Spherical affine cones for maximal reductive subgroups in exceptional cases

Inaugural-Dissertation

zur
Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität zu Köln

vorgelegt von

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aus Münster

Hundt Druck, Köln

2013

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Tag der Disputation: 22. Januar 2013

Kurzzusammenfassung

Für eine komplexe, einfach zusammenhängende einfache algebraische Gruppe G von exzeptionellem Typ und eine maximale parabolische Untergruppe $P \subset G$ klassifizieren wir alle Tripel (G, P, H), so dass $H \subset G$ eine maximal-reduktive Untergruppe ist, die sphärisch auf G/P operiert.

Weiter bestimmen wir die branching rules $\mathrm{res}_H^G(V_{k\omega_i}^*)$, wobei $k \in \mathbb{N}$ und ω_i das Fundamentalgewicht ist, das zu P assoziiert ist und bestimmen die kombinatorischen Invarianten der sphärischen affinen Kegel über G/P.

Abstract

Given a complex simply connected simple algebraic group G of exceptional type and a maximal parabolic subgroup $P \subset G$, we classify all triples (G, P, H) such that $H \subset G$ is a maximal reductive subgroup acting spherically on G/P.

In addition we derive branching rules for $\mathrm{res}_H^G(V_{k\omega_i}^*)$, $k \in \mathbb{N}$, where ω_i is the fundamental weight associated to P and find the combinatorial invariants for the spherical affine cones over G/P.

Danksagungen

Ich möchte mich herzlich bei Prof. Dr. Peter Littelmann für die Betreuung und die Vergabe dieses schönen Themas bedanken und dafür, dass er mir diese Arbeit ermöglicht hat. Mein weiterer herzlicher Dank gilt Dr. Stéphanie Cupit-Foutou, die diese Arbeit mitbetreut hat.

Danken möchte ich für zahlreiche hilfreiche Diskussionen auch Dr. Guido Pezzini, Dr. Inka Klostermann und Dr. Ghislain Fourier.

Außerdem danke ich meiner Familie und besonders meiner Frau Julia, die mich beim Schreiben dieser Arbeit liebevoll begleitet hat.

Contents

1 Introduction

Given a reductive algebraic group G, a reductive subgroup H and some irreducible G-module V, then V is also a H-module in a natural way. An obvious problem is to find branching rules that describe the decomposition of the H-module V into irreducible components.

In general it is not easy to calculate these decompositions. Thus it is nice to have simple combinatorial rules that describe the decomposition. A famous example is the Littlewood-Richardson rule for decomposition of tensor products. Further examples are known branching rules for the cases where $G \supset H$ is given by $GL_n(\mathbb{C}) \supset GL_{n-1}(\mathbb{C})$ or $Sp_n(\mathbb{C}) \supset Sp_{n-1}(\mathbb{C})$.

In this thesis we study this problem in the situation where G is a complex simply connected simple algebraic group of exceptional type. The subgroup structure of these groups has been studied in great detail and we want to consider maximal reductive subgroups of G. The maximal closed connected subgroups are listed in Theorem 1 of [Sei91]. These groups are either semisimple or parabolic. So the maximal reductive subgroups are easily obtained by adding the Levi factors of the maximal parabolic groups which are maximal reductive in G to the list of maximal semisimple subgroups. The modules V that we consider are those having a multiple of a fundamental weight as highest weight.

We approach the problem by working with spherical varieties, which are defined to be G-varieties that contain an open orbit of a Borel B of G. We consider the flag variety G/P where P is a maximal parabolic subgroup of G. Of special interest to us are the flag varieties of that form, that are H-spherical. The property of being spherical can also be described in a representation-theoretic way. Namely a normal affine G-variety is spherical if and only if its coordinate ring is a multiplicity-free G-module [VK78]. Let $\widehat{Y}$ denote the affine cone over G/P. Then due to the multiplicity-freeness of spherical varieties the flag variety is H-spherical if and only if all restrictions of the homogeneous components of the coordinate ring of $\widehat{Y}$ to H are multiplicity-free. These homogeneous components are exactly the irreducible submodules of the coordinate ring $\mathbb{C}[\widehat{Y}]$ and they are of shape $(V_{k\omega_i})^*$. In the case of sphericity we can derive branching rules for these modules.

Spherical varieties also form a class of varieties that is interesting in its own. Their origin lies in the study of homogeneous spaces which are important objects from different points of view. For example we can realize representations of linear algebraic groups in spaces of sections of homogeneous spaces.

Normal G-equivariant compactifications of homogeneous spaces and more generally their normal G-equivariant embeddings (i. e. normal G-varieties that contain G/H as an open orbit) were studied by Luna and Vust in [LV83] for an algebraically closed ground field of characteristic 0. One important invariant of such an embedding is given by its *complexity* which is defined to be the codimension of a generic B-orbit. So in this sense spherical varieties are the simplest embeddings for homogeneous spaces.

For the spherical varieties the theory of Luna and Vust is very well developed. Following their work the classification of spherical varieties was completed recently and can be carried out completely in combinatorial terms. The main invariants for this classification are the B-invariant but not G-invariant prime divisors of X and the G-invariant valuations of the function field $\mathbb{C}(X)$. This classification is divided into two steps. The first step is to classify all embeddings of a given spherical homogeneous space G/H. This is achieved by attaching a collection of convex polyhedral cones (called colored cones) to the variety. The second step is to classify G/H among all spherical homogeneous spaces G/H for a given connected reductive group G. This classification is done combinatorially by attaching to G/H a 5-tuple of invariants which is called its homogeneous spherical datum. Hence the classification of the spherical affine cones naturally leads to the determination of their combinatorial invariants.

The main results of this thesis are the following: For complex simply connected simple algebraic groups G of exceptional type we classify triples (G, P, H) where P is a maximal parabolic subgroup and H a maximal reductive subgroup of G, such that the H-varieties G/P are spherical. Furthermore we derive branching rules for the simple G-submodules of the coordinate ring of the affine cones in the spherical cases. The results are summarized in Table 3.1. A flag variety G/P is H-spherical if and only if the branching rules for the corresponding modules V are given in the table.

Further we find the spherical affine cones and its open orbits in the classification of spherical varieties by calculating their combinatorial invariants. These results are summarized in Table 4.1 and Table 4.2.

This thesis is organized as follows. In the second chapter we recall the basic definitions and properties of spherical varieties. We also explain how they are classified by colored cones and homogeneous spherical data.

In the third chapter we carry out the classification of the triples (G, P, H) such that H acts spherically on G/P.

Finally Chapter 4 is devoted to the computation of the colored cones and homogeneous spherical data of the spherical affine cones that were classified in Chapter 3.

Notation

We work over the field of complex numbers. G always denotes a simply connected simple algebraic group. Within G we choose a Borel subgroup B, a maximal torus T and thereby define a set $S = \{\alpha_1, \dots, \alpha_r\}$ of simple roots which are labeled according to Bourbaki-notation. The system of roots of G is denoted by Φ, the system of positive roots of G is denoted by Φ^+ and $(a_1, \dots, a_r)$

stands for the root $\alpha = \sum_{i=1}^{r} a_i\alpha_i$ and the support $\mathrm{supp}\,(\alpha)$ of α is the set of simple roots such that $a_i \neq 0$. Further X_α denotes a non-trivial element of the root space associated to α and U_α denotes the root group. Let $X(T)$ be the set of weights related to T and $X(T)^+$ the set of dominant weights. The irreducible G-module of highest weight $\lambda \in X(T)^+$ is denoted by V_λ. The fundamental weights of G are $\omega_1, \ldots, \omega_r$ and $\omega_1^*, \ldots, \omega_r^*$ are the fundamental weights such that $(V_{\omega_i})^* = V_{\omega_i^*}$, where $(V_{\omega_i})^*$ is the dual of V_{ω_i}. If we write $k\omega_i$, then $k \in \mathbb{N}$.

Let H denote a reductive subgroup of G with root system Φ_H and analogous to G we use the notation $(b_1, \ldots, b_s)_H := \sum_{i=1}^{s} b_i\beta_i$ where $S_H = \{\beta_1, \ldots, \beta_s\}$ is a set of simple roots of Φ_H given by the Borel subgroup $B_H = B \cap H$. The fundamental weights of H are denoted by $\lambda_1, \ldots, \lambda_s$, if H is semisimple. When H is a Levi subgroup, $\lambda_1, \ldots, \lambda_s$ denote the fundamental weights of the semisimple part of H.

2 Spherical varieties

We will now present some basic definitions on spherical varieties and explain how they are classified in terms of the Luna-Vust theory.

The spherical varieties can be classified combinatorially by attaching certain invariants to them. In general the classification of spherical is subdivided into two parts. On the one hand we want to find all spherical varieties that contain a given spherical open G-orbit. The second step is to find a classification for all spherical G-orbits of a given connected reductive group.

1 Basic definitions and properties of spherical varieties

We start by recalling some basic definitions and properties of spherical varieties that lead to the introduction of the combinatorial objects that are used for their classification. There are various surveys on spherical varieties and in this section we follow [Kno91] and [Pez10].

We will work over the field of complex numbers although many of the following results hold for any algebraically closed field.

1.1 Basic invariants and different characterizations of sphericity

Definition 2.1: A G-variety X is called *spherical* if X is normal and contains an open B-orbit.

If x is a point of the open B-orbit of X then we can consider the orbit Gx which is also open in X. If we denote the stabilizer of x in G by H then Gx is isomorphic to the homogeneous space G/H. The variety X is called an *embedding* of the homogeneous space G/H.

Further we say $H \subset G$ is a *spherical subgroup* if G/H is a spherical G-variety.

As for any G-variety we can define the B-semiinvariant rational functions

$$\mathbb{C}(X)^{(B)} = \{f \in \mathbb{C}(X)\backslash\{0\} \mid b.f = \chi_f(b)f,\ \forall b \in B\}$$

where χ_f denotes a character of B (i. e. a morphism $\chi_f : B \to \mathbb{C}^*$ of algebraic groups) corresponding to f.

We denote the weight lattice of B by $X(B)$ and consider the morphism $\varphi : \mathbb{C}(X)^{(B)} \to X(B)$ sending each function in $\mathbb{C}(X)^{(B)}$ to its B-weight.

Definition 2.2:

i) The image of φ is denoted by $\Lambda(X)$. It is a free abelian group, called the *weight lattice of* X.

ii) The rank of $\Lambda(X)$ is defined to be the *rank of* X, denoted by $\mathrm{rk}(X)$.

iii) We denote the dominant weights by $X^+(B)$. Then $\Lambda^+(X) = X^+(B) \cap \Lambda(X)$ is called the *weight monoid of* X.

iv) We denote the $\mathbb{Q}$-vector space $\mathrm{Hom}(\Lambda(X), \mathbb{Q})$ by $N(X)$. If $x \in N(X), \mu \in \Lambda(X)$, we write $\langle x, \mu \rangle$ for $x(\mu)$.

All these objects that were just defined are G-birational invariants of X and depend only on the open G-orbit of X.

A first characterizing property of spherical varieties can be given by their number of B-orbits.

Theorem 2.3 ([Kno95, Cor. 2.6])**:** *Let X be a spherical G-variety. Then B has finitely many orbits on X.*

Proof: If $Y \subset X$ is a B-stable closed subvariety then Y is also spherical by the previous theorem.

Suppose that there are infinitely many B-orbits on X and take $Z \subset X$ minimal with the property that it has an infinite number of B-orbits. Since Y is also spherical there is a dense orbit $B.y \subset Y$. So an irreducible component of $Y \backslash B.x$ must have an infinite number of B-orbits, contradicting the minimality of Y. □

There is also an important connection to representation theory due to Vinberg and Kimelfeld [VK78]. We say that a G-module M is *multiplicity-free* if for every dominant weight λ, the multiplicity of the highest weight module $V(\lambda)$ in M is at most 1.

Theorem 2.4 ([Bri97, Thm. 2.1])**:** *Let X be a normal quasi-projective G-variety. Then the following properties are equivalent:*

i) X is spherical.

ii) For any G-line bundle $\mathscr{L}$ on X, the G-module $H^0(X, \mathscr{L})$ is multiplicity-free.

If X is quasi-affine ii) can be replaced by

ii') $\mathbb{C}[X]$ is multiplicity-free.

Furthermore sphericity is indicated by the field of B-invariant rational functions due to a result by Rosenlicht [Ros63].

Theorem 2.5: *Let X be a normal quasi-projective G-variety. Then the following properties are equivalent:*

i) X is spherical.

ii) The rational B-invariant functions are constant, i. e. $\mathbb{C}(X)^B = \mathbb{C}$.

1.2 Local structure of a spherical variety

For any normal G-variety there is the Brion-Luna-Vust Local Structure Theorem [BLV86] describing an open neighborhood of a P-orbit where P is a certain parabolic subgroup of G.

Now we want to analyze the local structure in the case that the variety is spherical.

Definition 2.6: Let X be a spherical G-variety and $Y \subset X$ a G-orbit. We define the following set:

$$X_{Y,G} := \{x \in X \mid \overline{Gx} \supset Y\}$$

Proposition 2.7: *The set* $X_{Y,G}$ *is* G*-stable and open in* X*. The orbit* Y *is its only closed orbit.*

Proof: Suppose $x \in X_{Y,G}$. Then $\overline{Gx} \supset Y$. If $y = gx$ with $g \in G$, then $\overline{Gy} = \overline{Gx} \supset Y$.

To prove that $X_{Y,G}$ is open in X observe that if $x \in X \backslash X_{Y,G}$ then $\overline{Gx} \subset X \backslash X_{Y,G}$. And since there are only finitely many G-orbits, $X \backslash X_{Y,G}$ is closed.

Finally if Gx is a closed orbit of $X_{Y,B}$ then we have $Gx = \overline{Gx} \supset Y$, hence $Y = Gx$. □

Definition 2.8: A spherical G-variety is called *simple* if it contains a unique closed G-orbit.

The preceding proposition shows that any spherical G-variety can be covered by open simple spherical G-varieties. That means that the classification of spherical varieties can be settled in two steps. First we classify simple embeddings and then study how to patch these varieties together. So the simple embeddings will be the starting point for the combinatorial classification of embeddings for a given spherical homogeneous space.

Consider now the set

$$X_{Y,B} := \{x \in X \mid \overline{Bx} \supset Y\}.$$

It is a subset of $X_{Y,G}$.

Denote the B-stable prime divisors of X by $\mathcal{D}(X)$ and let P be the subgroup of G that stabilizes $X_{Y,B}$, i. e.

$$P = \{g \in G \mid gX_{Y,B} = X_{Y,B}\}.$$

Since we have $B \subset P$, P is a parabolic subgroup of G .

For later use we also define the set

$$\Delta(X) := \{D \in \mathcal{D}(X) \mid D \text{ is } B\text{-stable but not } G\text{-stable}\}.$$

The elements of $\Delta(X)$ are called *colors of X* and will play an important role in the classification of spherical varieties. Note that the notion of colors is birationally invariant if we identify the colors of X with the closures of the colors of G/H in X and therefore we regard $\Delta(X)$ as a subset of $\Delta(G/H)$.

Theorem 2.9 ([Bri97, Prop. 2.2, Thm. 2.3])**:** *With the previous notation we have:*

i) The set $X_{Y,B}$ is B-stable affine and open in X and $X_{Y,B} \cap Y \neq \varnothing$ is the open B-orbit of Y.

ii) The complement of $X_{Y,B}$ in X is the union of the B-stable prime divisors that do not contain Y.

$$X_{Y,B} = X \setminus \bigcup_{\substack{D \in \mathcal{D}(X) \\ D \not\supset Y}} D.$$

iii) There exists a Levi-subgroup $L \subset P$ and a closed subvariety $S \subset X_{Y,B}$ which is L-stable, affine and L-spherical such that the morphism

$$\begin{aligned} P^u \times S &\to X_{Y,B} \\ (g,s) &\mapsto g.s \end{aligned}$$

is a P-equivariant isomorphism. Here, the action of P on $P^u \times S$ is given by $p.(g,s) = (vlgl^{-1}, ls)$ if $p = vl$ with $v \in P^u$ and $l \in L$.

We want to illustrate the local structure with an example.

Example. Consider $G = \mathrm{SL}_2$ with its natural action on $\mathbb{C}^2$. This action has two orbits: $\mathcal{O} = \{(x,y) \in \mathbb{C}^2 \mid (x,y) \neq (0,0)\}$ and $Y = \{(0,0)\}$ which is the unique closed orbit. The Borel consisting of upper triangular matrices has three orbits: $\mathcal{O}_1^B = \{(x,y) \in \mathbb{C}^2 \mid y \neq 0\}$, $\mathcal{O}_2^B = \{(x,y) \in \mathbb{C}^2 \mid x \neq 0,\, y = 0\}$ and Y.

The open B-orbit is $\mathcal{O}_1^B$ and if we denote the unipotent upper triangular matrices by U, the open G-orbit can be identified with G/U which is equivariantly isomorphic to $\mathbb{C}^2 \setminus \{(0,0)\}$.

In this case we have $X_{Y,B} = \{(x,y) \in \mathbb{C}^2 \mid y \neq 0\}$. The stabilizer of $X_{Y,B}$ is B with Levi-decomposition $U \cdot T$ where T is the maximal torus of G consisting of diagonal matrices. In this case we can choose $S := \{(0,y) \in \mathbb{C}^2 \mid y \neq 0\}$ and get an isomorphism

$$\begin{aligned} U \times S &\to X_{Y,B} \\ \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\right) &\mapsto \begin{pmatrix} ay \\ y \end{pmatrix} \end{aligned}$$

The variety S is affine, T-stable and T-spherical.

1.3 G-invariant valuations and classification of simple embeddings

Let X be an embedding of a spherical homogeneous space G/H. Then we have a G-equivariant map $G/H \to X$ identifying G/H with the open orbit of X. This map induces an isomorphism between the fields of rational functions $\mathbb{C}(G/H)$ and $\mathbb{C}(X)$.

Definition 2.10: A *discrete valuation* of $\mathbb{C}(X)$ is a map $\nu : \mathbb{C}(X)^* \to \mathbb{Q}$ satisfying the following properties:

i) $\nu(f_1 \cdot f_2) = \nu(f_1) + \nu(f_2)$ for any $f_1, f_2 \in \mathbb{C}(X)^*$.

ii) $\nu(f_1 + f_2) \geq \min\{\nu(f_1), \nu(f_2)\}$ for any $f_1, f_2 \in \mathbb{C}(X)^*$ such that $f_1 + f_2 \neq 0$.

iii) $\nu(\mathbb{C}^*) = \{0\}$

iv) The image of ν is a discrete subgroup of $\mathbb{Q}$.

A discrete valuation is called *G-invariant* if $\nu(f) = \nu(gf)$ for any $g \in G$ and $f \in \mathbb{C}(X)^*$. The set of G-invariant valuations is denoted by $\mathcal{V}(X)$.

Let D be a prime divisor of X. We denote the corresponding discrete valuation of $\mathbb{C}(X)$ by ν_D.

Now let $\lambda \in \Lambda(X)$ and let $f_\lambda \in \mathbb{C}(X)^{(B)}$ be of weight λ. Since $\mathbb{C}(X)^B = \mathbb{C}$, the function f_λ is uniquely determined up to multiplication by a scalar. So we can define a map

$$\begin{array}{rrcl} \rho : & \{\text{discrete valuations}\} & \to & N(X) \\ & \nu & \mapsto & (\lambda \mapsto \nu(f_\lambda)) \end{array}$$

We will also regard ρ as a map from the set of prime divisors to $N(X)$ by setting $\rho(D) = \rho(\nu_D)$.

In general, ρ is not an injective map. But if we restrict ρ to elements of $\mathcal{V}(X)$, it becomes injective. So, a G-invariant discrete valuation depends only on its values on B-semiinvariant functions of $\mathbb{C}(X)$ and we can identify $\mathcal{V}(X)$ with a subset of $N(X)$.

Theorem 2.11 ([LV83, Prop. 7.4])**:** *Let X be a spherical G-variety. Then the restricted map*

$$\rho_{|\mathcal{V}(X)} : \mathcal{V}(X) \to N(X)$$

is injective.

This identification encodes one interesting property of the open G-orbit G/H of X by the following proposition which is due to Pauer.

Proposition 2.12 ([Kno91, Cor. 6.2])**:** *The G-orbit G/H of a spherical G-variety X contains a maximal unipotent subgroup U of G if and only if $\mathcal{V}(X) = N(X)$.*

Definition 2.13: A spherical G-variety with the property given in the preceding proposition are is called *horospherical*.

2 Classification of embeddings for a spherical homogeneous space

2.1 Classification of simple embeddings

At this point we can turn to the classification of simple embeddings.

Definition 2.14: Let X be a simple spherical G-variety. Define

$$\Delta_Y(X) := \{D \in \Delta(X) \mid D \supset Y\}$$

and

$$\mathcal{V}_X := \{D \in \mathcal{D}(X) \mid D \text{ is } G\text{-invariant}\}.$$

Proposition 2.15 ([Bri97, Prop. 3.2.1])**:**

i) Let X be a simple spherical G-variety with closed G-orbit Y. Then $f \in \mathbb{C}[X_{Y,B}]$ if and only if f is regular on the open G-orbit and $v_D(f) \geq 0$ for all $D \in \Delta_Y(X) \cup \mathcal{V}_X$.

ii) A simple embedding is uniquely determined by the pair $(\Delta_Y(X), \mathcal{V}_X)$.

Proof: Since X is simple we have $X = X_{Y,G}$ and we know that $X \backslash X_{Y,B}$ is the union of those B-stable divisors that do not contain Y and hence are not G-stable. $X_{Y,B}$ contains the open B-orbit Bx_0 and $X_{Y,B}$ is the union of B-stable prime divisors that contain Y. Such a divisor is either a color or G-stable. Since $X_{Y,B}$ is normal a rational function f is regular if and only if it is regular on Bx_0 and does not have a pole along the prime divisors.

If there is another embedding X' that defines the same sets then $\mathbb{C}[X_{Y,B}]$ and $\mathbb{C}[X'_{Y,B}]$ are the same subrings of $\mathbb{C}(X)$. So there is a G-equivariant rational map $\varphi : X \dashrightarrow X'$ and its restriction to $X_{Y,B}$ is an isomorphism. Since $X = GX_{Y,B}$ and $X' = GX'_{Y,B}$ the map φ is an isomorphism. □

We want to classify simple varieties by cones in $N(X)$ satisfying some properties. We fix some definitions concerning cones.

Definition 2.16:

i) A *cone* in a $\mathbb{Q}$-vector space V is a subset which is stable under multiplication by elements in $\mathbb{Q}_{\geq 0}$ and addition.

ii) A cone C is called *strictly convex* if it does not contain a linear subspace, i. e. $C \cap (-C) = \{0\}$.

iii) A cone C is *polyhedral* if $C = \mathbb{Q}_{\geq 0}v_1 + \ldots + \mathbb{Q}_{\geq 0}v_n$ for some vectors $v_i \in V$.

iv) The *dual cone* $C^\vee$ is the set

$$C^\vee = \{f \in V^* \mid f(v) \geq 0 \text{ for all } v \in C\},$$

where V^* denotes the dual vector space of V.

v) A *face of a cone* C is a subset $F \subset C$ of the form

$$F = \{v \in C \mid f(v) = 0\} \text{ for some } f \in C^\vee.$$

vi) The *relative interior* C° of a cone C is the complement of all proper faces.

vii) An *extremal ray* of a cone C is a face of dimension 1.

By Theorem 2.11, we can identify $\mathcal{V}_X$ with a subset of $N(X)$. Furthermore the colors of X can be identified with colors of G/H. So both sets have a description in terms of G/H.

We can associate a cone to a simple spherical G-variety X as follows.

Definition 2.17: Let X be a simple spherical G-variety. We define $\mathcal{C}(X) \subset N(X)$ to be the cone generated by $\mathcal{V}_X$ and $\rho(\Delta_Y(X))$. The pair $(\mathcal{C}(X), \Delta_Y(X))$ is called the *colored cone of* X.

Proposition 2.18 ([Bri97, Prop. 3.2.2])**:** *A simple embedding X is uniquely determined by its colored cone.*

Proof: The half-lines $\mathbb{Q}_+\nu$ where $\nu \in \mathcal{V}_X$ are exactly the extremal rays of $\mathcal{C}(X)$ that do not contain an element of $\rho(\Delta_Y(X))$ [Kno91, Lemma 2.4]. So $\mathcal{V}_X$ can be recovered from $\mathcal{C}(X)$. □

The pair $(\mathcal{C}(X), \Delta_Y(X))$ turns out to be a colored cone also in the sense of the following combinatorial definition.

Definition 2.19: A colored cone in $N(G/H)$ is a pair $(\mathcal{C}, \Delta)$ where $\mathcal{C} \subset N(G/H)$ and $\Delta \subset \Delta(G/H)$ such that

i) $\mathcal{C}$ is a strictly polyhedral cone generated by $\rho(\Delta)$ and a finite number of elements in $\mathcal{V}(G/H)$,

ii) the relative interior $\mathcal{C}^\circ$ intersects $\mathcal{V}(G/H)$,

iii) $0 \notin \rho(\Delta)$.

Theorem 2.20 ([LV83, Prop. 8.10])**:** *The map $X \mapsto (\mathcal{C}(X), \Delta_Y(X))$ defines a bijection between isomorphism classes of simple embeddings of G/H and colored cones in $N(G/H)$.*

Furthermore the colored cones also encode the inclusion of orbits if we look at their faces.

Definition 2.21: A *face of a colored cone* $(\mathcal{C}, \Delta)$ is a colored cone $(\mathcal{C}', \Delta')$ where $\mathcal{C}'$ is a face of $\mathcal{C}$, and $\Delta' = \Delta \cap \rho^{-1}(\mathcal{C}')$.

Let X be a (not necessarily simple) embedding of G/H and let Y be a G-orbit. Then we define the colored cone of the simple embedding $X_{Y,G}$ by $(\mathcal{C}(X_{Y,G}), \Delta_Y(X_{Y,G}))$.

