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In Economics, cynicism is a proof of maturity.

(Conventional Wisdom)

Cynicism is an excellent talent. It allows to look into the future.

(Universal Truth)

CONTENTS

0. INTRODUCTION

This work is titled *Essays in Equilibrium Finance*. Topics described by the term Finance deal with issues that are broadly related to the valuation of assets. An equilibrium-perspective is one which not only looks at an individual agent and his specific optimization problem, but includes the feedback effects generated by the interactions with other agents. Both words together describe a common denominator of the three chapters contained in the dissertation at hand, but there is more. The chapters are connected by a deeper link that I will lay down in this introduction.

The first chapter, Open-Loop Equilibria and Perfect Competition in Option Exercise Games, a joint work with Professor Kerry Back published in the Review of Financial Studies (Back and Paulsen (2009)), is concerned with the optimal exercise and valuation of growth options within a partial equilibrium setting. A finite number of firms have to decide about their production capacities. When a firm invests and expands its capacities by a marginal unit, it receives the marginal cash flow generated by that unit in exchange. The price at which firms sell decreases with total production in the market but also fluctuates with a common stochastic factor. So does the value of the marginal cash-flow. It might rise in which case a firm might want to invest more or it might decrease so that the firm regrets having invested at all and would like to disinvest. However, investment is irreversible in the model; a firm cannot undo past investments and regain its cost.

It is well known (e.g. see Dixit and Pindyck (1994)) that irreversibility creates an option-like feature. Analogously to exercising an American call option by paying the strike and receiving the price of the underlying stock in exchange, the firm pays the investment cost and receives the present value of the marginal cash flow generated by that unit. It is usually not optimal for a monopolistic firm to exercise its investment option when its option is 'at the money', i.e when the present value of the marginal cash flow net of investment cost is zero. If the firm invests, it gets as much as it loses. If it waits with its decision a bit longer, however, the net present value exceeds zero with some chance while not investing still yields a non-negative return should the value go down. Delaying the investment decision therefore yields a strictly positive expected return at the point where the net present value is zero just as for an American call option.

This is why it was argued (e.g. see Dixit and Pindyck (1994)) that there is an additional opportunity cost of investing which has to be accounted for. An investing firm not only pays the direct investment cost but also scraps the option to invest the same unit at a later time. Proper accounting considers the value of this option and investment is not undertaken until its return, i.e. the value of the marginal cash flow, outweighs both: the direct cost of investment and the indirect cost in form of giving up the option of waiting to invest. As the option value usually rises in volatility, investment tends to be increasingly delayed when volatility rises.

But what if there is more than one firm competing for market shares? If demand increases and one firm delays its investment leaving its investment option on the table for a while, another firm might jump in, invest and thereby 'steal' the option. If a firm invests, on the other hand, it might even deter its competitors from investing themselves. This consideration suggests that there is in incentive to invest early for two reasons: To preempt other firms' investment and to preempt other firms' preemption. Intuition proofs right. Competing firms invest earlier in equilibrium. With the number of firms going to infinity, the value of the option of waiting to invest approaches zero. In the limit, investment is undertaken as soon as its net present value reaches zero.

These results are not new. In fact they already appear in Grenadier (2002). However, it turned out that, in this reference, the formulation of the strategies and the proof of an equilibrium contained substantial errors. Our contribution is to lay down these defects and provide a rigorous proof for the statement that the strategies - properly reinterpreted - form an open-loop equilibrium. As open loop strategies lack subgame perfectness, we further show that perfect competition forms a subgame perfect equilibrium already for two firms. So even though the investment game is Cournot in nature - the strategic variables are capacities - it leads to a Bertrand like outcome. This is due to the extreme incentive for preemption. Firms preemptively invest to avoid other firms preemptions. My own contribution, in particular, lies in reinterpreting the strategies, providing the rigorous proof as well as giving a counter-examples to Grenadier (2002).

Though, with perfect competition, firms invest when the net present value is zero, this does not imply that increased volatility does not lead to delayed investment. On the contrary, delaying investment when volatility increases is welfare maximizing (in the sense of maximizing consumer surplus net of investment cost) and thus efficient. This is quite intuitive noting that a profit maximizing firm is analogous to a welfare maximizing central planner.

With perfect competition, efficiency is maintained in the market equilibrium. With increasing volatility, the threshold price which triggers investment into additional capacities increases as well. The equilibrium price process plays an important role thereby (see also Leahy (1993)), an insight that can be made intuitive by the following reasoning: Prices in equilibrium follow a geometric Brownian motion reflected at some threshold. If prices rise too much, they reach the threshold at which investment has a zero net present value and investment pushes the price down again. If volatility increases, so does the probability in which prices are low. But at the threshold, the average price must be such that firms make zero profits so that lower prices have to be compensated for by some upside. The upside occurs in form of a higher threshold at which prices are reflected, i.e. in form of delayed investment. Thereby the price dynamics - more precisely the higher reflection threshold and the increase of the probability of low prices - reconciles investment delays with zero profits for firms in the partial equilibrium setup.

Also in general equilibrium, the central planner's problem is a 'monopolistic' one. The planner maximizes utility over all admissible investment paths just as a firm maximizes profits. As it is usually optimal for a monopolistic firm to utilize the option premium waiting to invest by delaying investment when volatility rises, it is natural to hypothesize that it also is for a welfare maximizing central planner. But how would a delay reconcile with perfect competition and zero profits for firms?

The answer from above is peculiar to a partial equilibrium setting in which prices can be set into relation to prices of alternative goods. In a general equilibrium setting with a single representative consumption good, however, there are no relative prices whose dynamics can comprise an option premium. Hence, the argument cannot be extended from partial to general equilibrium. So how can an option premium of waiting to invest materialize in general equilibrium, then?

This question is addressed in the second chapter, Optimal Timing of Aggregate Investment and the Yield Curve, within a stylized general equilibrium model with irreversible investment. It turns out that the argument given above is not precisely correct. While it is true that with a single consumption good there are no relative prices within a particular instant of time, i.e. there are no intra-period prices, prices can still be related on their intertemporal dimension. It are precisely the dynamics of intertemporal prices, i.e. interest rates and future prices, that reconcile investment delays with zero profits in the context of the model. Longer term interest rates and futures on wages contain the expected growth-effect of optimally exercised growth options, rendering current investment opportunities unprofitable whenever a delay is efficient. In this sense, the term-structure of future prices reflects the option premium of waiting and leads to optimal delay in investment.

Interestingly, this mechanism is similar to what Keynes termed the 'speculative motive' for money demand and liquidity preference. First note that for liquidity preference to play an economic role, physical capital cannot be perfectly liquid. For if there is perfect liquidity on the asset-supply side, demand for liquidity is always satisfied. So to even give a meaning to the term, it is a prerequisite to capture the fact that physical capital is less liquid than money or short term assets, i.e. that it is a long term matter. In Chapter 2 this is done by assuming that that investment is irreversible[1].

[1] Keynes bears testimony to the importance of illiquid physical capital: *It is by reason of the existence of durable equipment that the economic future is linked to the present. It is, therefore, consonant with, and agreeable to, our broad principles of thought, that the expectation of the future should affect the present through the demand price for durable equipment. (Keynes, 1936, page 145)*

In Keynes' General Theory, the interesting component[2] of liquidity preference, the interest rate elastic component, stems out of the speculative motive. Keynes defines the speculative motive as 'the object of securing profit from knowing better than the market what the future will bring forth (Keynes (1936, page 170))', for example an 'individual, who believes that future rates of interest will be above the rates assumed by the market, has a reason for keeping actual liquid cash' (Keynes (1936, page 170)). In other words, speculative money demand is due to speculation on higher long term interest rates and lower bond prices. This fits to the intuition which drives the argument in Chapter 2. Here, investors speculate on better future investment opportunities which is why they demand short term rather than long term assets, should long term yields be too low compared to rates that can be rationally expected for the future. Thereby, short term rates can drop below zero although the marginal product of capital is strictly positive. But while Keynes speculative motive is due to believes which deviate from market expectations and therefore based on either a market inefficiency or on heterogeneous expectations, the model in Chapter 2 relies on a no-arbitrage condition as connection between the long rate an short rate dynamics. It is based on rational expectations and therefore consistent with neo-classical thinking.

Thinking about Keynes' theory and the speculative motive in particular, naturally leads to questions linked to liquidity preference. Can the model rationalize a liquidity trap? What exactly is a liquidity trap? Is monetary policy really powerless when short term rates are near or below zero, or are there remedies? Regrettably, these questions are far too ambitious to be answered based on the model of Chapter 2. More elementary questions have to be answered first. It is the believe of the author that while introducing a role for money in an ad hoc way (e.g. by postulating a cash-in advance-constraint or putting money in the utility function) allows to produce numbers, but does not contribute to a meaningful monetary analysis as long as the role that money plays in the real world is not cleared. So what exactly is money (what is liquidity)?

The third and last chapter, 'Why Fiat Money is a Safe Asset', attempts to make one first step towards answering this question. It asks: Why do people exchange real goods against a piece of paper that neither provides intrinsic utility nor (unlike in Keynes times) constitutes a claim on a real good such as gold? Why is money a safe asset whose value people (can) rely upon?. In the model presented in Chapter 3, a slightly extended version published in Economics Letters (Paulsen (2012)), money is 'safe': Fiat money has strictly positive value in the unique trembling hand equilibrium. This holds as each bank note is both: a witness for the existence of some agent in the economy with debt, backed by collateral, and the only matter that allows the debtor to settle her debt. Debtors fear to lose the collateral and compete with each other for not defaulting, i.e. they compete for money. This creates money demand

[2] Without interest rate elasticity of money demand Keynes theory would collapse to the 'classic' quantity theory.

and thereby ensures positive money value. As not only a single but all debtors in the economy demand money, idiosyncratic shocks to solvency wash out. This makes fiat money a safe asset.

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1. OPEN-LOOP EQUILIBRIA AND PERFECT COMPETITION IN OPTION EXERCISE GAMES

Abstract

The investment boundaries defined by Grenadier (2002) for an oligopoly investment game determine equilibria in open loop strategies. As closed loop strategies, they are not equilibria, because any firm by investing sooner can preempt the investments of other firms and expropriate the growth options. The perfectly competitive outcome is produced by closed loop strategies that are mutually best responses. In this equilibrium, the option to delay investment has zero value, and the simple NPV rule is followed by all firms.

JEL classification: C73, D43, D92, G31, L13
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1.1 Introduction

This paper analyzes oligopoly investment under uncertainty, assuming capital investment is irreversible and capital stocks are instantaneously adjustable upward at a fixed price of capital. This oligopoly model is analyzed by Baldursson (1998) and Grenadier (2002). A similar model is analyzed by Leahy (1993) and Baldursson and Karatzas (1996) under the assumption of perfect competition and by Abel and Eberly (1996) and Merhi and Zervos (2005) under the assumption of monopoly. The investment policies described by all of these authors are singular, meaning that the investment rate is zero almost everywhere and undefined when investment occurs. The oligopoly model has important implications for the value of the option to delay investment – and hence the cost of ignoring this option and using the simple NPV rule for project choice – and may also be useful for understanding the dynamics of risk and return in equilibrium (see Novy-Marx (2007)).

The equilibrium concept in Baldursson (1998) is Nash, and strategies are stochastic processes adapted to the exogenous process that influences demand. This is an "open loop" concept, in the sense that there is no feedback from the investment of any firm to the investment of any other firm. It appears that Grenadier presents an equilibrium in closed loop strategies, but this is misleading. We show that his equilibrium is also open loop.

The distinction between open loop and closed loop (or feedback) strategies is well understood in the context of deterministic oligopoly investment games. See Fershtman and Muller (1984), Reynolds (1987), Tirole (1988), or Fudenberg and Tirole (1992). Equilibria in open loop strategies are unattractive because they fail subgame or Markov perfection. Open loop strategies are commitments to invest, depending on the history of demand in the stochastic context, regardless of the investments of other firms, even though there is no device in the game to make such commitments credible. For example, in an open loop equilibrium, if one firm deviates to invest more than the equilibrium strategy specifies, driving the price down, other firms ignore this and continue to invest as they would have. This is inconsistent with subgame perfection.[1]

There are difficulties in even defining the game in closed loop form. To do so would seem to require an extension of Simon and Stinchcombe's (Simon and Stinchcombe, 1989) analysis of deterministic continuous-time games with finite action sets to stochastic continuous-time games with continuum action sets. However, it is possible to show that the closed loop "trigger strategies" of Grenadier (2002) are not mutually best responses. By investing earlier, any firm can preempt the investments of other firms. To do so reduces the aggregate value of growth options but allows the preempting firm to expropriate growth options. We show that this tradeoff favors preemption.

Trigger strategies employing the perfectly competitive trigger (i.e., following the sim-

[1] More precisely, it is inconsistent with subgame perfection if each firm observes the output price and hence has at least partial information about other firms' outputs.

ple NPV rule) are mutually best responses. If one firm's strategy is to invest enough to ensure that aggregate industry capital equals the capital stock of a perfectly competitive industry, then any other firm might as well employ the same strategy. This is an extreme form of Reynolds's (Reynolds, 1987) observation in a deterministic model with quadratic adjustment costs that closed loop equilibria involve higher steady-state capital stocks than open loop equilibria, because "the preemptive or strategic element of investment behavior in the feedback Nash equilibrium influences the long run market outcome."

Perfect competition is of course also the outcome of Bertrand competition, so one might conjecture that playing the perfectly competitive trigger is implicitly competition in prices. We believe that this is the wrong interpretation. The game is one of competition in quantities (capital stocks). However, modeling time as continuous means that firms can instantaneously respond to others' investment choices. The basic assumption of the model is that arbitrarily large investments can be made instantaneously with no adjustment cost other than the fixed price of capital. Thus, at each instant in time, the game can be viewed as one of Stackelberg competition, in which each firm chooses its investment with all other firms instantaneously following. Naturally, each firm aspires to be the Stackelberg leader. A stable point, perhaps the only stable point, of this joint Stackelberg leadership is perfect competition.

The first author would like to note that his prior sole-authored working paper on this topic was excessively critical of Grenadier's concept of a myopic firm. That concept is indeed useful — to derive an open loop equilibrium. We prove that conditions similar to those in Grenadier's Proposition 3 are sufficient conditions for an open loop equilibrium.

The proof of open loop equilibrium is in Section 2. Section 3 discusses the difficulties with defining the game in closed loop form, the fact that the trigger strategies of Grenadier (2002) are not best responses to each other, and the fact that playing the perfectly competitive triggers are mutually best responses. Section 4 briefly concludes.

1.2 Open Loop Equilibrium

There are n firms in the industry. There is a constant required rate of return r. The capital stock of firm i at date t is denoted by Q_{it}, and we set $Q_{-it} = \sum_{j \neq i} Q_{jt}$. The cost of a unit of capital is normalized to 1. Capital does not depreciate, and investment is irreversible.

Consider a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and a one-dimensional standard Brownian motion B on the probability space. Let X be a solution of a stochastic differential equation

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t\,. \tag{1.1}$$

Assume $\sigma(X_t) \neq 0$ for all t almost surely. Define the running maximum

$$X_t^* = \max_{0\leq s\leq t} X_s\,.$$

Assume that the operating cash flow rate of firm i at date t depends on X_t, Q_{it} and Q_{-it}. Denote it by $\pi(X_t, Q_{it}, Q_{-it})$. Assume that π is twice continuously differentiable in (x, q_i, q_{-i}), increasing in x, and concave in q_i.
Denote marginal operating cash flow by

$$\zeta(x, q_i, q_{-i}) = \frac{\partial}{\partial q_i}\pi(x, q_i, q_{-i}) .$$

Assume $\zeta_{q_{-i}} \geq \zeta_{q_i}$ and $\zeta_{q_{-i}}(x, 0, q_{-i}) \leq 0$, where the subscripts denote partial derivatives. Though it is not required, we could take

$$\pi(x, q_i, q_{-i}) = P(x, q_i + q_{-i})q_i - C(q_i)$$

for some functions P and C. In that case, $P_q \leq 0$ and $C'' \geq 0$ imply $\zeta_{q_{-i}} \geq \zeta_{q_i}$. Assume further that ζ is increasing in x and that the integrability constraint

$$\mathbb{E}\int_0^\infty e^{-rt} \sup_{a \leq q_{-i} \leq b} |\zeta(X_t, q_i, q_{-i})| dt < \infty . \tag{1.2}$$

is satisfied for each fixed triple (q_i, a, b).[2]
Denote the initial capital stock of firm i by q_{i0}. The set of admissible open loop strategies of firm i is

$$\mathcal{A}(q_{i0}) = \{(Q_{it})_{t\geq 0} \mid \text{nondecreasing, left-continuous, } \mathcal{F}_t\text{-adapted}, Q_{i0} \geq q_{i0}\} . \tag{1.3}$$

If each firm $j \neq i$ plays an open loop strategy, then the stochastic process Q_{-i} is an exogenous $\mathcal{F}_t$-adapted process from the point of view of firm i. Firm i chooses $Q_i \in \mathcal{A}(q_{i0})$ to maximize

$$\Pi(Q_i, Q_{-i}) = \mathbb{E}\int_0^\infty e^{-rt}\left[\pi(X_t, Q_{it}, Q_{-it})\, dt - dQ_{it}\right] . \tag{1.4}$$

An open loop equilibrium — i.e., a Nash equilibrium in open loop strategies — is an n-tuple $(Q_1^*, \ldots, Q_n^*)$ of admissible strategies such that, for each i,

$$Q_i^* \in \operatorname{argmax}_{Q_i \in \mathcal{A}(q_{i0})} \Pi(Q_i, Q_{-i}^*) . \tag{1.5}$$

The function m described in the proposition below should be interpreted as a marginal value function (marginal with respect to q_i). It is also the value function of the optimal stopping problem (1.9) defined below. The hypotheses of the proposition are similar to those in Grenadier's Proposition 3, though Grenadier's assumptions regard the functions

$$(x, q) \mapsto m(x, q, (n-1)q) \qquad \text{and} \qquad q \mapsto X(q, (n-1)q) ,$$

i.e., the values of m and X along a ray in the (q_i, q_{-i}) domain, whereas our assumptions concern m and X on their entire domains. The conclusion differs regarding the nature of equilibrium — open loop instead of closed loop. The proposition is proven in Appendix 1.5.

[2] The integrability constraint is used to deduce convergence of expectations in the proof of Proposition 1. It can be replaced by $\zeta \geq 0$, using the monotone convergence theorem in the proof.

Proposition 1.2.1. *Suppose there exist functions $m(x, q_i, q_{-i})$ and $X(q_i, q_{-i})$ satisfying the following conditions:*

1. *$X(q_i, q_{-i})$ is differentiable in q_{-i} and continuous in q_i.*
2. *$X(q_i, q_{-i})$ and $X(q_i, (n-1)q_i)$ are increasing in q_i.*
3. *m is bounded from below for each fixed pair (q_i, q_{-i}), twice continuously differentiable with respect to x, and once continuously differentiable with respect to (q_i, q_{-i}).*
4. *m is monotonically increasing in x for $x \leq X(q_i, q_{-i})$.*
5. *m solves the PDE*
$$\mu m_x + \frac{1}{2}\sigma^2 m_{xx} - rm + \zeta = 0 \tag{1.6}$$
on the region $x < X(q_i, q_{-i})$.
6. *$m(X(q_i, q_{-i}), q_i, q_{-i}) = 1$ (value matching).*
7. *$m_x(X(q_i, q_{-i}), q_i, q_{-i}) = 0$ (smooth pasting).*

Then the following are true:

(A) Myopic Optimality. If $Q_{jt} = q_{j0}$ for all $j \neq i$ and all $t \geq 0$, then
$$Q_{it} = \inf\{q_i \geq q_{i0} \mid X_t^* \leq X(q_i, q_{-i0})\} \tag{1.7}$$
maximizes (1.4) over $Q_i \in \mathcal{A}(q_{i0})$.

(B) Symmetric Open Loop Equilibrium. Suppose $q_{i0} = q_{j0}$ for all i and j. Then $(Q_1^*, \ldots, Q_n^*)$ is an open loop equilibrium, where, for each i,
$$Q_{it}^* = \inf\{q_i \geq q_{i0} \mid X_t^* \leq X(q_i, (n-1)q_i)\}\ . \tag{1.8}$$

Assumptions 3–7 ensure that the function m fulfills the criteria of a Hamilton-Jacobi-Bellman verification theorem for the optimal stopping problem
$$\min_{\tau} \mathbb{E}\left[\int_0^{\tau} e^{-rt}\zeta(X_t, q_i, q_{-i})dt + e^{-r\tau}\right]\ . \tag{1.9}$$

More precisely, m is the value function of this stopping problem, and the hitting time of the boundary $X(q_i, q_{-i})$ is an optimal stopping rule. The optimal stopping problem can be interpreted as the problem of a firm to optimally install a unit of capital under the myopic assumption that rival firms will never do so and that no further unit can be installed. In the formulation (1.9), the firm minimizes the opportunity cost (the foregone marginal cash flow) of not investing plus the discounted cost of investing.

The smooth pasting condition deserves a comment. Heuristically it can be derived by the envelope theorem. Namely, let $M(x, y)$ be the expected value in (1.9) when $X_0 = x$

and the stopping time is the first hitting time of y. By the value matching condition, $M(y, y) = 1$, so differentiating this with respect to y yields $M_x(y, y) + M_y(y, y) = 0$. However, if y^* is optimal, then $M_y(\cdot, y^*)$ should equal zero. So $M_x(y^*, y^*) = 0$ at the optimal boundary y^*.

There is a large literature on the connection between singular stochastic control problems and optimal stopping problems, starting with Karatzas and Shreve (1984). The equivalence between the control problem and the stopping problems here is the same as in Theorem 1 of Bank (2006). What is new about our proof, as far as we know, is the solution of the stopping problems in the presence of the exogenous singular process Q_{-i}.

Now we apply Proposition 1.2.1 to the linear model studied by Baldursson and Karatzas (1996) and the constant elasticity example considered by Grenadier (2002). Grenadier and Baldursson present the value of $X(q_i, q_{-i})$ when $q_{-i} = (n-1)q_i$. The entire function $X(q_i, q_{-i})$ is presented below.

Constant Elasticity

Assume $\pi(x, q_i, q_{-i}) = P(x, q_i + q_{-i})q_i$ with $P(x, q) = xq^{-1/\gamma}$. Assume $\gamma > 1$. Assume X is a geometric Brownian motion with parameters μ and σ; i.e., $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$ with a slight abuse of notation. Assume $r > \mu$ and $\beta > \gamma$, where

$$\beta = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}}{\sigma^2}. \tag{1.10}$$

These restrictions on the parameters imply that the model satisfies our assumptions regarding π. We have

$$\zeta(x, q_i, q_{-i}) = \left(1 - \frac{q_i}{\gamma(q_i + q_{-i})}\right) x(q_i + q_{-i})^{-1/\gamma}.$$

Proposition 1.2.2. *In the constant elasticity model, there is a unique pair (m, X) satisfying the conditions of Proposition 1,*[3] *and*

$$X(q_i, q_{-i}) = \frac{\beta}{\beta - 1}\left(\frac{\gamma}{\gamma - q_i/(q_i + q_{-i})}\right)(r - \mu)(q_i + q_{-i})^{\frac{1}{\gamma}}, \tag{1.11a}$$

$$m(x, q_i, q_{-i}) = \frac{\zeta(x, q_i, q_{-i})}{r - \mu} - \frac{\zeta(X(q_i, q_{-i}), q_i, q_{-i})}{(r - \mu)\beta}\left(\frac{x}{X(q_i, q_{-i})}\right)^{\beta}. \tag{1.11b}$$

Proof. The general solution to the PDE (1.6) is

$$m(x, q_i, q_{-i}) = \frac{\zeta(x, q_i, q_{-i})}{r - \mu} + Ax^{\beta} + Bx^{\beta'}$$

with β given by (1.10), and $\beta' = 1 - \beta - 2\mu/\sigma^2 < 0$. Assumptions 3 and 4 imply m is bounded in x on the interval $(0, X(q_i, q_{-i}))$, so $B = 0$. Solving conditions 6 and 7 in the unknowns A and $X(q_i, q_{-i})$ yields (1.11). □

[3] Uniqueness of m is on the domain $\{(x, q_i, q_{-i}) \mid x \leq X(q_i, q_{-i})\}$.

Linear Demand

Assume $\pi(x, q_i, q_{-i}) = P(x, q_i+q_{-i})q_i - cq_i$, with $P(x, q) = x - bq$, for constants b and c. Assume X is a geometric Brownian motion with parameters μ and σ. Assume $\beta > 2$, where β is defined in (1.10). Then the model satisfies our assumptions regarding π. We have $\zeta(x, q_i, q_{-i}) = x - 2bq_i - bq_{-i} - c$.

Proposition 1.2.3. *In the linear model, there is a unique pair (m, X) satisfying the conditions of Proposition 1.2.1, and*

$$X(q_i, q_{-i}) = \frac{\beta}{\beta - 1}\left(\frac{r-\mu}{r}\right)(c + r + 2bq_i + bq_{-i}), \tag{1.12a}$$

$$m(x, q_i, q_{-i}) = \frac{-2bq_i - bq_{-i} - c}{r} + \frac{x}{r-\mu} - \frac{x^\beta}{(r-\mu)\beta X(q_i, q_{-i})^{\beta-1}}. \tag{1.12b}$$

Proof. The general solution to the PDE (1.6) is:

$$m(x, q_i, q_{-i}) = \frac{\zeta(x, q_i, q_{-i})}{r} + Ax^\beta + Bx^{\beta'}$$

with β as given in (1.10) and $\beta' = 1 - \beta - 2\mu/\sigma^2 < 0$. For the same reasons as in the constant elasticity case, $B = 0$. Solving the smooth pasting and the value matching condition for A and X yields (1.12). □

1.3 Closed Loop Strategies and Best Responses

First, we admit we do not know how to define this game in closed loop form. There are substantial complications in doing so. If the capital stock processes were absolutely continuous instead of singular, one would view the investment rate $z_{it} = dQ_{it}/dt$ as the decision variable of firm i at date t. If each z_{it} were required to depend on the history of $(X, Q_1, \ldots, Q_n)$ prior to t in a sufficiently regular way, then the capital stock processes

$$Q_{it} = q_{i0} + \int_0^t z_{it}\, dt$$

would be well defined. With singular controls, one could view the action of firm i at any date t as being the Lebesgue-Stieltjes differential dQ_{it} of its capital stock process Q_i; however, these differentials are meaningful only in integrated form. An alternate view is that the action of firm i at date t is its total capital Q_{it}, chosen subject to the constraint that capital is irreversible: $Q_{it} \geq \sup_{s<t} Q_{is}$. However, this suffers from the general problem with continuous-time games that what seem to be well-defined strategies may not produce well-defined outcomes (see Simon and Stinchcombe, 1989). For example, the formula $Q_{it} = \sup_{s<t} Q_{is}$ seems to specify Q_{it} as a function of the history of play prior to t; yet, every nondecreasing left-continuous process Q_i satisfies the formula. Likewise, the formulas $Q_{it} = \lim_{s\uparrow t} Q_{js}$ and $Q_{jt} = \lim_{s\uparrow t} Q_{is}$ seem to define each firm's capital stock at time t as a function of the other firm's prior investments, but these formulas are satisfied by every left-continuous $Q_i = Q_j$. Thus,

formulas such as these — and one could give an arbitrary number of similar examples — are very far from uniquely specifying how the game is to be played. In order to define the game, some rules must be constructed to allow one to map such formulas, or whatever strategies are allowed, into unique outcomes. Simon and Stinchcombe (1989) accomplish this for deterministic games with finite action sets. Generalizing their work to the present context, and then finding equilibria, would seem to be substantial tasks.

Though we do not know how to define strategies in general, there are some combinations of decision rules that clearly produce well-defined outcomes. Grenadier's Proposition 1 states: "Each firm's investment strategy is characterized by increasing output incrementally whenever $X(t)$ rises to the trigger function $\bar{X}(q_i, Q_{-i})$."[4] Though Grenadier's statement is not a precise description of a strategy, it seems reasonable to take its meaning to be

$$Q_{it} = \inf \left\{ q_i \geq q_{i0} \,\middle|\, \sup_{0 \leq s \leq t} [X_s - X(q_i, Q_{-is})] \leq 0 \right\}. \tag{1.13}$$

Given a stochastic process Q_{-i}, this defines Q_i as the smallest nondecreasing process such that $X_t \leq X(Q_{it}, Q_{-it})$ for all t. Note that Q_{it} is allowed to depend on the contemporaneous Q_{-it}. This seems reasonable for all $t > 0$ if we restrict to left-continuous paths.[5] Decision rules of the form (1.13) do not necessarily produce well-defined outcomes. For example, in a two-firm game, if both firms play (1.13) for $X(q_i, q_{-i}) = q_i + q_{-i}$, then the division of output between the two firms is not defined. However, if all firms play (1.13) for the open loop equilibrium investment boundaries in the constant elasticity and linear examples — i.e., for X defined in (1.11a) or (1.12a) — then the capital paths of all firms are well defined. In fact, the strategies (1.13) with the open loop equilibrium investment boundaries produce the open loop equilibrium capital processes.

