Thermodynamic analogies in economics and finance: instability of markets

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Instability of Markets

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Abstract

Interest in thermodynamic analogies in economics is older than the idea of von Neumann to look for market entropy in liquidity, advice that was not taken in any thermodynamic analogy presented so far in the literature. In this paper we go further and use a standard strategy from trading theory to pinpoint why thermodynamic analogies necessarily fail to describe financial markets, in spite of the presence of liquidity as the underlying basis for market entropy. Market liquidity of frequently traded assets does play the role of the 'heat bath', as anticipated by von Neumann, but we are able to identify the no-arbitrage condition geometrically as an assumption of translational and rotational invariance rather than (as finance theorists would claim) an equilibrium condition. We then use the empirical market distribution to introduce an asset's entropy and discuss the underlying reason why real financial markets cannot behave thermodynamically: financial markets are unstable, they do not approach statistical equilibrium, nor are there any available topological invariants on which to base a purely formal statistical mechanics. After discussing financial markets, we finally generalize our result by proposing that the idea of Adam Smith's Invisible Hand is a falsifiable proposition: we suggest how to test nonfinancial markets empirically for the stabilizing action of The Invisible Hand.

1. Entropy, Disorder, and Formal Thermodynamic Analogies

Attempts at neo-classical equilibrium economic analogies with thermodynamics go back to Guilluame [1] and Samuelson [2]. Von Neumann apparently believed that thermodynamic formalism could potentially be useful in computer theory, for formulating a description of intelligience, and was interested in the possibility of a thermodynamics of economics. But presented with Guillaume's work, he criticized it on the basis of the misidentification of a quantity as entropy [3].

Much more recently, Smith and Foley [4] have presented a much more careful mapping of neo-classical equilibrium theory onto an apparently formal thermodynamics. The neo-classical equilibrium analog of the zeroth law requires identifying price as an intensive variable (not necessarily as temperature), and quantities of assets held are treated as extensive variables. Starting with utility maximization as the fundamental principle, analogs of thermodynamic potentials were defined purely formally by constructing Legendre transforms. A quantity that they identified as 'entropy' was constructed as a Legendre transform on utility, with utility maximization being interpreted as analogous to entropy maximization for a closed mechanical system in thermodynamic equilibrium (entropy maximization is not the second law, but is a deduction from the second law).

There are problems with the identification of entropy as utility, and we will discuss that in some detail. The real difficulty is that utility maximization in particular, and neo-classical equilibrium theory in general do not describe any real market, even as a zeroth order approximation. In real markets the physically observable variables, price p and quantity z of an asset held, treated by Smith and Foley fluctuate randomly, and utility does not appear at all, whereas the entropy of an asset presents itself to us directly: the entropy can be constructed from an observable quantity, namely, the histograms that define the market distribution.

We therefore follow a different but related path in this paper: we ignore the neo-classical model and utility altogether, and start instead with the same *observable* intensive and extensive quantities that appear in Smith and Foley, the prices and quantities of assets held. These quantities then appear in a Legendre transform in a

finance theory model that does describe real markets to a good first approximation. We use that Legendre transform and the underlying market fluctuations (the heat bath) as a basis for analyzing the very interesting question whether a formal thermodynamics of a financial market can be constructed. Finance is admittedly a special case of economics per se: it is the part of economics where data are both reliable and abundant, where trades can be studied on any time scale from a second to months or years. Good statistics are available for such studies up to about ten years. We will return to economics in general, where good statistics on many different time scales are generally not available, in parts 4 and 6. Our viewpoint is that financial markets are not tangential to economic theory, financial markets are presently the very best place to test both old and new ideas about economic theory.

