-i)A_i(t)^2\\
&A_i(\tau + t_i(0)) = 0;\\
&B_i(\tau + t_i(0)) = 0.
\end{aligned} \tag{24}$$

Observe that for every product i the value function is different, as coefficients are different due to different boundary conditions and $N-i$ term in the second equation.
This is a system of first order equations which can be readily solved. First the solution for $A_i(t)$ coefficient as a function of emergence time $t_i(0)$ is obtained:

$$A_i(t) = \frac{1}{(r+\beta)}(1 - e^{(r+\beta)(t-t_i(0)-\tau)}) \tag{25}$$

Substitution of this into the equation for $B_i(t)$ term yields the second coefficient as function of emergence time and the position of the product within the products range, i:

$$\dot{B_i}(t) = rB_i(t) - \frac{1}{2}\gamma^2(N-i)\Big(\frac{1}{(r+\beta)}(1 - e^{(r+\beta)(t-t_i(0)-\tau)})\Big)^2$$

$$\begin{aligned}
B_i(t) = {}& \frac{\gamma^2(N-i)}{r(r+\beta)^2}\times\\
&\times\Big(\frac{r}{\beta}e^{(r+\beta)(t-(\tau+t_i(0)))} - \frac{1}{2(r+\beta)}e^{2(r+\beta)(t-(\tau+t_i(0)))} - \frac{(r+\beta)^2}{\beta(r+2\beta)}e^{r(t-(\tau+t_i(0)))} + \frac{1}{2}\Big).
\end{aligned} \tag{26}$$

These calculations provide the form of the value function for quality growth for every product i:

$$V^{pat}(q_i,t) = \frac{1}{(r+\beta)}(1-e^{(r+\beta)(t-t(0)_i-\tau)}) \times q_i(t)+$$
$$+\frac{\gamma^2(N-i)}{r(r+\beta)^2}\times$$
$$\times\Big(\frac{r}{\beta}e^{(r+\beta)(t-(\tau+t_i(0)))} - \frac{1}{2(r+\beta)}e^{2(r+\beta)(t-(\tau+t_i(0)))} - \frac{(r+\beta)^2}{\beta(r+2\beta)}e^{r(t-(\tau+t_i(0)))} + \frac{1}{2}\Big). \tag{27}$$

The resulting coefficients being inserted into the first order condition (23) yields optimal investments into quality growth which now do depend on time but only within the limits of the patent's length τ ($t \in [t_i(0); t_i(0)+\tau]$):

$$g_i^{pat}(t) = \gamma\sqrt{(N-i)}\Big(\frac{1-e^{(r+\beta)(t-t_i(0)-\tau)}}{(r+\beta)}\Big) \tag{28}$$

Finally one obtains ODE for quality growth:

$$\dot{q_i(t)} = \gamma^2(N-i)\Big(\frac{1-e^{(r+\beta)(t-t_i(0)-\tau)}}{(r+\beta)}\Big) - \beta q_i(t);$$
$$q_i(t_i(0)) = 0. \tag{29}$$

which is the first-order linear ODE with the unique solution:

$$q_i^{pat}(t) =$$
$$= \frac{\gamma^2(N-i)}{(r+\beta)(r+2\beta)\beta}\times$$
$$\times\Big(\beta(e^{-(r+\beta)(t_i(0)-\tau-t)} - e^{(r+\beta)(t-t_i(0)-\tau)}) - (r+2\beta)(e^{-\beta(t-t_i(0))}-1)\Big) \tag{30}$$

As it can be seen from (30), quality growth for each product now depends on time, but only within the boundaries of the patent length and from the patent length itself, and from the time of emergence of the product, $t_i(0)$. One may treat this solution as suitable for *any* product i, including the boundary product $i = n(t)$. However, to define the location of the evolution path for the certain product in the product's space and in the overall quality improving process, one has to define the time of emergence from the variety expansion part of the problem. Observe also, that the solution for infinite-time case, (10) is the limiting case of this quality evolution path with $\tau \to \infty$.

Indeed, this may be demonstrated by taking the respective limit of the (30).