If all firms play (1.13) for an increasing $X(\cdot)$ and the paths are well defined, then any firm can preempt the investments of other firms by investing aggressively itself. The following proposition shows that the open loop equilibrium boundary in the linear model does not define a closed loop equilibrium, because preemption is a profitable deviation (the strategies (1.14) and (1.15b) are the strategies asserted by Grenadier to constitute an equilibrium in the linear model).

Proposition 1.3.1. *Suppose X is a geometric Brownian motion with drift μ and volatility σ. Assume $\pi(x, q_i, q_{-i}) = (X - b(q_i + q_{-i}) - c)q_i$ for constants $b > 0$ and $c \geq 0$. Assume $q_{i0} = q_{j0}$ for all i and j, and define $q_0 = \sum_{i=1}^n q_{i0}$. Assume $\beta > 2$,*

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[4] $\bar{X}(q_i, Q_{-i})$ equals the myopic trigger $X(q_i, Q_{-i})$ from Proposition 1.2.1 (see Grenadier's Proposition 2).

[5] If $X_0 > X(q_{i0}, q_{-i0})$, then (1.13) implies a jump at time 0. It allows this jump to depend on simultaneous jumps of other firms. This seems unreasonable, but one could view it as a reduced form for nearly instantaneous reactions. Simon and Stinchcombe (1989) discuss this issue.

where β *is defined in* (1.10). *Assume for all* $j \neq i$ *that*

$$Q_{jt} = \inf\left\{ q_j \geq q_{j0} \,\middle|\, \sup_{0\leq s\leq t}\left[X_s - \frac{\beta}{\beta-1}\left(\frac{r-\mu}{r}\right)(c+r+2bq_j+bQ_{-js})\right] \leq 0 \right\}. \tag{1.14}$$

Define

$$\tau = \inf\left\{ t \geq 0 \,\middle|\, X_t^* \geq \frac{\beta}{\beta-1}\left(\frac{r-\mu}{r}\right)\left(c+r+\frac{(n+1)bq_0}{n}\right)\right\}.$$

There exists $\alpha > 1$ *such that the open loop strategy*

$$Q_{it}^{\alpha} = \begin{cases} q_{i0} & \text{for } t \leq \tau\,, \\ \inf\left\{ q_i \geq \alpha q_{i0} \mid X_t^* \leq \frac{\beta}{\beta-1}\left(\frac{r-\mu}{r}\right)\left(c+r+\frac{n+\alpha}{\alpha}bq_i\right)\right\} & \text{for } t > \tau\,, \end{cases} \tag{1.15a}$$

produces higher expected discounted cash flows for firm i *than does the closed loop strategy*

$$Q_{it} = \inf\left\{ q_i \geq q_{i0} \,\middle|\, \sup_{0\leq s\leq t}\left[X_s - \frac{\beta}{\beta-1}\left(\frac{r-\mu}{r}\right)(c+r+2bq_i+bQ_{-is})\right] \leq 0 \right\}. \tag{1.15b}$$

Proof. The unique n–tuple $(Q_1,\ldots,Q_n)$ of stochastic processes satisfying (1.14) and (1.15b) is the open loop equilibrium $(Q_1^*,\ldots,Q_n^*)$ defined in Proposition 1 and Proposition 1.2.3. Let $\alpha \geq 1$. The unique n–tuple $(Q_1,\ldots,Q_n)$ of stochastic processes satisfying (1.14) and (1.15a) is $(Q_1^{\alpha},\ldots,Q_n^{\alpha})$, where Q_i^{α} is defined in (1.15a) and

$$(\forall\, j \neq i) \quad Q_{jt}^{\alpha} = \begin{cases} q_{j0} & \text{for } t \leq \tau\,, \\ Q_{it}^{\alpha}/\alpha & \text{for } t > \tau\,. \end{cases} \tag{1.16}$$

To see this, note that the equality of the Q_j for $j \neq i$ implies $Q_{-jt} = (n-2)Q_{jt} + Q_{it}^{\alpha}$. Making this substitution in (1.14), we have

$$Q_{jt} = \max\left\{ q_{j0}, \sup_{0\leq s\leq t} \frac{1}{nb}\left[\frac{\beta-1}{\beta}\left(\frac{r}{r-\mu}\right)X_s - c - r - bQ_{is}^{\alpha}\right]\right\}. \tag{17a}$$

Moreover, (1.15a) implies, for $t > \tau$,

$$Q_{it}^{\alpha} = \alpha \max\left\{ q_{i0}, \frac{1}{(n+\alpha)b}\left[\frac{\beta-1}{\beta}\left(\frac{r}{r-\mu}\right)X_t^* - c - r\right]\right\}. \tag{17b}$$

Substituting (17b) in (17a) yields (1.16).
Note that, for $\alpha = 1$, $Q_j^{\alpha} = Q_j^*$ for all $j = 1,\ldots,n$. Define $F(\alpha) = \Pi(Q_i^{\alpha}, Q_{-i}^{\alpha})$, where $\Pi(\cdot)$ is the expected discounted cash flow defined in (1.4). The claim is that $F(\alpha) > F(1)$ for some $\alpha > 1$. We show in Appendix B that the right-hand derivative of F at $\alpha = 1$ is positive. Thus, $F(\alpha) > F(1)$ for all sufficiently small $\alpha > 1$. □

The preemption strategy (1.15a) involves a limited amount of preemption: jumping to a market share of $\alpha/(n+\alpha-1)$ and maintaining that market share forever. For some parameter values in the linear model, expropriating all of the growth options is a profitable deviation from (1.15b). To explain what it means to expropriate all of the growth options, consider the constant elasticity example and the boundary (1.11a). Note that

$$\frac{q_i}{q_i+q_{-i}} \to 0 \qquad \Rightarrow \qquad X(q_i, q_{-i}) \to \frac{\beta}{\beta-1}(r-\mu)(q_i+q_{-i})^{\frac{1}{\gamma}} . \tag{1.18}$$

The limit of $X(q_i, q_{-i})$ displayed in (1.18) is the perfectly competitive investment boundary defined by Leahy (1993). It is the boundary at which a firm with infinitesimal market share would invest. Thus, if some firm j plays (1.13), i.e.,

$$Q_{jt} = \inf\left\{ q_j \geq q_{j0} \;\middle|\; \sup_{0\leq s\leq t} \left[X_s - X(q_j, Q_{-js})\right] \leq 0 \right\} , \tag{1.19}$$

then the behavior of firm j will approach that of a perfectly competitive firm as its market share decreases. If firm i invests sufficiently aggressively that it deters the investments of other firms, then the market shares of other firms will gradually decline toward zero, and their behavior under the decision rule (1.19) with boundary (1.11a) will approach that of perfect competition. Thus, aggregate output and price will approach the perfectly competitive output and price, and the value of industry growth options will eventually be destroyed. In exchange for this diminution of industry growth options, the preempting firm can expropriate all growth options to itself. We have calculated, though it is not presented here, that expropriating all of the growth options is a profitable deviation from (1.15b) in the linear model for some parameter values. Paulsen (2006) shows that preempting for a finite period of time is a profitable deviation in the constant elasticity model for some parameter values.

The perfectly competitive boundary is immune to preemption. Suppose, in the constant elasticity example, that each firm plays

$$Q_{it} = \inf\left\{ q_i \geq q_{i0} \;\middle|\; \sup_{0\leq s\leq t} \left[X_s - \frac{\beta}{\beta-1}(r-\mu)(q_i+Q_{-is})^{\frac{1}{\gamma}}\right] \leq 0 \right\} . \tag{1.20}$$

Given symmetric initial conditions and initial industry capital q_0, equation (1.20) holds for each i whenever the Q_i are nondecreasing processes such that industry capital $Q_t = \sum_{i=1}^n Q_{it}$ satisfies

$$Q_t = \inf\left\{ q \geq q_0 \mid X_t^* \leq \frac{\beta}{\beta-1}(r-\mu)q^{\frac{1}{\gamma}} \right\} .$$

Thus, (1.20) suffers from the problem discussed in the first paragraph of this section: It does not produce well-defined individual firm capital processes. However, it does produce well-defined individual firm values, which is the key requirement for choosing among strategies. Because industry growth options have zero value when investing at the perfectly competitive boundary, it does not matter how growth is distributed among the firms. Moreover, if all other firms play (1.20), then it is optimal for each individual firm to play (1.20), because the price process is unaffected by an individual firm playing (1.20) when other firms also play (1.20), and the investments from playing (1.20) are zero NPV when the price process is taken as given. The only suboptimal decision a firm could make when other firms play (1.20) is to invest *before* the perfectly competitive boundary is reached, and this does not occur when a firm plays (1.20). Thus, the strategies (1.20) are mutually best responses.

Though the strategies (1.20) are choices of quantities, not prices, the outcome is like Bertrand in that the "economic value added" of each firm is zero. Related to this is another feature the equilibrium shares with Bertrand: the strategies are weakly dominated. Investing zero at all times is as valuable as making zero NPV investments, and it is superior to playing (1.20) if other firms play investment strategies that are less aggressive than (1.20).

1.4 Conclusion

Open loop equilibria have an extreme Cournot nature: Each firm optimizes taking the entire output process of each other firm as given. They fail subgame perfection, because if a firm invests aggressively the game will reach a node from which the given output processes of other firms will not be part of a Nash equilibrium starting from that node. The closed loop strategies (1.13) have the potential to form subgame perfect equilibria, because each firm reacts to the investments of others. These strategies have a Stackelberg flavor, because all firms react to the investments of any firm like Stackelberg followers, and hence each firm is like a Stackelberg leader. A stable point of this joint Stackelberg leadership is perfect competition.

The closed loop strategies (1.13) employing the myopic (open loop equilibrium) boundary do not form an equilibrium, because any firm by investing more can cause other firms to invest less, like a Stackelberg leader, and hence expropriate some of the growth options. We proved that this preemptive investment is a profitable deviation in the linear model. Paulsen (2006) shows the same for the constant elasticity model, for some parameter values.

It is an open question whether the perfectly competitive boundary is the unique boundary such that closed loop strategies of the form (1.13) are impervious to preemption. If so, the perfectly competitive boundary would be the unique boundary such that the strategies (1.13) could constitute an equilibrium.

It seems likely that there would be other closed-loop equilibria, if the strategy spaces and mapping from strategy n–tuples to outcomes could be specified. There should be equilibria in punishment strategies, in which firms invest less than the perfectly competitive amount and threaten to punish any firm that deviates. These strategies are *not* of the form: invest when $X_t = X(Q_{it}, Q_{-it})$ for an increasing function X, because strategies of this form prescribe less investment when competitors invest more and hence do not allow for punishment.

Appendix

1.5 Proof of Proposition 1.2.1

Define

$$\bar{m}(x, q_i, q_{-i}) = \begin{cases} m(x, q_i, q_{-i}) & \text{if } x \le X(q_i, q_{-i}) \,, \\ 1 & \text{otherwise} \,. \end{cases} \tag{1.5.1}$$

Lemma 1.5.1. *$\bar{m}_{q_{-i}}(x, q_i, q_{-i}) = 0$ for $x \ge X(q_i, q_{-i})$.*

Proof. By the value-matching condition, we have

$$\bar{m}(X(q_i, q_{-i}), q_i, q_{-i}) = 1 \,.$$

Differentiating this equation with respect to q_{-i} yields

$$\bar{m}_x(X(q_i, q_{-i}), q_i, q_{-i}) X_{q_{-i}}(q_i, q_{-i}) + \bar{m}_{q_{-i}}(X(q_i, q_{-i}), q_i, q_{-i}) = 0 \,.$$

As the first term vanishes by the smooth-pasting condition, we get

$$\bar{m}_{q_{-i}}(X(q_i, q_{-i}), q_i, q_{-i}) = 0 \,.$$

This proves the claim for $x = X(q_i, q_{-i})$. For $x > X(q_i, q_{-i})$, $\bar{m}(x, q_i, z) = 1$ for z in a neighborhood of q_{-i} by definition of $\bar{m}$. Hence, differentiation yields the result. □

Lemma 1.5.2. *We have (for all $x \neq X(q_i, q_{-i})$)*

$$-r\bar{m} + \mu\bar{m}_x + \frac{1}{2}\sigma^2\bar{m}_{xx} \ge -\zeta$$

with equality for $x < X(q_i, q_{-i})$.

Proof. For $x < X(q_i, q_{-i})$ equality holds by Assumption 5 of Proposition 1.2.1. Assumptions 4 and 7 imply that $m_{xx} \le 0$ for $x = X(q_i, q_{-i})$. Therefore

$$-rm + \mu m_x \ge -\zeta$$

at $x = X(q_i, q_{-i})$. Using the smooth pasting condition we get

$$-r\bar{m} = -rm \ge -\zeta$$

at $x = X(q_i, q_{-i})$. Note that ζ increases in x while $\bar{m}$ remains constant, so

$$-r\bar{m} + \mu\bar{m}_x + \frac{1}{2}\sigma^2\bar{m}_{xx} = -r\bar{m} + 0 + 0 \ge -\zeta$$

for $x > X(q_i, q_{-i})$. □

Consider the myopic problem indexed by $y = q_{i0}$ and $z = q_{-i0}$:

$$\max_{Q_i \in \mathcal{A}(y)} \quad \mathbb{E}\int_0^\infty e^{-rt}\left(\pi(X_t, Q_{it}, z)\, dt - dQ_{it}\right) \,. \tag{1.5.2}$$

Related to this problem is the following optimal stopping problem: Choose a stopping time τ to maximize

$$\mathbb{E}\left[\int_\tau^\infty e^{-rt}\zeta(X_t, y, z)\, dt - e^{-r\tau}\right] = \mathbb{E}\left[\int_\tau^\infty e^{-rt}\left(\zeta(X_t, y, z) - r\right)\, dt\right] \,. \tag{1.5.3}$$

To interpret the optimal stopping problem, notice that a small investment ϵ at time τ increases expected discounted revenues by approximately

$$\epsilon \cdot \mathbb{E} \int_{\tau}^{\infty} e^{-rt} \zeta(X_t, y, z)\, dt$$

and has expected discounted cost equal to $\epsilon \cdot \mathbb{E}\left[e^{-r\tau}\right]$. The optimal stopping time is therefore the optimal time to make a investment of size $\epsilon \approx 0$. Subtracting $\mathbb{E}\left[\int_0^\infty e^{-rt}\zeta(X_t, y, z)\, dt\right]$ and multiplying by (-1) converts the problem of maximizing (1.5.3) to the following equivalent problem:

$$\min_{\tau} \quad \mathbb{E}\left[\int_0^\tau e^{-rt}\zeta(X_t, y, z)\, dt + e^{-r\tau}\right]. \tag{1.5.4}$$

This can be interpreted as minimizing the opportunity cost of a unit of capital.
We also consider the equilibrium problem indexed by $y = q_{i0}$ and $z = q_{-i0}$. In this problem, we assume the aggregate capital of firms $j \neq i$ is

$$L_{zt} = \inf\left\{q \geq z \mid \max_{0 \leq s \leq t} X_s \leq X(q/(n-1), q)\right\}. \tag{1.5.5}$$

The optimization problem for firm i that we study is:

$$\max_{Q_i \in \mathcal{A}(y)} \quad \mathbb{E}\int_0^\infty e^{-rt}\left(\pi(X_t, Q_{it}, L_{zt})\, dt - dQ_{it}\right). \tag{1.5.6}$$

The related optimal stopping problem is: Choose a stopping time τ to maximize

$$\mathbb{E}\left[\int_\tau^\infty e^{-rt}\zeta(X_t, y, L_{zt})\, dt - e^{-r\tau}\right] = \mathbb{E}\left[\int_\tau^\infty e^{-rt}\left(\zeta(X_t, y, L_{zt}) - r\right)\, dt\right], \tag{1.5.7}$$

which is equivalent to:

$$\min_{\tau} \quad \mathbb{E}\left[\int_0^\tau e^{-rt}\zeta(X_t, y, L_{zt})\, dt + e^{-r\tau}\right]. \tag{1.5.8}$$

For any y and z, define

$$\tau_{yz} = \inf\{t \mid X_t > X(y, z)\}. \tag{1.5.9}$$

Lemma 1.5.3. *τ_{yz} solves the myopic stopping problem* (1.5.4).

Proof. By an approximation argument as in Øksendal (2002) (see Theorem 10.4.1), we can assume that $\bar{m}$ is twice continuously differentiable with respect to x. Let τ be an arbitrary stopping time. Applying Itô's rule to $e^{-r(t\wedge\tau)}\bar{m}(X_{t\wedge\tau}, y, z)$ yields:

$$\begin{aligned} e^{-r(t\wedge\tau)}\bar{m}(X_{t\wedge\tau}, y, z) = \bar{m}(X_0, y, z) &+ \int_0^{t\wedge\tau} e^{-rs}\bar{m}_x \sigma\, dB_s \\ &+ \int_0^{t\wedge\tau} e^{-rs}(-r\bar{m} + \mu\bar{m}_x + \frac{1}{2}\sigma^2\bar{m}_{xx})\, ds. \end{aligned} \tag{1.5.10}$$

Applying Lemma 1.5.2 to (1.5.10), we get

$$\begin{aligned} e^{-r(t\wedge\tau)}\bar{m}(X_{t\wedge\tau}, y, z) \quad \geq \quad & \bar{m}(X_0, y, z) + \int_0^{t\wedge\tau} e^{-rs}\bar{m}_x\sigma\, dB_s \\ & - \int_0^{t\wedge\tau} e^{-rs}\zeta_s\, ds, \end{aligned}$$

with equality for $\tau \leq \tau_{yz}$. We cannot directly take expectations on both sides as we do not know whether the stochastic integral is a martingale or just a local martingale. So let $\tau_k \leq k$ be a localizing sequence for the stochastic integral. That is, $\tau_k \uparrow \infty$ and the stopped integrals are martingales. Taking expectations on both sides and using Doob's optional sampling theorem yields, for each k,

$$\mathbb{E}\left[e^{-r(\tau_k\wedge\tau)}\bar{m}(X_{\tau_k\wedge\tau}, y, z)\right] \quad \geq \quad \bar{m}(X_0, y, z) - \mathbb{E}\left[\int_0^{\tau_k\wedge\tau} e^{-rs}\zeta_s\, ds\right].$$

Observe that $\bar{m}$ is bounded from above by 1 and bounded from below by Assumption 3 in Proposition 1.2.1, whereas the integrals involving ζ are uniformly integrable by the integrability assumption (1.2). Taking the limit $k \to \infty$ we get

$$\begin{aligned}\bar{m}(X_0, y, z) \quad &\leq \quad \lim_{k\to\infty}\mathbb{E}\left[e^{-r(\tau_k\wedge\tau)}\bar{m}(X_{\tau_k\wedge\tau}, y, z)\right] + \lim_{k\to\infty}\mathbb{E}\left[\int_0^{\tau_k\wedge\tau} e^{-rs}\zeta_s\, ds\right] \\ &= \quad \mathbb{E}\left[e^{-r\tau}\bar{m}(X_\tau, y, z)\right] + \mathbb{E}\left[\int_0^{\tau} e^{-rs}\zeta_s\, ds\right],\end{aligned}$$

or

$$\begin{aligned}\bar{m}(X_0, y, z) \quad &\leq \quad \mathbb{E}\left[\int_0^{\tau} e^{-rs}\zeta_s\, ds\right] + \mathbb{E}\left[e^{-r\tau}\bar{m}(X_\tau, y, z)\right] \\ &\leq \quad \mathbb{E}\left[\int_0^{\tau} e^{-rs}\zeta_s\, ds\right] + \mathbb{E}\left[e^{-r\tau}\right],\end{aligned}$$

with equality for $\tau = \tau_{yz}$.

□

Lemma 1.5.4. *If $y = z/(n-1)$, then τ_{yz} solves the equilibrium stopping problem (1.5.8).*

Proof. We proceed as in the proof of Lemma 1.5.3 with the difference that now $dL_{zt} \neq 0$. Let τ be an arbitrary stopping time. Applying Itô's rule to $e^{-r(t\wedge\tau)}\bar{m}(X_{t\wedge\tau}, y, L_{z,t\wedge\tau})$ yields:

$$\begin{aligned}e^{-r(t\wedge\tau)}\bar{m}(X_{t\wedge\tau}, y, L_{t\wedge\tau,z}) = \bar{m}(X_0, y, z) + \int_0^{t\wedge\tau} e^{-rs}\bar{m}_x\sigma\, dB_s \\ + \int_0^{t\wedge\tau} e^{-rs}(-r\bar{m} + \mu\bar{m}_x + \frac{1}{2}\sigma^2\bar{m}_{xx})\, ds + \int_0^{t\wedge\tau} e^{-rs}\bar{m}_{q_{-i}}\, dL_{zs}\,. \qquad (1.5.11)\end{aligned}$$

Note that L increases only when $X_t = X(L_{zt}/(n-1), L_{zt})$. We have $L_{zt}/(n-1) \geq z/(n-1) = y$. By monotonicity of X, it follows that L increases only when $X_t \geq X(y, L_{zt})$. In this case, Lemma 1 implies

$$\bar{m}_{q_{-i}}(X_t, y, L_{zt})\, dL_{zt} = 0\,. \qquad (1.5.12)$$

Applying (1.5.12) and Lemma 2 to (1.5.10), we get

$$\begin{aligned}e^{-r(t\wedge\tau)}\bar{m}(X_{t\wedge\tau}, y, L_{z,t\wedge\tau}) \quad &\geq \quad \bar{m}(X_0, y, z) + \int_0^{t\wedge\tau} e^{-rs}\bar{m}_x\sigma\, dB_s \\ &\quad - \int_0^{t\wedge\tau} e^{-rs}\zeta_s\, ds\,,\end{aligned}$$

with equality for $\tau \leq \tau_{yz}$. As in the proof of Lemma 1.5.3 we take a localizing sequence $\tau_k \leq k$. Taking expectations on both sides yields, for each k:

$$\mathbb{E}\left[e^{-r(\tau_k\wedge\tau)}\bar{m}(X_{\tau_k\wedge\tau}, y, L_{z,\tau_k\wedge\tau})\right] \geq \bar{m}(X_0, y, z) - \mathbb{E}\left[\int_0^{\tau_k\wedge\tau} e^{-rs}\zeta_s\, ds\right].$$

Observe that $\bar{m}$ is bounded from above by 1 and $\zeta(X_s, y, L_{zs}) \leq \zeta(X_s, 0, L_{zs}) \leq \zeta(X_s, 0, z)$ which is integrable by assumption (1.2). So applying Fatou's lemma yields:

$$\begin{aligned}\bar{m}(X_0, y, z) \quad &\leq \quad \limsup_{k\to\infty}\mathbb{E}\left[e^{-r(\tau_k\wedge\tau)}\bar{m}(X_{\tau_k\wedge\tau}, y, L_{z,\tau_k\wedge\tau})\right] + \limsup_{k\to\infty}\mathbb{E}\left[\int_0^{\tau_k\wedge\tau} e^{-rs}\zeta_s\, ds\right] \\ &\leq \quad \mathbb{E}\left[e^{-r\tau}\bar{m}(X_\tau, y, L_{z\tau})\right] + \mathbb{E}\left[\int_0^{\tau} e^{-rs}\zeta_s\, ds\right]; \qquad (1.5.13)\end{aligned}$$

i.e.,

$$\begin{aligned}\bar{m}(X_0,y,z) &\leq \mathbb{E}\left[\int_0^\tau e^{-rs}\zeta_s\,ds\right]+\mathbb{E}\left[e^{-r\tau}\bar{m}(X_\tau,y,L_{z\tau})\right]\\ &\leq \mathbb{E}\left[\int_0^\tau e^{-rs}\zeta_s\,ds\right]+\mathbb{E}\left[e^{-r\tau}\right].\end{aligned}$$

Note that $L_{zt}=z$ for $t\leq\tau_{yz}$ when $z=(n-1)y$, so for $\tau=\tau_{yz}$ the value $\bar{m}$ is also bounded from below by Assumption 3 in Proposition 1.2.1. Using this and the integrability assumption (1.2) the right-hand side in (1.5.13) converges in L^1 and we get equality for $\tau=\tau_{yz}$. □

The next lemma shows, for a firm with capital stock y, that it is optimal to wait at least until X_t hits $X(y,(n-1)y)$, even if other smaller firms invest earlier.

Lemma 1.5.5. *Suppose $y>z/(n-1)$. For any stopping time τ,*

$$\mathbb{E}\left[\int_0^\tau e^{-rt}\zeta(X_t,y,L_{zt})\,dt+e^{-r\tau}\right]\geq\mathbb{E}\left[\int_0^{\hat\tau} e^{-rt}\zeta(X_t,y,L_{zt})\,dt+e^{-r\hat\tau}\right],$$

where $\hat\tau=\tau\vee\tau_{y,(n-1)y}$.

Proof. For $k=1,2,\ldots$, define the following convex combinations of y and $z/(n-1)$:

$$y_k=y\left[1-\left(\frac{n-1}{n}\right)^{k-1}\right]+\frac{z}{n-1}\left(\frac{n-1}{n}\right)^{k-1}.$$

Define $z_k=(n-1)y_k$. Note $z_1=z$. Set $\tau_0=0$ and $\tau_k=\tau_{y_k,z_k}$ for $k\geq 1$. Note that $\tau_k\uparrow\tau_{y,(n-1)y}$. We will first show, for $k\geq 1$,

$$\tau_{k-1}\leq t\leq\tau_k\quad\Rightarrow\quad\zeta(X_t,y_k,z_k)\geq\zeta(X_t,y,L_{zt}).\tag{1.5.14}$$

In the case $k=1$, (1.5.14) follows from the fact that $L_{zt}=z_1=z$ for $t\leq\tau_1$ and the fact that $y_1<y$ and ζ is decreasing in its second argument.
For $k>1$, consider any $t\in[\tau_{k-1},\tau_k]$. Note that $y_k=y+z_{k-1}-z_k\leq y+L_{zt}-z_k$. Hence $\zeta(X_t,y_k,z_k)\geq\zeta(X_t,y+L_{zt}-z_k,z_k)$. Moreover,

$$\begin{aligned}&\zeta(X_t,y+L_{zt}-z_k,z_k)-\zeta(X_t,y,L_{zt})\\ &\qquad=\int_0^{z_k-L_{zt}}\left[\zeta_{q_{-i}}(X_t,y-u,L_{zt}+u)-\zeta_{q_i}(X_t,y-u,L_{zt}+u)\right]du\geq 0,\end{aligned}$$

the inequality following from $\zeta_{q_{-i}}\geq\zeta_{q_i}$ and the fact that $z_k\geq L_{zt}$ for $t\leq\tau_k$. Thus, (1.5.14) holds for all k.
For $k\geq 0$, define $\hat\tau_k=\tau\vee\tau_k$. Note that $\hat\tau_k\uparrow\hat\tau$. Using (1.5.14) and the optimality of τ_k in the myopic problem starting from (y_k,z_k) (see Lemma 1.5.3), we obtain

$$\begin{aligned}0&\geq\mathbb{E}\left[1_{\{\hat\tau_{k-1}<\tau_k\}}\left(\int_{\hat\tau_{k-1}}^{\tau_k}e^{-rt}\zeta(X_t,y_k,z_k)\,dt+e^{-r\tau_k}-e^{-r\hat\tau_{k-1}}\right)\right]\\ &\geq\mathbb{E}\left[1_{\{\hat\tau_{k-1}<\tau_k\}}\left(\int_{\hat\tau_{k-1}}^{\tau_k}e^{-rt}\zeta(X_t,y,L_{zt})\,dt+e^{-r\tau_k}-e^{-r\hat\tau_{k-1}}\right)\right]\\ &=\mathbb{E}\left[\int_{\hat\tau_{k-1}}^{\hat\tau_k}e^{-rt}\zeta(X_t,y,L_{zt})\,dt+e^{-r\hat\tau_k}-e^{-r\hat\tau_{k-1}}\right].\end{aligned}\tag{1.5.15}$$

The equality (1.5.15) follows from the fact that $\{\hat{\tau}_{k-1} < \tau_k\} = \{\hat{\tau}_{k-1} < \hat{\tau}_k\}$ and the fact that $\tau_k = \hat{\tau}_k$ on this event. When we add the right-hand sides of (1.5.15) from $k = 1$ to $k = \ell$ for any ℓ, we obtain

$$0 \geq \mathbb{E}\left[\int_{\tau}^{\hat{\tau}_\ell} e^{-rt}\zeta(X_t, y, L_{zt})\,dt + e^{-r\hat{\tau}_\ell} - e^{-r\tau}\right],$$

or

$$\mathbb{E}\left[\int_0^{\tau} e^{-rt}\zeta(X_t, y, L_{zt})\,dt + e^{-r\tau}\right] \geq \mathbb{E}\left[\int_0^{\hat{\tau}_\ell} e^{-rt}\zeta(X_t, y, L_{zt})\,dt + e^{-r\hat{\tau}_\ell}\right].$$

The claim now follows by taking the limit $\ell \to \infty$, using the fact that the set

$$\left\{\int_0^{\hat{\tau}_\ell} e^{-rt}\zeta(X_t, y, L_{zt})\,dt \,\middle|\, \ell \in \mathbb{N}\right\}$$

of random variables is uniformly integrable due to the integrability assumption (1.2).