We will explain in part 4 why the Smith-Foley quantity identified as entropy cannot be identified as entropy in the sense of either statistical physics or information theory: in neo-classical equilibrium theory there is no analog of the heat bath, with disorder/uncertainty in the form of random fluctuations, only perfect order and complete information about all future markets and orders placed in those markets. Simply stated, their utility-based 'entropy' cannot represent disorder because disorder is not allowed in neo-classical equilibrium theory, nor are money and financial markets allowed in neo-classical theory for the same reason. We see this as the fatal flaw in an otherwise seemingly perfect mathematical mapping. We assert that one cannot find entropy in utility, one must look for entropy in disorder, as did Boltzmann, Gibbs, and Shannon. That is the basis for von Neumann's earlier observation about the need to search for entropy in liquidity. Samuelson's attempted thermal analogies can be criticized on the very same ground.

We will therefore begin by identifying a financial analog of the heat bath, and will write down the stochastic equations describing the empirically-observed Brownian motion in that bath. We will then define physically what we mean by reversible trading, in close analogy with the idea of approximately reversible processes in laboratory thermodynamics. We will next make use of a trading strategy that is defined by a Legendre transform in order to ask if that transform and the underlying liquidity bath lead to an

analogy with thermal equilibrium. Our Legendre transform is perfectly analogous to the ones treated formally by Smith and Foley, but the observable variables (price, quantity) fluctuate randomly because we deal with a real market, a market with entropy representing fundamental disorder.

Our attempt to build a complete thermal analogy fails, but the way that it fails is both interesting and instructive. By carefully defining what our physical (meaning measureable/observable) quantities mean, we are led for the first time to a geometric definition of the no-arbitrage condition. We show explicitly why 'no-arbitrage' is not, as both finance theorists and economists claim, an 'equilibrium' condition. Second, the market entropy (defined in part 5), reflecting the liquidity bath, does not and can not appear via a Legendre transform on the extensive quantities that we use in parts 3 and 4, showing that trying to define formal thermodynamics via Legendre transforms is not enough: you can't reach the market entropy via transformations of variables unless you start with a quantity that is derived from entropy/disorder in the first place. Actually, we already know this lesson from classical thermodynamics since the time of Clausius: it is necessary to include 'heat' along with work in the energy in order to arrive at entropy (there is no physical analog of heat, representing underlying disorder, in the quantities used by Smith and Foley). We then go further in part 5 and identify market instability as the fundamental underlying reason why thermodynamic analogies are impossible in finance. Finally, in part 6 we offer an extention of our reasoning to economics as a whole in the form of a challenge.

A word before going further. Equilibrium is not always essential for the identification of an abstract, formal thermodynamics, and neither is a heat bath, but a correct identification of entropy as disorder *is* necessary. E.g., entropy and formal thermodynamics exist [5] and have been constructed mathematically for the symbollic dynamics representing chaotic dynamical systems with generating partitions [6], but in this case the entropy is simply the Boltzmann entropy of the symbol sequences corresponding to a single Liapunov exponent, and therefore correctly describes disorder in the usual Boltzmann-Gibbs sense [7]. The underlying chaotic dynamical system is driven-dissipative and is far from equilibrium (one can illustrate the formal thermodynamics via a tent, logistic or Henon map, e.g.), but the entropy and

thermodynamics are based on time-independent quantities and therefore do not contradict the nonequilibrium nature of the underlying dynamics that gives rise to the invariant symbollic dynamics. Another way to say it is that the formal thermodynamics and entropy are based on the topological invariants and generating partition of the dynamical system [6,7], and not at all on the time-evolution from initial conditions. In what follows, as in [4], we address the question of trying to use an economic (finance) model 'directly' to construct a thermodynamics, because topological invariants (and generating partitions) do not exist in financial data in particular, or in economic phenomena more generally. That is, we simply apply empirically-based market dynamics to the same variables treated statically by Smith and Foley, but without the unnecessary introduction of empirically unobservable quantites like utility.

2. Liquidity and Approximately Reversible Trading

We define an approximately reversible trade as one where you can reverse your buy or sell order over a very short time interval, on the order of a few seconds or ticks, with only very small percentage losses. This is in analogy with approximately reversible processes in laboratory experiments in thermodynamics. This assumption requires a financial analog for the heat bath in thermodynamics.