As a result of limited life-cycle of the product development, each product's quality decays to zero after the time of expiration of the patent, as Figure 4 shows. To produce the quality evolution paths at this Figure the same set of parameters has been used as for infinite-time case above but with $\tau = 1$ to make the dynamics more clear. It has to be noted, that every next product

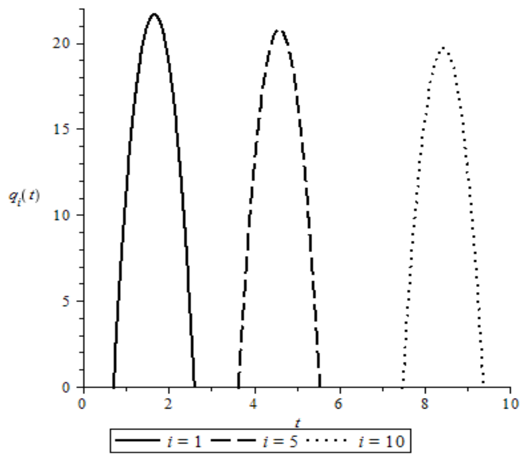

Figure 4: Quality innovations under limited life-cycles for different products i

has slightly lower maximal quality, then all the preceding products. This can be seen from the form of the solution (30): the greater is the index of the product i, the lower is its quality:

$$\frac{\partial q(i,t)}{\partial i} < 0. \tag{31}$$

For displayed on Figure 4 i values it also can be observed even with small changes of i.

At any given point in time the innovator has under his/her control the range (bounded continuum) of products to develop. This range, $n(t) - n(t - \tau)$, denotes those products which are already introduced to the market and their life-cycle (patent) have not yet expired. Further on this quantity is referred to as *effective range of products*. It is defined from the evolution of variety expansion path as well as emergence times of all the products development. This creates the link from variety expansion to the quality development, which is stronger then for the infinite-time case above. As it is discussed further in the paper, the process of out-dating of older products significntly changes innovator's behaviour.

4.3 *Solution for variety expansion*

The final step of the solution of quality growth problem is the calculation of the value function. Then we take this value function with zero quality level as an input for variety expansion problem in the same way as it has been done for infinite-time horizon model above. Applying the same logic as for infinite-time horizon model, one may note that for variety expansion problem only the value of quality growth model at zero time is relevant because the agent estimates his potential profit from the expansion of the range of products available for him to develop only and this is done at the moment of the emergence of this good, $t_i(0)$. Hence one need to know only $V^{pat}(q_i,t)_{|q_i=0,t=t_i(0)} = V^{pat}(0,\tau)_{n(t)}$, which is:

$$V^{pat}(0,\tau)_{n(t)} = \frac{\gamma^2(N-n(t))}{r\beta(r+2\beta)(r+\beta)^2}\times$$

$$\times\Big(r(r+2\beta)e^{-(r+\beta)\tau} - \frac{1}{2}r\beta e^{-2(r+\beta)\tau} - (r+\beta)^2 e^{-r\tau} + \frac{1}{2}\beta(r+2\beta)\Big). \tag{32}$$

Also note that the time horizon for variety expansion model is infinite. Then the HJB equation for this part is of the same form as in previous section (14).

Assuming the same quadratic form of the value function one have the system of algebraic equations on the coefficients. which might be readily solved. Denote

$$V(\tau) = \frac{V^{pat}(0,\tau)_{n(t)}}{(N-n(t))} = \frac{\gamma^2}{r\beta(r+2\beta)(r+\beta)^2}\times$$

$$\times\Big(r(r+2\beta)e^{-(r+\beta)\tau} - \frac{1}{2}r\beta e^{-2(r+\beta)\tau} - (r+\beta)^2 e^{-r\tau} + \frac{1}{2}\beta(r+2\beta)\Big). \tag{33}$$

This does not depend on $n(t)$.

Now the HJB equation for variety expansion problem takes the form:

$$rV^{pat}(n) = \max_{u(\bullet)}\Big\{\alpha u(t)\times V(\tau) - \frac{1}{2}u(t)^2 + \alpha u(t)\times\frac{\partial V_n}{\partial n}\Big\}. \tag{34}$$

Assuming quadratic form of the value function for this problem one have first order condition for the optimal control which depends on value function for quality problem:

$$u^{pat}(t) = \alpha\Big(V(\tau)(N-n(t)) + 2Cn(t) + F\Big) \tag{35}$$

with $V^{ass}(n) = Cn(t)^2 + Fn(t) + E$. Note, that in this part of the problem coefficients are not time-varying.

