A System-theoretic Approach to Multi-Agent Models

INAUGURAL-DISSERTATION
zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität zu Köln

vorgelegt von
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aus Hamburg

2012

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Tag der mündlichen Prüfung:	15. Juni 2012

Abstract

A system-theoretic model for cooperative settings is presented that unifies and extends the models of classical cooperative games and coalition formation processes and their generalizations. The model is based on the notions of system, state and transition graph. The latter describes changes of a system over time in terms of actions governed by individuals or groups of individuals. Contrary to classic models, the presented model is not restricted to acyclic settings and allows the transition graph to have cycles.

Time-dependent solutions to allocation problems are proposed and discussed. In particular, Weber's theory of randomized values is generalized as well as the notion of semi-values. Convergence assertions are made in some cases, and the concept of the Cesàro value of an allocation mechanism is introduced in order to achieve convergence for a wide range of allocation mechanisms. Quantum allocation mechanisms are defined, which are induced by quantum random walks on the transition graph and it is shown that they satisfy certain fairness criteria. A concept for Weber sets and two different concepts of cores are proposed in the acyclic case, and it is shown under some mild assumptions that both cores are subsets of the Weber set.

Moreover, the model of non-cooperative games in extensive form is generalized such that the presented model achieves a mutual framework for cooperative and non-cooperative games. A coherency to welfare economics is made and to each allocation mechanism a social welfare function is proposed.

Kurzzusammenfassung

Ein systemtheoretisches Modell für kooperative Situationen wird vorgestellt, welches die klassischen Modelle kooperativer Spiele und Koalitionsbildungsprozesse und deren Verallgemeinerungen vereinigt und erweitert. Das Modell basiert auf den Begriffen System, Zustand und Übergangsgraph. Letzterer beschreibt die Veränderung eines Systems in Form von Aktionen, welche von Individuen oder Gruppen von Individuen beherrscht werden. Im Gegensatz zu klassischen Modellen, ist das vorgestellte Modell nicht auf azyklische Situationen beschränkt und erlaubt es dem Übergangsgraphen auch Kreise zu enthalten.

Zeitabhängige Lösungen zu Allokationsproblemen werden vorgeschlagen und diskutiert. Insbesondere wird Webers Theorie der randomisierten Werte, ebenso wie der Begriff des Halbwerts, verallgemeinert. In manchen Fällen werden Konvergenzaussagen getroffen und das Konzept des Cesàro-Werts eines Allokationsmechanismus wird eingeführt, um Konvergenz einer großen Menge von Allokationsmechanismen zu erreichen. Quantenallokationsmechanismen, welche von Quantenirrfahrten auf dem Übergangsgraphen induziert werden, werden definiert und es wird gezeigt, dass diese gewissen Fairnesskriterien genügen. Ein Konzept für Weber-Mengen und zwei verschiedene core-Konzepte werden im azyklischen Fall vorgeschlagen und es wird unter schwachen Voraussetzungen gezeigt, dass beide cores Teilmengen der Weber-Menge sind.

Überdies wird das klassische Modell nicht-kooperativer Spiele in extensiver Form verallgemeinert, so dass das vorgestellte Modell einen gemeinsamen Rahmen für kooperative und nicht-kooperative Spiele bildet. Ein Zusammenhang zur Wohlfahrtsökonomie wird hergestellt und zu jedem Allokationsmechanismus wird eine gesellschaftliche Wohlfahrtsfunktion vorgeschlagen.