Proposition 2.22 ([Bri97, Prop. 3.4])**:** *The map* $Z \mapsto (\mathcal{C}(X_{Z,G}), \Delta_Z(X_{Z,G}))$ *is a bijection between the set of orbits* $Z \subset X$ *that contain* Y *in their closure and faces of* $(\mathcal{C}(X_{Y,G}), \Delta_Y(X_{Y,G}))$.

Theorem 2.20 settles the problem of classifying simple spherical varieties. These include affine spherical varieties for example.

Lemma 2.23: *Every affine spherical* G*-variety is simple.*

Proof: Let X be an affine spherical G-variety. Any G-invariant regular function is constant on the open G-orbit and thus on X. Hence we have $\mathbb{C}[X//G] = \mathbb{C}[X]^G = \mathbb{C}$, so $X//G$ is a point and it follows that there is a unique closed orbit. □

2.2 Classification of spherical embeddings

As a spherical variety is the union of simple spherical varieties, they are classified by a collection of colored cones.

Definition 2.24: A *colored fan* $\mathcal{F}$ in $N(G/H)$ is a collection of colored cones such that

i) every face of $\mathcal{C} \in \mathcal{F}$ is an element of $\mathcal{F}$,

ii) if $\mathcal{C}_1, \mathcal{C}_2$ are two elements of $\mathcal{F}$, then $\mathcal{C}_1^\circ \cap \mathcal{C}_2^\circ = \varnothing$, i. e. the relative interiors of the colored cones in $\mathcal{F}$ do not intersect.

For an embedding X of a spherical homogeneous space G/H we define its colored fan by

$$\mathcal{F}(X) = \{(\mathcal{C}(X_{Y,G}), \Delta_Y(X_{Y,G})) \text{ where } Y \text{ runs through all } G\text{-orbits of } X\}.$$

Theorem 2.25 ([Kno91, Thm. 3.3])**:** *Let* G/H *be a spherical homogeneous space. Then there is a bijection between the morphism classes of embeddings and colored fans in* $N(G/H)$ *given by* $X \mapsto \mathcal{F}(X)$.

Under this isomorphism the open orbit corresponds to the colored cone given by $(0, \varnothing)$ which is a face of any colored cone in $\mathcal{F}(X)$.

3 Classification of spherical homogeneous spaces

3.1 Wonderful varieties and spherical roots

Now we have a classification for the embeddings of a spherical homogeneous space G/H. For a full classification of spherical varieties it remains to classify spherical subgroups of a given group G. In order to do so we take a closer look at $\mathcal{V}(X)$.

Theorem 2.26 ([Bri97, Thm. 4.1], [Kno96, Cor. 7.1])**:** *Let X be a spherical G-variety. The valuation cone $\mathcal{V}(X)$ is a polyhedral convex cone which can be described by*

$$\mathcal{V}(X) = \{x \in N(X) \mid \langle x, \sigma_i \rangle \leq 0,\ i = 1, \dots, n\}$$

where σ_i are primitive elements in $\Lambda(X)$ that are linearly independent.

The elements σ_i are called *spherical roots* of X and can also be defined using the language of wonderful varieties.

Note that the horospherical varieties are exactly those where the set of spherical roots is empty.

Definition 2.27: Let X be a G-variety. The variety is called *wonderful* (of rank r) if it fulfills the following properties:

i) X is smooth and complete.

ii) X contains an open orbit X° whose complement is the union of r smooth G-invariant divisors $D_1, \dots, D_r$ which have normal crossings and non-empty intersection.

iii) For $x, y \in X$ we have

$$\{i \mid x \in D_i\} = \{j \mid y \in D_j\} \Leftrightarrow Gx = Gy.$$

If X is wonderful of rank 1, then let z be its unique point fixed by B^- (the opposite Borel). Then z is an element of the closed orbit Z of X. Consider the space $\mathrm{T}_z X / \mathrm{T}_z Z$ which is a vector space of dimension 1. It is also a T-module and its weight σ is defined to be the spherical root of X.

Now if X is an arbitrary spherical G-variety with open orbit G/H, we can consider the set of wonderful subvarieties of rank 1 of any embedding of G/H and the union of the corresponding spherical roots. Then this set equals the set of the σ_i above (cp. [Lun01]). We denote the set of spherical roots of X by Σ_X. It is a subset of $\Sigma(G)$ which denotes the union of spherical roots of any wonderful G-variety of rank 1. These varieties were classified by Ahiezer [Ahi83]. In particular $\Sigma(G)$ is finite. Namely for a simply connected reductive group (not necessarily simple) $\sigma \in \Sigma(G)$ if and only if it is a spherical root of a simple factor of G or of a product of two simple factors according to the first column of Table 2.1 [Tim11, p. 192].

Table 2.1: Spherical roots of G

G	$\Sigma(G)$	
A_l	$\alpha_i + \ldots + \alpha_j$ $(A_{j-i+1}, i \leq j \leq l)$	$2\alpha_i$ (A_1)
	$\alpha_i + \alpha_j$ $(A_1 \times A_1, i \leq j-2)$	$\frac{1}{2}(\alpha_1 + \alpha_3)$ $(A_1 \times A_1, l = 3)$
	$\alpha_{i-1} + 2\alpha_i + \alpha_{i+1}$ $(A_3, 1 < i < l)$	$\frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)$ $(A_3, l = 3)$

Table 2.1: Spherical roots of G

G	$\Sigma(G)$	
B_l	$\alpha_i + \ldots + \alpha_j$ $(A_{j-i+1}, i \leq j < l)$	α_l (A_1) $\quad 2\alpha_i$ (A_1)
	$\alpha_i + \alpha_j$ $(A_1 \times A_1, i \leq j-2)$	$\frac{1}{2}(\alpha_1 + \alpha_3)$ $(A_1 \times A_1, l = 3, 4)$
	$\alpha_{i-1} + 2\alpha_i + \alpha_{i+1}$ $(A_3, 1 < i < l-1)$	$\frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)$ $(A_3, l = 4)$
	$\alpha_i + \ldots + \alpha_l$ $(B_{l-i+1}, i < l)$	$2(\alpha_i + \ldots + \alpha_l)$ $(B_{l-i+1}, i < l)$
	$\alpha_{l-2} + 2\alpha_{l-1} + 3\alpha_l$ (B_3)	$\frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3)$ $(B_3, l = 3)$
C_l	$\alpha_i + \ldots + \alpha_j$ $(A_{j-i+1}, i \leq j < l)$	α_l (A_1) $\quad 2\alpha_i$ (A_1)
	$\alpha_i + \alpha_j$ $(A_1 \times A_1, i \leq j-2)$	$\alpha_{i-1} + 2\alpha_i + \alpha_{i+1}$ $(A_3, 1 < i < l-1)$
	$\alpha_i + \sum_{k=i+1}^{l-1} 2\alpha_k + \alpha_l$ $(C_{l+i+1}, i < l)$	$2(\alpha_{l-1} + \alpha_l)$ (C_2)
D_l	$\alpha_{i_1} + \ldots + \alpha_{i_k}$ $(A_k, k \geq 1)$	$2\alpha_i$ (A_1) $\quad \alpha_i + \alpha_j$ (A_1)
	$\alpha_{i_1} + 2\alpha_{i_2} + \alpha_{i_3}$ (A_3)	$\frac{1}{2}(\alpha_{i_1} + 2\alpha_{i_2} + \alpha_{i_3})$ $(A_3, l = 4)$
	$\sum_{k=i}^{l-2} 2\alpha_k + \alpha_{l-1} + \alpha_l$ $(D_{l-i+1}, i < l-1)$	
	$\sum_{k=i}^{l-2} \alpha_k + \frac{1}{2}(\alpha_{l-1} + \alpha_l)$ $(D_{l-i+1}, i < l-1)$	
	$\frac{1}{2}(\alpha_1 + \alpha_3)$ $(A_1 \times A_1, l = 4)$	$\frac{1}{2}(\alpha_1 + \alpha_4)$ $(A_1 \times A_1, l = 4)$
E_l	$\alpha_{i_1} + \ldots + \alpha_{i_k}$ $(A_k, k \geq 1)$ $\quad 2\alpha_i$ (A_1)	$\alpha_i + \alpha_j$ $(A_1 \times A_1)$
	$2\alpha_{i_1} + \ldots + 2\alpha_{i_{k-2}} + \alpha_{i_{k-1}} + \alpha_{i_k}$ $(D_k, k \geq 3)$	
F_4	α_i (A_1) $\quad 2\alpha_i$ (A_1)	$\alpha_i + \alpha_j$ $(A_1 \times A_1, i \leq j-2)$
	$\alpha_i + \alpha_{i+1}$ $(A_2, i \neq 2)$	$\alpha_2 + \alpha_3$ (B_2)
	$2(\alpha_2 + \alpha_3)$ (B_2)	$\alpha_1 + \alpha_2 + \alpha_3$ (B_3)
	$2(\alpha_1 + \alpha_2 + \alpha_3)$ (B_3)	$\alpha_1 + 2\alpha_2 + 3\alpha_3$ (B_3)
	$\alpha_2 + 2\alpha_3 + \alpha_4$ (C_3)	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ (F_4)
G_2	α_i (A_1), $\quad 2\alpha_i$ (A_1), $\quad \alpha_1 + \alpha_2$ (G_2),	$2\alpha_1 + \alpha_2$ (G_2), $\quad 4\alpha_1 + 2\alpha_2$ (G_2)
$X_l \times Y_m$	$\alpha_i + \alpha'_j$ $(A_1 \times A_1)$,	$\frac{1}{2}(\alpha_l + \alpha'_m)$ $(A_1 \times A_1, X = Y = C, l, m \geq 1)$

The name 'spherical roots' stems from the fact that they form a root system (called *little root system*) and $\mathcal{V}(X)$ is the negative Weyl chamber of its Weyl group [Bri90, Cor. 3.5].

In case that X is a quasi-affine variety there is a close connection between the spherical roots and the regular functions $\mathbb{C}[X]$ [Kno96]. If we consider its decomposition as a G-module $\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda^+(X)} \mathbb{C}[X]_\lambda$, then in general this does not define a graduation. The spherical roots are given by the deviation of the multiplication in $\mathbb{C}[X]$ from a graduation.

Theorem 2.28 ([Kno96, Thm. 1.3])**:** *Let X be a quasi-affine spherical G-variety and $\mathbb{C}[X]$ the regular functions on X. Let $m : \mathbb{C}[X] \otimes \mathbb{C}[X] \to \mathbb{C}[X]$ be the G-equivariant morphism induced by the multiplication in $\mathbb{C}[X]$.*

Then the saturation of the submonoid $\langle \lambda + \mu - \nu \in \Lambda(X) : m_\nu^{\lambda,\mu} \neq 0 \rangle$ (where $m_\nu^{\lambda,\mu}$ is the morphism $\mathbb{C}[X]_\lambda \otimes \mathbb{C}[X]_\mu \to \mathbb{C}[X]_\nu$) is free and the generators are a basis of Σ_X.

3.2 Homogeneous spherical data and spherical systems

Similar to the classification of embeddings, there is a classification of spherical homogeneous spaces by associating some combinatorial invariants to them. This classification goes back to Luna's work [Lun01].

Before stating the bijection we need some more terminology concerning colors and simple roots.

We say that *a color D is moved by a simple root α of G* if the minimal parabolic associated to α (which we denote by P_α) is not contained in the stabilizer of D, i. e. there exist $x \in D$ and $g \in P_\alpha$ such that $g.x \notin D$.

For any simple root α, let $\Delta(\alpha)$ denote the set of colors that are moved by α and let $S^p(X)$ be the set of simple roots that do not move any color. Then the following cases can occur.

Lemma 2.29 ([Lun97, §3.2, §3.4])**:** *For any simple root α we have $|\Delta(\alpha)| \leq 2$ and exactly one of the cases below occurs.*

i) $\Delta(\alpha) = \varnothing$, *i. e.* $\alpha \in S^p(X)$.

ii) $\Delta(\alpha) = \{D^+, D^-\}$. *In this case* $\alpha \in \Sigma(X)$ *and we have*

$$\rho(D^+) + \rho(D^-) = \alpha^\vee{}_{|\Lambda(X)}.$$

iii) $\Delta(\alpha) = \{D\}$ *and* $2\alpha \in \Sigma(X)$. *In this case we have*

$$\rho(D) = \frac{1}{2}\alpha^\vee{}_{|\Lambda(X)}.$$

iv) $\Delta(\alpha) = \{D\}$ *and* $2\alpha \notin \Sigma(X)$. *In this case we have*

$$\rho(D) = \alpha^\vee{}_{|\Lambda(X)}.$$

According to the above lemma we can write the set of simple roots as the disjoint union of four subsets by setting

- $S^a := \{\alpha \in S \mid \alpha \in \Sigma(X)\}$,
- $S^{2a} := \{\alpha \in S \mid 2\alpha \in \Sigma(X)\}$,
- $S^b := S \backslash \{S^a \cup S^{2a} \cup S^p\}$,

where S^p is short for $S^p(X)$. We say that a simple root is of type $a, 2a, b$ or p if it is an element of the corresponding set.

A similar partition of the set of colors can be achieved by the following lemma.

Lemma 2.30 ([Lun01, Prop. 3.2])**:** *Every color is moved by a unique simple root with the following two exceptions:*

a) $\alpha, \beta \in S^a(X)$ *with* $|\Delta(\alpha) \cup \Delta(\beta)| = 3$.

b) α, β *are orthogonal simple roots with* $\alpha + \beta \in \Sigma(X)$ *or* $\frac{1}{2}(\alpha + \beta) \in \Sigma(X)$. *In this case we have* $\Delta(\alpha) = \Delta(\beta)$ *and* $\alpha, \beta \in S^b$.

Note that if a color is moved by two different simple roots, then these roots are of the same type. So we define the type of a color by the type of the simple root(s) moving it and we denote the corresponding sets by $\Delta^a(X)$, $\Delta^{2a}(X)$ and $\Delta^b(X)$ respectively.

The preceding two lemmas show that the colors of types $2a$ and b as well as their images in $N(X)$ can be recovered from the colors of type a and the set $S^p(X)$. Thus only these are needed for the combinatorial classification.

It turns out that $(\Lambda(X), S^p(X), \Sigma(X), \Delta^a(X))$ forms a homogeneous spherical datum in the sense of the following definition due to Luna (cp. [Tim11]):

Definition 2.31: A collection $(\Lambda, S^p, \Sigma, \Delta^a)$ where Λ is a sublattice of $X(T)$, $S^p \subset S$, $\Sigma \subset \Sigma(G) \cap \Lambda$ is a set of linearly independent set of indivisible vectors in Λ and Δ^a is a set together with a map $\rho : \Delta^a \to \Lambda^*$ satisfying the following axioms is called a *homogeneous spherical datum for G*.

(A1) $\langle \rho(D), \sigma \rangle \leq 1$, $\forall D \in \Delta^a$, $\sigma \in \Sigma$ and equality holds if and only if $\sigma = \alpha \in \Sigma \cap S$ and $D = D_\alpha^\pm$ where $D_\alpha^+, D_\alpha^- \in \Delta^a$ are two distinct elements depending on α.

(A2) $\rho(D_\alpha^+) + \rho(D_\alpha^-) = \alpha^\vee$ on Λ for any $\alpha \in \Sigma \cap S$.

(A3) $\Delta^a = \{D_\alpha^\pm \mid \alpha \in \Sigma \cap S\}$

(Σ1) If $\alpha \in \frac{1}{2}\Sigma \cap S$, then $\langle \alpha^\vee, \Lambda \rangle \subset 2\mathbb{Z}$ and $\langle \alpha^\vee, \Sigma \backslash \{2\alpha\} \rangle \leq 0$.

(Σ2) If $\alpha, \beta \in S$ are two orthogonal roots and $\alpha + \beta \in \Sigma \cup 2\Sigma$, then $\alpha^\vee = \beta^\vee$ on Λ.

(S) $\langle \alpha^\vee, \Lambda \rangle = 0$, $\forall \alpha \in S^p$ and the pair (σ, S^p) comes from a wonderful variety of rank 1 for any $\sigma \in \Sigma$.

A *spherical system* is a homogeneous spherical datum where $\Lambda = \mathbb{Z}\Sigma$ and thus it is represented by a triple (S^p, Σ, Δ^a).

As already stated, wonderful varieties of rank 1 were classified. They are uniquely determined by their spherical root σ and the set $S^p(X)$. So, condition (S) means that there exists a wonderful variety X of rank 1 with spherical root σ and $S^p(X) = S^p$. This condition is equivalent to having the inclusions

$$S^{pp}(\sigma) \subset S^p \subset S^p(\sigma)$$

where $S^p(\sigma)$ denotes the set of simple roots that are orthogonal to σ and $S^{pp}(\sigma)$ is the set of simple roots

- $S^p(\sigma) \cap \text{supp}(\sigma) \backslash \alpha_r$ if $\sigma = \alpha_1 + \ldots + \alpha_r$ has support of type B_r,
- $S^p(\sigma) \cap \text{supp}(\sigma) \backslash \alpha_1$ if $\sigma = \alpha_1 + \ldots + \alpha_r$ has support of type C_r,

– $S^p \cap \operatorname{supp}(\sigma)$ in all other cases

(cp. [BL11]). Here, we say $\alpha_1 + \ldots + \alpha_r$ has support of type B_r (resp. C_r), if the underlying Dynkin diagram of $a_1, \ldots, a_r$ is of type B_r (resp. C_r).

Recently the classification of spherical varieties was completed by the following theorem.

Theorem 2.32: *For any connected reductive group G we have the bijections:*

$$\left\{ \begin{array}{c} \textit{spherical homogeneous} \\ G\textit{-spaces} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{homogeneous spherical} \\ \textit{data for } G \end{array} \right\}$$

$$\{ \ \textit{wonderful } G\textit{-varieties} \ \} \longleftrightarrow \{ \ \textit{spherical systems for } G \ \}$$

This theorem takes it origin in the work of Luna and his paper [Lun01]. Luna also showed that the first bijection follows from the second one and that it suffices to consider groups of adjoint type.

Luna completed the proof for groups having semisimple factors of type A and conjectured that the bijection holds for any connected reductive group (also known as the *Luna conjecture*).

The uniqueness part, i. e. that any wonderful variety is uniquely determined by its spherical system, was proven by Losev in [Los09].

First the conjecture was proven to hold in some special cases. For example it was shown by Bravi and Pezzini that the conjecture holds for groups with simple factors of type A and C or A and D in [Pez03] and [BP05].

Now there are two independent proofs of the whole conjecture. One is also by Bravi and Pezzini who reduced the problem to a subclass of so called primitive spherical systems and showed that any such system is geometrically realizable, i. e. is the spherical system of a wonderful variety (see [BP11a] and [BP11b]).

A second independent proof which takes another approach was achieved by Cupit-Foutou in [CF09].

3 Classification of spherical affine cones and branching rules

From now on G is a complex simply connected simple algebraic groups of exceptional type.

1 Main results and outline of proof

Now we summarize the results and give an outline of the proof. In this chapter we derive the branching rules stated in the following table. Further we show that if $\mathrm{res}^G_H(V_{k\omega_i})$ is given in the table, then $G/P_{\omega_i^*}$ is a spherical H-variety. Conversely, if a maximal reductive subgroup $H \subset G$ does not appear in the table, then the varieties G/P_{ω_i} are not H-spherical.

Note that for the subgroups $D_5 \times \mathbb{C}^* \subset E_6$ and $E_6 \times \mathbb{C}^* \subset E_7$ the weight of the $\mathbb{C}^*$-action depends on the embedding of $\mathbb{C}^*$. The embedding that we chose is given in the corresponding sections.

Table 3.1

G	H	ω	$\mathrm{res}^G_H(V_\omega)$
G_2	A_2	$k\omega_1$	$\bigoplus_{a_1+a_2\leq k} V_{a_1\lambda_1+a_2\lambda_2}$
		$k\omega_2$	$\bigoplus_{a_1+a_2+a_3=k} V_{(a_1+a_3)\lambda_1+(a_2+a_3)\lambda_2}$
F_4	B_4	$k\omega_1$	$\bigoplus_{a_1+a_2=k} V_{a_1\lambda_2+a_2\lambda_4}$
		$k\omega_2$	$\bigoplus_{a_1+\ldots+a_5=k} V_{(a_1+a_2)\lambda_1+(a_3+a_4)\lambda_2+(a_1+a_5)\lambda_3+(a_2+a_4)\lambda_4}$
		$k\omega_3$	$\bigoplus_{a_1+\ldots+a_5=k} V_{(a_1+a_5)\lambda_1+a_2\lambda_2+a_3\lambda_3+(a_4+a_5)\lambda_4}$
		$k\omega_4$	$\bigoplus_{a_1+a_2\leq k} V_{a_1\lambda_1+a_2\lambda_4}$
E_6	$A_5 \times A_1$	$k\omega_1$	$\bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_4} \otimes V_{a_1\lambda_6}$
		$k\omega_6$	$\bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_2+a_2\lambda_4+a_3\lambda_5} \otimes V_{a_3\lambda_6}$
	F_4	$k\omega_1$	$\bigoplus_{a_1\leq k} V_{a_1\lambda_4}$
		$k\omega_2$	$\bigoplus_{a_1+a_2=k} V_{a_1\lambda_1+a_2\lambda_4}$

Table 3.1

G	H	ω	$\mathrm{res}^G_H(V_\omega)$
		$k\omega_3$	$\bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_3+a_3\lambda_4}$
		$k\omega_5$	$\bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_3+a_3\lambda_4}$
		$k\omega_6$	$\bigoplus_{a_1\leq k} V_{a_1\lambda_4}$
	C_4	$k\omega_1$	$\bigoplus_{a_1+2a_2+2a_3=k} V_{a_1\lambda_2+a_2\lambda_4}$
		$k\omega_6$	$\bigoplus_{a_1+2a_2+2a_3=k} V_{a_1\lambda_2+a_2\lambda_4}$
	$D_5\times\mathbb{C}^*$	$k\omega_1$	$\bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_5}\otimes V_{-2a_1+a_2+4a_3}$
		$k\omega_2$	$\bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_2+a_2\lambda_4+a_3\lambda_5}\otimes V_{3a_2-3a_3}$
		$k\omega_3$	$\bigoplus_{a_1+\ldots+a_6=k} V_{(a_1+a_6)\lambda_1+a_2\lambda_2+a_3\lambda_3+a_4\lambda_4+(a_5+a_6)\lambda_5}\otimes V_{2a_1-4a_2+2a_3-a_4+5a_5-a_6}$
		$k\omega_5$	$\bigoplus_{a_1+\ldots+a_6=k} V_{(a_1+a_6)\lambda_1+a_2\lambda_2+a_3\lambda_3+(a_4+a_6)\lambda_4+a_5\lambda_5}\otimes V_{-2a_1+4a_2-2a_3-5a_4+a_5+a_6}$
		$k\omega_6$	$\bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_4}\otimes V_{2a_1-a_2-4a_3}$
E_7	A_7	$k\omega_7$	$\bigoplus_{2a_1+a_2+2a_3+a_4=k} V_{a_2\lambda_2+a_3\lambda_4+a_4\lambda_6}$
	$E_6\times\mathbb{C}^*$	$k\omega_1$	$\bigoplus_{a_1+a_2+a_3\leq k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6}\otimes V_{2a_1-2a_3}$
		$k\omega_2$	$\bigoplus_{a_1+a_2+a_3+2a_4+a_5+a_6+a_7=k} V_{a_1\lambda_1+(a_2+a_7)\lambda_2+a_3\lambda_3+a_4\lambda_4+a_5\lambda_5+a_6\lambda_6}\otimes V_{-a_1+3a_2+a_3-a_5+a_6-3a_7}$
		$k\omega_7$	$\bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_1+a_2\lambda_6}\otimes V_{-a_1+a_2+3a_3-3a_4}$
	$D_6\times A_1$	$k\omega_7$	$\bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6}\otimes V_{a_1\lambda_7}$

To obtain the previous table we shall adapt the proof of Proposition 4.4 in [FL10] by Feigin and Littelmann. But first we introduce some additional notation.

Let $P_i \supset B$ denote the maximal parabolic subgroup of G associated to the fundamental weight ω_i. We shall consider the natural action of H on the projective varieties $Y = G/P_i$. The affine cone over Y is denoted by $\widehat{Y}$ and the stabilizer of $\overline{1} \in G/P_i$ is denoted by $H_{\overline{1}}$. The group $H_{\overline{1}}$ is a parabolic subgroup of H. Its opposite parabolic subgroup in H is denoted by Q. Furthermore let Q^u be its unipotent radical and let L be the Levi-subgroup $H_{\overline{1}} \cap Q$ with Borel subgroup B_L defined by the simple roots of H that appear in L. If we consider the orbit $O = H.\overline{1} \simeq H/H_{\overline{1}}$ with normal bundle $\mathcal{N}$ having fiber N at $\overline{1}$ then N has the structure of an L-module since $L \subset H_{\overline{1}}$.

If no confusion can arise we write P instead of P_i from now on.

The proof is divided into two parts. First we determine in which cases Y is a spherical H-variety. This part of the proof is conducted in four steps.

Step 1: The Borel subgroup B_H is a subgroup of P, it is contained in the stabilizer $H_{\overline{1}}$ of $\overline{1} \in Y$. Thus $H_{\overline{1}}$ is a parabolic subgroup of H and we can apply the Brion-Luna-Vust Local Structure Theorem [BLV86] to get the following proposition. This theorem sates that there exists a locally closed affine subvariety $Z \subset Y$ such that $\overline{1} \in Z$, Z is stable under the action of L, $Q^u.Z$ is open in Y and the canonical map $Q^u \times Z \to Q^u.Z$ is an isomorphism of varieties.

We have the following proposition.

Proposition 3.1: *The variety Y is H-spherical if and only if Z is a spherical L-variety.*

Proof: Assume Z is spherical, i. e. a Borel subgroup of L has a dense orbit in Z. Let B_L be the Borel subgroup $B_H \cap L \subset L$ and let B_L^- be the opposite Borel subgroup. Then $B_H^- = Q^u B_L^-$ is a Borel subgroup of H. Let $z \in Z$ be an element such that $B_L^-.z$ is dense in Z. Since $Q^u.Z$ is dense in Y, so is $B_H^-.z = Q^u(B_L^-.z)$. Hence Y is a spherical H-variety.