□

Lemma 1.5.6. *If $y \geq z/(n-1)$, then $\tau_{y,(n-1)y}$ solves the equilibrium stopping problem* (1.5.8).

Proof. For convenience, denote $\tau_{y,(n-1)y}$ by τ^*. For $y = z/(n-1)$ the statement follows from Lemma 1.5.4. Suppose $y > z/(n-1)$. By virtue of the previous lemma, we can restrict the search for optimal stopping times to those times τ satisfying $\tau \geq \tau^*$. For such a stopping time, the value achieved in (1.5.8) is

$$\begin{aligned}\mathbb{E}\left[\int_0^{\tau} e^{-rt}\zeta(X_t, y, L_{zt})\,dt + e^{-r\tau}\right] &= \mathbb{E}\left[\int_0^{\tau^*} e^{-rt}\zeta(X_t, y, L_{zt})\,dt + e^{-r\tau^*}\right] \\ &\quad + e^{-r\tau^*}\mathbb{E}\left[\int_{\tau^*}^{\tau} e^{-r(t-\tau^*)}\zeta(X_t, y, L_{zt})\,dt + e^{-r(\tau-\tau^*)}\right]. \end{aligned} \tag{1.5.16}$$

Thus, minimizing (1.5.8) is equivalent to minimizing

$$\mathbb{E}\left[\int_{\tau^*}^{\tau} e^{-r(t-\tau^*)}\zeta(X_t, y, L_{zt})\,dt + e^{-r(\tau-\tau^*)} \mid \mathcal{F}_{\tau^*}\right]. \tag{1.5.17}$$

Recall that $\tau^* = \inf\{t | X_t > X(y, (n-1)y)\}$. Thus,

$$X_{\tau^*} = X(y, (n-1)y) = \max_{0 \leq s \leq \tau^*} X_s\,.$$

This implies that

$$L_{z\tau^*} = \inf\{q \geq z \mid X_{\tau^*} \leq X(q/(n-1), q)\} = (n-1)y\,.$$

Minimizing (1.5.17) is therefore equivalent to minimizing (1.5.8) given $z = (n-1)y$, and the solution of this is $\tau_{y,(n-1)y}$; i.e., the minimum value of (1.5.17) is attained at $\tau = \tau^*$.

□

The following lemma completes the proof of the symmetric open loop equilibrium. If all firms $j \neq i$ choose the processes Q_j^* defined in (1.8), then $Q_{-i} = L_z$ defined in (1.5.5), where $z = q_{-i0}$. For convenience, set $\tau_y = \tau_{y,(n-1)y}$. Note that Q_i^* defined in (1.8) and τ_y satisfy

$$\tau_y = \inf\{t \geq 0 \mid Q_{it}^* \geq y\}\,, \tag{1.5.18}$$

for $y > q_{i0}$, meaning that τ_y is the investment time of unit number y for the capital process Q_i^*.

Lemma 1.5.7. *Assume $q_{i0} = q_{j0}$ for all i and j and Q_j^* is given by* (1.8) *for $j \neq i$. Then Q_i^* defined in* (1.8) *maximizes* (1.4) *on $\mathcal{A}(q_{i0})$.*

Proof. Let $\xi \in A$ be an arbitrary admissible control. We can assume that $\mathbb{E}\int_0^\infty e^{-rt}\,d\xi_t < \infty$ as otherwise the firm value would be $-\infty$. Integrating by parts and applying the monotone convergence theorem shows that this implies $\mathbb{E}\int_0^\infty e^{-rt}\xi_t\,dt < \infty$, which implies further that $\mathbb{E}\left[\lim_{t\to\infty} e^{-rt}\xi_t\right] = 0$. For $y \geq q_{i0}$, define

$$\tau_y^\xi = \inf\{t \geq 0 | \xi_t \geq y\}\,,$$

the investment time of unit number y. Then

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-rt}(\pi(X_t,\xi_t,L_{zt})\,dt - \,d\xi_t)\right]\\
&= \mathbb{E}\left[\int_0^\infty e^{-rt}(\pi(X_t,\xi_t,L_{zt}) - r(\xi_t - q_{i0}))\,dt - \lim_{t\to\infty} e^{-rt}\xi_t\right]\\
&= \mathbb{E}\left[\int_0^\infty \left(e^{-rt}\pi(X_t,q_{i0},L_{zt}) + \int_{q_{i0}}^{\xi_t} e^{-rt}(\pi_{q_i}(X_t,y,L_{zt}) - r)\,dy\right)\,dt\right]\\
&= \mathbb{E}\left[\int_0^\infty e^{-rt}\pi(X_t,q_{i0},L_{zt})\,dt + \int_{q_{i0}}^\infty \int_{\tau_y^\xi}^\infty e^{-rt}(\pi_{q_i}(X_t,y,L_{zt}) - r)\,dt\,dy\right]\,,
\end{aligned}$$

where we integrated by parts to obtain the first equality and changed the order of integration to obtain the third. Recalling the definition $\zeta = \pi_{q_i}$, and applying Fubini and the optimality of τ_y for $y > q_{i0}$, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-rt}(\pi(X_t,\xi_t,L_{zt})\,dt - \,d\xi_t)\right] - \mathbb{E}\left[\int_0^\infty e^{-rt}\pi(X_t,q_{i0},L_{zt})\,dt\right]\\
&= \int_{q_{i0}}^\infty \mathbb{E}\left[\int_{\tau_y^\xi}^\infty e^{-rt}(\zeta(X_t,y,L_{zt}) - r)\,dt\right]\,dy\\
&\leq \int_{q_{i0}}^\infty \mathbb{E}\left[\int_{\tau_y}^\infty e^{-rt}(\zeta(X_t,y,L_{zt}) - r)\,dt\right]\,dy
\end{aligned}$$

with equality if $\tau_y^\xi = \tau_y$ (equivalently, if $\xi = Q_i^*$).

□

To prove the myopic optimality, note that the investment times

$$\tau_{yq_{-i0}} = \inf\{t \mid X_t > X(y,q_{-i0})\} = \inf\{t \geq 0 \mid Q_{it} \geq y\} \tag{1.5.19}$$

of the myopic strategy Q_i are optimal stopping times for the myopic stopping problems (see Lemma 1.5.3). The same proof as for Lemma 1.5.7 shows that Q_i is an optimal investment strategy when the rival firms hold their capital stocks constant.

1.6 Proof of Proposition 4

Set $P_t^\alpha = X_t - b\sum_{j=1}^n Q_{jt}^\alpha$. We have

$$F(\alpha) = \mathbb{E}\int_{[0,\tau)} e^{-rt}[(P_t^\alpha - c)Q_{it}^\alpha - dQ_{it}^\alpha] + \mathbb{E}\int_{[\tau,\infty)} e^{-rt}[(P_t^\alpha - c)Q_{it}^\alpha - dQ_{it}^\alpha]\,.$$

The first term is independent of α. We want to take the derivative of the second term with respect to α. For $t \geq \tau$, we have

$$Q_{it}^\alpha = \inf\{q_i \geq \alpha q_{i0} \mid X_t^* \leq A + Bq_i\}\,,$$

where

$$\begin{aligned}
A &= \frac{\beta}{\beta-1}\left(\frac{r-\mu}{r}\right)(c+r)\,,\\
B &= \frac{\beta}{\beta-1}\left(\frac{r-\mu}{r}\right)\left(\frac{n+\alpha}{\alpha}\right)b\,.
\end{aligned}$$

Also, for $t > \tau$,

$$P_t^\alpha - c = X_t - C - DQ_{it}^\alpha\,,$$

where

$$C = c\,,$$
$$D = \left(\frac{n+\alpha-1}{\alpha}\right) b\,.$$

We show below that

$$\mathbb{E}\int_{[\tau,\infty)} e^{-r(t-\tau)}((P_t^\alpha - c)Q_{it}^\alpha\,dt - dQ_{it}^\alpha) = -(\alpha-1)q_{i0} + \alpha q_{i0}\left(\frac{X_\tau}{r-\mu} - \frac{C + D\alpha q_{i0}}{r}\right) + \frac{X_\tau}{B}\left[\frac{(rB - 2rD + 2\mu D)(A + B\alpha q_{i0})}{r(r-\mu)(\beta-2)B} + \frac{2AD - BC - rB}{r(\beta-1)B}\right]\left(\frac{A + B\alpha q_{i0}}{X_\tau}\right)^{1-\beta}\,. \quad (1.6.1)$$

The right-hand derivative of this with respect to α at $\alpha = 1$ (computed using Mathematica and Maple) is

$$\frac{(n-1)\left[(n+1)^2\beta(\beta-1)b^2q_{i0}^2 + 2(n+1)(r+c)\beta b q_{i0} + 2(r+c)^2\right]}{(n+1)^3(\beta-1)(\beta-2)rb} > 0\,.$$

It remains to verify (1.6.1). Define

$$L_t = \log\left(\frac{A + BQ_{it}}{A + Bq_{i0}}\right)\,,$$
$$Z_t = \log\left(\frac{A + Bq_{i0}}{X_t}\right)\,,$$
$$Y_t = \log\left(\frac{A + BQ_{it}}{X_t}\right)\,.$$

Note that Z is a Brownian motion with drift, and $dZ_t = -\left(\mu - \frac{1}{2}\sigma^2\right)\,dt - \sigma\,dB_t$. We have

$$A + BQ_{it} = \max(A + Bq_{i0}, X_t^*)\,.$$

It follows that

$$L_t = \max\left(0, \max_{0\le s\le t} -Z_s\right)\,.$$

Hence, $Y_t = L_t + Z_t$ is a Brownian motion (with drift) reflected at zero — see, e.g., Harrison (1985). Moreover, L increases only when $Y = 0$, and Y_s is a sufficient statistic for the $\mathcal{F}_s$–conditional distribution of the increment $L_t - L_s$ for any $t > s$. Note that $L_t - Z_t = 2L_t - Y_t$. Also,

$$\mathbb{E}\int_0^\infty e^{-rt}e^{-(Z_t - Z_0)}\,dt = \frac{1}{r-\mu}\,.$$

We have

$$X_t = X_0 e^{-(Z_t - Z_0)}\,,$$
$$Q_{it} = \frac{A + BQ_{i0}}{B}e^{L_t - L_0} - \frac{A}{B}\,, \quad L_0 = 0\,,$$
$$X_tQ_{it} = \frac{X_0(A + BQ_{i0})}{B}e^{(L_t - Z_t)-(L_0 - Z_0)} - \frac{AX_0}{B}e^{-(Z_t - Z_0)}\,,$$
$$Q_{it}^2 = \frac{A^2}{B^2} - \frac{2A(A + BQ_{i0})}{B^2}e^{L_t - L_0} + \left(\frac{A + BQ_{i0}}{B}\right)^2 e^{2(L_t - L_0)}\,,$$
$$dQ_{it} = \frac{A + BQ_{i0}}{B}e^{L_t - L_0}\,dL_t\,.$$

We will calculate the following below:

$$h_1(y) = \mathbb{E}\left[\int_s^\infty e^{-r(t-s)+L_t-L_s}\,dt \mid Y_s = y\right],$$
$$h_2(y) = \mathbb{E}\left[\int_s^\infty e^{-r(t-s)+(2L_t-Y_t)-(2L_s-Y_s)}\,dt \mid Y_s = y\right],$$
$$h_3(y) = \mathbb{E}\left[\int_s^\infty e^{-r(t-s)+2L_t-2L_s}\,dt \mid Y_s = y\right],$$
$$h_4(y) = \mathbb{E}\left[\int_s^\infty e^{-r(t-s)+L_t-L_s}\,dL_t \mid Y_s = y\right].$$

In terms of these functions,

$$\mathbb{E}\int_0^\infty e^{-rt}Q_{it}\,dt = \frac{A+BQ_{i0}}{B}h_1(Y_0) - \frac{A}{rB},$$
$$\mathbb{E}\int_0^\infty e^{-rt}X_tQ_{it}\,dt = \frac{X_0(A+BQ_{i0})}{B}h_2(Y_0) - \frac{AX_0}{(r-\mu)B},$$
$$\mathbb{E}\int_0^\infty e^{-rt}Q_{it}^2\,dt = \frac{A^2}{rB^2} - \frac{2A(A+BQ_{i0})}{B^2}h_1(Y_0) + \left(\frac{A+BQ_{i0}}{B}\right)^2 h_3(Y_0),$$
$$\mathbb{E}\int_0^\infty e^{-rt}\,dQ_{it} = \frac{A+BQ_{i0}}{B}h_4(Y_0).$$

It is shown below that

$$h_1(y) = \frac{1}{r} + \frac{1}{r(\beta-1)}e^{-\beta y}, \tag{1.6.2a}$$
$$h_2(y) = \frac{1}{r-\mu} + \frac{1}{(r-\mu)(\beta-2)}e^{(1-\beta)y}, \tag{1.6.2b}$$
$$h_3(y) = \frac{1}{r} + \frac{2}{r(\beta-2)}e^{-\beta y}, \tag{1.6.2c}$$
$$h_4(y) = \frac{1}{\beta-1}e^{-\beta y}. \tag{1.6.2d}$$

Straightforward algebra then yields (1.6.1).
To calculate h_1–h_4, use the fact that each of the following is a martingale, and hence the ds and dL_s terms of its differential vanish:

$$\int_0^s e^{-rt+L_t}\,dt + e^{-rs+L_s}h_1(Y_s),$$
$$\int_0^s e^{-rt+2L_t-Y_t}\,dt + e^{-rs+2L_s-Y_s}h_2(Y_s),$$
$$\int_0^s e^{-rt+2L_t}\,dt + e^{-rs+2L_s}h_3(Y_s),$$
$$\int_0^s e^{-rt+L_t}\,dL_t + e^{-rs+L_s}h_4(Y_s).$$

We have

$$dY = -\left(\mu - \frac{1}{2}\sigma^2\right)dt - \sigma\,dB + dL,$$

so the ds terms vanishing implies

$$1 - rh_1 - \left(\mu - \frac{1}{2}\sigma^2\right) h_1' + \frac{1}{2}\sigma^2 h_1'' = 0\,, \tag{1.6.3a}$$

$$1 - (r-\mu)h_2 - \left(\mu + \frac{1}{2}\sigma^2\right) h_2' + \frac{1}{2}\sigma^2 h_2'' = 0\,, \tag{1.6.3b}$$

$$1 - rh_3 - \left(\mu - \frac{1}{2}\sigma^2\right) h_3' + \frac{1}{2}\sigma^2 h_3'' = 0\,, \tag{1.6.3c}$$

$$-rh_4 - \left(\mu - \frac{1}{2}\sigma^2\right) h_4' + \frac{1}{2}\sigma^2 h_4'' = 0\,. \tag{1.6.3d}$$

Equating the coefficients of dL to zero at $y = 0$ yields the boundary conditions:

$$\begin{aligned} h_1(0) + h_1'(0) &= 0\,, \\ h_2(0) + h_2'(0) &= 0\,, \\ 2h_3(0) + h_3'(0) &= 0\,, \\ 1 + h_4(0) + h_4'(0) &= 0\,. \end{aligned}$$

Boundary conditions at $y = \infty$ are obtained by noting that $L_t - L_s \downarrow 0$ pointwise as $y = Y_s \uparrow \infty$ and using the dominated convergence theorem. This yields

$$\begin{aligned} \lim_{y\to\infty} h_1(y) &= \frac{1}{r}\,, \\ \lim_{y\to\infty} h_2(y) &= \frac{1}{r-\mu}\,, \\ \lim_{y\to\infty} h_3(y) &= \frac{1}{r}\,, \\ \lim_{y\to\infty} h_4(y) &= 0\,. \end{aligned}$$

The general solutions of the differential equations (1.6.3) subject to the boundary conditions at infinity are

$$\begin{aligned} h_1(y) &= \frac{1}{r} + A_1 e^{-\beta y}\,, \\ h_2(y) &= \frac{1}{r-\mu} + A_2 e^{(1-\beta)y}\,, \\ h_3(y) &= \frac{1}{r} + A_3 e^{-\beta y}\,, \\ h_4(y) &= A_4 e^{-\beta y}\,. \end{aligned}$$

for constants A_i. Imposing the boundary conditions at zero yields (1.6.2).

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2. OPTIMAL TIMING OF AGGREGATE INVESTMENT AND THE YIELD CURVE

Abstract

How does the optimal exercise timing of growth options reconcile with a perfect competition environment in which any delay in investment will be exploited by competitors? To answer this question, I solve a stylized general equilibrium model with irreversible investment in closed form. Current investment opportunities compete with future ones and waiting for better information has an option value in the sense that increased volatility indeed leads to a delay in investment. The delay reconciles with zero profits for firms via anticipated increases in future prices. Longer term interest rates and futures on wages already contain the expected growth-effect of optimally exercised growth options, rendering current investment opportunities unprofitable. In this sense, the term-structure of future prices reflects the option premium of waiting and leads to optimal delay in investment.

JEL classification: D92, E22, E44, G12
Keywords: Irreversible Investment, Term Structure, Option Premium, Capital Accumulation, General Equilibrium

2.1 Introduction

> *The most important confusion concerning the meaning and significance of the marginal efficiency of capital has ensued on the failure to see that it depends on the prospective yield of capital, and not merely on its current yield. This can be best illustrated by pointing out the effect on the marginal efficiency of capital of an expectation of changes in the prospective cost of production [...]. The output from equipment produced to-day will have to compete, in the course of its life, with the output from equipment produced subsequently, perhaps at a lower labor cost, perhaps by an improved technique [...] In so far as such developments are foreseen as probable, or even as possible, the marginal efficiency of capital produced to-day is appropriately diminished* (J.M. Keynes in Keynes (1936, p. 141))

The purpose of this article is to study a model which captures the fact that investment is a long term concern: current investment opportunities compete with future ones and the future plays an important role in determining the marginal efficiency of capital. So the gain from learning more about it and about alternative investment opportunities might be higher than the current yield of capital, that is the opportunity cost of not investing. Thus it might be optimal to delay investment and wait for more and better information even at the opportunity cost of high current yields, i.e. it may be optimal to utilize the so called option value of waiting to invest. I ask i) whether this can be the case in general equilibrium and ii), if so, how does the delay materialize in a market economy with perfect competition in which, if nothing else changes, any delay in investment and unused profit opportunity would be exploited by competitors.

To this end, I solve a stylized general equilibrium model with irreversible investment in closed form. The model is essentially a simple real-business-cycle (RBC) model or stochastic Ramsey model with three deviations. First, investment is assumed to be irreversible; once investment costs are paid, they cannot be be regained by disinvesting if economic conditions turn out to be unprofitable. This makes investment a long-term concern and the decision to invest or not invest becomes dependent on the 'prospective yield of capital' rather than its 'current yield'. Second, the stochastics do not enter total factor productivity (TFP) but investment cost. TFP affects new and old invest in the same way. Hence, with stochastic TFP, there would be no distinction in the productivity of already installed capital and marginal capital. If volatility increases, then the probability of bad states with low consumption and binding irreversibility constraint does so too. This leads to an incentive to hedge against these states by investing more instead of less. In a very simple stochastic Ramsey model with log utility and linear technology investment occurs no later than with reversible investment as shown in Paulsen (2009). With stochastic investment cost, however, only productivity per fixed unit of investment cost is stochastic. This captures the idea of stochastic *marginal* productivity ('marginal efficiency of capital') rather than stochastic *average* productivity. Third, I assume the investment cost to be in form of labor input instead of units of the consumption good. This is equivalent to say that the consumption good and the capital good are produced by two distinct production functions, whereat the one for the capital good is linear and labor is the only input. This assumption is made for two reasons. First, without stochastic TFP, output is temporarily fixed. If output had to be split between consumption and investment, both could not co-move as in typical business cycles. Second, the assumption avoids a complementarity of investment today and investment tomorrow. Namely with only one production function, investment increases output and therefore lowers the cost of future investment in terms of foregone marginal utility from consumption. This additional benefit can decrease the option value of waiting to invest (see Paulsen (2009) for the case of a very simple model). Last but not least, the assumption makes the model analytically tractable.

It turns out that uncertainty about the future indeed leads to a delay in aggregate investment. If a delay becomes optimal, longer term future prices will rise. This is because a delay will only be optimal if future investment opportunities are expected to be more profitable than todays. Long term futures, i.e. interest rates at which banks are willing to lend and future prices on labor, anticipate the optimally exercised growth options and incorporate their potential growth effect, rendering currently available investment opportunities unprofitable. In this sense, long term factor prices comprise an option value of delayed investment.

This story suits to a statement by board members of Wells Fargo in January 2010 in which they say that the current spread between longer and shorter term interest rates - though high - is not sufficient for Wells Fargo. For lending out they would demand an even higher long term rate which compensates for the risk of increasing long rates:

> *... actually this as [sic!] the classic short-term view of the business or long-term view of the business. 400 basis points or something like that, which we make in the carry trade today, is very attractive. But we think it's the wrong decision long-term because we think the bias is for higher rates, not for lower rates, and we're willing to wait for that to happen. We think that's the better trade.* (John Stumpf, CEO Wells Fargo & Company, January 20, 2010)[1]
>
> *...we are effectively giving up 400 basis points today for possibly a year or so, maybe plus or minus, to avoid the potential risk of a larger number of basis points for 30 years. So the last thing we want to do is get stuck with securities at these low levels of interest rates.* (Howard Atkins, CFO Wells Fargo & Company, January 20, 2010)[2]

The remaining part of the paper is structured as follows. Section 2 reviews the related literature. Section 3 takes a viewpoint from the perspective of Keynes' General Theory. Section 4 presents the formal model. Section 5 solves for an equilibrium. Section 6 solves for investment dynamics, long term interest rates and stock prices in closed form and analyzes option premia. Finally, section 7 concludes.

[1] Wells Fargo & Company Q4 2009 Earnings Call Transcript January 20, 2010
[2] Loc. cit.

2.2 Related Literature

Classical general equilibrium models like the Arrow-Debreu model (Arrow and Debreu (1954)) or the Radner model (Radner (1972)) assume exogenous asset endowments and do not allow accumulation of capital. In standard general equilibrium models with capital accumulation, e.g. Cox, Ingersoll and Ross Cox et al. (1985a,b) consumer's (or firms in the place of consumers) can invest into a production technology with stochastic return but they are free to disinvest and consume the capital stock whenever they want; investment is reversible.

The irreversible investment literature, on the other hand, mainly focuses on monopolistic firms (e.g. Dixit and Pindyck (1994); Pindyck (1988); Riedel and Su (2011)) and partial equilibrium analysis (e.g. Back and Paulsen (2009); Baldursson and Karatzas (1996); Leahy (1993)). Only few papers deal with irreversibility in a general equilibrium context. Sargent (1979) proves existence of a solution in a discrete time model and then provides numerical computations of investment rates. Olson (1989) also shows existence of a solution in a discrete time context and proves some stationarity properties of the investment process.

In Coleman (1997) ans Jamet (2004) multisector models without aggregate uncertainty are considered. Though there are some non-trivial effects due to sector specific uncertainty, effects on aggregates remain small. Coleman (1997) numerically computes a two sector model with a regime change. Depending on the state of nature, sector one or sector two is more productive. When the regime switches, investment is shifted from one sector to the other so that aggregate investment remains basically constant. However, as both sectors may be differently developed and thus experience different marginal returns of capital, there are some effects on the interest rate. A productivity shift from a more developed sector to a less developed one leads to an increase in the interest rate that decays with investment into this sector. Additionally, Coleman finds that *because of a desire to smooth consumption, with irreversible investment a rise in uncertainty concerning the future return to capital tends to lead to more current investment and a lower real interest rate.* In Jamet (2004), uncertainty is concentrated on and independent between infinitely many intermediate sectors. Hence, aggregate uncertainty vanishes by the law of large numbers leading to deterministic wealth and a constant interest rate.

Chiarolla and Haussmann (2009) show existence of an equilibrium in a general equilibrium environment with irreversible investments. In their else very general model, they assume that firms do not maximize shareholder value but net present value (i.e. expected profits are computed with respect to the physical measure rather than the risk adjusted one).

There are some papers that deal numerically with investment irreversibilities in a general equilibrium setting. Veracierto (2002) numerically analyzes a RBC model with irreversibility constraints on a plant level. He finds that *aggregate fluctuations are ba-*

sically the same under fully flexible or completely irreversible investment. Bloom et al. (2009) study the impact of uncertainty in a calibrated dynamic stochastic equilibrium model with idiosyncratic risks, fixed cost and a partial irreversibility constraint. They find "that increases in uncertainty lead to large drops in economic activity. This occurs because a rise in uncertainty makes firms cautious, leading them to pause hiring and investment. It also reduces the reallocation of capital and labor across firms, leading to large falls in productivity growth." In all these articles, it is the productivity parameter on which uncertainty enters the economy. In this article, however, productivity is deterministic and it are investment cost that vary stochastically. Or, equivalently, it is stochastic *marginal* productivity that fluctuates.

Bilbiee Ghironi and Melitz (Bilbiie et al., 2007) construct a model with focus on endogenous product variety. Besides stochastic productivity shocks their model also contains stochastic entry cost, but they assume the irreversibility constraint to be never binding in order to simplify their analysis.

The paper most closely related to the one at hand is Leahy (1993). It studies the effects of investment irreversibilities in a perfectly competitive setting. The author finds that critical prices that trigger investment are the same for a firm facing perfect competition and for an infinitesimal small monopolistic firm. The equilibrium price process is a geometric Brownian motion reflected at some threshold. That is, if prices rise too much, they reach the threshold at which investment has a zero net present value and investment into additional capacities pushes the price down. If volatility increases, so does the probability in which prices are low. But as the average price must be such that firms make zero profits, lower prices have to be compensated for by some upside. This occurs in form of a higher threshold at which prices are reflected, i.e. in form of delayed investment. In particular, increased uncertainty leads to delayed investment (in the sense of a higher price trigger), which is also efficient from a welfare perspective. Both, investment delay and zero profits for firms reconcile via the price dynamics, which - in this sense - absorb the option premium of waiting to invest.

However, the mere existence of time varying intra-period prices makes the set-up a partial equilibrium one. In general equilibrium with a single representative consumption good, there are no such intra-period prices and the argument cannot be sustained. (Besides wages, which indeed will play an important role as long as labor enters the production function). Instead of intra-period prices, however, it are inter-period prices, namely interest rates whose dynamics reconcile zero profits with a delay in investment. Just as in the partial equilibrium setting of Leahy (1993), where prices are expected to be lower in the future when delay becomes optimal, in the general equilibrium set-up of this article, inter-temporal prices (e.g. interest rates) are expected to rise in such a situation.

From the technical viewpoint, this article is closely related to the paper "On Irreversible Investment" by Riedel and Su (2011). They explicitly solve the irreversible investment problem of a monopolistic firm with Cobb Douglas production function

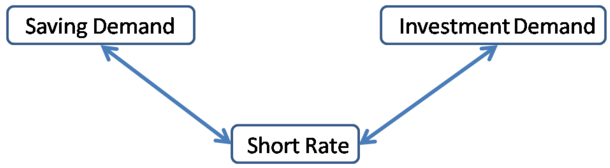

Fig. 2.1: The reversible case

and general multiplicative stochastic demand modeled by an exponential Lévy process. The equilibrium problem of the article at hand in central planner's form turns out to be equivalent to such a monopolistic firm's problem but with stochastic investment cost instead of stochastic demand. Beside this difference, all methods are contained in Riedel-Su so that I do not provide anything novel from the mathematical point of view; I rely on the methods exposed in Riedel and Su (2011) to solve the central planner's problem.

I remark that Hugonnier, Morellec and Sundaresan author a working paper titled 'Irreversible Investment in General Equilibrium' (Hugonnier et al. (2005)). However, in their model, the agent consumes the capital stock[3]. Irreversibility enters through another feature, namely the opportunity of a once and for all transformation of the capital stock, which is completely different to the setting in this article.

2.3 A Keynesian perspective

Though the model at hand is neoclassical, it shares essential features described by Keynes in his 'General Theory' (Keynes, 1936). Similar to Keynes' liquidity preference schedule, it highlights a trade-off between liquid short term assets and illiquid long term investment opportunities. Indeed, liquidity preference is a kind of *financial dual* to technological liquidity (e.g. investment reversibility). In equilibrium, net-financial investment equals physical investment so that (net-)financial liquidity is constrained by technological liquidity. In the extreme case of reversible investment, there is no need and no role for liquidity preference. Any demand could be easily fulfilled by the capital stock; illiquidity does not exist. If, however, investment is irreversible, then liquidity preference affects investment behavior and, vice versa, any preference over potential *timings* of investment can be interpreted as preference over asset portfolios with different maturities and therefore as a liquidity preference. Exactly as the investors face a trade-off between between investing today and investing tomorrow that potentially involves an option value of waiting to invest, bond traders face a trade-off of when to perform maturity transformation, of when to buy long term bonds. This may involve an option value of waiting for lower bond prices in the future similar to what Keynes described as the speculative motive for liquidity preference.