Contents

1 Preface

Whenever multiple *agents* (*e.g.* persons, companies, groups of individuals,...) interact in a given economic or sociological context, these agents make decisions to take certain actions that will influence the welfare of each individual, as well as the welfare of the whole group. There is a basic conflict between the interests of individual agents which are guided by self-interest and the interests of the group of agents as a whole. The description and analysis of these interactions, this conflict and the behavior of the agents by mathematical models are primary objectives of game theory. By performing several actions, the agents mutually generate a social surplus or social costs. Finding fair allocations of these surpluses or costs is a principal purpose of cooperative game theory.
Cooperative processes and allocation processes naturally have a dynamic flavor, but mathematical models for their analysis are often based on set theoretic frameworks or acyclic combinatorial structures that are inherently static. Precisely this is the initial point of this thesis. By understanding cooperation and allocation as dynamic processes, it presents a new formalism for studying allocation problems and cooperative behavior. This formalism is based on a simple observation: not the agents, but the actions governed by them, are the essential objects which cause surpluses or costs in a cooperative environment.

In Chapter 2 classical and recent models of cooperative game theory and coalition formation processes are reviewed. By giving certain examples of cooperative settings, main points of criticism are figured out, that will serve as motivation for a more dynamic and non-acyclic approach to cooperative topics. Models for cooperative games on precedence structures ([34], [46]), on combinatorial structures ([11], [13], [14], [15], [30]), on closed set-systems ([19], [28], [32]), models for multi-choice games ([49], [50], [44]), the first system-theoretic (but still acyclic) approach [37], as well as models for coalition formation are discussed. This motivates the wish to give a unifying model for cooperative settings.
Chapter 3 picks up these points of criticism and this wish and introduces our model. First, the terms *system* and *state* are defined and basic properties of these objects are

derived. After this we investigate evolution processes of systems and find a strong relation to a graph, the so called *transition graph*, whose vertices are states and whose arcs are actions governed by agents to bring the system from one state to another. This leads us to the general notions of a time dependent *cooperation system* and cooperative games on cooperation systems. We prove that all mentioned classical models yield special cases of cooperation systems. Hence cooperation systems yield a unifying framework for cooperative settings.

The set of all cooperative games on classical structures is a vector space. A certain basis - the *unanimity basis* - of this vector space is a tool to apply the theory of linear algebra to cooperative games such that the elements of this basis have a *nice* game-theoretic interpretation. We generalize the notion of unanimity games to general cooperation systems and give interpretations of these games in Chapter 4. This is done by embedding the discussion into the framework of incidence algebras.

Chapter 5 is dedicated to first solutions to the allocation problem of cooperative games on cooperation systems. The notion of allocation mechanisms is developed as a function that assigns to each cooperative game on a cooperation system a time-dependent valuation of the actions governed by the agents. Linear allocation mechanisms are studied and characterized and the non-negativity axiom is introduced, which is a generalization of the classical monotonicity axiom. Moreover, the classical concept of efficiency is generalized and linear and efficient allocation mechanisms are characterized as certain flows on the transition graph. We argue that a fair allocation mechanism should distribute values in the same ratio to certain arcs if the system is in the same state more than once. We call this property ratio fairness, and linear, efficient and ratio fair allocation mechanisms are seen to correspond to generalized random walks on the transition graph. Therefore, we call them *randomized*. This yields a generalization of Weber's seminal theory of classical linear values [91]. *Via* this correspondence we are able relate non-negative, linear, ratio fair and efficient allocation mechanisms to the notion of entropy. A generalization of the classical Shapley value [82] is seen to be the unique allocation mechanism of this type with maximal entropy. We give some basic convergence results of randomized allocation mechanisms that are strongly related to well-known facts of the convergence theory of random walks. The chapter is closed by several concrete examples of allocation mechanisms and also a generalization of the famous Banzhaf value [8] is given.

Given two cooperative settings, we consider in Chapter 6 how they could be jointly modeled. The strong relation of cooperation systems to graphs again shows to be worthwhile since transition graphs of jointly modeled settings are seen to be Cartesian products of transition graphs of the single cooperation systems. We give an approach to construct a random walk on the Cartesian product of two graphs out of given random walks on its factors. This approach directly leads to an application of the theory developed in Chapter 5 and we are able to give a concrete description of randomized allocation mechanisms on product systems that are induced by randomized allocation mechanisms on the single factors.