The assumption of adequate liquidity (we assume liquidity to include the requirement of 'depth of market' here) is analogous to the assumption of a heat bath in thermodynamics: reversibly removing a small number of shares from the market does not affect the market price of the asset traded, just as reversibly removing a small amount of energy from the heat bath doesn't affect to temperature, to zeroth order. We state here the assumptions central for attempting the financial analogy with thermodynamics. One is that transaction costs are negligible. Another is that the 'liquidity bath' is large enough that borrowing the money, selling the call and buying the stock are possible approximately instantaneously (during a few ticks in the market) without affecting the price of either the stock or call, or the interest rate r. That is, the desired margin purchase is assumed to be possible approximately reversibly in real time through your

discount broker on your Mac or PC. This will not be possible if the number of shares involved is too large, or if the market crashes or balks (because bid/ask limit orders have too large a spread) at selling ϕ shares approximately reversibly. The 'liquidity bath' dries up in either case. The assumption of 'no market impact' (meaning enough liquidity) during trading is an approximation that is limited to very small trades in a heavily-traded market and is easily violated when, e.g., Deutsche Bank takes a very large position in Mexican Pesos or Swedish Crowns (see Dacorogna et al [8] for a discussion of the effects of large limit orders on asset prices).

In the theory of risk free hedges [9,10] in continuous time finance, approximately reversible trading is always implicitly assumed but is seldom stated explicitly. There, one can start with the replicating, self-financing hedging strategy and derive both the delta-hedge and the option pricing pde [10] simultaneously. The equations of the replicating, self-financing hedge have the structure of a Legendre transform. That, plus the ideas of reversibility and heat bath described above, permit us to ask if an underlying thermal analogy is possible. Next, we define the Legendre transform and also the equation of Brownian motion. Both require the assumption of a 'big enough' liquidity bath.

3. The Replicating Self-Financing Hedge

Consider a dynamic hedging strategy (ϕ,ψ) defined as follows [10]. Suppose you're short a European call at price $C(p,K,t,T)$ where K and T are the strike price and time. To cover your bet that the underlying stock price will drop, you simultaneously buy ϕ shares of the stock at price p by borrowing ψ dollars from the broker (the stock is bought on margin, e.g.). In general, the strategy consists of holding $\phi(p,t)$ shares of stock at price p, a risky asset, and $\psi(p,t)$ shares of a money market fund at initial price m=1 Euro/share, a riskless asset (with fixed interest rate r) at all times $t \leq T$ during the bet, where T is the strike time. At the initial time t_o the call is worth

$$C(p_o, t_o) = \phi_o p_o + \psi_o m_o$$

(1)

where $m_o=1$ Euro. This is the initial condition, and the idea is to replicate this balance at all later times $t \leq T$ without injecting any new money into the portfolio. Assuming that (ϕ,ψ) are twice differentiable functions (as would be needed for a thermodynamics analogy), the portfolio is self-financing if, during dt,

$$d\phi p + d\psi m = 0$$

(2)

so that

$$dC = \phi dp + \psi dm$$

(3)

where $dm=rmdt$. In (3), dp is a stochastic variable, and $p(t+dt)$ and $C(t+dt)$ are unknown and random at time t when $p(t)$ and $C(p,t)$ are observed. Viewing C as a function of (p,m), eqn. (3) tells us that

$$\phi = \frac{\partial C}{\partial p}$$

(4)

This is the delta-hedge condition. Next, we want the portfolio in addition to be 'replicating', meaning that the functional relationship

$$C(p,t) = \phi(p,t)p + \psi(p,t)m$$

(5)

holds for all later (p,t) up to expiration, and p is the known price at time t (for a stock purchase, we can take p to be the ask price). Equation (5) expresses the idea that holding the stock plus money market in the combination (ϕ,ψ) is equivalent to holding the call. The strategy (ϕ,ψ), if it can be constructed, defines a 'synthetic call': the call at price C is synthesized by holding a certain number $\phi>0$ shares of stock and $\psi<0$ of money market at each instant t and price $p(t)$. These conditions, combined with Ito's lemma, predict the option pricing equation and therefore the price C of the call

[10,11]. Equations (2-5) are standard finance theory and form the basis for attempting the thermal analogy in part 4.