[3] capital stock =wealth in their terminology

Although irreversibility of investment is key for Keynes General Theory, modern standard macroeconomic models as the real business cycle model or New Keynesian models (text-book models) usually assume investment to be reversible; if investment turns out to be bad tomorrow, it is possible to disinvest and regain its cost. This reversibility assumption implies that capital is not afflicted with long term risks. To invest or not is only a short term consideration: As soon as the current marginal product of capital exceeds the short term interest rate, investment becomes profitable. In equilibrium, the marginal return on capital must equal the short term interest rate and the capital stock instantaneously adjusts to fluctuations in the former. In other words, capital demand is infinitely elastic to the short rate which equilibrates savings and investments, schematically illustrated in Figure 2.1. In particular, the short rate is always positive whenever there exist investment opportunities with positive marginal return. Thus, with reversible investment, an increase in uncertainty in the economy via lowering the interest rate by people's precautionary savings motive leads to a higher capital stock and higher investments. So whenever uncertainty increases (and there is no negative productivity shock) one should observe increased investment activity, not less. Keynes criticizes the reversibility assumption which is implicit when the marginal efficiency of capital is thought of being tied to current profits. It would destroy the link between the future (and the expectations thereof) with today's investment:

> The mistake of regarding the marginal efficiency of capital primarily in terms of the current yield of capital equipment, which would be correct only in the static state where there is no changing future to influence the present, has had the result of breaking the theoretical link between to-day and to-morrow. Even the rate of interest is, virtually, a current phenomenon; and if we reduce the marginal efficiency of capital to the same status, we cut ourselves off from taking any direct account of the influence of the future in our analysis of the existing equilibrium. The fact that the assumptions of the static state often underlie present-day economic theory, imports into it a large element of unreality. (Keynes, 1936, page 145)

According to Keynes, it are durable equipments that connect the future to the present.

> It is by reason of the existence of durable equipment that the economic future is linked to the present. It is, therefore, consonant with, and agreeable to, our broad principles of thought, that the expectation of the future should affect the present through the demand price for durable equipment. (Keynes, 1936, page 145)

If the marginal efficiency of capital depends on its prospective yield rather than its current yield, investment demand becomes tied to the long term interest rate. Savings, on the other hand, might be motivated by short term rather than long term considerations, for instance as a precautionary measure to hedge against a decline in economic activity, i.e. savings might be in the form of short term assets, e.g. money, earning a low or even zero interest rate. To link savings and investment, one therefore has to think about the link between the short term and the long term interest rate, i.e. about maturity transformation.

At this point comes Keynes' main contribution[4], the introduction of the liquidity schedule, into play. The liquidity preference schedule describes the demand for money for a given level of output and long term interest rate. It can therefore be interpreted as short cut formulation of a financial market in the form of a postulated relationship between the interest rate on money savings (zero) and long term rates, i.e. as an ad-hoc model of maturity transformation.

Because of liquidity preference, investment demand, which depends on the long rate, is only indirectly linked to saving demand. Both might vary independently if the linkage, i.e. if liquidity preference, changes. In Keynes words: "the scale of investment fluctuate[s] for reasons quite distinct (a) from those which determine the propensity of the individual to save out of a given income" (Keynes, 1937, page 218). The interest rate on money is fixed to zero and therefore cannot adjust to fluctuations in liquidity preference. Hence, according to Keynes General Theory, there might be excess savings in the form of money holdings, which vanish in equilibrium by a reduction of output Y.

The model at hand is similar in the following sense. In line with Keynes argument about durable equipments, I assume investment to be irreversible which immediately implies that investment becomes a long term concern. Investors have to take the whole future into consideration and capital installed today has to compete with future investments which might be into different and more profitable technologies. For this reason, the long term interest rate becomes the appropriate reference value for investment demand. On the other hand, savings today and savings tomorrow only differ by the short term return. Hence, saving demand refers to the short rate and savings are without loss of generality in the form of short term assets. Short term assets resemble money in Keynes theory[5].

Taken both parts together, investment demand is sensitive to the long rate while saving demand reacts to the short rate. This is illustrated in Figure 2.2. But both, the short rate and the long rate, cannot vary freely. In equilibrium they are linked by a no arbitrage condition resembling Keynes liquidity preference schedule: The long rate must equal a weighted average of future short rates. Note that this is not a level-to-level-correspondence; the long rate is not solely determined by the short rate itself but its *dynamics*!

This connection is a form of rigidity. Namely changes in the level of the short rate do not change the long rate as long as the dynamics are not sufficiently affected and if

[4] see also Harrod (1946)

[5] The principle difference to Keynes is that the return on short term assets can vary and equilibrate saving and investment demand at any given level of output Y, while money In Keynes world bears a fixed interest of zero. However, equilibrium might necessitate a negative short rate.

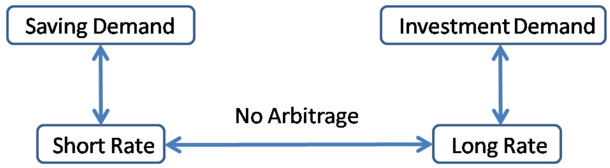

Fig. 2.2: The irreversible case

the dynamics change, the levels do not necessarily change. It differs from wage and price rigidities in the sense that it is not an ad hoc market imperfection, but a "neo classical rigidity"

Similarly, in Keynes theory a lower long term interest rate leads to higher money demand. There, the interest elasticity of money demand is due to the speculative motive 'the object of securing profit from knowing better than the market what the future will bring forth' (Keynes, 1936, page 170) ,for example an 'individual, who believes that future rates of interest will be above the rates assumed by the market, has a reason for keeping actual liquid cash' (Keynes, 1936, page 170). In other words, speculative money demand is due to speculation on higher long term interest rates and lower bond prices. This fits to the intuition which drives the argument in this article. Here, investors speculate on better future investment opportunities which is why they would demand short term rather than long term assets if long term yields were too low compared to rates that can be rationally expected for the future. But while Keynes speculative motive is due to believes which deviate from market expectations and therefore based on either a market inefficiency or on heterogeneous expectations, this model relies on a no-arbitrage condition as connection between the long rate an short rate dynamics. It is based on rational expectations and therefore consistent with neo-classical thinking.

Of course, an excess demand for short term assets cannot persist in equilibrium; speculative demand must vanish. Long term and short term interest rates in equilibrium are such that first there is no demand for short term assets and second there does not exist an arbitrage opportunity. This leads to a spread between the long and the short rate. In equilibrium, today's long term interest rate already contains expected future yields which might lift them to a level too high for current investment opportunities to be profitable. Equilibrium short rates, on the other hand, might turn negative due to a shrinking economy as consequence of the lack of investment even though the marginal product of capital is strictly positive. A situation which is usually interpreted as liquidity trap.

2.4 Model

Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. The model economy consists of firms and households which I describe in the following sections.

2.4.1 Firms

Operating Profits

A firm receives the operating cash flow

$$\pi_t(l_t) \triangleq Al_t^{1-\alpha} - w_t l_t$$

per unit of capital. Here l_t is the amount of labor hired by the firm at the wage rate w_t. The constant and deterministic number $A \in \mathbb{R}$ describes the productivity. As this return is homogeneous in the firm's capital stock, each (arbitrarily small) unit of capital can be thought of as a firm of its own and it is not necessary to keep track of the firms' sizes. In particular, there is no distinction between investment of additional capital units and entry of new firms.

Entry Cost

Before earning profits, a firm incurs stochastic investment (entry) cost in form of γ_t units of labor. This might be justified by thinking of two separate production functions. One for the consumption good and one for capital, where the latter is approximated by a linear one[6].

Once entry cost are paid, productivity is deterministic and constant over time. This differs from many macroeconomic models, for instance the standard real business cycle models and variants thereof, which assume stochastic levels of productivity A_t. In these models, a productivity shock affects the whole capital stock equally. Stochastic entry cost, however, imply that only for new installable capital output per unit cost is stochastic. They model stochastic *marginal* productivity rather than *average* productivity and therefore capture the 'fickle and highly unstable marginal efficiency of capital' (Keynes, 1936, p. 204). Note that stochastic cost require irreversibility of investment as otherwise one could take advantage of increasing cost.

Remark 2.4.1. *Note that k_t denotes capital measured in efficiency units as each unit is associated with a fixed productivity A. Hence, γ_t are not the unit cost of physical capital and cannot be identified by a time series of investment good prices which have a rather low volatility. Justiniano et al. (2010) estimate the volatility of the investment specific technology shock to be 6 % per quarter.*

I model entry cost by an exponential Lévy process. Later, I will restrict to a geometric Brownian motion:

Assumption 2.4.2. *Entry cost are given by an exponential Lévy process*

$$\gamma_t \triangleq \exp(Y_t)$$

where Y_t is a Lévy process, i.e. a process with stationary and independent increments.

[6] Two separated production functions exclude that capacities built up to increase consumption today are used for investments tomorrow. Separate production functions avoid this complementarity between today's and tomorrow's investment.

1. (geometric Brownian motion) Let $d\gamma_t = \mu\gamma_t dt + \sigma\gamma_t dB_t$ be given by a Brownian motion. In this case, the return per investable unit, i.e marginal productivity, fluctuates continuously.
2. (exponential Poisson process) Let $Y_t = -N_t$ be the negative of a Poisson process. Then cost will decrease stochastically. A decrease in cost might be interpreted as a decrease in uncertainty about which products will be useful and thus a reduction in false and useless investments or simply as technological progress.

Profit Maximization and Present Value

Once entry cost are paid, firms are indistinguishable. By profit maximization, they receive the operating revenue.

$$\begin{aligned}\pi_t &= \max_{l_t} \underbrace{\left(Al_t^{1-\alpha} - w_t l_t\right)}_{=\pi_t(l)} \\ &= \alpha A^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{w_t}\right)^{\frac{1-\alpha}{\alpha}}\end{aligned}$$

per capital unit. Capital depreciates at the constant rate δ. Let $d\rho_t$ be the dividend stream of the money market[7].
With this notation, an investor computes a firm's present value at time t as

$$s_t \triangleq \mathbb{E}^{\mathbb{Q}}\left[\int_t^\infty e^{-\int_t^u d\rho_r} e^{-\delta(u-t)}\pi_u du \mid \mathcal{F}_t\right]$$

under some risk-neutral pricing measure $\mathbb{Q}$ (equivalent martingale measure). Let I_t be the measure of firms that have entered the market up to time t, i.e. the cumulative investment (entry) process. As capital depreciates, the aggregate capital stock k_t evolves according to

$$dk_t = -\delta k_t dt + dI_t$$

Entry Cost Financed by Issuing Stocks

Firms finance the entry cost by issuing stocks. Let s_t be the market price of a stock for a firm with one unit of capital. As new firms can be generated by the cost of $\gamma_t w_t$, that is the amount of labor employed in capital production times the price of labor (the wage rate), free entry requires $s_t \leq \gamma_t w_t$ in equilibrium. Firms do not enter the market when they expect negative profits, i.e. $s_t = \gamma_t w_t$ whenever $dI_t > 0$. Together, we have

$$(\gamma_t w_t - s_t)\, dI_t = 0$$

[7] If ρ_t was absolutely continuous, one could write $d\rho_t = r_t dt$ where r_t is the short rate. However, as it turns out, ρ_t will become singular.

as a measure on $\Omega \times \mathbb{R}_+$, i.e. for $\mathbb{P}$-almost-all fixed ω as a measure on $\mathbb{R}_+$.

Remark 2.4.3. *I would prefer firms to be financed by long term bonds, but as firms with different entry time face different entry cost, this would introduce heterogeneity into firms cash flows. To avoid this, I assume firms to issue stocks but allow households to finance stocks by long term bonds. Investment behavior is thus 'outsourced' to the household sector. However, as the model is frictionless, this assumption does not change equilibrium outcomes.*

2.4.2 Households

Felicity

The infinitely lived representative agent receives utility from consumption and disutility from working.

$$\begin{aligned} u(c,l) &\triangleq \frac{c^{1-\theta}}{1-\theta} - \beta l \\ &= u(c) - \beta l \end{aligned}$$

with a slight abuse of notation.

Asset Market Participation

In addition to labor income, a household receives income from ownership of firms and savings in the money market. Denote by ϕ_t^s the amount invested in the money market[8] and by ϕ_t^e the numbers of stocks, where one stock is a claim on a firm with one unit of capital installed[9]. Let s_t be the price of a stock. Then the households budget constraint is:

$$c_t dt + s_t d\phi_t^e + d\phi_t^s \leq w_t dL_t + \phi_t^e \pi_t dt + \phi_t^s d\rho_t - \delta s_t \phi_t^e dt$$

Here L_t denotes cumulative labor. That is total expenditures for consumption ($c_t dt$), new stocks ($s_t d\phi_t^e$) and new bonds $d\phi_t^s$ cannot exceed income. Income is the sum of labor income ($w_t dL_t$), dividend payments ($\phi_t^e \pi_t dt$), interest income ($\phi_t^s d\rho_t$) minus depreciating equity ($-\delta s_t \phi_t^e dt$).

Assume that the portfolio processes ϕ_t^s and ϕ_t^e are adapted to the filtration, i.e. households have to trade on the basis of current information. Two further, technical conditions are needed. First, assume the portfolio processes to be of finite variation. This avoids the definition of a self-financing portfolio[10]. Second, we have to exclude doubling-strategies and Ponzi games. For this, define the value of the portfolio (ϕ^e, ϕ^s) or the consumers wealth by

[8] The money market is a security with dividend payments $d\rho_t$ such that the security's price is constant at 1.

[9] Note that by this convention, the number of outstanding stocks issued by a single firm depreciates with its capital stock.

[10] The assumption can be made without loss of generality as any infinite variation portfolio-strategy can be be approximated by finite variation processes.

$$W_t^{(\phi^e,\phi^s)} \triangleq \phi_t^e s_t + \phi_t^s$$

and the gains from this portfolio as

$$\begin{aligned} dG_t^{(\phi^e,\phi^s)} &\triangleq e^{-\rho_t}\phi_t^e\left(ds_t - \delta s_t dt\right) + e^{-\rho_t}\phi_t^e \pi_t dt + e^{-\rho_t}\phi_t^s d\rho_t - e^{-\rho_t} W_t^{(\phi^e,\phi^s)} d\rho_t \\ &= e^{-\rho_t}\phi_t^e\left(ds_t - \delta s_t dt\right) + e^{-\rho_t}\phi_t^e \pi_t dt - e^{-\rho_t}\phi_t^e s_t d\rho_t \end{aligned}$$

Note that the gain process cumulates gains from changing stock prices (net of depreciation), dividends,interest income and borrowing cost discounted by the interest process. To exclude Ponzi-schemes, assume that only limited losses are allowed, i.e. assume that there exists a uniformly integrable process, which might well depend on the initial portfolio, which bounds the gain process from below. Formally:

$$\exists X_t \in \mathcal{L}^1(\mathbb{R},\mathbb{Q}) \text{ uniformly (in } t\text{) integrable, such that } G_t \geq -X_t \quad \mathbb{Q}-\text{a.s.} \tag{2.4.1}$$

In addition, assume non-negative terminal wealth

$$\liminf_{t\to\infty} e^{-\rho_t} W_t^{(\phi^e,\phi^s)} \geq 0 \quad \mathbb{Q}-\text{a.s.} \tag{2.4.2}$$

Households Problem

The households problem is

Definition 2.4.4. *Given an initial portfolio $(\phi^s_{0-}, \phi^e_{0-})$ and processes w_t, ρ_t, s_t, π_t, the households problem (HP) is*

$$\begin{aligned} &\max_{c_t, L_t, \phi_t^e, \phi_t^s} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\frac{c_t^{1-\theta}}{1-\theta}dt - \beta dL_t\right)\right] \\ &s.t. \quad c_t dt + s_t d\phi_t^e + d\phi_t^s \leq w_t dL_t + \phi_t^e \pi_t dt + \phi_t^s d\rho_t - \delta s_t \phi_t^e dt \\ &L_t \quad \textit{is non-decreasing} \\ &\phi_t^e, \phi_t^s \in \mathcal{A} \triangleq \{\phi_t \mid \textit{ adapted, of finite variation and s.t. (2.4.1) and (2.4.2)}\} \end{aligned} \tag{2.4.3}$$

where ρ without a time subscript is the household's discount factor, while ρ_t denotes cumulative interest payments as introduced in the preceding section.

2.5 Solution

In this section, I first define an equilibrium. Then the associated central planner's problem is formulated and solved. Finally, the solution to the latter is used to derive an equilibrium for the market economy.

2.5.1 Equilibrium

An equilibrium is

Definition 2.5.1. *An equilibrium for the initial capital k_{0-} is a tuple, consisting of a consumption process c_t, labor process L_t, a wage process w_t, a pricing measure $\mathbb{Q}$, an interest process ρ_t, a capital process k_t and portfolio processes ϕ^e_t and ϕ^s_t such that*

1. *(The stock market clears)*
$$k_t = \phi^e_t$$
2. *(Investment is irreversible) k_t solves*
$$dk_t = -\delta k_t dt + dI_t$$
for some non-decreasing right-continuous process I_t
3. *(The bond market clears) ρ_t is predictable and*
$$\phi^s_t = 0$$
4. *(Labor market clears) Given the wage w_t*
$$dL_t = k_t \arg\max_l \pi_t(l)dt + \gamma_t dI_t$$
5. *(The investment market clears)*
$$s_t \triangleq \mathbb{E}^{\mathbb{Q}}\left[\int_t^\infty e^{-\int_t^s d\rho_r} e^{-\delta(s-t)}\pi_s ds \mid \mathcal{F}_t\right] \leq \gamma_t w_t \quad \mathbb{P}\otimes dt\text{-}a.s.$$
where $\pi_s = \max_l \pi_s(l)$ and
$$(\gamma_t w_t - s_t)dI_t = 0 \text{ as a measure on } \Omega \times \mathbb{R}_+$$
6. *(Households maximize utility) Given w_t, ρ_t, s_t and π_t and the initial portfolio $(\phi^s_{0-}, \phi^e_{0-}) = (0, k_{0-})$, the processes $c_t, L_t, \phi^s_t, \phi^e_t$ solve (HP).*

Thus, an equilibrium demands that the whole capital stock is held as equity by households (1.), the capital stock depreciates with rate δ and grows with the investment process I_t (2.). In equilibrium all bonds net out to zero (3.). Condition (4.) requires that total labor is given by the sum of labor employed in existing firms and in the investment process. (5.) is a no-arbitrage/ free entry condition. It states that the stock price can never exceed the price at which new firms can be created ($\gamma_t w_t$) and both are equal whenever new firms enter the market. In addition all processes must be such that households maximize utility (6.) .

2.5.2 Central Planner's Version

I now combine the firms and the households problem into one single central planner's problem. For this, let the economy wide production function be $F(l, k) = Ak^\alpha l^{1-\alpha}$.

Central Planner's Version I

Definition 2.5.2. *The central planner's problem is*

$$\begin{array}{lll} \text{(CP I)} & \max_{l_t, I_t} & \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(u(c_t, l_t)dt - \beta\gamma_t dI_t\right)\right] \\ & s.t. & c_t \leq F(l_t, k_t) \\ & & dk_t = -\delta k_t dt + dI_t \\ & & I_t \text{ is non-decreasing, right-continuous, adapted} \end{array}$$

Central Planner's Version II

I now simplify the planner's problem to a standard singular control problem. At first, since l_t can be chosen flexibly, let

$$V(k) \triangleq \max_l u(Ak^\alpha l^{1-\alpha}, l)$$

i.e.

$$V(k) = \frac{\psi}{\alpha\bar{\theta}\nu} k^{\alpha\bar{\theta}\nu}$$

with $\bar{\theta} = 1 - \theta$ and

$$\nu = \frac{1}{\alpha + \theta(1-\alpha)}, \quad \phi = \left(\frac{1-\alpha}{\beta}\right)^\nu, \quad \psi = \alpha\bar{\theta}\nu\left(\frac{1}{\bar{\theta}}\phi^{(1-\alpha)\bar{\theta}} - \beta\phi\right)A^{\bar{\theta}\nu}$$

The problem simplifies to

Definition 2.5.3. *The reformulated central planner's problem is*

$$\begin{array}{llll} \text{(CP II)} & \max_{I_t} & \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(V(k_t)dt - \beta\gamma_t dI_t\right)\right] & (2.5.1) \\ & s.t. & dk_t = -\delta k_t dt + dI_t & (2.5.2) \\ & & I_t \text{ is non-decreasing, right-continuous, adapted} & \end{array}$$

This is now an irreversible investment problem in standard form which can be solved by standard singular control methods (see Back and Paulsen (2009)) or by the Riedel-Su method (Riedel and Su (2011)). I proceed by the Riedel-Su method. For this, I first derive first order conditions in a heuristic way and then show that they are sufficient.

FONC

Note that from investing one marginal unit at time s, the central planner receives the marginal utility

$$M_s \triangleq \mathbb{E}\left[\int_s^\infty e^{-(\rho+\delta)(t-s)} V_k(k_t^*)dt \middle| \mathcal{F}_s\right] \quad (2.5.3)$$

As he is free to do so at cost $\beta\gamma_s$, the first order necessary conditions for optimality (FONC) are

$$M_s \leq \beta\gamma_s \quad \mathbb{P}\otimes ds\text{-almost surely} \tag{2.5.4}$$

with equality whenever $dI_t > 0$, i.e. more precisely

$$(\beta\gamma_s - M_s)\, dI_s = 0 \text{ as a measure on } \Omega \times \mathbb{R}_+ \tag{2.5.5}$$

I now show that the thus heuristically derived FONCs are sufficient (and omit to verify that these conditions are indeed necessary for an optimum).

FONC are Also Sufficient

By the concavity of the functional $I_t \mapsto \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(V(k_t)dt - \beta\gamma_t dI_t\right)\right]$, the necessary conditions are also sufficient as the following theorem shows:

Theorem 2.5.4. *Let I_t^* be an investment process such that with*

$$k_t^* \triangleq \int_0^t e^{-\delta(t-s)} dI_s^* + e^{-\delta t} k_{0-}^* \tag{2.5.6}$$

the FONC (2.5.4) and (2.5.5) are fulfilled for M defined by (2.5.3). Then I_t^* is optimal for (CP II)

The proof is in the appendix.

Simplification of FONC

Hence, we are interested in a process I_t^* fulfilling the FONC. I will now rewrite the FONC as a backward equation which simplifies the problem as will become clear below.

Lemma 2.5.5. *Let l_t be a stochastic process, such that*

$$\mathbb{E}\left[\int_s^\infty e^{-(\rho+\delta)t} V_k(e^{-\delta t} \max_{s\leq\tau\leq t} l_\tau e^{\delta\tau})dt \,\middle|\, \mathcal{F}_s\right] = e^{-(\rho+\delta)s}\beta\gamma_s \quad \mathbb{P}\otimes ds - a.s. \tag{2.5.7}$$

Then the process $k_t^* \triangleq e^{-\delta t} \max\left[\max_{\tau\leq t} l_\tau e^{\delta\tau}, k_{0-}\right]$ is an optimal capital stock, i.e. $dI_t^* \triangleq dk_t^* + \delta k_t^* dt$ solves the FONC (2.5.4) and (2.5.5) and thus the maximization problem (CP II).

The result is easy to interpret. The optimal investment process is the minimal process that keeps capital k_t above the process l_t, i.e. such that $k_t \geq l_t$ for all t. Here, l_t can be interpreted as the capital stock that would be optimal if the irreversibility constraint were not binding in the current instant. If $l_t \geq k_t$ it indeed does not bind, investment occurs and capital is held above l_t. If, however, $l_t < k_t$ the planner would like to disinvest but cannot due to the irreversibility constraint. Capital depreciates until $l_t \geq k_t$ and it is invested again. The proof is in the appendix.

By the Lemma, we are interested in a solution to the backward equation (2.5.7). Finding such a solution turns out to be surprisingly simple.

Solution

Theorem 2.5.6. *Let* $l_t = \kappa \gamma_t^{-\frac{1}{\theta\nu}}$ *with* $\kappa = \left(\frac{\psi}{\beta(\rho+\delta(1-\theta\nu))}\right)^{\frac{1}{\theta\nu}} \mathbb{E}\left[e^{\underline{G}_\tau}\right]^{\frac{1}{\theta\nu}}$*, where* τ *is an independent from* $\underline{G}_t \triangleq \min_{s\leq t}(Y_s - Y_0 - \theta\nu\delta s)$ *exponentially distributed random-variable with parameter* $\rho + \delta(1-\theta\nu)$*. Then* l_t *fulfills (2.5.7). In particular* $k_t^* \triangleq e^{-\delta t}\max\left[\max_{\tau\leq t} l_\tau e^{\delta\tau}, k_{0-}\right]$ *is an optimal capital stock, i.e.* $dI_t^* \triangleq dk_t^* + \delta k_t^* dt$ *solves (CP II).*

The proof is given in the appendix.

Remark 2.5.7. *Note that the base capacity* l_t *is identical to the the base capacity* $\tilde{l}_t$ *without depreciation if we substitute* $\tilde{\rho} \triangleq \rho + \delta(1-\theta\nu)$ *and* $\tilde{Y}_t \triangleq Y_t - \theta\nu\delta t$*.*

Later I will assume that γ_t is a geometric Brownian motion. In this case, Remark 2.5.7 helps to explicitly compute κ.

Explicit Solution in Case of Geometric Brownian Motion

Theorem 2.5.8. *Let* $d\gamma_t = \mu\gamma_t dt + \sigma\gamma_t dB_t$ *be a geometric Brownian motion.*

1. *If* $\delta = 0$*, then*

$$\mathbb{E}\left[e^{\underline{G}_\tau}\right] = 1 - \frac{\sigma^2}{\mu + \frac{1}{2}\sigma^2 + \sqrt{2\rho\sigma^2 + (\mu - \frac{1}{2}\sigma^2)^2}}$$

2. *For* $\delta \neq 0$

$$\mathbb{E}\left[e^{\underline{G}_\tau}\right] = 1 - \frac{\sigma^2}{\mu - \theta\nu\delta + \frac{1}{2}\sigma^2 + \sqrt{2(\rho + \delta(1-\theta\nu))\sigma^2 + (\mu - \frac{1}{2}\sigma^2 - \theta\nu\delta)^2}}$$

The proof is in the appendix

2.5.3 Market Equilibrium

Definition of Equilibrium Processes

In the last subsection, I computed the social planner solution I_t^* to the problem at hand. In this section, I extend this solution to an equilibrium for the market economy as defined in Definition 2.5.1 in the usual way. That is, I construct $c_t^*, L_t^*, w_t^*, \mathbb{Q}^*, \rho_t^*, \phi_t^{e*}, \phi_t^{s*}$ such that with k_t^* given by (2.5.6) conditions 1) to 6) of Definition 2.5.1 are fulfilled.

For this, let l_t^* as be utility maximizing amount of labor per capital unit, i.e.

$$l_t^* \triangleq \arg\max_l u(F(l, k_t^*), l)/k_t^* \tag{2.5.8}$$

In particular, output and consumption is given by

$$c_t^* \triangleq F(k_t^* l_t^*, k_t^*) = A k_t^* (l_t^*)^{1-\alpha}$$

Cumulative labor is then the integral over labor used for the production of consumption good and investment up to time t.

$$dL_t^* \triangleq k_t^* l_t^* dt + \gamma_t dI_t^* \tag{2.5.9}$$

Total amount of equity must be given by the capital stock, i.e.

$$\phi_t^{e*} \triangleq k_t^*$$

and bonds net out to zero

$$\phi_t^{s*} \triangleq 0$$

Define wages as the rate of substitution between leisure and consumption

$$w_t^* \triangleq \frac{\beta}{u_c(c_t^*)} \tag{2.5.10}$$

Let interest payments ρ_t^* be a predictable process such that

$$e^{\rho_t^*} e^{-\rho t} u_c(c_t^*)$$

is a local martingale, i.e. in case of γ_t being a geometric Brownian motion then

$$d\rho_t^* \triangleq -d\log\left(e^{-\rho t} u_c(c_t^*)\right) = (\rho - \alpha\theta\nu\delta)\, dt + \alpha\theta\nu \frac{1}{k_t^*} dI_t^* \tag{2.5.11}$$

as the right hand side is continuous and therefore predictable. If, however, γ_t contains jumps, this process would no longer be feasible. In this case the jumps have to be replaced by its compensator[11].

Define the pricing $\mathbb{Q}^*$ measure by

$$d\mathbb{Q}^* \triangleq e^{\int_0^s d\rho_u^*} e^{-\rho s} \frac{u_c(c_s^*)}{u_c(c_t^*)} \cdot d\mathbb{P} \quad \text{on } \mathcal{F}_s$$

Note that $\lim_{s\to\infty} e^{\int_0^s d\rho_u^*} e^{-\rho s} \frac{u_c(c_s^*, l_s^*)}{u_c(c_t^*, l_t^*)}$ exists due to the supermartingale convergence (e.g. see Kallenberg (2001)) theorem and therefore the definition extends to $\mathcal{F}_\infty = \bigcup \mathcal{F}_t$. In particular

$$\begin{aligned} s_t^* &= \mathbb{E}^{\mathbb{Q}^*}\left[\int_t^\infty e^{-(\rho_s^* - \rho_t^*) - \delta(s-t)} \pi_s(l_s^*) ds \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^\infty e^{-(\rho+\delta)(s-t)} \frac{u_c(c_s^*, l_s^*)}{u_c(c_t^*, l_t^*)} \pi_s\left(l_s^*\right) ds \middle| \mathcal{F}_t\right] \end{aligned}$$

[11] Applying Lévy's decomposition theorem and writing

$$\gamma_t = \exp(at + \sigma B_t + \int_{[0,t]\times\mathbb{R}} z\left(N(dz, dt) - \nu(dz)dt\right)) \tag{2.5.12}$$

for some Brownian motion B_t, Poisson measure N with compensator ν and $a, \sigma \in \mathbb{R}$, one can obtain an explicit formula.