If on the other hand Y is H-spherical, then $B_H^-.y = Q^u(B_L^-).y$ is open in Y for some $y \in Y$. Since $Q^u.Z$ is open in Y we can assume that $y \in Z$. Now if $Q^u(B_L^-.y)$ is dense in Y it follows that $B_L^-.y$ is dense in Z. □

Step 2:

Lemma 3.2: *As an L-module the fiber N is isomorphic to $T_{\overline{1}}Z$.*

Proof: We have

$$T_{\overline{1}}(Q^u \times Z) = T_{\overline{1}}Y$$

since $Q^u.Z$ is dense in Y. Furthermore we have

$$T_{\overline{1}}(Q^u \times Z) = \operatorname{Lie} Q^u \oplus T_{\overline{1}}Z$$

and since $T_{\overline{1}}Y = T_{\overline{1}}(H/H_y) \oplus N$ it follows

$$T_{\overline{1}}(H/H_y) \oplus N = \operatorname{Lie} Q^u \oplus T_{\overline{1}}Z.$$

Now $T_{\overline{1}}(H/H_y) \simeq \operatorname{Lie} H/\operatorname{Lie} H_y \simeq \operatorname{Lie} Q^u$ since Q is the parabolic subgroup opposite to H_y and so it follows that $N \simeq T_{\overline{1}}Z$ (as L-module). □

Step 3:

Proposition 3.3: *The action of L on Z is spherical if and only if the action of L on N is spherical.*

Proof: Suppose that the action of L on Z is spherical. Then $\mathbb{C}[Z]^L = \mathbb{C}$. And then by a corollary of Luna's Slice Theorem ([Slo89, Kor. 5.4]) we get that the action of L on Z is isomorphic to the L-module N.

If on the other hand N is L-spherical than $\mathbb{C}[N]^L = \mathbb{C}$ and by Luna's Slice Theorem ([Slo89, §4]) we get that also $\mathbb{C}[Z]^L = \mathbb{C}$ and then we can apply the same arguments to get that N and Z are isomorphic as L-varieties (cp. [Lit94, 2.2]). □

Step 4: It remains to compute N and to check in which cases it is a spherical L-module. Note that we have

$$N \simeq (\mathrm{Lie}\, G/\mathrm{Lie}\, P_i)/(\mathrm{Lie}\, H/\mathrm{Lie}\, H_{\overline{1}}).$$

So if $\Phi_H \subset \Phi$, then we can describe N as the root spaces that occur in $T_{\overline{1}}Y = \mathrm{Lie}\, G/\mathrm{Lie}\, P_i$ but not in $T_{\overline{1}}(H/H_{\overline{1}})$. These are all the root spaces $\mathbb{C}X_\alpha$ such that α is negative and $\mathbb{C}X_\alpha \not\subset \mathrm{Lie}P_i$ as well as $\mathbb{C}X_\alpha \not\subset \mathrm{Lie}\, H$.

The second part is to compute the restrictions of the G-modules $V_{k\omega_i^*}$ to H. It is well-known ([VP72, Thm. 2]) that

$$\mathbb{C}[\widehat{Y}] = \bigoplus_{k\geq 0} V_{k\omega_i^*}$$

where $V_{k\omega_i^*}$ corresponds to the homogeneous functions of degree k on $\widehat{Y}$. In order to derive branching rules for $V_{k\omega_i^*}$ we need to determine the U_H-invariants of $V_{k\omega_i^*}$, where U_H is the unipotent radical of B_H.

Because $\widehat{Y}$ is a spherical $(H \times \mathbb{C}^*)$-variety and because $U_H = U_{H\times\mathbb{C}^*}$, we know from Lemma 1 in [Lit94] that the ring $\mathbb{C}[\widehat{Y}]^{U_H}$ is a polynomial ring with some set of generators f_j of degree d_j, $1 \leq j \leq s$, where s is the number of generators. Thus we have the following branching rules in this situation.

Theorem 3.4: *Let η_j denote the weight of f_j with respect to H and suppose G/P_i is a spherical H-variety. Then we get*

$$\mathrm{res}_H^G(V_{k\omega_i^*}) = \bigoplus_{a_1d_1+\ldots+a_sd_s=k} V_{a_1\eta_1+\ldots+a_s\eta_s}.$$

We need to compute the number of generators, i.e. the dimension of $\mathbb{C}[\widehat{Y}]^{U_H}$. Let U_L be the unipotent radical of $B_L = B \cap L$.

Proposition 3.5: *We have*

$$\dim \mathbb{C}[\widehat{Y}]^{U_H} = \dim N - \dim(\text{generic } U_L\text{-orbit}) + 1.$$

Proof: We know that $\dim \mathbb{C}[\widehat{Y}]^{U_H} = \mathrm{trdeg}\ \mathbb{C}(\widehat{Y})^{U_H}$ and by a theorem of Rosenlicht we know that $\mathrm{trdeg}\ \mathbb{C}(\widehat{Y})^{U_H} = \dim \widehat{Y} - \dim(\text{generic } U_H\text{-orbit})$ (paragraph II.4.3.E in [Kra84, p. 143]).

So the proposition is an immediate corollary of the following lemma. □

Lemma 3.6: *Let Y, N, U_L and U_H be defined as above. Let O_1 be a generic U_H-orbit in Y and O_2 be a generic U_L-orbit in N. Then*

$$\dim Y - \dim O_1 = \dim N - \dim O_2.$$

Proof: Let $O \subset Y$ be the open subset of X such that $\dim U_H.x$ is maximal for all $x \in O$ (i. e. $U_H.x$ is a generic orbit). We have $O \cap Q^u.Z \neq \emptyset$, because $Q^u.Z$ is open and dense in Y.

Let $x = qz$ be an element in $O \cap Q^u.Z$. We know that $U_H = U_L.Q^u = Q^u.U_L$. So we have $U_H.x = U_H.(qz) = U_L Q^u(qz) = U_L Q^u.z = U_H.z$ and we can assume that $U_H.x$ is a generic U_H-orbit in Y with $x \in Z$.

Suppose y is an element of the stabilizer $(U_H)_x$ of x. Then we have $y = q.u$ for some $q \in Q^u$, $u \in U_L$. So it follows from the Local Structure Theorem that $q = \mathrm{id}$ and $ux = x$. Thus we get $(U_H)_x = (U_L)_x$.

With $\dim Y = \dim Z + \dim Q^u$ (Local Structure Theorem) we get

$$\begin{aligned}
\dim Y - \dim U_H.x &= \dim Q^u + \dim Z - \dim U_H.z \\
&= \dim Z - (\dim U_H.x - \dim Q^u) \\
&= \dim Z - (\dim U_H - \dim (U_H)_x - \dim Q^u) \\
&= \dim Z - (\dim U_H - \dim Q^u - \dim (U_L)_x) \\
&= \dim Z - (\dim U_L - \dim (U_L)_x) \\
&= \dim Z - \dim U_L.x.
\end{aligned}$$

□

2 The maximal reductive subgroups of the exceptional groups

We want to list all maximal reductive subgroups of the exceptional algebraic groups. G. Seitz listed all maximal closed connected subgroups in arbitrary characteristics. We recall his results for the case that the ground field is $\mathbb{C}$ ([Sei91], Thm. 1).

Theorem 3.7: *Let G be a simple algebraic group of exceptional type and let X be maximal among the proper closed connected subgroups of G. Then either X contains a maximal torus of G or X is semisimple and the pair (G, X) is given below. Moreover, maximal subgroups of each type exist and are unique up to conjugacy in $Aut(G)$.*

G	X *simple*	X *not simple*
G_2	A_1	
F_4	A_1	$A_1 \times G_2$
E_6	$A_2,\ G_2,\ F_4,\ C_4$	$A_2 \times G_2$
E_7	$A_1,\ A_2$	$A_1 \times A_1,\ A_1 \times G_2,\ A_1 \times F_4,\ G_2 \times C_3$
E_8	$A_1,\ B_2$	$A_1 \times A_2,\ G_2 \times F_4$

Since the maximal subgroups that do not contain a maximal torus are semisimple they are also maximal reductive subgroups of G.

It remains to identify the maximal reductive subgroups that are contained in a maximal subgroup of maximal rank. These groups fall in two categories. Some are the maximal parabolic subgroups of G and the others are so called subsystem subgroups. There is an algorithm (cp. paragraph no. 17 of [Dyn57] or [BdS49]) that determines these subgroups: Start with the Dynkin diagram of G and adjoin the smallest root δ to obtain the extended Dynkin diagram. By removing a node from the extended diagram you arrive at the Dynkin diagram of a subgroup of G. By Theorem 5.5 and the subsequent remark in [Dyn57] these groups are maximal. Since they are semisimple they are also maximal reductive.

To complete the list we need to consider the maximal parabolic subgroups of G. Any reductive subgroup of a parabolic can be assumed to be a subgroup of its Levi factor by Theorem 1 in [LS96]. By considering the Dynkin diagrams it is transparent that the Levi subgroups need not be maximal reductive but can be subgroups of a subsystem subgroup. A simple case by case check shows that there are only two Levi groups, that are maximal reductive.

Summarizing this we have the following maximal reductive subgroups containing a maximal torus.

G	subsystem subgroups	Levi subgroups
G_2	$A_2,\ A_1 \times A_1$	
F_4	$A_1 \times C_3,\ A_2 \times A_2,\ A_3 \times A_1,\ B_4$	
E_6	$A_5 \times A_1,\ A_2 \times A_2 \times A_2$	$D_5 \times \mathbb{C}^*$
E_7	$D_6 \times A_1,\ A_5 \times A_2,\ A_3 \times A_3 \times A_1,\ A_7$	$E_6 \times \mathbb{C}^*$
E_8	$A_1 \times E_7,\ A_2 \times E_6,\ A_3 \times D_5,\ A_4 \times A_4$ $A_5 \times A_2 \times A_1,\ A_7 \times A_1,\ D_8,\ A_8$	

3 The exceptional group of type G_2

In this and each of the following sections let $\mathfrak{b}$ be the Lie algebra of B_L and $\mathfrak{u}$ the Lie algebra of U_L according to the case under consideration.

Now we consider the simply connected simple algebraic group G of type G_2. The long roots of its root system form a subsystem of type A_2 and we consider the subsystem subgroup H obtained in this way. The simple roots of H are given by

$$(1,0)_{A_2} = (3,1) \text{ and } (0,1)_{A_2} = (0,1).$$

Using the same methods as before we can prove:

Theorem 3.8: *The varieties G/P_1 and G/P_2 are H-spherical.*

Proof:
Case G/P_1: We compute

$$L = \langle T, U_{\pm(0,1)} \rangle.$$

and

$$N = \mathbb{C}X_{-(1,0)_{G_2}} \oplus \mathbb{C}X_{-(1,1)_{G_2}} \oplus \mathbb{C}X_{-(2,1)_{G_2}}.$$

If we define $X := X_{-(1,1)} + X_{-(2,1)}$ we have $[\mathfrak{b}, X] = N$, which shows that N is L-spherical. It follows that G/P_1 is a spherical H-variety.

Case G/P_2: In this case we can compute that $L = T$ and

$$N = \mathbb{C}X_{-(1,1)} \oplus \mathbb{C}X_{-(2,1)}.$$

The module N consists of two linearly independent root spaces and since T is 2-dimensional N is obviously L-spherical. That implies that G/P_2 is a spherical H-variety. □

Theorem 3.9: *Let G be of type G_2 and H of type A_2. Then we have the following branching rules:*

$$\text{i)} \quad \mathrm{res}^G_H(V_{k\omega_1}) = \bigoplus_{a_1+a_2 \leq k} V_{a_1\lambda_1 + a_2\lambda_2},$$

$$\text{ii)} \quad \mathrm{res}^G_H(V_{k\omega_2}) = \bigoplus_{a_1+a_2+a_3 = k} V_{(a_1+a_3)\lambda_1 + (a_2+a_3)\lambda_2}.$$

Remark. In G_2 the fundamental weights are self-dual.

Proof: i) We use "LiE" to compute the restriction of V_{ω_1} and get

$$\mathrm{res}^G_H(V_{\omega_1}) = \mathbb{C} \oplus V_{\lambda_1} \oplus V_{\lambda_2}.$$

Let f_0, f_1, f_2 be highest weight vectors of these representations. We need to show that $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by these elements, i. e. we need to show that the dimension of $\mathbb{C}[\widehat{Y}]^{U_H}$ is 3.

By considering $X_{-(1,0)} \in N$ we immediately see that the U_L-orbit of this element is of codimension 2. Thus $\dim \mathbb{C}[\widehat{Y}]^{U_H} = 3$ and since we have already found three algebraically independent elements the branching rules follow immediately.

ii) We use "LiE" to compute

$$\operatorname{res}^G_H(V_{\omega_2}) = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_1+\lambda_2}.$$

Let f_1, f_2, f_3 be highest weight vectors of these modules. We know that L is the maximal torus in this case and so U_L is just the identity. A generic orbit in N is of dimension 0. And since N is 2-dimensional, its codimension is 2. That means a generic U_H-orbit has codimension 3 in $\widehat{Y}$ and that is also the dimension of $\mathbb{C}[\widehat{Y}]^{U_H}$. We have already found three linearly independent elements which form a generating set. The branching rules follow immediately. □

Proposition 3.10: *The varieties G/P_i are not spherical H-varieties if H is any other maximal reductive subgroup of G_2.*

Proof: We have the following maximal reductive subgroups besides A_2: $A_1 \times A_1$ and A_1. If we compute the dimensions of a Borel subgroup in each case and the dimensions of G/P_i we obtain:

	G/P_1	G/P_2
dim	5	5

and

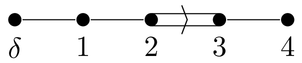
H	$A_1 \times A_1$	A_1
$\dim B_H$	4	2

So $\dim B_H < \dim G/P_i$, $i = 1, 2$ for these subgroups. □

4 The exceptional group of type F_4

In this section let G be the group of type F_4.

Let H be the subgroup of type B_4 in G. This is a subsystem subgroup so from the Dynkin diagram of F_4 we pass on to the extended Dynkin diagram by adding the smallest root δ to the system of simple roots.



By removing the simple root α_4 we obtain a root-subsystem of type B_4 and thus we find the corresponding subgroup $H \subset G$.

Explicitly we can choose the roots

$$(1,0,0,0)_{B_4} = (0,1,2,2), \quad (0,1,0,0)_{B_4} = (1,0,0,0),$$
$$(0,0,1,0)_{B_4} = (0,1,0,0), \quad (0,0,0,1)_{B_4} = (0,0,1,0),$$

which form a set of simple roots of a root subsystem of type B_4 in F_4.

We have the following theorem:

Theorem 3.11: *The varieties G/P_i, $i = 1, \ldots, 4$, are spherical H-varieties.*

Proof: We need to check that N is a spherical L-module in each case.

Case G/P_1: In this case we have

$$L = \langle T, U_{\pm(0,1,2,2)}, U_{\pm(0,1,0,0)}, U_{\pm(0,0,1,0)}, U_{\pm(0,1,1,0)}, U_{\pm(0,1,2,0)} \rangle$$

and

$$N = \mathbb{C}X_{-(1,2,3,1)} \oplus \mathbb{C}X_{-(1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1)}.$$

The Borel subgroup B_L of L obviously contains the maximal torus T of G. Since N consists of four root spaces with linearly independent roots and T is 4-dimensional we know that there is a dense B_L-orbit in N. Hence N is L-spherical and that implies that G/P_1 is H-spherical.

Case G/P_2: Here we have

$$L = \langle T, U_{\pm(1,0,0,0)}, U_{\pm(0,0,1,0)} \rangle.$$

We compute N in the same way as in the previous case and get

$$N = \mathbb{C}X_{-(0,1,1,1)} \oplus \mathbb{C}X_{-(0,1,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \mathbb{C}X_{-(1,2,2,1)} \oplus \mathbb{C}X_{-(1,2,3,1)}.$$

We check the sphericity on the level of Lie algebras. Consider the element

$$X := X_{-(1,1,2,1)} + X_{-(0,1,2,1)} + X_{-(1,1,1,1)} + X_{-(1,2,3,1)}$$

in N. Then $[\mathfrak{b}, X] = N$. That means that N is a spherical L-variety and therefore G/P_2 is a spherical H-variety.

Case G/P_3: We get

$$N = \mathbb{C}X_{-(0,0,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1)} \oplus \mathbb{C}X_{-(0,1,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \mathbb{C}X_{-(1,2,2,1)} \oplus \mathbb{C}X_{-(1,2,3,1)}.$$

If we consider

$$X := X_{-(1,2,3,1)} + X_{-(1,2,2,1)} + X_{-(1,1,1,1)} + X_{-(0,1,2,1)} \in N$$

we have that $[\mathfrak{b}, X] = N$, i.e. N is a spherical L-variety and that means that G/P_3 is a spherical H-variety.

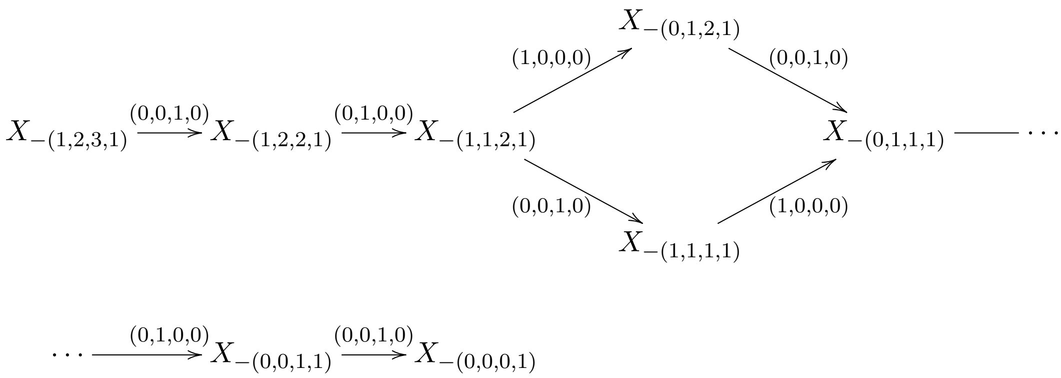
Case G/P_4: In this case we have

$$\begin{aligned} L = \langle T, & U_{\pm(1,0,0,0)}, U_{\pm(0,1,0,0)}, U_{\pm(0,0,1,0)}, \\ & U_{\pm(1,1,0,0)}, U_{\pm(0,1,1,0)}, U_{\pm(1,1,1,0)}, \\ & U_{\pm(0,1,2,0)}, U_{\pm(1,1,2,0)}, U_{\pm(1,2,2,0)} \rangle \end{aligned}$$

and

$$\begin{aligned} N = & \mathbb{C}X_{-(0,0,0,1)} \oplus \mathbb{C}X_{-(0,0,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1)} \oplus \\ & \mathbb{C}X_{-(0,1,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,1)} \oplus \\ & \mathbb{C}X_{-(1,2,2,1)} \oplus \mathbb{C}X_{-(1,2,3,1)}. \end{aligned}$$

The module N has the following structure.



We have $L = \mathbb{C}^* \times SO_7$ and N is an irreducible L-module of dimension 8. There exists only one such module which is the Spin_7-module. That N is a spherical L-module was proven by Victor Kac [Kac80, Thm. 3, p. 208]. It follows that G/P_4 is a spherical H-module. $\square$

The spherical cases imply the following branching rules.

Theorem 3.12: *Let G be of type F_4 and H of type B_4. Then we have the* *following branching rules:*

$$\text{i)}\quad \operatorname{res}^G_H(V_{k\omega_1}) = \bigoplus_{a_1+a_2=k} V_{a_1\lambda_2+a_2\lambda_4},$$

$$\text{ii)}\quad \operatorname{res}^G_H(V_{k\omega_2}) = \bigoplus_{a_1+\ldots+a_5=k} V_{(a_1+a_2)\lambda_1+(a_3+a_4)\lambda_2+(a_1+a_5)\lambda_3+(a_2+a_4)\lambda_4},$$

$$\text{iii)}\quad \operatorname{res}^G_H(V_{k\omega_3}) = \bigoplus_{a_1+\ldots+a_5=k} V_{(a_1+a_5)\lambda_1+a_2\lambda_2+a_3\lambda_3+(a_4+a_5)\lambda_4},$$

$$\text{iv)}\quad \operatorname{res}^G_H(V_{k\omega_4}) = \bigoplus_{a_1+a_2\leq k} V_{a_1\lambda_1+a_2\lambda_4}.$$

Remark. In F_4 the fundamental weights are self-dual.

Proof:
i): Standard computations yield

$$\operatorname{res}^G_H(V_{\omega_1}) = V_{\lambda_2} \oplus V_{\lambda_4}.$$

Let now $f_1, f_2 \in V_{\omega_1}$ be highest weight vectors of V_{λ_2} and V_{λ_4} respectively. We will show that $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by these degree 1 elements. We know that $\mathbb{C}[\widehat{Y}]^{U_H}$ is a polynomial ring. The grading and weights of f_1 and f_2 imply that they are algebraically independent. To rule out the possibility that there are generators of degree two or higher we need to show that the Krull dimension of $\mathbb{C}[\widehat{Y}]^{U_H}$ is 2.

Thus we need to find a generic U_L-orbit in N and compute its codimension. Since we have found 2 algebraically independent elements in $\mathbb{C}[\widehat{Y}]^{U_H}$, we already know that the codimension must be at least 2.

Consider the Lie algebra $\mathfrak{l}$ of L. From above we know that the Lie algebra $\mathfrak{u}$ of U_L, is

$$\mathfrak{u} = \mathbb{C}X_{(0,1,2,2)} \oplus \mathbb{C}X_{(0,1,0,0)} \oplus \mathbb{C}X_{(0,0,1,0)} \oplus \mathbb{C}X_{(0,1,1,0)} \oplus \mathbb{C}X_{(0,1,2,0)}.$$

Define $X := X_{-(1,2,3,1)} \in N$. Then

$$[X_{(0,1,2,2)}, X] = 0, \qquad [X_{(0,1,0,0)}, X] = 0,$$
$$[X_{(0,0,1,0)}, X] = X_{-(1,2,2,1)}, \qquad [X_{(0,1,1,0)}, X] = X_{-(1,1,2,1)},$$
$$[X_{(0,1,2,0)}, X] = X_{-(1,1,1,1)},$$

which shows that the orbit of X is of dimension 3. Thus a generic orbit has dimension at least 3 with codimension at most 1. By Proposition 3.5 we know that in this case $\dim \mathbb{C}[\widehat{Y}]^{U_H} \leq 2$. But since we have found two generators the dimension is exactly 2 and the restriction rules follow.

ii): In this case we need to find generators of $\mathbb{C}[\widehat{Y}]^{U_H}$. One can use the software "LiE" to compute

$$\mathrm{res}_H^G(V_{\omega_2}) = V_{\lambda_1+\lambda_3} \oplus V_{\lambda_1+\lambda_4} \oplus V_{\lambda_2} \oplus V_{\lambda_2+\lambda_4} \oplus V_{\lambda_3}.$$

Let $f_1, \ldots, f_5$ be highest weight vectors of these irreducible modules.

Consider $X := X_{-(1,1,2,1)} + X_{-(1,2,3,1)} \in N$ and let $\mathfrak{u}$ be the Lie-algebra of U_L the unipotent radical of B_L. The stabilizer of this element is just 0, which means that the dimension of a generic U_L-orbit is 2 with codimension 4. This implies that the codimension of a generic U_H-orbit in $\widehat{Y}$ is 5. Thus $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by its degree 1 elements and the assertion follows.

iii): We need to find generators of $\mathbb{C}[\widehat{Y}]^{U_H}$. One can use "LiE" to compute

$$\mathrm{res}_H^G(V_{\omega_3}) = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus V_{\lambda_1+\lambda_4}.$$

Let $f_1, \ldots, f_5$ be highest weight vectors of these irreducible modules.

Consider $X := X_{-(1,1,1,1)} + X_{-(1,2,2,1)} \in N$ and take an element $u \in \mathfrak{u}$ with $u = aX_{(1,0,0,0)} + bX_{(0,1,0,0)} + cX_{(1,1,0,0)}$. Then

$$\begin{aligned} & [u, X] = 0 \\ \Rightarrow \quad & \quad = aX_{-(0,1,1,1)} + bX_{-(1,1,2,1)} + c(X_{-(0,1,2,1)} + X_{-(0,0,1,1)}) \\ \Rightarrow \quad & a = b = c = 0 \Rightarrow u = 0 \end{aligned}$$

and hence a generic U_L-orbit has dimension 3 with codimension 4. That means that $\mathbb{C}[\widehat{Y}]^{U_H}$ is of dimension 5 and generated by the elements f_i.

iv): In this case we need to find generators of $\mathbb{C}[\widehat{Y}]^{U_H}$. We use "LiE" to compute

$$\mathrm{res}_H^G(V_{\omega_4}) = \mathbb{C} \oplus V_{\lambda_1} \oplus V_{\lambda_4}.$$

Let $f_1, \ldots, f_3$ be highest weight vectors of these irreducible modules.

Consider $X := X_{-(1,2,3,1)}$. We know that for

$$X_{(1,0,0,0)}, X_{(0,1,0,0)}, X_{(1,1,0,0)} \in \mathfrak{u}$$

we have

$$[X_{(1,0,0,0)}, X] = [X_{(0,1,0,0)}, X] = [X_{(1,1,0,0)}, X] = 0$$

and

$$\begin{aligned} [X_{(0,0,1,0)}, X] &= X_{-(1,2,2,1)}, & [X_{(0,1,1,0)}, X] &= X_{-(1,1,2,1)}, \\ [X_{(0,1,2,0)}, X] &= X_{-(1,1,1,1)}, & [X_{(1,1,1,0)}, X] &= X_{-(0,1,2,1)}, \\ [X_{(1,1,2,0)}, X] &= X_{-(0,1,1,1)}, & [X_{(1,2,2,0)}, X] &= X_{-(0,0,1,1)} \end{aligned}$$

and thus the generic stabilizer is at most of dimension 3. The generic orbit is at least of dimension 6 and thus its codimension is at most 2. This means that a generic U_H-orbit in $\widehat{Y}$ is of dimension less or equal to 3.

Since we have found 3 algebraically independent elements the dimension of $\mathbb{C}[\widehat{Y}]^{U_H}$ is exactly 3 and this finishes the proof. □

Proposition 3.13: *The varieties G/P_i are not spherical H-varieties if H is any other maximal reductive subgroup of F_4.*

Proof: We have the following maximal reductive subgroups besides B_4: $A_1 \times C_3$, $A_2 \times A_2$, $A_3 \times A_1$, $A_1 \times G_2$ and A_1. If we compute the dimensions of a Borel subgroup in each case and the dimensions of G/P_i we obtain:

	G/P_1	G/P_2	G/P_3	G/P_4
dim	15	20	20	15

H	$A_1 \times C_3$	$A_2 \times A_2$	$A_3 \times A_1$	$A_1 \times G_2$	A_1
$\dim B_H$	14	10	11	10	2

So we have $\dim B_H < \dim G/P_i$ for $i = 1, \ldots, 4$ in each case. □

5 The exceptional group of type E_6

Now we turn to the group of type E_6. First we calculate the dimensions of the Borel subgroups of the maximal reductive subgroups as well as the dimensions of G/P_i for $i = 1, \ldots, 6$.

H	$A_5 \times A_1$	$A_2 \times A_2 \times A_2$	$D_5 \times \mathbb{C}^*$	$A_2 \times G_2$	G_2	A_2	F_4	C_4
$\dim B_H$	22	15	26	13	8	5	28	20

and

	G/P_1	G/P_2	G/P_3	G/P_4	G/P_5	G/P_6
dim	16	21	25	29	25	16

.

Thus we get the following proposition.

Proposition 3.14: *Let G be the simply connected simple algebraic group of type E_6 and let H be a maximal reductive subgroup of type $A_2 \times A_2 \times A_2$, $A_2 \times G_2$, G_2 or A_2.*
Then G/P_i is not H-spherical for $i = 1, \ldots, 6$.

Proof: In these cases we have $\dim B_H < \dim G/P_i$ for $i = 1, \ldots, 6$. □

Now we consider the remaining groups and first we start with the subsystem subgroup of type $A_5 \times A_1$.

Theorem 3.15: *Let G be the simply connected simple algebraic group of type E_6 and let H be the maximal reductive subgroup of type $A_5 \times A_1$. Then G/P_1 and G/P_6 are spherical H-varieties. The varieties $G/P_2, \ldots, G/P_5$ are not H-spherical.*

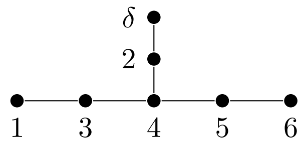
Proof: The dimension of a Borel subgroup of a group of type $A_5 \times A_1$ is 22. Since we have $\dim G/P_3 = 25$, $\dim G/P_4 = 29$, $\dim G/P_5 = 25$ these varieties cannot be spherical.

We know that $\omega_2^* = \omega_2$ in type E_6. Now if G/P_2 was a spherical H-variety, $\mathrm{res}_H^G(V_{k\omega_2})$ would be multiplicity-free for all $k \in \mathbb{N}$ by what has been said above. But with "LiE" we compute

$$\mathrm{res}_H^G(V_{4\omega_2}) = \ldots \oplus 2(V_{2\lambda_3} \otimes V_{3\lambda_6}) \oplus \ldots$$

which means that there are multiplicities in this case.

To prove that G/P_1 and G/P_6 are spherical H-varieties we proceed as in the cases above. We will show how H is embedded in G. For doing so we consider the extended Dynkin diagram of type E_6 again by adding the smallest root δ to the simple roots. Now omitting the root α_2 we obtain the embedding of $A_5 \times A_1$ in E_6.



Explicitly we choose the following set of simple roots:

$$\begin{array}{ll}
(1,0,0,0,0,0)_{A_5\times A_1} = (1,0,0,0,0,0) & (0,1,0,0,0,0)_{A_5\times A_1} = (0,0,1,0,0,0) \\
(0,0,1,0,0,0)_{A_5\times A_1} = (0,0,0,1,0,0) & (0,0,0,1,0,0)_{A_5\times A_1} = (0,0,0,0,1,0) \\
(0,0,0,0,1,0)_{A_5\times A_1} = (0,0,0,0,0,1) & (0,0,0,0,0,1)_{A_5\times A_1} = (1,2,2,3,2,1)
\end{array}$$

Case G/P_1: We compute

$$\begin{aligned} L = \langle T, & U_{\pm(0,0,1,0,0,0)}, U_{\pm(0,0,0,1,0,0)}, U_{\pm(0,0,0,0,1,0)}, U_{\pm(0,0,0,0,0,1)}, \\ & U_{\pm(0,0,1,1,0,0)}, U_{\pm(0,0,0,1,1,0)}, U_{\pm(0,0,0,0,1,1)}, \\ & U_{\pm(0,0,1,1,1,0)}, U_{\pm(0,0,0,1,1,1)}, U_{\pm(0,0,1,1,1,1)} \rangle \end{aligned}$$

and

$$\begin{aligned} N = & \mathbb{C}X_{-(1,1,1,1,0,0)} \oplus \mathbb{C}X_{-(1,1,1,1,1,0)} \oplus \mathbb{C}X_{-(1,1,1,2,1,0)} \oplus \\ & \mathbb{C}X_{-(1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,1,0)} \oplus \mathbb{C}X_{-(1,1,1,2,1,1)} \oplus \\ & \mathbb{C}X_{-(1,1,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,2,2,1)} \oplus \\ & \mathbb{C}X_{-(1,1,2,3,2,1)}. \end{aligned}$$

Now let $X := X_{-(1,1,2,3,2,1)} + X_{-(1,1,1,1,1,1)}$. We have

$$[\mathfrak{h}, X] = \langle X_{-(1,1,2,3,2,1)},\ X_{-(1,1,1,1,1,1)} \rangle,$$

since the roots are linearly independent. Next we compute

$$\begin{array}{ll} [X_{(0,0,0,1,0,0)}, X] = X_{-(1,1,2,2,2,1)} & [X_{(0,0,0,1,1,0)}, X] = X_{-(1,1,2,2,1,1)} \\ [X_{(0,0,1,1,0,0)}, X] = X_{-(1,1,1,2,2,1)} & [X_{(0,0,1,1,1,0)}, X] = X_{-(1,1,1,2,1,1)} \\ [X_{(0,0,0,1,1,1)}, X] = X_{-(1,1,2,2,1,0)} & [X_{(0,0,1,1,1,1)}, X] = X_{-(1,1,1,2,1,0)} \\ [X_{(0,0,0,0,0,1)}, X] = X_{-(1,1,1,1,1,0)} & [X_{(0,0,0,0,1,1)}, X] = X_{-(1,1,1,1,0,0)} \end{array}$$

and these computations show that we have ten linearly independent vectors in $[\mathfrak{b}, X] \Rightarrow [\mathfrak{b}, X] = N \Rightarrow N$ is a spherical L-module. Hence G/P_1 is a spherical H-variety.

Case G/P_6: The H-sphericity of G/P_6 is an immediate corollary of the following theorem which states that $\mathbb{C}[\widehat{Y}]$ is multiplicity free. □

Theorem 3.16: *Let G be the simply connected simple algebraic group of type E_6 and let $H \subset G$ be the maximal reductive subgroup of type $A_5 \times A_1$.*

Then we have the following branching rules:

$$\begin{aligned} &i) \quad \operatorname{res}^G_H(V_{k\omega_6}) = \bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_4} \otimes V_{a_1\lambda_6}, \\ &ii) \quad \operatorname{res}^G_H(V_{k\omega_1}) = \bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_2+a_2\lambda_4+a_3\lambda_5} \otimes V_{a_3\lambda_6}. \end{aligned}$$

Remark. In E_6 we have $\omega_1^* = \omega_6$, $\omega_2^* = \omega_2$, $\omega_3^* = \omega_5$ and $\omega_4^* = \omega_4$.

Proof: ii) With "LiE" we compute

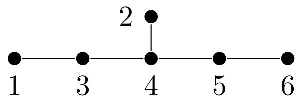
$$\begin{aligned} \operatorname{res}^G_H(V_{\omega_6}) &= (V_{\lambda_2} \otimes \mathbb{C}) \oplus (V_{\lambda_5} \otimes V_{\lambda_6}), \\ \operatorname{res}^G_H(V_{2\omega_6}) &= (V_{2\lambda_2} \otimes \mathbb{C}) \oplus (V_{\lambda_2+\lambda_5} \otimes V_{\lambda_6}) \oplus (V_{2\lambda_5} \otimes V_{2\lambda_6}) \oplus (V_{\lambda_4} \otimes \mathbb{C}). \end{aligned}$$

There are at least two generators of degree 1 and of weights $(\lambda_2, 0)$ and (λ_5, λ_6) and one generator of degree 2 and of weight $(\lambda_4, 0)$ for $\mathbb{C}[\widehat{Y}]^{U_H}$ with $Y = G/P_1$. In the proof of the previous theorem we have found an element $X \in N$ with a U_L-orbit of codimension 2. So it follows that $\dim \mathbb{C}[\widehat{Y}]^{U_H} = 3$ and the branching rules follow immediately.

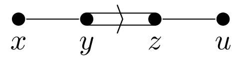
i) Theses branching rules follow directly from ii) by noting that $\lambda_1^* = \lambda_5$, $\lambda_2^* = \lambda_4$ and $\lambda_6^* = \lambda_6$. □

Theorem 3.17: *Let G be the simply connected simple algebraic group of type E_6 and let H be the maximal reductive subgroup of type F_4. Then G/P_i, $i \neq 4$, are spherical H-varieties. The variety G/P_4 is not H-spherical.*

Proof: If we have the Dynkin diagrams



and



of E_6 and F_4, then we have an embedding of the simple Lie-algebra F_4 in E_6 by choosing the following root vectors

$$\begin{aligned} X_x &:= X_{(0,1,0,0,0,0)}, \quad X_z := \frac{1}{\sqrt{2}}(X_{(0,0,1,0,0,0)} + X_{(0,0,0,0,1,0)}) \\ X_y &:= X_{(0,0,0,1,0,0)}, \quad X_u := \frac{1}{\sqrt{2}}(X_{(1,0,0,0,0,0)} + X_{(0,0,0,0,0,1)}) \end{aligned}$$

([Dyn57, p. 258, Table 24] with different numbering of the Dynkin diagrams). Now we consider the associated algebraic subgroup of E_6.

Case G/P_1: We compute

$$N = \mathbb{C}X_{-(1,1,1,2,2,1)}.$$

So N is obviously L-spherical and thus G/P_1 is H-spherical.

Case G/P_6: The H-sphericity of $Y = G/P_6$ is an immediate corollary of the following theorem which states that $\mathbb{C}[\widehat{Y}]$ is multiplicity free.

Case G/P_2: In this case we get

$$\begin{aligned} N =& \mathbb{C}X_{-(0,1,0,1,1,0)} \oplus \mathbb{C}X_{-(0,1,0,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1,1,1)} \oplus \\ & \mathbb{C}X_{-(0,1,1,2,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)_{E_6}}. \end{aligned}$$

If we define $X := X_{-(1,1,1,2,2,1)}$ then we have:

$$\begin{aligned} &[X_{(0,0,0,1)_{F_4}}, X] = X_{-(0,1,1,2,2,1)}, \quad [X_{(0,0,1,1)_{F_4}}, X] = X_{-(0,1,1,2,1,1)}, \\ &[X_{(0,1,1,1)_{F_4}}, X] = X_{-(0,1,1,1,1,1)}, \quad [X_{(0,1,2,1)_{F_4}}, X] = X_{-(0,1,0,1,1,1)}, \\ &[X_{(0,1,2,2)_{F_4}}, X] = X_{-(0,1,0,1,1,0)}. \end{aligned}$$

With $[\mathfrak{h}, X] = \mathbb{C}X$ we get $[\mathfrak{b}, X] = N$ and it follows that N is a spherical L-module.

Case G/P_3: In this case we get

$$\begin{aligned} N =& \mathbb{C}X_{-(0,0,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,1,1)} \oplus \\ & \mathbb{C}X_{-(0,1,1,2,2,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1)}. \end{aligned}$$

Set $X := X_{-(1,1,1,2,2,1)} + X_{-(0,1,1,2,1,1)}$. Then we have

$$[\mathfrak{h}, X] = \mathbb{C}X_{-(1,1,1,2,2,1)} \oplus \mathbb{C}X_{-(0,1,1,2,1,1)},$$

since the roots of the root vectors defining X are linearly independent. Furthermore we have

$$[X_{(0,0,0,1)_{F_4}}, X] = X_{-(0,1,1,2,2,1)}, \quad [X_{(1,0,0,0)_{F_4}}, X] = X_{-(0,1,1,1,1,1)},$$
$$[X_{(1,1,0,0)_{F_4}}, X] = X_{-(0,0,1,1,1,1)}.$$

So $[\mathfrak{b}, X] = N \Rightarrow N$ is a spherical L-module and this implies that G/P_3 is H-spherical.

Case G/P_5: The H-sphericity of $Y = G/P_5$ is an immediate corollary of the following theorem which states that $\mathbb{C}[\widehat{Y}]$ is multiplicity free. □

We can derive branching rules in the cases where G/P_i is a spherical H-variety.

Theorem 3.18: *Let G be the simple simply connected algebraic group of type E_6 and H be the subgroup of type F_4.*

Then we have the branching rules:

$$\begin{aligned}
i)\quad \mathrm{res}^G_H(V_{k\omega_1}) &= \bigoplus_{a_1 \leq k} V_{a_1\lambda_4}, \\
ii)\quad \mathrm{res}^G_H(V_{k\omega_2}) &= \bigoplus_{a_1+a_2=k} V_{a_1\lambda_1+a_2\lambda_4}, \\
iii)\quad \mathrm{res}^G_H(V_{k\omega_3}) &= \bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_3+a_3\lambda_4}, \\
iv)\quad \mathrm{res}^G_H(V_{k\omega_5}) &= \bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_3+a_3\lambda_4}, \\
v)\quad \mathrm{res}^G_H(V_{k\omega_6}) &= \bigoplus_{a_1 \leq k} V_{a_1\lambda_4}.
\end{aligned}$$

Proof: v) In this case we work with $Y = G/P_1$. With “LiE” we compute

$$\mathrm{res}^G_H(V_{\omega_6}) = \mathbb{C} \oplus V_{\lambda_4}.$$

Since N is 1-dimensional in this case, each U_L-orbit is 0-dimensional with codimension 1. So $\dim \mathbb{C}[\widehat{Y}]^{U_H} = 2$ and $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by its degree-1-elements. The branching rules follow.

i) Theses branching rules follow directly from v) by noting that $\omega_1 = \omega_6^*$ and $\lambda_i^* = \lambda_i$.

ii) In this case we work with $Y = G/P_2$. With “LiE” we compute

$$\mathrm{res}^G_H(V_{\omega_2}) = V_{\lambda_1} \oplus V_{\lambda_4},$$

so there are two generators of degree 1. The module N is of dimension 6 and we have seen that $X_{-(1,1,1,2,2,1)} \in N$ is an element such that $U_L.X$ is of dimension 5. So $\dim \mathbb{C}[\widehat{Y}]^{U_H} \leq 2$ and hence $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by its degree-1-elements. The branching rules follow immediately.

iv) In this case we work with G/P_3. With "LiE" we compute

$$\mathrm{res}_H^G(V_{\omega_5}) = V_{\lambda_1} \oplus V_{\lambda_3} \oplus V_{\lambda_4},$$

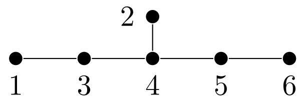
so again there are 3 generators of degree 1. The module N is of dimension 5 and $X_{-(1,1,1,2,2,1)_{E_6}} + X_{-(0,1,1,2,1,1)_{E_6}}$ is an element of N with a 3-dimensional U_L-orbit (cp. proof of previous theorem). So $\dim \mathbb{C}[\widehat{Y}]^{U_H} \leq 3$. It follows that $\mathbb{C}[\widehat{Y}]^{U_H}$ is generated by its degree-1-elements and so the branching rules follow.

iii) These branching rules follow directly from v) by noting that $\omega_3 = \omega_5^*$ and $\lambda_i^* = \lambda_i$. □

Theorem 3.19: *Let G be the simply connected simple algebraic group of type E_6 and let H be the maximal reductive subgroup of type C_4. Then G/P_1 and G/P_6 are spherical H-varieties. The varieties $G/P_2, \dots, G/P_5$ are not H-spherical.*

Proof: That $G/P_2, \dots, G/P_5$ are not H-spherical follows by dimension reasons.

For the other two cases we consider the Dynkin diagrams



and

of E_6 and C_4 respectively. Then the simple Lie-algebra of type C_4 is embedded into the simple Lie-algebra of type E_6 by choosing the following root vectors:

$$X_x := \frac{1}{\sqrt{2}}(X_{(0,1,1,1,0,0)} + X_{(0,1,0,1,1,0)}), \quad X_y := \frac{1}{\sqrt{2}}(X_{(1,0,0,0,0,0)} + X_{(0,0,0,0,0,1)})$$
$$X_z := \frac{1}{\sqrt{2}}(X_{(0,0,1,0,0,0)} + X_{-(0,0,0,0,1,0)}), \; X_u := X_{(0,0,0,1,0,0)}$$

(cp. [Dyn57, p. 258, Table 24]). Now we consider the associated subgroup H of G.

Case G/P_1: We compute

$$N = \mathbb{C}X_{-(1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,2,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,1,1)} \oplus$$
$$\mathbb{C}X_{-(1,1,2,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,3,2,1)}.$$

We define $X := X_{-(1,1,2,3,2,1)} + X_{-(1,1,1,1,1,1)}$. Then we have

$$[\mathfrak{h}, X] = \mathbb{C}X_{-(1,1,2,3,2,1)} \oplus \mathbb{C}X_{-(1,1,1,1,1,1)}.$$

Further we get

$$[X_{(0,0,0,1)_{C_4}}, X] = X_{-(1,1,2,2,2,1)}, \quad [X_{(0,0,1,1)_{C_4}}, X] = X_{-(1,1,2,2,1,1)}$$
$$[X_{(0,0,2,1)_{C_4}}, X] = X_{-(1,1,1,2,1,1)}.$$

This implies that $[\mathfrak{b}, X]$ contains five linearly independent vectors of $N \Rightarrow [\mathfrak{b}, X] = N$. Hence N is L-spherical.

Case G/P_6: The H-sphericity of $Y = G/P_6$ is an immediate corollary of the following theorem which states that $\mathbb{C}[\widehat{Y}]$ is multiplicity free. □

From the spherical cases we can derive the following branching rules:

Theorem 3.20: *Let G be the simply connected simple algebraic group of type E_6 and H be the subgroup of type C_4.*

Then we have the following branching rules:

$$\text{i)}\quad \operatorname{res}^G_H(V_{k\omega_1}) = \bigoplus_{a_1+2a_2+2a_3=k} V_{a_1\lambda_2+a_2\lambda_4},$$
$$\text{ii)}\quad \operatorname{res}^G_H(V_{k\omega_6}) = \bigoplus_{a_1+2a_2+2a_3=k} V_{a_1\lambda_2+a_2\lambda_4}.$$

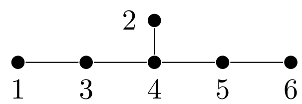
Proof:
ii) Here we are in the case $Y = G/P_1$. With "LiE" we compute

$$\operatorname{res}^G_H(V_{\omega_6}) = V_{\lambda_2} \quad \text{and} \quad \operatorname{res}^G_H(V_{2\omega_6}) = \mathbb{C} \oplus V_{2\lambda_2} \oplus V_{\lambda_4}.$$

So there is one generator of degree 1 and two of degree 2 in $\mathbb{C}[\widehat{Y}]^{U_H}$. From the calculations in the proof of the previous theorem we know that $X_{-(1,1,2,3,2,1)} + X_{-(1,1,1,1,1,1)}$ is an element of N whose U_L-orbit is of codimension 2. Hence $\dim \mathbb{C}[\widehat{Y}]^{U_H} \leq 3$. But since we have already found three generators we know that $\dim \mathbb{C}[\widehat{Y}]^{U_H} = 3$. The branching rules follow immediately.

i) Theses branching rules follow directly from ii) by noting that $\omega_1 = \omega_6^*$, $\lambda_2^* = \lambda_2$ and $\lambda_4^* = \lambda_4$. □

Next we consider the Levi subgroup H of G that is obtained by omitting the simple root α_1. From the Dynkin diagram of E_6 we see that H is the group $D_5 \times \mathbb{C}^*$.



Theorem 3.21: *Let G be the simply connected simple algebraic group of type E_6 and let H be the Levi subgroup $D_5 \times \mathbb{C}^*$.*
Then G/P_i is a spherical H-variety for $i \neq 4$. The variety G/P_4 is not H-spherical.

Proof: This is proven in [Lit94]. □

Theorem 3.22: *Let G be the simply connected simple algebraic groups of type E_6 let $H \subset G$ be the Levi subgroup $D_5 \times \mathbb{C}^*$. Then we have the following* *branching rules.*

$$\begin{aligned}
&\textit{i)} \quad \mathrm{res}^G_H(V_{k\omega_1}) = \bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_5} \otimes V_{-2a_1+a_2+4a_3},\\
&\textit{ii)} \quad \mathrm{res}^G_H(V_{k\omega_2}) = \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_2+a_2\lambda_4+a_3\lambda_5} \otimes V_{3a_2-3a_3},\\
&\textit{iii)} \quad \mathrm{res}^G_H(V_{k\omega_3}) = \bigoplus_{a_1+\ldots+a_6=k} \begin{matrix} V_{(a_1+a_6)\lambda_1+a_2\lambda_2+a_3\lambda_3+a_4\lambda_4+(a_5+a_6)\lambda_5} \otimes \\ V_{2a_1-4a_2+2a_3-a_4+5a_5-a_6} \end{matrix},\\
&\textit{iv)} \quad \mathrm{res}^G_H(V_{k\omega_5}) = \bigoplus_{a_1+\ldots+a_6=k} \begin{matrix} V_{(a_1+a_6)\lambda_1+a_2\lambda_2+a_3\lambda_3+(a_4+a_6)\lambda_4+a_5\lambda_5} \otimes \\ V_{-2a_1+4a_2-2a_3-5a_4+a_5+a_6} \end{matrix},\\
&\textit{v)} \quad \mathrm{res}^G_H(V_{k\omega_6}) = \bigoplus_{a_1+a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_4} \otimes V_{2a_1-a_2-4a_3}.
\end{aligned}$$

Proof: From paragraph 1.4 in [Lit94] we get the following branching rules.

$$\begin{aligned}
&\text{i)} \quad \mathrm{res}^G_H(V_{k\omega_1}) = \bigoplus_{a_1+a_2+a_3=k} V_{(a_3-a_1-a_2)\omega_1+a_2\omega_3+a_1\omega_6},\\
&\text{ii)} \quad \mathrm{res}^G_H(V_{k\omega_2}) = \bigoplus_{a_1+a_2+a_3+a_4=k} V_{-(a_1+2a_2)\omega_1+a_3\omega_2+a_2\omega_3+a_1\omega_5},\\
&\text{iii)} \quad \mathrm{res}^G_H(V_{k\omega_3}) = \bigoplus_{a_1+\ldots+a_6=k} V_{\substack{-(2a_2+a_3+a_5+2a_6)\omega_1+a_5\omega_2+(a_4+a_6)\omega_3\\+a_3\omega_4+a_2\omega_5+(a_1+a_6)\omega_6}},\\
&\text{iv)} \quad \mathrm{res}^G_H(V_{k\omega_5}) = \bigoplus_{a_1+\ldots+a_6=k} V_{\substack{-(a_1+2a_3+a_4+2a_5+a_6)\omega_1+(a_5+a_6)\omega_2+a_4\omega_3\\+a_3\omega_4+a_2\omega_5+(a_1+a_6)\omega_6}},\\
&\text{v)} \quad \mathrm{res}^G_H(V_{k\omega_6}) = \bigoplus_{a_1+a_2+a_3=k} V_{-(a_2+a_3)\omega_1+a_2\omega_2+a_1\omega_6}.
\end{aligned}$$

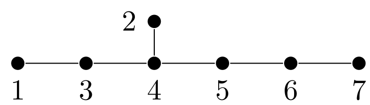
We fix the embedding of $\mathbb{C}^*$ by the coweight $3\omega_1^\vee = 4\alpha_1^\vee + 3\alpha_2^\vee + 5\alpha_3^\vee + 6\alpha_4^\vee + 4\alpha_5^\vee + 2\alpha_6^\vee$. Then the fundamental weights are

$$\begin{aligned}
\lambda_1 &= \omega_6 - \frac{1}{2}\omega_1, & \lambda_2 &= \omega_5 - \omega_1,\\
\lambda_3 &= \omega_4 - \frac{3}{2}\omega_1, & \lambda_4 &= \omega_2 - \frac{3}{4}\omega_1,\\
\lambda_5 &= \omega_3 - \frac{5}{4}\omega_1, & \lambda_6 &= \frac{1}{4}\omega_1.
\end{aligned}$$

Thus we get the branching rules in the theorem. □

6 The exceptional group of type E_7

Let G be of type E_7 with the following Dynkin Diagram.



For this group there are only a few cases of sphericity as we will see. As we did in the last section we start by calculating the dimensions of the Borel subgroups of the maximal reductive subgroups as well as the dimensions of G/P_i for $i = 1, \ldots, 7$.

We have

	G/P_1	G/P_2	G/P_3	G/P_4	G/P_5	G/P_6	G/P_7
dim	33	42	47	53	50	42	27

.

For the Borel subgroups B_H we have:

H	A_7	$E_6 \times \mathbb{C}^*$	$A_3 \times A_3 \times A_1$	$A_5 \times A_2$	$D_6 \times A_1$	$A_1 \times A_1$
$\dim B_H$	35	43	20	25	38	4

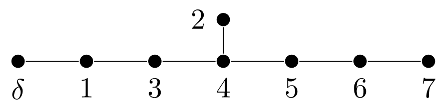
H	$A_1 \times G_2$	$G_2 \times C_3$	$A_1 \times F_4$	A_1	A_2
$\dim B_H$	10	20	30	2	5

So we can rule out a lot of cases by dimension comparison.

Proposition 3.23: *Let G be the simply connected simple algebraic group of type E_7. If H is a maximal reductive subgroup of type $A_3 \times A_3 \times A_1$, $A_5 \times A_2$, $A_1 \times A_1$, $A_1 \times G_2$, $G_2 \times C_3$, A_1 or A_2, then G/P_i is not a spherical H-variety for $i = 1, \ldots, 7$.* □

Proof: In these cases we have $\dim B_H < \dim G/P_i$ for $i = 1, \ldots, 7$. □

Now we turn to the remaining subgroups and start with the subgroup of type A_7. This is a subsystem subgroup so we add the smallest root δ to the simple roots and consider the extended Dynkin diagram.



By omitting the simple root α_2 we obtain the embedding of the root system A_7 into E_7. Explicitly we get

$$(1,0,0,0,0,0,0)_{A_7} = (1,0,0,0,0,0,0),\ (0,1,0,0,0,0,0)_{A_7} = (0,0,1,0,0,0,0),$$
$$(0,0,1,0,0,0,0)_{A_7} = (0,0,0,1,0,0,0),\ (0,0,0,1,0,0,0)_{A_7} = (0,0,0,0,1,0,0),$$
$$(0,0,0,0,1,0,0)_{A_7} = (0,0,0,0,0,1,0),\ (0,0,0,0,0,1,0)_{A_7} = (0,0,0,0,0,0,1),$$
$$(0,0,0,0,0,0,1)_{A_7} = (1,2,2,3,2,1,0).$$

Now we consider the corresponding subsystem subgroup H.

Theorem 3.24: *Let G be the simply connected simple algebraic group of type E_7 and H the maximal reductive subgroup of type A_7. Then G/P_7 is a spherical H-variety whereas G/P_i is not H-spherical for $i \neq 7$.*

Proof: By dimension comparison G/P_i can only be spherical for $i = 1$ or $i = 7$. We know that for E_7 we have $\omega_i^* = \omega_i$. And with LiE we compute

$$\operatorname{res}_H^G(V_{4\omega_1}) = \ldots \oplus 2V_{\lambda_4} \oplus \ldots .$$

This shows that we have multiplicities in this case and G/P_1 is not a spherical H-variety.

For G/P_7 we use the same methods as above. We compute

$$\begin{aligned} N = &\mathbb{C}X_{-(0,1,0,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,1,1,1,1)} \oplus \\ &\mathbb{C}X_{-(0,1,1,2,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,2,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1,1)} \oplus \\ &\mathbb{C}X_{-(1,1,2,2,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,2,1)} \oplus \\ &\mathbb{C}X_{-(1,1,2,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,1,2,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,3,2,1,1)} \oplus \\ &\mathbb{C}X_{-(1,1,2,2,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,3,2,2,1)} \oplus \mathbb{C}X_{-(1,1,2,3,3,2,1)}. \end{aligned}$$

Define $X := X_{-(1,1,2,3,3,2,1)} + X_{-(1,1,2,2,1,1,1)} + X_{-(0,1,0,1,1,1,1)}$. The roots of the root-vectors in X are linearly independent. Thus we get that

$$[\mathfrak{h}, X] := \langle X_{-(1,1,2,3,3,2,1)_{E_7}}, X_{-(1,1,2,2,1,1,1)_{E_7}}, X_{-(0,1,0,1,1,1,1)_{E_7}} \rangle$$

and further

$$\begin{aligned} &[X_{(0,0,1,0,0,0,0)}, X] = X_{-(1,1,1,2,1,1,1)}, && [X_{(0,0,0,0,1,0,0)}, X] = X_{-(1,1,2,3,2,2,1)}, \\ &[X_{(1,0,1,0,0,0,0)}, X] = X_{-(0,1,1,2,1,1,1)}, && [X_{(0,0,1,1,0,0,0)}, X] = X_{-(1,1,1,1,1,1,1)}, \\ &[X_{(0,0,0,1,1,0,0)}, X] = X_{-(1,1,2,2,2,2,1)}, && [X_{(0,0,0,0,1,1,0)}, X] = X_{-(1,1,2,3,2,1,1)}, \\ &[X_{(1,0,1,1,0,0,0)}, X] = X_{-(0,1,1,1,1,1,1)}, && [X_{(0,0,1,1,1,0,0)}, X] = X_{-(1,1,1,2,2,2,1)}, \\ &[X_{(0,0,0,1,1,1,0)}, X] = X_{-(1,1,2,2,2,1,1)}, && [X_{(1,0,1,1,1,0,0)}, X] = X_{-(0,1,1,2,2,2,1)}, \\ &[X_{(0,0,1,1,1,1,0)}, X] = X_{-(1,1,1,2,2,1,1)}, && [X_{(1,0,1,1,1,1,0)}, X] = X_{-(0,1,1,2,2,1,1)}. \end{aligned}$$

This shows that $\dim[\mathfrak{b}, X] = 15 = \dim N \Rightarrow [\mathfrak{b}, X] = N \Rightarrow N$ is a spherical L-module. And thus G/P_7 is a spherical H-variety. □

Since G/P_7 is a spherical H-variety we can derive branching rules for $V_{k\omega_7^*} = V_{k\omega_7}$.

Theorem 3.25: *Let G be the simply connected simple algebraic group of type E_7 and H the maximal reductive subgroup of type A_7. Then*

$$\operatorname{res}_H^G(V_{k\omega_7}) = \bigoplus_{2a_1+a_2+2a_3+a_4=k} V_{a_2\lambda_2+a_3\lambda_4+a_4\lambda_6}.$$

Proof: With "LiE" we compute

$$\mathrm{res}_H^G(V_{\omega_7}) = V_{\lambda_2} \oplus V_{\lambda_6}.$$

So there are two generators of degree 1 of weight λ_2 and λ_6. Further we have

$$\mathrm{res}_H^G(V_{2\omega_7}) = \mathbb{C} \oplus V_{2\lambda_2} \oplus V_{2\lambda_6} \oplus V_{\lambda_2+\lambda_6} \oplus V_{\lambda_4},$$

which shows that there are 2 generators of degree 2 which are of weight 0 and λ_4. This shows that $\dim \mathbb{C}[\widehat{Y}]^{U_H} \geq 4$.

In the proof of the previous theorem we have found an $X \in N$ such that $U_L.X$ is of codimension 3. It follows that $\dim \mathbb{C}[\widehat{Y}]^{U_H} = 4$ and we have found four generators. The branching rules follow immediately. □

Next we will consider the Levi subgroup $E_6 \times \mathbb{C}^*$, which is obtained by omitting the simple root α_7 in the Dynkin diagram.

Theorem 3.26: *Let G be the simply connected simple algebraic group of type E_7 and $H \subset G$ the Levi subgroup of type $E_6 \times \mathbb{C}^*$. Then G/P_1 and G/P_7 are spherical H-varieties whereas G/P_i, $i = 2, \ldots, 6$ are not spherical H-varieties.*

Proof: This was proven in [Lit94]. □

We get the following branching rules from the spherical cases.

Theorem 3.27: *Let G be the simply connected simple algebraic group of type E_7 and H the Levi subgroup of type $E_6 \times \mathbb{C}^*$. Then we have the following branching rules.*

i) $$\mathrm{res}_H^G(V_{k\omega_1}) = \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6} \otimes V_{2a_1-2a_3},$$

ii) $$\mathrm{res}_H^G(V_{k\omega_2}) = \bigoplus_{\substack{a_1+a_2+a_3+2a_4+\\ a_5+a_6+a_7=k}} V_{a_1\lambda_1+(a_2+a_7)\lambda_2+a_3\lambda_3+a_4\lambda_4+a_5\lambda_5+a_6\lambda_6} \otimes V_{-a_1+3a_2+a_3-a_5+a_6-3a_7},$$

iii) $$\mathrm{res}_H^G(V_{k\omega_7}) = \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\lambda_1+a_2\lambda_6} \otimes V_{-a_1+a_2+3a_3-3a_4}.$$

Proof: From paragraph 1.4 in [Lit94] we get the following branching rules.

i) $$\mathrm{res}_H^G(V_{k\omega_1}) = \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\omega_1+a_2\omega_2+a_3\omega_6-(a_2+2a_3)\omega_7},$$

ii) $$\mathrm{res}_H^G(V_{k\omega_2}) = \bigoplus_{\substack{a_1+a_2+a_3+2a_4+\\ a_5+a_6+a_7=k}} V_{a_1\omega_1+(a_2+a_7)\omega_2+a_3\omega_3+a_4\omega_4+a_5\omega_5+a_6\omega_6-(a_1+a_3+2a_4+2a_5+a_6+2a_7)\omega_7},$$

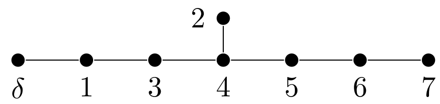
iii) $$\mathrm{res}_H^G(V_{k\omega_7}) = \bigoplus_{a_1+a_2+a_3+a_4=k} V_{a_1\omega_1+a_2\omega_6+(a_3-a_1-a_2-a_4)\omega_7}.$$

We fix the embedding of $\mathbb{C}^*$ by the coweight $2\omega_7^\vee = 2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 6\alpha_4^\vee + 5\alpha_5^\vee + 4\alpha_6^\vee + 3\alpha_7^\vee$. Then the fundamental weights are

$$\begin{aligned} \lambda_1 &= \omega_1 - \frac{2}{3}\omega_7, & \lambda_2 &= \omega_2 - \omega_7, \\ \lambda_3 &= \omega_3 - \frac{4}{3}\omega_7, & \lambda_4 &= \omega_4 - 2\omega_7, \\ \lambda_5 &= \omega_5 - \frac{5}{3}\omega_7, & \lambda_6 &= \omega_6 - \frac{4}{3}\omega_7, \\ \lambda_7 &= \frac{1}{3}\omega_7. \end{aligned}$$

Thus we get the branching rules in the theorem. □

Now we will turn to the subgroup of E_7 of type $D_6 \times A_1$. We will consider the extended Dynkin diagram of E_7 again by adding the smallest root δ to the simple roots.



If we omit the simple root α_6 we have a sub-diagram of type $D_6 \times A_1$ and consider the the corresponding subsystem subgroup. Explicitly we can choose the following simple roots:

$$\begin{aligned} &(1,0,0,0,0,0,0)_H = (0,1,1,2,2,2,1), \ (0,1,0,0,0,0,0)_H = (1,0,0,0,0,0,0), \\ &(0,0,1,0,0,0,0)_H = (0,0,1,0,0,0,0), \ (0,0,0,1,0,0,0)_H = (0,0,0,1,0,0,0), \\ &(0,0,0,0,1,0,0)_H = (0,1,0,0,0,0,0), \ (0,0,0,0,0,1,0)_H = (0,0,0,0,1,0,0), \\ &(0,0,0,0,0,0,1)_H = (0,0,0,0,0,0,1). \end{aligned}$$

Theorem 3.28: *Let G be the simply connected simple algebraic group of type E_7. If H is the subgroup of type $D_6 \times A_1$ then G/P_7 is a spherical H-variety and G/P_i is not a spherical H-variety for $i = 1, \dots, 6$.*

Proof: Dimension comparison shows that $G/P_2, \dots, G/P_6$ are not H-spherical. For G/P_1 we can compute the restriction of $V_{k\omega_1}$ (note that $\omega_i^* = \omega_i$ for E_7) with LiE and get

$$\mathrm{res}^G_H(V_{4\omega_1}) = \ldots \oplus 2(V_{2\lambda_6} \otimes V_{2\lambda_7}) \oplus \ldots \ .$$

Thus there are multiplicities in this case and we know that the H-variety G/P_1 is not H-spherical.

Case G/P_7: We compute

$$N = \mathbb{C}X_{-(0,0,0,0,0,1,1)} \oplus \mathbb{C}X_{-(0,0,0,0,1,1,1)} \oplus \mathbb{C}X_{-(0,0,0,1,1,1,1)} \oplus$$

$$\begin{aligned}&\mathbb{C}X_{-(0,1,0,1,1,1,1)} \oplus \mathbb{C}X_{-(0,0,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,0,1,1,1,1,1)} \oplus \\ &\mathbb{C}X_{-(0,1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(1,1,1,1,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,1,1,1)} \oplus \\ &\mathbb{C}X_{-(1,1,1,2,1,1,1)} \oplus \mathbb{C}X_{-(0,1,1,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,1,1,1)} \oplus \\ &\mathbb{C}X_{-(1,1,1,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,2,2,2,1,1)} \oplus \mathbb{C}X_{-(1,1,2,3,2,1,1)} \oplus \\ &\mathbb{C}X_{-(1,2,2,3,2,1,1)}.\end{aligned}$$

Now define $X := X_{-(1,2,2,3,2,1,1)} + X_{-(1,0,1,1,1,1,1)}$. The roots of these two root vectors are linearly independent and we have

$$[\mathfrak{h}, X] = \langle X_{-(1,2,2,3,2,1,1)}, X_{-(1,0,1,1,1,1,1)} \rangle$$

Further we have

$$\begin{aligned}
&[X_{(1,0,0,0,0,0,0)}, X] = X_{-(0,0,1,1,1,1,1)}, && [X_{(0,1,0,0,0,0,0)}, X] = X_{-(1,1,2,3,2,1,1)}, \\
&[X_{(1,0,1,0,0,0,0)}, X] = X_{-(0,0,0,1,1,1,1)}, && [X_{(0,1,0,1,0,0,0)}, X] = X_{-(1,1,2,2,2,1,1)}, \\
&[X_{(1,0,1,1,0,0,0)}, X] = X_{-(0,0,0,0,1,1,1)}, && [X_{(0,1,1,1,0,0,0)}, X] = X_{-(1,1,1,2,2,1,1)}, \\
&[X_{(0,1,0,1,1,0,0)}, X] = X_{-(1,1,2,2,1,1,1)}, && [X_{(1,1,1,1,0,0,0)}, X] = X_{-(0,1,1,2,2,1,1)}, \\
&[X_{(1,0,1,1,1,0,0)}, X] = X_{-(0,0,0,0,0,1,1)}, && [X_{(0,1,1,1,1,0,0)}, X] = X_{-(1,1,1,2,1,1,1)}, \\
&[X_{(1,1,1,1,1,0,0)}, X] = X_{-(0,1,1,2,1,1,1)}, && [X_{(0,1,1,2,1,0,0)}, X] = X_{-(1,1,1,1,1,1,1)}, \\
&[X_{(1,1,1,2,1,0,0)}, X] = X_{-(0,1,1,1,1,1,1)}, && [X_{(1,1,2,2,1,0,0)}, X] = X_{-(0,1,0,1,1,1,1)}.
\end{aligned}$$

So we have $\dim[\mathfrak{b}, X] = 16 = \dim N$. This implies that N is a spherical L-module and thus G/P_7 is a spherical H-variety. □

From the sphericity of G/P_7 we can derive branching rules for $V_{k\omega_7^*} = V_{k\omega_7}$.

Theorem 3.29: *Let G be the simply connected simple algebraic group of type E_7 and let H be a maximal reductive subgroup of type $D_6 \times A_1$. Then*

$$\operatorname{res}^G_H(V_{k\omega_7}) = \bigoplus_{a_1+2a_2+a_3=k} V_{a_1\lambda_1+a_2\lambda_2+a_3\lambda_6} \otimes V_{a_1\lambda_7}.$$

Proof: With “LiE” we compute

$$\operatorname{res}^G_H(V_{\omega_7}) = (V_{\lambda_1} \otimes V_{\lambda_7}) \oplus (V_{\lambda_6} \otimes \mathbb{C}).$$

So there are two generators of degree 1 with weights (λ_1, λ_7) and $(\lambda_6, 0)$. Further we have

$$\operatorname{res}^G_H(V_{2\omega_7}) = (V_{2\lambda_1} \otimes V_{2\lambda_7}) \oplus (V_{\lambda_1+\lambda_6} \otimes V_{\lambda_7}) \oplus (V_{2\lambda_6} \otimes \mathbb{C}) \oplus (V_{\lambda_2} \otimes \mathbb{C}).$$

Thus there is a further generator of degree 2 and weight λ_2 and therefore we know that $\dim \mathbb{C}[\widehat{Y}]^{U_H} \geq 3$.

In the proof of the previous theorem we have seen that there is an $X \in N$ such that $\dim U_H.X$ is of codimension 2. It follows that $\dim \mathbb{C}[\widehat{Y}]^{U_H} = 3$. The branching rules follow. □

The last maximal reductive subgroup of G where a sphericity of G/P_i can occur is the group H of type $A_1 \times F_4$. From the table with dimensions of G/P_i we know that only G/P_7 can be a spherical H-variety. But with LiE we compute

$$\mathrm{res}_H^G(V_{4\omega_7}) = \ldots \oplus 2(V_{4\lambda_1} \otimes V_{\lambda_5}) \oplus \ldots$$

and thus there are multiplicities in this case. We have shown:

Theorem 3.30: *Let G be the simply connected simple group of type E_7 and H the maximal subgroup of type $A_1 \times F_4$.*

Then G/P_i $(i = 1, \ldots, 7)$ is not a spherical variety. □

7 The exceptional group of type E_8

We start our computations again by calculating the dimensions of the Borel subgroups of the maximal reductive subgroups and the dimensions of G/P_i for $i = 1, \ldots, 8$.

H	$E_7{\times}A_1$	$E_6{\times}A_2$	$A_3{\times}D_5$	$A_4{\times}A_4$	$A_5{\times}A_2{\times}A_1$
$\dim B_H$	72	47	34	28	27

H	$A_7{\times}A_1$	D_8	A_8	$G_2{\times}F_4$	$A_2{\times}A_1$	C_2	A_1
$\dim B_H$	37	72	44	36	6	6	2

The dimensions of the varieties G/P_i $(i = 1, \ldots, 8)$ are:

	G/P_1	G/P_2	G/P_3	G/P_4	G/P_5	G/P_6	G/P_7	G/P_8
dim	78	92	98	106	104	97	83	57

By dimension comparison there are only two possibilities of sphericity. If we take the maximal reductive subgroup H_1 of type $E_7 \times A_1$ or the maximal reductive subgroup H_2 of type D_8, then the variety G/P_8 can be spherical for H_1 or H_2. But we can compute the following restrictions by using LiE

$$\mathrm{res}_{H_1}^G(V_{5\omega_8}) = \ldots \oplus 2(V_{1\lambda_1+2\lambda_7} \otimes V_{2\lambda_8}) \oplus \ldots,$$
$$\mathrm{res}_{H_2}^G(V_{4\omega_8}) = \ldots \oplus 2V_{\lambda_8} \oplus \ldots,$$

which show that there are multiplicities in these cases. So there are no spherical cases for G. We have shown:

Theorem 3.31: *Let G be the simply connected simple algebraic groups of type E_8. Let H be one of its maximal reductive subgroups.*

Then G/P_i $(i = 1, \ldots, 8)$ is not a spherical variety. □

This concludes the classification of the triples (G, P, H).

4 The Luna-Vust invariants for the spherical affine cones

We will now compute the combinatorial invariants that are attached to the spherical affine cones which were classified in the last chapter.

1 The colored cones of the spherical affine cones

Recall that all affine spherical varieties are simple and are thus classified by colored cones.

1.1 Overview of the colored cones for the spherical affine cones

The colored cones $(\mathcal{C}(\widehat{Y}), \Delta_Z(\widehat{Y}))$ of the spherical affine cones are of simple nature. If D is any color we know that it is given by a non constant homogeneous function. So in particular $0 \in D$. That means that $\Delta_Z(\widehat{Y})$ is the set of all colors. And $\mathcal{C}(\widehat{Y})$ is the cone generated by the set of $\rho(D)$ where D runs through all B-stable divisors of $\widehat{Y}$. In the following we denote colors by D_i whereas $D_j^{H\times\mathbb{C}^*}$ denotes a $H\times\mathbb{C}^*$-stable divisor.

We summarize the colored cones in the following table. In the first two columns we give the groups that we consider, in the third one the spherical affine cone and in the fourth columns we list the elements $\rho(D_i)$ and $\rho(D_j^{H\times\mathbb{C}^*})$. So the pair $(\mathcal{C}(\widehat{Y}_i), \Delta_Z(\widehat{Y}_i))$ can easily be recovered. $\mathcal{C}(\widehat{Y}_i)$ is the cone generated by $\rho(D_i)$ and $\rho(D_j^{H\times\mathbb{C}^*})$ and $\Delta_Z(\widehat{Y}_i)$ is the set consiting of the elements $\rho(D_i)$.

We define γ to be the dual basis element to $(0,1)$ in $X(B\times\mathbb{C}^*)$, i.e. γ is defined by $\gamma(\lambda_i, 0) = 0$ and $\gamma(0,1) = 1$.

Table 4.1: Colored cones

G	H	$\widehat{Y}_i$	$\rho\left(D_i^{(H\times\mathbb{C}^*)}\right)$
G_2	A_2	$\widehat{Y}_1$	$\rho(D_1) = \beta_1^\vee$, $\rho(D_2) = \beta_2^\vee$, $\rho(D^{H\times\mathbb{C}^*}) = -\beta_1^\vee - \beta_2^\vee + \gamma$
		$\widehat{Y}_2$	$\rho(D_1) = -\beta_2^\vee + \gamma$, $\rho(D_2) = -\beta_1^\vee + \gamma$, $\rho(D_3) = \beta_1^\vee + \beta_2^\vee - \gamma$

Table 4.1: Colored cones

G	H	$\widehat{Y}_i$	$\rho\left(D_i^{(H\times\mathbb{C}^*)}\right)$
F_4	B_4	$\widehat{Y}_1$	$\rho(D_1) = \beta_2^\vee, \quad \rho(D_2) = \beta_4^\vee$
		$\widehat{Y}_2$	$\rho(D_1) = \beta_1^\vee + \beta_2^\vee + \beta_3^\vee - \gamma,$
			$\rho(D_2) = -\beta_2^\vee - \beta_3^\vee + \gamma,$
			$\rho(D_3) = -\beta_3^\vee - \beta_4^\vee + \gamma,$
			$\rho(D_4) = \beta_2^\vee + \beta_3^\vee + \beta_4^\vee - \gamma,$
			$\rho(D_5) = -\beta_1 - \beta_2^\vee + \gamma$
		$\widehat{Y}_3$	$\rho(D_1) = -\beta_2^\vee - \beta_3^\vee - \beta_4^\vee + \gamma, \qquad \rho(D_2) = \beta_2^\vee,$
			$\rho(D_4) = -\beta_1^\vee - \beta_2^\vee - \beta_3^\vee + \gamma, \qquad \rho(D_3) = \beta_3^\vee,$
			$\rho(D_5) = \beta_1^\vee + \beta_2^\vee + \beta_3^\vee + \beta_4^\vee - \gamma$
		$\widehat{Y}_4$	$\rho(D_1) = \beta_1^\vee, \qquad \rho(D_2) = \beta_4^\vee,$
			$\rho(D^{H\times\mathbb{C}^*}) = -\beta_1^\vee - \beta_4^\vee + \gamma$
E_6	$A_5 \times A_1$	$\widehat{Y}_1$	$\rho(D_1) = \beta_1^\vee, \quad \rho(D_2) = \beta_2^\vee, \quad \rho(D_3) = \beta_4^\vee$
		$\widehat{Y}_6$	$\rho(D_1) = \beta_2^\vee, \quad \rho(D_2) = \beta_4^\vee, \quad \rho(D_3) = \beta_5^\vee$
	F_4	$\widehat{Y}_1$, $\widehat{Y}_6$	$\rho(D) = \beta_1^\vee, \quad \rho(D^{H\times\mathbb{C}^*}) = -\beta_1^\vee + \gamma$
		$\widehat{Y}_2$	$\rho(D_1) = \beta_1^\vee, \quad \rho(D_2) = \beta_4^\vee$
		$\widehat{Y}_3$, $\widehat{Y}_5$	$\rho(D_1) = \beta_1^\vee, \quad \rho(D_2) = \beta_3^\vee, \quad \rho(D_3) = \beta_4^\vee$
	C_4	$\widehat{Y}_1$, $\widehat{Y}_6$	$\rho(D_1) = \beta_2^\vee, \quad \rho(D_2) = \beta_4^\vee,$
			$\rho(D^{H\times\mathbb{C}^*}) = -\frac{1}{2}\beta_2^\vee + \beta_4^\vee + \frac{1}{2}\gamma$
	$D_5 \times \mathbb{C}^*$	$\widehat{Y}_1$	$\rho(D_1) = \beta_1^\vee, \quad \rho(D_2) = \beta_4^\vee,$
			$\rho(D^{H\times\mathbb{C}^*}) = -\beta_1^\vee - \beta_4^\vee + \gamma$
		$\widehat{Y}_2$	$\rho(D_1) = \beta_2^\vee, \quad \rho(D_2) = \beta_4^\vee,$
			$\rho(D_3) = \beta_5^\vee, \quad \rho(D^{H\times\mathbb{C}^*}) = -\beta_2^\vee - \beta_4^\vee - \beta_5^\vee + \gamma$
		$\widehat{Y}_3$	$\rho(D_1) = \frac{3}{4}\beta_1^\vee + \frac{1}{2}\beta_2^\vee - \frac{1}{4}\beta_3^\vee - \frac{5}{8}\beta_4^\vee + \frac{1}{8}\beta_5^\vee - \frac{1}{8}\gamma',$
			$\rho(D_2) = \beta_2^\vee, \qquad \rho(D_3) = \beta_3^\vee,$
			$\rho(D_4) = -\frac{1}{4}\beta_1^\vee + \frac{1}{2}\beta_2^\vee - \frac{1}{4}\beta_3^\vee + \frac{3}{8}\beta_4^\vee + \frac{1}{8}\beta_5^\vee - \frac{1}{8}\gamma',$
			$\rho(D_5) = \beta_5^\vee,$
			$\rho(D_6) = \frac{1}{4}\beta_1^\vee - \frac{1}{2}\beta_2^\vee + \frac{1}{4}\beta_3^\vee, + \frac{5}{8}\beta_4^\vee - \frac{1}{8}\beta_5^\vee + \frac{1}{8}\gamma'$

Table 4.1: Colored cones

G	H	$\widehat{Y}_i$	$\rho\left(D_i^{(H\times\mathbb{C}^*)}\right)$
		$\widehat{Y}_5$	$\rho(D_1) = \frac{3}{4}\beta_1^\vee + \frac{1}{2}\beta_2^\vee - \frac{1}{4}\beta_3^\vee + \frac{1}{8}\beta_4^\vee - \frac{5}{8}\beta_5^\vee + \frac{1}{8}\gamma'$, $\rho(D_2) = \beta_2^\vee, \quad \rho(D_3) = \beta_3^\vee, \quad \rho(D_4) = \beta_4^\vee$, $\rho(D_5) = -\frac{1}{4}\beta_1^\vee + \frac{1}{2}\beta_2^\vee - \frac{1}{4}\beta_3^\vee + \frac{1}{8}\beta_4^\vee + \frac{3}{8}\beta_5^\vee + \frac{1}{8}\gamma'$, $\rho(D_6) = \frac{1}{4}\beta_1^\vee - \frac{1}{2}\beta_2^\vee + \frac{1}{4}\beta_3^\vee - \frac{1}{8}\beta_4^\vee + \frac{5}{8}\beta_5^\vee - \frac{1}{8}\gamma'$
		$\widehat{Y}_6$	$\rho(D_1) = \beta_1^\vee, \quad \rho(D_2) = \beta_5^\vee$, $\rho(D^{H\times\mathbb{C}^*}) = -\beta_1^\vee - \beta_5^\vee + \gamma$
E_7	A_7	$\widehat{Y}_7$	$\rho(D_1) = \beta_2^\vee, \quad \rho(D_2) = \beta_4^\vee, \quad \rho(D_3) = \beta_6^\vee$, $\rho(D^{H\times\mathbb{C}^*}) = -\frac{1}{2}\beta_2^\vee - \frac{1}{2}\beta_4^\vee - \frac{1}{2}\beta_6^\vee + \frac{1}{2}\gamma$
	$E_6 \times \mathbb{C}^*$	$\widehat{Y}_1$	$\rho(D_1) = \beta_1^\vee, \quad \rho(D_2) = \beta_2^\vee, \quad \rho(D_3) = \beta_6^\vee$, $\rho(D^{H\times\mathbb{C}^*}) = -\beta_1^\vee - \beta_2^\vee - \beta_6^\vee + \gamma$
		$\widehat{Y}_2$	$\rho(D_1) = \beta_1^\vee$, $\rho(D_2) = \frac{1}{6}\beta^\vee + \frac{1}{2}\beta_2^\vee - \frac{1}{6}\beta_3^\vee + \frac{1}{6}\beta_5^\vee - \frac{1}{6}\beta_6^\vee + \frac{1}{6}\gamma'$, $\rho(D_3) = -\frac{1}{6}\beta^\vee + \frac{1}{2}\beta_2^\vee + \frac{1}{6}\beta_3^\vee - \frac{1}{6}\beta_5^\vee + \frac{1}{6}\beta_6^\vee - \frac{1}{6}\gamma'$, $\rho(D_4) = \beta_3^\vee, \quad \rho(D_5) = \beta_4^\vee, \quad \rho(D_6) = \beta_5^\vee$, $\rho(D_7) = \beta_6^\vee$
		$\widehat{Y}_7$	$\rho(D_1) = \beta_1^\vee, \quad \rho(D_1^{H\times\mathbb{C}^*}) = -\frac{1}{3}\beta_1 - \frac{2}{3}\beta_6^\vee + \frac{1}{6}\gamma' + \frac{1}{2}\gamma$, $\rho(D_2) = \beta_6^\vee, \quad \rho(D_2^{H\times\mathbb{C}^*}) = -\frac{2}{3}\beta_1 - \frac{1}{3}\beta_6^\vee - \frac{1}{6}\gamma' + \frac{1}{2}\gamma$
	$D_6 \times A_1$	$\widehat{Y}_7$	$\rho(D_1) = \beta_1^\vee, \quad \rho(D_2) = \beta_2^\vee, \quad \rho(D_3) = \beta_6^\vee$

1.2 Colors and $H \times \mathbb{C}^*$-invariant prime divisors of the spherical affine cones

We will now explain how we achieved the results on the colored cones.

Lemma 4.1: *The generators of $\mathbb{C}[\widehat{Y}]^{U_H}$ that are B-semiinvariant but not $H \times \mathbb{C}^*$-semiinvariant define pairwise distinct colors and every color is defined in this way.*

Proof: We know that $\widehat{Y}$ is factorial [VP72, Thm. 4]. So any divisor is given by an equation. Let $D = Z(f)$ be a B-stable prime divisor, where $Z(f)$ is our notation for the zero set of f. Consider the special open set $\widehat{Y}_f$. The function f is regular and invertible on $\widehat{Y}_f$ which is an affine B-variety.

In this situation we have $(b.f)(x) = \chi(b)f(x)$ for all $b \in B, x \in \widehat{Y}_f$ for some character χ of B (cp. [KKV89, §1]) which means that f is B-semiinvariant on $\widehat{Y}_f$ and thus on $\widehat{Y}$. Since D is a prime divisor f is a generator of $\mathbb{C}[\widehat{Y}]^{U_H}$.

We can apply the same argument to show that if D is $H \times \mathbb{C}^*$-invariant then f is $H \times \mathbb{C}^*$-semiinvariant, i. e. the weight of the semisimple part of $H \times \mathbb{C}^*$ is trivial.

Conversely a generator with the properties as stated in the theorem defines a B-stable but not $H \times \mathbb{C}^*$-stable prime divisor i. e. a color.

It remains to show that they are pairwise distinct. Suppose f_1 and f_2 are two generators that define the same color. Then we can consider $f = \frac{f_1}{f_2}$. The function f is regular, B_H-semiinvariant, and invertible. We consider the action of the commutator H' and as above we get that $(h.f)(x) = \chi(h)f(x)$ $(h \in H',\ x \in \widehat{Y})$ where χ is a character of H. But then $h.f = f$ since H' has no non-trivial characters. In particular $b.f = f$ for any b in the Borel of H'. So it follows that f_1 and f_2 have the same weight with regard to H'. Hence if there are no generators of the same H'-weight we are done.

The only exception is the case where G is of type E_7 and H of type $E_6 \times \mathbb{C}^*$. If we consider the spherical variety $\widehat{G/P_2}$ then we have a generator g_1 of weight $(\lambda_2, 3, 1)$ and another generators g_2 of weight $(\lambda_2, -3, 1)$.

We can rewrite these weights in terms of the fundamental weights of G and get $(\lambda_2, 3) = \omega_2$ and $(\lambda_2, -3) = \omega_2 - 2\omega_7$. We regard $\widehat{Y}$ as a subvariety of V_{ω_2} as described before and chosse a basis consiting of root vectors together with its dual basis. The generators are given by restricting the corresponding dual basis elements to $\widehat{Y}$.

Now we can consider a lowest weight vector $v_{-\omega_2} \in \widehat{Y}$ and get that $g_1(v_{-\omega_2}) \neq 0$ whereas $g_2(v_{-\omega_2}) = 0$. Hence the colors that are defined by these generators are distinct. □

Next as an example we will compute the elements $\rho(D_i^{(H\times\mathbb{C}^*)})$ for the spherical $H \times \mathbb{C}^*$-variety $\widehat{Y_1}$ where G is of type G_2 and H of type A_2.

In this case the generators of $\Lambda(\widehat{Y})$ were given by functions with weights $\nu_1 = (0, 1)$, $\nu_2 = (\lambda_1, 1)$ and $\nu_3 = (\lambda_2, 1)$. By the lemma the function of weight $(0, 1)$ corresponds to a $H \times \mathbb{C}^*$-invariant prime divisor $D^{H\times\mathbb{C}^*}$ whereas the other two functions define colors D_1 and D_2.

By definition of $\rho(D)$ we have

$$\begin{array}{ccc} \rho(D_1)(\lambda_1, 1) = 1 & \rho(D_2)(\lambda_1, 1) = 0 & \rho(D^{H\times\mathbb{C}^*})(\lambda_1, 1) = 0 \\ \rho(D_1)(\lambda_2, 1) = 0 & \rho(D_2)(\lambda_2, 1) = 1 & \rho(D^{H\times\mathbb{C}^*})(\lambda_2, 1) = 0 \\ \rho(D_1)(0, 1) = 0 & \rho(D_2)(0, 1) = 0 & \rho(D^{H\times\mathbb{C}^*})(0, 1) = 1 \end{array}$$

Solving these systems of equations we get:

$$\rho(D_1) = \beta_1^\vee, \quad \rho(D_2) = \beta_2^\vee, \quad \text{and} \quad \rho(D^{H\times\mathbb{C}^*}) = -\beta_1^\vee - \beta_2^\vee + \gamma.$$

We compute the remaining cases by solving the equations $\rho(D_i^{(H\times\mathbb{C}^*)})(\nu_j) = \delta_{ij}$ and get the results that are presented in Table 4.1.

2 Homogeneous spherical data for the spherical affine cones

Now that we have computed the colored cones for the affine spherical varieties it remains to find the invariants that classify the open orbit of these varieties.

2.1 Overview of the homogeneous spherical data

We present the spherical homogeneous data in the following table.

Since $\mathbb{C}(X)^{(B)} = \mathrm{Quot}\mathbb{C}[X]^{(B)}$ the weight lattice for a spherical affine cone is the lattice generated by the weights of the generators that were calculated before. For example if we consider the case again where G is of type G_2 and H of type A_2, then for $\widehat{Y_1}$ we get

$$\Lambda(\widehat{Y}) = \mathbb{Z}(\lambda_1, 0) + \mathbb{Z}(\lambda_2, 0) + \mathbb{Z}(0, 1).$$

The other cases are obtained in the same way and the weight lattices are given in Table 4.2.

In the subsequent subsections we will explain how we obtained the remaining invariants.

Table 4.2: Homogeneous spherical data

G	H	$\widehat{Y}_i$	$(\Lambda, S^p, \Sigma, \Delta^a)$	
G_2	A_2	$\widehat{Y}_1$	Λ	$\mathbb{Z}(\lambda_1, 1) \oplus \mathbb{Z}(\lambda_2, 1) \oplus \mathbb{Z}(0, 1)$
			S^p	$\varnothing$
			Σ	$\sigma = \beta_1 + \beta_2$
			Δ^a	$\varnothing$
		$\widehat{Y}_2$	Λ	$\mathbb{Z}(\lambda_1, 1) \oplus \mathbb{Z}(\lambda_2, 1) \oplus \mathbb{Z}(\lambda_1 + \lambda_2, 1)$
			S^p	$\varnothing$
			Σ	$\sigma_1 = \beta_1$ $\quad\sigma_2 = \beta_2$
			Δ^a	$\rho(D_1) = -\beta_2^\vee + \gamma,$ $\quad\rho(D_2) = -\beta_1^\vee + \gamma,$ $\rho(D_3) = \beta_1^\vee + \beta_2^\vee - \gamma$
F_4	B_4	$\widehat{Y}_1$	Λ	$\mathbb{Z}(\lambda_2, 1) \oplus \mathbb{Z}(\lambda_4, 1)$
			S^p	$\{\beta_1,\ \beta_3\}$
			Σ	$\sigma = \frac{1}{2}(\beta_1 + 2\beta_2 + \beta_3)$
			Δ^a	$\varnothing$

Table 4.2: Homogeneous spherical data

G	H	$\widehat{Y}_i$		$(\Lambda, S^p, \Sigma, \Delta^a)$
		$\widehat{Y}_2$	Λ	$\mathbb{Z}(\lambda_1,0)\oplus\mathbb{Z}(\lambda_2,0)\oplus\mathbb{Z}(\lambda_3,0)\oplus\mathbb{Z}(\lambda_4,0)\oplus\mathbb{Z}(0,1)$
			S^p	$\varnothing$
			Σ	$\sigma_1=\beta_1$ $\sigma_2=\beta_2$ $\sigma_3=\beta_3$ $\sigma_4=\beta_4$
			Δ^a	$\rho(D_1)=\beta_1^\vee+\beta_2^\vee+\beta_3^\vee-\gamma$, $\rho(D_2)=-\beta_2^\vee-\beta_3^\vee+\gamma$, $\rho(D_3)=-\beta_3^\vee-\beta_4^\vee+\gamma$, $\rho(D_4)=\beta_2^\vee+\beta_3^\vee+\beta_4^\vee-\gamma$, $\rho(D_5)=-\beta_1^\vee-\beta_2^\vee+\gamma$
		$\widehat{Y}_3$	Λ	$\mathbb{Z}(\lambda_1,0)\oplus\mathbb{Z}(\lambda_2,0)\oplus\mathbb{Z}(\lambda_3,0)\oplus\mathbb{Z}(\lambda_4,0)\oplus\mathbb{Z}(0,1)$
			S^p	$\varnothing$
			Σ	$\sigma_1=\beta_1$ $\sigma_2=\beta_4$ $\sigma_3=\beta_2+\beta_3$ $\sigma_4=\beta_3+\beta_4$
			Δ^a	$\rho(D_1)=-\beta_2^\vee-\beta_3^\vee-\beta_4^\vee+\gamma$, $\rho(D_4)=-\beta_1^\vee-\beta_2^\vee-\beta_3^\vee+\gamma$, $\rho(D_5)=\beta_1^\vee+\beta_2^\vee+\beta_3^\vee+\beta_4^\vee-\gamma$
		$\widehat{Y}_4$	Λ	$\mathbb{Z}(\lambda_1,1)\oplus\mathbb{Z}(\lambda_4,1)\oplus\mathbb{Z}(0,1)$
			S^p	$\{\beta_2,\beta_3\}$
			Σ	$\sigma_1=\beta_1+\beta_2+\beta_3+\beta_4$ $\sigma_2=\beta_2+2\beta_3+3\beta_4$
			Δ^a	$\varnothing$
E_6	$A_5\times A_1$	$\widehat{Y}_1$	Λ	$\mathbb{Z}(\lambda_2,0,1)\oplus\mathbb{Z}(\lambda_4,0,2)\oplus\mathbb{Z}(\lambda_5,\lambda_6,1)$
			S^p	$\{\beta_1,\ \beta_3\}$
			Σ	$\sigma_1=\beta_1+2\beta_2+\beta_3$ $\sigma_2=\beta_5+\beta_6$
			Δ^a	$\varnothing$
		$\widehat{Y}_6$	Λ	$\mathbb{Z}(\lambda_1,\lambda_6,1)\oplus\mathbb{Z}(\lambda_2,0,2)\oplus\mathbb{Z}(\lambda_4,0,1)$
			S^p	$\{\beta_3,\ \beta_5\}$
			Σ	$\sigma_1=\beta_3+2\beta_4+\beta_5$ $\sigma_2=\beta_1+\beta_6$
			Δ^a	$\varnothing$
	F_4	$\widehat{Y}_1$, $\widehat{Y}_6$	Λ	$\mathbb{Z}(\lambda_4,1)\oplus\mathbb{Z}(0,1)$
			S^p	$\{\beta_1,\ \beta_2,\ \beta_3\}$
			Σ	$\sigma=\beta_1+2\beta_2+3\beta_3+2\beta_4$
			Δ^a	$\varnothing$
		$\widehat{Y}_2$	Λ	$\mathbb{Z}(\lambda_1,1)\oplus\mathbb{Z}(\lambda_4,1)$
			S^p	$\{\beta_2,\ \beta_3\}$

Table 4.2: Homogeneous spherical data

G	H	$\widehat{Y}_i$		$(\Lambda, S^p, \Sigma, \Delta^a)$
			Σ	$\sigma = \beta_1 + \beta_2 + \beta_3$
			Δ^a	$\varnothing$
		$\widehat{Y}_3$, $\widehat{Y}_5$	Λ	$\mathbb{Z}(\lambda_1, 1) \oplus \mathbb{Z}(\lambda_3, 1) \oplus \mathbb{Z}(\lambda_4, 1)$
			S^p	$\{\beta_2\}$
			Σ	$\sigma_1 = \beta_1 + \beta_2 + \beta_3 \quad \sigma_2 = \beta_2 + 2\beta_3 + \beta_4$
			Δ^a	$\varnothing$
	C_4	$\widehat{Y}_1$, $\widehat{Y}_6$	Λ	$\mathbb{Z}(\lambda_2, 1) \oplus \mathbb{Z}(\lambda_4, 2) \oplus \mathbb{Z}(0, 2)$
			S^p	$\{\beta_1,\ \beta_3\}$
			Σ	$\sigma_1 = \beta_1 + 2\beta_2 + \beta_3 \quad \sigma_2 = 2(\beta_3 + \beta_4)$
			Δ^a	$\varnothing$
	$D_5 \times \mathbb{C}^*$	$\widehat{Y}_1$	Λ	$\mathbb{Z}(\lambda_1, 2, 1) \oplus \mathbb{Z}(\lambda_4, -1, 1) \oplus \mathbb{Z}(0, -4, 1)$
			S^p	$\{\beta_2,\ \beta_3,\ \beta_5\}$
			Σ	$\sigma = \beta_2 + 2\beta_3 + 2\beta_4 + \beta_5$
			Δ^a	$\varnothing$
		$\widehat{Y}_2$	Λ	$\mathbb{Z}(\lambda_2, 0, 1) \oplus \mathbb{Z}(\lambda_4, 3, 1) \oplus \mathbb{Z}(\lambda_5, -3, 1) \oplus \mathbb{Z}(0, 0, 1)$
			S^p	$\{\beta_1,\ \beta_3\}$
			Σ	$\sigma_1 = \beta_3 + \beta_4 + \beta_5 \quad \sigma_2 = \beta_1 + 2\beta_2 + \beta_3$
			Δ^a	$\varnothing$
		$\widehat{Y}_3$	Λ	$\mathbb{Z}(\lambda_1, -2, 1) \oplus \mathbb{Z}(\lambda_2, 4, 1) \oplus \mathbb{Z}(\lambda_3, -2, 1)$ $\oplus \mathbb{Z}(\lambda_4, -5, 1) \oplus \mathbb{Z}(\lambda_5, 1, 1) \oplus \mathbb{Z}(\lambda_1 + \lambda_4, 1, 1)$
			S^p	$\varnothing$
			Σ	$\sigma_1 = \beta_1 \quad \sigma_2 = \beta_4 \quad \sigma_3 = \beta_2 + \beta_3 \quad \sigma_4 = \beta_3 + \beta_5$
			Δ^a	$\rho(D_1) = \frac{3}{4}\beta_1^\vee + \frac{1}{2}\beta_2^\vee - \frac{1}{4}\beta_3^\vee - \frac{5}{8}\beta_4^\vee + \frac{1}{8}\beta_5^\vee - \frac{1}{8}\gamma'$, $\rho(D_4) = -\frac{1}{4}\beta_1^\vee + \frac{1}{2}\beta_2^\vee - \frac{1}{4}\beta_3^\vee + \frac{3}{8}\beta_4^\vee + \frac{1}{8}\beta_5^\vee - \frac{1}{8}\gamma'$, $\rho(D_6) = \frac{1}{4}\beta_1^\vee - \frac{1}{2}\beta_2^\vee + \frac{1}{4}\beta_3^\vee + \frac{5}{8}\beta_4^\vee - \frac{1}{8}\beta_5^\vee + \frac{1}{8}\gamma'$
		$\widehat{Y}_5$	Λ	$\mathbb{Z}(\lambda_1, 2, 1) \oplus \mathbb{Z}(\lambda_2, -4, 1) \oplus \mathbb{Z}(\lambda_3, 2, 1)$ $\oplus \mathbb{Z}(\lambda_4, -1, 1) + \mathbb{Z}(\lambda_5, 5, 1) + \mathbb{Z}(\lambda_1 + \lambda_5, -1, 1)$
			S^p	$\varnothing$
			Σ	$\sigma_1 = \beta_1 \quad \sigma_2 = \beta_5 \quad \sigma_3 = \beta_2 + \beta_3 \quad \sigma_4 = \beta_3 + \beta_4$
			Δ^a	$\rho(D_1) = \frac{3}{4}\beta_1^\vee + \frac{1}{2}\beta_2^\vee - \frac{1}{4}\beta_3^\vee + \frac{1}{8}\beta_4^\vee - \frac{5}{8}\beta_5^\vee + \frac{1}{8}\gamma'$, $\rho(D_5) = -\frac{1}{4}\beta_1^\vee + \frac{1}{2}\beta_2^\vee - \frac{1}{4}\beta_3^\vee + \frac{1}{8}\beta_4^\vee + \frac{3}{8}\beta_5^\vee + \frac{1}{8}\gamma'$, $\rho(D_6) = \frac{1}{4}\beta_1^\vee - \frac{1}{2}\beta_2^\vee + \frac{1}{4}\beta_3^\vee - \frac{1}{8}\beta_4^\vee + \frac{5}{8}\beta_5^\vee - \frac{1}{8}\gamma'$
		$\widehat{Y}_6$	Λ	$\mathbb{Z}(\lambda_1, -2, 1) \oplus \mathbb{Z}(\lambda_5, 1, 1) \oplus \mathbb{Z}(0, 4, 1)$

Table 4.2: Homogeneous spherical data

G	H	$\widehat{Y}_i$		$(\Lambda, S^p, \Sigma, \Delta^a)$
			S^p	$\{\beta_2,\ \beta_3,\ \beta_4\}$
			Σ	$\sigma = \beta_2 + 2\beta_3 + \beta_4 + 2\beta_5$
			Δ^a	$\varnothing$
E_7	A_7	$\widehat{Y}_7$	Λ	$\mathbb{Z}(\lambda_2, 1) \oplus \mathbb{Z}(\lambda_4, 2) \oplus \mathbb{Z}(\lambda_6, 1) \oplus \mathbb{Z}(0, 2)$
			S^p	$\{\beta_1,\ \beta_3,\ \beta_5,\ \beta_7\}$
			Σ	$\sigma_1 = \beta_1 + 2\beta_2 + \beta_3 \qquad \sigma_2 = \beta_3 + 2\beta_4 + \beta_5$ $\sigma_3 = \beta_5 + 2\beta_6 + \beta_7$
			Δ^a	$\varnothing$
	$E_6 \times \mathbb{C}^*$	$\widehat{Y}_1$	Λ	$\mathbb{Z}(\lambda_1, 2, 1) \oplus \mathbb{Z}(\lambda_2, 0, 1) \oplus \mathbb{Z}(\lambda_6, -2, 1) \oplus \mathbb{Z}(0, 0, 1)$
			S^p	$\{\beta_3,\ \beta_4,\ \beta_5\}$
			Σ	$\sigma_1 = 2\beta_2 + \beta_3 + 2\beta_4 + \beta_5$ $\sigma_2 = \beta_1 + \beta_3 + \beta_4 + \beta_5 + \beta_6$
			Δ^a	$\varnothing$
		$\widehat{Y}_2$	Λ	$\mathbb{Z}(\lambda_1, -1, 1) \oplus \mathbb{Z}(\lambda_2, 3, 1) \oplus \mathbb{Z}(\lambda_2, -3, 1) \oplus \mathbb{Z}(\lambda_3, 1, 1)$ $\oplus \mathbb{Z}(\lambda_4, 0, 2) \oplus \mathbb{Z}(\lambda_5, -1, 1) \oplus \mathbb{Z}(\lambda_6, 1, 1)$
			S^p	$\varnothing$
			Σ	$\sigma_1 = \beta_2 \qquad \sigma_2 = \beta_1 + \beta_3 \qquad \sigma_3 = \beta_3 + \beta_4$ $\sigma_4 = \beta_4 + \beta_5 \qquad \sigma_5 = \beta_5 + \beta_6$
			Δ^a	$\rho(D_2) = \frac{1}{6}\beta^\vee + \frac{1}{2}\beta_2^\vee - \frac{1}{6}\beta_3^\vee + \frac{1}{6}\beta_5^\vee - \frac{1}{6}\beta_6^\vee + \frac{1}{6}\gamma',$ $\rho(D_3) = -\frac{1}{6}\beta^\vee + \frac{1}{2}\beta_2^\vee + \frac{1}{6}\beta_3^\vee - \frac{1}{6}\beta_5^\vee + \frac{1}{6}\beta_6^\vee - \frac{1}{6}\gamma',$
		$\widehat{Y}_7$	Λ	$\mathbb{Z}(\lambda_1, -1, 1) \oplus \mathbb{Z}(\lambda_6, 1, 1) \oplus \mathbb{Z}(0, 3, 1) + \mathbb{Z}(0, -3, 1)$
			S^p	$\{\beta_2,\ \beta_3,\ \beta_4,\ \beta_5\}$
			Σ	$\sigma_1 = 2\beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + \beta_5$ $\sigma_2 = \beta_2 + \beta_3 + 2\beta_4 + 2\beta_5 + 2\beta_6$
			Δ^a	$\varnothing$
	$D_6 \times A_1$	$\widehat{Y}_7$	Λ	$\mathbb{Z}(\lambda_1, \lambda_7, 1) \oplus \mathbb{Z}(\lambda_2, 0, 2) \oplus \mathbb{Z}(\lambda_6, 0, 1)$
			S^p	$\{\beta_3,\ \beta_4,\ \beta_5\}$
			Σ	$\sigma = \beta_1 + \beta_7$
			Δ^a	$\varnothing$

2.2 The invariants $S^p(\widehat{Y})$ and $\Delta^a(\widehat{Y})$

To find the invariants $S^p(\widehat{Y})$ and $\Delta^a(\widehat{Y})$ we need to compute the sets $\Delta(\beta_i)$, i. e. we need to determine which colors are moved by the simple roots of H. Since the colors are given by generators of $\mathbb{C}[\widehat{Y}_i]^{U_H}$ having a non-trivial weight for the semisimple part of $H \times \mathbb{C}^*$ the color which is generated by such a generator f is moved exactly by the simple roots β_i such that the fundamental weight λ_i appears in the weight of f.

In the case of the maximal reductive subgroup H of type A_2 in G of type G_2 let $\widehat{Y}$ be the spherical affine cone $\widehat{G/P_1}$.

Here we have the colors given by generators of weight

$$\nu_1 = (\lambda_1, 1) \quad \text{and} \quad \nu_2 = (\lambda_2, 1).$$

So the first color is moved by β_1 and the second one by β_2. Hence

$$\Delta(\beta_1) = \{D_1\} \quad \text{and} \quad \Delta(\beta_2) = \{D_2\}.$$

That means we have $S^p(\widehat{Y}) = \varnothing$ as well as $\Delta^a(\widehat{Y}) = \varnothing$.

We compute the remaining cases in the same manner to achieve the following table which allows us to immediately acquire the invariants under consideration in this subsection. A color D_i in the table is defined by the i-th generator of $\Lambda(\widehat{Y})$ given in Table 4.2.

Table 4.3: The sets $\Delta(\beta_i)$

G	H	$\widehat{Y}_i$	$\Delta(\beta_i)$		
G_2	A_2	$\widehat{Y}_1$	$\Delta(\beta_1) = \{D_1\}$,	$\Delta(\beta_2) = \{D_2\}$	
		$\widehat{Y}_2$	$\Delta(\beta_1) = \{D_1, D_3\}$,	$\Delta(\beta_2) = \{D_2, D_3\}$	
F_4	B_4	$\widehat{Y}_1$	$\Delta(\beta_1) = \varnothing$,	$\Delta(\beta_2) = \{D_1\}$,	
			$\Delta(\beta_3) = \varnothing$,	$\Delta(\beta_4) = \{D_2\}$	
		$\widehat{Y}_2$	$\Delta(\beta_1) = \{D_1, D_2\}$,	$\Delta(\beta_2) = \{D_3, D_4\}$,	
			$\Delta(\beta_3) = \{D_1, D_5\}$,	$\Delta(\beta_4) = \{D_2, D_4\}$	
		$\widehat{Y}_3$	$\Delta(\beta_1) = \{D_1, D_5\}$,	$\Delta(\beta_2) = \{D_2\}$,	
			$\Delta(\beta_3) = \{D_3\}$,	$\Delta(\beta_4) = \{D_4, D_5\}$	
		$\widehat{Y}_4$	$\Delta(\beta_1) = \{D_1\}$,	$\Delta(\beta_2) = \varnothing$,	
			$\Delta(\beta_3) = \varnothing$,	$\Delta(\beta_4) = \{D_2\}$	
E_6	$A_5 \times A_1$	$\widehat{Y}_1$	$\Delta(\beta_1) = \varnothing$,	$\Delta(\beta_2) = \{D_1\}$,	$\Delta(\beta_3) = \varnothing$,
			$\Delta(\beta_4) = \{D_2\}$,	$\Delta(\beta_5) = \{D_3\}$,	$\Delta(\beta_6) = \{D_3\}$
		$\widehat{Y}_6$	$\Delta(\beta_1) = \{D_1\}$,	$\Delta(\beta_2) = \{D_2\}$,	$\Delta(\beta_3) = \varnothing$,
			$\Delta(\beta_4) = \{D_3\}$,	$\Delta(\beta_5) = \varnothing$,	$\Delta(\beta_6) = \{D_1\}$

Table 4.3: The sets $\Delta(\beta_i)$

G	H	$\widehat{Y}_i$	$\Delta(\beta_i)$
	F_4	$\widehat{Y}_1$, $\widehat{Y}_6$	$\Delta(\beta_1)=\varnothing$, $\Delta(\beta_2)=\varnothing$, $\Delta(\beta_3)=\varnothing$, $\Delta(\beta_4)=\{D_1\}$
		$\widehat{Y}_2$	$\Delta(\beta_1)=\{D_1\}$, $\Delta(\beta_2)=\varnothing$, $\Delta(\beta_3)=\varnothing$, $\Delta(\beta_4)=\{D_2\}$
		$\widehat{Y}_3$, $\widehat{Y}_5$	$\Delta(\beta_1)=\{D_1\}$, $\Delta(\beta_2)=\varnothing$, $\Delta(\beta_3)=\{D_2\}$, $\Delta(\beta_4)=\{D_3\}$
	C_4	$\widehat{Y}_1$, $\widehat{Y}_6$	$\Delta(\beta_1)=\varnothing$, $\Delta(\beta_2)=\{D_1\}$, $\Delta(\beta_3)=\varnothing$, $\Delta(\beta_4)=\{D_2\}$
	$D_5\times\mathbb{C}^*$	$\widehat{Y}_1$	$\Delta(\beta_1)=\{D_1\}$, $\Delta(\beta_2)=\varnothing$, $\Delta(\beta_3)=\varnothing$, $\Delta(\beta_4)=\{D_2\}$, $\Delta(\beta_5)=\varnothing$
		$\widehat{Y}_2$	$\Delta(\beta_1)=\varnothing$, $\Delta(\beta_2)=\{D_1\}$, $\Delta(\beta_3)=\varnothing$, $\Delta(\beta_4)=\{D_2\}$, $\Delta(\beta_5)=\{D_3\}$
		$\widehat{Y}_3$	$\Delta(\beta_1)=\{D_1,D_6\}$, $\Delta(\beta_2)=\{D_2\}$, $\Delta(\beta_3)=\{D_3\}$, $\Delta(\beta_4)=\{D_4,D_6\}$, $\Delta(\beta_5)=\{D_5\}$
		$\widehat{Y}_5$	$\Delta(\beta_1)=\{D_1,D_6\}$, $\Delta(\beta_2)=\{D_2\}$, $\Delta(\beta_3)=\{D_3\}$, $\Delta(\beta_4)=\{D_4\}$, $\Delta(\beta_5)=\{D_5,D_6\}$
		$\widehat{Y}_6$	$\Delta(\beta_1)=\{D_1\}$, $\Delta(\beta_2)=\varnothing$, $\Delta(\beta_3)=\varnothing$, $\Delta(\beta_4)=\varnothing$, $\Delta(\beta_5)=\{D_2\}$
E_7	A_7	$\widehat{Y}_7$	$\Delta(\beta_1)=\varnothing$, $\Delta(\beta_2)=\{D_1\}$, $\Delta(\beta_3)=\varnothing$, $\Delta(\beta_4)=\{D_2\}$, $\Delta(\beta_5)=\varnothing$, $\Delta(\beta_6)=\{D_3\}$, $\Delta(\beta_7)=\varnothing$
	$E_6\times\mathbb{C}^*$	$\widehat{Y}_1$	$\Delta(\beta_1)=\{D_1\}$, $\Delta(\beta_2)=\{D_2\}$, $\Delta(\beta_3)=\varnothing$, $\Delta(\beta_4)=\varnothing$, $\Delta(\beta_5)=\varnothing$, $\Delta(\beta_6)=\{D_3\}$
		$\widehat{Y}_2$	$\Delta(\beta_1)=\{D_1\}$, $\Delta(\beta_2)=\{D_2,D_3\}$, $\Delta(\beta_3)=\{D_4\}$, $\Delta(\beta_4)=\{D_5\}$, $\Delta(\beta_5)=\{D_6\}$, $\Delta(\beta_6)=\{D_7\}$
		$\widehat{Y}_7$	$\Delta(\beta_1)=\{D_1\}$, $\Delta(\beta_2)=\varnothing$, $\Delta(\beta_3)=\varnothing$, $\Delta(\beta_4)=\varnothing$, $\Delta(\beta_5)=\varnothing$, $\Delta(\beta_6)=\{D_2\}$
	$D_6\times A_1$	$\widehat{Y}_7$	$\Delta(\beta_1)=\{D_1\}$, $\Delta(\beta_2)=\{D_2\}$, $\Delta(\beta_3)=\varnothing$, $\Delta(\beta_4)=\varnothing$, $\Delta(\beta_5)=\varnothing$, $\Delta(\beta_6)=\{D_3\}$, $\Delta(\beta_7)=\{D_1\}$

2.3 Determination of spherical roots

Computing the spherical roots is more involved and we will do it mostly case-by-case.

2.3.1 Spherical roots with support of type A_1

By Lemma 2.29, the spherical roots with support of type A_1 are determined by the shape of $\Delta(\beta_i)$. A simple root β_i is a spherical root iff $|\Delta(\beta_i)| = 2$ and $2\beta_i$ is a spherical root iff $\Delta(\beta_i) = \{D\}$ with $\rho(D) = \frac{1}{2}\beta_i^\vee$. Thus Table 4.3 and the results from section 1.2 verify that the spherical roots of support A_1 are the ones given in Table 4.2.

2.3.2 Spherical roots with support different from type A_1

We are now going to determine the spherical roots of other type. This is conducted in two steps. First we determine a list of candidates for the spherical roots and then show that these are indeed spherical roots.

Step 1: Determination of candidates for spherical roots
From the axioms of a homogeneous spherical datum we know that the spherical roots $\Sigma(\widehat{Y})$ is a subset of $\Sigma(H) \cap \Lambda(\widehat{Y})$ consisting of linearly independent and indivisible vectors. The sets $\Sigma(H)$ and $\Lambda(\widehat{Y})$ are given in Table 2.1 and Table 4.1 and we can list the indivisible vectors in $\Sigma(H) \cap \Lambda(\widehat{Y})$. Note that if β is an element in $\Sigma(H)$ then this defines a possible spherical root if $(\beta, 0)$ is an element of $\Lambda(\widehat{Y})$ since we regard it as a weight of $H \times \mathbb{C}^*$.

We can reduce the list of candidates further by making use of the fact that they need to fulfill the axioms of a spherical homogeneous datum.

Example. We consider the case $F_4 \supset B_4$ and $\widehat{Y} = \widehat{G/P_3}$. Here $\Delta(\beta_1)$ and $\Delta(\beta_4)$ are of cardinality 2, so we know that β_1 and β_4 are spherical roots. Since $\Delta(\beta_2) = \{D_2\}$ and $\Delta(\beta_3) = \{D_3\}$ are of cardinality one with $\rho(D_2) = \beta_2^\vee$ and $\rho(D_3) = \beta_3^\vee$, there are no other spherical roots with support of type A_1.

Among the indivisible elements in $\Sigma(H) \cap \Lambda(\widehat{Y})$ with different support are the elements $\sigma_1 = \beta_1 + \beta_2 = \lambda_1 + \lambda_2 - \lambda_3$ and $\sigma_2 = (\beta_1 + \beta_2 + \beta_3) = \lambda_1 - \lambda_3 - 2\lambda_4$.

For the element σ_1 we consider the axiom (A1) of spherical homogeneous data. We have three colors of type a one of which is $\rho(D_5) = \beta_1^\vee + \beta_2^\vee + \beta_3^\vee + \beta_4^\vee - \gamma$. So we have $\langle \rho(D_5), \sigma_1 \rangle = 1$ and hence σ_1 is not a spherical root.

For the element σ_2 we have $S^p(\sigma_2) = \{\beta_2\}$ and $S^{pp} = \{\beta_2\}$. But since $S^p(\widehat{Y}) = \varnothing$ we have $S^{pp}(\sigma_2) \not\subset S^p(\widehat{Y})$ and hence condition (S) of spherical homogeneous data is not fulfilled. It follows that σ_2 cannot be a spherical root.

Step 2: Proving that all candidates are indeed spherical roots
Now we will verify the following proposition. In the following when we speak of a candidate for a spherical root, we always mean an element that was determined in step 1.

Proposition 4.2: *Let $\widehat{Y}$ be one of the spherical affine cones. All candidates for spherical roots are spherical roots.*

Proof: First we cover the cases where there is exactly one possible spherical root, i. e. one candidate and no spherical root with support of type A_1.

Proposition 4.3: *If G/P is horospherical, then G/P is not simple.*

Proof: Suppose G/P is a horospherical H-variety. Then $Y = \overline{H.[v]}$ for some $[v] \in G/P$ and for the stabilizer we have $H_{[v]} \supset U$, where U is a maximal unipotent subgroup of H. Recall that as an H-module we have the decomposition $V_{\omega_i} = V_{\eta_1} \oplus \ldots \oplus V_{\eta_s}$ according to our classification. Then $[v] = [v_{i_1} + \ldots + v_{i_j}]$ where the v_{i_k} are highest weight vectors of the H-module V_{ω_i}, so $v_{i_k} = v_{\eta_{i_k}}$.

Lemma 4.4: *In this situation $j > 1$.*

Proof: Suppose $j = 1$. Then $\exists \eta_{i_j}$ such that $Y = \overline{H.[v_{\eta_{i_j}}]} = H.[v_{\eta_{i_j}}]$ by the Borel's Fixed Point Theorem. It follows that $H.[v_{\omega_i}] = G.[v_{\omega_i}]$. But if $\operatorname{rank} G = \operatorname{rank} H$, then we can consider the positive root $\alpha = \alpha_1 + \ldots + \alpha_n$ which is not a root of H. Hence if $u \in U_\alpha$, then $u[v_{\omega_i}] \notin H.[v_{\omega_i}]$.

For the cases where the rank of G and H differs, we employ a dimension argument. Easy calculation lead to the following dimensions.

<u>G of type E_6, H of type F_4:</u>

	ω_1	ω_2	ω_3	ω_5	ω_6
$\dim G/P_{\omega_i}$	16	21	25	25	16
$\dim H/H_{[v_{\omega_i}]}$	15	15	20	20	15

<u>G of type E_6, H of type C_4:</u>

	ω_1	ω_6
$\dim G/P_{\omega_i}$	16	16
$\dim H/H_{[v_{\omega_i}]}$	11	11

So $H.[v_{\omega_i}] \neq G.[v_{\omega_i}]$ which is a contradiction. Hence $j \neq 1$ which proves the lemma. □(Lemma 4.4)

Consider the set $M = \{\eta_{i_1}, \ldots, \eta_{i_j}\}$ and its convex hull $\operatorname{Conv}(M) \subseteq X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

Denote the set of one-parameter-subgroups by $Y(T)$ and let $\langle \cdot, \cdot \rangle$ be the natural pairing $X(T) \times Y(T) \to \mathbb{Z}$.

Since $|M| \geq 2$ we can find two extremal points of the convex hull. Thus we find two elements $\eta_r, \eta_s \in M$ and $\mu_1, \mu_2 \in Y(T)$ such that $\langle \eta_r, \mu_1 \rangle = \min\{\langle \eta_{i_j}, \mu_1 \rangle\}$ and $\langle \eta_{i_j}, \mu_1 \rangle > \langle \eta_r, \mu_1 \rangle$ for all $i_j \neq r$ and the analogous for $\langle \eta_s, \mu_2 \rangle$.

Now consider $[v] = [v_{i_1} + \ldots + v_{i_k}]$ again. Without loss of generality assume that $r = i_1$, $s = i_2$ i. e. η_{i_1} and η_{i_2} are extremal points.

We have $\mu_1(s).[v] \in H.[v]$.

$$\begin{aligned}\mu_1(s).[v] &= [\eta_{i_1}(\mu_1(s))v_{\eta_{i_1}} + \ldots + \eta_{i_j}(\mu_1(s))v_{\eta_{i_j}}]\\ &= [s^{\overbrace{\langle\mu_1,\eta_{i_1}\rangle - \langle\mu_1,\eta_{i_1}\rangle}^{=0}} v_{\eta_{i_1}} + s^{\overbrace{\langle\mu_1,\eta_{i_2}\rangle - \langle\mu_1,\eta_{i_1}\rangle}^{>0}} v_{\eta_{i_2}} + \ldots\\ &\quad + s^{\overbrace{\langle\mu_1,\eta_{i_j}\rangle - \langle\mu_1,\eta_{i_1}\rangle}^{>0}} v_{\eta_{i_j}}]\end{aligned}$$

It follows $\lim_{s\to 0} \frac{1}{\eta_{i_1}(\mu_1(s))}\mu_1(s).[v] = [v_{\eta_{i_1}}] \in \overline{H.[v]} = Y$. And by the same argument also $[v_{\eta_{i_2}}] \in Y$. But then there are two closed orbits by Borel's Fixed Point Theorem. Hence Y is not simple. □(Prop. 4.3)

We can now use this results to find the spherical roots for the following cases $\widehat{Y_1}$ for G of type F_4 and H of type B_4 as well as $\widehat{Y_1}$, $\widehat{Y_2}$ and $\widehat{Y_6}$ for G of type E_6 and H of type $D_5 \times \mathbb{C}^*$. In these cases there is only one possible spherical root, namely the one that is listed in Table 4.2.

Now suppose that for one of the cases which were just stated $\widehat{Y_i}$ was a horospherical $H \times \mathbb{C}^*$-variety, i. e. $\Sigma_{\widehat{Y}} = \varnothing$. Then $\overline{H \times \mathbb{C}^*.v} = \widehat{Y}$ where the stabilizer of v contained a maximal unipotent subgroup. But then also $\overline{H.[v]} = Y$ where the stabilizer of $[v]$ in H also contained a maximal unipotent subgroup. Hence to prove that the possible spherical root is indeed a spherical root it remains to show that for all cases in question we have that Y is simple as an H-variety.

Since the closed H-orbits of $\mathbb{P}(V_{\omega_i})$ are exactly the H-orbits $G.[v_{\eta_i}]$ where η_i is a highest weight of V_{ω_i} considered as an H-module, it remains to show that v_{ω_i} is the only such element.

Proposition 4.5: *Let $H \subset G$ be a maximal reductive subgroup of G such that* $\operatorname{rank} G = \operatorname{rank} H$.

A highest weight vector $v_{\eta_j} \in G.v_{\omega_i}$ if and only if $\eta_j \in W\omega_i$, where W is the Weyl group of G.

Proof: By the Bruhat decomposition of G we know that

$$G.[v_{\omega_i}] = \bigcup_{w\in W} U[v_{w(\omega_i)}].$$

So an element of Y is of the form $u.v_{w(\omega_i)} = v_{w(\omega_i)} + v'$ where v' is sum of vectors having higher weights. But then, if $v' \neq 0$ than it cannot be a weight vector. So a highest weight vector $v_{\eta_j} \in G.v_{\omega_i}$ for H is of the form $v_{w(\omega_i)}$.

If on the other hand $\eta_j \in W.\omega_i$ than $v_{w(\omega_i)} \in G.v_{\omega_i}$ is a weight vector having weight η_j. Since the weight space is 1-dimensional it must be a highest weight vector. □(Prop. 4.5)

1) G of type E_6, H of type B_4:

Case $\widehat{G/P_1}$: For the case under consideration we have $\mathbb{P}(V_{\omega_1}) = \mathbb{P}(V_{\lambda_2} \oplus V_{\lambda_4})$. One calculates that $\lambda_2 = \omega_1$ and $\lambda_4 = \omega_3 - \omega_4$. We can use LiE to calculate that $\omega_3 - \omega_4$ is not an element of the W-orbit of ω_1. So $H.[v_{\omega_1}] = H.[v_{\lambda_2}]$ is the only closed orbit and thus Y is not simple. It follows that the only possible spherical root is indeed a spherical root.

2) G of type E_6, H of type F_4:

Cases $\widehat{G/P_1}$, $\widehat{G/P_2}$, $\widehat{G/P_6}$: Again there is only one possibility for the spherical roots which is given in Table 4.2.

Proposition 4.6: *If G is of type E_6 and H of type F_4, then the spherical H-varieties of the form G/P are simple.*

Proof: As we have seen elements of $G.[v_{\omega_i}]$ are of the form $[v_{w(\omega_i)} + v']$ where v' is a sum of weight vectors that have higher weights than $w(\omega_i)$.

The simple root vectors of the Lie algebra of H are the following.

$$X_{\beta_1} = X_{\alpha_2} \qquad X_{\beta_2} = X_{\alpha_4}$$
$$X_{\beta_3} = \frac{1}{\sqrt{2}}(X_{\alpha_3} + X_{\alpha_5}) \qquad X_{\beta_4} = \frac{1}{\sqrt{2}}(X_{\alpha_1} + X_{\alpha_6})$$

Now suppose that $v_{w(\omega_i)}+v'$ is a highest weight vector for H. Then $X_{\beta_i}.(v_{w(\omega_i)}+ v') = 0$ for $i = 1, \dots, 4$.

$X_{\beta_1}.(v_{w(\omega_i)} + v') = 0$ implies $X_{\alpha_2}.v_{w(\omega_i)} = 0$. Since otherwise we know that $X_{\beta_1}.v_{w(\omega_i)} = X_{\alpha_2}.v_{w(\omega_i)}$ is a weight vector of weight $w(\omega_i)+\alpha_2$. But this vector cannot cancel with a summand of $X_{\alpha_2}.v'$ since they all have higher weights and hence we would have $X_{\beta_1}.(v_{w(\omega_i)} + v') \neq 0$.

$X_{\beta_2}.(v_{w(\omega_i)} + v') = 0$ implies $X_{\alpha_4}.v_{w(\omega_i)} = 0$ by the same argument.

$X_{\beta_3}.(v_{w(\omega_i)} + v') = 0$ implies $X_{\alpha_3}.v_{w(\omega_i)} = 0$ and $X_{\alpha_5}.v_{w(\omega_i)} = 0$. This is the case, if this was not the case so for example $X_{\alpha_3}.v_{w(\omega_i)} \neq 0$, then this vector would have weight $w(\omega_i) + \alpha_3$ could not cancel with any other summand of $(X_{\alpha_3} + X_{\alpha_5}).(v_{w(\omega_i)} + v')$ since they all have different weights.

$X_{\beta_4}.(v_{w(\omega_i)} + v') = 0$ implies $X_{\alpha_1}.v_{w(\omega_i)} = 0$ and $X_{\alpha_6}.v_{w(\omega_i)} = 0$ by the same argument.

It follows that $X_{\alpha_i}.v_{w(\omega_i)} = 0$ for $i = 1, \dots, 6$. So $v_{w(\omega_i)}$ is the highest weight for G and we have $w(\omega_i) = \omega_i$. So it follows $v' = 0$ and $v_{w(\omega_i)} + v' = v_{\omega_i}$. Hence we have shown that there is a unique highest H-weight in the orbit $G.[v_{\omega_i}]$ and thus there is only one closed H-orbit in G/P which means that these varieties are simple. □

So, the only candidates are indeed spherical roots. For the other cases with one possible spherical root, this argument cannot be applied since they turn out not to be simple.

The proof of the proposition for the remaining cases is conducted case-by-case. Every case relies on Theorem 2.28. First we cover cases where there are generators of degree 2 in $\mathbb{C}[\widehat{Y}]^{U_H}$.

In the following we write $V_{\nu_1} \cdot V_{\nu_2} \subset \mathbb{C}[\widehat{Y}]$ for the module that is generated by elements of the form $fg \in \mathbb{C}[\widehat{Y}]$, where $f \in V_{\nu_1}$ and $g \in V_{\nu_2}$.

3) G of type E_6, H of type C_4:

<u>Case $\widehat{G/P_1}$:</u> There are two candidates

$$\begin{aligned} &\sigma_1 = (2\lambda_2 - \lambda_4, 0), \\ \text{and} \quad &\sigma_2 = (-2\lambda_2 + 2\lambda_4, 0). \end{aligned}$$

Further we have

$$\begin{aligned} &\operatorname{res}^G_{H\times\mathbb{C}^*}(V_{\omega_1^*}) = \operatorname{res}^G_{H\times\mathbb{C}^*}(V_{\omega_6}) = V_{(\lambda_2,1)} \\ \text{and} \quad &\operatorname{res}^G_{H\times\mathbb{C}^*}(V_{2\omega_1^*}) = \operatorname{res}^G_{H\times\mathbb{C}^*}(V_{2\omega_6}) = V_{(2\lambda_2,2)} \oplus V_{(\lambda_4,2)} \oplus V_{(0,2)}. \end{aligned}$$

If we consider $\mathbb{C}[\widehat{Y}]$ as a G-module we know that $V_{\omega_6} \cdot V_{\omega_6} = V_{2\omega_6}$ ([VP72]). So it follows

$$V_{(\lambda_2,1)} \cdot V_{(\lambda_2,1)} = V_{(2\lambda_2,2)} \oplus V_{(\lambda_4,2)} \oplus V_{(0,2)}.$$

Thus $(2\lambda_2, 0) = (2\lambda_2, 2) - (0, 2)$ is an element of $\langle \Sigma_{\widehat{Y}} \rangle_{\mathbb{N}}$. But then σ_1 and σ_2 must be spherical roots.

<u>Case $\widehat{G/P_6}$:</u> For this case the proof is exactly the same, one just interchanges ω_1 and ω_6.

4) G of type E_7, H of type A_7:

<u>Case $\widehat{G/P_7}$:</u> There are three candidates

$$\sigma_1 = (2\lambda_2 - \lambda_4, 0), \qquad \sigma_2 = (-\lambda_2 + 2\lambda_4 - \lambda_6, 0), \qquad \sigma_3 = (-\lambda_4 + 2\lambda_6, 0)$$

for spherical roots. We have

$$\begin{aligned} &\operatorname{res}^G_{H\times\mathbb{C}^*}(V_{\omega_7^*}) = V_{(\lambda_2,1)} \oplus V_{(\lambda_6,1)} \\ \text{and} \quad &\operatorname{res}^G_{H\times\mathbb{C}^*}(V_{2\omega_7^*}) \supseteq V_{(0,2)}. \end{aligned}$$

Again if considering $\mathbb{C}[\widehat{Y}]$ as a G-module we have $V_{\omega_7^*} \cdot V_{\omega_7^*} = V_{2\omega_7^*}$. It follows that $V_{(0,2)}$ is an irreducible component of $V_{(\lambda_2,1)} \cdot V_{(\lambda_2,1)}$, $V_{(\lambda_2,1)} \cdot V_{(\lambda_6,1)}$ or $V_{(\lambda_6,1)} \cdot V_{(\lambda_6,1)}$. And thus $(2\lambda_2, 0)$, $(2\lambda_6, 0)$ or $(\lambda_1 + \lambda_6, 0)$ is an element of $\langle \Sigma_{\widehat{Y}} \rangle_{\mathbb{N}}$. But only $\lambda_1 + \lambda_6 = \sigma_1 + \sigma_2 + \sigma_3$ can be written as a linear combination of the possible spherical roots with non-negative coefficients. This the proposition follows in this case.

5) G of type E_7, H of type $D_6 \times A_1$:

Case $\widehat{G/P_7}$: The only candidate for a spherical root is

$$\sigma = (2\lambda_1 - \lambda_2, 2\lambda_7, 0).$$

We have

$$\begin{aligned} \mathrm{res}^G_{H\times\mathbb{C}^*}(V_{\omega_7^*}) &= V_{(\lambda_1,\lambda_7,1)} \oplus V_{(\lambda_6,0,1)} \\ \mathrm{res}^G_{H\times\mathbb{C}^*}(V_{2\omega_7^*}) &\supset V_{(\lambda_2,0,2)} \end{aligned}$$

Since as G-modules we know that $V_{\omega_7^*} \cdot V_{\omega_7^*} = V_{2\omega_7^*}$ which implies that $V_{(\lambda_2,0,2)}$ is an irreducible component of one of the products of irreducible components in $V_{\omega_7^*}$. Thus

$$(2\lambda_1 - \lambda_2, 2\lambda_7, 0) \text{ or } (2\lambda_6 - \lambda_2, 0, 0) \text{ or } (\lambda_1 + \lambda_6 - \lambda_2, \lambda_7, 0)$$

is an element of $\langle \Sigma_{\widehat{Y}} \rangle_{\mathbb{N}}$. Since there is only one possible spherical root, the first case holds and σ is indeed a spherical root.

Now we turn to the remaining cases. The strategy for finding the spherical roots in these cases is as follows. Let $\widehat{Y}$ be one of the spherical affine cones and let $\{\sigma_1, \ldots, \sigma_s\}$ be the union of spherical roots with support of type A_1 and further candidates for spherical roots. Let σ_j be one fixed element of this set. We consider a suitable product $V_{\mu_1} \cdot V_{\mu_2} \subset \mathbb{C}[\widehat{Y}]$. By suitable we mean, that there exists a $\nu \in \Lambda(\widehat{Y})^+$, such that $\mu_1 + \mu_2 - \nu = \sum_{i=1}^s n_i \sigma_i$ where $n_i \in \mathbb{N}$ with $n_j > 0$.

Next we prove $V_\nu \subset V_{\mu_1} \cdot V_{\mu_2}$ by calculating the dimension of the weight space $(V_{\mu_1} \cdot V_{\mu_2})_\nu$ yielding that σ_j is a spherical root thanks to Thm. 2.28.

We will illustrate our strategy for finding the spherical roots by computing a few cases in detail.

6) G of type G_2, H of type A_2:

Case $\widehat{G/P_1}$: The only candidate for a spherical root is $\sigma = (\lambda_1 + \lambda_2, 0)$.

We need to show that the candidate σ is indeed a spherical root. The restrictions to $H \times \mathbb{C}^*$ of the simple components $V_{\omega_1^*}$ and $V_{\omega_2^*}$ are

$$\begin{aligned} \mathrm{res}^G_{H\times\mathbb{C}^*}(V_{\omega_1^*}) &= V_{(\lambda_1,1)} \oplus V_{(\lambda_2,1)} \oplus V_{(0,1)} \\ \mathrm{res}^G_{H\times\mathbb{C}^*}(V_{2\omega_1^*}) &= V_{(0,2)} \oplus V_{(\lambda_1,2)} \oplus V_{(\lambda_2,2)} \oplus V_{(\lambda_1+\lambda_2,2)} \\ &\quad \oplus V_{(2\lambda_1,2)} \oplus V_{(2\lambda_2,2)} \end{aligned}$$

If

$$V_{(\lambda_1,1)} \cdot V_{(\lambda_2,1)} \supset V_{(0,2)}$$

then $(\lambda_1, 1) + (\lambda_2, 1) - (0, 2) = (\lambda_1 + \lambda_2, 0)$ is an element of $\langle \Sigma_{\widehat{Y}} \rangle_{\mathbb{N}}$ and thus σ is a spherical root.

We have that

$$V_{(\lambda_1,1)} \cdot V_{(\lambda_2,1)} \subset (V_{(\lambda_1,1)} \otimes V_{(\lambda_2,1)}) \cap \mathbb{C}[\widehat{Y}]$$

and the right hand side equals $V_{(\lambda_1+\lambda_2,2)} \oplus V_{(0,2)}$. The module $V_{(\lambda_1+\lambda_2,2)}$ must be a simple component of the product and we need to prove that $V_{(0,2)}$ is a simple component as well.

We have $\dim(V_{\lambda_1+\lambda_2,2})_{(0,2)} = 2$. Thus it suffices to show that weight space of the weight $(0,2)$ in $V_{(\lambda_1,1)} \cdot V_{(\lambda_2,1)}$ is of dimension 3.

Just as we did before we consider the affine cone $\widehat{Y}$ embedded in the simple module V_{ω_1}. We indicate its structure below. The weight spaces are all 1-dimensional and we choose a weight vector v_ω for each one. The weights with respect to H are given in parenthesis. The weight vectors with upper index A form the H-representation V_{λ_1}, the ones indexed with B form the H-representation V_{λ_2} and the weight vector with index C the trivial representation.

$$
\begin{array}{cc}
v^A_{\omega_1} & (\lambda_1) \\
\downarrow{\scriptstyle 1} & \\
v^B_{-\omega_1+\omega_2} & (\lambda_2) \\
\downarrow{\scriptstyle 2} & \\
v^B_{2\omega_1-\omega_2} & (\lambda_1-\lambda_2) \\
\downarrow{\scriptstyle 1} & \\
v^C_0 & (0) \\
\downarrow{\scriptstyle 1} & \\
v^A_{-2\omega_1+\omega_2} & (-\lambda_1+\lambda_2) \\
\downarrow{\scriptstyle 2} & \\
v^A_{\omega_1-\omega_2} & (-\lambda_2) \\
\downarrow{\scriptstyle 1} & \\
v^B_{-\omega_1} & (-\lambda_1)
\end{array}
$$

If v_μ is a weight vector of V_λ having weight μ we denote the dual basis element by v^*_μ which is a weight vector of V_{λ^*} of weight $-\mu$. Restricting these functions to $\widehat{Y}$ we get regular functions on the affine cone.

We use these to construct functions of weight $(0,2)$ in $V_{(\lambda_1,1)} \cdot V_{(\lambda_2,1)}$. We set

$$\begin{aligned} f_1 &:= (v^B_{-\lambda_1})^* \cdot (v^A_{\lambda_1})^* \\ f_2 &:= (v^B_{\lambda_1-\lambda_2})^* \cdot (v^A_{-\lambda_1+\lambda_2})^* \\ f_3 &:= (v^B_{\lambda_2})^* \cdot (v^A_{-\lambda_2})^* \end{aligned}$$

The first factor of each f_i is an element of $V_{\lambda_1,1} \subset \mathbb{C}[\widehat{Y}]$, the second one of $V_{\lambda_2,1} \subset \mathbb{C}[\widehat{Y}]$. It remains to prove that they are linearly independent. So let $f = a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$. We choose suitable elements in $\widehat{Y}$ to show that all three coefficients are 0.

First note that all weights except 0 are in the W-orbit of ω_1 where W denotes the Weyl group of G_2. We set

$$v^1_1 = v^B_{\lambda_1-\lambda_2} \in G.v_{\omega_1}.$$

Then we apply an element $u \in U_{-\alpha_1}$ and get

$$v^1_2 = u.v_1 = v^B_{\lambda_1-\lambda_2} + c_1 v^C_0 + v^A_{-\lambda_1+\lambda_2}$$

with non-zero coefficients c_1, c_2. We have $f_1(v^1_2) = f_3(v^1_2) = 0$ and $f_1(v^1_2) \neq 0$. Thus it follows that $a_2 = 0$ and $f = a_1 f_1 + a_3 f_3$.

Next we set

$$v^2_1 = v^B_{\lambda_2} \in G.v_{\omega_1}$$

and apply an element $u \in U_{-(\alpha_1+\alpha_2)}$ to get

$$v^2_2 = v^B_{\lambda_2} + c_1 v^C_0 + c_2 v^A_{-\lambda_2}$$

with non-zero coefficients c_1, c_2. Now $f_1(v^2_2) = 0$ and $f_3(v^2_2) \neq 0$. Hence we have $f = a_1 f_1 = 0$. But since $\mathbb{C}[\widehat{Y}]$ is a UFD and f_1 is the product of two non-zero elements in $\mathbb{C}[\widehat{Y}]$ it follows that also $a_1 = 0$.

This concludes the proof that σ is a spherical root.

7) G of type F_4, H of type B_4:

<u>Case $\widehat{G/P_4}$:</u> We compute this case as a second example. The candidates in this case are

$$\begin{aligned} &\sigma_1 = (\lambda_1, 0) \\ \text{and} \quad &\sigma_2 = (-\lambda_1 + 2\lambda_4, 0). \end{aligned}$$

Rechall that

$$\begin{aligned} \mathrm{res}^G_{H\times\mathbb{C}^*}(V_{\omega_4^*}) &= V_{(0,1)} \oplus V_{(\lambda_1,1)} \oplus V_{(\lambda_4,1)} \\ \mathrm{res}^G_{H\times\mathbb{C}^*}(V_{2\omega_4^*}) &= V_{(0,2)} \oplus V_{(\lambda_1,2)} \oplus V_{(\lambda_4,2)} \oplus V_{(2\lambda_1,2)} \\ &\quad \oplus V_{(2\lambda_4,2)} \oplus V_{(\lambda_1+\lambda_4,2)} \end{aligned}$$

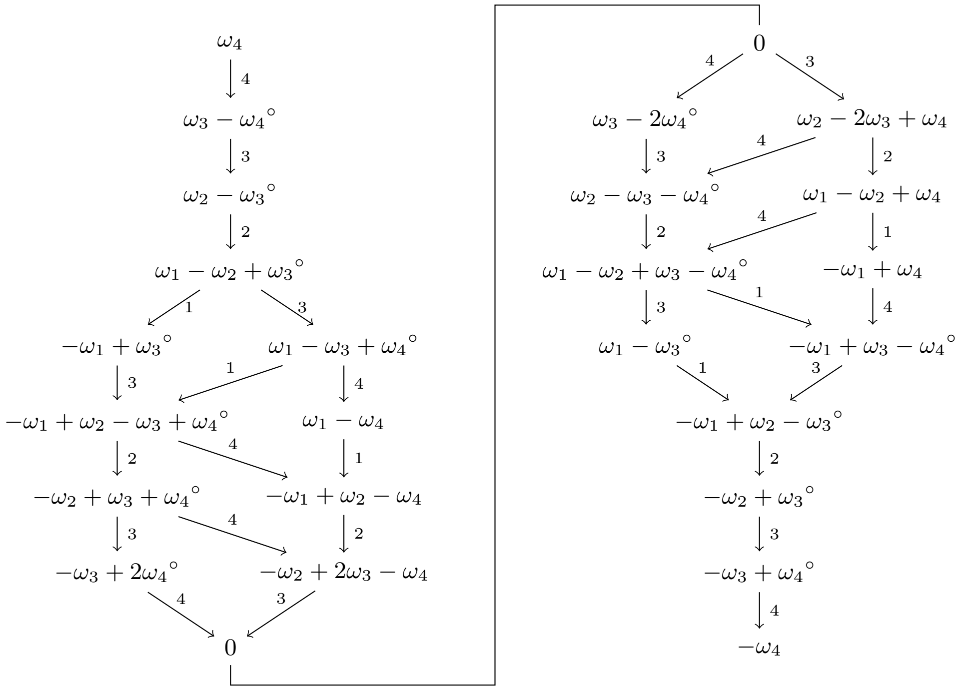
In the following we compute

$$V_{(\lambda_4,1)} \cdot V_{(\lambda_4,1)} = V_{(2\lambda_4,2)} \oplus V_{(\lambda_4,2)} \oplus V_{(0,2)}.$$

Then $(2\lambda_4, 0) = \sigma_1 + \sigma_2 \in \langle \Sigma_{\widehat{Y}} \rangle_{\mathbb{N}}$ and thus σ_1 and σ_2 are spherical roots.

Since $S^2(V_{(\lambda_4,1)}) = V_{(2\lambda_4,2)} \oplus V_{(\lambda_4,2} \oplus V_{(0,1)}$, we know that only these irreducible components can occur. We have $\dim(V_{(2\lambda_4,2)})_{(0,2)} = 6$, $\dim(V_{(\lambda_4,2)})_{(0,2)} = 1$. Thus we need to show that the weight space of $(0,2)$ in $V_{(\lambda_4,1)} \cdot V_{(\lambda_4,1)} = 8$.

We condsider the usual embedding of $\widehat{Y}$ in V_{ω_4} whose structure we indicate below. All weight spaces are one dimensional except that of weight 0, which is of dimension 2. The weight spaces that belong to $V_{(\lambda_4)}$ are indexed by a circle.



Now we considerfollowing functions in $V_{(\lambda_4,1)} \cdot V_{(\lambda_4,1)} \subset \mathbb{C}[\widehat{Y}]$.

$$\begin{aligned}
f_1 &= (v_{\omega_3-\omega_4})^*(v_{-\omega_3+\omega_4})^* \\
f_2 &= (v_{\omega_2-\omega_3})^*(v_{-\omega_2+\omega_3})^* \\
f_3 &= (v_{\omega_1-\omega_2+\omega_3})^*(v_{-\omega_1+\omega_2-\omega_3})^* \\
&\ \vdots \\
f_8 &= (v_{-\omega_3+2\omega_4})^*(v_{\omega_3-2\omega_4})^*
\end{aligned}$$

and set $f = \sum_{i=1}^{8} a_i f_i = 0$.

Note that all weights of V_{λ_4} are elements of the Weyl group orbit of ω_4. We set

$$v_1^1 = v_{\omega_3 - 2\omega_4} \in G.v_{\omega_4}.$$

Applying an element $u \in U_{\alpha_4}$ yields

$$v_2^1 = v_1^1 + c_1 v_0 + c_2 v_{-\omega_3+2\omega_4},$$

where v_0 is some weight vector of weight 0. We have $f_i(v_2^1) = 0$ for $i \neq 8$. Hence $a_8 = 0$.

Next we set

$$v_1^2 = v_{\omega_2 - \omega_3 - \omega_4} \in G.v_{\omega_4}$$

and apply $u \in U_{\alpha_3+\alpha_4}$, yielding

$$v_2^2 = v_1^2 + c_1 v_0 + c_2 v_{-\omega_2+\omega_3+\omega_4}.$$

Here $f_i(v_2^2) = 0$ for $i \neq 7$, hence $a_7 = 0$. We proceed this way and successively compute that $a_1 = \ldots = a_8 = 0$. It follows that $(V_{(\lambda_4,1)} \cdot V_{(\lambda_4,1)})_{(0,2)}$ is of dimension 8.

For the remaining cases we will state which product one needs to consider to obatain that a given candidate is a spherical root. For each case we used the same strategy as we did in the two example cases above.

Case $\widehat{G/P_3}$: In this case there are two candidates

$$\begin{aligned} &\sigma_3 = (-\lambda_1 + \lambda_2 + \lambda_3 - 2\lambda_4, 0) \\ \text{and} \quad &\sigma_4 = (-\lambda_2 + \lambda_3, 0). \end{aligned}$$

To prove that these two elements are spherical roots we consider the products $V_{(\lambda_2,1)} \cdot V_{(\lambda_3,1)}$ and $V_{(\lambda_1+\lambda_4,1)} \cdot V_{(\lambda_3,1)}$. One shows that the first of these two products contains a simple module $V_{(\lambda_1+\lambda_2+\lambda_4,2)}$ and the second one a simple module $V_{(\lambda_1+2\lambda_4,2)}$. So $\sigma_3 = (\lambda_2 + \lambda_3, 2) - (\lambda_1 + 2\lambda_4, 2)$ and $\sigma_4 = (\lambda_1 + \lambda_3 + \lambda_4, 2) - (\lambda_1 + \lambda_2 + \lambda_4, 2)$ are spherical roots.

8) G of type E_6, H of type $A_5 \times A_1$:

Case $\widehat{G/P_1}$: The candidates in this case are $\sigma_1 = (-\lambda_4 + 2\lambda_5, 2\lambda_6, 0)$ and $\sigma_2 = (2\lambda_2 - \lambda_4, 0, 0)$.

Here we compute the products

$$\begin{aligned} V_{(\lambda_2,0,1)} \cdot V_{(\lambda_2,0,1)} &= V_{(2\lambda_2,0,2)} \oplus V_{(\lambda_4,0,2)}, \\ V_{(\lambda_5,\lambda_6,1)} \cdot V_{(\lambda_5,\lambda_6,1)} &= V_{(2\lambda_5,2\lambda_6,2)} \oplus V_{(\lambda_4,0,2)}. \end{aligned}$$

It follows that $\sigma_1 = (2\lambda_5, 2\lambda_6, 2) - (\lambda_4, 0, 2)$ and $\sigma_2 = (2\lambda_2, 0, 2) - (\lambda_4, 0, 2)$ are spherical roots.

Case $\widehat{G/P_6}$: This case is almost identical to the previous one. The candidates are $\sigma_1 = (2\lambda_1 - \lambda_2, 2\lambda_6, 0)$, $\sigma_2 = (-\lambda_2 + \lambda_4, 0, 0)$. We compute

$$V_{(\lambda_1,\lambda_6,1)} \cdot V_{(\lambda_1,\lambda_6,1)} = V_{(2\lambda_1,2\lambda_6,2)} \oplus V_{(\lambda_2,0,2)}$$
$$V_{(\lambda_4,0,1)} \cdot V_{(\lambda_4,0,1)} = V_{(2\lambda_4,0,2)} \oplus V_{(\lambda_2,0,2)}.$$

Hence $\sigma_1 = (2\lambda_1, 2\lambda_6, 2) - (\lambda_2, 0, 2)$ and $\sigma_2 = (2\lambda_4, 0, 2) - (\lambda_2, 0, 2)$ are spherical roots.

9) G of type E_6, H of type F_4:

Cases $\widehat{G/P_3}$ and $\widehat{G/P_5}$: The candidates in this case are $\sigma_1 = (\lambda_1 - \lambda_4, 0)$ and $\sigma_2 = (-\lambda_1 + \lambda_3, 0)$.

We compute

$$V_{(\lambda_1,1)} \cdot V_{(\lambda_3,1)} \supset V_{(\lambda_3+\lambda_4,2)}$$
$$\text{and} \quad V_{(\lambda_3,1)} \cdot V_{(\lambda_3,1)} \supset V_{(\lambda_1+\lambda_3,2)}.$$

Hence $\sigma_1 = (\lambda_1 + \lambda_3, 2) - (\lambda_3 + \lambda_4, 2)$ and $\sigma_2 = (2\lambda_3, 2) - (\lambda_1 + \lambda_3, 2)$ are spherical roots.

10) G of type E_6, H of type $D_5 \times \mathbb{C}^*$:

Case $\widehat{G/P_1}$: The only candidate in this case is $\sigma = (-\lambda_1 + 2\lambda_4, 0, 0)$. We compute that

$$V_{(\lambda_4,-1,1)} \cdot V_{(\lambda_4,-1,1)} \supset V_{(\lambda_1,-2,2)}.$$

Hence $\sigma = (2\lambda_4, -2, 2) - (\lambda_1, -2, 2)$ is a spherical root.

Case $\widehat{G/P_2}$: The candidates in this case are $\sigma_1 = (-\lambda_2, +\lambda_4 + \lambda_5, 0, 0)$ and $\sigma_2 = (2\lambda_2 - \lambda_4 - \lambda_5, 0, 0)$.

We compute

$$V_{(\lambda_4,3,1)} \cdot V_{(\lambda_5,-3,1)} \supset V_{(\lambda_2,0,2)}$$
$$\text{and} \quad V_{(\lambda_2,0,1)} \cdot V_{(\lambda_2,0,1)} \supset V_{(\lambda_4+\lambda_5,0,2)}.$$

It follows that $\sigma_1 = (\lambda_4 + \lambda_5, 0, 2) - (\lambda_2, 0, 2)$ and $\sigma_2 = (2\lambda_2 - \lambda_4 - \lambda_5, 0, 0)$ are spherical roots.

Case $\widehat{G/P_3}$: The candidates in this case are $\sigma_3 = (-\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5, 0, 0)$ and $\sigma_4 = (-\lambda_2 + \lambda_3 - \lambda_4 + \lambda_5, 0, 0)$.

We compute

$$V_{(\lambda_3,-2,1)} \cdot V_{(\lambda_3,-2,1)} \supset V_{(\lambda_1+2\lambda_4,-4,2)}.$$

Hence $\sigma_3 + \sigma_4 = (2\lambda_3, -4, 2) - (\lambda_1 + 2\lambda_4, -4, 2) \in \langle \Sigma_{\widehat{Y}} \rangle_{\mathbb{N}}$ which implies that σ_3 and σ_4 are spherical roots.

Case $\widehat{G/P_5}$: The candidates in this case are $\sigma_3 = (-\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5, 0, 0)$ and $\sigma_4 = (-\lambda_2 + \lambda_3 + \lambda_4 - \lambda_5, 0, 0)$.

We compute

$$V_{(\lambda_3,2,1)} \cdot V_{(\lambda_3,2,1)} \supset V_{(\lambda_1+2\lambda_5,4,2)}.$$

Hence $\sigma_3 + \sigma_4 = (2\lambda_3, 4, 2) - (\lambda_1 + 2\lambda_5, 4, 2) \in \langle \Sigma_{\widehat{Y}} \rangle_{\mathbb{N}}$ which implies that σ_3 and σ_4 are spherical roots.

Case $\widehat{G/P_6}$: The only candidate in this case is $\sigma = (-\lambda_1 + 2\lambda_5, 0, 0)$.

We compute

$$V_{(\lambda_5,1,1)} \cdot V_{(\lambda_5,1,1)} \supset V_{(\lambda_1,2,2)}.$$

Hence $\sigma = (2\lambda_5, 2, 2) - (\lambda_1, 2, 2)$ is a spherical root.

11) G of type E_7, H of type $E_6 \times \mathbb{C}^*$:

Case $\widehat{G/P_1}$: The candidates in this case are $\sigma_1 = (-\lambda_1 + 2\lambda_2 - \lambda_6, 0, 0)$ and $\sigma_2 = (\lambda_1 - \lambda_2 + \lambda_6, 0, 0)$.

Here we compute

$$\begin{aligned} & V_{(\lambda_2,0,1)} \cdot V_{(\lambda_2,0,1)} \supset V_{(\lambda_1+\lambda_6,0,2)} \\ \text{and} \quad & V_{(\lambda_1,2,1)} \cdot V_{(\lambda_6,-2,1)} \supset V_{(\lambda_2,0,2)} \end{aligned}$$

It follows that $\sigma_1 = (2\lambda_2, 0, 2) - (\lambda_1 + \lambda_6, 0, 2)$ and $\sigma_2 = (\lambda_1 + \lambda_6, 0, 2)$ are spherical roots.

Case $\widehat{G/P_2}$: The candidates in this case are

$$\begin{aligned} \sigma_2 &= (\lambda_1 + \lambda_3 - \lambda_4, 0, 0), \\ \sigma_3 &= (-\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5, 0, 0), \\ \sigma_4 &= (-\lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6, 0, 0) \\ \sigma_5 &= (-\lambda_4 + \lambda_5 + \lambda_6, 0, 0). \end{aligned}$$

To show that these are spherical roots we compute

$$\begin{aligned} & V_{(\lambda_2,3,1)} \cdot V_{(\lambda_2,-3,1)} \supset V_{(\lambda_1+\lambda_6,0,2)}, \\ & V_{(\lambda_3,1,1)} \cdot V_{(\lambda_3,1,1)} \supset V_{(\lambda_2+\lambda_5,2,2)}, \\ \text{and} \quad & V_{(\lambda_5,-1,1)} \cdot V_{(\lambda_5,-1,1)} \supset V_{(\lambda_2+\lambda_3,-2,2)}. \end{aligned}$$

It follows that $2\sigma_1 + \sigma_3 + \sigma_4 = (2\lambda_2, 0, 2) - (\lambda_1 + \lambda_6, 0, 2) \in \langle \Sigma_{\widehat{Y}} \rangle_{\mathbb{N}}$, hence σ_3 and σ_4 are spherical roots.

Further $\sigma_2 + \sigma_3 = (2\lambda_3, 2, 2) - (\lambda_2 + \lambda_5, 2, 2) \in \langle \Sigma_{\widehat{Y}} \rangle_{\mathbb{N}}$, hence σ_2 is a spherical root.

Lastly $\sigma_4 + \sigma_5 = (2\lambda_5, -2, 2) - (\lambda_2 + \lambda_3, -2, 2)$, hence $\sigma_4 + \sigma_5 \in \langle \Sigma_{\widehat{Y}} \rangle_{\mathbb{N}}$. So also σ_5 is a spherical root.

Case $\widehat{G/P_7}$: The candidates in this case are $\sigma_1 = (2\lambda_1 - \lambda_6, 0, 0)$ and $\sigma_2 = (-\lambda_1 + 2\lambda_6, 0, 0)$.

We compute

$$\begin{aligned} &V_{(\lambda_1,-1,1)} \cdot V_{(\lambda_1,-1,1)} \supset V_{(\lambda_6,-2,2)} \\ \text{and} \quad &V_{(\lambda_6,1,1)} \cdot V_{(\lambda_6,1,1)} \supset V_{(\lambda_1,2,2)}. \end{aligned}$$

It follows that $\sigma_1 = (2\lambda_1, -2, 2) - (\lambda_6, -2, 2)$ and $\sigma_2 = (2\lambda_6, 2, 2) - (\lambda_1, 2, 2)$ are spherical roots.

This concludes the proof of Proposition 4.2. □

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Teilpublikationen:

B. Niemann, *Spherical affine cones in exceptional cases and related branching rules*, arXiv:1111.3823