The stochastic differential equation (sde) for the stock price is given by

$$dp = Rpdt + \sigma(p, t)pdB$$

(6)

and represents Brownian motion in the liquidity bath. Here, B(t) defines a Wiener process, with <dB>=0 and <dB^2>=dt, [10,11] one easily obtains the sde

$$dC = (\dot{C} + \sigma^2 p^2 C'' / 2)dt + C'dp$$

(7)

From these equations then follows the option-pricing partial differential equation (pde) with diffusion coefficient $\sigma^2(p,t)p^2$. the restriction to lognormal distributions or standard Black-Scholes (B-S) theory is neither implied nor necessary [11]. In B-S theory one has σ=constant, but the empirical distribution of returns suggests an entirely different and unrelated functional form for $\sigma^2(p,t)$ [12,13]. We are therefore not assuming ordinary Brownian motion of the simple Einstein variety in the logarithmic return $x=\ln(p(t)/p_o)$, although our reasoning applies to that model as well.

4. The geometric meaning of 'no-arbitrage'

Equations (3), (4) and (5) with stochastic variables (C,p) define a Legendre transform. This is the formal motivation for discussing a thermal anlogy. The variables p and m are intensive, (ϕ,ψ) are extensive variables, and so the call price C is an extensive variable, analogous to a thermodynamic potential. The attempt to identify p and m as analogous to a specific pair of intensive thermodynamic variables, like pressure and temperature, is both uninteresting and unnecessary, because all extensive variables obey a zeroth law in thermal equilibrium.

The interesting parts of the attempted analogy are, first, that the assumption of adequate liquidity is analogous to the heat bath: reversibly removing a small number of shares from the market does not affect the market price, just as reversibly removing a small amount of energy from the heat bath doesn't affect to temperature, to zeroth order. Second, following the zeroth law of thermodynamics, the absence of arbitrage possibilities would then have to be analogous to thermal equilibrium, where there are no correlations: one can't get something for nothing out of the heat bath because of the second law. Likewise, arbitrage is impossible systematically in the absence of correlations (this is essentially the content of the efficient market hypothesis, that there exist no correlations that can be exploited for profit).

In finance theory [14] the no arbitrage condition is called an 'equilibrium' condition. We will now try to make that analogy precise. First, some oversimplified statistical thermodynamics. In a system in thermal equilibrium there is spatial homogeneity and isotropy, the temperature is the same throughout the system independent of position. The same time-dependent temperature fluctuations may be observed at any point in the system. Such fluctuations are described by an asymptotically stationary stochastic process like

$$d\Delta T = -\beta\Delta T dt + \sigma dB(t)$$

(8)

with the mean square fluctuation at long times given by

$$\sigma^2 = \left\langle \Delta T^2 \right\rangle = k T_{eq}{}^2 / c_V$$

(9)

and an equilibrium density of fluctuations (for $\beta\Delta t \ggg 1$) about $\Delta T = T - T_{eq}$ given by the stationary distribution

$$f(\Delta T) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\Delta T^2 / \sigma^2}$$

(10)

Whether in economic theory 'equilibrium' is taken to mean statistical equilibrium or, more generally, a steady state, is irrrelevant as we will point out in part 5 below. Both statistical equilibria (entropy maxima, for a closed system) and stable steady states require an asymptotically-stationary stochastic process. Now for the question whether the finance/thermo analog can be carried to completion.

First, we can see easily that the no-arbitrage condition does not guarantee market equilibrium, which is defined by vanishing total excess demand for an asset. The excess demand $\varepsilon(p,t)$ is defined by $dp/dt = \varepsilon(p,t)$ and is given by the right hand of the sde (6) as drift plus noise. Consider two spatially-separated markets with two different price distributions for the same asset. If enough traders go long in one market and short in the other, then the market price distributions can be brought into agreement. However, if there is positive excess demand for the asset then the average price of the asset will continue increasing with time, so that there is no equilibrium.