Thus Defined Processes are Equilibrium

I now show that the thus defined processes form an equilibrium. Conditions 1), 2) and 3) are fulfilled by definition of the processes.
It remains to show 4), 5) and 6)

Theorem 2.5.9. *Let the components of the central planner's value function be finite:*

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} V(k_t^*)dt\right] + \mathbb{E}\left[\int_0^\infty e^{-\rho t}\beta\gamma_t dI_t\right] < \infty \tag{2.5.13}$$

Then conditions 4),5), 6) of definition 2.5.1 are fulfilled. That is the labor market clears, the investment market clears and households maximize utility.

The proof is in the appendix.

Remark 2.5.10. *Note that the interest process $d\rho_t$ consists of two components. An absolutely continuous component which might be negative if depreciation is high and discounting is low and an increasing component related to investment. If investment is singular (e.g. if γ_t is a diffusion) also the latter component is singular. Singular money market returns might be interpreted as 'fees' and 'premia' payed on debt and money holdings. Alternatively, in an economy with money, singular real returns might be caused by singular movements in the price level.*

2.6 Analysis in Case of Geometric Brownian Motion

From now on, assume that γ_t is given by a geometric Brownian motion.

Assumption 2.6.1. *Entry cost are given by a geometric Brownian motion*

$$d\gamma_t \triangleq \mu\gamma_t dt + \sigma\gamma_t dB_t \tag{2.6.1}$$

Note that the optimal capital stock is decreasing in volatility σ for fixed entry cost, but increasing in the long run due to increasing probability of low entry cost.

Theorem 2.6.2 (Optimal Capital Stock)**.** *Let $\rho + \delta(1-\theta\nu) > 0$. Then $k_t^* = \kappa\gamma_t^{-\frac{1}{\theta\nu}}$, the capital stock that was optimal if the irreversibility constraint were currently non-binding, is decreasing in σ (for fixed γ_t).*

Proof. By Theorem 2.5.6

$$\kappa = \left(\frac{\psi}{\beta\left(\rho + \delta(1-\theta\nu)\right)}\right)^{\frac{1}{\theta\nu}} \mathbb{E}\left[e^{\underline{G}_\tau}\right]^{\frac{1}{\theta\nu}}$$

where τ is an independent from $\underline{G}_t \triangleq \min_{s\leq t} Y_s - Y_0 - \theta\nu\delta s$ exponentially distributed random-variable with parameter $\rho + \delta(1-\theta\nu)$. Hence σ affects κ only via $\mathbb{E}\left[e^{\underline{G}_\tau}\right]$

which is decreasing in σ. Namely let $\sigma^2 < \tilde{\sigma}^2$. Set $\hat{\sigma}^2 \triangleq \tilde{\sigma}^2 - \sigma^2$. We can write

$$\begin{aligned} Y_t^{\tilde{\sigma}} &= (\mu - \frac{1}{2}\tilde{\sigma}^2)t + \tilde{\sigma}\tilde{B}_t \\ &= \underbrace{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t}_{=Y_t^\sigma} + \hat{\sigma}\hat{B}_t - \frac{1}{2}\hat{\sigma}^2 t \\ &= Y_t^\sigma + \hat{\sigma}\hat{B}_t - \frac{1}{2}\hat{\sigma}^2 t \end{aligned}$$

where $\hat{B}$ and B are independent Brownian motions and $\tilde{B} = \left(\sigma B + \hat{\sigma}\hat{B}\right)/\tilde{\sigma}$ is a Brownian Motion. With this, we get

$$\begin{aligned} \mathbb{E}\left[e^{\underline{\tilde{G}_\tau}}\right] &= \mathbb{E}\left[\min_{s\leq\tau} e^{Y_s^{\tilde{\sigma}} - \delta\theta\nu s}\right] \\ &= \mathbb{E}\left[\min_{s\leq\tau} e^{Y_s^\sigma - \delta\theta\nu s + \hat{\sigma}\hat{B}_s - \frac{1}{2}\hat{\sigma}^2 s}\right] \\ &< \mathbb{E}\left[e^{Y_{\tau_\sigma}^\sigma - \delta\theta\nu\tau_\sigma + \hat{\sigma}\hat{B}_{\tau_\sigma} - \frac{1}{2}\hat{\sigma}^2\tau_\sigma}\right] \end{aligned}$$

where τ_σ is the minimizer of $Y_s^\sigma - \delta\theta\nu s$ under the constraint $\tau_\sigma \leq \tau$, i.e.

$$Y_{\tau_\sigma}^\sigma - \delta\theta\nu\tau_\sigma = \min_{s\leq\tau}\left(Y_s^\sigma - \delta\theta\nu s\right)$$

The inequality is strict, as the r.v. exceeds 1 with non-zero probability whereas the minimum on the left hand side is bounded by 1 (for $s = 0$). As Y^σ and $\hat{B}$ are independent and τ_σ is measurable with respect to $\sigma(Y^\sigma)$, we get

$$\begin{aligned} & \mathbb{E}\left[e^{Y_{\tau_\sigma}^\sigma - \delta\theta\nu\tau_\sigma + \hat{\sigma}\hat{B}_{\tau_\sigma} - \frac{1}{2}\hat{\sigma}^2\tau_\sigma}\right] \\ =\ & \mathbb{E}\left[\mathbb{E}\left[e^{Y_{\tau_\sigma}^\sigma - \delta\theta\nu\tau_\sigma + \hat{\sigma}\hat{B}_{\tau_\sigma} - \frac{1}{2}\hat{\sigma}^2\tau_\sigma}\middle|\tau_\sigma\right]\right] \\ =\ & \mathbb{E}\left[\mathbb{E}\left[e^{Y_{\tau_\sigma}^\sigma - \delta\theta\nu\tau_\sigma}\middle|\tau_\sigma\right]\mathbb{E}\left[e^{\hat{\sigma}\hat{B}_{\tau_\sigma} - \frac{1}{2}\hat{\sigma}^2\tau_\sigma}\middle|\tau_\sigma\right]\right] \\ =\ & \mathbb{E}\left[\mathbb{E}\left[e^{Y_{\tau_\sigma}^\sigma - \delta\theta\nu\tau_\sigma}\middle|\tau_\sigma\right]\cdot 1\right] \\ =\ & \mathbb{E}\left[e^{Y_{\tau_\sigma}^\sigma - \delta\theta\nu\tau_\sigma}\right] \\ =\ & \mathbb{E}\left[e^{\underline{G}_\tau}\right] \end{aligned}$$

□

In particular there is a delay in investment even though from the central planner's perspective the fundamentals, namely productivity and cost, remain constant. What changes with the volatility are the fundamental's *dynamics*. This delay is due to the optimal exercise strategy of growth options and the option premium of waiting to invest. If cost fluctuate, there is a gain in waiting and speculating on even lower cost. This is the content of the next section. But before, I analyze the impact of volatility on the long run behavior of the capital stock.

Theorem 2.6.3. *The long run dynamics of the capital stock k_t^* fulfill*

1. *) Let $\delta\theta\nu - \mu + \frac{1}{2}\sigma^2 \geq 0$ then*

$$\lim_{t\to\infty} \frac{1}{t} \log k_t^* = \frac{1}{2\theta\nu}\sigma^2 - \frac{1}{\theta\nu}\mu \quad \mathbb{P}a.s.$$

2. *) Let $\delta\theta\nu - \mu + \frac{1}{2}\sigma^2 + \frac{\sigma^2}{2\theta\nu} \geq 0$ then*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[k_t^*\right] = \frac{1}{\theta\nu}\left(\frac{1}{2}\sigma^2 - \mu\right) + \frac{\sigma^2}{2\theta^2\nu^2}$$

In particular the long run growth of the capital stock is a.s. (and in expectation) increasing in volatility σ.

Hence, the long run capital stock increases in volatility. The intuition is that the more volatile entry cost are, the more probable are low entry cost which will lead to growth while the set on which entry cost explode has vanishing probability.

2.6.1 Option Premia

In this subsection, I show that the delay in investment is due to an increase in the option premium of waiting to invest. I assume $\delta = 0$ for two reasons. First, for simplicity and second it allows to give a precise definition of the option premium.

Definition of Option Premium and Connection to Optimal Stopping

Definition 2.6.4. *Let τ_y be the time of investment of the y-th capital unit, i.e. $\tau_y \triangleq \inf\{t > 0 | k_t^* \geq y\}$. The option premium is the expected discounted difference between the flow of marginal utility generated by the y-th unit of capital and its cost at τ_y.*

$$OP_t(y, \gamma_t) \triangleq \mathbb{E}\left[\int_{\tau_y}^{\infty} e^{-\rho(s-t)} V_k(y) ds - e^{-\rho(\tau_y - t)} \beta\gamma_{\tau_y} \middle| \mathcal{F}_t\right]$$

To motivate this definition note that - interchanging the order of integration - we can write the value of the social planner as

$$\begin{aligned}
& \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}\left(V(k_t^*)dt - \beta\gamma_t dk_t^*\right)\right] \\
= & \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}\left(\int_0^{k_t^*} V_k(y) dy dt - \beta\gamma_t dk_t^*\right)\right] \\
= & \mathbb{E}\left[\int_{k_{0-}^*}^{\infty}\left(\int_{\tau_y}^{\infty} e^{-\rho t} V_k(y) dt - e^{-\rho\tau_y}\beta\gamma_{\tau_y}\right) dy\right] + \int_0^{k_{0-}^*}\int_0^{\infty} e^{-\rho t} V_k(y) dt dy \\
= & \underbrace{\int_0^{k_{0-}^*}\int_0^{\infty} e^{-\rho t} V_k(y) dt dy}_{\text{installed capital}} + \underbrace{\int_{k_{0-}^*}^{\infty} \mathbb{E}\left[\int_{\tau_y}^{\infty} e^{-\rho t} V_k(y) dt - e^{-\rho\tau_y}\beta\gamma_{\tau_y}\right] dy}_{\text{growth options}}
\end{aligned}$$

where τ_y is the installment time of capital unit y, i.e. $\tau_y \triangleq \inf\{t \mid k^*_t \geq y\}$. That is the central planner's is the sum of the utility flow stemming from already installed capital and the value of all growth options. Note that as k^* maximizes the value function, τ_y is an optimal stopping time for

$$\max_{\tau} \mathbb{E}\left[\int_{\tau}^{\infty} e^{-\rho t} V_k(y)dt - e^{-\rho\tau_y}\beta\gamma_{\tau_y}\right] \tag{2.6.2}$$

for dy-almost all y, $\mathbb{P}$-a.s. For if it were not, replacing τ_y with optimal stopping times would lead to a higher utility[12]. Hence, the optimal investment process maximizes the flow of utility generated by the y-th unit net of disutility due to installment cost. The value of the stopping problem is exactly $OP_t(y, \gamma_t)$. It is the value one obtains by not-stopping now, i.e. the continuation value and can therefore be interpreted as an opportunity cost to exercising the option. Consequently, it is optimal to exercise the option when the return in form of the stream of marginal utility outweighs both, the direct exercise cost and the opportunity cost due the scrapped opportunity to exercise later. I.e. when

$$\int_{t}^{\infty} e^{-\rho(s-t)} V_k(y)ds \geq \beta\gamma_t + OP_t(y, \gamma_t) \tag{2.6.3}$$

in which case (2.6.3) holds with equality by the definition of the option premium. Note that the left hand side is deterministic and does not depend on volatility σ. Neither does the current value γ_t. Hence any delay must be due to an increase in the option premium (rather than a decrease in marginal utility). The Option premium can be explicitly computed.

Theorem 2.6.5. *Let $R(y) = \int_0^\infty e^{-\rho s} V_k(y)ds$ be the present value of the flow of marginal utility obtained from capital unit y. Then:*

1.
$$OP(\gamma) = \left(\frac{\gamma}{\gamma^*}\right)^{\eta} \frac{R(y)}{1-\eta}, \quad \gamma^* = \frac{R(y)\eta}{\beta(\eta - 1)} \tag{2.6.4}$$

with η being the negative root of

$$\mu\eta + \frac{1}{2}\sigma^2\eta\,(\eta - 1) - \rho = 0 \tag{2.6.5}$$

In particular $OP > 0$

2. *The option premium is increasing in uncertainty:*

$$\frac{\partial OP}{\partial \sigma^2} > 0$$

[12] Note that replacing the installment times τ_y with optimal stopping times for (2.6.2) is feasible as the optimal stopping times are increasing in y due to the concavity of V. This fact is proven in the appendix.

3. *Even for $\mu = \tilde{\mu} + \sigma^2$, i.e. if marginal productivity $1/\gamma_t$ is a martingale and μ increases in σ, the optimal capital stock is decreasing in σ as long as $\mu = \tilde{\mu} + \sigma^2 \geq -\rho$*

Proof. As the installment times of unit y are optimal stopping times for (2.6.2), it holds

$$OP(\gamma) = \max_{\tau} \mathbb{E}\left[e^{-\rho\tau}\left(R(y) - \beta\gamma_\tau\right) \mid \gamma_0 = \gamma\right]$$

Hence, OP solves

$$\begin{aligned} \mu\gamma OP' + \frac{1}{2}\sigma^2\gamma^2 OP'' - \rho OP &= 0 \quad \text{(Euler Equation)} \\ OP(\gamma^*) &= R(y) - \beta\gamma^* \quad \text{(Value Matching)} \\ OP'(\gamma^*) &= -\beta \quad \text{(Smooth Pasting)} \end{aligned}$$

Trying a solution of the form $OP(\gamma) = c\gamma^\eta$ one arrives at 1). For 2) and 3) differentiate (2.6.5) with respect to σ^2. This yields $\frac{\partial\eta}{\partial\sigma^2} > 0$. Differentiating (2.6.4) with respect to η yields

$$\frac{\partial OP}{\partial\eta} = \frac{\left(\frac{\gamma}{\gamma^*}\right)^\eta \gamma^*\beta\left(\ln(\gamma) - \ln(\gamma^*)\right)}{-\eta} \geq 0$$

as $\gamma \geq \gamma^*$. Hence also $\frac{\partial OP}{\partial\sigma^2} = \frac{\partial OP}{\partial\eta}\frac{\partial\eta}{\partial\sigma^2} \geq 0$ □

Reconciled with Zero Profits via Rise in Long Rate and Wages

The question at hand is how the delay is realized in the market interpretation of the economy. In a market equilibrium, firms must make zero profits, i.e. it holds $s_t \leq \gamma_t w_t$. Thus, there can only be a delay in investment if the fundamentals as seen by the market change. Investment which has been profitable must become unprofitable and this can only be due to a change in fundamentals. Note that a firm's profit evaluation contains three further parameters, $\mathbb{Q}^*, w^*$ and ρ^* which are influenced by σ. These are 'market fundamentals' which are artificial from the standpoint of the social planner. Indeed, this section shows that it are increases in the longer term interest rates and an increase in expected future wages (future prices on labor) which render current investment opportunities unprofitable when volatility increases.
To see this, write the stock price as

$$\begin{aligned} s_t^* &= \mathbb{E}^{\mathbb{Q}^*}\left[\int_t^\infty e^{-(\rho_s^* - \rho_t^*)} e^{-\delta(s-t)} \pi_s^* ds \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}^*}\left[\int_t^\infty e^{-(\rho_s^* - \rho_t^*)} e^{-\delta(s-t)} \max_l \left(Al^{1-\alpha} - w_s^* l\right) ds \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}^*}\left[\int_t^\infty e^{-(\rho_s^* - \rho_t^*)} e^{-\delta(s-t)} \left(w_s^*\right)^{\frac{\alpha-1}{\alpha}} \pi ds \middle| \mathcal{F}_t\right] \end{aligned} \quad (2.6.6)$$

with $\pi = \alpha A^{\frac{1}{\alpha}}(1-\alpha)^{\frac{1-\alpha}{\alpha}}$. Hence

$$\begin{aligned} s_t^* &= \pi \int_t^\infty e^{-\delta(s-t)} \mathbb{E}^{\mathbb{Q}^*}\left[e^{-(\rho_s^*-\rho_t^*)} \left(w_s^*\right)^{\frac{\alpha-1}{\alpha}} \Big| \mathcal{F}_t \right] ds \\ &= \pi \int_t^\infty e^{-\delta(s-t)} B_{t,s} \hat{w}_{t,s}^\alpha ds \end{aligned} \tag{2.6.7}$$

where

$$B_{t,s} \triangleq \mathbb{E}^{\mathbb{Q}^*}\left[e^{-(\rho_s^*-\rho_t^*)} \Big| \mathcal{F}_t \right]$$

is the time-t price of a zero coupon bond with maturity $t-s$ and

$$\hat{w}_{t,s}^\alpha \triangleq B_{t,s}^{-1} \mathbb{E}^{\mathbb{Q}^*}\left[e^{-(\rho_s^*-\rho_t^*)} {w_s^*}^{1-\frac{1}{\alpha}} \Big| \mathcal{F}_t \right]$$

is the price of a future contract on ${w_s^*}^{-\frac{1}{\alpha}}$ units of labor each sold at the spot price w_s^*.

Equation (2.6.7) expresses s_t^* as an index of bond prices and futures on wages. The index is a metric of the term structure of future prices. As the stock price decreases in volatility due to Theorem 2.6.2 in combination with investment market clearing (see Definition 2.5.1, number 5.) also the index decreases. So the term structure must change and interest rates and futures on wages measured by the metric of the index have to increase. We record this observation in a theorem

Theorem 2.6.6. *At the investment trigger $\gamma = \gamma^*$ the entity*

$$s_t^* = \pi \int_t^\infty e^{-\delta(s-t)} B_{t,s} \hat{w}_{t,s}^\alpha ds$$

is decreasing in volatility σ.

Proof. We have to show that the stock pürice s_t^* decreases in volatility σ. To see this suppose $\sigma_0 < \sigma_1$ and assume we are at the investment trigger for $\sigma = \sigma_0$, i.e. $\gamma_t = \gamma^*(\sigma_0)$ and $s_t(\sigma_0) = \gamma^*(\sigma_0) w_t^*$. As $k_t^*(\sigma)$ is decreasing in volatility by Theorem 2.6.2 so is $\gamma^*(\sigma)$. In particular $s_t(\sigma_0) = \gamma^*(\sigma_0) w_t^* > \gamma^*(\sigma_1) w_t^* \geq s_t(\sigma_1)$ where the last inequality is due to the equilibrium condition (Definition 2.5.1, number 5). □

Note that with l_t being the long rate (i.e. the fixed coupon payed by a perpetual bond with price 1) at time t by definition

$$l_t^{-1} = \int_t^\infty B_{t,s} ds \tag{2.6.8}$$

Hence for $\alpha = 1$ and $\delta = 0$ the index coincides (up to scaling by π) with the inverse long rate. That is for parameters close to these, longer term interest rates increase. Both, the long rate and the stock price can be computed in closed form. This is done in the following theorems.

Theorem 2.6.7. *The inverse long rate*

$$l^{-1}(k_t^*, \gamma_t) = \mathbb{E}^{\mathbb{Q}^*}\left[\int_t^\infty e^{-(\rho_s^* - \rho_t^*)} \cdot 1ds \middle| \mathcal{F}_t\right] \tag{2.6.9}$$

is explicitly given by

$$l^{-1}(k,\gamma) \;=\; \frac{1}{\rho - \alpha\theta\nu\delta}\left(1 - \frac{\alpha}{\alpha - c}\left(\frac{k}{\kappa\gamma^{-\frac{1}{\theta\nu}}}\right)^{c\theta\nu}\right) \quad \text{for} \quad k \geq \kappa\gamma^{-\frac{1}{\theta\nu}} \tag{2.6.10}$$

where c is the smallest root[13] *of*

$$(c-\alpha)\theta\nu\delta - c\mu + \rho + \frac{1}{2}\sigma^2(c - c^2) = 0 \tag{2.6.11}$$

with $\upsilon = \theta\nu$.

Proof. The proof is in the appendix. □

Note that the inverse long rate, i.e the price of a perpetual bond, consists of two components. It is the sum of the inverse short rate in times without investment $1/(\rho - \alpha\theta\nu\delta)$ and a negative 'growth component' reflecting the impact of future capital expansions. The latter depends on volatility σ, but the sign of the relation ship depends on the parameters and arguments.

Theorem 2.6.8. *Let*

$$\begin{aligned} s_t^* \;&=\; \mathbb{E}^{\mathbb{Q}^*}\left[\int_t^\infty e^{-(\rho_s^* - \rho_t^*) - \delta(s-t)}\pi_s^* ds \middle| \mathcal{F}_t\right] \\ &=\; u_c(c_t^*)^{-1}\mathbb{E}^{\mathbb{P}}\left[\int_t^\infty e^{-(\rho+\delta)(s-t)}u_c(c_s^*)\pi_s^* ds \middle| \mathcal{F}_t\right] \end{aligned}$$

be the stock price. The stock price in labor units s_t^*/w_t^* is explicitly given by

$$\frac{s_t^*}{w_t^*} = \pi\beta^{-\frac{1}{\alpha}}\frac{A^{-\frac{\theta}{\alpha}\nu}\phi^{-(1-\alpha)\frac{\theta}{\alpha}}}{\rho + \delta - \delta\theta\nu}\left(1 - \frac{1}{1-C}\left(\frac{k}{\kappa\gamma^{-\frac{1}{\theta\nu}}}\right)^{C\theta\nu}\right)k^{-\theta\nu} \quad \text{for} \quad k \geq \kappa\gamma^{-\frac{1}{\theta\nu}} \tag{2.6.12}$$

where C is the negative root[14] *of*

$$\theta\nu(C-1)\delta - C\mu + \rho + \delta + \frac{1}{2}\sigma^2(C - C^2) = 0 \tag{2.6.13}$$

Proof. The proof is in the appendix. □

Just as the inverse long rate, the price consists of two components. The net present value of the current cash flow and the negative influence of future capital expansions which increase wages and therefore decrease profits.

[13] i.e. $c \triangleq \frac{-2\mu+\sigma^2+2\delta\theta\nu-\sqrt{8(\rho-\alpha\delta\theta\nu)\sigma^2+(2\mu-\sigma^2-2\delta\theta\nu)^2}}{2\sigma^2}$

[14] i.e. $C \triangleq \frac{-2\mu+\sigma^2+2\delta\upsilon-\sqrt{8(\rho+\delta-\delta\theta\nu)\sigma^2+(2\mu-\sigma^2-2\delta\upsilon)^2}}{2\sigma^2}$

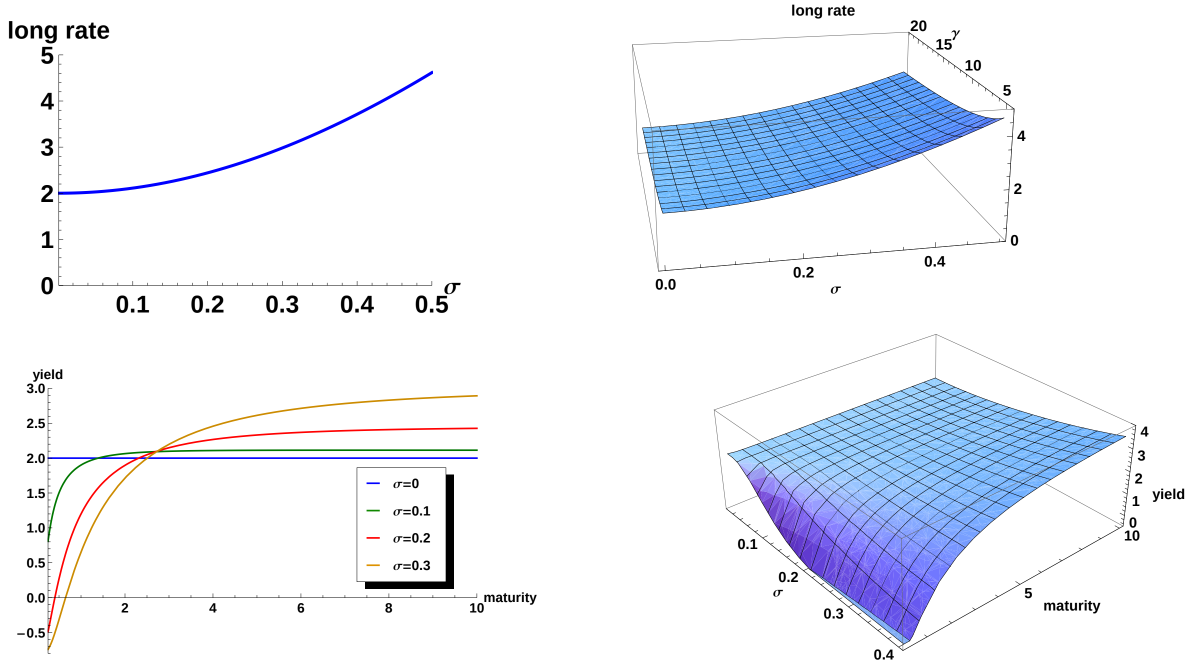

Fig. 2.3: Long term interest rate and term spread are increasing in volatility: Parameter values: $\beta = A = K = 1$, $\alpha = 1/3$, $\theta = 2$, $\rho = 0.02$, $\mu = 0$, $\delta = 0.07$, $\gamma = 4.35$

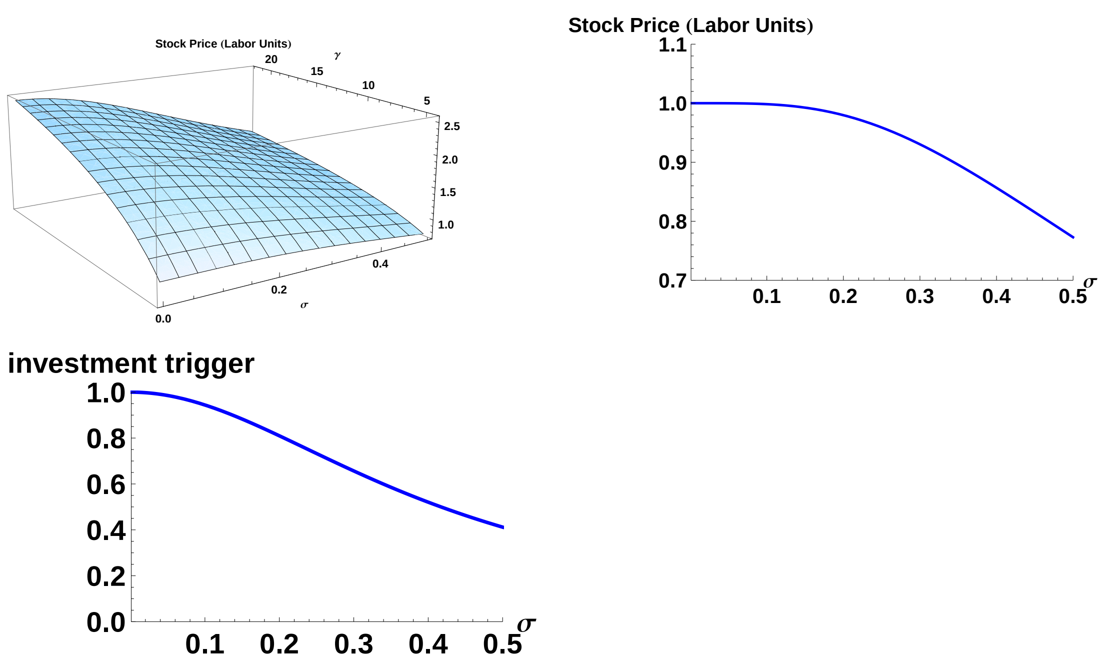

Fig. 2.4: The critical investment cost and stock prices in labor units decrease in volatility

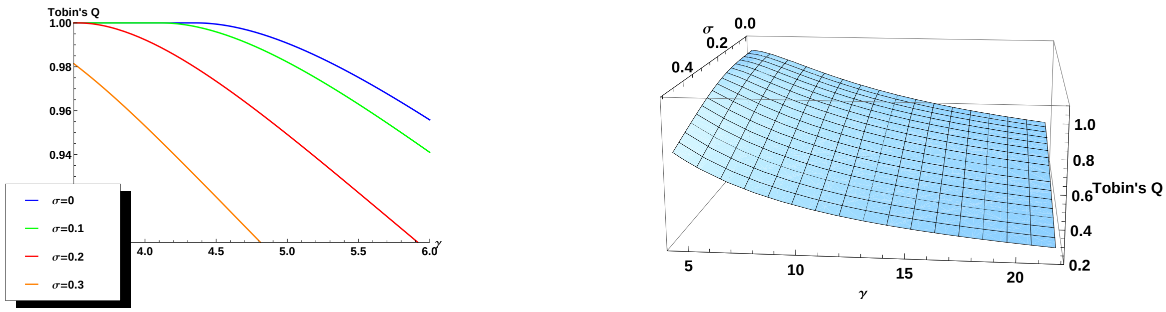

Fig. 2.5: Tobin's Q is decreasing in volatility and cost. Tobin's Q is defined as the ratio of the market price relative to the reproduction price of stocks. In this case this ratio is $\frac{s_t^*}{\gamma_t w_t^*}$

Figure 2.3 shows the long term interest rate in dependence of the volatility σ for the parameter values: $\beta = A = K = 1$, $\alpha = 1/3$, $\theta = 2$, $\rho = 0.02$, $\mu = 0$, $\delta = 0.07$, $\gamma = 4.35$. Here, γ is chosen such that with $\sigma = 0$, the economy is in the steady state. As can be seen, the long rate is increasing in volatility. The higher the volatility, the higher the probability of future capital increases at lower cost. This capital expansion is anticipated by an increase in the long term interest rate which renders current investment opportunities unprofitable even though neither productivity nor cost of investment ('fundamentals') have changed. With σ not the fundamentals itself but their dynamics change. As current investment opportunities have to compete with future ones that are superior in expectation they become unprofitable.
With positive volatility, investment stops and the short term interest rate falls to the level determined by the discount rate ρ and capital depreciation δ. With higher volatility early investment becomes less and less probable so that yields become low for longer maturities.
The lower graph in figure 2.4 shows the investment trigger, that is the critical cost which makes investment optimal, relative to the trigger value with $\sigma = 0$. With increasing σ the trigger decreases, that is lower and lower cost are demanded for investment to be undertaken. The graph above shows how the stock price (measured in labor units) varies with volatility. Tobin's Q (figure 2.6.1, the ratio of the market price relative to the reproduction price of stocks, falls in γ and in σ. The first is not surprising. If reproduction cost increase, the ratio falls even though the market price (see figure 2.4) rises. If volatility rises, so does the probability of cheap future competition so that the market price and thus Tobin's Q fall.