The system of algebraic equations for coefficients C, F, E is solved in the same way as before without any changes: insert expression for $u^{opt}(t)$ into the (34) above and regroup coefficients at equal powers of $n(t)$. Hence one arrives to the system of 3 equations with three unknown coefficients, which has a straightforward solution. Substitution for these coefficients into the first order condition (35) yields optimal investments into variety expansion process as a function of $V(\tau), n(t)$:

$$u^{pat}(t) = \frac{2\alpha r(N - n(t))V(\tau)}{r + \sqrt{4\alpha^2 rV(\tau) + r^2}} \tag{36}$$

Then dynamic constraint (2) yields the first-order ODE for $n(t)$:

$$\dot{n(t)} = \frac{2\alpha^2 rV(\tau)}{r + \sqrt{4\alpha^2 r \times V(\tau) + r^2}}(N - n(t)) \tag{37}$$

which has the solution

$$n^{pat}(t) = N + e^{-\frac{2\alpha^2 rtV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}}(n_0 - N) \tag{38}$$

Which is of the same structure as the solution for infinite-time horizon model above.

The shape of dynamics of variety expansion is also similar and demonstrates convergence of evolution paths with different initial ranges, as it is demonstrated by the Figure 5 using the same parameter set as above, with $\tau = 1$, except for varying initial range. Compare the intensity of the process with those of the infinite-time model above, at Figure 2.

The last point which is necessary to obtain the full characterization of dynamics of the model is the emergence time, $t_i(0)$ for all products $i \in \mathbf{N}$. This time is just an inverse function of variety expansion process, since it is defined from the condition $i = n(t)$ in each case. It is calculated by substitution i for n and $t_i(0)$ for t into the variety expansion and then finding the inverse Formally:

$$t_i(0) : i \to f(i);$$

$$f(i) = n(t)^{-1}|_{n=i};$$

$$t_i(0) = -ln\Big(\frac{(N - i)}{(N - n_0)}\Big) \times \frac{r + \sqrt{4\alpha^2 rV(\tau) + r^2}}{2\alpha^2 rV(\tau)} \tag{39}$$

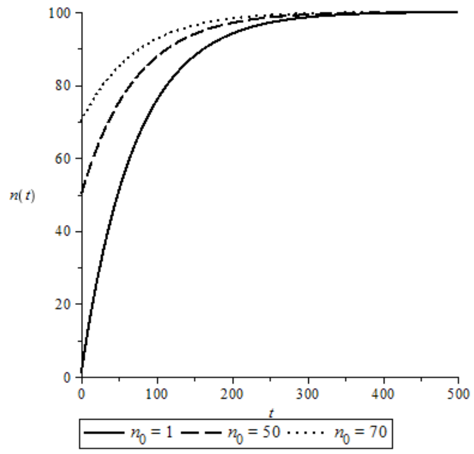

Figure 5: Products variety expansion for products with limited life-cycles

This function, demonstrated on Figure 6, shows, that the higher is the index of a product, i, the more time is needed for the introduction of the next product after this one. This is the direct consequence of the slowing rates of variety expansion, as Figure 5 shows. As a result, *density* of quality

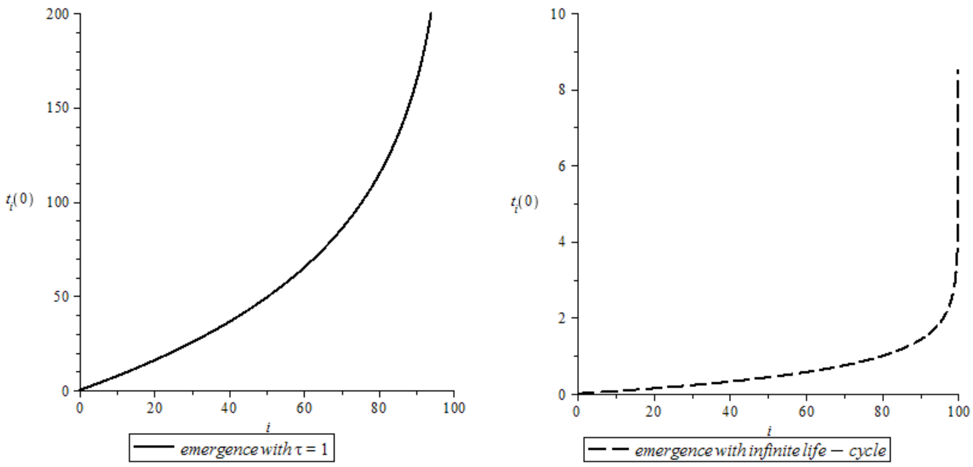

Figure 6: Time of emergence as function of product position in the product's space.