Next, in order to understand the main point, consider a spatial distribution of markets with different price distributions at each location, e.g., N.Y., Frankfurt, London, Tokyo, … . That is, the price distribution depends on both position and time. It is easy to formulate the no arbitrage condition in the language of statistical physics. In a physical system in thermal equilibrium the average temperature is constant throughout the system, is independent of location. The temperature fluctuations at each point in the system

obey a stationary process. The no arbitrage condition requires spatial homogeneity and isotropy of the price distribution (to within transaction, shipping and customs fees, and taxes). An example of an arbitrage opportunity would represented by two different markets where in the distribution of the second market theprice variable is uniformly shifted by a constant δp relative to the first market. A 'no arbitrage' condition is therefore equivalent to translational and rotational invariance of the price distribution. But the financial market price distribution is not stationary (we explain this assertion in the next section), so that equilibrium is not achieved even if arbitrage is impossible. A collection of markets with arbitrage possibilities can be formulated via a master equation where the distribution of prices is not spatially homogeneous, but varies (in average price, e.g.) from market to market.

To make closer contact with the reasoning of Smith and Foley [4], where temperature and pressure analogs are identified from utility maximization via Legendre transforms and the zeroth law of thermodynamics is applied, we note the following. In physics, we define the empirical temperature t of an equilibrium system with energy E and volume V (one could equally-well use any other pair of extensive variables), where for any of n mentally-constructed subsystems of the equilibrium system we have

$$t = f(E_1, V_1) = \ldots f(E_n, V_n)$$

(11)

This condition, appled to systems in thermal contact with each other, reflects the historic origin of the need for an extra, non-mechanical variable called temperature. As we have stated above, instead of temperature one can as well take any other intensive variable, e.g., pressure or chemical potential. The economic analog of equilibrium would then be the absence of arbitrage possibilities, that there is only one price of an asset in n different markets

$$p = f(\phi_1, \psi_1) = \ldots f(\phi_n, \psi_n)$$

(12)

This is a standard neo-classical condition for a unique, time-independent price $p=f(z)$ where z is the quantity of an asset held at

price p that would follow from utility maximization. We would need to be able to derive a condition like (12) in order to come into partial agreement with the ideas of Smith and Foley. However, we cannot derive the condition (12) from our model (6), which is not neo-classical and yields a t-dependent average price. The reason that we cannot derive such a condition is that we cannot allpy the idea of statistical equilibrium to the market variables.

Now for our central criticism of the paper by Smith and Foley [4]. They have proposed a thermodynamic interpretation of the standard neo-classical relationship p=f(z) that follows from utility maximization. In their discussion a quantity labeled as entropy is formally defined in terms of Legendre-transformed utility, but the quantity so-defined cannot represent disorder/uncertainty because there is no liquidity, no analog of the heat bath, in neo-classical equilibrium theory. The ground for our assertion has been established within the context of neo-classical theory. Kirman [15] has pointed out, following Radner's proof of noncomputability of neo-classical equilibria under slight uncertainty [16], that demand for money (liquidity demand) does not appear in neo-classical equilibrium theory, where the future is completely determined for all times [17]. Kirman speculates very that liquidity demand arises from uncertainty (disorder). This seems to be a reasonable speculation. The paper by Bak et al [18] is motivated by the fact that a standard neo-classical economy is a pure barter economy (the model used by Smith and Foley), where price p is merely a label because the absolute value of money is undetermined. In the model by Bak et al money appears, but markets are still cleared at the end of each day[1]. In real markets, market clearing occurs seldom and, when it does, it is generally accompanied by unsatisfied demand (one can think of the fish market in Chioggia, e.g.).

The absence of entropy representing disorder in neo-classical equilibrium theory can be contrasted with thermodynamics of physical systems in the following way: for a market let us define the economic efficiency of an asset as

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[1] In a standard neo-classical economy there is no capital accumulation, no financial market, and also no production of goods. Ref. [18] is interesting, it introduces a model of bounded rationality via local interactions between agents.

$$e = \min\left(\frac{D}{S}, \frac{S}{D}\right)$$

(13)

where S and D are net supply and net demand for that market. In standard neo-classical equilibrium the efficiency is 100%, $e=1^2$, whereas the second law of thermodynamics via the heat bath prevents 100% efficiency in thermodynamic machines. That is, the neo-classical market equilibrium condition $e=1$ is not a thermodynamic efficiency, unless we would be able to interpret it as the zero (Kelvin) temperature result of an unknown thermodynamic theory (100% efficiency of a machine is thermodynamically possible only at zero absolute temperature). One cannot derive a noncontradictory Carnot efficiency $e<1$ from the formalism of Smith and Foley without assuming zero absolute temperature because, in neo-clasical equilibrium theory, the market efficiency is stuck at 100%.

5. Instability of Financial Markets

How do real markets behave? Are stationary processes indicated by nonfinancial economic data? We as physicists know that financial markets are not near equilibrium, but finance theorists nevertheless sometimes try to analyze the data by using transformations to variables that assume an underlying stationary process. But is stationarity possible for financial markets, empirically-seen?

We have discovered [12,13] that the empirical distribution for interday trading of bonds and foreign exchange, written in terms of returns $x=\ln(p(t)/p(t_o))$, is exponential

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[2] Efficiency $e<1$ occirs in asymmetric information theory (see Ackerlof [19] and Stiglitz [20]), but the method is still restricted to dynamic equilibria via expected utility maximization (there is no entropy, no statistical equilibrium in the model).

$$f(x,t) = \begin{cases} \dfrac{v^2}{\gamma + v} e^{-v(x - R\Delta t)}, & x > R\Delta t \\[3mm] \dfrac{\gamma^2}{\gamma + v} e^{\gamma(x - R\Delta t)}, & x < R\Delta t \end{cases}$$

(14)

excepting extreme returns. Here, $v = 1/b\sqrt{\Delta t}$ and $\gamma = 1/b'\sqrt{\Delta t}$ with b and b′ constants and with R assumed constant for interday trading (this condition is not essential, R can be taken to be t-dependent). The time-dependence of γ and v guarantees that the global (or average volatility) obeys $\sigma^2 = \langle D(x,t)\rangle \Delta t = c\Delta t$ with c constant, and where the average is over the density f(x,t). The diffusion coefficient ('local volatility') generating this distribution via the Fokker-Planck equation

$$\frac{\partial f(x,t)}{\partial t} = -R\frac{\partial f(x,t)}{\partial x} + \frac{1}{2}\frac{\partial^2 (D(x,t)f(x,t))}{\partial x^2}$$

(15)

or corresponding sde

$$dx = Rdt + \sqrt{D(x,t)}dB(t)$$

(16)

is found to be

$$D(x,t) = \begin{cases} b^2(1 + v(x - R\Delta t)), & x > R\Delta t \\ b'^2(1 - \gamma(x - R\Delta t)), & x < R\Delta t \end{cases}$$

(17)

for small to moderate returns. This model will now be used to discuss the stability of financial markets, and is easily generalized

to describe the observed fat tails in the empirical returns density f by adding a term on the order of $((x-R\Delta t)/\Delta t^{1/2})^2$ to the diffusion coefficient (17) [13].

The entropy of the market distribution, which we introduce next, plays no role in the attempted thermodynamic analogy represented by (2) and (5). The Gibb's entropy S(t) of the empirical returns distribution with density f(x,t) is simply

$$S(t) = -\int_{-\infty}^{\infty} f(x,t) \ln f(x,t) dx$$

(18)

and, so constructed, correctly represents the disorder of the market for the asset under consideration. Statistical equilibrium is impossible because, according to (14), this entropy is always increasing. The entropy S(t) can never reach a maximum because the empirical density f(x,t) spreads diffusively without bound, describing the loss of information about market returns (or prices) as time goes on.

More generally, either of two separate conditions is enough to prevent entropy maximization, or market equilibration: the time-dependent diffusion coefficient D(x,t), and the lack of finite upper and lower bounds on x. E.g., with a t-independent volatility D(x) and expected stock rate of return R(x), the solution of (15) describing statistical equilibrium

$$f(x) = \frac{C}{D(x)} e^{2\int \frac{R(x)}{D(x)} dx}$$

(19)

obviously exists mathematically, but there is no guarantee that this solution can be reached dynamically by the time evolution of an arbitrary distribution. We will return to this point via an enlightening example below, where a particle in a harmonic potential is completely delocalized by a simple price-dependent diffusion coefficient. However, if we impose constraints whereby the particle with position $x(t) = \ln(p(t)/p(0))$ is confined by two walls, $p_{min} \leq p \leq p_{max}$ (e.g., by price controls), time evolution to statistical equilibrium could always be achieved. This is not what is taught in standard economics texts [17]. In Nakamura [21], the

stabilizing Invisible Hand is misinterpreted as the no-arbitrage condition, but as we have shown in part 4 above, the absence of arbitrage opportunities has nothing to do with either market stability or equilibrium.

Similar words can be asserted about the condition for a steady state, where one would (following Onsager and Machlup) introduce a quantity related to entropy that is maximized. To derive a steady state distribution, we rewrite (12) as

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial j(x,t)}{\partial x}$$

$$j(x,t) = R(x,t)f(x,t) - \frac{1}{2}\frac{\partial(D(x,t)f(x,t))}{\partial x}$$

(20)

The condition for a steady state is that j=constant, leading to a solution that differs with (20) but, again, one where t-independent R and D are necessary. Here, the process described by (16) must be stationary, so that the mean and variances of x are necessarily constants. Stationary processes describe steady states and vice-versa, with equilibrium given by the special case where j=0.

To make the contrast between real markets and equilibrium statistical physics sharper, we remind the reader that a Brownian particle in equilibrium in a heat bath has a velocity distribution that is Maxwellian. The sde that describes the approach of a nonequilibrium distribution of particle velocities to statistical equilibrium is the Smoluchowski-Ornstein-Uhlenbeck sde [22] for the particle's velocity

$$dv = -\beta v dt + \sigma dB$$

(21)

The distribution of positions X is also generated by a Fokker-Planck equation, but subject to boundary conditions that confine the particle to a finite volume V so that the equilibrium

distribution of positions is constant and describes time-independent fluctuations about statistical equilibrium. The corresponding stochastic process is stationary, the same as equation (8) above. Nothing comparable is possible with the model sde (16) that generates the empirical financial market distribution.

We can see this by comparing the stationary process (21) with the Osborne-Black-Scholes lognormal pricing model. In (21) statistical equilibrium is reached because the particle is confined by a harmonic potential in velocity space, $U(v)=\beta v^2/2$. Every physicist knows that a harmonic potential means bounded motion. Stated in terms of the price variable, the Osborne lognormal pricing model [23,24] is defined by

$$dp = \mu p dt + \sigma p dB$$
(22)

With negative expected returns $\mu < 0$ the only difference with (21) is that the diffusion coefficient is proportional to 'velocity' p, so on the basis of our experience in statistical physics we would expect that the harmonic price potential

$$U(p) = \frac{|\mu|}{2}p^2$$

(23)

leads to confined motion (localization in p-space) and a corresponding approach to statistical equilibrium. That is, we would expect that (22) defines a stationary process when $\mu < 0$. But the physics intuition is completely wrong here. There is indeed a statistical equilibrium distribution for (22), for all values of μ, but the lognormal distribution (the Green function for (22)) cannot approach it as time goes to infinity so long as p can become infinite. That is, the particle is not bound by the potential (23) but is delocalized! The delocalization is reflected by the fact that the lognormal distribution spreads without limit as t increases, never reaching statistical equilibrium. Otherwise said, the moments of the lognormal distribution are t-dependent and do not approach constants asymptotically. This nonintuitive example shows the danger of 'existence proofs' for equilibria that ignore dynamics

(see Kaniovski [25] for an example of an existence proof that ignores dynamics).

Financial markets cannot be described even to zeroth order by a stationary distribution, but all neo-classical thinking (standard textbook economic theory [17]) is based on the implicit assumption of stable equilibrium. Because financial data are nonstationary, neo-classical economists have introduced the misleading notion of 'temporary price equilibria' [26] in order to try to save the underlying wrong ideas in the face of the empirical data. We have shown elsewhere [11] that the notion of 'temporary price equilibria' is self-contradictory: in that definition there is an artificially- and arbitrarily-defined 'excess demand' that is made to vanish, whereas the actual excess demand $\varepsilon(p)$ defined correctly by $dp/dt=\varepsilon(p)$ by drift plus white noise in (6) above appears in that model but never vanishes. Qualitatively, temporary price equilibria amount to assuming that each particle location in Brownian motion is an 'equilirbium' that is interrupted by a 'shock', but in real markets the Brownian motion (the 'shocks') occurs on the time scale of one second, so that the language makes no sense at all, physically.

We have pointed out elsewhere [27] that there are at least 5 separate ideas of 'equilibrium' in economics and finance, 4 of which are completely wrong. Here, we add out a 6th wrong definition of 'equilibrium': Fama [28] misidentified market averages as describing 'market equilibrium', in spite of the fact that those averages are time-dependent. The only dynamically acceptable definition of equilibrium is that price p is constant, $dp/dt=0$, respecting the equilibrium requirement of vanishing excess demand. In stochastic theory this is generalized (as in statistical physics and thermodynamics) to mean that all average values are time-independent, so that <p>=constant and, furthermore, all moments of the price (or returns) distribution are time-independent. This would correspond to a state of statistical equilibrium defined by (19) or a steady state, where prices would fluctuate with constant variance about constant average values (with vanishing excess demand on the average), but this state has never been observed in data obtained from real markets, nor is it predicted by any model that describes real markets empirically-correctly. This takes us to the last point.

6. The Invisible Hand is a Falsifiable Proposition

Statistical equilbrium of financial markets is impossible, empirically-seen. Financial markets cannot be correctly described as stationary processes. Adam Smith's Invisible Hand, which is assumed by economists to be of general validity, is an assumption that markets are in or near stable equilibrium, reaquiring the implicit assumption of a stationary process. Consider any market anywhere in the world. With unfilled limit orders the excess demand $\varepsilon(p,t)$, defined by $dp/dt = \varepsilon(p,t)$, cannot vanish. Market stability would be represented by the state of statistical equilibrium, or at least a stable steady state, where the price $p(t)$ would satisfy the condition for a stationary stochastic process. Standard economic theory has it wrong: apparently, one cannot have *both* unregulated/deregulated markets *and* stability simultaneously. One might at best have either stable equilibrium or total lack of regulations/constraints, but not both. In between lies a whole spectrum of other possibilities based on the regulation of otherwise free markets, from less to more unstable[4].

We assumed in the paragraph above that examples of stable markets are not likely to be found. It is necessary to pose this assertion as a scientific challenge: Is it possible to find examples of stability in nonfinancial markets? One could search for market stability by using the following method: choose any nonfinancial market with data adequate for determining the time development of the empircal price distribution (we can study the time-development of the finance distribution accurately because we have accurate data over the last twelve or so years). If the distribution is stationary or approaches stationarity asymptotically, then The Invisible Hand stabilizes the market. With a stationary process the variance approaches a constant as initial correlations die out, equilibrium markets are not volatile. For diffusive markets like financial ones, in contrast, the variance is always increasing with time Δt. The question of the existence of Adam Smith's Invisible Hand can be removed from politics and ideology, it can be decided scientifically.

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[4] For real examples of market instability vs. the neo-classical equilibrium prescriptions enforced by the World Bank and the IMF as conditions for loans, see Stiglitz [29], in particular pp. 101-3 and 128-30.

We expect that empirical evidence for stationary markets cannot be found anywhere on the globe. This provides a direct challenge to economists, who advise and direct national treasury departments and international agencies like the World Bank and IMF on the basis of the neo-classical model of equilibria, and where stability is assumed without having been demonstrated.

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