2.7 Conclusion

I solved a general equilibrium model with irreversible capital accumulation. The model is such that a delay in investment is optimal if entry cost fluctuate. This is due to the option value of investing later at lower cost. While in the partial equilibrium setting of Leahy (1993) it are the dynamics of intra-period prices that reconcile a delay with zero profits, in the general equilibrium model of this article this role is absorbed by the dynamics of inter-period prices. That is, in a market setting, the delay occurs via an increase in longer term future prices which anticipate better future investment opportunities, rendering current projects unprofitable. In this sense, the long term rates and futures on labor comprise an option value of waiting to invest. This is similar to Keynes speculative motive which lifts the long term rate at which investors are willing to lend due to speculation on higher future rates. Depending on the parameter values, the speculative demand (in the sense of a lack of long term asset supply) might drive the equilibrium short rate below zero and - if zero is a lower bound - the economy out of equilibrium.

Appendix

Proof of Theorem 2.5.4: Let I_t be an arbitrary investment process and $k_{0-} = k^*_{0-}$. Define

$$k_t \triangleq \int_0^t e^{-\delta(t-s)} dI_s + e^{-\delta t} k_{0-}$$

We have to show that

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(V(k_t)dt - \beta\gamma_t dI_t\right)\right] - \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(V(k^*_t)dt - \beta\gamma_t dI^*_t\right)\right] \leq 0$$

By the concavity of V, we have

$$V(k_t) - V(k^*_t) \leq V_k(k^*_t)(k_t - k^*_t) \tag{2.7.1}$$

Hence

$$\begin{aligned}
& \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(V(k_t)dt - \beta\gamma_t dI_t\right)\right] - \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(V(k^*_t)dt - \beta\gamma_t dI^*_t\right)\right] \\
= \ & \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(V(k_t) - V(k^*_t)\right)dt - \int_0^\infty e^{-\rho s}\beta\gamma_s\left(dI_s - dI^*_s\right)\right] \\
\leq \ & \mathbb{E}\left[\int_0^\infty e^{-\rho t} V_k(k^*_t)(k_t - k^*_t)dt - \int_0^\infty e^{-\rho s}\beta\gamma_s\left(dI_s - dI^*_s\right)\right]
\end{aligned}$$

Now,

$$\begin{aligned}
& \mathbb{E}\left[\int_0^\infty e^{-\rho t} V_k(k^*_t)(k_t - k^*_t)dt - \int_0^\infty e^{-\rho s}\beta\gamma_s\left(dI_s - dI^*_s\right)\right] \\
= \ & \mathbb{E}\left[\int_0^\infty e^{-\rho t} V_k(k^*_t)\int_0^t e^{-\delta(t-s)}\left(dI_s - dI^*_s\right)dt - \int_0^\infty e^{-\rho s}\beta\gamma_s\left(dI_s - dI^*_s\right)\right] \\
= \ & \mathbb{E}\left[\int_0^\infty e^{-\rho s}\int_s^\infty e^{-(\rho+\delta)(t-s)} V_k(k^*_t)dt\left(dI_s - dI^*_s\right) - \int_0^\infty e^{-\rho s}\beta\gamma_s\left(dI_s - dI^*_s\right)\right]
\end{aligned}$$

where we switched the order of integration. By optional projection (see (Kallenberg, 2001, Corollary 19.19)) we can write this as

$$\begin{aligned}
= \ & \mathbb{E}\left[\int_0^\infty e^{-\rho s}\left(\mathbb{E}\left[\int_s^\infty e^{-(\rho+\delta)(t-s)} V_k(k^*_t)dt \middle| \mathcal{F}_s\right] - \beta\gamma_s\right)\left(dI_s - dI^*_s\right)\right] \\
= \ & \mathbb{E}\left[\int_0^\infty e^{-\rho s}\left(M_s - \beta\gamma_s\right)\left(dI_s - dI^*_s\right)\right]
\end{aligned}$$

But $M_s - \beta\gamma_s \leq 0$ and $(M_s - \beta\gamma_s)\, dI^*_s = 0$ $\mathbb{P}$-a.s. by the FONC. Hence

$$\mathbb{E}\left[\int_0^\infty e^{-\rho s}\left(M_s - \beta\gamma_s\right)\left(dI_s - dI^*_s\right)\right] \leq 0 \tag{2.7.2}$$

which finishes the proof.

□

Proof of Lemma 2.5.5: Note that

$$\begin{aligned}
e^{-(\rho+\delta)s} M_s &= \mathbb{E}\left[\int_s^\infty e^{-(\rho+\delta)t} V_k(k^*_t)dt \middle| \mathcal{F}_s\right] \leq \mathbb{E}\left[\int_s^\infty e^{-(\rho+\delta)t} V_k(e^{-\delta t}\max_{\tau \leq t} l_\tau e^{\delta\tau})dt \middle| \mathcal{F}_s\right] \\
&\leq \mathbb{E}\left[\int_s^\infty e^{-(\rho+\delta)t} V_k(e^{-\delta t}\max_{s\leq\tau\leq t} l_\tau e^{\delta\tau})dt \middle| \mathcal{F}_s\right] = e^{-(\rho+\delta)s}\beta\gamma_s
\end{aligned}$$

where we had made use of the fact that $V_{kk} \leq 0$. Note that this equation holds with equality whenever $dI_s^* > 0$ as in this case a new maximum is attained at s and $l_s^* = \max_{\tau \leq s} l_\tau e^{\delta\tau} = k_t^*$. Multiplying both sides with $e^{(\rho+\delta)s}$ yields the result. □

Proof of Theorem 2.5.6: Recall $V_k(k) = \psi k^{-\theta\nu}$ and $\gamma_t = \exp(Y_t)$ where Y_t is a Lévy process. We try a base capital stock of the form $l_t = \kappa\gamma_t^{-\lambda}$ for some $\kappa, \lambda \in \mathbb{R}$. For such a capital stock

$$
\begin{aligned}
e^{-(\rho+\delta)s}\beta &\overset{!}{=} \gamma_s^{-1}\mathbb{E}\left[\int_{t=s}^{\infty} e^{-(\rho+\delta)t} V_k(e^{-\delta t}\max_{s\leq\tau\leq t} l_\tau e^{\delta\tau})dt \,\middle|\, \mathcal{F}_s\right] \\
&= e^{-Y_s}\mathbb{E}\left[\int_{s}^{\infty} e^{-(\rho+\delta)t}\psi\left(e^{-\delta t}\max_{s\leq\tau\leq t} l_\tau e^{\delta\tau}\right)^{-\theta\nu} dt \,\middle|\, \mathcal{F}_s\right] \\
&= \psi\kappa^{-\theta\nu}\mathbb{E}\left[\int_{s}^{\infty} e^{-(\rho+\delta)t}\min_{s\leq\tau\leq t} e^{\theta\nu\lambda Y_\tau - Y_s - \theta\nu\delta(\tau-t)}dt \,\middle|\, \mathcal{F}_s\right] \\
&= \psi\kappa^{-\theta\nu}e^{-(\rho+\delta)s}\mathbb{E}\left[\int_{s}^{\infty} e^{-(\rho+\delta)(t-s)}\min_{s\leq\tau\leq t} e^{Y_\tau - Y_s - \theta\nu\delta(\tau-t)}dt \,\middle|\, \mathcal{F}_s\right]
\end{aligned}
$$

where we had chosen $\lambda \triangleq (\theta\nu)^{-1}$. As $Y_\tau - Y_s$ are stationary and independent from $\mathcal{F}_s$ by the Lévy property, we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_{s}^{\infty} e^{-(\rho+\delta)(t-s)}\min_{s\leq\tau\leq t} e^{Y_\tau - Y_s - \theta\nu\delta(\tau-t)}dt \,\middle|\, \mathcal{F}_s\right] && (2.7.3) \\
=\ &\mathbb{E}\left[\int_{0}^{\infty} e^{-(\rho+\delta)t}\min_{0\leq\tau\leq t} e^{Y_\tau - Y_0 - \theta\nu\delta(\tau-t)}dt\right] && (2.7.4) \\
=\ &\mathbb{E}\left[\int_{0}^{\infty} e^{-(\rho+\delta(1-\theta\nu))t}\min_{0\leq\tau\leq t} e^{G_\tau}dt\right] && (2.7.5) \\
=\ &\mathbb{E}\left[\int_{0}^{\infty} e^{-(\rho+\delta(1-\theta\nu))t}e^{\underline{G}_t}dt\right] && (2.7.6)
\end{aligned}
$$

where $G_t \triangleq Y_t - Y_0 - \theta\nu\delta t$ is also a Lévy process and

$$\underline{G}_t \triangleq \min_{s\leq t} G_s$$

With this, we can solve for κ and get

$$\kappa^{\theta\nu} = \beta^{-1}\psi\mathbb{E}\left[\int_{0}^{\infty} e^{-(\rho+\delta(1-\theta\nu))t}e^{\underline{G}_t}dt\right]$$

We can write this

$$
\begin{aligned}
\kappa^{\theta\nu} &= \beta^{-1}\psi\frac{1}{\rho+\delta(1-\theta\nu)}\mathbb{E}\left[\int_{0}^{\infty}(\rho+\delta(1-\theta\nu))\,e^{-(\rho+\delta(1-\theta\nu))t}e^{\underline{G}_t}dt\right] \\
&= \beta^{-1}\psi\frac{1}{\rho+\delta(1-\theta\nu)}\mathbb{E}\left[e^{\underline{G}_\tau}\right]
\end{aligned}
$$

where τ is an exponentially distributed random variable with parameter $\rho+\delta(1-\theta\nu)$ independent from $\underline{G}_t$. □

Proof of Lemma 2.5.8: By Remark 2.5.7 it is sufficient to show 1. Let $Y_t=(\mu-\frac{1}{2}\sigma^2)t+\sigma B_t$ such that $\gamma_t=\exp(Y_t)$. Let

$$\underline{Y}_t \triangleq \min_{s\leq t} Y_s \tag{2.7.7}$$

Note that $\mathbb{P}\left[\underline{Y}_t\leq y\right]=\mathbb{P}\left[T_y\leq t\right]$ where T_y is the first passage time of y. With this notation:

$$\begin{aligned}
\mathbb{E}\left[e^{\underline{Y}_t}\right] &= \int_0^1 \mathbb{P}\left[e^{\underline{Y}_t}>x\right]dx \\
&= \int_{-\infty}^0 e^y\mathbb{P}\left[\underline{Y}_t>y\right]dy \\
&= \int_{-\infty}^0 e^y\left(1-\mathbb{P}\left[T_y\leq t\right]\right)dy \\
&= 1-\int_{-\infty}^0 e^y\mathbb{P}\left[T_y\leq t\right]dy
\end{aligned} \tag{2.7.8}$$

With this, one computes:

$$\begin{aligned}
\mathbb{E}\left[e^{\underline{Y}_\tau}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{\underline{Y}_\tau}\middle|\tau\right]\right] \\
&= \int_0^\infty \rho e^{-\rho t}\mathbb{E}\left[e^{\underline{Y}_t}\right]dt \\
&= \int_0^\infty \rho e^{-\rho t}\left(1-\int_{-\infty}^0 e^y\mathbb{P}\left[T_y\leq t\right]dy\right)dt \text{ by equation (2.7.8)} \\
&= 1-\int_{-\infty}^0 e^y\int_0^\infty \rho e^{-\rho t}\mathbb{P}\left[T_y\leq t\right]dtdy \\
&= 1-\int_{-\infty}^0 e^y\mathbb{E}\left[e^{-\rho T_y}\right]dy \quad \text{(by integration by parts)}
\end{aligned} \tag{2.7.9}$$

Equation (2.7.9) reduces the problem to the computation of $\mathbb{E}\left[e^{-\rho T_y}\right]$, the moment generating function of T_y, which can be done by applying a standard trick. Note that

$$M_t\triangleq e^{\theta Y_t-at},\quad a=a(\theta)=\theta(\mu-\frac{1}{2}\sigma^2)+\frac{1}{2}\theta^2\sigma^2 \tag{2.7.10}$$

is a martingale by the choice of a. Hence, by the optional stopping Lemma (e.g. see Karatzas and Shreve (1998)), we have

$$\begin{aligned}
\mathbb{E}\left[M_{T_y}\right] &= \mathbb{E}\left[\lim_{n\to\infty} M_{T_y\wedge n}\right] \\
&= \lim_{n\to\infty}\mathbb{E}\left[M_{T_y\wedge n}\right] \text{ (by dominated convergence for } y,\theta <\leq 0) \\
&= 1 \quad \text{(by optional stopping)}
\end{aligned}$$

Therefore

$$\begin{aligned}
& 1=\mathbb{E}\left[M_{T_y}\right]=\mathbb{E}\left[e^{\theta y-aT_y}\right] \\
\Leftrightarrow\ & \mathbb{E}\left[e^{-aT_y}\right]=e^{-\theta y}
\end{aligned} \tag{2.7.11}$$

Take θ as the negative root of $a=(\theta)=\rho$ (for the positive one, the integrals do not converge), i.e

$$\theta=-\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)-\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2\rho}{\sigma^2}} \tag{2.7.12}$$

Using (2.7.11) in (2.7.9), we get

$$\begin{aligned}\mathbb{E}\left[e^{\frac{Y}{\tau}}\right] &= 1-\int_{-\infty}^{0} e^{y}\mathbb{E}\left[e^{-\rho T_y}\right]dy \\ &= 1-\int_{-\infty}^{0} e^{(1-\theta)y}dy \\ &= 1-\frac{1}{1-\theta} \\ &= 1-\frac{1}{\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2\rho}{\sigma^2}}+\frac{\mu}{\sigma^2}+\frac{1}{2}} \qquad (2.7.13)\end{aligned}$$

which can be simplified to the formula given in the main body.

□

Proof of Theorem 2.5.9: 4.) Let $l = \arg\max_{l'} u(F(l', k^*), l')/k^*$. Hence

$$F_l(lk^*, k^*) = \frac{\beta}{u_c(F(lk^*, k^*))} = w^*$$

As F is homogeneous of degree one:

$$F_l(l, 1) = F_l(lk^*, k^*) = w^*$$

so that $l = \arg\max_l (F(l, 1) - wl) = \arg\max_l \pi(l)$.

5.) Notice that $\pi(l^*) = F(l^*, 1) - w^* l^* = F(l^*, 1) - F_l(l^*, 1)l^* = F_k(l^*, 1) = F_k(l^* k^*, k^*)$ as $w^* = \frac{\beta}{u_c(c^*)} = F_l(l^*, 1)$ and $F(l, k) = F_l l + F_k k$. In particular

$$\begin{aligned}s_t^* &= \mathbb{E}^{\mathbb{Q}^*}\left[\int_t^\infty e^{-\int_t^s d\rho_u^*} e^{-\delta(s-t)}\pi_s\left(l_s^*\right)ds\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^\infty e^{-(\rho+\delta)(s-t)}\frac{u_c(c_s^*)}{u_c(c_t^*)}\pi_s\left(l_s^*\right)ds\middle|\mathcal{F}_t\right] \\ &= \frac{1}{u_c(c_t^*)}\mathbb{E}\left[\int_t^\infty e^{-(\rho+\delta)(s-t)}u_c(F(l_s^* k_s^*, k_s^*))F_k(l_s^* k_s^*, k_s^*)ds\middle|\mathcal{F}_t\right] \\ &= \frac{1}{u_c(c_t^*)}\mathbb{E}\left[\int_t^\infty e^{-(\rho+\delta)(s-t)}V_k(k_t^*)ds\middle|\mathcal{F}_t\right] \\ &\leq \frac{\beta\gamma_t}{u_c(c_t^*)} = w_t^*\gamma_t\end{aligned}$$

with equality whenever $dI_t^* > 0$ by the FONC. □

6.) First note that the budget constraint (2.4.3) is fulfilled with equality. That is

$$c_t^* dt + s_t^* d\phi_t^{e*} + d\phi_t^{s*} = w_t^* dL_t^* + \phi_t^{e*}\pi_t^* dt + \phi_t^{s*} d\rho_t^* - \delta s_t^* \phi_t^{e*} dt$$

Further

$$dG_t^* = e^{-\rho_t^*} k_t^* \left(ds_t^* - \delta s_t^* dt + \pi_t^* dt - s_t^* d\rho_t^*\right) \qquad (2.7.14)$$

Hence, integrating the ds_t^* term by parts

$$\begin{aligned} G_t^* &= G_0^* + e^{-\rho_s^*} k_s^* s_s^* \,|_0^t - \int_0^t e^{-\rho_s^*} s_s^* dk_s^* + \int_0^t e^{-\rho_s} k_s^* \left(\pi_s^* ds - \delta s_s^* ds \right) \quad \text{(I. by parts)} \\ &= G_0^* - \int_0^t e^{-\rho_s^*} s_s^* dI_s^* + \int_0^t \left(\int_s^t e^{-\rho_u^*} e^{-\delta(u-s)} \pi_u^* du + e^{-\rho_t^*} e^{-\delta(t-s)} s_t^* \right) dI_s^* \quad \text{(by (2.5.6))} \\ &\quad + k_{0-}^* \int_0^t e^{-\rho_u^*} e^{-\delta u} \pi_u^* du \\ &= \mathbb{E}^{\mathbb{Q}^*} \left[G_\infty^* \mid \mathcal{F}_t \right] \end{aligned}$$

where the latter equality obtains as all terms converge by monotone convergence to random variables that lie in $\mathcal{L}^1(\mathbb{Q}^*)$ by the boundedness assumption on the planner's value function 2.5.13 and by the fact that $s_t^* = \mathbb{E}^{\mathbb{Q}^*}\left[\int_t^\infty e^{-(\rho_s^* - \rho_t^*) - \delta(s-t)} \pi_s^* ds\right]$ In particular, G_t^* is uniformly integrable so that $\phi^{e,*}$ and $\phi^{s,*}$ are admissible. Further,

$$\mathbb{E}\left[G_\infty^*\right] = \mathbb{E}\left[G_\infty^* \mid \mathcal{F}_0\right] = G_0^* = e^{-\rho_0} s_0^* k_{0-}^* \tag{2.7.15}$$

which we need later.

Now let $c_t, L_t, \phi_t^e, \psi_t^e$ be an alternative admissible household's choice for fixed $(w^*, s^*, \pi^*, \rho^*, \mathbb{Q}^*)$. It is to show

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(u(c_t) dt - \beta dL_t\right)\right] - \mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(u(c_t^*) dt - \beta dL_t^*\right)\right] \le 0$$

By concavity of u we have $u(c) - u(c^*) \le u_c(c^*)(c - c^*)$. In particular

$$\begin{aligned} &\mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t) dt\right] - \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(c_t^*) dt\right] \\ &\le \mathbb{E}\left[\int_0^\infty e^{-\rho t} u_c(c_t^*)(c_t - c_t^*) dt\right] \end{aligned} \tag{2.7.16}$$

By the budget constraint:

$$c_t dt \quad \le \quad -s_t^* d\phi_t^e - d\phi_t^s + w_t^* dL_t + \phi_t^e \pi_t^* dt + \phi_t^s d\rho_t^* - \delta s_t^* \phi_t^e dt \tag{2.7.17}$$

Using (2.7.17) and $w_t^* = \beta / u_c(c_t^*)$ yields cancellation of the dL term:

$$\begin{aligned} &\mathbb{E}\left[\int_0^\infty e^{-\rho t} \left(u_c(c_t^*) c_t dt - \beta dL_t\right)\right] \\ &\le u_c(c_0^*) \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^\infty e^{-\rho_t^*} \left(-s_t^* d\phi_t^e + \phi_t^e \pi_t^* dt - \phi_t^e \delta s_t^* dt - d\phi_t^s + \phi_t^s d\rho_t^*\right)\right] \end{aligned} \tag{2.7.18}$$

with equality for $\phi^{e,*}, \phi^{s,*}, L^*$.

By integration by parts, we get

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^\infty e^{-\rho_t^*} \left(-s_t^* d\phi_t^e + \phi_t^e \pi_t^* dt - \phi_t^e \delta s_t^* dt - d\phi_t^s + \phi_t^s d\rho_t^*\right)\right] \\ = \; &\mathbb{E}^{\mathbb{Q}^*}\left[-e^{-\rho_t^*} \phi_t^e s_t^* - \phi_t^s \Big|_0^\infty + \int_0^\infty e^{-\rho_t^*} \left(\phi_t^e \pi_t^* dt + \phi_t^e \left(ds_t^* - \delta s_t^* dt\right) - \phi_s^e s_t^* d\rho_t^*\right)\right] \\ = \; &W_{0-}^{(\phi^e, \phi^s)} - \mathbb{E}^{\mathbb{Q}^*}\left[\lim_{t \to \infty} e^{-\rho_t^*} W_t^{(\phi^e, \phi^s)}\right] + \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^\infty \phi_t^e e^{\delta t} dN_t,\right] \text{ by (2.4.2)} \end{aligned} \tag{2.7.19}$$

where

$$N_t \quad \triangleq \quad \int_0^t e^{-\rho_s^* - \delta s} \pi_s^* ds + e^{-\rho_t^* - \delta t} s_t^*$$

is a $\mathbb{Q}^*$-martingales as

$$s_t^* \quad = \quad \mathbb{E}^{\mathbb{Q}^*}\left[\int_t^\infty e^{-\int_t^s d\rho_u^* - \delta(s-t)}\pi_s^* ds \middle| \mathcal{F}_t\right]$$

Note that

$$\phi_t^e e^{\delta t} dN_t = dG_t^{(\phi^e,\phi^s)}$$

so that due to admissibility (2.4.1), we can apply Fatou's Lemma and get

$$\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^\infty \phi_t^e e^{\delta t} dN_t\right] \leq 0 \tag{2.7.20}$$

so that (2.7.19) is smaller than zero:

$$\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^\infty e^{-\rho_t^*}\left(-s_t^* d\phi_t^e + \phi_t^e \pi_t^* dt - \phi_t^e \delta s_t^* dt - d\phi_t^s + \phi_t^s d\rho_t^*\right)\right] \leq 0 \tag{2.7.21}$$

By (2.7.15) we get for $\phi^{e,*}, \phi^{s,*}, L^*$

$$\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^\infty \phi_t^{e,*} e^{\delta t} dN_t\right] = \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^\infty dG_t^{(\phi^{e,*},\phi^{s,*})}\right] = 0 \tag{2.7.22}$$

Further,

$$\mathbb{E}^{\mathbb{Q}^*}\left[\lim_{t\to\infty} e^{-\rho_t^*} W_t^{(\phi^{e,*},\phi^{s,*})}\right] = 0 \tag{2.7.23}$$

as $W_t^{(\phi^{e,*},\phi^{s,*})}$ is bounded by the continuation value of the planner's value function which is finite by (2.5.13). Inserting both, (2.7.22) and (2.7.23) into (2.7.19) implies

$$\mathbb{E}^{\mathbb{Q}^*}\left[\int_0^\infty e^{-\rho_t^*}\left(-s_t^* d\phi_t^{e,*} + \phi_t^{e,*}\pi_t^* dt - \phi_t^{e,*}\delta s_t^* dt - d\phi_t^{s,*} + \phi_t^{s,*} d\rho_t^*\right)\right] = 0 \tag{2.7.24}$$

In particular, combining (2.7.16),(2.7.18), and (2.7.21) and (2.7.24) yields

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(u(c_t)dt - \beta dL_t\right)\right] - \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(u(c_t^*)dt - \beta dL_t^*\right)\right] \\
&\leq u_c(c_0^*)^{-1}\left(W_{0-}^{(\phi^e,\phi^s)} - W_{0-}^{(\phi^{e*},\phi^{s*})}\right) \\
&\leq 0
\end{aligned}$$

as was to show.

Proof of Theorem 2.6.3. First, as $k_t^* = e^{-\delta t}\max\left[\kappa \max_{s\leq t} e^{\delta s}\gamma_s^{-\frac{1}{\theta\nu}}, k_{0-}\right]$ it holds for $k_t^* \geq k_{0-}$

$$\begin{aligned}
&\frac{1}{t}\log k_t^* \\
&= \frac{1}{t}\log\kappa - \delta + \frac{1}{t}\max_{s\leq t}\left(\delta s - \frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right)s - \frac{1}{\theta\nu}\sigma B_s\right) \\
&\leq \frac{1}{t}\log\kappa + \frac{1}{\theta\nu}\left(\frac{1}{2}\sigma^2 - \mu\right) - \min_{s\leq t}\frac{1}{t}\frac{1}{\theta\nu}\sigma B_s
\end{aligned}$$

In particular $\lim_{t\to\infty}\frac{1}{t}\log k_t^* \leq \frac{1}{\theta\nu}\frac{1}{2}\sigma^2 - \frac{1}{\theta\nu}\mu$ by the strong law of large numbers. On the other hand,

$$\begin{aligned}
&\frac{1}{t}\log\kappa - \delta + \frac{1}{t}\max_{s\leq t}\left(\delta s - \frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right)s - \frac{1}{\theta\nu}\sigma B_t\right) \\
&\geq \frac{1}{t}\log\kappa - \frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right) - \frac{1}{t}\frac{1}{\theta\nu}\sigma B_s
\end{aligned}$$

which implies $\lim_{t\to\infty} \frac{1}{t}\log k_t^* \geq \frac{1}{\theta\nu}\frac{1}{2}\sigma^2 - \frac{1}{\theta\nu}\mu$. This is 1). Second,

$$
\begin{aligned}
& \frac{1}{t}\log \mathbb{E}\left[k_t^*\right] \\
= \; & \frac{1}{t}\log\kappa - \delta + \frac{1}{t}\log\mathbb{E}\left[e^{\max_{s\leq t}\left(\delta s - \frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right)s - \frac{1}{\theta\nu}\sigma B_s\right)}\right] \\
\leq \; & \frac{1}{t}\log\kappa - \delta + \frac{1}{t}\log\mathbb{E}\left[\left\|e^{\max_{s\leq t}\left(\delta s - \frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right)s - \frac{1}{\theta\nu}\sigma B_s\right)}\right\|^p\right]^{\frac{1}{p}} \quad \text{for } p>1 \text{ by Jensen's inequality} \\
\leq \; & \frac{1}{t}\log\kappa - \delta + \frac{1}{t}\log\frac{p}{p-1}E\left[\left\|e^{\delta t - \frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right)t - \frac{1}{\theta\nu}\sigma B_t}\right\|^p\right]^{\frac{1}{p}} \quad \text{by Doob's inequality} \\
= \; & \frac{1}{t}\log\kappa - \delta + \frac{1}{t}\log\frac{p}{p-1} + \delta - \frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right) + \frac{p\sigma^2}{2\theta^2\nu^2}
\end{aligned}
$$

In particular, letting $t\to\infty$ and $p\to 1$ one gets $\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[k_t^*\right] \leq -\frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right) + \frac{\sigma^2}{2\theta^2\nu^2}$. To get the reverse inequality note that

$$
\begin{aligned}
& \frac{1}{t}\log\mathbb{E}\left[k_t^*\right] \\
\geq \; & \frac{1}{t}\log\kappa - \delta + \frac{1}{t}\log\mathbb{E}\left[e^{\delta t - \frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right)t - \frac{1}{\theta\nu}\sigma B_t}\right] \\
= \; & \frac{1}{t}\log\kappa - \delta + \frac{1}{t}\left(\delta t - \frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right)t + \frac{\sigma^2}{2\theta^2\nu^2}t\right) \\
= \; & \frac{1}{t}\log\kappa - \frac{1}{\theta\nu}\left(\mu - \frac{1}{2}\sigma^2\right) + \frac{\sigma^2}{2\theta^2\nu^2}
\end{aligned}
$$

□

Lemma 2.7.1. *Let τ_y be optimal for (2.6.2). Then almost surely for arbitrary $y' > y$ holds $\tau_{y'} \geq \tau_y$. In particular τ_y is invertible in the sense that with $I_t \triangleq \sup\{y \in \mathbb{Q} \mid \tau_y \leq t\}$, holds $\tau_y = \inf\{t \in \mathbb{Q} \mid I_t \geq y\}$.*

Proof. Suppose not. Then there exists a set $A \in \mathcal{F}$ with $\mathbb{P}[A] > 0$ such that there exists $y' > y$ with $\tau_{y'} < \tau_y$ on A. Without loss of generality we can assume $A \in \mathcal{F}_{\tau_{y'}} \cap \mathcal{F}_{\tau_y}$ for otherwise we could choose $A \triangleq \{\tau_{y'} < \tau_y\}$ which belongs to $\mathcal{F}_{\tau_{y'}} \cap \mathcal{F}_{\tau_y}$. Define

$$\hat{\tau}_y \triangleq \tau_{y'}1_A + \tau_y 1_{A^C}$$

which then is also a stopping time. As will be shown, $\hat{\tau}_y$ will be strictly better than τ_y for y, a contradiction. Namely

$$
\begin{aligned}
& \mathbb{E}\left[\int_{\hat{\tau}_y}^{\infty} V_k(y)dt - e^{-\rho\hat{\tau}_y}\beta\gamma_{\hat{\tau}_y}\right] - \mathbb{E}\left[\int_{\tau_y}^{\infty} V_k(y)dt - e^{-\rho\tau_y}\beta\gamma_{\tau_y}\right] \\
= \; & \mathbb{E}\left[\int_{\hat{\tau}_y}^{\tau_y} V_k(y)dt - e^{-\rho\hat{\tau}_y}\beta\gamma_{\hat{\tau}_y} + e^{-\rho\tau_y}\beta\gamma_{\tau_y}; A\right] \\
= \; & \mathbb{E}\left[\int_{\tau_{y'}}^{\tau_y} V_k(y)dt - e^{-\rho\tau_{y'}}\beta\gamma_{\tau_{y'}} + e^{-\rho\tau_y}\beta\gamma_{\tau_y}; A\right] \\
> \; & \mathbb{E}\left[\int_{\tau_{y'}}^{\tau_y} V_k(y')dt - e^{-\rho\tau_{y'}}\beta\gamma_{\tau_{y'}} + e^{-\rho\tau_y}\beta\gamma_{\tau_y}; A\right] \\
= \; & \mathbb{E}\left[\int_{\tau_{y'}}^{\infty} V_k(y')dt - e^{-\rho\tau_{y'}}\beta\gamma_{\tau_{y'}}\right] - \mathbb{E}\left[\int_{\tau_y}^{\infty} V_k(y')dt - e^{-\rho\tau_y}\beta\gamma_{\tau_y}\right] \\
\geq \; & 0
\end{aligned}
$$

Here the strict inequality is due to the strict concavity of V and $\tau_y > \tau_y'$ on A. The last inequality holds as $\tau_{y'}$ is optimal for y'. Hence τ_y is not optimal.

□

Proof of Theorem 2.6.7. Guess that $L(k,\gamma) \triangleq l^{-1}u_c(F(k)) = \mathbb{E}^{\mathbb{P}}\left[\int_0^\infty e^{-\rho t}u_c(F(k_t^*))dt\right]$ is of the form

$$L(k,\gamma) = a_1 k^{b_1} + a_2 k^{b_2}\gamma^c \tag{2.7.25}$$

for some real numbers $a_1, a_2, b_1, b_2, c \in R$. $L(k,\gamma)$ solves

$$u_c(F(k_t^*)) - \delta k_t^* L_k + \mu\gamma_t L_\gamma + \frac{1}{2}\sigma^2\gamma_t^2 L_{\gamma\gamma} - \rho L + L_k dI_t^* = 0 \tag{2.7.26}$$

As

$$u_c(F(k)) = A^{-\theta\nu}\phi^{-(1-\alpha)\theta}k^{-\alpha\nu\theta}$$

this implies

$$\begin{aligned} b_1 &= -\alpha\nu\theta \\ a_1 &= \frac{A^{-\theta\nu}\phi^{-(1-\alpha)\theta}}{\rho - \alpha\delta\theta\nu} \end{aligned}$$

and c is the smaller root of

$$b_2\delta - c\mu + \rho + \frac{1}{2}\sigma^2(c - c^2) = 0 \tag{2.7.27}$$

When γ reaches the investment trigger $\left(\frac{\kappa}{k}\right)^\upsilon$, where $\upsilon = \theta\nu$, the capital stock increases. Hence

$$0 = L_k(k, \left(\frac{\kappa}{k}\right)^\upsilon) = a_1 b_1 k^{b_1 - 1} + a_2 b_2 k^{b_2 - 1}\left(\frac{\kappa}{k}\right)^{c\upsilon}$$

which implies

$$\begin{aligned} b_1 - 1 &= b_2 - 1 - \upsilon c \\ a_1 b_1 + a_2 b_2 \kappa^{c\upsilon} &= 0 \end{aligned}$$

Or

$$b_2 = b_1 + \upsilon c \tag{2.7.28}$$

$$a_2 = -\frac{a_1 b_1}{b_2 \kappa^{c\upsilon}} \tag{2.7.29}$$

Solving (2.7.27) and (2.7.28) for b_2 and c yields the given expression for c. Altogether, L solves (2.7.26) and, as $c < 0$, $\lim_{\gamma\to\infty} L(k,\gamma) < \infty$ A verification theorem shows that L chosen this way is indeed equal to $l^{-1}(k,\gamma)u_c(F(k))$ (see e.g. Kallenberg (2001, Theorem 19.6)). Hence, inserting all coefficients:

$$\begin{aligned} l^{-1}(k,\gamma) &= u_c(F(k))^{-1}\left(a_1 k^{b_1} + a_2 k^{b_2}\gamma^c\right) \\ &= \frac{1}{\rho - \alpha\theta\nu\delta}\left(1 - \frac{\alpha}{\alpha - c}\left(\frac{k}{\kappa\gamma^{-\frac{1}{\theta\nu}}}\right)^{c\theta\nu}\right) \end{aligned}$$

□

Proof of Theorem 2.6.8. We have

$$\frac{s_t^*}{w_t^*} = \pi\beta^{-\frac{1}{\alpha}}\mathbb{E}^{\mathbb{P}}\left[\int_t^\infty e^{-(\rho+\delta)(s-t)}u_c(c_s^*)^{\frac{1}{\alpha}}ds\middle|\mathcal{F}_t\right]$$

by (2.6.6) so it amounts to compute the expected value. However, in the proof of Theorem 2.6.7 a similar expectation was computed. Indeed, if one substitutes $\rho + \delta$ for ρ and θ/α for θ, both expectations are identical. Note that we do not change the investment trigger $\kappa\gamma^{\frac{-1}{\upsilon}}$ as the investment process still depends on the original θ.

□

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3. WHY FIAT MONEY IS A SAFE ASSET

Abstract

This paper presents a model in which (1) fiat money has strictly positive value in the unique trembling hand equilibrium. This holds as each bank note is both: (a) a witness for the existence of some agent in the economy with debt, backed by collateral, and (b) the only matter that allows the debtor to settle her debt. The fear to lose the collateral creates future money demand by the debtor and thereby ensures positive money value. (2) Money is a safe asset as not only a single but all debtors in the economy demand money so that idiosyncratic shocks to solvency wash out. By this mechanism, fiat money is essentially equivalent to large securitized pools of debt which (3) can implement the pooling allocation even if pooling itself is infeasible.

JEL classification: E40, E50

Keywords: Fiat Money, Securitization, Safe Asset, Collateral

3.1 Introduction

Money lies at the core of economics. Nowadays, in the western world all currencies are fiat money regimes. A Dollar bill is a piece of paper that does neither provide intrinsic utility nor is it a claim that can be converted at the central bank to gold or other real goods. But why then does anybody accept a Dollar bill as a medium of exchange? More precisely, I ask: (1) Is the value of fiat money bounded away from zero and if so (2) why? (3) Is the value stable in the sense that it survives backward induction and rational expectation arguments? (4) Is money a safe asset in the sense that the money value is attached with little risk and (5) if so why? Finally, (6) can the existence of money implement allocations which could not be reached without it?

I stress that I do *not* deal with the question of why people invest in a non-interest-bearing asset, and I only touch the question of why people trade in an asset which would be redundant in a perfectly frictionless world with question (6). A tiny amount of friction or transaction cost will solve these questions. Even more, compared to the question of why people can reasonably rely on the value of a piece of paper called money, *provided they want to do so (for whatever reason)*, these question are of lower order importance. Note that (a) also short term credit bears only little interest if at all and (b) one could easily think of interest bearing checking accounts and one would not assume anything fundamental to change. Indeed, some checking accounts pay interests. If, however, people would not accept money as a medium of exchange, the world would presumably look radically different. I therefore concentrate on questions (1) to (6) posed above.

However elementary these questions are, surprisingly no adequate answer seems to exist. Several models are set up to explain why fiat money has value, most of which can be grouped into two categories. First, there are those that rely on an infinite horizon argument in the sense that money has value today because it is expected to have value tomorrow which is justified because it is expected to have value the day after tomorrow and so on. To this category belong

1. Samuelson's OLG model (Samuelson, 1958)
2. Every explanation relying on trust or social agreement
3. Search theoretic models (e.g. the Kiyotaki-Wright models (Kiyotaki and Wright, 1989, 1993))

The second category of models consists of those which rely on some exogenous factor, e.g. models in which

1. money value (prices) is assumed to be exogenous (e.g. Grandmont and Younes (1972)).
2. an exogenous agency such as the government forces agents to demand money, e.g. in order to pay taxes (c.t. Lerner (1947))

According to the author's opinion, with exception of the Lerner model, neither kind of models is convincing when it comes to justifying a positive money value. Infinite horizon models essentially build upon bubble-like arguments as in these models the transaction value of money differs from its fundamental value - zero - and this difference is only sustained if there always exists another person or another time point at which money can be expected to have value. As in a Ponzi Scheme, the potato is only passed on. Accordingly, none of these arguments survives in a setting with only finitely many time points (agents) and in all these models there also exists a non-monetary equilibrium in which money has zero value, the fundamental or non-bubble solution. For instance in the Kiyotaki-Wright model (Kiyotaki and Wright, 1989), no monetary equilibrium would exist were time to be finite. However, I believe that a good explanation for positive money value shall not hinge upon the infiniteness assumption. Exogenous factors, on the other hand, tend to be artificial and their exogeneity has to be justified.

To the best of my knowledge there is only one series of models by Dubey, Geanakoplos (e.g. Dubey and Geanakoplos (1992)) and, at an earlier stage, Shubik and Wilson (Shubik and Wilson, 1977) that does not match to these two categories. In Dubey and Geanakoplos (1992) multiple commodities and endowments lead to an incentive to trade. Before trade starts, a (central) bank lends out units of paper which are intrinsically useless. The paper has to be returned to the bank after the trading period. If not, there is a default for which the agent will be punished in form of disutility. The default punishment creates money demand at the time after trade even in the absence of follow-up periods. However, for money to enter the equilibrium, there must be some mechanism why agents demand money in the first place. In Dubey and Geanakoplos (1992) this is done by additionally imposing a cash-in-advance (CIA) constraint. Also in this model, zero money value is a possible equilibrium outcome.

The model of the article at hand is similar except for two essential differences. First, agents demand money in first place not for transactional reasons, but because of its store of value function. Technically, this is realized by considering two periods and agents with different time preferences. This has three advantages. First, it allows to get rid of the ad hoc CIA constraint[1]. Second, the introduction of a second period separates money creation (borrowing) from money annihilation (redemption) which allows to show that money necessarily has value once it has entered the world (e.g. in $t = 0.5$). Third, it stresses the asset role of money and allows to draw an analogy to a securitized pool of debt. Namely, as money is demanded not only by a single

[1] Note that a two-period setting in which money serves as only asset is similar to a CIA constraint but at the same time different in essential aspects. In both settings, trade, be it in different goods or in the same good at different time points, is restricted to be conducted by the use of money. Within a two-period setting, however, there is the potential risk that money loses value in the second period and the holder of the money bill goes away empty-handed. Further, in a two-period setting, the infeasibility of direct intertemporal trades (i.e. of credit) can be justified by informational frictions. For instance, it might be impossible to specify the quality of a good on paper and therefore to write a credit contract while it will be possible to observe the quality as soon as one touches and perhaps even tests the good. Arguments like these cannot be used to justify a CIA constraint.

debtor but by all debtors in the economy, idiosyncratic shocks to debtors solvency wash out, leading to a securitization effect making money a safe asset. This result hinges upon the asset role of money.

Second, instead of modeling the default punishment in form of an ad-hoc disutility, the agent is forbidden to consume in case she defaults. So agents with debt demand money in order to repay, because if they do not, they lose their endowments and cannot consume. By doing so, future endowments can be interpreted as collateral backing debt and agents redeem their debt (if they have sufficient endowments) because they do not want to lose the collateral. This interpretation, namely the fear to lose collateral as incentive to repay debt and therefore for money demand, is easier established than if the repayment stems from an incentive to avoid some abstract disutility. It is motivated by the fact that in the real world every income above a minimum level is seized if debt is not redeemed and technologies that generate endowments (capital) often can be used as collateral.

Note that the model resembles reality in the sense that most central banks, be it the European Central Bank (ECB), the Bank of England (BoE) or the Bank of Japan (BoJ)[2] do not simply print money, they lend it out. Money only enters the world in form of a credit that has to be repaid with interests on top of it. If repayment does not occur, there is a default and the debtors collateral is seized. A fact which is not captured by any model in which money exists without a corresponding liability ('helicopter money'). The model at hand shows that these ingredients are sufficient to ensure positive money value. The debtor's collateral - or rather the debtor's disincentive to lose the collateral - indirectly 'backs' fiat money even though a dollar bill itself does not constitute a direct claim on the collateral.

Of course, the argument is not orthogonal to the models belonging to the categories described above. First, the value of money relies on the assumption that there is an institution that enforces the default punishment. This is similar to the viewpoint that it is the obligation to pay taxes which establishes money demand. Second, the securitization effect only holds if a large number of agents take on debts in form of paper money. Hence, also in this model, at least for the second result, a form of social coordination is necessary.

[2] Though the Federal Reserve System (FED) does not primarily lend money to financial intermediaries (it buys treasuries), it fits into the framework. Buying treasuries is roughly the same as directly lending to the state, so the initial debtor is the government and the only way it can redeem her debt is by collecting taxes. Hence, there is no difference whether one regards money demand as stemming out of the governments obligation and capacity to repay debts or as stemming out of her capability to collect taxes. In this sense, the Lerner explanation might be seen as the US version of the model at hand.

3.2 The Model

There is a unit mass of consumers $i \in [0,1]$, living for two time periods $t \in \{0,1\}$. Half of the agents are early consumers and half of them are late consumers in the sense of a higher or lower discount rate. To be more precise, let $\delta \in (0,1)$ and

$$\delta_i \triangleq \begin{cases} \delta & \text{if } i \leq \frac{1}{2} \\ 1 & \text{else} \end{cases} \tag{3.2.1}$$

so that all agents $i \leq 1/2$ are early consumers. Endowments in $t = 0$ are $e_{i,0} = 1$ units of a single perishable consumption good for each agent i.
At $t = 1$, each consumer receives an exogenous endowment of $\alpha e_{i,1}$. Here, $\alpha : \Omega \to (0, \bar{\omega}]$ is an aggregate shock and $e_{i,1} : \Omega \to (0, \bar{\omega}]$ is an idiosyncratic shock[3]. Assume $\mathbb{E}[\alpha e_{i,1}] \leq 1$ to avoid dealing with a corner solution.

I assume that all shocks are independent and all idiosyncratic shocks are identically distributed. Denote the distribution function of $e_{i,1}$ by F. Assume that F is absolutely continuous with density f.

Ex ante (period 0) utility is the simple discounted present value of consumption, i.e.

$$U_i = c_{i,0} + \delta_i \mathbb{E}[c_{i,1}] \tag{3.2.2}$$

In addition, in $t = 0$, there is a (central) bank lending out $M \in \mathbb{R}^+$ units of paper. Note that paper does not appear in the consumers utility function; it is intrinsically useless. I interpret M as (fiat) money. If a consumer borrows d units of money at rate R, her debt in $t = 1$ is Rd. She is free to choose the amount of money π she repays to the bank. However, if the consumer does not repay her debt, she will not be able to consume in $t = 1$. This could be because there is a legal system in the background enforcing repayment or because endowments serve as collateral for debt. This acts like but is different to a punishment for defaulting in form of disutility as assumed in Dubey and Geanakoplos (1992). Let $\bar{d} \in \mathbb{R}^+$ be an exogenously (i.e. by the (central) bank) given upper borrowing constraint.

Consumers can make money deposits, also denoted by d, paying the return r. The notation is so that $d > 0$ denotes deposits and $d < 0$ debt. In general, for money to have positive value, $r < R$. In that case, more money has to be returned than there exists in the economy. Hence not everybody can repay its debt and some agents will default. This creates competition among the debtors for being not one of the defaulters which, in turn, results in a positive money value. Note that for $r = 1$ money does not bear any interest.

As the central bank's choice variables r, R, $\bar{d}$ (M will be determined endogenously) are predetermined in period 0, I treat the central bank as exogenous and do not

[3] Note that by assumption $\mathbb{P}[\alpha_t e_{i,t} \leq 0] = 0$. This is made for a technical reason to avoid having to give a meaning to a trade of 'nothing' against 'nothing'.

specify her preferences. She might, for instance, want to establish a certain price level, inflation or nominal interest rate. In equilibrium she will make zero profits (no more money than she lent out will be returned), so it is also possible to regard her as merged with a perfectly competitive banking sector. See the fourth section for a detailed description.

Accordingly, the consumers' choice variables are elements of the space

$$\mathcal{S} \triangleq \{(d(u,r,R,M,p_0), \pi(d,e_1,\alpha,p_1,r,R,M)) \in \mathbb{R}^2 \mid \text{ s.t. } d \geq -\bar{d}, \pi \geq 0\} \quad (3.2.3)$$

where I dropped the index i for notational simplicity. Here, $d = d^+ - d^-$ are deposits (=purchased bonds if $d > 0$ and promises to repay $-Rd$ if $d < 0$) and $\pi \in \mathbb{R}^+$ are repayments in $t = 1$. The choices may depend on (u_i, r, R, M, p_0) and $(d_i, e_{i,1}, \alpha, p_1, r, R, M)$ resp. where u_i are *i.i.d.* random variables allowing mixed strategies and p_t is the price level in t. This also pins down the information structure.

Consumption in period t is then determined by the budget constraints and the default constraint. That is
Budget constraint at $t = 0$:

$$c_0 + d/p_0 = e_0, \quad c_0 \geq 0 \quad (3.2.4)$$

Budget constraint at $t = 1$:

$$c_1 + \pi/p_1 = \alpha e_1 + d^+ r/p_1 \quad \text{if } \pi \geq Rd^- \;, \quad c_1 \geq 0 \quad (3.2.5)$$

Default constraint:

$$c_1 = 0 \quad \text{if } \pi < Rd^- \quad (3.2.6)$$

Consumers maximize utility (3.2.2) subject to the constraints (3.2.4),(3.2.5) and (3.2.6):

$$\begin{aligned} &\max_{(d_i,\pi_i)\in\mathcal{S}} \mathbb{E}\left[c_{i,0} + \delta_i c_{i,1}\right] \quad \text{s.t.} \\ &c_{i,0} + d_i/p_0 = e_{i,0} \\ &c_{i,1} + \pi_i/p_1 = \alpha e_{i,1} + d_i^+ r/p_1 \quad \text{if } \pi_i \geq Rd_i^- \\ &c_{i,1} = 0 \quad \text{else} \end{aligned} \quad (3.2.7)$$

Note that given the preferences and allocations, the first best solution (if $\mathbb{E}\left[\alpha e_{1,i}\right] = e_{i,0} = 1$) would be to let the early consumers consume all endowments in $t = 0$ and the late consumers all endowments in $t = 1$. A credit market (potentially combined with redistribution to achieve equal weights) could implement this allocation but might be - and in this model is - infeasible, e.g. because the consumption good is ex-ante not contractible. This could be because the quality of the good cannot be specified on paper or simply because it is not known which goods are available in the future. The opportunity to borrow money, i.e. of a different and intrinsically useless commodity, will serve in parts as a substitute for a credit market in the consumption good.
An equilibrium for this economy is

Definition 3.2.1. *An equilibrium is a tuple* $\mathcal{E} = (d_i, \pi_i, r, R, p_t, \bar{d}, M), t \in \{0,1\}, i \in [0,1]$*, where* $r, R, \bar{d}, M \in \mathbb{R}$*,* $(d_i, \pi_i) \in \mathcal{S}$ *and* p_t *are random variables revealed in* t *such that*

1. $(d_i, \pi_i) \in \mathcal{S}$ *maximizes consumers utility*
2. $\int d_i^- di = M$ *(money market clearing:* M *is lent out)*
3. $\int d_i^+ di = M$ *(lent out money serves as deposit)*
4. $\int \pi_i di \leq rM \quad$ *a.s. (money market clearing in* $t = 1$ *)*
5. $\int c_{i,t} di \leq \int \alpha_t e_{i,t} di \quad$ *a.s. for* $t \in \{0,1\}$ *(goods market clearing)*

I explicitly allow $p = \infty$ *for a non-monetary equilibrium, but for* $r < R$ *this will not be an equilibrium outcome.*

In order to proceed by backward induction, I define an $t = 1$-equilibrium:

Definition 3.2.2. *Given the information at* $t = 1$*, i.e.* $r, R, \bar{d}$ *and the distribution of debt levels and endowments* $d_i, e_{i,t}$*,* $(i \in [0,1])$ *a* $t = 1$*-equilibrium is a tuple* $(c_{i,1}, \pi_i, p_1)$ *such that*

1. *consumers maximize utility subject to the budget and default constraint, i.e.*

$$c_{i,1} = \max\{c_{i,1} \mid \ (c_{i,1}, \pi_i) \text{ fulfill (3.2.5) and (3.2.6) } \} \quad \forall i \in [0,1]$$

2. $\int \pi_i di \leq rM$ *(money market clearing in* $t = 1$ *)*
3. $\int c_{i,1} di \leq \int \alpha e_{i,1} di$ *(goods market clearing)*

So far monetary policy is not specified. Under monetary policy I understand any restriction imposed on the parameters $r, R, M, \bar{d}, p_0, p_1$. As will be seen, an equilibrium imposes three restrictions (money market equilibrium in $t = 0, 1$ and saving (deposit) market equilibrium), so three further constraints can be imposed by monetary policy. Here, I assume that monetary policy determines r, R, and $\bar{d}$. Money supply M and prices will then become an equilibrium outcome.

3.3 Solution

I first show that the second period price level is finite (money has positive value) and only depends on the realization of the aggregate shock α (money is safe) in any equilibrium. I then show existence of an equilibrium and that there is a one to one map between monetary equilibria and equilibria with debt pools.

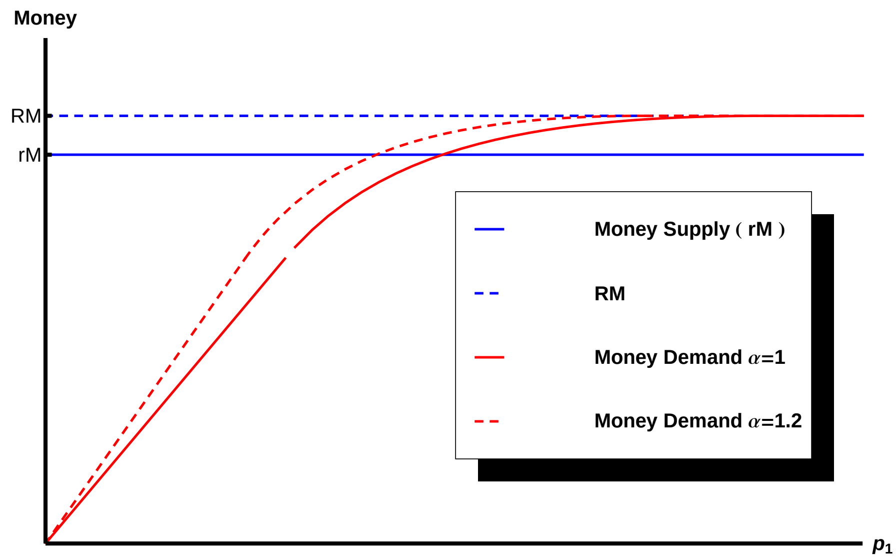

Fig. 3.1: Money Demand and Supply

3.3.1 The monetary equilibrium

I proceed by backward induction. Assume the economy is in period $t = 1$ and there already is some distribution of debt/deposits d_i, $i \in [0,1]$ and uncertainty about second period endowments $\alpha e_{i,1}$ is revealed. Clearly, optimal repayment is[4]

$$\pi_i = \min\{\alpha e_{i,1} p_1, R d_i^-\} \tag{3.3.1}$$

As a consequence, the price level p_1 is finite and money always has positive value if there is at least one defaulter, which is the case if and only if $r < R$.

Theorem 3.3.1. *Given $r, R, d_i, e_{i,1}$, ($i \in [0,1]$), suppose that $r < R$ and $M \triangleq \int d_i^- di = \int d_i^+ di > 0$. Then $p_1 < \infty$ in any $t = 1$-equilibrium.*

Proof. Suppose on the contrary $p_1 = \infty$. Then, by (3.3.1), $\pi_i = R d_i^-$ for all i with $e_{i,1} > 0$ and, as $\mathbb{P}[e_{i,1} \leq 0] = 0$, for almost all i. In particular

$$\int \pi_i di = R \int d_i^- di = RM > rM$$

A contradiction to the definition of an equilibrium. □

Money has positive value as, due to $r < R$, some agents necessarily default and lose their collateral. This creates competition for not being among the defaulters, which, in turn, ensures a positive money value in $t = 1$. The argument is illustrated in Figure 3.1. Money demand is increasing in the price level p_1. This is as for low prices

[4] Here, I assume that repayment is as much as possible in case of default. Zero payments were an alternative possibility that lead to the same level of utility for the defaulting agents, but this would imply that some resources are wasted. Hence for an equilibrium to exist, incentives for intertemporal trade would have to outweigh the deadweight loss of defaults.

debtors default and only demand the money equivalent of their endowments which is zero in the extreme case of $p_1 = 0$. On the other hand, for large prices, debtors demand RM, the whole amount owed to the bank as in this case they can repay all debts outstanding and avoid the seizure of their consumption goods. If $r = R$ nobody is forced to default, and a continuum of equilibria emerges. Prices were indetermined and $p_1 = \infty$ would be a possible equilibrium outcome. However, money is scarce in the sense that only $rM < RM$ units are available. This drives down the price for consumption goods (i.e. drives up the price of money) to a level at which money demand is reduced by a sufficient amount of defaulters to match money supply. If aggregate endowments are high (e.g. the dashed $alpha = 1.2$ line), debtors have more resources to compete with and the price level is low as can be seen in the figure.

Given finiteness of p_1, it is easy to derive a $t = 1$-equilibrium.

Theorem 3.3.2. *Given $r, R, d_i, e_{i,1}, (i \in [0,1])$ suppose that $r < R$ and $M \triangleq \int d_i^- di = \int d_i^+ di > 0$. Then there exists a unique $t = 1$-equilibrium. In particular, p_1 is uniquely determined by*

$$rM = \int \pi_i di \tag{3.3.2}$$

Proof. By the definition of an equilibrium

$$\int \pi_i di \leq rM$$

Suppose

$$\int \pi_i di < rM \tag{3.3.3}$$

This is excluded by Walras law. Namely, as consumers maximize utility, it holds $c_{i,1} + \pi_i/p_1 = \alpha e_{i,1} + r d_i^+/p_1$. Integration over agents $i \in [0,1]$ yields

$$\begin{aligned} \int c_{i,1} di + \frac{1}{p_1} \int \pi_i di &= \int \alpha e_{i,1} di + \frac{r}{p_1} \int d_i^+ di \\ &= \int \alpha e_{i,1} di + \frac{r}{p_1} M \end{aligned}$$

Hence, (3.3.3) would imply

$$\int c_{i,1} di > \int \alpha e_{i,1} di$$

which is a contradiction to goods-market-clearing. So the condition has to hold with equality. As $\pi_i = \min\{\alpha e_{i,1} p_1, R d_i^-\}$ is monotonic in p_1 (and not for all i, $\pi_i = R d_i^-$ as this would conflict with money market clearing (see Theorem 3.3.1) and F is absolutely continuous), there exists a unique p_1 that solves (3.3.2). Then π_i as in (3.3.1) and $c_{i,1} = \alpha e_{i,1} - \pi_i/p_1$ together with p_1 form a $t = 1$ equilibrium. □

Note that the price level p_1 only depends on the realization of the aggregate shock α due to the law of large numbers (LLN). To see this note that, applying LLN

$$\int \pi_i di = \mathbb{E}\left[\int \pi_i di \mid \alpha\right] = \int_{[0,1]} \int_{(0,\bar{\omega}]} \min\{\alpha e p_1, R d_i^-\} dF(e) di \tag{3.3.4}$$

This is exactly the securitization effect which makes money a safe asset.

Corollary 3.3.3. *In particular, due to the law of large numbers, p_1 only depends on the initial debt levels d_i and the aggregate shock α but not on the realization of the idiosyncratic endowment shocks $e_{i,1}$. Further, p_1 is anti-proportional to α, i.e. $p_1(\alpha) = \bar{p}_1/\alpha$ with $\bar{p}_1 = p_1(\alpha = 1)$*

Having established finiteness of the price level p_1 and a $t = 1$ equilibrium, it is easy to go one (time-) step back and derive a full equilibrium. In $t = 0$, each deposit, each Dollar note, is a witness for some debt and therefore for some future money demand which, in turn, guarantees purchasing power. Due to idiosyncratic risk, individual collateral might turn out to be worthless, but - as will be shown below - money is demanded by the whole pool of debtors leading to a securitization effect. This is what makes money a safe asset. For a full equilibrium p_0 and d_i $i \in [0, 1]$ have to be determined.

Theorem 3.3.4. *There are two types of equilibria in $t = 0$. Namely, given monetary policy $(r, R, \bar{d})$ with $r < R$*
1) there are equilibria in which $M = 0$ and therefore $d_i = 0$ for all i. One of in which $p_0 = p_1 = \infty$ (a.s.).
2) there is an (up to the distribution of d_i^+) unique equilibrium for which $M > 0$. In this equilibrium $M = \frac{1}{2}\bar{d}$. Prices p_1 are uniquely determined by Theorem 3.3.2 and by $p_0 = 1/\mathbb{E}[r/p_1]$. Further, $d_i = -\bar{d}$ for $i \leq 1/2$ and $d_i \geq 0$ arbitrary for $i \geq 1/2$ as long as $\int_{(1/2,1]} d_i di = \frac{1}{2}\bar{d}$ and $d_i \leq p_0$, e.g. $d_i = \bar{d}$ for $i > 1/2$.

The proof is in the appendix.

The result with finite second period prices p_1 only holds true if $r < R$. Intuitively $\frac{r}{R}$ is the ratio of debt served and total debt outstanding. If $\frac{r}{R} \uparrow 1$ less and less agents default. Competition among debtors reduces and prices rise. Nonetheless, prices are bounded as long as second period endowments are bounded from below. This can be inferred from Figure 3.1. If money supply rM (the blue line) increases towards RM, prices rise but stay bounded as maximum money demand RM is already achieved at a finite price level. The following Corollary makes the argument formal.

Corollary 3.3.5. *Moreover, if $\underline{\omega} = \operatorname{essinf} \alpha e_{i,1} > 0$, then p_1 is (independent of r) bounded away from infinity as long as $r < R$ in the monetary equilibrium.*

$$p_1 \leq \frac{R\bar{d}}{\underline{\omega}} \tag{3.3.5}$$

The proof is in the appendix.

As is stated in Theorem 3.3.4 there are two types of equilibria, one of in which $M = 0$ and agents do not make use of money. However, once money has entered the world, the unique equilibrium is the monetary one in which prices are finite (see Theorem 3.3.2). I now argue that the non-monetary equilibrium is implausible. Two ways to do so come into my mind.

First, one could abandon the assumption that agents are infinitesimally small. The problem with the setup so far is that if one agent deviates from the $M = 0$ equilibrium by taking debt $\bar{d}$, the amount of money entering the world is infinitesimally small and does not change equilibrium entities. In particular, prices could stay at infinity. One way to avoid this would be to have a large but finite population of agents. The drawback would be more tedious algebra. Another way would be to alter the equilibrium definition in a way that it deals with infinitesimal small entities (e.g. by switching to a counting measure when only finitely many agents hold debt/deposits). This would require a more technical definition that had to be justified.

I therefore follow a different argument by showing that the $M = 0$ equilibrium is instable in a precisely defined sense. For this, I show that the monetary equilibrium is the unique trembling hand equilibrium. Let

$$\begin{aligned} \mathcal{S}^{\varepsilon} \triangleq\; & \{(d(u, r, R, M, p_0), \pi(d, e_1, \alpha, p_1, r, R, M)) \in \mathbb{R}^2 \mid \\ & \text{s.t. } d \geq -\bar{d}, \pi \geq 0 \text{ and } \mathbb{P}\left[d \leq -\varepsilon\right] \geq \varepsilon\} \end{aligned} \tag{3.3.6}$$

be the space of mixed strategies in which the probability that an agent borrows more than ε units of money is at least ε.

Definition 3.3.6. *An equilibrium $\mathcal{E}$ is a trembling hand (or stable) equilibrium, if there exists a series of equilibria $\mathcal{E}^{\varepsilon}$ where the strategies for the players $i \leq \frac{1}{2}$ are restricted to lie in $\mathcal{S}^{\varepsilon}$ so that $\mathcal{E}^{\varepsilon} \to \mathcal{E}$ for $\varepsilon \to 0$ (point wise)*

Here, I do not allow trembling in repayments π for simplicity and I do not force trembling into $d \geq 0$ as well as late consumers $i \geq 1/2$ to tremble at all in order to illustrate that the money value does not depend on forcing agents to accept money - not even with a vanishing probability. Hence there is a slight deviation from the original definition of a trembling hand perfect equilibrium (see e.g. (Mas-Colell et al., 1995)) which makes the definition at hand less restrictive.

Theorem 3.3.7. *Given r, R and $\bar{d}$, the second case in Theorem 3.3.4 describes the unique[5] trembling hand equilibrium. Even more, every equilibrium $\mathcal{E}^{\varepsilon}$ out of the converging sequence already coincides with the final equilibrium (up to the distribution d_i^+ of deposits among late consumers)*

[5] Uniqueness refers to p_0, p_1, M, d_i^- but not to the distribution of d_i^+ among late consumers

Proof. Clearly, the second case in Theorem 3.3.4 fulfills Definition 3.3.6. To show uniqueness, let $\mathcal{E}^\varepsilon = (d_i, \pi_i, r, R, p_t, \bar{d}, M)$ be an equilibrium within the converging sequence of a trembling hand equilibrium. As $M = \int d_-^- di \geq \varepsilon^2/2 > 0$ by the law of large numbers, it holds $p_1 < \infty$ by Theorem 3.3.1. But then necessarily $p_0 = 1/\mathbb{E}[r/p_1] = \bar{p}_1/(r\mathbb{E}[\alpha]) < \infty$, as otherwise late consumers would either borrow money up to their limit and buy goods, driving up p_0 or sell endowments against money, driving down p_0. But for the early consumers more debt is always better than less debt, so they borrow up to their limit $\bar{d}$ (see Lemma 3.6 in the Appendix for details). This also determines π via (3.3.1). □

The preceding Theorem illustrates that when money is created (and be it only by an infinitesimally small amount), then money will necessarily have positive value.

3.3.2 Equilibrium Allocations

Equilibrium allocations are influenced by monetary policy. More precisely, they are parameterized by the ratio $\frac{r}{R}$, the ratio of debt paid back to total debt outstanding.

Corollary 3.3.8. *Let $r < R$. The allocation of consumption goods across time and states in the monetary equilibrium only depends on the ratio $\frac{r}{R}$. In particular, $\bar{d}$ only affects price levels but not the real allocation.*

The proof is contained in the appendix.

3.3.3 Pools of Debt

I now compute equilibria with pools of debt and show that there is a one to one map to monetary equilibria. The point is that the equilibrium conditions turn out to be the same.

A pool is a real asset with the parameters $\bar{d}^p, R^p$ where $\bar{d}^p$ is the maximum debt and R^p is the interest rate to be paid on this debt. In $t = 1$, the pool collects all outstanding debts, resp. all endowments,

$$\pi_i^p = \min\{\alpha e_{i,1}, R^p d_i^{p-}\} \tag{3.3.7}$$

and distributes them to the depositors who then earn an interest of

$$r^p \triangleq \frac{\int \pi_i^p di}{\int d_i^{+p} di} \tag{3.3.8}$$

per unit of deposit. Note that in this case, unlike in the monetary equilibrium from the preceding section, the return on deposits depends on the aggregate shock while - in the absence of default - the debt service does not. To establish an exact equivalence also in the case with aggregate shock, I therefore slightly change the definition of a pool and allow the interest rate R^p to depend on the aggregate shock α.

The consumers problem is

$$\max_{d_i^p \geq -\bar{d}^p} e_{i,0} - d_i^p + \delta_i \mathbb{E}\left[\max\{0, e_{i,1} - R^p d_i^{p-} + r^p d_i^{p+}\}\right] \quad (3.3.9)$$

A pool-equilibrium is:

Definition 3.3.9. *A pool-equilibrium is a tuple $(r^p, R^p, \bar{d}^p, d_i^p)$ where d_i^p is a distribution of debt/deposits such that (3.3.8) holds, d_i^p solves (3.3.9) and deposits net out to zero.*

$$\int d_i^p di = 0$$

Theorem 3.3.10 (Equivalence of monetary and pool-equilibria)**.** *For any monetary equilibrium (i.e. in which $p < \infty$ which is the case if and only if there almost surely exists a default), there exists a pool-equilibrium with $r^p < R^p$ in which agents consume the same bundles and, vice versa, for any non-trivial (i.e. $d^p \not\equiv 0$) pool-equilibrium in which $r^p < R^p$ and R^p is proportional to the aggregate shock α, there exists a monetary equilibrium in which agents consume the same bundles.*

Proof. With $R^p = Rp_0/p_1$, $\bar{d}^p = \bar{d}/p_0$, $r^p = p_0/p_1 r$ and $d_i^p = d_i/p_0$ the choice sets, choices and outcomes in the pool setting are the same as in the monetary setting. For the other direction suppose $(r^p, R^p, \bar{d}^p, d_i^p)$ is a pool-equilibrium. Let p_0 and p_1 be arbitrary, $R = p_1 R^p/p_0$, $d_i = p_0 d_i^p$, $r = r^p p_1/p_0$, $\bar{d} = p_0 \bar{d}^p$ and $M = \int d_i^+ di$. □

While the real allocations in monetary equilibria are parametrized by the ratio $\frac{r}{R}$, the ratio of debt repaid to total debt outstanding, in other words credit quality, pool-equilibria are parametrized by $\bar{d}^p$, the debt limit in real terms.

3.4 Extension

Instead of modeling only the central bank lending out (up to) M units of money and setting the interest rates r, R and the debt limit $\bar{d}$, it is possible to extend the model to a competitive banking system which endogenously determines R and $\bar{d}$ given the inputs r and M from the central bank and absorbs all losses. Here, I only give a sketchy analysis as this shall not be a major focus of the paper.

Let there be continuum $j \in [0, 1]$ of banks of measure 1 that play a Bertrand like game to attract borrowers. Before $t = 0$, say in $t = 0_-$, each bank j decides about their lending policy, i.e. the rates R_j and borrowing limit $\bar{d}_j$ they want to offer to consumers. Banks refinance through the central bank which allows each bank to borrow up to $2\mathcal{M}$ units of money per (unit of) borrower they attract. The factor scaling is chosen to keep 2 units of money per early consumer. Banks borrow from the central bank and pass the money on to their clients. Banks have to redeem the money in $t = 1$ without interest. Banks choose (R_j, d_j) to maximize profits

$$\Pi_j = \left(\mathbb{E}\left[\min\{e_1\alpha p_1, R_j d_j\}\frac{p_0}{p_1}\right] - d_j\right) \times B$$

Here, the first term is the expected difference between (inflation adjusted) money units redeemed and the money lent out per borrower. The second term, $B =$ meas(borrowers), denotes the measure of borrowers a bank attracts. One can write

$$\Pi_j = (\rho(d_j, R_j) - 1)\, d_j \times B \tag{3.4.1}$$

where

$$\rho(d_j, R_j) \triangleq \frac{\mathbb{E}\left[\min\{e_1 \alpha p_1, R_j d_j\}\frac{p_0}{p_1}\right]}{d_j} \tag{3.4.2}$$

is the expected real interest rate paid by an agent who approaches a bank with policy (d_j, R_j). In $t = 0$ agents observe the banks' credit policies, approach a bank of their choice[6] to borrow and or/set deposits, trade and consume. Let, for simplicity, the interest rate on deposits be set to $r = 1$ by the central bank. As before, agents observe endowments in $t = 1$, trade, redeem their debt or default and consume.

I now informally derive an equilibrium for this setting. As all debtors have an incentive to take as much debt as possible, debt limits $\bar{d}_j$ and interest rates R_j have to be equal across banks, say to $(\bar{d}, R)$ in equilibrium. Define $M = \bar{d}/2$ as the measure of total debt. In an efficient solution no early consumer would be allowed to consume in $t = 1$, hence all debtors will have to default. This, together with the equilibrium condition (3.3.2), implies

$$\frac{1}{2}\bar{d} = \int_{[0,1]} \min\{\alpha e_{i,1} p_1, R d_i\} di = \int_{[0,1/2]} \alpha e_{i,1} p_1 di = \frac{1}{2}\alpha \mathbb{E}\left[e_1\right] p_1 \tag{3.4.3}$$

the last equality being due to the law of large numbers, which pins down p_1 as

$$p_1 = \frac{\bar{d}}{\alpha \mathbb{E}\left[e_1\right]} \tag{3.4.4}$$

Every debtor defaults iff $R\bar{d} \geq \operatorname{ess\,sup} p_1 \alpha e_1 = \operatorname{ess\,sup} \bar{d} e_1 / \mathbb{E}\left[e_1\right]$. This gives a lower bound for R (so the assertion that every bank has to set the same interest rate R_j was not correct. R_j just has to lie over a certain threshold, higher interest rate do not matter as the agent defaults anyway). Now any debt limit $\bar{d} \leq 2\mathcal{M}$ could be an equilibrium. Note, however, that the price level is maximal for $\bar{d} = 2\mathcal{M}$ so that the money value is bounded from below for all equilibria. Also, $\bar{d} = 2\mathcal{M}$ is the only stable equilibrium in the sense that if banks earn a however small amount of profits, they want to lend out the maximum amount. The following Theorem records the preceding arguments. A detailed proof can be found in the appendix.

Theorem 3.4.1. *Let $\bar{d} \leq 2\mathcal{M}$ be arbitrary. Define $R^* \triangleq \operatorname{ess\,sup} e_1 / \mathbb{E}\left[e_1\right]$. Then*

1. *Any collection $(\bar{d}, R_j)$ with $R_j \geq R^*$ arbitrary for $j \in [0, 1]$ and the remaining values as in Theorem 3.3.4 for the monetary policy $(1, R^*, \bar{d})$ is an equilibrium.*

[6] If a bank faces more customers than it can serve, randomly chosen customers will have to approach other banks, possibly one of second choice. But this does not happen in equilibrium where every bank will choose the same credit policy.

2. *Vice versa, any equilibrium is of the form above, i.e.* $(\bar{d}, R_j)$ *with* $R_j \geq R^*$ *for any bank* j *that attracts some customers.*

3.5 Money has Pooling Property even if Pooling is Infeasible

Note that for a pooling contract, the repayment obligation (3.3.7) has to be specified ex ante in period $t = 0$ in terms of the consumption good, while for the monetary equilibrium the repayment is an ex post equilibrium outcome in $t = 1$. This is interesting as it allows for a monetary equilibrium to implement a pool-equilibrium even in situations where the pooling equilibrium is infeasible. Think, for instance, of a situation in which contracts such as (3.3.7) cannot be written. Such a restriction might be due to:

Uncontractable Quality: Suppose that endowments are of different quality and one good with quality q provides the same utility as q goods with quality 1, so that the consumer is indifferent between quantity and quality. Suppose further that quality cannot be contracted, but can be observed an instant before purchase, i.e. in period $t = 1$ all qualities are known. For instance, it might be difficult to specify the quality of a car in advance, but a short drive will reveal how valuable it is. In this case it is impossible to write future contracts about the delivery of x units of a certain fixed quality in $t = 1$ as delivery might occur with goods of vanishing quality. In this case, writing the pooling contract $\pi_i^p \triangleq \min\{\alpha e_{i,1}, R^p d_i^{p-}\}$ where d_i^{p-} is quality weighted quantity would be infeasible. Therefore, no equilibrium with debt pools exists. With money, however, there is an equilibrium outcome leading to the same results as if pooling was possible. In this sense, money is a device to pool debt even in cases in which debt pools are not available. The same holds for:

Different products: Suppose that in $t = 1$ there are different goods than in $t = 0$ and it is unknown which type of products will be available. In this case, it would not be possible to write future contracts unless it is possible to specify all future scenarios. Money, however, would still work as a pooling device. In this sense, money is a way to overcome incompleteness of credit markets.

This result also sheds some light on the question why in former times gold was the preferred medium of exchange. Gold has the property that its quality can be easily described and verified. There is no difficulty in writing contracts in terms of gold.

3.6 Conclusion

This article was motivated by a series of questions that can now be answered conditional on the model being a sufficient close approximation to the real world.

(1) Is the value of fiat money bounded away from zero and if so (2) why?

The answer is yes. Money is a witness for monetary debt which is backed by collateral. The fear to lose the collateral creates money demand and ensures positive money value.

(3) Is the value stable in the sense that it survives backward induction and rational expectation arguments?

Yes. The last periods money value is unique and positive. The equilibrium is derived by backward induction. Even more: It is the unique trembling hand equilibrium.

(4) Is money a safe asset in the sense that the money value is attached with little risk and (5) if so why?

Yes. As money is not only demanded by a single but by all debtors, idiosyncratic shocks to solvency wash out and only aggregate uncertainty remains equivalently to a large securitized pool of debt.

(6) Can the existence of money implement allocations that were not reachable without it?

Yes. Scenarios in which money implements the debt pooling equilibrium while ex ante pooling contracts are infeasible, e.g. if quality is not contractible, are imaginable and can be easily incorporated in the model at hand.

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Appendix

Proof of Theorem 3.3.4. 1) Clearly, $p_1 = p_0 = \infty$ implies that no one has an incentive to take on debt or make deposits. Hence, $d_i = 0$ for all i and thereby $M = 0$ is an equilibrium. However, also other prices might support $M = 0$ equilibrium outcome. For this, note that in any equilibrium the real interest rate, if well defined (i.e. with finite prices)), received on deposits has to equal 1 to make late consumers maximize utility, i.e. $\mathbb{E}[rp_0/p_1] = 1$. However, early consumers with debt d then have to pay an expected real interest rate of

$$\rho(d) = \mathbb{E}\left[\min\{\frac{e_1\alpha p_1}{d}, R\}\frac{p_0}{p_1}\right]$$

which is decreasing in d but might still be more than $1/\delta$ for $d = \bar{d}$ if p_1 is low in relation to R. In this case $d_i = 0$ would be optimal and market clearing is trivially fulfilled.
2) First, I show that the equilibrium described in the theorem is indeed an equilibrium. To see this, note that $\mathbb{E}\left[r\frac{p_0}{p_1}\right] = 1$, i.e. the expected real interest paid on deposits is one. This makes late consumers indifferent between the level of deposits. That early consumers are maximizing utility is a consequence of the next lemma, which also shows that $d_i = \bar{d}$ has to hold for $i \leq 0.5$ in every equilibrium. It remains to show that an appropriate distribution of $d_i \geq 0 \quad i \in (0.5, 1]$ exists, i.e. one in which $\int d_i^+ di = \bar{d}/2$ and $d_i \leq p_0$, the latter condition excluding negative consumption. Such a distribution exists, e.g. $d_i = \bar{d}$. To see this, note that

$$\begin{aligned} \frac{r}{2p_1}\bar{d} &= \frac{rM}{p_1} \\ &= \frac{1}{p_1}\int \pi_i di \\ &\leq \int_{[0,1/2]} \alpha e_{i,1} di \end{aligned}$$

In particular, by the law of large numbers over $e_{i,1}$ and application of expectations

$$\begin{aligned} \bar{d}r\frac{1}{p_1} &\leq \mathbb{E}[e_1\alpha \mid \alpha] \quad \text{and hence} \\ \bar{d}r\mathbb{E}\left[\frac{1}{p_1}\right] &\leq \mathbb{E}[\mathbb{E}[e_1\alpha \mid \alpha]] \leq 1 \quad \text{by assumption on period 1 endowments} \end{aligned}$$

Therefore,

$$\bar{d}\frac{1}{p_0} = \bar{d}r\mathbb{E}\left[\frac{1}{p_1}\right] \leq 1 \tag{3.6.1}$$

which had to be shown.
It has just been shown that given $r, R, \bar{d}$, $M = \frac{1}{2}\bar{d}$, $d_i = -\bar{d}$ for $i \leq 0.5$ and $d_i = \bar{d}$ for $i > 0.5$, π_i defined by (3.3.1), p_1 defined by Theorem 3.3.4, i.e. by $\int \min\{\alpha e_{i,1}p_1, R\bar{d}\}di = rM$, together with $p_0 = 1/(\mathbb{E}[r/p_1])$ forms an equilibrium. During the argument only the distribution of d^+ exhibited some degrees of freedom. So the equilibrium above is - up the distribution of d^+ - the only one for which $M > 0$.

□

Lemma A.1 Let p_1 be the price in a $t = 1$-equilibrium with $M > 0$ and $p_0 = 1/\mathbb{E}[r/p_1]$. Let

$$\rho(d) \triangleq \frac{\mathbb{E}\left[\min\{e_1\alpha p_1, Rd\}\frac{p_0}{p_1}\right]}{d} \tag{3.6.2}$$

be the real interest rate to be paid by an agent with debt d. Then, first, $\rho(d)$ is decreasing in the debt level d and, second, $\rho(\bar{d}) \leq 1$. In particular, $d_i = -\bar{d}$ for all $i \in [0, 1/2]$ in equilibrium.

Proof. To see (1), write $\rho(d) \triangleq \mathbb{E}\left[\min\{\frac{e_1\alpha p_1}{d}, R\}\frac{p_0}{p_1}\right]$. For (2), note that $f(d) \triangleq \mathbb{E}\left[\min\{e\alpha p_1, Rd\}\frac{p_0}{p_1}\right]$ is concave and $f(0) = 0$. Hence, for all $d_i^- \leq \bar{d}$

$$f(d_i^-) \geq \left(1 - \frac{d_i^-}{\bar{d}}\right) f(0) + \frac{d_i^-}{\bar{d}} f(\bar{d}) = \frac{d_i^-}{\bar{d}} f(\bar{d})$$

Integration over i yields

$$\int f(d_i^-)di \geq \frac{f(\bar{d})}{\bar{d}} \int d_i^- di \tag{3.6.3}$$

However, by the definition of an equilibrium and (3.3.1)

$$r \int d_i^- d_i = \int \min\{\alpha e_{i,1} p_1, R d_i^-\} di$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[\frac{p_0}{p_1} r \int d_i^- d_i\right] &= \mathbb{E}\left[\frac{p_0}{p_1} \int \min\{\alpha e_{i,1} p_1, R d_i^-\} di\right] \\
&= \int f(d_i^-)di \\
&\geq \frac{f(\bar{d})}{\bar{d}} \int d_i^- di \quad \text{by (3.6.3)} \qquad (3.6.4)
\end{aligned}$$

Dividing through $\int d_i^- d_i$ and using that late consumers are unrestricted and have to be indifferent between their level of deposits, (i.e. $1 = \mathbb{E}\left[\frac{rp_0}{p_1}\right]$) leads to

$$1 = \mathbb{E}\left[\frac{rp_0}{p_1}\right] \geq \frac{f(\bar{d})}{\bar{d}} = \rho(\bar{d}) \tag{3.6.5}$$

This proves the second claim.
Now Let $U(d)$ be the utility of a debtor with initial debt d. It has to be shown that for $d < \bar{d}$ holds $U(d) < U(\bar{d})$ (debtors borrow up to their limit). To see this, note that

$$\begin{aligned}
U(d) &= e_0 + \frac{d}{p_0} + \delta \mathbb{E}\left[\max\{0, \alpha e_1 - \frac{Rd}{p_1}\}\right] \\
&= e_0 + \frac{d}{p_0} + \delta \left(\mathbb{E}\left[\alpha e_1\right] - \mathbb{E}\left[\min\{\alpha e_1 p_1, Rd\}\frac{1}{p_1}\right]\right) \\
&= e_0 + \frac{d}{p_0} + \delta \left(\mathbb{E}\left[\alpha e_1\right] - \frac{\rho(d)d}{p_0}\right) \\
&= e_0 + \frac{d}{p_0}\left(1 - \delta\rho(d)\right) + \delta \mathbb{E}\left[\alpha e_1\right] \qquad (3.6.6)
\end{aligned}$$

which takes a maximum for the debt level $d = \bar{d}$ by the properties of $\rho(d)$ just shown and the assumption $\delta < 1$. □

Proof of Corollary 3.3.5. To understand the price bound on p_1, note that as long as $r < R$ there has to be at least one defaulter. That is $\alpha e_{i,1} p_1 < R\bar{d}$ for at least one $e_{i,1}$. Equivalently

$$p_1 < \frac{R\bar{d}}{\alpha e_{i,1}}$$

Maximizing over all possible outcomes $\alpha e_{i,1}$ yields the result

□

Proof of Corollary 3.3.8. By Theorem 3.3.4 holds $d_i = \bar{d}$ for all $i \leq 1/2$. Combining this with (3.3.2) implies that p_1 is determined as the unique solution to

$$\frac{1}{2} r\bar{d} = rM = \int_{[0,1/2]} \min\{\alpha e_{i,1} p_1, R\bar{d}\} di \tag{3.6.7}$$

$$\Leftrightarrow \quad \frac{1}{2}\bar{d} = M = \int_{[0,1/2]} \min\{\alpha e_{i,1} \frac{p_1}{r}, \frac{R\bar{d}}{r}\} di \tag{3.6.8}$$

Now assume that $\bar{d}$ and r vary with r/R held constant. It is to show that real allocations do not change. From (3.6.8) can be inferred that p_1 is proportional to $r\bar{d}$, $p_0 = 1/\mathbb{E}\left[r/p_1\right]$ is proportional to $\bar{d}$ and real debt services $\min\{\alpha e_1, \frac{R\bar{d}}{p_1}\}$ do not alter. In particular, the real allocation does not change. Vice versa, let R/r vary. Then as long as not everybody defaults, i.e. as long as as long as $R\bar{d} < p_1 \operatorname{ess\,sup} \alpha e_1$, the entity p_1/r and therefore the real allocation has to change. □

Proof of Theorem 3.4.1. For 1), the only thing to show is that banks maximize profits, all other things already being proven with Theorem 3.3.4. First note that

$$\begin{aligned}
\bar{d}\rho(\bar{d}, R^*) &= \mathbb{E}\left[\min\{e_1\alpha p_1, R^*\bar{d}\}\frac{p_0}{p_1}\right] \\
&= \mathbb{E}\left[\min\{e_1\bar{p}_1, R^*\bar{d}\}\frac{p_0}{p_1}\right] \\
&= \mathbb{E}\left[\min\{e_1\bar{p}_1, R^*\bar{d}\}\right]\mathbb{E}\left[\frac{p_0}{p_1}\right] \\
&= \mathbb{E}\left[\min\{e_1\alpha p_1, R^*\bar{d}\}\right]\cdot 1 \\
&= 2\int_{[1/2]} \pi_i di \quad \text{(law of large numbers)} \\
&= \bar{d}
\end{aligned}$$

so that $\rho(\bar{d}, R^*) = 1$. Suppose now that one bank deviates from $(\bar{d}, R^*)$ and sets a menu (d_j, R_j). Clearly, lowering credit quality $d_j > \bar{d}$ only produces more defaults but not more revenues even for $R_j > R^*$ by the fact that ρ is decreasing in d_j (see Lemma A.1) but not increasing in R_j for $R \geq R^*$ (definition of R^*). Neither can it be improving to only set a lower interest rate $d_j = \bar{d}, R_j < R^*$. So suppose, $d_j < \bar{d}$. Let $\rho(d)$ be the real interest rate to be paid by an agent with debt d as in Lemma A.1. As $\rho(\bar{d}) \leq 1$ and early consumers utility is strictly increasing in the debt level (see also (3.6.6)), the bank has to set R_j such that $\rho(d_j, R_j) < 1$, which would lead to negative profits by (3.4.1).

(2) It is enough to show that $R_j < R^*$ cannot occur in equilibrium (if bank j were to attract some customers) as in this case an argument as in the proof of (1) shows that $d_j = \bar{d}$ is constant among all banks. So suppose there exists a bank attracting borrowers with policy $R_j < R^*$. As banks cannot make any profits (note that average monetary profits have to be zero as banks owe all money that exists in the economy to the central bank. So by pure profit maximization, without even referring to competition, banks cannot make profits.) it has to hold $\rho(d_j, R_j) = 1$. But for given ρ, consumers prefer strictly more to less debt (see (3.6.6) in the proof of Lemma A.1), so $d_j = \bar{d}$ has to be constant among all banks (that attract customers) in equilibrium. But then, if $R_j < R^*$, not every early consumer defaults in the second period (by definition of R^*). Hence, lowering credit quality $d_j > \bar{d}$ and choosing R_j such that $\rho(d_j, R_j) = 1 + \epsilon$ would be a profitable deviation for some small $\epsilon > 0$ in which a bank would make positive profits. □

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