In Chapter 7, our view is peeled away from randomized allocation mechanisms and the class of not necessarily efficient allocation mechanisms is studied. Linear and ratio fair allocation mechanisms are seen to be induced by certain matrices. This leads us to a generalization of classical semi-values [26]. The second part of this chapter introduces the Cesàro value of an allocation mechanism, which is the Cesàro mean of its values over time. By applying the classical ergodic theorem of linear algebra together with the characterization of linear and ratio fair allocation mechanisms, we achieve the convergence of all Cesàro values of allocation mechanisms which are induced by a matrix with norm less or equal to one. In particular, the Cesàro values of randomized allocation mechanisms are convergent.

Chapter 8 introduces the term of quantum allocation mechanisms. These are allocation mechanisms which are induced by certain unitary evolution processes, so called quantum random walks, on the transition graph of a cooperation system. It is shown that quantum allocation mechanisms yield examples of linear and efficient but not ratio fair allocation mechanisms. Moreover, by applying an ergodic theorem of Faigle and Schönhuth [38] it is proven that - even if quantum allocation mechanisms are not ratio fair - their Cesàro values converge.

Besides allocation mechanisms the set-valued solution concept of the *core* is widely studied in cooperative game theory. Chapter 9 proposes two different generalizations of the classical core for acyclic cooperation systems. Marginal-worth vectors are defined in our general context. The convex hull of all marginal-worth vectors is classically known as the Weber-set of a cooperative game due to a result of Weber [91], who proved that the core is always a subset of the Weber-set. Both proposed core concepts are seen to have this property, and we prove generalized versions of Weber's theorem with the aid of a greedy-type algorithm.

Chapter 10 takes a step beyond cooperative aspects. Non-cooperative games in extensive form are generalized and are seen to be tuples of cooperative games on cooperation systems. Equilibrium points are characterized in graph theoretic terms. A relation between mixed strategies and random walks on a tensor product of certain graphs is exposed in the case of a two player non-cooperative game. A social welfare function measures a certain social utility with respect to individual utility functions. Motivated by a model for coalition formation in societies [33] a bridge to welfare economics is build. Social welfare functions are seen to arise as special cooperative games on a cooperation system. The developed theory of allocation mechanisms yields answers to the question, how a certain social welfare value should be allocated in a fair way. The basic theory of social welfare and social choice is recalled. And a possibility to transport fairness criteria of social welfare functions to fairness criteria of allocation mechanisms and vice versa is pointed out.

Acknowledgment First and foremost I want to thank Ulrich Faigle for giving me the possibility to write this thesis under his supervision. In order of appearance in my academical life I want to thank my teachers: Olaf von Grudzinski for developing my basic mathematical skills during the first year of my studies and for teaching me over-pedantic correctness and the ambition for maximal generality in mathematics. Bernd Stellmacher for releasing my stringent view on mathematics and teaching me the principle "*Die Hauptsache ist, man versteht, was gemeint ist!*" and for encouraging me to do a doctorate. Ulrich Faigle for introducing me to various topics of applied mathematics and especially to cooperative game theory. Moreover, I want to thank him for teaching me to look at things in ways no one did before. Even if I often lacked comprehension of his thoughts at first, finally his ideas proved to be right in the majority of cases. It was a pleasure to learn from him even if it was not always an easy path. It was even more a pleasure to work with him and I hope that this thesis and the time under his guidance is a basis for joint-work in the future. But I am most thankful for the several kicks in the back at the right time.
I want to thank Rainer Schrader for co-supervising this thesis and for taking the time to answer all my questions and to dry all tears the life of a doctoral candidate has down the pat. His speed of understanding, if I asked him something, impressed me over and over again. I am grateful for Alexander Schönhuth inviting me to the CWI Amsterdam, for helping me to understand quantum random walks a bit more and for also co-supervising this thesis. Moreover, I thank him for reading huge parts of this thesis in advance and for pointing out some mistakes and ambiguities. Our conversations and his support encouraged me and were an essential part in finishing this thesis.
I also want to thank my colleagues in the work group: Maximilian Heyne who was a fellow sufferer from the beginning. This thesis would not exist without our endless conversations (mathematical or not) and the regular visits of the "Zuckerhexe". Birgit Engels for providing me a smooth start in the work group and for several mutual smoking breaks. Oliver Schaudt and Vera Weil for several discussions and especially Vera Weil for giving me reasons to smirk again and again.
Last but not least I want to thank my parents for their constant support and love during

