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Endogenous income taxes in OLG economies

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Abstract

This paper introduces fiscal increasing returns, through endogenous labor income tax rates as in Schmitt-Grohe and Uribe (1997), into the overlapping generations model with endogenous labor and consumption in both periods of life (for example, Cazzavillan and Pintus (2004)). We show that under numerical calibrations of the parameters, in particular a reasonable share of first period consumption over the wage income, local indeterminacy can easily occur with small distortionary taxes, provided that the elasticity of capital-labor substitution is less than the share of capital in total income and the wage elasticity of the labor supply is large enough. More important is the fact that increasing the size of tax distortions enlarges the range of values of the consumption–to–wage ratio associated with multiple equilibria, because of two conflicting effects on savings that operate through wage and interest rate.

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JEL: C62, E32

1. Introduction

In this paper, we consider a two periods overlapping generations model with endogenous labor, consumption in both periods of life and fiscal increasing returns coming from endogenous labor income taxes. Cazzavillan and Pintus (2004) have pointed out that intertemporal substitution in consumption is a critical element to make expectation-driven fluctuations disappear in the context of LOG economies if the ratio between savings and wage is reasonably low. But in this paper, we show that indeterminacy can appear in their framework if there are fiscal increasing returns caused by endogenous labor income tax rates as in Schmitt-Grohe and Uribe (1997), since endogenous labor income taxes can increase the ratio between savings and wage. We prove that with tax distortions and realistic calibrations for the fundamentals, indeterminacy can easily occur in a CES economy provided that the elasticity of the labor supply is large enough and the elasticity of the input substitution is less than the share of capital in total income.

Since Reichlin (1986), the Diamond (1965) one-sector overlapping generations model augmented to include endogenous labor supply, external effects and/or fiscal increasing returns has become a popular framework to analyze expectations driven business cycles.[1] Unlike those early works that focus on a particular case without first period consumption, recent works such as Cazzavillan and Pintus (2004, 2006) and Lloyd-Braga et al. (2007), consider a life-cycle utility function which is first, separable between consumption and leisure, and second, linearly homogenous with respect to young and old consumptions. The main contribution of these papers is to analyze the relationship between external effects and indeterminacy under the framework with consumption in both periods.

[1] For example, Cazzavillan (2001) and Gokan (2009a, 2009b).

Our paper instead discusses the relationship between fiscal increasing returns and indeterminacy under the framework with consumption in both periods. Particularly, we concentrate on the focal case where fiscal increasing returns come from endogenous labor income taxes and show that local indeterminacy easily occurs with reasonable steady state labor income tax rates provided that the elasticity of capital-labor substitution is less than the share of capital in total income and the elasticity of labor supply is large enough.

The paper is organized as follows. Section 2 sets up the model. In Section 3, we establish the existence of a normalized steady state. Section 4 contains the derivation of the characteristic polynomial and presents the geometrical method used for the local dynamic analysis and our main results on local indeterminacy. Section 5 gathers some concluding comments. All the proofs are gathered in a final appendix.

2. The model

As in Cazzavillan and Pintus (2004), we consider a competitive, non-monetary, overlapping generations model with production. The model involves a unique perishable good, which can be either consumed or saved as investment. Identical competitive firms all face the same technology. Identical households live for two periods. The agent consumes in both periods, supplies labor and saves when young. When old, her saved income is rented as physical capital to the firm.

Assuming additively separable preferences, the household born at time $t \geq 0$ maximizes her lifetime utility

$$\max_{c_{1t},\, \lambda_t,\, c_{2t+1}} \left[U_1(c_{1t}/B) - U_3\left(\lambda_t\right) + \beta U_2\left(c_{2t+1}\right)\right]$$

subject to the constraints

$$c_{1t} + z_t = \left(1 - \tau_t\right) \Omega_t \lambda_t \tag{1}$$

$$c_{2t+1} = R_{t+1} z_t \tag{2}$$

$$c_{1t} \geq 0,\ c_{2t+1} \geq 0,\ \overline{\lambda} \geq \lambda_t \geq 0, \text{ for all } t \geq 0,$$

where λ_t, c_{1t} and z_t are labor, consumption and saving, respectively, of the individual of the young generation, c_{2t+1} is the consumption of the same individual when old, and $\Omega_t > 0$ and $R_{t+1} > 0$ are the real wage at time t and the gross interest rate at time $t+1$. Moreover, τ_t, $\beta \in (0,1)$, $B > 0$ and $\overline{\lambda}$ are the labor income tax rate, the discount factor, a scaling parameter and the maximum amount of labor supply, respectively.

The preferences satisfy the following condition as in Cazzavillan and Pintus (2004).

Assumption 1. The functions $U_1(c/B)$, $U_3(\lambda)$ and $U_2(c)$ are defined and continuous on the set R_+. Moreover, they are C^r, for r large enough, on the set R_{++}, with $U_1'(c/B) > 0$, $U_2'(c) > 0$, $U_3'(\lambda) > 0$, $U_1''(c/B) < 0$, $U_2''(c) < 0$, $U_3''(\lambda) > 0$. $\lim_{\lambda \to \overline{\lambda}} U_3'(\lambda) = +\infty$, where $\overline{\lambda} > 1$, and $\lim_{\lambda \to 0} U_3'(\lambda) = 0$. In addition, $0 < R_1(c/B) \equiv -(c/B)U_1''(c/B)/U_1'(c/B) < 1$, $0 < R_2(c) \equiv -cU_2''(c)/U_2'(c) < 1$, and $R_3(\lambda) \equiv \lambda U_3''(\lambda)/U_3'(\lambda) > 0$.

The conditions $0 < R_1(c/B) < 1$ and $0 < R_2(c) < 1$ are used to ensure that consumption and leisure are gross substitutes, and the saving function is increasing in R. For example, we can assume that $U_1(c/B) = \frac{(c/B)^{1-\alpha_1}}{1-\alpha_1}$, $U_2(c_2) = \frac{(c_2)^{1-\alpha_2}}{1-\alpha_2}$ and $U_3(\lambda) = \frac{\lambda^{1+\alpha_3}}{1+\alpha_3}$. When $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ and $0 < \alpha_3$, these equations satisfy assumption 1.

When the solution of the households maximization problem is interior, the first order conditions are

$$U_1'(c_{1t}/B)/B = \beta R_{t+1} U_2'(c_{2t+1}) = U_3'(\lambda_t) / [(1-\tau_t)\Omega_t]. \tag{3}$$

Using the first order conditions, the current consumption can be written as follows

$$c_{1t} = B\left(U_1'\right)^{-1}\left(\frac{BU_3'\left(\lambda_t\right)}{\left(1-\tau_t\right)\Omega_t}\right), \tag{4}$$

and the savings of the young agent born at time t are

$$z_t = \left(1-\tau_t\right)\Omega_t\lambda_t - B\left(U_1'\right)^{-1}\left(\frac{BU_3'\left(\lambda_t\right)}{\left(1-\tau_t\right)\Omega_t}\right). \tag{5}$$

Multiplying both terms of the last equality in Eqs. (3) by z_t yields

$$\beta U_2'\left(c_{2t+1}\right)c_{2t+1} = \frac{z_t U_3'\left(\lambda_t\right)}{\left(1-\tau_t\right)\Omega_t}, \text{ or, } R_{t+1}z_t = u_2^{-1}\left(\frac{z_t U_3'\left(\lambda_t\right)}{\beta\left(1-\tau_t\right)\Omega_t}\right), \tag{6}$$

where $u_2(c_{2t+1}) = U_2'\left(c_{2t+1}\right)c_{2t+1}$ is an increasing function in c_{2t+1}.

The perishable output (y) is produced using capital (k) and labor (λ),

$$y = AF\left(k,\lambda\right) = Alf\left(a\right),$$

where $a = k/\lambda$ and $A > 0$ is a scaling factor. The competitive factor market implies that the real wage rate and the real gross rate of return on capital are

$$\Omega\left(a\right) \equiv A\left[f\left(a\right) - af'(a)\right] = A\omega\left(a\right),\ R\left(a\right) \equiv Af'\left(a\right) + 1 - \delta = A\rho\left(a\right) + 1 - \delta, \tag{7}$$

where $1 \geq \delta \geq 0$ is the constant depreciation rate of capital. If we consider the CES production function, the reduced production function can be given by

$$
\begin{aligned}
f(a) &= A(sa^{-\eta}+1-s)^{-\frac{1}{\eta}} \text{ if } \eta \neq 0, \\
&= Aa^{s} \qquad\qquad\quad \text{if } \eta = 0,
\end{aligned}
$$

where $\eta > -1$ determines the elasticity of input substitution through $\sigma = 1/(1+\eta)$, while $0 < s < 1$ governs the share of capital income in production.

As in Schmitt-Grohe and Uribe (1997) and Gokan (2006), at each point in time, the government finances its **constant** expenditure through labor income taxes, i.e.,

$$
g = \tau_t \Omega(a_t)\lambda_t > 0. \tag{8}
$$

Using the fact that at the equilibrium $k_{t+1} = z_t$ holds, we can easily derive the dynamical system characterizing equilibrium paths of $(k_t,\, a_t)$.

$$
k_{t+1} = \Omega(a_t)\frac{k_t}{a_t} - B(U_1')^{-1}\left(\frac{BU_3'(\frac{k_t}{a_t})\frac{k_t}{a_t}}{\Omega(a_t)\frac{k_t}{a_t} - g}\right) - g, \tag{9}
$$

$$
R(a_{t+1})k_{t+1} = u_2^{-1}\left\{\frac{k_{t+1}(\frac{k_t}{a_t})U_3'\left(\frac{k_t}{a_t}\right)}{\beta[\Omega(a_t)\frac{k_t}{a_t} - g]}\right\}. \tag{10}
$$

3. Steady state existence

A steady state is a pair $(k^*,\, a^*)$ such that.

$$
k^* = A\omega\left(a^*\right)\frac{k^*}{a^*} - B(U_1')^{-1}\left(\frac{BU_3'(\frac{k^*}{a^*})k^*}{A\omega\left(a^*\right)k^* - ga^*}\right) - g,
$$

$$A\rho\left(a^{*}\right)+1-\delta=\frac{1}{k^{*}}u_{2}^{-1}\{\frac{k^{*2}U_{3}'\left(\frac{k^{*}}{a^{*}}\right)}{\beta[A\omega\left(a^{*}\right)k^{*}-ga^{*}]}\}. \tag{11}$$

To simplify the algebra, we follow Cazzavillan and Pintus (2004) and use the parameters A and B to normalize the steady state.

Proposition 1. *Under the assumptions on the utility and production functions, $(k^{*},a^{*})=(1,1)$ is a normalized steady state (NSS) of the dynamic system (9) and (10) if and only if g is not too large, $A^{*}\omega\left(1\right)>g+1$, $\beta u_{2}[\frac{\rho(1)}{\omega(1)}(1+g)+1-\delta]<U_{3}'(1)$ and $\lim_{c\rightarrow 0}cU_{1}'(c)<\frac{A^{*}\omega(1)-1-g}{A^{*}\omega(1)-g}U_{3}'(1)$, where A^{*} is the unique solution of $A\rho\left(1\right)+1-\delta=u_{2}^{-1}\{\frac{U_{3}'(1)}{\beta[A\omega(1)-g]}\}$.*

Proof. See the Appendix A.1. ■

Multiplicity of steady states can arise in our model. For brevity, we just analyze the local dynamics around the NSS.[2]

4. Local dynamics analysis

Let us linearize the dynamic system (9) and (10) around the NSS (1,1). We shall define ε_{Ω} and ε_{R} as the elasticities of the functions $\Omega(a)$ and $R(a)$ evaluated at the NSS. In addition, let $\theta\equiv\frac{\Omega(a^{*})}{a^{*}}=\Omega(1)=A^{*}\omega\left(1\right)>g+1$, $R_{1}\equiv R_{1}(\frac{c_{1}^{*}}{B^{*}})$, $R_{2}\equiv R_{2}(c_{2}^{*})$, and $R_{3}\equiv R_{3}(1)$. Then, we have the following proposition.

Proposition 2. *The linearized dynamics generated by the two-dimensional system (9) and (10) around the NSS are determined by the determinant D and the trace T of the Jacobian matrix associated with Eqs. (9) and (10).*

$$dk_{t+1}=[\theta+\frac{\theta-1-g}{R_{1}}(R_{3}-\frac{g}{\theta-g})]dk_{t}+[\theta(\varepsilon_{\Omega}-1)-\frac{\theta-1-g}{R_{1}}(R_{3}+\frac{\theta\varepsilon_{\Omega}-g}{\theta-g})]da_{t}, \tag{12}$$

[2] Thanks to Yoichi Gokan for pointing this out to us.

$$
\begin{aligned}
|\varepsilon_R| da_{t+1} &= -\{\frac{R_3}{1-R_2} - \frac{g}{(\theta-g)(1-R_2)} + \frac{R_2}{1-R_2}[\theta + \frac{\theta-1-g}{R_1}(R_3 - \frac{g}{\theta-g})]\}dk_t + \\
&\quad \{\frac{R_3}{1-R_2} + \frac{\theta\varepsilon_\Omega - g}{(\theta-g)(1-R_2)} - \frac{R_2}{1-R_2}[\theta(\varepsilon_\Omega - 1) - \frac{\theta-1-g}{R_1}(R_3 + \frac{\theta\varepsilon_\Omega - g}{\theta-g})]\}da_t
\end{aligned} \tag{13}
$$

where $g = \tau^{NSS}\Omega(1) = \tau^{NSS}\theta$ *and* $\tau^{NSS} \in (0,1)$ *is the steady state labor income tax rate. Moreover, the expressions of* D *and* T *are:*

$$
\begin{aligned}
T &= \frac{1}{|\varepsilon_R|(1-R_2)}\left\{R_3 + \frac{\theta\varepsilon_\Omega - g}{\theta-g} - R_2\left[\theta(\varepsilon_\Omega - 1) - \frac{\theta-1-g}{R_1}\left(R_3 + \frac{\theta\varepsilon_\Omega - g}{\theta-g}\right)\right]\right\} + \\
&\quad \theta + \frac{\theta-1-g}{R_1}(R_3 - \frac{g}{\theta-g}),
\end{aligned} \tag{14}
$$

$$
D = \frac{\theta\varepsilon_\Omega(1+R_3)}{|\varepsilon_R|(1-R_2)}. \tag{15}
$$

The way to analyze the local stability of the normalized steady state is to study the variation of the trace T and the determinant D, i.e. the sum and the product, respectively, of the roots of the characteristic polynomial $Q(\pi) = \pi^2 - T\pi + D$, in the (T, D) plane when some parameters are made vary continuously. There is a local eigenvalue which is equal to $+1$ when $1 - T + D = 0$. It is represented by the line (AC) in Fig. 1. Moreover, one eigenvalue is equal to -1 when $1 + T + D = 0$. That is to say, (T, D) lies on the line (AB). Finally, the two eigenvalues are complex conjugate of modulus 1, when (T, D) belongs to the segment [BC] of equation $D = 1$, $|T| \leq 2$. Since both characteristic roots are equal to zero when both T and D are 0, then, by continuity, they have modulus less than one if and only if (T, D) lies in the interior of the triangle ABC, which is defined by $|T| < |1+D|$, $|D| < 1$. In this case, the steady state is locally indeterminate given that the unique predeterminate variable is k. If $|T| > |1 + D|$, the stationary state is a saddle-point. Finally, in the complementary region $|T| < |1 + D|, |D| > 1$, the steady state is a source.

The diagram below can also be used to study local bifurcations. When the point (T, D) crosses the interior of the segment [BC], a *Hopf bifurcation* will occur. If, instead, the point crosses the line

(AB), one eigenvalue goes through -1. In that case, a *flip bifurcation* will occur. Finally, when the point crosses the line (AC), one eigenvalue goes through $+1$, one expects an exchange of stability between $(1,1)$ and another steady state through a *transcritical bifurcation*.

As in Cazzavillan and Pintus (2004), we focus on two parameters, the elasticity of capital–labor substitution (σ) and the relative curvature of the second-period utility function R_2. To be more precise, we shall first fix the technology, i.e. θ, the elasticities ε_Ω and ε_R, as well as R_1 and R_3, and make R_2 vary continuously in the open interval $(0,1)$. This means that we will consider the parametrized curve $(T(R_2),\ D(R_2))$ when R_2 lies in the interval $(0,1)$. From the expressions of D and T given in the above proposition, one sees that $(T(R_2),\ D(R_2))$ describes a half-line Δ which starts from the point $(T_0(\sigma), D_0(\sigma))$ for $R_2 = 0$, where $T_0(\sigma)$ is the trace in (14) and $D_0(\sigma)$ is the determinant in (15) when $R_2 = 0$. In addition, the slope of Δ is

$$\frac{D'(R_2)}{T'(R_2)} = \frac{\theta\varepsilon_\Omega\,(1+R_3)}{\frac{\theta\varepsilon_\Omega}{\theta-g}\,(\theta-1-g)\left(\frac{1}{R_1}-1\right)+\theta+R_3(1+\frac{\theta-1-g}{R_1})-\frac{g}{\theta-g}(1+\frac{\theta-1-g}{R_1})} \tag{16}$$

and does not depend on R_2.

Using the same method as in Cazzavillan and Pintus (2004, p. 464), we express the elasticities ε_Ω and ε_R as functions of the depreciation rate δ, the share of capital in total income $0 < s(a) = a\rho(a)/f(a) < 1$, and the elasticity of capital–labor substitution $\sigma(a) \geq 0$. It is easy to find that

$$\varepsilon_\Omega = \frac{s(a)}{\sigma(a)} \text{ and } |\varepsilon_R| = \mu\,(a)\,\frac{1-s\,(a)}{\sigma\,(a)}, \tag{17}$$

where $\mu\,(a) \equiv \frac{s(a)\theta(a)}{s(a)\theta(a)+(1-s(a))(1-\delta)} \in (0,1]$ (see Cazzavillan and Pintus 2004, footnote 3 on p. 464). Moreover, the coordinates of the origin of the half-line $\Delta(\sigma)$ as functions of the elasticity parameter

σ are:

$$T_0(\sigma) = \theta + \frac{\theta - 1 - g}{R_1}\left(R_3 - \frac{g}{\theta - g}\right) + \frac{\sigma\left(R_3 - \frac{g}{\theta-g}\right) + \frac{s\theta}{\theta-g}}{\mu(1-s)},$$

$$D_0(\sigma) = \frac{s\theta(1+R_3)}{\mu(1-s)} \geq 0,$$

where $s = s(a^*)$, $\theta = \theta(a^*)$, $\mu = \mu(a^*) = \frac{s(a^*)\theta(a^*)}{s(a^*)\theta(a^*)+(1-s(a^*))(1-\delta)}$, and $\sigma = \sigma(a^*)$. We can easily see that the the slope of the half-line $\Delta(\sigma)$ is $\frac{s\theta(1+R_3)}{\sigma\left[\theta+\left(1+\frac{\theta-1-g}{R_1}\right)\left(R_3-\frac{g}{\theta-g}\right)\right]+\frac{s\theta(\theta-1-g)}{\theta-g}\frac{1-R_1}{R_1}}$.

Assumption 2. $R_3 > \frac{g}{\theta-g} - \frac{\theta}{1+\frac{\theta-1-g}{R_1}} = \frac{\tau^{NSS}}{1-\tau^{NSS}} - \frac{\theta}{1+\frac{\theta-1-\theta\tau^{NSS}}{R_1}}$. It corresponds to the case of small distortionary labor income tax rates, that is, τ^{NSS} not large. This condition can be met for a sufficiently high R_3 (if labor supply elasticity is finite), so that the slope of Δ is positive.[3]

To understand the main results, it is useful to relate the parameter θ and τ^{NSS} to the consumption–to–wage ratio. It is easy to show that $c_1/\Omega\lambda = \frac{\theta(1-\tau^{NSS})-1}{\theta}$. From this equation, one can recover the results by Cazzavillan and Pintus (2004) when $\tau^{NSS} = 0$.

If s and θ are kept fixed and σ is regarded as an independent parameter, we find that as σ increases from zero to $+\infty$, the point $(T_0(\sigma), D_0(\sigma)$ moves along a flat half-line Δ_1. More precisely, $T_0(\sigma)$ increases from a finite number to $+\infty$ along the flat line (Δ_1), but $D_0(\sigma)$ doesn't change. In addition, $\Delta(\sigma)$ pivots rightward and it has a positive slope when $\sigma = 0$, and horizontal when $\sigma = +\infty$, but the origin $(T_0(\sigma), D_0(\sigma)$ moves to the right along the line Δ_1.

In order to get local indeterminacy, first, we need that $D_0(\sigma) < 1$, which requires that s and θ are small enough, i.e., a sufficiently low share of capital in total income and a sufficiently low ratio of consumption while young to saving ($c_1^* = \theta(1-\tau^{NSS}) - 1$ in the NSS). As Cazzavillan and Pintus (2004) point out, the latter requirement is crucial to local indeterminacy. Adding endogenous labor income tax rate will be *helpful* to local indeterminacy since in our case, $c_1/\Omega\lambda = \frac{\theta(1-\tau^{NSS})-1}{\theta}$ is

[3] In Lloyd-Braga et al. (2007), they assume that capital externalities are almost zero, $s \leq 1/2$, $\gamma \geq 1$ and $\alpha \geq \alpha_1$ (see, assumption 5 in their paper) to ensure that the slope of Δ is positive.

smaller than that in Cazzavillan and Pintus (2004). Second, we should impose other requirements as in Cazzavillan and Pintus (2004, paragraphs 1 and 2 on p. 466).

Following Cazzavillan and Pintus (2004), we consider the case (I) where $D_0(\sigma) < 1$, $T_0(0) < 1 + D_0(\sigma)$, $slope_\Delta(\sigma) > slope_\Delta(\bar{\sigma})$ and the latter ($slope_\Delta(\bar{\sigma})$) is bigger than 1. Here $\bar{\sigma}$ is the value of σ such that the line Δ_1 intersects the line *(AC)*. It is easy to know that the half-line $\Delta(\sigma)$ intersects the interior of the segment BC for σ in $(0, \sigma_H)$, where σ_H is the value of σ such that $\Delta(\sigma)$ goes through C. Then we know that, for all σ in $(\bar{\sigma}, \sigma_H)$, the half-line Δ intersects not only the line *(AC)* at $R_2 = R_{2T}$, but also the segment BC at $R_2 = R_{2H}$. When σ moves beyond σ_H, Δ will not cross the interior of the segment BC, but it can cross the line AC up to $\sigma = \sigma_T$, where σ_T is the value of σ such that the $slope_\Delta(\sigma)$ is one. When $\sigma > \sigma_T$, the $slope_\Delta(\sigma)$ is less than one. We provide these parameters here.[4]

$$
\begin{aligned}
R_{2H} &= 1 - (1 + R_3)\,\theta\chi(\theta) > 0, \text{ where } \chi(\theta) = \frac{s\theta + (1-s)(1-\delta)}{(1-s)\,\theta}. \\
\sigma_H &= s\frac{(1 - R_{2H})\,R_1\chi_1 - \frac{\theta\chi(\theta)R_1}{\theta - g} - \theta\chi(\theta)\,R_{2H}\frac{\theta-1-g}{\theta-g} + \theta\chi(\theta)\,R_{2H}R_1}{\chi(\theta)\,R_1R_3 - \frac{g\chi(\theta)R_1}{\theta-g} + \theta\chi(\theta)\,R_1R_{2H} + \chi(\theta)\,R_{2H}\,(\theta-1-g)\left(R_3 - \frac{g}{\theta-g}\right)}, \\
\text{where } \chi_1 &\equiv 2 - \theta - \frac{\theta-1-g}{R_1}\left(R_3 - \frac{g}{\theta-g}\right). \\
\sigma_T &= \frac{s\theta\left[1 + R_3 - \left(1 - \frac{1}{\theta-g}\right)\left(\frac{1}{R_1} - 1\right)\right]}{\theta + \left(R_3 - \frac{g}{\theta-g}\right)\left(1 + \frac{\theta-1-g}{R_1}\right)}. \\
\bar{\sigma} &= \frac{s\,(\theta-1-g)\left(\frac{\theta\chi(\theta)}{\theta-g} - \frac{F_1}{R_1}\right) + s\theta\chi(\theta)\,R_3 + s\,(1-\theta)}{F_1\chi(\theta)}, \text{ where } F_1 = R_3 - \frac{g}{\theta-g}. \\
R_{2T} &= \frac{\chi_2 + \frac{\sigma\chi(\theta)F_1}{s} - \theta\chi(\theta)\,R_3 - \theta\chi(\theta)\,\frac{\theta-1-g}{\theta-g}}{\chi_2 + \frac{\chi(\theta)}{s}\left[\theta\,(s-\sigma) - \frac{\theta-1-g}{R_1}\left(\frac{s\theta}{\theta-g} + \sigma F_1\right)\right]}, \text{ where } \chi_2 = \theta - 1 + F_1\frac{\theta-1-g}{R_1}.
\end{aligned}
$$

[4]For how to derive these parameters, see the appendix A.2. in Cazzavillian and Pintus (2004). It means that σ_H is the solution of $T(R_{2H}) = 2$; σ_T is the solution of slope$_\Delta(\sigma)$= 1; R_{2H} is the solution of $D(R_2) = 1$; R_{2T} solves $T(R_2) = 1 + D(R_2)$.

In fact, four possible dynamics in case (I) are the same as in Cazzavillian and Pintus (Fig. 1–2004, pp. 463, 466) except that the critical values of the independent parameter σ and the bifurcation parameter R_2 are different from those in their model. We summarize these results in the following theorem.

Theorem 1. *Let $(a^*, k^*) = (1, 1)$ be a normalized steady state which is set according to the procedure outlined in proposition 1. Then, under assumptions 1, 2, and those stated in the appendix A.2, the following holds.*

(i) $0 < \sigma < \overline{\sigma}$: the steady state $(1, 1)$ is a sink for $R_2 < R_{2H}$, undergoes a Hopf bifurcation at $R_2 = R_{2H}$, and becomes a source for $R_2 > R_{2H}$;

(ii) $\overline{\sigma} < \sigma < \sigma_H$: the steady state $(1, 1)$ is a saddle for $R_2 < R_{2T}$, undergoes a transcritical bifurcation at $R_2 = R_{2T}$, becomes a sink for $R_{2T} < R_2 < R_{2H}$, undergoes a Hopf bifurcation at $R_2 = R_{2H}$, and becomes a source for $R_2 > R_{2H}$;

(iii) $\sigma_H < \sigma < \sigma_T$: the steady state $(1, 1)$ is a saddle for $R_2 < R_{2T}$, undergoes a transcritical bifurcation at $R_2 = R_{2T}$, and becomes a source for $R_2 > R_{2T}$;

(iv) $\sigma > \sigma_T$: the steady state $(1, 1)$ is a saddle for all R_2 in the open interval $(0, 1)$.

Proof. See Appendix A.2. ■

We then turn to analyze the case (II) where the origin $(T_0(0), D_0(0))$ lies outside the triangle ABC and the slope of the half-line $\Delta(\sigma)$ is steeper than that of the line connecting the origin with the point C. This means that $T_0(0) > 1 + D_0(0)$, $D_0(0) < 1$, $1 < T_0(0) < 2$ and $slope_\Delta(0) > \frac{1-D_0(0)}{2-T_0(0)}$. Similar to Cazzavillan and Pintus (2004), we have the same theorem 2 except that the critical values of the independent parameter σ and the bifurcation parameter R_2 are different from those in their model.

Theorem 2. *Let $(a^*, k^*) = (1, 1)$ be a steady state which is set according to the procedure outlined*

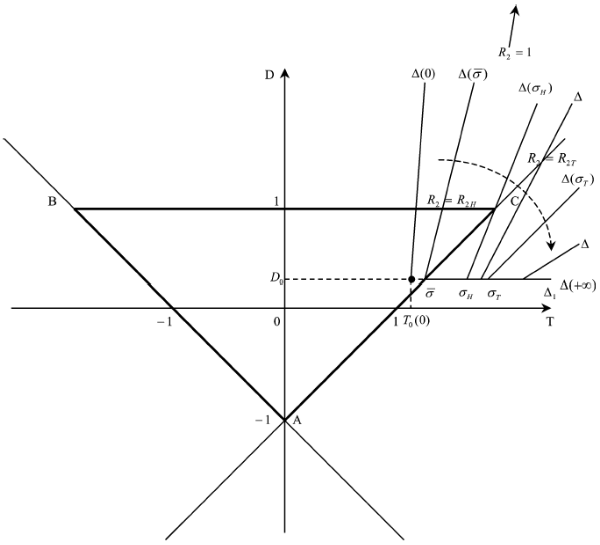

Fig. 1.

in proposition 1. Then, under Assumptions 1, 2, and those stated in the appendix A.3, the following results hold.

(i) $0 < \sigma < \sigma_H$*: the steady state* $(1,1)$ *is a saddle for* $R_2 < R_{2T}$*, undergoes a transcritical bifurcation at* $R_2 = R_{2T}$*, becomes a sink for* $R_{2T} < R_2 < R_{2H}$*, undergoes a Hopf bifurcation at* $R_2 = R_{2H}$*, and becomes a source for* $R_2 > R_{2H}$*;*

(ii) $\sigma_H < \sigma < \sigma_T$*: the steady state* $(1,1)$ *is a saddle for* $R_2 < R_{2T}$*, undergoes a transcritical bifurcation at* $R_2 = R_{2T}$*, and becomes a source for* $R_2 > R_{2T}$*;*

(iii) $\sigma > \sigma_T$*: the steady state* $(1,1)$ *is a saddle for all* R_2 *in the open interval* $(0,1)$*.*

Perhaps the reader is interested in studying the impact of small labor income tax rates on the conditions leading to local indeterminacy, as shown in Figure 1 (or Theorem 1). The lemma 1 in the appendix shows that if $\frac{1}{1-\tau^{NSS}} < \theta < \theta_1$ holds, indeterminacy can arise. Here θ_1 is a critical value above which local indeterminacy can not arise. The next proposition will show that θ_1 can be increasing in the level of labor income tax rates (τ^{NSS}) provided that τ^{NSS} is not too large. Then, increasing the size of distortionary taxes from zero can enlarge the range of the values of θ that are

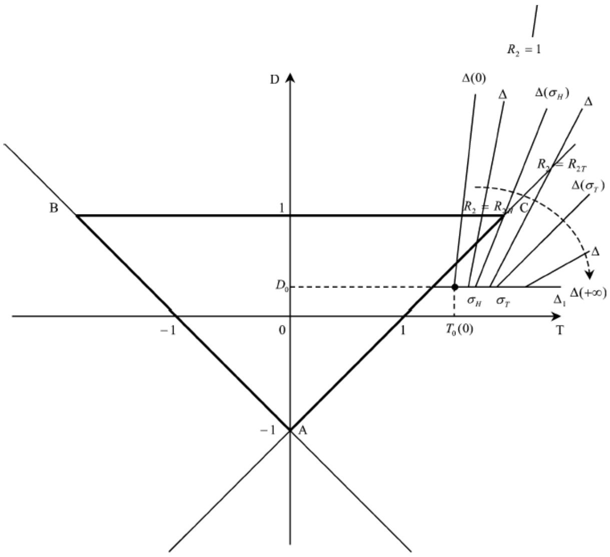

Fig. 2.

compatible with local indeterminacy.

Proposition 3. *Under the assumptions of Theorem 1, the critical lower bound, θ_1 above which local indeterminacy can not arise, is increasing in the level of labor income tax rates provided that the distortionary tax rate (τ^{NSS}) is not too large. Moreover, $R_3 > \frac{\tau^{NSS}}{1-\tau^{NSS}} - \frac{\theta}{1+\frac{\theta-1-\theta\tau^{NSS}}{R_1}}$ will be met if the utility function in the first period of life is close enough to logarithmic ($R_1 = 1$) and τ^{NSS} is not too large.*

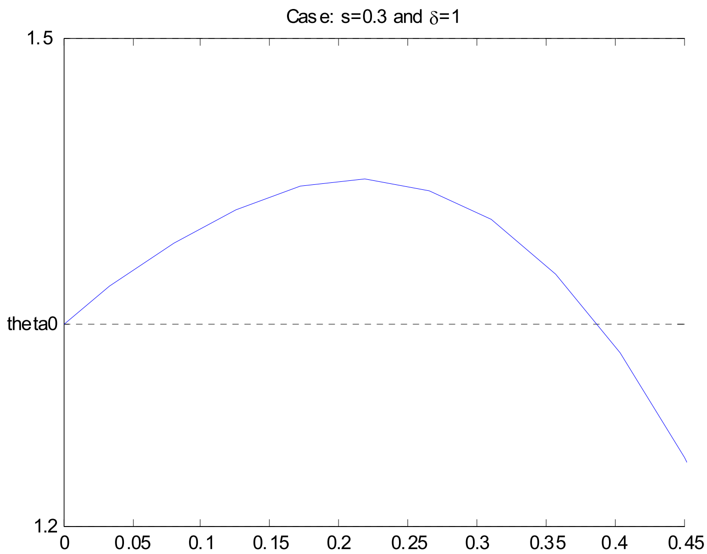

Figure 3. The Case: $s = 0.3$ and $\delta = 1$. θ_1 (the vertical axis) is not a monotone function of τ^{NSS}.

The following numerical example shows how the share of wage devoted to savings has to be large for local indeterminacy to arise (θ should be small) and how endogenous labor income tax rates are helpful to local indeterminacy: this is consistent with recent works, for example, Schmitt-Grohe and Uribe (1997) and Gokan (2006).[5] In particular, Schmitt-Grohe and Uribe (1997) have shown that, in a standard neoclassical growth model, the indeterminacy condition obtained in their fiscal increasing returns model has a close correspondence with the one obtained in the productive increasing returns model of Benhabib and Farmer (1994). However, we show that, in an overlapping generations model with endogenous labor and consumption in both periods of life, the indeterminacy condition obtained in our model may also have a close correspondence with the one obtained in the productive increasing returns model of Lloyd-Braga, Nourry and Venditti (2007): both labor externalities and endogenous labor income tax rates are helpful to local indeterminacy.[6]

We illustrate, using numerical examples, our main results that increasing steady state labor income tax rates may enlarge the range of parameter values (σ_H) associated with multiple equilibria. To fix ideas and ease comparisons with Cazzavillan and Pintus (2004), we set $s = 0.3$ and $\delta = 1$, where full capital depreciation is perfectly consistent with the time period implied by the OLG setting, whereas the chosen value of the capital share in total output is close to the one that Schmitt-Grohe and Uribe (1997) use. We further assume that τ^{NSS} can take the values of 0.1, 0.12, 0.14, 0.16 and 0.18. These values can imply the lower and upper bounds on θ (i.e., $\frac{1}{1-\tau^{NSS}}$ and θ_1). The values of R_1 and R_3 must belong to the relevant intervals defined in lemma 1. And we assume that $R_1 = 0.99$ and $R_3 = 0.62$.[7] Similar to Cazzavillan and Pintus (2004), we can show that total consumption, including consumption by the old agents, has to be less than 45% of output in the case

[5] For a given level of θ, indeterminacy is more likely, the larger the steady state labor income tax rates.

[6] The reader can compare our indeterminacy result with those in Lloyd-Braga et al. (2007).

[7] For how to select these proper values of R_1 and R_3, see the matlab programs which are available upon request.

of Fig. 1.

Considering the elasticity of capital–labor substitution, we find that the condition $\sigma < \sigma_H$, which is necessary to get endogenous fluctuations, places a upper bound on σ. It is easy to find that $\sigma_H < \sigma_T$. Numerical examples show that $\sigma_T > s$ and, therefore, that $\sigma_H < s < \sigma_T$. This suggests that σ_H may be below the capital share, and that σ_T may be above the capital share. In fact, we illustrate that, irrespective of the values for R_1 and R_3, σ_H decreases when θ increases, for a given τ^{NSS} and; σ_H increases when τ^{NSS} increases, for a given θ.[8] The former conclusion has already been found by Cazzavillan and Pintus (2004). While the latter shows that endogenous labor income tax rates are helpful to local indeterminacy.

σ_H	$\tau^{NSS} = 0.1$	$\tau^{NSS} = 0.12$	$\tau^{NSS} = 0.14$	$\tau^{NSS} = 0.16$	$\tau^{NSS} = 0.18$
$\theta = 1.22$	0.215	0.227	0.238	0.249	0.260
$\theta = 1.25$	0.188	0.200	0.212	0.223	0.233
$\theta = 1.30$	0.135	0.148	0.159	0.171	0.181
$\theta = 1.35$	0.071	0.083	0.094	0.105	0.115

Table 1.

We are now in a position to intuitively explain why endogenous labor income tax rates are helpful to local indeterminacy. Cazzavillan and Pintus (2004) have already shown that when intertemporal substitution in consumption across periods is introduced, endogenous fluctuations require very low values of the propensity to consume out of wage income of the young generation (in our model, $(1-\tau^{NSS}) - \frac{1}{\theta}$). In addition, endogenous fluctuations require elasticities of capital–labor substitution that are well below the share of capital in total income. We find that (1) for a given θ, adding labor

[8] Again, R_1 and R_3 must belong to the relevant intervals defined in lemma 1.

income tax rates (τ^{NSS}) will make the ratio of consumption while young to saving smaller, thus making sunspots more likely to occur and; (2) for a given θ, adding tax rates will make the upper bound on σ associated with multiple equilibria (σ_H) larger although this bound is still less than the share of capital in total income. To be more precise, endogenous fluctuations arise due to the interaction of two conflicting effects: when the capital stock increases, it leads to an increase in wage rate and, therefore, an increase in savings which leads the capital stock in the next period to be higher. However, capital accumulation is followed by a decrease in the real interest rate that will depress savings and/or capital accumulation. In other words, the initial wage increase will be offset by a decrease of the real interest rate. In our model, there is one force which tends to strengthen the conflicting effects of wage and interest rate movements: increasing labor income tax rates makes smaller the share of consumption out of wage income in the first period of life, thus making the rise in savings larger. Different from Cazzavillan and Pintus (2006), increasing tax rates can *not* change the sensitivity of the interest rate with respect to variations in the capital stock (the elasticity of R with respect to k is $\varepsilon_{R,k} = \mu(a)\frac{s(a)-1}{\sigma(a)} < 0$ and does *not* depend on τ^{NSS}). Considering these two reasons, it is expected that the larger labor income tax rates, the higher the impact of the wage variation on savings (that is, the lower the consumption–to–wage ratio) that is required for cyclical equilibria to occur.

5. Concluding Remarks

We study a version of Diamond's OLG model modified to allow for consumption in both periods and endogenous labor income tax rates. We have shown that local indeterminacy of the steady state prevails, when income tax rates are not too large, as long as the fraction of young-age consumption out of wage income is small enough. More importantly, we found that increasing the size of tax distortions increases the range of values of the consumption-to-wage ratio associated with multiple equilibria. We related this result to the fact that adding labor income tax rates (τ^{NSS}) will make the ratio of consumption while young to saving smaller, thus making sunspots more likely to occur. This explains why endogenous income taxes are helpful to local indeterminacy in the OLG model with first-period consumption and it is also useful to understand why labor income tax rates can lead to local indeterminacy in the infinite-horizon model (Schmitt-Grohe and Uribe, 1997).

References

[1] Benhabib, J., Farmer, R.E.A., 1994. Indeterminacy and increasing returns. Journal of Economic Theory 63, 19–41.

[2] Cazzavillan, G., 2001. Indeterminacy and endogenous fluctuations with arbitrarily small externalities. Journal of Economic Theory 101, 133–157.

[3] Cazzavillan, G., Pintus, P., 2004. Robustness of multiple equilibria in OLG economies. Review of Economic Dynamics 7, 456–475.

[4] Cazzavillan, G., Pintus, P., 2006. Capital externalities in OLG economies. Journal of Economic Dynamics & Control 30, 1215–1231.

[5] Diamond, P., 1965. National debt in a neoclassical growth model. American Economic Review 55, 1126–1150.

[6] Gokan, Y., 2006. Dynamic effects of government expenditure in a finance constrained economy. Journal of Economic Theory 127, 323–333.

[7] Gokan, Y., 2009a. Dynamic effects of government budgetary policies in Reichlin's overlapping generations model with externalities. Mimeo, Ritsumeikan University.

[8] Gokan, Y., 2009b. Macroeconomic Instability, government budgetary policy and overlapping generations economy. Mimeo, Ritsumeikan University.

[9] Lloyd-Braga, T., Nourry, C., Venditti, A., 2007. Indeterminacy in dynamic models: When Diamond meets Ramsey. Journal of Economic Theory 134, 513 – 536.

[10] Reichlin, P., 1986. Equilibrium cycles in an overlapping generations economy with production. Journal of Economic Theory 40, 89–102.

[11] Schmitt-Grohe, S., Uribe, M., 1997. Balanced-budget rule, distortionary taxes and aggregate instability. Journal of Political Economy 105, 976-1000.

6. Appendix:

A.1. Proof of Proposition 1

If $(k^*, a^*) = (1, 1)$ is a normalized steady state of the dynamic system (9) and (10), we have the following: (c_1^* is the steady state of the first period consumption.)

$$A\omega\,(1) - g - 1 = B(U_1')^{-1}(\frac{BU_3'(1)}{A\omega\,(1) - g}) = c_1^* > 0, \tag{D-1}$$

$$A\rho\,(1) + 1 - \delta = u_2^{-1}\{\frac{U_3'\,(1)}{\beta[A\omega\,(1) - g]}\}. \tag{D-2}$$

If g is not too large, $A > \frac{g+1}{\omega(1)}$ can make c_1^* larger than zero. It is easy to find that $\beta[A\omega\,(1) - g]u_2[A\rho\,(1)+1-\delta] = U_3'\,(1)$ and the LHS term is an increasing function of A. In order to have a unique A^* satisfying (D-2), we require that $\beta[A\omega\,(1)-g]u_2[A\rho\,(1)+1-\delta]|_{A=\frac{g+1}{\omega(1)}} < U_3'\,(1)$. It is equivalent to $\beta u_2[\frac{\rho(1)}{\omega(1)}(1+g)+1-\delta] < U_3'(1)$. We can easily get B^* from (D-1) after we pin down the unique A^* from (D-2). In particular, we can rewrite (D-1) as follows: $\frac{A\omega(1)-g-1}{B}U_1'(\frac{A\omega(1)-g-1}{B}) = \frac{A\omega(1)-g-1}{A\omega(1)-g}U_1'(1)$. It is easy to see that $\frac{A\omega(1)-g-1}{B}U_1'(\frac{A\omega(1)-g-1}{B})$ is a decreasing function of B. In order to have the unique B^*, we should impose the restriction: $\lim_{c\to 0}cU_1'(c) < \frac{A^*\omega(1)-1-g}{A^*\omega(1)-g}U_3'(1)$.

A.2. Proof of Theorem 1

Lemma 1. *Let* $\frac{1}{1-\tau^{NSS}} < \theta < \theta_1 = \frac{\Upsilon+\sqrt{\Upsilon^2-4\phi\frac{(1-\delta)^2}{1-\tau^{NSS}}}}{2\phi}$*, where* $\Upsilon \equiv \frac{(2-\delta)(1-s)(1-\tau^{NSS})-2s(1-\delta)-(1-s)(1-\delta)\tau^{NSS}}{(1-s)(1-\tau^{NSS})}$ *and* $\phi \equiv \frac{(1-s)^2(1-\tau^{NSS})^2-s(1-s)(1-\tau^{NSS})+s^2+s\tau^{NSS}(1-s)}{(1-s)^2(1-\tau^{NSS})}$*. Moreover, we assume that* $R_1 > \overline{R}_1$ *and* $\overline{R}_3 < R_3 < \overline{\overline{R}}_3$*, where* $\overline{R}_3 = \frac{\theta-1-\theta\chi(\theta)+\frac{\chi(\theta)}{1-\tau^{NSS}}-\left[\theta(1-\tau^{NSS})-1\right]\frac{\tau^{NSS}}{1-\tau^{NSS}}}{1+\theta\chi(\theta)-\theta(1-\tau^{NSS})}$*,* $\overline{\overline{R}}_3 = \frac{1-\theta\chi(\theta)}{\theta\chi(\theta)}$ *and* $\overline{R}_1 = \frac{\left[\theta(1-\tau^{NSS})-1\right]\left(R_3-\frac{\tau^{NSS}}{1-\tau^{NSS}}\right)}{1+\theta\chi(\theta)(1+R_3)-\theta-\frac{\chi(\theta)}{1-\tau^{NSS}}}$*, with* $\frac{s}{\mu(1-s)} = \chi(\theta)$*. Then we have the following results: the origin* $(T_0(0), D_0(0))$ *lies inside the ABC triangle and the half line* $\Delta(\sigma)$ *intersects the interior of the segment BC at* $\sigma = 0$ *(*$T_0(0) < 1 + D_0(0)$*,* $D_0(0) < 1$*). Moreover, we have* $slope_\Delta(0) > slope_\Delta(\overline{\sigma}) > slope_\Delta(\sigma_H) > slope_\Delta(\sigma_T) = 1$*.*

Proof. *Similar to Cazzavillan and Pintus (2004),* $D_0(0) < 1$ *is satisfied iff* $0 < R_3 < \frac{1-\theta\chi(\theta)}{\theta\chi(\theta)} \equiv \overline{\overline{R}}_3$*, where* $\chi(\theta) = \frac{s\theta+(1-s)(1-\delta)}{(1-s)\theta}$ *and* $\mu = \frac{s}{(1-s)\chi(\theta)}$*. This requires that* $\theta < \overline{\theta} \equiv \frac{\delta(1-s)}{s}$ *and* $s < \frac{\delta}{(1-\tau^{NSS})^{-1}+\delta} \le \frac{1}{2}$ *as* $\theta > 1+g$*.*

$T_0(0) < 1 + D_0(0)$ *is satisfied iff* $R_1 > \overline{R}_1 \equiv \frac{\left[\theta(1-\tau^{NSS})-1\right]\left(R_3-\frac{\tau^{NSS}}{1-\tau^{NSS}}\right)}{1+\theta\chi(\theta)(1+R_3)-\theta-\frac{\chi(\theta)}{1-\tau^{NSS}}}$ *with* $R_3 > \widetilde{R}_3 = \frac{\theta-1-\chi(\theta)\left(\theta-\frac{1}{1-\tau^{NSS}}\right)}{\theta\chi(\theta)}$*. Since* $R_1 > 1$*, we need that* $\overline{R}_1 < 1$*, which is equivalent to*

$$R_3 > \overline{R}_3 = \frac{\theta-1-\theta\chi(\theta)+\frac{\chi(\theta)}{1-\tau^{NSS}}-\left[\theta(1-\tau^{NSS})-1\right]\frac{\tau^{NSS}}{1-\tau^{NSS}}}{1+\theta\chi(\theta)-\theta(1-\tau^{NSS})},$$

where $1+\theta\chi(\theta)-\theta(1-\tau^{NSS}) > 0$*.* $1+\theta\chi(\theta)-\theta(1-\tau^{NSS}) > 0$ *holds iff* $\theta < \overline{\overline{\theta}} \equiv \frac{(2-\delta)(1-s)}{(1-\tau^{NSS})(1-s)-s}$*. It is easy to verify that if* $\delta > \frac{2s}{(1-\tau^{NSS})(1-s)}$*, the binding upper bound on* θ *is* $\overline{\overline{\theta}}$*, as* $\overline{\overline{\theta}} < \overline{\theta}$*. Otherwise, if* $\delta < \frac{2s}{(1-\tau^{NSS})(1-s)}$*, the binding upper bound on* θ *is* $\overline{\theta}$*, as* $\overline{\overline{\theta}} > \overline{\theta}$*. In addition,* $\overline{R}_3 > \widetilde{R}_3$*. Then we have that* $D_0(0) < 1$ *and* $T_0(0) < 1 + D_0(0)$ *iff* $R_1 > \overline{R}_1$ *and* $\overline{R}_3 < R_3 < \overline{\overline{R}}_3$*, provided that either* $\theta < \overline{\theta}$*, when* $\delta < \frac{2s}{(1-\tau^{NSS})(1-s)}$*, or* $\theta < \overline{\overline{\theta}}$*, when* $\delta > \frac{2s}{(1-\tau^{NSS})(1-s)}$*. The inequality* $\overline{R}_3 < R_3 < \overline{\overline{R}}_3$ *holds iff* *the polynomial holds.*

$$P_1(\theta) = \phi\theta^2 - \Upsilon\theta + \frac{(1-\delta)^2}{1-\tau^{NSS}} < 0,$$

with $\phi = \frac{(1-s)^2(1-\tau^{NSS})^2 - s(1-s)(1-\tau^{NSS}) + s^2 + s\tau^{NSS}(1-s)}{(1-s)^2(1-\tau^{NSS})}$ *and* $\Upsilon = \frac{(2-\delta)(1-s)(1-\tau^{NSS}) - 2s(1-\delta) - (1-s)(1-\delta)\tau^{NSS}}{(1-s)(1-\tau^{NSS})}$. *In addition,* $P_1(\theta)$ *has a root in* $(\frac{1}{1-\tau^{NSS}}, \overline{\theta})$, *which is* $\theta_1 = \frac{\Upsilon + \sqrt{\Upsilon^2 - 4\phi\frac{(1-\delta)^2}{1-\tau^{NSS}}}}{2\phi}$. *And* $P_1(\theta) < 0$ *holds for all* $\theta \in (\frac{1}{1-\tau^{NSS}}, \theta_1)$. *When* $\delta > \frac{2s}{(1-\tau^{NSS})(1-s)}$, $\frac{1}{1-\tau^{NSS}} < \theta_1 < \overline{\overline{\theta}} < \overline{\theta}$ *can hold for properly chosen parameters. A numerical example is* $\tau^{NSS} = 0.1$, $\delta = 1$ *and* $s = 0.3$.

Following Cazzavillan and Pintus (2004), it is easy to show that $slope_{\Delta}(0) > slope_{\Delta}(\overline{\sigma}) > slope_{\Delta}(\sigma_H) > slope_{\Delta}(\sigma_T) = 1$. ■

A.3. Proof of Theorem 2.

It needs the following lemma.

Lemma 2. *Let* $\frac{1}{1-\tau^{NSS}} < \theta < \theta_2 = \frac{y + \sqrt{y^2 - 4\frac{(1-\delta)}{(1-\tau^{NSS})}}}{2}$, *with* $y = 2 - \frac{s}{(1-s)(1-\tau^{NSS})}$ *and assume that either* $R_3 < \overline{\overline{R}}_3 = \frac{1-\theta\chi(\theta)}{\theta\chi(\theta)}$, *if* $\frac{1}{1-\tau^{NSS}} < \theta < \theta_1$, *or* $R_3 < \overline{\overline{\overline{R}}}_3 = \frac{2-\theta-\frac{\chi(\theta)}{1-\tau^{NSS}}}{[\theta(1-\tau^{NSS})-1]} + \frac{\tau^{NSS}}{1-\tau^{NSS}}$, *if* $\theta_1 < \theta < \theta_2$. *Moreover, we assume that* $R_1 > \overline{\overline{\overline{\overline{R}}}}_1 = \frac{[\theta(1-\tau^{NSS})-1]\left\{(1+R_3)\theta\left[R_3 - \frac{\tau^{NSS}}{1-\tau^{NSS}} - \frac{\chi(\theta)}{1-\tau^{NSS}}\right] + \frac{1}{1-\tau^{NSS}}\right\}}{(1+R_3)\theta[2-\theta(1+\chi(\theta))] + \frac{\theta(1-\tau^{NSS})-1}{1-\tau^{NSS}}}$. *Then we have* $T_0(0) > 1 + D_0(0)$, $D_0(0) < 1$, $1 < T_0(0) < 2$ *and* $slope_{\Delta}(0) > \frac{1-D_0(0)}{2-T_0(0)}$. *In other words, the origin* $(T_0(0), D_0(0))$ *lies outside the triangle ABC and the slope of the half-line* $\Delta(\sigma)$ *is steeper than that of the line connecting the origin with the point C.*

Proof. We require that $R_3 < \overline{\overline{R}}_3 = \frac{1-\theta\chi(\theta)}{\theta\chi(\theta)}$ in order to get $D_0(0) < 1$. If $(T_0(0), D_0(0))$ lies outside the triangle ABC, it implies that $T_0(0) > 1 + D_0(0)$. This inequality holds iff

$$R_1 < \overline{R}_1 = \frac{\left[\theta(1-\tau^{NSS}) - 1\right]\left(R_3 - \frac{\tau^{NSS}}{1-\tau^{NSS}}\right)}{1 + \theta\chi(\theta)(1+R_3) - \theta - \frac{\chi(\theta)}{1-\tau^{NSS}}},$$

with $R_3 > \widetilde{R}_3 = \frac{\theta - 1 - \chi(\theta)\left(\theta - \frac{1}{1-\tau^{NSS}}\right)}{\theta\chi(\theta)}$. $\overline{R}_1 > 0$ implies that $R_3 > \frac{\tau^{NSS}}{1-\tau^{NSS}}$. $T_0(0) < 2$ iff $R_1 > \overline{\overline{\overline{R}}}_1 \equiv \frac{\left[\theta(1-\tau^{NSS})-1\right]\left(R_3 - \frac{\tau^{NSS}}{1-\tau^{NSS}}\right)}{2-\theta-\frac{\chi(\theta)}{1-\tau^{NSS}}}$, with $2 - \theta - \frac{\chi(\theta)}{1-\tau^{NSS}} > 0$, i.e., $P_2(\theta) = \theta^2 - y\theta + (1-\delta)/\left(1-\tau^{NSS}\right) <$

0, where $y = 2 - \frac{s}{(1-s)(1-\tau^{NSS})}$. It is easy to know that the polynomial $P_2(\theta)$ has one root in $\left((1-\tau^{NSS})^{-1}, +\infty\right)$, which is $\theta_2 = \frac{y+\left[y^2-4(1-\delta)/\left(1-\tau^{NSS}\right)\right]^{1/2}}{2}$ and less than $\overline{\theta}$. This means that if $\left(1-\tau^{NSS}\right)^{-1} < \theta_2 < \overline{\theta}$, we have $T_0(0) < 2$.

Since $R_1 \in (0,1)$, $\overline{\overline{\overline{\overline{R}}}}_1 < 1$ implies that $R_3 < \overline{\overline{\overline{\overline{R}}}}_3 = \frac{2-\theta-\frac{\chi(\theta)}{1-\tau^{NSS}}}{\theta(1-\tau^{NSS})-1} + \frac{\tau^{NSS}}{1-\tau^{NSS}}$. $\overline{\overline{\overline{\overline{R}}}}_3$ is another upper bound on R_3. Notice that $\overline{\overline{\overline{\overline{R}}}}_3(\theta)$ and $\overline{\overline{R}}_3(\theta)$ are decreasing in θ. To be accurate, $\overline{\overline{R}}_3(\theta)$ goes down from $\left[1-\chi\left((1-\tau^{NSS})^{-1}\right)/(1-\tau^{NSS})\right] / \left[\chi\left((1-\tau^{NSS})^{-1}\right)/(1-\tau^{NSS})\right]$ to $\left[1-\theta_2\chi(\theta_2)\right]/\theta_2\chi(\theta_2)$, whereas $\overline{\overline{\overline{\overline{R}}}}_3(\theta)$ goes down from $+\infty$ to $\frac{\tau^{NSS}}{1-\tau^{NSS}}$. In order to have a unique θ in the interval $\left((1-\tau^{NSS})^{-1}, \theta_2\right)$ such that $\overline{\overline{\overline{\overline{R}}}}_3 = \overline{\overline{R}}_3$, we require that $\overline{\overline{R}}_3(\theta_2) > \frac{\tau^{NSS}}{1-\tau^{NSS}}$, or, $\delta > \frac{s\theta_2}{1-s} + \tau^{NSS}$, which can hold for small τ^{NSS}. The unique θ satisfying $\overline{\overline{\overline{\overline{R}}}}_3 = \overline{\overline{R}}_3$, is the same θ_1 obtained in lemma 1. As a result, one has $\overline{\overline{\overline{\overline{R}}}}_3 > \overline{\overline{R}}_3$, for θ in $\left((1-\tau^{NSS})^{-1}, \theta_1\right)$, and $\overline{\overline{\overline{\overline{R}}}}_3 < \overline{\overline{R}}_3$, for θ in (θ_1, θ_2). Put it differently, when θ is fixed in $\left((1-\tau^{NSS})^{-1}, \theta_2\right)$, there is only one upper bound on R_3, which is either $\overline{\overline{R}}_3$, when $\theta \in \left((1-\tau^{NSS})^{-1}, \theta_1\right)$, or $\overline{\overline{\overline{\overline{R}}}}_3$, when $\theta \in (\theta_1, \theta_2)$.

Another condition is that $slope_{\Delta}(0) > (1-D_0(0))/(2-T_0(0))$. It implies that $R_1 > \overline{\overline{\overline{\overline{R}}}}_1 = \frac{[\theta(1-\tau^{NSS})-1]\left\{(1+R_3)\theta\left[R_3-\frac{\tau^{NSS}}{1-\tau^{NSS}}-\frac{\chi(\theta)}{1-\tau^{NSS}}\right]+\frac{1}{1-\tau^{NSS}}\right\}}{(1+R_3)\theta[2-\theta(1+\chi(\theta))]+\frac{\theta(1-\tau^{NSS})-1}{1-\tau^{NSS}}}$. It is easy to see that, as long as $R_3 < \overline{\overline{\overline{R}}}_3 = \frac{2-\theta-\frac{\chi(\theta)}{1-\tau^{NSS}}}{[\theta(1-\tau^{NSS})-1]} + \frac{\tau^{NSS}}{1-\tau^{NSS}}$, $\overline{\overline{\overline{\overline{R}}}}_1 < 1$. Furthermore, $\overline{\overline{\overline{\overline{R}}}}_1 > \overline{\overline{\overline{R}}}_1$ holds for $\theta \in \left((1-\tau^{NSS})^{-1}, \theta_2\right)$. The reason is that sign$(\overline{\overline{\overline{\overline{R}}}}_1 - \overline{\overline{\overline{R}}}_1)$ =sign$\left(\left(2-\theta-\frac{\chi(\theta)}{1-\tau^{NSS}}\right) - \left(R_3 - \frac{\tau^{NSS}}{1-\tau^{NSS}}\right)\left[\theta\left(1-\tau^{NSS}\right)-1\right]\right)$. If the conditions $D_0(0) < 1$ and $R_3 < \overline{\overline{\overline{R}}}_3 \equiv \frac{2-\theta-\frac{\chi(\theta)}{1-\tau^{NSS}}}{\theta(1-\tau^{NSS})-1} + \frac{\tau^{NSS}}{1-\tau^{NSS}}$ hold, $\overline{\overline{\overline{\overline{R}}}}_1 > \overline{\overline{\overline{R}}}_1$. It follows that $R_1 > \overline{\overline{\overline{\overline{R}}}}_1$ implies that $T_0(0) < 2$ and $slope_{\Delta}(0) > \frac{1-D_0(0)}{2-T_0(0)}$ hold as long as $D_0(0) < 1$ and $R_3 < \overline{\overline{\overline{R}}}_3$ hold. We are done. ■