innovations, $q_i(t)$, is decreasing with time, as Figure 4 shows: the distance between evolution paths of technologies for products in the beginning of the products range is shorter, then in the end of it. Note, however, that this is not the effect of the finiteness of life-cycles of products but rather of their heterogeneity and is the same as in the benchmark model described above. However, shorter life-cycles lead to slower emergence of products. Compare two graphs of the Figure 6 to see that.

5 Analysis of the model

5.1 *Compensation effect vs potential profit effect*

Now one may ask whether the introduction of patents stimulate quality growth and variety expansion processes or not relative to infinite-time version of the model.

First consider effects of the length of life-cycles on the variety expansion. A priori one may encounter two opposite effects in this part of the model. The first one should be negative: the shorter is the length of patent (life-cycle), the lesser is the range of products effectively at the agent's disposal at each point in time. This range is given by $n^{pat}(t) - n^{pat}(t-\tau)$ since only products introduced during this time are covered by patents at the time t. Then to maximize the range of products under control at each point in time the agent should invest more in variety expansion with shortening patent length. This effect is referred to as *compensation effect*, because the agent has to compensate the decrease in the effective product's range with additional investments into variety expansion.

At the same time shorter length of the life-cycle limits agent's opportunities to develop products' qualities and thus, decreases incentives to develop new products. This effect is referred to as *potential profit effect*, as it is the changes in potential profit expected from new product, which creates it.
To formally define these two effects, consider first the derivative of the effective product's range with respect to the length of the life-cycle, which includes both effects:

$$\frac{\partial[n^{pat}(t) - n^{pat}(t-\tau)]}{\partial\tau} = -\frac{2\alpha r^2(r + \sqrt{4\alpha^2 rV(\tau) + r^2} + 2\alpha^2 V(\tau))}{\sqrt{4\alpha^2 rV(\tau) + r^2} \times (r + \sqrt{4\alpha^2 rV(\tau) + r^2})^2} \times$$
$$(N - n_0)\frac{dV(\tau)}{d\tau} e^{-\frac{2\alpha rV(\tau)}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}t}((t-1)e^{\frac{2\alpha rV(\tau)\tau}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}} - 1 - t). \tag{40}$$

The sign of this derivative depends on the sign of expression

$$\mathbf{A} = ((t-1)e^{\frac{2\alpha rV(\tau)\tau}{r+\sqrt{4\alpha^2 rV(\tau)+r^2}}} - 1 - t). \tag{41}$$

This last may be positive or negative depending on relative size of the value function $V(\tau)$. It depends on the length of the life-cycle, τ. For longer life-cycles it is greater then one and the subsequent expression (41) is positive for almost all t's, yielding negative derivative sign, for shorter life-cycles it is negative for most t's, yielding positive derivative sign. Observe also that since this expression depends on time there is always some initial period when it is negative for $t \to 0$ and always positive for $t \to \infty$. In effect this means that changes in patents' length may influence the effective range of products in different directions. This last phenomena is illustrated in Figure 7.

The effective product's range is first increasing with patent's length,

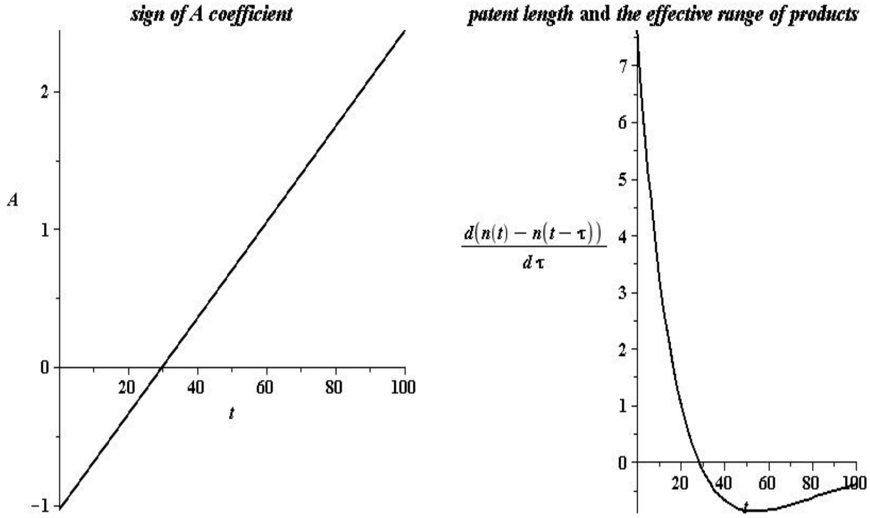

Figure 7: Varying sign of the effect of patents length

but afterwards it decreases. This point to the fact that this effective range is subject to effects of the length of the life-cycle. To consider them, decompose the above derivative:

$$(42) \qquad \frac{\partial[n^{pat}(t) - n^{pat}(t-\tau)]}{\partial\tau} = \frac{\partial n^{pat}(t)}{\partial\tau} - \frac{\partial n^{pat}(t-\tau)}{\partial\tau} = PPE + CE.$$

We identify the first component of (42) with the potential profit effect (PPE) and the second - with compensation effect (CE). Shorter patent length means that the agent is able to derive less profit from the usage of the given product which in turn lowers his incentives to invest into the variety expansion. This is the first effect, as this directly influences the total range $n(t)$ through rate of investments. To observe it, consider the derivative $\frac{\partial n^{pat}(t)}{\partial\tau}$ which amounts to:

$$PPE =$$
$$= \frac{\partial n^{pat}(t)}{\partial \tau} = (n_0 - N) \times \frac{\partial [e^{-X(\tau)t}]}{\partial \tau} = (N - n_0) e^{-X(\tau)t} \times \frac{\partial [X(\tau)]}{\partial \tau} \times t > 0;$$
(43)
$$X(\tau) > 0.$$

Here $X(\tau)$ denotes some expression of exogenous model's parameters and is function of τ only, not of time t.
This effect is quiet standard and describes how the evolution of products range changes with changes in the length of the life-cycle of every product in this range. The longer is the life-cycle, the more incentives the agent has to invest into the introduction of new products. It is the decreasing function of time (since the form of variety expansion path),but always positive.

The second effect is calculated in a similar fashion by substituting $t - \tau$ for t:

$$CE = -\frac{\partial n^{pat}(t-\tau)}{\partial \tau} =$$
(44)
$$= -(n_0 - N) \times \frac{\partial [e^{-X(\tau)(t-\tau)}]}{\partial \tau} = (N - n_0) e^{-X(\tau)(t-\tau)} \times (X(\tau) + (t-\tau)\frac{\partial [X(\tau)]}{\partial \tau}).$$

Unlike the first component this one is (almost always) negative. As time flows it becomes bigger in size and gradually offsets the influence of the first effect. This is illustrated by the Figure 8

It can be seen from this figure that as length of the life-cycle grows, the effective range is growing at initial stage and decreases afterwards. The time at which the compensation effect outperforms the potential profit effect does not depend on the parameter τ and is defined from the very form of the variety expansion process.

Economic intuition behind this result is clear: at initial stage of development there are a lot of opportunities to develop new versions of the basic product for the monopolist. Hence, increase in the length of the life-cycle gives him/her higher possibilities to develop all these new products and derive profit from them. Thus potential profit effect is high. At the same time the increase in this length affects negatively the rate of investments into variety expansion, but this effect is not significant. As time flows, more products are introduced into the market, but effective range decreases, as the variety expansion process slows down. At this point the effect of expected profit from all of the new products in the additional range from the increase in the

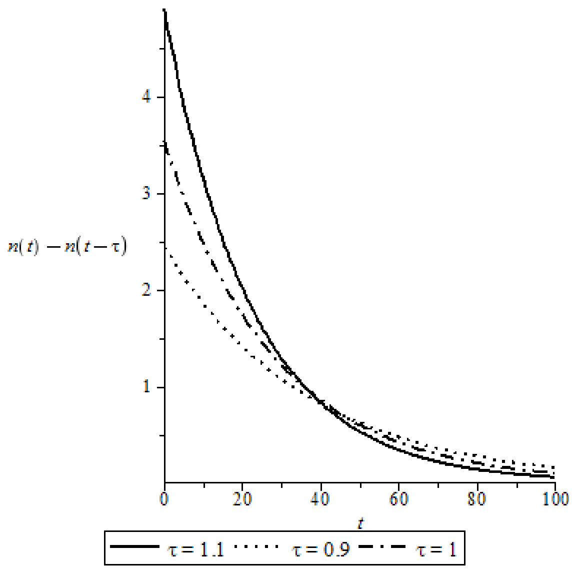

Figure 8: Changes in the effective products ranges in time

length of patent wears down, as there is lesser mass of products in this range. Simultaneously it becomes less important to sustain the given effective range, as it increases from the length of the patent, hence compensation effect is larger.

It has to be noted, that the above discussion does not mean that the variety expansion process may negatively depend on the length of the life-cycle. As it has been seen, its derivative is always positive. Hence the process of variety expansion is always boosted by the increasing length of the life-cycle. This happens because the effective products range is a characteristic of a speed of variety expansion, not of its overall level. Thus with longer life-cycles the level (stock) of products variety is always increasing, while the rate of their introduction and as a result, the effective patented range, not always increases but only at the initial stage of development while decreasing afterwards. Infinite-time horizon model may be considered as a patent model with infinite patent length in this respect. It can be shown that the patent model is equivalent to the infinite-time horizon model with $\tau \to \infty$. The comparison of $n(t)$ dynamics with the same initial range for infinite-time horizon and patent model at Figure 9 illustrates the ideas of higher stock and lower dynamics.

In this figure the infinite-time variety has higher level then limited life-cycle variety at every point, but the intensity of addition of new products is

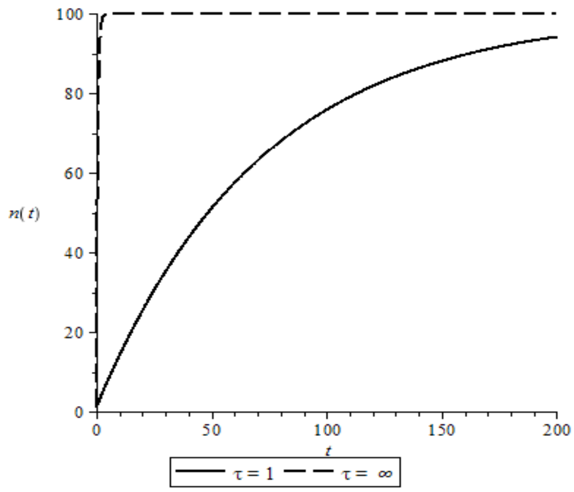

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Figure 9: Finite and infinite life-cycles and variety expansion

higher for infinite time case at the beginning, while lower afterwards.

Quality growth essentially depends on the length of the life-cycle of products also. The longer the life-cycle, the closer patent's model quality dynamics is to the infinite-time one. The quality growth displays only the potential profit effect as long as one consider single product quality investments: the longer the life-cycle of the product, the higher is the maximal attainable quality of this product and thus the higher is the expected stream of profits from the development of this product. Hence,

$$\frac{\partial q_i^{pat}(t)}{\partial \tau} > 0. \tag{45}$$

This can be checked by directly computing the derivative of (30) w.r.t. to τ. There is no ambiguity in the effect of the length of the life-cycle onto the development of every separate product i from the effective range of products.

However there are two different effects on the aggregate level of quality development. Observe that at any given time t there is a mass of products under the control of the innovator which qualities might be developed. This mass is given by the effective products range defined above. The level of aggregate (across products) quality development is then given by the quantity, denoted $\mathbf{Q}$:

$$\mathbf{Q} = \int_{n(t-\tau)}^{n(t)} q_i(t)di. \tag{46}$$

We proceed in the same fashion as for the case of variety expansion: calculate the derivative and decompose it. For this we use usual rules of integration and derivative w.r.t. to the parameter. Expressions for this case are very cumbersome and not displayed. The quantity being computed is:

$$\frac{\partial \mathbf{Q}}{\partial \tau} = \int\limits_{n(t-tau)}^{n(t)} \frac{\partial q_i^{pat}(t)}{\partial \tau} di. \tag{47}$$

where we make use of the interchange of the order of differentiation and integration due to Fubini's theorem (and its extensions).

Surprisingly enough, the effect of the products life-cycle on this aggregate measure of quality growth follows the same pattern as for variety expansion. The derivative sign is initially positive but changes to negative at the mature stage of products range development (for higher t). The compensation effect for qualities influences the range of the integral itself, that is, the total quantity of quality improving innovations.

Indeed the total effect of changes in patents length on the overall qualities development is defined from 2 sources: potential profit effect for quality of every single product within the effective range and by the scale of the effective range itself. It is known from above discussion, that this effective range tends to shrink along increase of the life-cycle's length for mature stages of development; thus the total range of quality investments shrinks also. The effect of the range's length outweighs the effect of potential profit for every single product and thus the overall behaviour of the $\mathbf{Q}$ is determined by the changes in effective range of products. This is illustrated at Figure 10.

It should be noted, that for any $\tau < \infty$ quality growth for every separate product is less then the maximal level for infinite-time case. Figure 11 displays quality innovations for the finite ($\tau = 50$) an infinite life-cycles for products $i = 1, i = 50$ with the same parameter values as before (left picture) and the influence of the changes in life-cycle length on the quality development of any single product (picture to the right, $i = 1$).

6 Discussion

In this paper an extension of the basic infinite-time model of (Bondarev 2012) which allows to consider finite-time life-cycles (patents) of products together with the infinite-time process of products' generation is developed. Every product has an effective life-time within which it has a substantial demand associated with it and hence is capable of generating profit for the innovator.

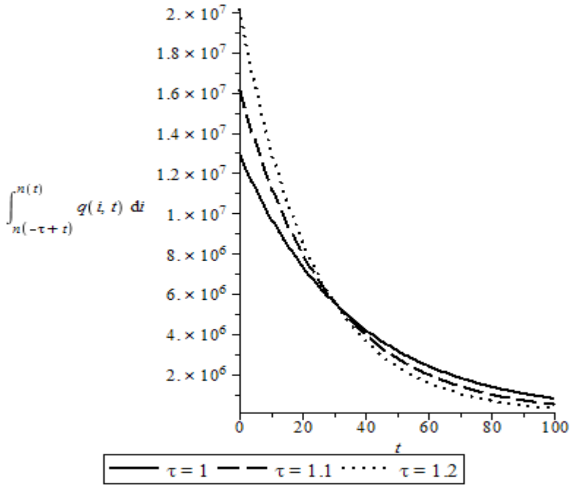

Figure 10: Finite and infinite life-cycles and overall quality innovations

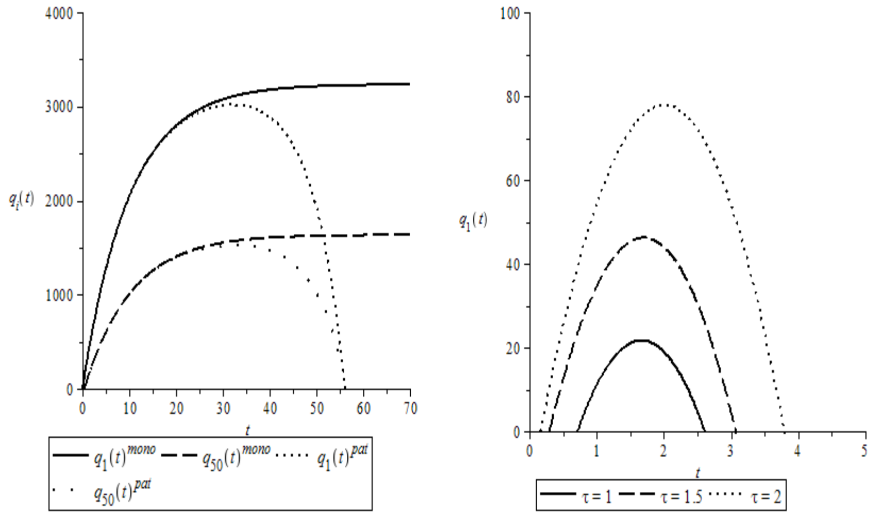

Figure 11: Effect of the length of the patent on the quality development of separate products

Although not very much products literally disappear from the market if to take some reasonable scope of analysis, rather big portion of existing variety of products is renewed within some periods of time. This means one has to formulate a logic of behaviour of an innovating agent which would combine infinite planning horizon with finite life-time of products.

In some respect the presented model is close to the strand of literature on sequential innovations, since it allows for finite life-times of products and thus explicitly the 'sequence' of innovations is introduced in a continuous way. Apart from the patenting problem this may be viewed as a stream of finitely living innovations, as in (Chang 1995) which is the closest paper. In this paper Chang assumes two competing firms which are innovators and in two periods of time where in the first period firm 1 is introducing some new product and in period two the other firm is introducing another product which is an improved version of the first product, introduced previously. In this respect this describes cumulative innovations pattern. In current model the process of variety expansion is going in a cumulative way, since the next product cannot be introduced without introducing the previous one. At the same time smooth time structure and infinite-time horizon for this process are assumed. Moreover every of the introduced products has its own new dimension of quality. Despite all these differences the general idea of cumulative innovations looks like the same. With finite lifetimes of these products one has some notion of creative destruction here in the sense of Schumpeter but with the difference that it is not the creation of new product which destroys the previous one, but the time-structure f the model itself.

To incorporate the results of this paper into the literature on optimal patents duration and scope one might treat τ as the patent's length, although identical across products. In the literature on patent's length two different distinctive notions came into being since the work (Nordhaus 1967). They are the scope (breadth) of the patent and effective length of patent versus its statutory length.

Statutory length of the patent is the length of patent explicitly granted to the innovator whereas effective length is the actual time during which the patent is effective - that is prevents others from using or producing the same product, (Scotchmer 1996). In the given model this distinction is not that explicit, as there is only one parameter, τ, which is assumed to be the statutory patent's length. At the same time one has the notion of effective mass of products and this is varying over time and does depend on the patent's length. This effective mass, $n(t) - n(t - \tau)$, might be treated as the effective protection. It has been seen that while the rate of variety expansion is slowing down over time this effective mass is decreasing and statutory patent's length may have effects of different directions on it. It is not true that the stronger is the protection the larger is this effective mass of products. Rather the direction of the effect is defined by the state of the industry: whether it is young (low products range) or mature (high products range). The same is true for the overall development of all these products. At the same time for every single product the effect of the patent's length is unambiguous: the longer is the patent; the higher is the quality being developed for every given product.

It seems that such an effect may cause under-provision of variety expansion in favour of quality growth for already introduced products when longer patents are introduced in the mature industry, while for some industries variety expansion is overextended while quality of existing products is underdeveloped. As an example, one may consider the open software development, where quality is much more underdeveloped in favour of the great variety of products and this may actually happen because of the lack of patents which would stimulate the opposite distribution of investments.

The other feature of patenting literature is taking into account not only the length (duration) but also the breadth of the patent, e.g. how much close products it may cover. In the given framework it would mean that the single patent is granted upon the invention of the product $n(t_0)$ to all products $n(t_0) + \epsilon$. This would make no difference in the given model since patents are granted automatically and with no costs. But observe that in the absence of competitors the notion of the breadth of patent cannot be relevant here. Alternatively one may tract the given patenting pattern as that of a maximal possible breadth, since all of the products in $n(t)$ range are not substitutes to each other. Close enough products are already grouped within the single $q_i(t)$ dynamics and thus the patent on the product i covers all these products already. If one would like to consider lesser breadth of patents that may be done by granting shorter length of patents for each i since then not all the range of possible products associated with this invention should be covered. In an effect it would mean lesser range of qualities (substitutable products) to be developed by the innovator. In this respect length of the patent, τ, implicitly defines the breadth of the patent also.

To conclude note that the suggested model allows for some extensions into the field of patenting. Namely, its rather easy to consider patents of variable length, as in (Cornelli and Schankerman 1999). For that one has to replace τ parameter by some continuous function to obtain the same kind of modification with respect to patents as one may has with respect to investment efficiencies in $\gamma(\bullet)$ function. This function might be considered even non-monotonic. The only difficulty one would have then is with obtaining explicit solution for variety expansion dynamics. Another immediate extension is to allow for some patenting costs and include them into the variety expansion dynamics like a constant (for constant patents) or a function of a number of product (for variable patents' lengths). Last extension which might be of interest is to consider the response of the dynamics under patents to different market structures.

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