the last years - or better: during my whole life.

Cologne, April 2012.

2 Models for cooperative games and coalition formation

This chapter recalls the model of classical cooperative games in Section 2.1 and some of its generalizations as a basis for the introduction of our model in the next chapter. Also the allocation problem is stated in Section 2.1 and discussed by means of some examples. Solutions to the allocation problem are not proposed in this chapter, and we will concentrate on finding solutions in later chapters.
In Section 2.2, we give examples of previously proposed set-theoretical generalizations to the classical model. Amongst others we highlight the approach of Faigle and Kern [34] to generalize cooperative games to games on precedence structures. After that, in Section 2.3, we will present the model of Faigle and the author [37] that took a first step towards a more general view on cooperative game theory and was a main part of inspiration for this thesis.
In Section 2.4, we discuss points of criticism of the mentioned models by means of an example in order to give reasons for further generalizations and to define tasks for our approach that will be stated in the next chapter.

2.1 Classical cooperative games

The classical model of cooperative games (first proposed by von Neumann and Morgenstern [70]) assumes that arbitrary subsets of *agents* can join to form *coalitions* and create values in a given economic context. Formally a *cooperative game* is a pair (N, v) consisting of a finite set of *players* (or agents) and a *characteristic function* $v : 2^N \to \mathbb{R}$, that assigns to each *coalition* $S \subseteq N$ of players its *value* $v(S)$. Usually one assumes that v is normalized; *i.e.*: $v(\emptyset) = 0$.

2.1.1 The allocation problem

Given a cooperative game (N, v) in some economic context, the players in N create the value $v(N)$. The allocation problem is to distribute the jointly generated value among the players in a *fair* way. With other words: the allocation problem is the problem of assigning to each cooperative game (N, v) a payoff vector, $x \in \mathbb{R}^N$, in a way such that this allocation is *fair* and allocates a certain jointly generated value to the players. Usually one assumes that the value $v(N)$ is allocated, *i.e.*: $x(N) := \sum_{i \in N} x_i = v(N)$. Such an allocation vector is called *efficient*.
Since *fairness* is a very subjective concern, there are many different ideas of solution concepts to the allocation problem proposed in the literature (*cf.* Section 5.1.1).

We give two examples of cooperative games and the associated allocation problems:

Example 2.1.1 (Government formation) *Assume that in an elected parliament there are representatives of five parties: a conservative party C, a social democratic party S, a liberal party L, an ecologic party G and a Marxistian party M. The outcome of the election does not admit that one party could govern on its own. Thus the parliament goes into coalition negotiations. To be more concrete, assume the following outcome of an election (fictive data for an election to the "Deutsche Bundestag"). There are 598 seats in the parliament. The distribution of seats is as follows:*

C 198 *seats*
S 202 *seats*
G 81 *seats*
L 23 *seats*
M 95 *seats*

Assume one wants to model the distribution of power among the parties for the coalition negotiations by a cooperative game. Therefore let $v : 2^{\{C,S,L,G,M\}} \to \mathbb{R}$ *and set the value* *of a coalition to* 1 *if it is up to governance (i.e.: if it is capable to govern with a majority of at least* 299 *seats) and* 0 *else wise. Set* $N := \{C,S,L,G,M\}$*. A payoff vector* $x \in \mathbb{R}^N$ *with* $\sum_{i\in N} x_i = v(N) = 1$ *reflects a distribution of power of the parties. Intuitively one would say,* x *is fair for this game if its power distribution gives account to the real power of the party, i.e.: the power of a party X should depend on the possible coalitions, which are up to governance and contain X. Therefore, the power of L should be much lesser than that of S.*

Example 2.1.2 (Gain distribution) *Christy, Peter and Bob like cinema movies. The local cinema offers the opportunity to buy tickets in advance without any compound to a fixed film. A single ticket costs 10 Euros. But the cinema also offers a quantity discount. If you buy three tickets at once, you get one ticket for free, and if you buy 5 tickets at once, you get 2 tickets for free. The budgets of Christy, Peter and Bob are as follows:*

$$\begin{aligned} \textit{Christy}\ 25\ \textit{Euros} \\ \textit{Peter}\ 35\ \textit{Euros} \\ \textit{Bob}\ 20\ \textit{Euros}. \end{aligned}$$

It is obvious that they should put all their money together in order to get tickets for 80 Euros, which will result in a whole of 11 cinema tickets. But how should those 11 tickets be distributed among Christy, Peter and Bob? One possible distribution could be as follows: Christy and Peter both get 4 tickets and Bob gets 3 tickets. This seems to be fair, but Peter could get this 4 tickets by spending only 30 of his 35 Euros alone on tickets, without cooperating with Christy and Bob.

2.2 Set-theoretic generalizations

Consider Example 2.1.1 again. As long as there is no further information, this modeling seems adequate. But in reality, the political business is not as simple as mentioned in this example. For instance, assume that the conservative party C has given its electorates the promise, not to contract a coalition with the Marxistian party M. In order to valuate the power distribution of the parties in this example, *all* possible coalitions were considered. But with the election pledge of C, not to build a coalition with M, this model does not reflect the reality. Arithmetically this possibility is feasible, party politically it is not.
This example gives a reason for generalizing the classical model of cooperative games. The classical model is not adequate for many real situations, since not in every situation

all coalitions of players in 2^N are feasible. In the literature there are many generalizations that aim in one way or another for a generalization of the classical model of the form: given a set of feasible coalitions $\mathcal{F} \subseteq 2^N$ and a valuation function $v : \mathcal{F} \to \mathbb{R}$ the triple $(N, \mathcal{F}, v)$ is called a cooperative game. In the rest of this section we will give a short overview over these approaches.

2.2.1 Precedence constraints

In 1992 Faigle and Kern [34] considered partially ordered player sets $(N, \leq)$. Interpreting the partial order, such that the presence of j in a coalition enforces also the presence of i in this coalition, if $i \leq j$, they call a coalition $S \subseteq N$ *feasible* if for all $j \in S$ and all $i \leq j$ also $i \in S$ holds. By setting $\mathcal{F}_{\leq}$ as the set of feasible coalitions induced by the partial order $\leq$, a cooperative game on a partially ordered player set $(N, \leq)$ is a tuple, $(N, \leq, \mathcal{F}_{\leq}, v)$, where v is a valuation function from $\mathcal{F}_{\leq}$ to $\mathbb{R}$.
Another approach to games with precedence constraints is presented by Gilles, Owen and v.d. Brink [46]. They represent the precedence constraints by a directed graph with vertex set N and arc set $\{ij \in V \times V | i < j\}$. Faigle and Kern [34] argue, that by investigating the transitive hull of this graph, one can assume without loss of generality, that the precedence order in [46] is already partially ordered.

2.2.2 Combinatorial coalition structures

Other approaches for generalizations of classical cooperative games investigated different combinatorial structures as underlying sets of feasible coalitions. Bilbao *et al.* studied cooperative games on convex geometries [11], matroids [15] and antimatroids [2]. Recently the models on certain combinatorial structures were generalized to so called augmenting systems [14]. All these models yield special cases of *greedoids*:

Selection structures and greedoids

A *selector* is a map, $\sigma : 2^N \to 2^N$, with the property:

$$\sigma(S) \subseteq N \setminus S, \text{ for all } S \subseteq N.$$

An *ordered selection* is a sequence, $\pi := p_1...p_k$, of players with the property:

$$p_i \in \sigma(\{p_1, ..., p_{i-1}\}) \quad (1 \leq i \leq k).$$

The underlying set of players of an ordered selection is called *selection*. Let $\mathcal{S}$ denote the set of all selections induced by σ.

A *greedoid* is a pair (N, σ) with the property: for all $S, S' \in \mathcal{S}$

$$|S| < \left|S'\right| \Rightarrow \sigma(S) \cap S' \neq \emptyset$$

holds. A general theory of greedoids could be found in the book of Korte, Lovász and Schrader [56] (without any applications to game theory). A model of cooperative games on selection structures was proposed by Faigle and Peis in [30]. Since the above mentioned models are all based on greedoids, we will only discuss how greedoids fit into our model and with that, we implicitly argue for all models on combinatorial structures.

2.2.3 Closed set-systems

Aside the approaches that involve certain combinatorial structures, there are models on (more or less) arbitrary set-systems with different properties to be *closed* in some sense. For instance Faigle [28], as well as v. d. Brink *et al.* [19], investigated union closed set systems as sets of feasible coalitions and studied different generalizations of solution concepts on them.
Relaxations of the assumption of union closedness can for instance be found in the work of Faigle, Grabisch and Heyne [32].

2.3 A first system-theoretic approach

In [37] Faigle and the author took a step beyond the set-theoretic viewpoint on cooperative games. By understanding cooperative situations as sequences of actions, which are performed by the players, that shift the cooperative situation from one state to another, we developed a model for cooperation that has some sort of system-theoretic spirit. For instance, in a classical cooperative situation, player $i \in N$ takes the *action* to join a coalition S and therefore i takes the cooperative situation from state S to state $S \cup i$.
Formally we define a *game system* to be a tuple, $(N, V, A, \mathcal{A}, s)$, consisting of a finite set of players N, a finite set of states V, a set of actions $A \subseteq V \times V$, which is partitioned into the actions that the single players can perform: $\mathcal{A} = (A_1, ..., A_n)$ and of a starting state $s \in V$. We make the following assumptions on the tuple $(N, V, A, \mathcal{A}, s)$ if we call it a game system:

(a) s is the unique starting state

(b) The situation is acyclic, *i.e.*: there is no possibility to take the system back to a state, it previously was in.

A cooperative game on a game system $(N, V, A, \mathcal{A}, s)$ is a valuation of the states: $v : V \to \mathbb{R}$ s.t. $v(s) = 0$. First of all, note that all above stated generalizations of cooperative games are acyclic. We give greedoids as examples of game systems:

Example 2.3.1 (Greedoids as game systems) *Let (N, σ) be a greedoid and let $\mathcal{S}$ be the set of selections induced by σ. By setting*

$$V := \mathcal{S} \text{ and } A_i := \{(S, S \cup \{i\}) | S \in \mathcal{S}, i \in \sigma(S)\} \text{ for } i \in N,$$

$(N, V, A, \mathcal{A}, s)$ is a game system with $\mathcal{A} := \bigcup_{i \in N} A_i$. Therefore, all the above mentioned models for games on combinatorial structures yield special cases for games on game systems.

2.4 Criticism of these models

In this short section we discuss some points of criticism of the above mentioned models. Since we showed in Section 2.3, that all mentioned models (except those on closed set systems) yield special cases of the model in 2.3, we will only critically discuss this model. First we will extend Example 2.1.2:
Assume that Christy, Peter and Bob are regular cinema visitors. And assume that the budgets in Example 2.1.2 are their monthly cinema budgets. So, every month they are able to buy eleven cinema tickets. Assume the cinema plays a new movie each week. Therefore, none of the three is interested in watching more than four films per month. In January Christy, