Levy subordinator model: A two parameter model of default dependency

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"There is no "right" model. The best you can do is pick a model that mimics the most important behavior of the underlyer in your market. Then add perturbations if necessary."
– Emanuel Derman in "Modeling the Volatility Smile".

Abstract

The May 2005 crisis and the recent credit crisis have indicated to us that any realistic model of default dependency needs to account for at least two risk factors, firm-specific and catastrophic. Unfortunately, the popular Gaussian copula model has no identifiable support to either of these. In this article, a two parameter model of default dependency based on the Lévy subordinator is presented accounting for these two risk factors. Subordinators are Lévy processes with non-decreasing sample paths. They help ensure that the loss process is non-decreasing leading to a promising class of dynamic models. The simplest subordinator is the Lévy subordinator, a maximally skewed stable process with index of stability 1/2. Interestingly, this simplest subordinator turns out to be the appropriate choice as the basic process in modeling default dependency. Its attractive feature is that it admits a closed form expression for its distribution function. This helps in automatic calibration to individual hazard rate curves and efficient pricing with Fast Fourier Transform techniques. It is structured similar to the one-factor Gaussian copula model and can easily be implemented within the framework of the existing infrastructure. As it turns out, the Gaussian copula model can itself be recast into this framework highlighting its limitations. The model can also be investigated numerically with a Monte Carlo simulation algorithm. It admits a tractable framework of random recovery. It is investigated numerically and the implied base correlations are presented over a wide range of its parameters. The investigation also demonstrates its ability to generate reasonable hedge ratios.

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Modeling dependent events and their correlations has remained a challenging issue. An understanding of its implications is needed for pricing derivative instruments referencing a collection of credit names, and it is a subject of interest outside the realm of credit derivatives as well. Many models have been developed for pricing the correlation products, but the market standard has remained the Gaussian copula model in spite of all the criticisms it received for its alleged role in the recent credit crisis. As has been emphasized by many, it has been well-known that the Gaussian copula model has serious limitations and is inadequate as a model of default dependency.

Major attraction of the Gaussian copula model is its simplicity and tractability. It can easily be calibrated to individual hazard rate curves. It can be formulated in closed form providing a semi-analytical framework for pricing. It admits efficient pricing with recursive methods or Fast Fourier Transform techniques. As we will see in this article, there exists another simple and tractable model similar in architecture that also enjoys these properties. Unlike the Gaussian copula model, it is a dynamical two-parameter model capable of offering a reasonable explanation of the correlation smile. The two parameters provide the two measures necessary to assess dependency risk, a measure of correlation and that of the likelihood of a catastrophe. The model is based on the Lévy subordinator, an $\alpha = 1/2$ stable process maximally skewed to the right, whose distribution function is expressible in closed form and is known as the Lévy distribution. Though it is inevitable that, with a model of such few parameters, there is bound to exist a residual smile, the ability to capture the smile characteristics will be helpful in sensitivity analysis and stress testing.

Issues with the Gaussian copula model have been addressed before. Brigo, Pallavicini and Torresetti [2010] provide a discussion of its limitations and an account of the developments in this field. They also note that, since the start of the credit crisis, the probability mass associated to a catastrophic or armageddon event, i.e. the default of the entire pool of credit references, has increased dramatically. The need for such a catastrophic scenario while pricing the super-senior tranche was noted earlier by many authors, see for instance Balakrishna [2009]. However, in many of the models, such a scenario needs to be enforced, somewhat artificially. An attractive feature of the subordinator models discussed here is that such a scenario arises naturally as a consequence of a drift term that is well-known to be a natural component of the dynamics of subordinators.

During the earlier May 2005 crisis, a so-called correlation dislocation is said to have taken place. As has been pointed out by many, this is attributable to increased firm-specific risk. The May 2005 crisis and the recent credit crisis have thus indicated to us that any realistic model of default dependency needs to account for at least two risk factors, firm-specific and catastrophic. Unfortunately, the Gaussian copula model has no identifiable support to either of these. The two risk factors necessitate introduction of at least two parameters into any realistic model of default dependency as in the models discussed in this article.

Pricing models are helpful in computing hedges. Delta-hedges are sensitivities to the underlyings and are relatively more important. When the underlyings are credit default swaps as in our case, sensitivities are determined with respect to the individual hazard rates, assuming an environment wherein the parameters governing default dependency are relatively stable. Hazard rates are intensity-like variables having dimension of inverse time. Many of the pricing models have other intensity-like variables in them. It is not obvious while computing delta-hedges whether such variables need to be bumped along with the hazard rates. An important virtue of the Gaussian copula model is that it has no such additional intensity-like variables. As we will see in this article, the subordinator models discussed here share this virtue as they too have no additional intensity-like variables.

Models of the volatility smile have taught us that an explanation of the smile alone is not a guarantee for obtaining satisfactory hedge ratios. Local volatility models, though capable of providing a perfect fit to the smile, are criticized for giving rise to hedge corrections inconsistent with typical market behavior. Models respecting some concept of stationarity have been pursued to obtain better hedge ratios. Within the context of the correlation smile, subordinator models discussed here attempt to achieve a similar goal. Numerical investigation of the Lévy subordinator model demonstrates its ability to generate reasonable hedge ratios under a wide range of its parameters.

In Balakrishna [2007] and some of the literature in the field, it is found that the modeled loss distribution displays one or more bumps along its tail. Even if such a distribution is able to reproduce the market prices providing an explanation of the correlation smile, it is not immediately obvious whether the bumps are a realistic feature of the distribution or an artifact of the model. Models capable of reproducing the market prices without such bumps, even if less accurate, can potentially give rise to better behaved prices and sensitivities. As it turns out, the Lévy subordinator model presented here exhibits no such bumps along the tail of its default probability distribution.

The article is organized as follows. To start with, a brief review of subordinators is presented in section 1. In section 2, a one-factor framework of default dependency is formulated starting with the de Finetti theorem from probability theory. Two classes of models arise naturally that are termed type-I and type-II. These models formulated on an infinitely large homogeneous collection to start with are further extended in section 3 to be applicable to finite heterogeneous collections and to general hazard rate curves. Type-I models are well-known that include many of the reduced form models that have a systemic component to their dynamics. Type-II models are formulated as new that include the subordinator models introduced here but it turns out, as discussed in section 4, that the popular copula models can also be recast into this framework. Section 5 discusses type-II models governed by stable subordinators, in particular the Lévy subordinator model based on the $\alpha = 1/2$ stable subordinator. Section 6 discusses some niceties of the $\alpha = 1/2$ choice found numerically as appropriate. An attractive feature of the Lévy subordinator model is the possibility of semi-analytical pricing that is discussed in section 7. Section 8 discusses the large homogeneous pool approximation. Section 9 is a short presentation of an efficient pricing technique based on the Fast Fourier Transform. The model can also be investigated numerically with a Monte Carlo simulation algorithm as discussed in section 10. As is now well appreciated, random recovery is helpful in better pricing of the senior tranches, and section 11 presents a tractable framework of random recovery. The effect of initial conditions is analyzed in section 12. Section 13 discusses default contagion within the present framework. Some possible extensions such as intensity based modeling are discussed in section 14. Section 15 concludes with a discussion and a brief summary. Figures 1-16 present the results of a numerical investigation into the model's implications.

1 A Brief on Subordinators

Lévy processes, and hence subordinators, is a well researched branch of mathematics. For the sake of completeness, the following gives a brief review of subordinators.

A stochastic process is an indexed family of random variables. A continuous-time stochastic process is such a process indexed over continuous time. A Lévy process is a continuous-time stochastic process starting as zero that has independent and stationary increments, and is stochastically continuous. Independence is a statement that increments over disjoint time-intervals are independent random variables. Stationarity is a statement that increment over any time-interval is distributed with its time-dependence only on the length of the time-interval. Stochastic continuity means that jumps are random and rare, that the probability of a jump occurring at a given time is zero. A realization or a sample path of a stochastic process is a sampling of each of the random variables in the family. Subordinators are real-valued Lévy processes with non-decreasing sample paths.

Given a subordinator $X(t)$, its Laplace transform, or equivalently its Laplace exponent $\eta(u)$, is given by

$$e^{-t\eta(u)} = \mathrm{E}\left\{e^{-uX(t)}\right\}, \quad u \geq 0, \tag{1}$$

where $\mathrm{E}\{\}$ denotes expectation value. The specific time-dependence assumed for the Laplace transform above is a consequence of the properties of the subordinator as a Lévy process. It follows that the Laplace exponent of a sum of two independent subordinators is the sum of their Laplace exponents. An important result for Lévy Processes is the Lévy-Khintchine formula. In the case of subordinators, it gives for the Laplace exponent

$$\eta(u) = bu + \int_0^\infty \lambda(dy)\left(1 - e^{-uy}\right). \tag{2}$$

Here $b \geq 0$ is called the drift coefficient that contributes a non-negative drift bt to $X(t)$ so that $X(t) \geq bt$ for all t. $\lambda(dy)$ is called the Lévy measure that is required to satisfy $\int_0^\infty \lambda(dy)\min(y, 1) < \infty$. It is also true that any function of the above form is the Laplace exponent of a subordinator.

An important subclass of subordinators are stable subordinators. Their Laplace exponent is $\eta(u) = au^\alpha + bu$ for some constant a and index of stability $\alpha \in (0, 1)$, obtainable from the Lévy measure $a\alpha\left[\Gamma(1-\alpha)\right]^{-1} y^{-1-\alpha}dy$. Stable subordinators are also a subclass of stable processes having index of stability $\alpha \in (0, 1)$ and maximally skewed to the right, that is, their skew parameter set to one. It follows that stable subordinators (more generally stable processes) feature an additive property, that is if $X(t)$ and $Y(t)$ are two independent stable subordinators with index of stability α (and parameters b_X, a_X and b_Y, a_Y), then $Z(t) = pX(t) + qY(t)$ is also a stable subordinator with index of stability α (having $b_Z = pb_X + qb_Y$ and $a_Z = p^\alpha a_X + q^\alpha a_Y$).

Inversion of the Laplace transform gives us the probability density function of the random variable $X(t)$ at time t, or equivalently its cumulative distribution function $g_t(x)$ given by

$$g_t(x) = \mathrm{E}\left\{1_{X(t)\leq x}\right\}, \tag{3}$$

where $1_{\{\dots\}}$ is the indicator function. No closed form expression is available for $g_t(x)$ in general for the stable subordinators, except for the $\alpha = 1/2$ stable subordinator called the

Lévy subordinator[1]. In the case of the Lévy subordinator that has $\eta(u) = a\sqrt{u} + bu$, the distribution is known as the Lévy distribution and is given by

$$\begin{aligned} g_t(x) &= 2N\left(-at/\sqrt{2(x-bt)}\right), \\ \partial_x g_t(x) &= \frac{1}{2\sqrt{\pi}} at(x-bt)^{-3/2} e^{-\frac{1}{4}(at)^2/(x-bt)}, \end{aligned} \tag{4}$$

where $N()$ is the cumulative standard normal distribution function. This includes a non-negative drift component bt discussed above so that $g_t(x)$ and $\partial_x g_t(x)$ can be taken to be zero for $x < bt$.

Lévy distribution (4) is also the first passage time distribution of a Brownian motion over time variable $x \geq bt$ with the barrier set at $at/\sqrt{2}$. The first passage time distribution of a Brownian motion with drift rate $c\sqrt{2}$, that is of a Gaussian process, is also available in closed form and is known as the inverse Gaussian distribution. The associated subordinator is the inverse Gaussian subordinator that has $\eta(u) = a\left(\sqrt{u+c^2} - c\right) + bu$ and

$$\begin{aligned} g_t(x) &= N\left(-at/z + cz\right) + e^{2act} N\left(-at/z - cz\right), \\ \partial_x g_t(x) &= \frac{2}{\sqrt{2\pi}} ate^{act} z^{-3} e^{-\frac{1}{2}\left((at)^2 z^{-2} + c^2 z^2\right)}, \quad z = \sqrt{2(x-bt)}. \end{aligned} \tag{5}$$

Though not a stable subordinator, inverse Gaussian subordinator is useful as the natural extension of the Lévy subordinator. Its Lévy measure is that of the Lévy subordinator damped exponentially with $e^{-c^2 y}$. Other stable subordinators are also generalized in this way with an exponential damping called tempering of the Lévy measure.

Stable subordinators feature a scaling property such that $(at)^{-1/\alpha}(X(t) - bt)$ is independent of t in distribution, that is $g_t(x)$ is a function of the combination $(at)^{-1/\alpha}(x - bt)$. This scaling property is evident in the behavior of the tail of their distributions. The long tail of the distribution of a stable subordinator $X(t)$ obeys a power-law decay with

$$g_t(x+bt) \to 1 - at\left[\Gamma(1-\alpha)\right]^{-1} x^{-\alpha}, \quad \text{for large } x. \tag{6}$$

At the very short end, the log-distribution exhibits a power-law behavior with

$$-\ln g_t(x+bt) \to (1-\alpha)\left(\alpha(at)^{1/\alpha}\right)^{\alpha/(1-\alpha)} x^{-\alpha/(1-\alpha)}, \quad \text{for small } x. \tag{7}$$

A consequence of the power-law decay for large x is that stable subordinators have both infinite mean and infinite variance.

The scaling property enables one to express the distribution function $g_t(x)$ for any stable subordinator in terms of that of a standardized random variable Z. Random values of Z can be generated using Kanter's method (special case of the Chambers-Mallows-Stuck method for a stable distribution) from two independent random numbers: an exponentially distributed

[1] In the literature, one sometimes finds the term "Lévy subordinator" used for all subordinators. As in Applebaum [2005], it is used here just for the $\alpha = 1/2$ stable subordinator. Similarly, the term "Lévy distribution" is used here just for its distribution.

W with unit mean and a uniformly distributed $\theta \in (0, \pi)$ (see for instance Zolotarev [1986]). For the stable subordinator standardized to yield $\eta(u) = u^\alpha$, this can be obtained as

$$Z = \frac{\sin(\alpha\theta)}{\sin(\theta)} \left[\frac{\sin((1-\alpha)\theta)}{W \sin(\theta)} \right]^{(1-\alpha)/\alpha}. \tag{8}$$

The distribution function $f_\alpha(z)$ of Z thus computed depends on just α and hence can be used at any time t to obtain $g_t(x) = f_\alpha(z)$ given $z = (at)^{-1/\alpha}(x - bt), x > bt$.

The most basic subordinator is the Poisson process having $\eta(u) = \lambda(1 - e^{-u})$. It has unit-size jumps occurring at intensity λ. The form of the Laplace exponent in (2) indicates that a subordinator should be constructible from Poisson processes with varying jump sizes. Lévy-Itô decomposition theorem applied to subordinators provides us with such a construction that reads, in the differential form,

$$dX(t) = bdt + \int_{y=0}^{\infty} y dN(\lambda(dy), t). \tag{9}$$

Here $N(\lambda(dy), t)$ is a Poisson process of intensity $\lambda(dy)$ associated with the interval $(y, y + dy)$ and $dN(\lambda(dy), t)$ is its increment over the time interval $(t, t + dt)$. Poisson increments associated with disjoint t and disjoint y intervals are independent random variables. If $N(\lambda(dy), t)$ jumps up by one at time t, $dN(\lambda(dy), t)$ causes $X(t)$ to jump up by y at time t. Over an infinitesimal time interval $(t, t + dt)$, we have

$$\mathrm{E}\left\{\exp\left[-uydN(\lambda(dy), t)\right]\right\} = \exp\left[-dt\lambda(dy)\left(1 - e^{-uy}\right)\right]. \tag{10}$$

This follows simply on noting that $dN(\lambda(dy), t)$ takes values zero and one with probabilities $1 - dt\lambda(dy)$ and $dt\lambda(dy)$ respectively, irrespective of the value of $N(\lambda(dy), t)$. It is now straightforward to obtain the Laplace exponent (2) summing up contributions arising from disjoint t and disjoint y intervals.

2 One Factor Formulation

It is instructive to proceed formulating the model starting with an infinitely large homogeneous collection of credit names. This offers an intuitive insight into its structure that has a basis in probability theory due to a theorem attributed to de Finetti. The model then evolves as a natural extension of this formulation.

Given an infinite homogeneous collection of credit names, consider a configuration of its defaulted and undefaulted states at some future time t. Let us say our interest is not in the actual assignment of states among the names, but only on the fraction of names in the defaulted states. The collection being infinite, this fraction ν can take any value from zero to one. The configuration is thus characterized by just one common risk factor that is identifiable with the fraction ν.

Given such a configuration, the assignments of fraction ν names defaulted and the rest undefaulted are all equally possible. The probability of finding a given name defaulted is ν. Because the collection is infinite, the probability of finding a second name defaulted given that the first one has been is also ν, so that the probability of finding both the names defaulted is ν^2. More generally, the probability of finding a set of j names defaulted and k names undefaulted is $\nu^j(1-\nu)^k$. In other words, the states can be treated as independent variables. If the configuration under consideration itself has a probability density function $\partial_\nu F_t(\nu)$, $F_t(\nu)$ being the cumulative default distribution function, the probability of finding j names defaulted and k names undefaulted can hence be written as

$$P_{[j,k]}(t) = \int_0^1 d\nu \partial_\nu F_t(\nu) \nu^j (1-\nu)^k. \tag{11}$$

This is a one-factor formulation since, given a value of just one variable ν, defaults get treated as independent random variables. This intuitive result has a basis in probability theory due to a theorem attributed to de Finetti. It is sometimes helpful to express $F_t(\nu)$ in terms of a random variable $\mathcal{V}(t)$ taking values in $[0,1]$ as

$$F_t(\nu) = \mathrm{E}\left\{1_{\mathcal{V}(t)\leq\nu}\right\}. \tag{12}$$

Random variables $\mathcal{V}(t)$ for all $t>0$ with $\mathcal{V}(0)=0$ can be viewed together as defining a stochastic process that we may denote for simplicity as $\mathcal{V}(t)$ itself.

Some general characteristics of $F_t(\nu)$ can be inferred to start with. It is of course a non-decreasing function of ν. As the May 2005 crisis has indicated, there can be firm-specific contributions to defaults and hence, in a realistic model, one expects a minimum value $\nu_{\min}(t)$ for ν below which $F_t(\nu)=0$. This is because firm-specific contributions are mutually independent and hence, in an infinitely large homogeneous collection, one expects at least a fraction of names equal to the firm-specific default probability to have defaulted. Further, it has been usual to consider $F_t(\nu)\to 1$ as $\nu\to 1$. But, as the recent credit crisis has indicated, there can be a non-zero probability for all the names in the collection to have defaulted by time t so that $F_t(\nu)$ should be allowed to tend to some $F_{\max}(t)<1$ as $\nu\to 1$. For ν in-between, $F_t(\nu)$ is expected to be a decreasing function of t. Note that $\mathcal{V}(t)$ can not be decreasing as a function of t since we do not allow for recovery of defaulted names. Given two times t_1 and t_2, $t_1<t_2$, we have

$$F_{t_1}(\nu) - F_{t_2}(\nu) = \mathrm{E}\left\{1_{\mathcal{V}(t_1)\leq\nu} - 1_{\mathcal{V}(t_2)\leq\nu}\right\} = \mathrm{E}\left\{1_{\mathcal{V}(t_1)\leq\nu,\mathcal{V}(t_2)>\nu}\right\} - \mathrm{E}\left\{1_{\mathcal{V}(t_1)>\nu,\mathcal{V}(t_2)\leq\nu}\right\}. \tag{13}$$

For a non-decreasing stochastic process $\mathcal{V}(t)$, the last expectation above is zero. Assuming that there are nonzero contributions to the first term, as is usually the case, we thus have $F_{t_1}(\nu) > F_{t_2}(\nu)$ for ν in-between.

Formulation (11) has all the complexities of the model bundled into one common function $F_t(\nu)$. We may proceed with it by modeling $\mathcal{V}(t)$, but there is an interesting and more flexible alternate formulation that emphasizes individual behavior. Let us rewrite (11) as

$$P_{[j,k]}(t) = \int_0^1 dF [p_t(F)]^j [1-p_t(F)]^k, \tag{14}$$

where $p_t(F)$ is the inverse of $F_t(\nu)$ defined by $p_t(F_t(\nu))=\nu$, $\nu\geq\nu_{\min}(t)$. Because $F_t(\nu)$ is now an integration variable, both of its t and ν dependences have been conveniently dropped. F can be viewed as a uniformly distributed random variable. Note that all the functional intricacies are now bundled into the conditional individual default probability $p_t(F)$.

Generic characteristics of $F_t(\nu)$ discussed above imply similar ones for $p_t(F)$. Equivalently, they can be inferred from F viewed as an indicator of economic conditions, with higher F corresponding to less favorable circumstances. Conditional individual survival probability $q_t(F) = 1 - p_t(F)$ is a non-increasing function of F for all $t > 0$. With $F = 1$ corresponding to the worst case scenario, that of total collapse with all the names defaulting, we have $q_t(1) = 0$. A non-zero probability of such a scenario implies that $q_t(F) = 0$ for some $F \geq F_{\max}(t)$. At the $F = 0$ end, the common variables are ineffective in causing defaults so that $q_t(0)$ is firm-specific. As noted earlier, $F_t(\nu)$ for ν in-between is a decreasing function of t. Consequently, for F in-between, $q_t(F)$ is decreasing as a function of t, starting at one and ending up at zero as t runs from zero to infinity.

The characteristics of $q_t(F)$ suggest that $1-q_t(F)/q_t(0)$ can be viewed as the cumulative distribution function of a random variable $\Phi_i(t)$ taking values in $[0, 1]$. In other words,

$$q_t(F) = q_t(0)\mathrm{E}\left\{1_{\Phi_i(t)\geq F}\right\}. \tag{15}$$

Random variables $\Phi_i(t)$ for all $t > 0$ can be viewed together as defining a stochastic process, denoted for simplicity as $\Phi_i(t)$ itself. It is a non-increasing process with $\Phi_i(0) = 1$ and $\Phi_i(\infty) = 0$. There is one such independent stochastic process for each name in the collection, hence the name-subscript. In the homogeneous collection under discussion here, they are identically distributed. For the unconditional individual default probability $P(t)$, or equivalently its survival counterpart $Q(t) = 1 - P(t)$, we then have

$$Q(t) = \int_0^1 dF q_t(F) = q_t(0)\mathrm{E}\left\{\int_0^1 dF 1_{\Phi_i(t)\geq F}\right\} = q_t(0)\mathrm{E}\left\{\Phi_i(t)\right\}. \tag{16}$$

Satisfying this ensures that the model gets calibrated to individual hazard rate curves.

Models based on formulation (11) are referred to here as type-I models. Note that there is only one stochastic process $\mathcal{V}(t)$ in its one-factor formulation modeling the common factor. Many reduced form models belong to this class. Intensity based models that have a systemic component to their stochastic default intensities are type-I models. Because $\mathcal{V}(t)$ is directly related to the loss process (for uniform recovery rates), many loss process models can also be viewed as type-I models.

In contrast, formulation (14) appears new. Here the collection, though homogeneous, has one independent stochastic process $\Phi_i(t)$ for each of its names. A model of these $\Phi_i(t)$s is referred to here as a type-II model. These models are in some sense like structural models. It turns out that the popular copula models can be reformulated as belonging to this class. The reformulation highlights their unnaturalness modeling $\Phi_i(t)$s as static objects. Interestingly, as we will discover, there also exist new models, perhaps more promising, that model $\Phi_i(t)$s as genuinely dynamic stochastic processes.

3 Finite Heterogeneous Collection

The previous discussion was confined to an infinite homogeneous collection leading to two formulations of default dependency. The formulations can be considered to be applicable as such to a finite homogeneous collection under the assumption that the latter can be extended to an infinite one. The two formulations result in different models depending on the choices made for $\mathcal{V}(t)$ or $\Phi_i(t)$s. The following introduces further extensions of the two homogeneous versions to heterogeneous collections.

Consider type-I formulation (11) to start with. It is convenient to work with $\Lambda(t) = -\ln[1-\mathcal{V}(t)]$, a non-decreasing process taking values in $[0,\infty]$ with $\Lambda(0) = 0$ and $\Lambda(\infty) = \infty$. The joint survival probability $Q_\Omega(t)$ for a list of names in Ω can then be expressed as

$$Q_\Omega(t) = \mathrm{E}\left\{e^{-\sum_{i\in\Omega}\Lambda_i(t)}\right\}. \tag{17}$$

A name-subscript i has been attached to $\Lambda(t)$ to make it applicable to a heterogeneous collection. Since we had only on process $\mathcal{V}(t)$ to start with, expectation $E\{\}$ is taken with respect to a common process $X(t)$ in a one factor formulation. All $\Lambda_i(t)$s are considered to be driven by $X(t)$, for instance as $\Lambda_i(t) = a_i X(t) + b_i t$ for some parameters a_is and b_is. Expression (17) indicates that defaults are independent given a realization of $X(t)$. For the unconditional individual survival probability $Q_i(t)$, we now have

$$Q_i(t) = \mathrm{E}\left\{e^{-\Lambda_i(t)}\right\}. \tag{18}$$

Satisfying this ensures that the model gets calibrated to individual hazard rate curves.

Type-II formulation (14) can be similarly generalized to a finite heterogeneous collection. Here too, it is convenient to work with $\Lambda_i(t) = -\ln\Phi_i(t)$ for each name, a non-decreasing stochastic process taking values in $[0,\infty]$ with $\Lambda_i(0) = 0$ and $\Lambda_i(\infty) = \infty$. The conditional default and survival probabilities $p_t(F)$ and $q_t(F) = 1 - p_t(F)$ are now denoted with name-subscripts as $p_i(F,t)$ and $q_i(F,t)$ respectively. Integration variable F may now be viewed simply as a uniformly distributed common factor. In terms of $\Lambda_i(t)$, $q_i(F,t)$ reads

$$q_i(F,t) = q_i(0,t)\mathrm{E}\left\{1_{\Lambda_i(t)\le -\ln F}\right\}. \tag{19}$$

For the unconditional individual survival probability $Q_i(t)$ we then have

$$Q_i(t) = q_i(0,t)\mathrm{E}\left\{e^{-\Lambda_i(t)}\right\}. \tag{20}$$

Satisfying this ensures that the model gets calibrated to individual hazard rate curves. The joint survival probability $Q_\Omega(t)$ for a list of names in Ω can be expressed as

$$Q_\Omega(t) = \int_0^1 dF \prod_{i\in\Omega} q_i(F,t) = \left[\prod_{i\in\Omega} q_i(0,t)\right] \mathrm{E}\left\{e^{-\mathrm{Max}_{i\in\Omega}\Lambda_i(t)}\right\}, \tag{21}$$

where $\mathrm{Max}_{i\in\Omega}$ picks up the largest $\Lambda_i(t)$ in Ω. This follows from the fact that $\Lambda_i(t)$s are independent stochastic processes. It is interesting to note that this defines the model with no reference to the common factor that has been integrated away. Dependency is built into the combination $\mathrm{Max}_{i\in\Omega}\Lambda_i(t)$.

The two apparently equivalent formulations have resulted in very different models of default dependency, especially at the heterogeneous level. Expression (17) emphasizes conditional independence of defaults. In contrast, expression (21) indicates conditional maximum dependency, a supplier-consumer kind of dependency. To understand it better, consider a realization of the $\Lambda_i(t)$ processes. Within the context of the realization, (21) suggests that the joint survival probability of a list of names in Ω is $e^{-\mathrm{Max}_{i\in\Omega}\Lambda_i(t)}$ (here and below, it is assumed that they have survived their firm-specific risk factors). Consider further just two names in Ω say 1 and 2, and a realization with $\Lambda_1(t) \leq \Lambda_2(t)$. The probability of both the names having survived during $(0,t)$ is $e^{-\Lambda_2(t)}$. Since the survival probability of name 2 irrespective of the state of name 1 is also $e^{-\Lambda_2(t)}$, this implies that it is not possible to have name 1 defaulted and name 2 survived. If name 1 has defaulted, then name 2 should have defaulted as well. More generally, without loss of generality, consider a realization with the ordering $\Lambda_1(t) \leq \Lambda_2(t) \leq \cdots$. In this ordering, if name i is known to have defaulted during $(0,t)$, all names labeled $j > i$ should have defaulted as well. That is, if a common crisis has resulted in name i defaulting, then all names known to be more vulnerable to the crisis in a realization should have defaulted as well. Stated this way, maximum dependency appears to be more realistic than the conditional independence formulation of expression (17). Besides, some contagion effects appear to be already in place.

The $F-$formulation can be viewed as realizing the $\Lambda_i(t)$ processes further to obtain an assignment of defaulted and undefaulted states. This follows after re-introducing the $F-$integral as

$$e^{-\mathrm{Max}_{i\in\Omega}\Lambda_i(t)} = \int_0^1 dF \prod_{i\in\Omega} 1_{\Lambda_i(t)\leq -\ln F}. \tag{22}$$

F can be sampled from a uniform distribution. Given a value for F, a name is in defaulted or undefaulted state at time t depending on whether $\Lambda_i(t)$ is above $-\ln F$ or not. Thus, in some sense, $\Lambda_i(t)$ can be viewed as the amount of impact a common crisis has on a name and $-\ln F$ as the minimum amount of impact needed to lead to a default. In an infinite homogeneous collection, because $\Lambda_i(t)$s are independent and identically distributed, the fraction of $\Lambda_i(t)$s below $-\ln F$, and hence the fraction $1-\nu$ of names in undefaulted states at time t (conditional on surviving their firm-specific risk factors) is expected to agree with the cumulative distribution function of $\Lambda_i(t)$, namely $\mathrm{E}\left\{1_{\Lambda_i(t)\leq -\ln F}\right\}$. This is consistent with our earlier discussion in section 2 and realizes $\mathcal{V}(t)$ as $p_t(F)$. Note that $\Lambda_i(t)$ is assumed here to keep evolving even after default, but does not get counted below $-\ln F$ after default (consider $-\ln F$ time-independent as in the following).

In the case of type-II models, a straightforward extension of (21) to joint distribution of default times is

$$\begin{aligned} \mathrm{Prob}(\tau_i > t_i, \tau_j > t_j, \cdots) &= \int_0^1 dF\, [q_i(F,t_i) q_j(F,t_j)\cdots] \\ &= [q_i(0,t_i) q_j(0,t_j)\cdots]\, \mathrm{E}\left\{e^{-\mathrm{Max}(\Lambda_i(t_i),\Lambda_j(t_j),\cdots)}\right\}, \end{aligned} \tag{23}$$

where τ_is are random default times. The resulting model can be formulated as a first passage model with the crossing of barrier $-\ln F$ by the non-decreasing $\Lambda_i(t)$ triggering default of the i^{th} credit name, conditional on surviving firm-specific risk factors. F is then a random variable uniformly distributed and a possible interpretation is that of $\Lambda_i(t)$ as an intrinsic age process and $-\ln F$ as a common age limit or, as noted above, $\Lambda_i(t)$ as the amount of impact the common crisis has on a name and $-\ln F$ as the minimum amount of impact needed to lead to a default.

A further generalization to a multi-factor joint distribution of default times is

$$\begin{aligned}\text{Prob}(\tau_i > t_i, \tau_j > t_j, \cdots) &= \int \mathcal{D}C\,(F_1, F_2, \cdots)\,[q_i(F_i, t_i)q_j(F_j, t_j)\cdots] \\ &= [q_i(0, t_i)q_j(0, t_j)\cdots]\,\text{E}\left\{C(\Phi_i(t_i), \Phi_j(t_j), \cdots)\right\}, \qquad (24)\end{aligned}$$

where $C()$ is a copula and $\mathcal{D}$ is a short notation for the $n-$dimensional differential, n being the number of factors. Similar distribution has been discussed in Schönbucher and Schubert [2001] within the context of an intensity model, but in the absence of firm-specific risk factors. Recall that copula is a joint distribution of uniformly distributed random variables. Since $F_i, F_j, \cdots$ are uniformly distributed, their joint distribution is given by a copula. To keep things simple, we have kept the copula as time-independent. For simplicity of presentation, we have also considered maximum number of factors, that is as many Fs as there are names in the collection. Fewer factors can be handled with suitable restrictions on the copula. The one-factor case is recovered with the maximally dependent copula $C(F_1, F_2, \cdots) = \text{Min}(F_1, F_2, \cdots)$.

Like the one-factor formulation, the above multi-factor version, when all the t_is are identical to say t, has a basis in probability theory. Consider again an infinitely large collection as in section 2, but now heterogeneous having finitely many, say n, types of names. Its homogeneous subcollections comprising of each name types are also taken to be infinitely large. A configuration of its defaulted and undefaulted states at time t is now characterized by n kinds of fractions, $\nu_i, i = 1, \cdots, n$. Arguing as before, we can express the joint survival probability $Q_{ij\cdots}(t)$ for a list of names in $\{i, j, \cdots\}$ as

$$Q_{ij\cdots}(t) = \int \mathcal{D}F_t(\nu_1, \nu_2, \cdots)\,[(1-\nu_i)(1-\nu_j)\cdots], \qquad (25)$$

where $F_t(\nu_1, \nu_2, \cdots)$ is the joint cumulative distribution function of the ν_is. Next, as usual in copula based dependency modeling, introduce individual survival probabilities $q_i(F_i, t)$ satisfying $q_i(F_t^{(i)}(\nu_i), t) = 1 - \nu_i$ where $F_t^{(i)}(\nu_i)$ is the i^{th} marginal cumulative distribution function. Changing the integration variables to $F_i = F_t^{(i)}(\nu_i)$, we then obtain the multi-factor formulation (24) for identical t_is, though under a scenario where the names are assumed immersed in an infinitely extended pool. The assumption of extendibility to an infinite pool may not always hold, but the type-II formulation with its separation of dependency from the marginals makes its applicability appear more natural.

4 Copula Models

Interestingly, the popular copula models can also be recast into the present formulation. This is instructive as it highlights their unnaturalness, in particular their static nature.

In the present setting, a copula model is just a type-II model that appears as an attempt at constructing $\Phi_i(t)$s with the right properties. Stochasticity of $\Phi_i(t)$ arises from a single real-valued random variable, one for each name in the collection. Time-dependence of $\Phi_i(t)$ results from a time-dependent mean of the random variable. To recover the standard results, let us express $\Phi_i(t)$ as

$$\Phi_i(t) = 1 - N_Y\left(\frac{1}{\sqrt{\rho}}\left(K_i(t) - \sqrt{1-\rho}Z_i\right)\right). \qquad (26)$$

Z_is, one for each name in the collection, are independent random variables. $N_Y()$ is the cumulative distribution function of some random variable Y. $K_i(t)$ is a function of time to be determined. ρ is a correlation parameter as will become evident below. Assuming that there are no firm-specific risk factors, and expressing F as $F = 1 - N_Y(Y)$ in terms of the random variable Y independent of all the Z_is, we have

$$\begin{aligned} p_i(F,t) &= \mathrm{E}\left\{1_{\Phi_i(t)\le F}\right\} = \mathrm{E}\left\{1_{\frac{1}{\sqrt{\rho}}\left(K_i(t)-\sqrt{1-\rho}Z_i\right)\ge Y}\right\} \\ &= \mathrm{E}\left\{1_{Z_i\le\frac{1}{\sqrt{1-\rho}}\left(K_i(t)-\sqrt{\rho}Y\right)}\right\} = N_{Z_i}\left(\frac{1}{\sqrt{1-\rho}}\left(K_i(t)-\sqrt{\rho}Y\right)\right). \end{aligned} \qquad (27)$$

$N_{Z_i}()$ is the cumulative distribution function of Z_i. The unconditional individual default probability $P_i(t)$ can now be expressed as

$$P_i(t) = \mathrm{E}\left\{1 - \Phi_i(t)\right\} = \mathrm{E}\left\{1_{Y\le\frac{1}{\sqrt{\rho}}\left(K_i(t)-\sqrt{1-\rho}Z_i\right)}\right\} = \mathrm{E}\left\{1_{X_i\le K_i(t)}\right\} = N_{X_i}(K_i(t)), \qquad (28)$$

where $X_i = \sqrt{1-\rho}Z_i + \sqrt{\rho}Y$. $N_{X_i}()$ is the cumulative distribution function of X_i. This determines $K_i(t)$ to be $N_{X_i}^{-1}(P_i(t))$. It is an increasing function of time resulting in a decreasing time-dependence for $\Phi_i(t)$.

These are the standard results known for one-factor copula models. Default time copula results from expression (23) for joint distribution of default times. Z_is, and hence X_is, are usually taken to be identically distributed. Then $N_{Z_i}()$s, and similarly $N_{X_i}()$s, are the same for all the names in the collection. Y and Z_is, and hence X_is, are usually normalized to have zero mean and unit variance. ρ is then the correlation between two different X_is. In the case of the Gaussian copula model, Y, Z_i and X_i are all standard normal random variables, and hence N_Y, N_{Z_i} and N_{X_i} are all identical to the cumulative standard normal distribution function. In other copula models, N_Y and N_{Z_i} may be known by choice, but then N_{X_i} is not guaranteed to be easily computable.

It is also possible to recover multi-factor copula models in the present setting. As discussed earlier, a multi-factor type-II model can be extended to introduce joint distribution of default times by expression (24). The Φ_i-construct is as before, but in terms of name-subscripted Ys and ρs. For simplicity of presentation, this considers maximum number of factors, that is as many Ys as there are names in the collection. Assuming no firm-specific risk factors, that is $q_i(0,t_i) = 1$, the joint survival probability (24) can be rewritten as

$$\mathrm{Prob}(\tau_i > t_i, \tau_j > t_j, \cdots) = \mathrm{E}\left\{C\left(\Phi_i(t_i), \Phi_j(t_j), \cdots\right)\right\} = \mathrm{E}\left\{1_{U_i\le\Phi_i(t_i)}1_{U_j\le\Phi_j(t_j)}\cdots\right\}. \qquad (29)$$

In the last step, uniformly distributed random variables $U_i, U_j, \cdots$ are introduced to express the copula $C()$ as their joint distribution function. Now, expressing U_is in terms of random variables Y_is as $U_i = 1 - N_{Y_i}(Y_i)$, we have

$$\mathrm{Prob}(\tau_i > t_i, \tau_j > t_j, \cdots) = \mathrm{E}\left\{1_{X_i\ge K_i(t_i)}1_{X_j\ge K_j(t_j)}\cdots\right\}, \qquad (30)$$

where $X_i = \sqrt{1-\rho_i}Z_i + \sqrt{\rho_i}Y_i$. Restricting to one name, we find $K_i(t) = N_{X_i}^{-1}(P_i(t))$. Writing $X_i \geq K_i(t_i)$ in terms of the uniform random variable $N_{X_i}(X_i)$ as $N_{X_i}(X_i) \geq P_i(t_i)$, we note that the above is in fact a survival copula of default times.

It is well-known that copula models are static models. That they lack dynamics is evident from the $\Phi_i(t)-$construct that has just one random variable Z_i generating the whole process $\Phi_i(t)$. A sample path of $\Phi_i(t)$ is fully determined given a sample of Z_i. Its time-dependence is due to $K_i(t)$ alone. It is not straightforward to accommodate a term structure of correlations. Besides, as already noted, these models have no identifiable support for firm-specific risk. There is no support for catastrophic risk either. One may attempt to include these risk factors along the lines presented here, but better models can be constructed with the help of subordinators that we next turn to.

5 Stable Subordinator Models

To proceed, it is helpful to invoke some concept of stationarity, that the model looks similar at some future time given conditions similar to as it exists today. This is easier to do at the individual levels since, at the collective level, there can be dependency information available from defaults that needs to be incorporated. Consider first that the names involved have flat hazard rate curves. For the unconditional individual survival probability $Q_i(t)$, a flat hazard rate curve implies $Q_i(t_2) = Q_i(t_1)Q_i(t_2 - t_1)$ given two times t_1 and t_2 such that $0 < t_1 < t_2$. Assuming it holds similarly for any firm-specific component and writing

$$\mathrm{E}\left\{e^{-\Lambda_i(t_2)}\right\} = \mathrm{E}\left\{e^{-\Lambda_i(t_1)}\mathrm{E}\left\{e^{-(\Lambda_i(t_2)-\Lambda_i(t_1))}|\Lambda_i(t_1)\right\}\right\}, \tag{31}$$

we note that a stationarity concept can be naturally accommodated with a $\Lambda_i(t)$ that has independent and stationary increments. This is as expected since it is well-known that the most natural choice for $\Lambda_i(t)$ incorporating stationary is a Lévy process. In our case, $\Lambda_i(t)$ is further expected to be a subordinator, that is a Lévy process involving only non-decreasing sample paths. Realizations of $\Lambda_i(t)$ then contribute a non-increasing factor $e^{-\Lambda_i(t)}$ to the unconditional individual survival probability.

Though type-II models are more promising and are the main focus of the article, a little discussion of type-I models is helpful to put them in perspective. Type-I models were formulated in a heterogeneous setting in section 3. Consider the individual processes $\Lambda_i(t)$ modeled as $\Lambda_i(t) = a_iX(t) + b_it$ for some non-negative constants a_is and b_is in terms of a common process $X(t)$, a subordinator having Laplace exponent $\eta(u)$. This gives for the unconditional individual survival probability

$$Q_i(t) = \mathrm{E}\left\{e^{-\Lambda_i(t)}\right\} = e^{-t(\eta(a_i)+b_i)}. \tag{32}$$

This generates a constant hazard rate $\eta(a_i) + b_i$ leading to a heterogeneous model with flat hazard rate curves. Reduced models of this kind and their intensity based extensions have been well studied in the literature (see for instance Balakrishna [2007]). Here the drift term b_it generates the firm-specific component. However, to generate a nonzero probability of a total collapse, a nonzero probability of $X(t) = \infty$ needs to be enforced, somewhat artificially. Besides, though time-dependent a_is and b_is can be helpful, it is not straightforward to extend the model to allow for general hazard rate curves while retaining some concept of stationarity. Replacing $X(t)$ by a time-changed subordinator $X(\phi(t))$ given some non-decreasing function of time $\phi(t)$ can be helpful, but allowing for $\phi(t)$ to be name-dependent leads to both conceptual and practical issues.

Type-II models are better suited for the purpose since they are formulated independently at the individual levels to start with. If $\Lambda_i(t)$ has Laplace exponent $\eta_i(u)$ and the firm-specific hazard rate is b_i so that $q_i(0,t) = e^{-tb_i}$, we get for the unconditional individual survival probability

$$Q_i(t) = q_i(0,t)\mathrm{E}\left\{e^{-\Lambda_i(t)}\right\} = e^{-t(b_i+\eta_i(1))}. \tag{33}$$

This generates a constant hazard rate $b_i + \eta_i(1)$ leading to a heterogeneous model with flat hazard rate curves. Unlike in type-I models, firm-specific component here arises from $q_i(0,t)$ and as we will see later, drift component of $\Lambda_i(t)$ has a different role to play, to generate a nonzero probability of a total collapse. As for extending this to a general hazard rate curve, there can be various approaches, some of them possibly offering an explanation of its time-dependence. If our interest is just accommodating the curve, the easiest thing to do is to make some of the parameters time-dependent. This can be done independently at the individual level making it applicable to heterogeneous collections. Note that all model intricacies here are built into $q_i(F,t)$, in particular $\Lambda_i(t)$. We may thus simply consider some of the parameters of $\Lambda_i(t)$ as time-dependent. An equivalent but more convenient approach is to work with a time-changed subordinator, time-changed on an individual basis with regard to an intrinsic time-like variable $\theta_i(t)$ given by

$$\theta_i(t) = -\ln Q_i(t). \tag{34}$$

Since the individual survival probability $Q_i(t)$ is a non-increasing function of time, $\theta_i(t)$ is appropriate to play the role of time. This approach provides a simple and neat extension of the stationarity concept to general hazard rate curves.

Specifically, consider subordinators $X_i(t)$, independent ones for each of the names in the collection, identically distributed for simplicity with Laplace exponent $\bar{\eta}(u)$ (the bar on η is discussed below). Let us set

$$q_i(0,t) = e^{-(1-\bar{\eta}(1))\theta_i(t)}, \quad \Lambda_i(t) = X_i(\theta_i(t)). \tag{35}$$

Now, the hazard rate curves are automatically calibrated to, since

$$Q_i(t) = q_i(0,t)\mathrm{E}\left\{e^{-\Lambda_i(t)}\right\} = e^{-(1-\bar{\eta}(1))\theta_i(t)}\mathrm{E}\left\{e^{-X_i(\theta_i(t))}\right\} = e^{-\theta_i(t)}. \tag{36}$$

As for the conditional survival probability $q_i(F,t)$, we have

$$q_i(F,t) = q_i(0,t)\mathrm{E}\left\{1_{X_i(\theta_i(t))\leq -\ln F}\right\} = e^{-(1-\bar{\eta}(1))\theta_i(t)}g_{\theta_i(t)}(-\ln F), \tag{37}$$

where $g_t(x)$ is the cumulative distribution function of $X_i(t)$. Note that we require $\bar{\eta}(1) \leq 1$ so that the firm-specific survival probability $q_i(0,t)$ is non-increasing as a function of t.

Let us now confine ourselves to modeling with subordinators whose distributions have power-law tails. In particular, we will be dealing with stable subordinators reviewed briefly in section 1. A rationale for this choice can be noted from expression (21) for the joint survival probability. It contains the combination $\text{Max}_{i\in\Omega}\Lambda_i(t)$, the maximum of independent, and say identically distributed, Λ_i-processes. This is also the combination that is central to extreme value theory. Distributions having power-law tails give rise naturally to extreme value distributions in an appropriate limit. Note that the extreme events we are concerned with here occur for large Λ_is, necessitated by large $-\ln F$ or small F which is the short-end behavior of $F_t(\nu)$ in the infinite homogeneous pool. A dependent default when its likelihood is small, that is when ν is close to $\nu_{\min} = 1 - e^{-(1-\bar{\eta}(1))\theta(t)}$, is in fact an extreme event.

It turns out that type-II models using stable subordinators can be calibrated reasonably well to market data on CDOs (see Balakrishna [2010]). Interestingly, applicable index of stability α turns out to be just 1/2. The $\alpha = 1/2$ stable subordinator called the Lévy subordinator has Laplace exponent $\bar{\eta}(u) = \bar{\sigma}\sqrt{u} + \bar{\mu}u$ for some parameters $\bar{\sigma}$ and $\bar{\mu}$ (the bars indicate that the parameters can be time-dependent and are defined for the period $(0, t)$ distinguishing them from the instantaneous ones introduced below). Its cumulative distribution function is known as the Lévy distribution and is available in closed form,

$$g_t(x) = 2N\left(-\bar{\sigma}t/\sqrt{2(x-\bar{\mu}t)}\right), \quad x > \bar{\mu}t, \tag{38}$$

where $N()$ is the cumulative standard normal distribution function. This leads to a two-parameter model with $\bar{\sigma}$ as one of the parameters along with a non-negative drift rate $\bar{\mu}$ so that $g_t(x) = 0$ for $x < \bar{\mu}t$. With the Lévy subordinator chosen for $X_i(t)$, the above distribution gives us for the conditional survival probability

$$q_i(F,t) = 2e^{-(1-(\bar{\sigma}+\bar{\mu}))\theta_i(t)}N\left(-\bar{\sigma}\theta_i(t)/\sqrt{-2(\ln F + \bar{\mu}\theta_i(t))}\right). \tag{39}$$

Consistency requirement $\bar{\eta}(1) \le 1$ here reads $\bar{\sigma} + \bar{\mu} \le 1$.

Due to the presence of a non-negative drift component $\bar{\mu}t$ in the subordinator $X_i(t)$, the distribution $g_t(x)$ gets set to zero for $x < \bar{\mu}t$. This introduces a positive drift $\bar{\mu}\theta_i(t)$ to $\Lambda_i(t)$ that forces $q_i(F,t)$ to zero for $F > e^{-\bar{\mu}\theta_i(t)}$. In the infinite homogeneous pool, this implies that $F_t(\nu)$ tends to $F_{\max}(t) = e^{-\bar{\mu}\theta(t)}$ as $\nu \to 1$. This realizes the possibility envisioned earlier of a probability mass $1 - F_{\max}(t)$ at $\nu = 1$, that is, a non-zero probability $1 - F_{\max}(t)$ of all the names in the pool defaulting by time t, $\bar{\mu}$ controlling the likelihood of such a catastrophe. Recall also that $F_t(\nu) = 0$ for ν below $\nu_{\min}$. Both the probability mass and the minimum ν increase with t starting from zero.

Parameters $\bar{\sigma}$ and $\bar{\mu}$ can be both name and time dependent. Time-dependence can be conveniently expressed as $\theta_i(t)$-dependence. It is subject to $q_i(F,t)$ decreasing with respect to $\theta_i(t)$ so that, suppressing name and $t-$dependence for simplicity, $\bar{\sigma}\theta$, $\bar{\mu}\theta$ and $(1-(\bar{\sigma}+\bar{\mu}))\theta$ should all be non-decreasing with respect to θ. This can be expressed in terms of instantaneous or local parameters σ and μ as

$$\sigma \ge 0, \quad \mu \ge 0, \quad \sigma + \mu \le 1, \quad \text{where } \sigma \equiv d(\bar{\sigma}\theta)/d\theta, \quad \mu \equiv d(\bar{\mu}\theta)/d\theta. \tag{40}$$

Satisfying this is sufficient to ensure similar constraints for $\bar{\sigma}$ and $\bar{\mu}$. It is thus possible to have consistent $\bar{\sigma}$ and $\bar{\mu}$ term-structures, for instance with piecewise constant σ and μ. One may find it convenient to work with $\gamma \equiv \sigma + \mu$ and $\kappa \equiv \mu/(\sigma + \mu)$ (similarly for barred parameters) both of which are allowed to range from zero to one. Note that γ is the fraction of the hazard rate that is attributable to systemic risk factors, fraction $1-\gamma$ being firm-specific. Of the fraction γ, a further fraction κ is attributed to catastrophic scenarios.

As noted earlier, $F_t(\nu)$ in the infinite homogeneous pool can be obtained by solving $q_t(F) = 1-\nu$ for $F = F_t(\nu)$. In the Lévy subordinator model, when $\bar{\sigma}+\bar{\mu} \leq 1$ as is required for consistency, the minimum ν below which $F_t(\nu) = 0$ is

$$\nu_{\min}(t) = 1 - e^{-(1-(\bar{\sigma}+\bar{\mu}))\theta(t)}. \tag{41}$$

This increases with t starting from zero at $t = 0$. For ν above $\nu_{\min}(t)$, $F_t(\nu)$ is

$$F_t(\nu) = \exp\left\{-\bar{\mu}\theta(t) - \frac{1}{2}(\bar{\sigma}\theta(t))^2\left[N^{-1}\left(\frac{1}{2}(1-\nu)e^{(1-(\bar{\sigma}+\bar{\mu}))\theta(t)}\right)\right]^{-2}\right\}, \quad \nu \geq \nu_{\min}(t). \tag{42}$$

Note that $\int_0^1 d\nu F_t(\nu) = 1 - \int_0^1 dF_t(\nu)\nu = e^{-\theta(t)}$ as expected. Further, $F_t(\nu) \to e^{-\bar{\mu}\theta(t)}$ as $\nu \to 1$ so that there is a probability mass at $\nu = 1$ as observed earlier, suggesting a finite probability $1 - e^{-\bar{\mu}\theta(t)}$ of a total collapse.

6 Why $\alpha = 1/2$

An intriguing aspect of the model is the choice $\alpha = 1/2$ for the index of stability of the stable subordinator. Calibration results indicate this as appropriate so that the Lévy subordinator can be considered to be the basic model of $\Lambda_i(t)$ processes. This is somewhat analogous to the Brownian motion being the choice as the basic model of many economic processes. Perhaps this has something to do with fact that Lévy distribution is dual to the half-normal distribution or that it is the first passage time distribution of the Brownian motion. Though the following discussion does not answer why $\alpha = 1/2$ is appropriate, it may become clearer from the discussion that the question is better phrased as to why $1/\alpha = 2$, a choice that has some obvious niceties and appealness associated with it. Note that $p_t(F)$ and $F_t(\nu)$ are inverses of each other and hence an index in the range $[1/2, 1]$ applicable to $p_t(F)$ has an effective applicability to $F_t(\nu)$ in the range $[1, 2]$. Modeling $F_t(\nu)$ directly as arising from a stable process is expected to result in an index in the range $[1, 2]$ but requiring a cutoff to prevent negative moves, as borne out by a study in Balakrishna [2009].

The long tail of the distribution of a stable subordinator $X(t)$ obeys a power-law decay as in (6). Thus for F small (dropping the name-subscripts), we have

$$\begin{aligned} q_t(F) &= q_t(0)g_{\theta(t)}(-\ln F) \\ &\to q_t(0) - q_t(0)\sigma\theta(t)\left[\Gamma(1-\alpha)\right]^{-1}(-\ln F)^{-\alpha}. \end{aligned} \tag{43}$$

Here and in the following, for simplicity, model parameters are taken to be constants so that $\bar{\sigma} = \sigma$ and $\bar{\mu} = \mu$. Recall that $q_t(0) = e^{-(1-\eta(1))\theta(t)} = 1 - \nu_{\min}(t)$. This suggests that for $\nu > \nu_{\min}(t)$ but close to $\nu_{\min}(t)$, that is for small $(\nu - \nu_{\min}(t))/(1-\nu_{\min}(t)) > 0$, we have

$$-\ln F_t(\nu) \to (\sigma\theta(t))^{1/\alpha}\left[\Gamma(1-\alpha)\right]^{-1/\alpha}\left[(\nu-\nu_{\min}(t))/(1-\nu_{\min}(t))\right]^{-1/\alpha}. \tag{44}$$

This is the expected form arising from extreme value theory considerations with its relevant index $1/\alpha$. For $\alpha = 1/2$, the index is two. Similar behavior is obtained for small $\theta(t)$ since $g_{\theta(t)}(-\ln F)$ is a function of the combination $(\sigma\theta(t))^{-1/\alpha}(-\ln F - \mu\theta(t))$,

$$-\ln F_t(\nu) \to \mu\theta(t) + (\sigma\theta(t))^{1/\alpha}\left[\Gamma(1-\alpha)\right]^{-1/\alpha}\nu^{-1/\alpha}. \tag{45}$$

Again for $\alpha = 1/2$, the index is two and $F_t(\nu)$ is regular in $\theta(t)$ as $\theta(t) \to 0$, consistent with the closed form solution (42).

The log-distribution of a stable subordinator $X(t)$ exhibits power-law at the very short end as in (7). This suggests that for ν very close to one, we have

$$-\ln F_t(\nu) \to \mu\theta(t) + \alpha(\sigma\theta(t))^{1/\alpha}(1-\alpha)^{(1-\alpha)/\alpha}\left[-\ln(1-\nu)\right]^{-(1-\alpha)/\alpha}. \tag{46}$$

This gives a fat long tail as a function of $-\ln(1-\nu)$ with a tail-index of $1/\alpha - 1$. For $\alpha = 1/2$, this index is one. Thus, $-\ln F_t(\nu)$ as a function of $-\ln[(1-\nu)/(1-\nu_{\min}(t))]$ features a power-law at the long and short ends with indices of one and two respectively.

Let us next look at default correlation over period $(0, t)$ for small $\theta(t)$. We will assume here that $\alpha \in (1/2, 1)$. The two-point survival probability over period $(0, t)$ in the homogeneous case can be expressed as

$$\int_0^1 dF[q_t(F)]^2 = [q_t(0)]^2 e^{-\mu\theta(t)} \int_0^\infty dx x^{-\alpha-1} f(x) e^{-(\sigma\theta(t))^{1/\alpha}x}, \tag{47}$$

where $f(x) = x^{\alpha+1} d(h(x))^2/dx$ and $h(x) = g_{\theta(t)}((\sigma\theta(t))^{1/\alpha}x + \mu\theta(t))$ (independent of t for a stable subordinator). When $\theta(t)$ is small, it can be shown that (see Balakrishna [2009])

$$\int_0^1 dF[q_t(F)]^2 \simeq 1 + (\mu - 2)\theta(t) + c(\sigma\theta(t))^{1/\alpha}, \tag{48}$$

for some constant c. This leads to a two-point default probability $\simeq \mu\theta(t) + c(\sigma\theta(t))^{1/\alpha}$ and a default correlation $\simeq \mu + c\sigma(\sigma\theta(t))^{(1-\alpha)/\alpha}$. For $\alpha \in (1/2, 1)$, the exponent of the last term above is $1/\alpha \in (1, 2)$. As $\alpha \to 1/2$, this exponent tends to two and the next correction term in the expansion becomes significant. It can be shown that in this limit, that is in the Lévy subordinator model for which $h(x) = 2N(-1/\sqrt{2x})$, the two-point default probability $\simeq \mu\theta(t) - (2\sigma^2/\pi)(\theta(t))^2 \ln\theta(t)$ and default correlation $\simeq \mu - (2\sigma^2/\pi)\theta(t)\ln\theta(t)$. The default correlation remains nonzero at μ as $\theta(t) \to 0$ resulting in what may be called the instantaneous default correlation. It leads to a tail dependence for the loss distribution that has become necessary recently for better pricing of the senior tranches.

7 Semi-Analytical Pricing

An attractive feature of the Lévy subordinator model is that it admits a closed form expression for $q_i(F, t)$. This make it possible to price CDOs semi-analytically. The procedure is identical to that used in the Gaussian copula model. The following is a brief review of the steps involved.

Consider the loss process $L(t)$. Its contribution at time t to tranche $[a, b]$ with attachment point a and detachment point b can be expressed as a call-spread, or equivalently as $b - a$ minus the put-spread so that the loss per tranche size is

$$L(t)_{[a,b]} = 1 - \frac{1}{b-a} \left[\max(b - L(t), 0) - \max(a - L(t), 0) \right]. \tag{49}$$

If the cumulative distribution function of $L(t)$ is $H_t(x)$ at time t, the expected loss per tranche size is

$$\bar{L}(t)_{[a,b]} = 1 - \frac{1}{b-a} \int_0^b dx (dH_t(x)/dx) \left[\max(b - x, 0) - \max(a - x, 0) \right]. \tag{50}$$

After a partial integration, this can be written as

$$\bar{L}(t)_{[a,b]} = 1 - \frac{1}{b-a} \int_a^b dx H_t(x). \tag{51}$$

Loss distribution is usually taken to be a discrete distribution, in which case more care can be exercised in deriving this. If discrete, $H_t(x)$ is flat in-between successive points giving appropriate interpolations at $x = a$ and $x = b$.

Given the expected loss per tranche size, the default leg of a tranche can be priced per tranche size as

$$\mathrm{DL}_{[a,b]} = \int_0^T D(t) d\bar{L}(t)_{[a,b]}, \tag{52}$$

where T is the maturity and $D(t)$ is the discount factor for the time period $(0, t)$. Similarly the premium leg at unit spread can be priced per tranche size as

$$\mathrm{PL}_{[a,b]} = \sum_{i=1}^{N_\delta} \delta_i(t_i) D(t_i) \left[1 - \bar{L}(t_i)_{[a,b]} \right] + \mathrm{PL}'_{[a,b]}, \tag{53}$$

where $\delta_i(t_i)$ is the accrual factor for the period (t_{i-1}, t_i), $t_{N_\delta} = T$ and N_δ is the number of periods. $\mathrm{PL}'_{[a,b]}$ is the contribution from accrued interest payments made upon default,

$$\mathrm{PL}'_{[a,b]} = \sum_{i=1}^{N_\delta} \int_{t_{i-1}}^{t_i} \delta_i(t) D(t) d\bar{L}(t)_{[a,b]}, \tag{54}$$

where $\delta_i(t)$ is the accrual factor for the partial period covering (t_{i-1}, t). Given the leg values, fair spread can be obtained by dividing the default leg by the premium leg, after taking care of any upfront payments.

Integrations can be performed over a sufficiently fine time-grid. Time-steps making up the grid can be as wide as the periods themselves for efficient pricing, and hence the factors multiplying the increments $d\bar{L}(t)_{[a,b]}$ are evaluated at mid-points of the time-steps. The super senior tranche can be priced like other tranches along with a part of the premium leg of notional that is a fraction R of the total notional outstanding, or, if recovery rates are nonuniform, sum of fractions R_i of the individual notionals outstanding.

8 Large Homogeneous Pool

Because the Lévy subordinator model is structured similar to the Gaussian copula model, efficient pricing techniques of the latter can be directly employed in the present case. One of them is the large homogeneous pool approximation that can be a useful tool since it admits an explicit expression for the loss distribution.

We have already discussed an infinitely large homogeneous pool and its cumulative distribution function $F_t(\nu)$. It can also be approached starting from a finite collection of n names. The joint default probability that k or less number of names are in the defaulted state at time t and the rest are not is given by

$$P_{\{k\}}(t) = \sum_{j=0}^{k} \binom{n}{j} \int_0^1 dF \, [p_t(F)]^j \, [1 - p_t(F)]^{n-j} \,. \tag{55}$$

For an infinitely large homogeneous pool of names, that is as $n \to \infty$, it is well-known that, by the law of large numbers, the above simplifies to

$$F_t(\nu) = P_{\{\nu n\}}(t) = \int_0^1 dF 1_{p_t(F) \le \nu}, \tag{56}$$

where $\nu = k/n$ is the fraction of names in the defaulted state at time t. This indicates that $F_t(\nu)$ can be obtained by summing up the region of F over which $p_t(F) \le \nu$. We have considered $p_t(F)$ to be an increasing function of F. Hence, $F_t(\nu)$ can be obtained by solving $p_t(F) = \nu$, or equivalently $q_t(F) = 1 - \nu$, for $F = F_t(\nu)$, in agreement with our earlier result in an infinitely large homogeneous pool.

In an infinitely large homogeneous pool with a uniform recovery rate R, the loss distribution $H_t(x)$ is given explicitly by $F_t(x/(1-R))$ so that the expected loss per tranche size (51) for a tranche $[a, b]$ can be computed as

$$\bar{L}(t)_{[a,b]} = 1 - \frac{1}{\nu_b - \nu_a} \int_{\nu_a}^{\nu_b} d\nu F_t(\nu), \tag{57}$$

where $\nu_a = a/(1-R)$ and $\nu_b = b/(1-R)$. Because $F_t(\nu) = 0$ for $\nu \le \nu_{\min}(t)$, the expected loss becomes 100% of the tranche size once $\nu_{\min}(t)$ crosses ν_b, if b is small enough for this to occur within the maturity of the trade. This leads to overpricing of the equity tranches. Finite n offers better pricing by smoothening out the small ν behavior.

9 Finite Pool with FFT

Large homogeneous pool approximation yields fast results, but at the expense of accuracy. As is well-known, many of the factor models admit efficient pricing for finite pools with recursive methods or Fast Fourier Transform (FFT) techniques. Being structured similar to the Gaussian copula model, the present model can be handled analogously. The following outlines the steps involved in computing with FFT.

To obtain the loss distribution for a finite collection of n names, consider the loss variable at time t conditional on F given by

$$\mathcal{L}(F,t) = \sum_{i=1}^{n} L_i \xi_i(F,t), \tag{58}$$

where $\xi_i(F,t)$ is the conditional default indicator at time t and $L_i = (1 - R_i)w_i$, R_i being the recovery rate and w_i the fraction of the total pool notional associated with the i^{th} name. Though not explicitly shown, L_i can be dependent on both F and t (as in section 11 on random recovery rates). Default indicators being independent conditional on F, the above has the characteristic

$$\mathrm{E}\left\{e^{iu\mathcal{L}(F,t)}\right\} = \prod_{m=1}^{n} \left[q_m(F,t) + p_m(F,t)e^{iuL_m}\right], \tag{59}$$

where $i = \sqrt{-1}$. This characteristic is the Fourier transform of the density function of the loss distribution (conditional on F unless mentioned otherwise). Hence, the loss distribution can be obtained by inverting it using FFT techniques. The result can be used to compute the expected loss per tranche size for a tranche with attachment point a and detachment point b according to

$$\bar{L}(F,t)_{[a,b]} = 1 - \frac{1}{b-a}\int_a^b dx H_t(F,x), \tag{60}$$

where $H_t(F,\cdot)$ is the cumulative loss distribution function.

FFT requires discretization of u. Discretization is straightforward if L_i's are uniform at L across the collection ($L = (1-R)/n$ if R_is are uniform). Inversion then yields the loss distribution at loss-points $j = 0, \cdots, n$ in units of L. This gives the default probability density $P_{[j]}(F,t)$, the sum of products of various combinations of j of the $p_i(F,t)$s and $n-j$ of the $q_i(F,t)$s. Consider it extended up to $j = N-1 \geq n$ by padding with zeros where N is a power of 2, as is usually done for an efficient FFT. In this case, (59) reads

$$\sum_{j=0}^{N-1} P_{[j]}(F,t)e^{i\omega jk} = \prod_{m=1}^{n} \left[q_m(F,t) + p_m(F,t)e^{i\omega k}\right], \quad k = 0, \cdots, N-1, \tag{61}$$

where $\omega = 2\pi/N$. This can easily be computed and inverted using FFT techniques to obtain $P_{[j]}(F,t), j = 0, \cdots, n$, and hence its cumulative counterpart $G_t(F,\nu)$ (that corresponds to $H_t(F, jL)$) where $\nu = j/n$ is the fraction of names in the defaulted state. Expected loss per tranche size is then

$$\bar{L}(F,t)_{[a,b]} = 1 - \frac{1}{\nu_b - \nu_a}\int_{\nu_a}^{\nu_b} d\nu G_t(F,\nu), \tag{62}$$

where $\nu_a = a/(nL)$, $\nu_b = b/(nL)$, and $G_t(F,\nu)$ is flat in-between successive $\nu-$points. Integration of $\bar{L}(F,t)_{[a,b]}$ over F gives $\bar{L}(t)_{[a,b]}$, the unconditional expected loss per tranche size. This result can be used to price the CDO tranches as discussed in section 7. Integration over F can be performed numerically using the Gauss-Legendre quadrature formula. It is efficient to perform the integration after the prices are computed conditional on F.

10 Monte Carlo Pricing

Though the Lévy subordinator model can be handled semi-analytically as detailed above, a Monte Carlo simulation algorithm can be a useful tool to price non-standard products. It can also be useful for pricing standard tranches as it is found to be efficient, accurate and easily implementable, and does not involve discretization of time. The following algorithm can be viewed as simulating the model defined by expression (23) or simply as a method of computing the integrals involved in the semi-analytical pricing. Efficiency of the algorithm can be improved substantially by using quasi random sequences such as Sobol sequences to generate each of the independent uniform random numbers.

The algorithm reads as follows.

1. Draw a uniformly distributed random number F and n independent uniformly distributed random numbers $u_i, i = 1, \cdots, n$.

2. For each credit name i, first determine whether it defaults before the time horizon T by checking if $q_i(F, T) < u_i$ where $q_i(F, .)$ is given in equation (39). If so, solve the equation $q_i(F, t_i) = u_i$ for $\theta_i(t_i)$. Determine default time t_i of credit name i by a table look up into its hazard rate curve.

3. Given the default times before the time horizon, price the instrument. For the next scenario, go to step 1.

4. Average all the prices thus obtained to get a price for the instrument.

Given a scenario of default times, it is straightforward to price the CDOs. One proceeds processing the defaults one by one, starting from the first up to maturity, picking up payments by the default leg, switching to the next tranche whenever a tranche gets wiped out, at the same time computing the premium legs per unit spread for all the surviving tranches. Whenever a default leg pays out the loss amount, the notional of that tranche gets reduced by the same amount, and the notional of the super senior tranche gets reduced by the recovery amount (when the super senior is the only survivor, it gets treated like a default swap). The leg values can be added across tranches to obtain those for the index default swap. Fair spreads can be computed given the leg values at the end of the simulation.

11 Random Recovery Rates

It has become apparent, especially during the recent crisis, that random recovery rate is helpful in better pricing of the senior tranches. Random recovery rates have been discussed by Andersen and Sidenius [2004], and recently by Amraoui and Hitier [2008] and Krekel [2008] in response to the recent crisis within the context of the Gaussian copula model. Here, let us consider a similar approach with an emphasis on tractability, and randomness of recovery rates arising from a decreasing dependence on F. Because we are concerned with just one individual name in this section, name-subscripts are omitted. Further, because time-dependence of $q_t(F)$ is only through its dependence on $\theta(t)$, all time-dependences in this section can be expressed conveniently as a dependence on $\theta(t)$ if desired.

Consider a possibly time-dependent recovery rate $R(t)$ used in building the hazard rate curve to start with. We may refer to $R(t)$ as the instantaneous recovery rate. The expected loss contribution during an infinitesimal time interval $(t, t + dt)$ is $(1 - R(t))dP(t)$ where

$P(t) = 1 - e^{-\theta(t)}$ is as before the default probability over the period $(0, t)$. If $R(t)$ were constant, the accumulated expected loss at time t is $(1 - R)P(t)$. If $R(t)$ is time-dependent and one prefers working with the expected loss, it is convenient to introduce a recovery rate $\bar{R}(t)$ such that the expected loss at time t is $(1 - \bar{R}(t))P(t)$. The two are related by $d[(1 - \bar{R}(t))P(t)] = (1 - R(t))dP(t)$ or $d[\bar{R}(t)P(t)] = R(t)dP(t)$, that solves to

$$\bar{R}(t) = \frac{1}{1 - e^{-\theta(t)}} \int_0^{\theta(t)} d\phi(s) e^{-\phi(s)} R(s). \tag{63}$$

Hence $\bar{R}(t)$ is the expected recovery rate conditional on default during $(0, t)$ that we may refer to as the period recovery rate.

Consider now a model of the kind implied by expression (23). Let $R(F, t)$ and $\bar{R}(F, t)$ be the conditional equivalents of $R(t)$ and $\bar{R}(t)$. $R(F, t)$ is the recovery rate to be used in the Monte-Carlo algorithm presented earlier and $\bar{R}(F, t)$ is the recovery rate for use in the semi-analytical pricing. A model of $R(F, t)$ is required to satisfy

$$\int_0^1 dF R(F, t) \partial_{\theta(t)} p(F, t) = R(t) e^{-\theta(t)}. \tag{64}$$

This ensures that the conditional expected loss contribution integrates to the unconditional one. In terms of the period recovery rate $\bar{R}(F, t)$, the requirement is

$$\int_0^1 dF \bar{R}(F, t) p(F, t) = \bar{R}(t) \left(1 - e^{-\theta(t)}\right). \tag{65}$$

Given a model of $R(F, t)$ satisfying (64), $\bar{R}(F, t)$ satisfying above can be obtained recursively during semi-analytical pricing by integrating the relation

$$\partial_t \left(\bar{R}(F, t) p(F, t)\right) = R(F, t) \partial_t (p(F, t)). \tag{66}$$

This is the conditional equivalent of $d(\bar{R}(t)P(t)) = R(t)dP(t)$. If $\bar{R}(F, t)$ is modeled directly, it provides us with $R(F, t)$ for use in the Monte-Carlo algorithm. Modeling $\bar{R}(F, t)$ directly helps us avoid integrating it during semi-analytical pricing. However, it is better to model $R(F, t)$ that guarantees a consistent framework for both $R(F, t)$ and $\bar{R}(F, t)$, helpful in generating a consistent term structure of, say, piecewise constant model parameters.

A simple tractable model of $R(F, t)$ can be constructed as comprising of a firm-specific component R_0 and a systemic component R_s such that

$$R(F, t)\lambda(F, t) = R_0 \widehat{\lambda}(t) + R_s \widetilde{\lambda}(F, t). \tag{67}$$

Here $\lambda(F, t) = -\partial_t(\ln q(F, t))$ is conditional hazard rate, $\widehat{\lambda}(t) = -\partial_t(\ln q(0, t))$ is its firm-specific component and $\widetilde{\lambda}(F, t) = \lambda(F, t) - \widehat{\lambda}(t)$ is the systemic one. One expects $\lambda(F, t)$ to be an increasing function of F ranging from $\widehat{\lambda}(t)$ at $F = 0$, to ∞ as $F \to F_{\max}(t)$ (this can be verified in the Lévy subordinator model). The above implies

$$R(F, t) = R_s + (R_0 - R_s) \frac{\widehat{\lambda}(t)}{\lambda(F, t)}. \tag{68}$$

Assuming $R_0 > R_s$, this decreases from R_0 to R_s as F runs from zero to $F_{\max}(t)$. It is in line with one's expectation of an inverse relationship between recovery rates and default rates[2]. Requirement (64) gives

$$R(t) = R_s + (R_0 - R_s)\,\frac{\widehat{\lambda}(t)}{d_t\theta(t)} = R_0 - \gamma(R_0 - R_s), \tag{69}$$

where γ is introduced through $\widehat{\lambda}(t) = (1-\gamma)d_t\theta(t)$ (in the subordinator model $\gamma = \sigma + \mu$). This is a simple linear relationship between $R(t)$ and γ. An interesting feature of this recovery model is that time-independence of model parameters R_0, R_s and γ implies or is consistent with a flat $R(t)$. A term-structure expected for γ implies a time-dependence for $R(t)$. From a calibration perspective however, while building the hazard rate curve, it is inconvenient to consider $R(t)$ as a dependent variable. If so, one could consider an implied time-dependence for R_0 or R_s arising from a given $R(t)$ and a term-structure expected for γ.

The above model of random recovery is quite general, but it assumes that the systemic component is a constant. One would expect the later to be random contributing effectively with a fixed R_s. If desired, a generalization can be constructed with a $\widetilde{R}(F,t)$ having a decreasing $F-$dependence satisfying $R(F,t)\lambda(F,t) = R_0\widehat{\lambda}(t) + \widetilde{R}(F,t)\widetilde{\lambda}(F,t)$ so that

$$R(F,t) = R_0 - \left(R_0 - \widetilde{R}(F,t)\right)\left(1 - \frac{\widehat{\lambda}(t)}{\lambda(F,t)}\right). \tag{70}$$

With this choice, requirement (64) reads

$$\int_0^1 dF\,\widetilde{R}(F,t)\partial_{\theta(t)}\widetilde{q}(F,t) = -\gamma R_s e^{-\bar{\gamma}\theta(t)}, \tag{71}$$

where $\widetilde{q}(F,t) = q(F,t)/q(0,t)$ and $R_s = (R(t) - (1-\gamma)R_0)/\gamma$. A tractable choice for $\widetilde{R}(F,t)$ in the subordinator model is, for some $\chi > 0$ (and $F \le F_{\max}(t)$),

$$\widetilde{R}(F,t) = R_0 - (R_0 - R_1)\left(\frac{F}{F_{\max}(t)}\right)^{\chi}. \tag{72}$$

This and hence $R(F,t)$ decreases from R_0 to R_1 (assuming $R_0 > R_1$) as F runs from zero to $F_{\max}(t)$. Though not explicitly shown, R_0, R_1 and χ can be time-dependent. Requirement (71) can now be evaluated to obtain

$$\frac{\eta(1+\chi)}{1+\chi}e^{-\bar{c}\theta(t)} = \gamma\frac{R_0 - R_s}{R_0 - R_1}, \quad \text{where} \quad \bar{c} = \bar{\eta}(1+\chi) - \bar{\eta}(1) - \bar{\mu}\chi. \tag{73}$$

Here $\eta(u)$ and $\bar{\eta}(u)$ are the Laplace exponents of the subordinator with local and period parameters respectively ($\eta(u) = \sigma\sqrt{u} + \mu u$ and $\bar{c} = \bar{\sigma}\left(\sqrt{1+\chi} - 1\right)$ for the Lévy Subordinator). Given $R_0 \ge R_1$ (and $\chi \ge 0$), we have $R_0 \ge R(t) \ge R_s \ge R_1$. The above relation can be used to determine say χ (or R_0) given $R(t)$ and other model parameters. If R_s is given, relation (69) can be used to relate $R(t)$ and γ as discussed earlier.

[2] Altman, Brady, Resti and Sironi [2005] provide empirical evidence for the inverse relationship. Given their data, a linear regression of recovery rates and inverse default rates gives $R_s \simeq 28\%$ and $(R_0 - R_s)\widehat{\lambda} \simeq$ 27bp (assuming constant $\widehat{\lambda}$) with a goodness of fit measure $R^2 \simeq 60\%$ The lowest default rate used is 84bp so that $\widehat{\lambda} <$ 84bp, implying $R_0 > 60\%$.

12 Initial Conditions

We have assumed that all the $\Lambda_i(t)$ processes start off as $\Lambda_i(0) = 0$. In one possible interpretation of $\Lambda_i(t)$ as the amount of negative economic impact on the name, this implies that conditions prior to time zero are assumed to be completely favorable. In this section, let us explore the consequences of relaxing this assumption.

The joint survival probability $Q_\Omega(t)$ over the period (t_0, t) for a list of names in Ω is given in (21) for zero initial conditions, that is $\Lambda_i(t_0) = 0$ for all i, $t_0 = 0$ being today. Let assume that it holds good for some earlier time $t_0 < 0$ given a realization of the $\Lambda_i(t)$ processes from t_0 to today. Now, the joint survival probability over the period $(0, t)$ conditional on survival up to today, denoted for simplicity as $Q_\Omega(t)$ itself, can be written down as

$$Q_\Omega(t) = e^{\Lambda_\Omega(0)} \left[\prod_{i\in\Omega} q_i(0,t)\right] \mathrm{E}\left\{e^{-\mathrm{Max}_{i\in\Omega}(\Lambda_i(t))}\right\} = e^{\Lambda_\Omega(0)} \int_0^1 dF \prod_{i\in\Omega} q_i(Fe^{\Lambda_i(0)}, t), \tag{74}$$

where $\Lambda_\Omega(0) = \mathrm{Max}_{i\in\Omega}(\Lambda_i(0))$ is the largest $\Lambda_i(0)$ and

$$q_i(F,t) = q_i(0,t)\mathrm{E}\left\{1_{\Lambda_i(t)-\Lambda_i(0)\leq -\ln F}\right\}. \tag{75}$$

As before, $q_i(0,t)$ and $q_i(F,t)$ are functions defined from today onwards with $q_i(0,0) = q_i(F,0) = 1$. Note that $Q_\Omega(t)$ depends only on the differences $\Lambda_\Omega(0) - \Lambda_i(0)$. To see this, note that $q_i(Fe^{\Lambda_i(0)}, t) = 0$ for $F > e^{-\Lambda_\Omega(0)}$ for the name with $\Lambda_i(0) = \Lambda_\Omega(0)$. The integrand is hence zero for $F > e^{-\Lambda_\Omega(0)}$ so that the integration variable can be changed to $F' = Fe^{\Lambda_\Omega(0)}$ that ranges from zero to one. Because $Q_\Omega(t)$ depends only on the differences, an overall impact does not contribute to it; only the relative values are relevant. This is as it should be since the names are all known to have survived up to today. The presence of initial conditions with relative values leads to higher joint survival probabilities. Calibration to individual hazard rate curves remains the same as before since, when there is only one name in Ω, $Q_\Omega(t)$ is independent of the initial condition.

Though the above is expressed as an integral over F, it is not a conditionally independent representation since the factor $e^{\Lambda_\Omega(0)}$ multiplying the integral is $\Omega-$dependent. One way to render it conditionally independent is to express the factor as

$$e^{\Lambda_\Omega(0)} = e^{\Lambda_M(0)} - \int_1^{e^{\Lambda_M(0)}} dG \prod_{i\in\Omega} 1_{\Lambda_i(0)\leq \ln G}, \tag{76}$$

where $\Lambda_M(0)$ is the largest $\Lambda_i(0)$ in the collection. Using this in (74) we obtain

$$Q_\Omega(t) = e^{\Lambda_M(0)} \int_0^1 dF \prod_{i\in\Omega} q_i(Fe^{\Lambda_i(0)}, t) - \int_0^1 dF \int_1^{e^{\Lambda_M(0)}} dG \prod_{i\in\Omega} q_i(Fe^{\Lambda_i(0)}, t) 1_{\Lambda_i(0)\leq \ln G}. \tag{77}$$

This is a weighted difference of two conditionally independent representations that can be useful in semi-analytical pricing in the presence of non-trivial initial conditions, but the presence of a double integral can make it computationally inefficient.

13 Default Contagion

We have modeled $\Lambda_i(t)$s as a set of independent stochastic processes evolving from time zero onwards. As our zero of time passes by, one would expect $\Lambda_i(t)$s to get realized. However, it is not obvious how this information can be extracted from the names. Let us assume in this section that the only information available from the names is their defaulted or undefaulted status. In such a scenario, one expects the hazard rate of a given name to jump up on every default information as it arrives.

A simple analysis in our one-factor formulation helps us to infer that such a contagion tendency exists at the probability level itself. Arguments are similar to those discussed in Balakrishna [2007]. Consider $\pi_n^{(\nu)}(t)$, the conditional probability of default during $(0,t)$ of say name n given the information that names $1,\cdots,\nu$ have defaulted,

$$\pi_n^{(\nu)}(t) = \frac{\int_0^1 dF\,[p_1(F,t)\cdots p_\nu(F,t)]\,p_n(F,t)}{\int_0^1 dF\,[p_1(F,t)\cdots p_\nu(F,t)]}. \tag{78}$$

This can be compared to $\pi_n^{(\nu-1)}(t)$, conditional probability of default without the default information about name ν. If $\pi_n^{(\nu)}(t) > \pi_n^{(\nu-1)}(t)$, it suggests that the likelihood of name n having defaulted increases with the number of names known to have defaulted. This check amounts to verifying

$$\int_0^1 dF \int_0^F dG w_{\nu-1}(F,t) w_{\nu-1}(G,t)\,[p_\nu(F,t) - p_\nu(G,t)]\,[p_n(F,t) - p_n(G,t)] > 0, \tag{79}$$

where $w_{\nu-1}(F,t) = p_1(F,t)\cdots p_{\nu-1}(F,t)$. This always holds under nontrivial circumstances since $p_i(F,t)$s are non-decreasing functions of F for all the names.

Now coming to jumps in the hazard rate due to default contagion, again for name n, consider the joint survival probability

$$Q(t_*) = \mathrm{Prob}(\tau_1 > t_1, \cdots, \tau_n > t_n), \tag{80}$$

where τ_is are random default times, $t_1 < \cdots < t_n$, t_* denotes dependence on $t_1,\cdots,t_n$ and for convenience the names are labeled according to the same order. The probability density (or intensity) that names $1,\cdots,\nu$ have defaulted respectively at times $t_1,\cdots,t_\nu$ and the rest have survived up to times $t_{\nu+1},\cdots,t_n$ is then given by

$$Q_{(\nu)}(t_*) = (-1)^\nu \frac{\partial^\nu Q(t_*)}{\partial t_1 \cdots \partial t_\nu}. \tag{81}$$

We may allow $\nu = 0$ as well in which case $Q_{(0)}(t_*) = Q(t_*)$. Given the default and the survival information defining it, the hazard rate for name n can now be written as

$$h_n^{(\nu)}(t_*) = -\partial_n \ln Q_{(\nu)}(t_*), \tag{82}$$

where $\partial_n = \partial/\partial t_n$. The jump in the hazard rate $h_n^{(\nu-1)}(t_*)$ due to name ν defaulting is, dropping $t-$arguments for simplicity,

$$\Delta_\nu h_n^{(\nu-1)} = h_n^{(\nu)} - h_n^{(\nu-1)} = -\partial_n \ln\left(\frac{Q_{(\nu)}}{Q_{(\nu-1)}}\right) = -\partial_n \ln\left(-\partial_\nu \ln Q_{(\nu-1)}\right). \tag{83}$$

If positive, this indicates that defaults are contagious, that for every credit name defaulting, the hazard rate of the name n being observed jumps up. To know this jump for any time t_n, given all the survival and default information up to time $t \leq t_n$, all of $t_{\nu+1}, \cdots, t_{n-1}$ should be set to t. If interested in this jump just after the last credit name ν has defaulted, t should be set to t_ν as well.

Hazard rate jumps are, as one would expect, proportional to default correlation. This can be seen by rewriting the jump as

$$\Delta_\nu h_n^{(\nu-1)} = \frac{Q_{(\nu-1)}}{Q_{(\nu)}} \left[\frac{\partial_n \partial_\nu Q_{(\nu-1)}}{Q_{(\nu-1)}} - \frac{\left(-\partial_n Q_{(\nu-1)}\right)}{Q_{(\nu-1)}} \frac{\left(-\partial_\nu Q_{(\nu-1)}\right)}{Q_{(\nu-1)}} \right] = \sqrt{\frac{-\partial_n Q_{(\nu-1)}}{Q_{(\nu)}}} \rho_{n\nu}^{(\nu-1)}, \quad (84)$$

where $\rho_{n\nu}^{(\nu-1)} \sqrt{dt_n dt_\nu}$ is the default correlation between credit names n and ν for defaults during infinitesimal intervals $(t_n, t_n + dt_n)$ and $(t_\nu, t_\nu + dt_\nu)$ respectively, given the conditions implicit in $Q_{(\nu-1)}$. For $\nu = 1$, this relates to default correlation in the absence of contagion and for $\nu > 1$ to default correlation in the middle of a contagion.

Consider now the extended one-factor model defined by (23) that gives for the joint survival probability,

$$Q(t_*) = \int_0^1 dF \left[q_1(F, t_1) \cdots q_n(F, t_n) \right]. \quad (85)$$

In this model, the probability density $Q_{(\nu)}(t_*)$ is

$$Q_{(\nu)}(t_*) = \int_0^1 dF w(F, t_*) \left[\lambda_1(F, t_1) \cdots \lambda_\nu(F, t_\nu) \right], \quad (86)$$

where $w(F, t_*) = q_1(F, t_1) \cdots q_n(F, t_n)$ and $\lambda_i(F, t_i) = -\partial_i(\ln q_i(F, t_i))$. This gives for the hazard rate

$$h_n^{(\nu)}(t_*) = \frac{\int_0^1 dF w(F, t_*) \left[\lambda_1(F, t_1) \cdots \lambda_\nu(F, t_\nu) \right] \lambda_n(F, t_n)}{\int_0^1 dF w(F, t_*) \left[\lambda_1(F, t_1) \cdots \lambda_\nu(F, t_\nu) \right]}. \quad (87)$$

This is again of the form $\pi_n^{(\nu)}(t)$ given in expression (78). Similar steps let us conclude that $h_n^{(\nu)}(t_*)$ will jump up on every default if $\lambda_i(F, t_i)$ is monotonic in F for all the names. As has been noted in section 11, the conditional hazard rate $\lambda_i(F, t_i)$ is expected to be a non-decreasing function of F (this can be verified in the Lévy subordinator model).

14 Intensity Modeling

We modeled the individual process $\Lambda_i(t)$ as a time-changed Lévy subordinator. An alternate approach to modeling it as a non-decreasing process is to express it as the time-integral of a non-negative stochastic process that in some sense can be interpreted as stochastic default intensity. However, as we see below, when driven by a stable subordinator, the resulting model is effectively a time-inhomogeneous stable subordinator model. Hence, our stable subordinator models can be considered to be sufficiently generic.

In the following, we are concerned with one credit name and hence the name-subscript is dropped from the results. Consider $\lambda(t)$ obeying the stochastic differential equation

$$d\lambda(t) = -m\lambda(t)dt + dS(t), \quad (88)$$

where $S(t)$ is a subordinator and m is the mean reversion rate (better referred to as the decay rate since the mean need not be finite). This can be solved to obtain

$$\begin{aligned}\lambda(t) &= \lambda(0)e^{-mt} + \int_0^t e^{-m(t-s)} dS(s), \\ \Lambda(t) &= \int_0^t \lambda(s)ds = \lambda(0)b(t) + \int_0^t b(t-s)dS(s),\end{aligned} \tag{89}$$

where $b(t) = (1 - e^{-mt})/m$. For simplicity, we have not considered time-changing the process here. Given above, one can determine the Laplace transform

$$e^{-\psi_t(u)} = \mathrm{E}\left\{e^{-u\Lambda(t)}\right\} = \exp\left\{-\lambda(0)b(t)u - \int_0^t \eta(ub(s))ds\right\}, \tag{90}$$

where $\eta(u)$ is the Laplace exponent of $S(t)$. If $S(t)$ is chosen to be the Lévy subordinator with $\eta(u) = \sigma\sqrt{u} + \mu u$, we have

$$\psi_t(u) = \left\{\lambda(0)b(t) + \mu \int_0^t b(s)ds\right\} u + \left\{\sigma \int_0^t \sqrt{b(s)}ds\right\} \sqrt{u}. \tag{91}$$

Note that this reduces to $t(\mu' u + \sigma' \sqrt{u})$ when the limit $m \to \infty$ is appropriately taken, that is with $\mu/m \to \mu'$ and $\sigma/\sqrt{m} \to \sigma'$ as $m \to \infty$. Hence, our earlier Lévy subordinator model is the $m \to \infty$ limit of this intensity model. Besides, $\psi_t(u)$ itself is of the form $p(t)u + q(t)\sqrt{u}$ for some parameters $p(t)$ and $q(t)$ so that the Laplace transform is that of a Lévy distribution. It can be generated by a time-inhomogeneous Lévy subordinator. Drift $p(t)$ is now stochastic since, when viewed as a dynamical model, it has a dependence on the present value of the stochastic process $\lambda(t)$.

There can be some reservations about intensity modeling in this type-II framework. Note that $\Lambda_i(t) = \int_0^t \lambda_i(s)ds$ is a predictable process and for t infinitesimally small can be written as $\approx \lambda_i(0)t$. Consider $\lambda_i(0)$ to be a fraction γ of the initial hazard rate $h_i(0)$ inferred from the hazard rate curve, fraction $1 - \gamma$ being the firm-specific component $\widehat{\lambda}_i(0)$. Fixing $\Lambda_i(t)$ deterministically may itself be questionable but, leaving that aside, consider the joint survival probability $Q_{12}(t)$ for two names 1 and 2. Assuming without loss of generality $h_1(0) \leq h_2(0)$ and hence $\lambda_1(0) \leq \lambda_2(0)$, we have $Q_{12}(t) \approx 1 - \widehat{\lambda}_1(0)t - \widehat{\lambda}_2(0)t - \lambda_2(0)t = 1 - (1-\gamma)h_1(0)t - h_2(0)t$. This leads to maximum dependability, a restrictive feature of the model. Instantaneous default correlation at time zero can be computed to be $\gamma\sqrt{h_1(0)/h_2(0)}$. This can turn out to be significant even under normal economic conditions giving rise to an unacceptable number of simultaneous defaults. These observations can also be made directly from our result above that involves $\lambda_i(0)$ contributing to the drift component that, as we have already noted, accounts for catastrophic scenarios.

Instead, it may be more appropriate to make $\theta_i(t)$ dynamic, say as the time-integral of some non-negative stochastic process $\lambda_i(t)$. Our earlier Lévy subordinator model can now be defined for a realization of $\lambda_i(t)$. Time-changing the subordinator is still with $\theta_i(t)$, but the latter is no longer given by (34). Instead, it leads to the individual survival probability $Q_i(t) = \mathrm{E}\left\{e^{-\theta_i(t)}\right\}$. A possible choice for the $\lambda_i(t)$ process is square-root diffusion that provides analytical results to compute $Q_i(t)$. A closed form expression is also available for the distribution of $\lambda_i(t)$, but computing joint default probabilities involves distribution of its time-integral making it harder pricing semi-analytically. Nevertheless, this approach is appealing as it accounts for stochasticity of hazard rates.

15 Discussions and Conclusions

Modeling the dependency structure of defaults or other such extreme events is a problem of multi-dimensional mathematical complexity. But the need for a simple and tractable solution is evident from the popularity of the Gaussian copula model. The model has remained the market standard for pricing correlation products in spite of all its limitations, very well-known in the field. It is the purpose of the article to demonstrate that there do exist other models, equally simple and tractable, that have better loss process dynamics and explanatory power. The Lévy subordinator model presented here is one such model offering a reasonable explanation of the correlation smile.

The Lévy subordinator model is a one-factor model driven by the Lévy subordinator, an $\alpha = 1/2$ stable process maximally skewed to the right. The distribution function of the Lévy subordinator is known in closed form as the Lévy distribution. The model shares many of the attractive features of the Gaussian copula model. It gets automatically calibrated to individual hazard rate curves. It can be used for pricing both semi-analytically by employing recursive methods or Fast Fourier Transform techniques, and via a Monte Carlo algorithm. Being structured similar to the Gaussian copula model, it can easily be implemented within the framework of the existing computational infrastructure. In fact, the only modifications needed are to use (39) for the conditional survival probability in place of a similar one of the Gaussian copula model and to integrate the conditional results over a uniform distributed common factor instead of a normally distributed one.

The model has just two parameters σ and μ, or equivalently $\gamma = \sigma + \mu$ and $\kappa = \mu/(\sigma + \mu)$. Both γ and κ are permitted to range from zero to one. γ is the fraction of hazard rate that is attributable to systemic factors. Of this fraction, a further fraction κ is attributed to catastrophic scenarios. When deep in crisis, γ may well tend close to its upper limit of one. When $\gamma = 1$, the model still has a freedom of one parameter, namely κ. This can be helpful in, at least qualitatively, accounting for in-crisis correlation smiles.

Figures 1-16 present the results of a numerical investigation into the model's implications. Results are for a homogeneous collection with flat hazard rates. Figures 1-4 plot the base correlations for various values of parameters under fixed recovery rates. Base correlations are implied by the Gaussian copula model for the Lévy subordinator model prices. Model prices and hedge ratios are computed for tranches [0,3]%, [3,7]%, [7,10]%, [10,15]% and [15,30]%. As can be seen from the figures, despite having only two parameters at its disposal, the model is capable of generating correlation smiles of various slope characteristics. Figures 5-8 present the hedge ratios under similar conditions. The hedge ratios appear reasonable, atleast qualitatively, with the right dependencies on the model parameters. Figures 9 and 10 show the effect of random recovery rates on base correlations and hedge ratios for different values of γ in the random recovery model of (68) with $R(t)$ given by (69). Figures 11 and 12 respectively plot the joint default probability distributions for different values of γ and κ parameters. Figure 13 plots them for a heterogeneous collection for different values of a parameter controlling heterogenity. Figure 14 plots the individual hedge ratios in a heterogeneous collection. Heterogeneity has a significant effect on the model behavior, especially on the the tail of the loss distribution. This can be understood with an $F_t(\nu)$ as the inverse of the average of the $p_t(F)$s in the heterogeneous collection.

We have considered the model parameters as uniform across the collection for simplicity. As already noted, they can be name-dependent (and also time-dependent). An attractive feature of the one-factor model is that the pool specifics are completely encoded into the names so that it becomes easier to combine pools to construct larger ones. For example, if we have two pools calibrated separately in the one-factor setup giving rise to two sets of parameters, we can price trades on a pool constructed out of names picked up from the two pools assuming applicability of the one-factor formulation. This is of course the first step since in general one would need the multi-factor formulation (24) with at least two factors and a copula describing their joint distribution.

We modeled the individual process $\Lambda_i(t)$ as a time-changed Lévy subordinator. Other subordinators can also be attempted that admit closed form solutions to their distributions such as the inverse Gaussian subordinator that is a natural extension of the Lévy subordinator. Though relatively less efficient, stable subordinators with α in the neighborhood of $1/2$ can also be tried using the method described in section (1) to obtain their distributions (see Figures 15 and 16). Alternately, the conditional survival probability $q_i(F, t)$ can be modeled directly, for instance as a mixture of Lévy distributions. Though these extensions were not found to be helpful in improving the fit to market data on CDOs in a preliminary study in Balakrishna [2010] (except to some extent stable subordinators with α near $1/2$), they may be helpful under different market conditions.

Though the model has been developed with an application to CDOs in mind, it could be useful in other disciplines that involve modeling a dependent set of events. Simplicity and tractability with its large homogeneous pool approximation, an efficient semi-analytical framework and a Monte Carlo algorithm makes the model an attractive choice.

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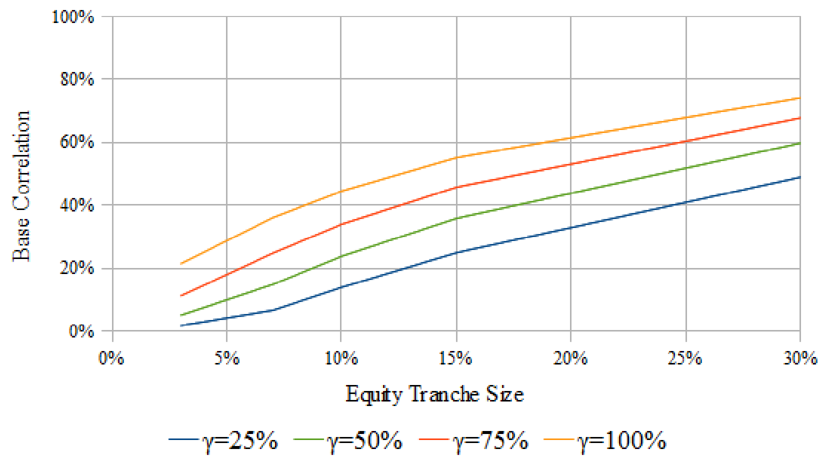

Figure 1: Base correlations for $\gamma = 25\%, 50\%, 75\%, 100\%$ with fixed $\kappa = 25\%$ and recovery rate $R = 40\%$. Maturity is 5 years and reference spread is at 100bp.

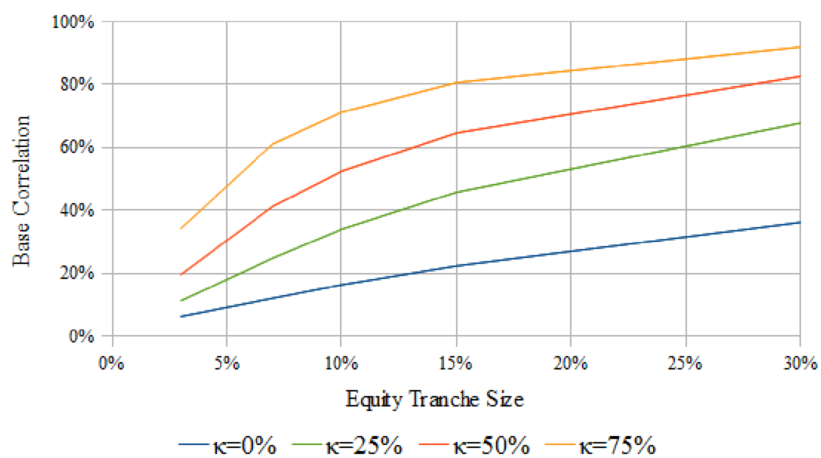

Figure 2: Base correlations for $\kappa = 0\%, 25\%, 50\%, 75\%$ with fixed $\gamma = 75\%$ and recovery rate $R = 40\%$. Maturity is 5 years and reference spread is at 100bp.

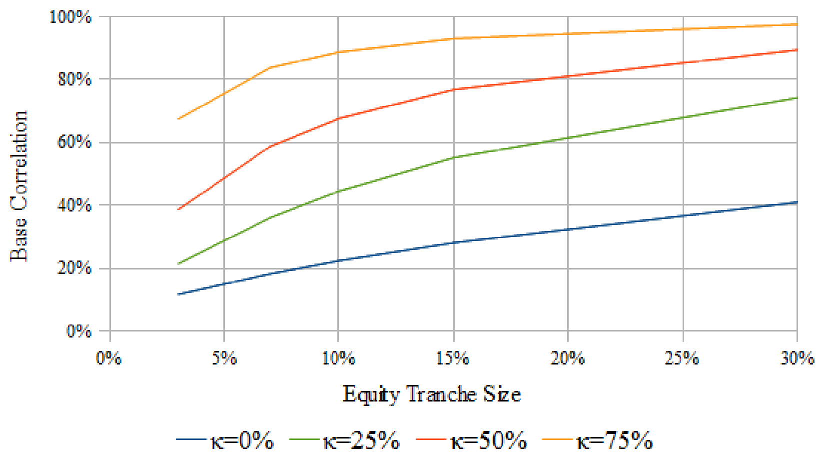

Figure 3: Base correlations for $\kappa = 0\%, 25\%, 50\%, 75\%$ with fixed $\gamma = 100\%$ and recovery rate $R = 40\%$. Maturity is 5 years and reference spread is at 100bp.

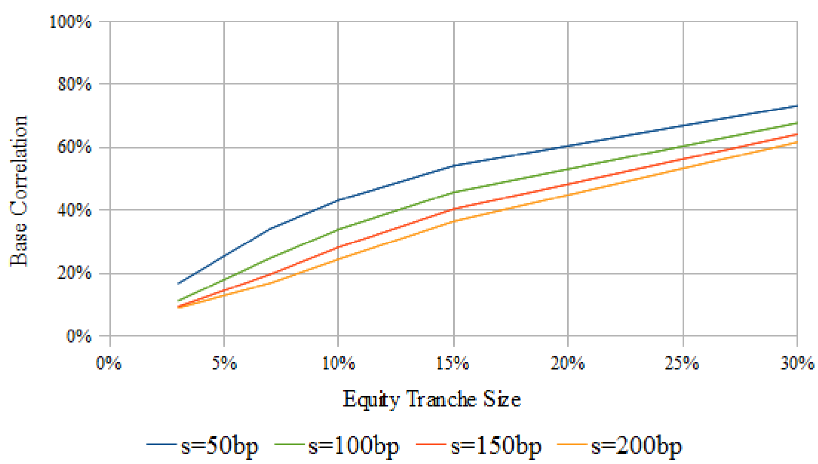

Figure 4: Base correlations for reference spreads $s = 50, 100, 150, 200$bp with fixed $\gamma = 75\%, \kappa = 25\%$ and recovery rate $R = 40\%$. Maturity is 5 years.

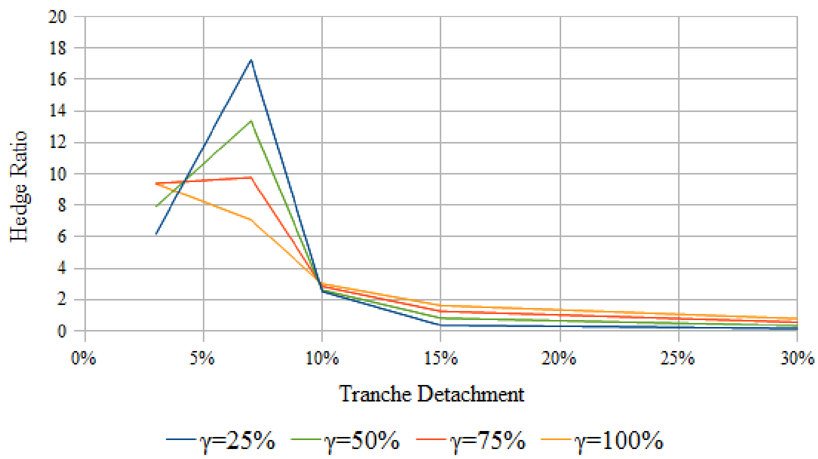

Figure 5: Hedge ratios for $\gamma = 25\%, 50\%, 75\%, 100\%$ with fixed $\kappa = 25\%$ and recovery rate $R = 40\%$. Maturity is 5 years and reference spread is at 100bp.

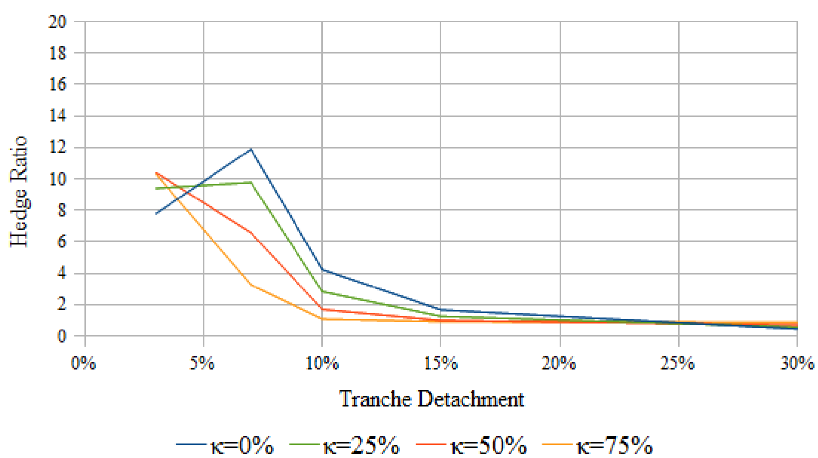

Figure 6: Hedge ratios for $\kappa = 0\%, 25\%, 50\%, 75\%$ with fixed $\gamma = 75\%$ and recovery rate $R = 40\%$. Maturity is 5 years and reference spread is at 100bp.

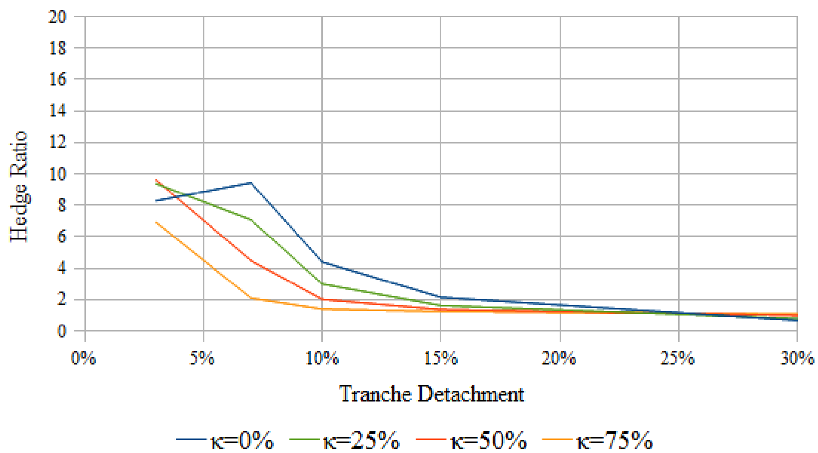

Figure 7: Hedge ratios for $\kappa = 0\%, 25\%, 50\%, 75\%$ with fixed $\gamma = 100\%$ and recovery rate $R = 40\%$. Maturity is 5 years and reference spread is at 100bp.

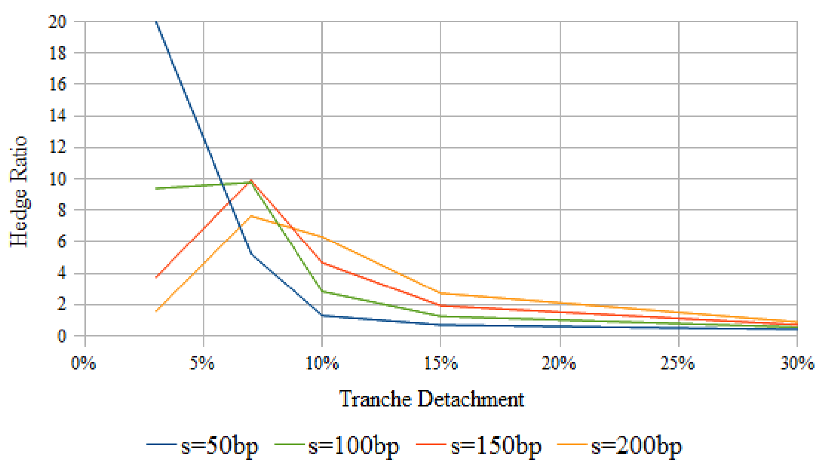

Figure 8: Hedge ratios for reference spreads $s = 50, 100, 150, 200$bp with fixed $\gamma = 75\%, \kappa = 25\%$ and recovery rate $R = 40\%$. Maturity is 5 years.

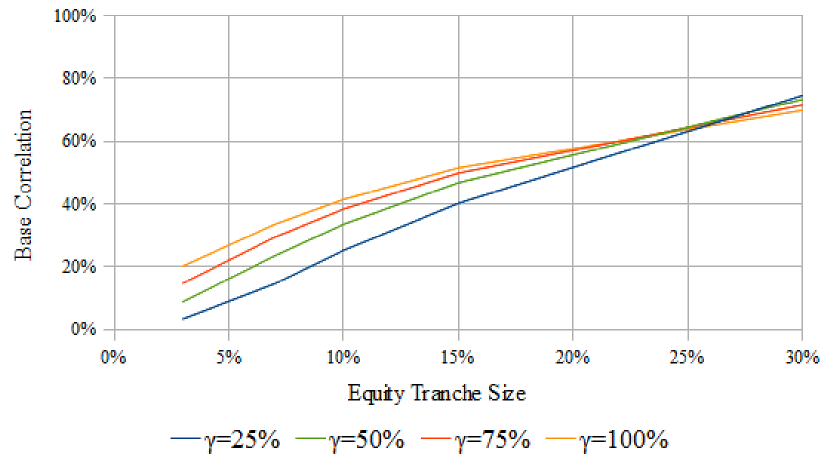

Figure 9: Base correlations for $\gamma = 25\%, 50\%, 75\%, 100\%$ with fixed $\kappa = 25\%$. Recovery rate is random as in (68) with $R_0 = 65\%, R_s = 25\%$ and $R = 55\%, 45\%, 35\%, 25\%$ respectively given by (69). Maturity is 5 years and reference spread is at 100bp.

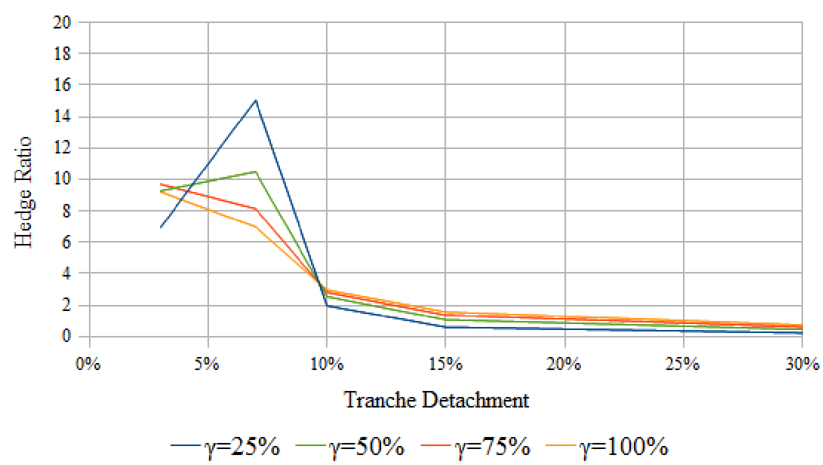

Figure 10: Hedge ratios for $\gamma = 25\%, 50\%, 75\%, 100\%$ with fixed $\kappa = 25\%$. Recovery rate is random as in (68) with $R_0 = 65\%, R_s = 25\%$ and $R = 55\%, 45\%, 35\%, 25\%$ respectively given by (69). Maturity is 5 years and reference spread is at 100bp.

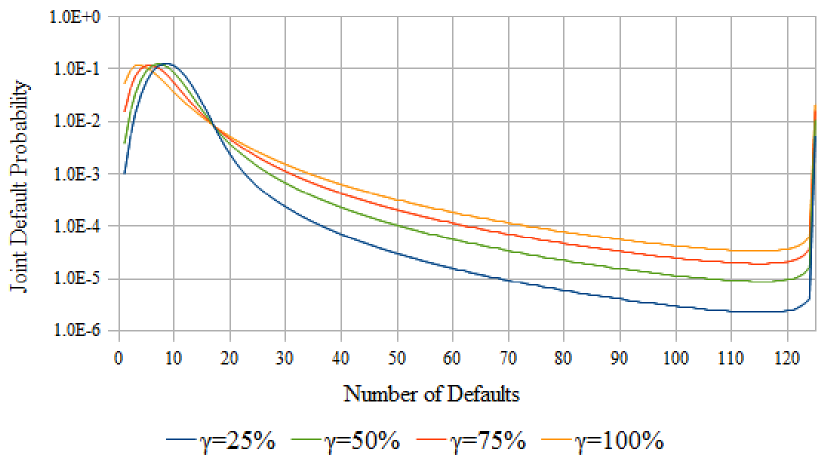

Figure 11: Joint default probability distributions for $\gamma = 25\%, 50\%, 75\%, 100\%$ with fixed $\kappa = 25\%$ and recovery rate $R = 40\%$. Maturity is 5 years and reference spread is at 100bp.

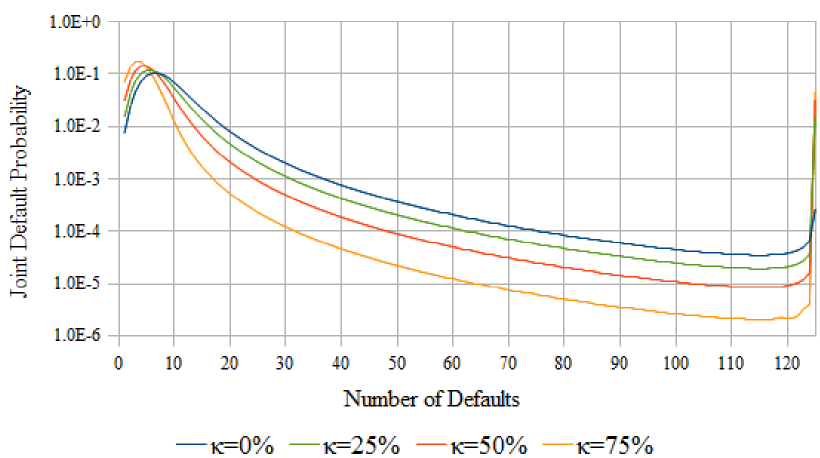

Figure 12: Joint default probability distributions for $\kappa = 0\%, 25\%, 50\%, 75\%$ with fixed $\gamma = 75\%$ and recovery rate $R = 40\%$. Maturity is 5 years and reference spread is at 100bp.

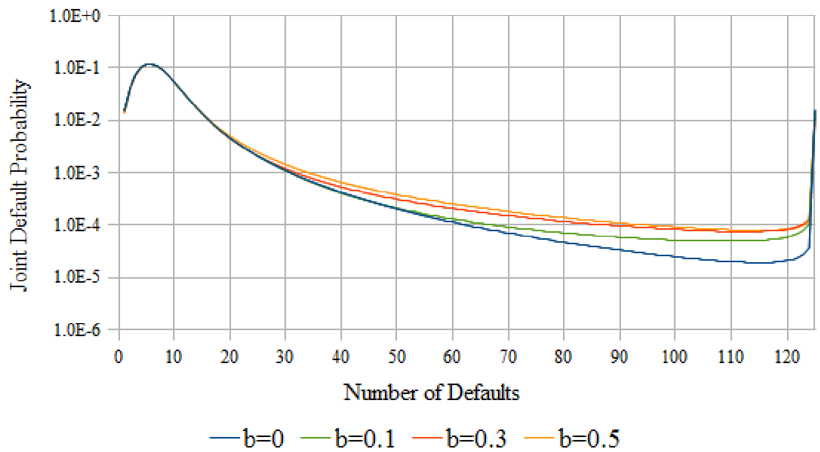

Figure 13: Joint default probability distributions for a heterogeneity parameter $b = 0, 0.1, 0.3, 0.5$ with fixed $\gamma = 75\%, \kappa = 25\%$ and recovery rate $R = 40\%$. Maturity is 5 years. The collection is heterogeneous with the individual spreads $a(1-b)\nu^{-b}$ where $\nu = i/n, i = 1, \cdots, n = 125$ and a chosen to yield an index spread of 100bp.

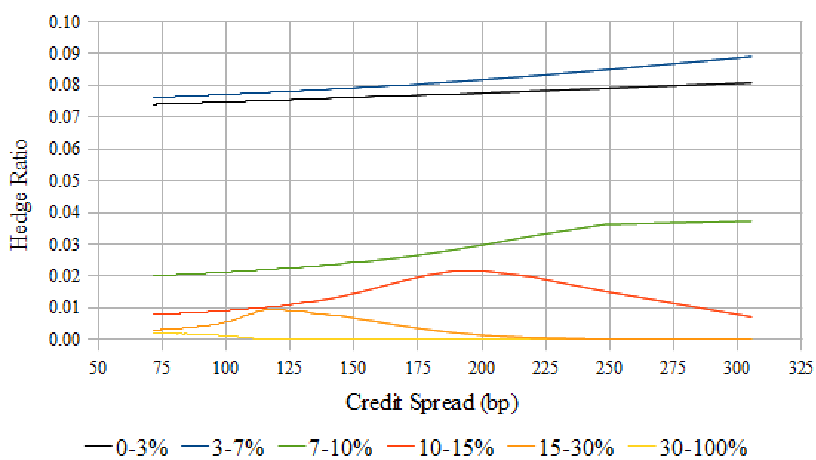

Figure 14: Individual hedge ratios for different tranches with fixed $\gamma = 75\%, \kappa = 25\%$ and recovery rate $R = 40\%$. Maturity is 5 years. The collection is heterogeneous with the individual credit spreads $0.7a\nu^{-0.3}$ where $\nu = i/n, i = 1, \cdots, n = 125$ and $a = 102.5$bp yielding an index spread of 100bp.

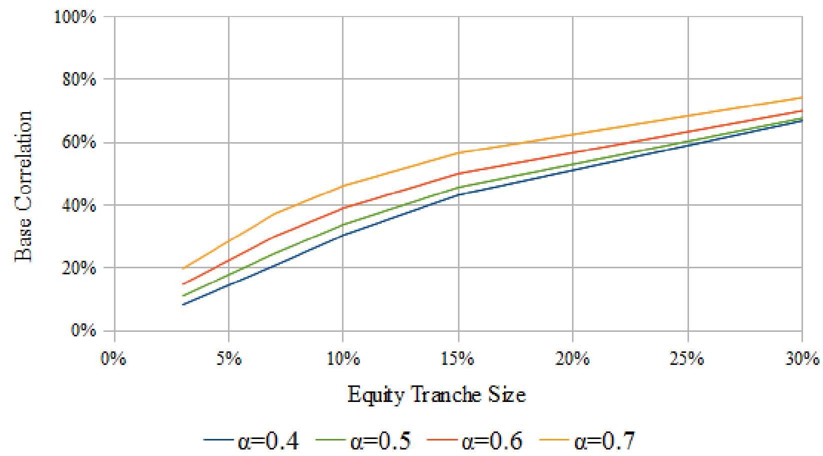

Figure 15: Base correlations for index of stability $\alpha = 0.4, 0.5, 0.6, 0.7$ with fixed $\gamma = 75\%, \kappa = 25\%$ and recovery rate $R = 40\%$. Maturity is 5 years and reference spread is at 100bp.

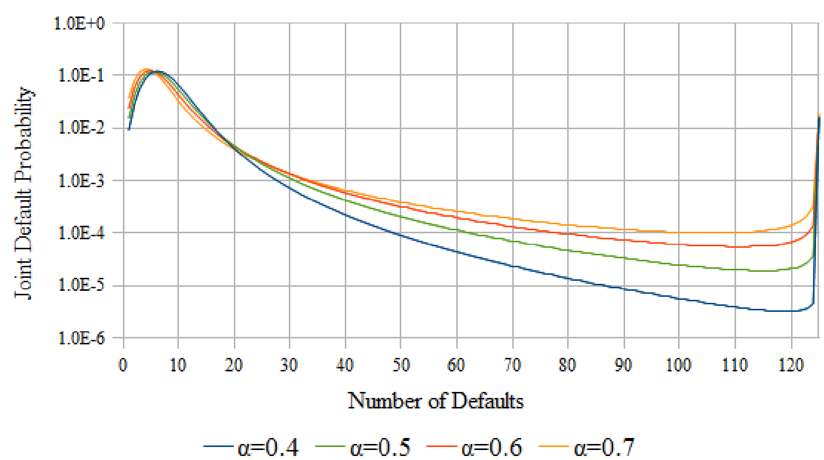

Figure 16: Joint default probability distributions for index of stability $\alpha = 0.4, 0.5, 0.6, 0.7$ with fixed $\gamma = 75\%, \kappa = 25\%$ and recovery rate $R = 40\%$. Maturity is 5 years and reference spread is at 100bp.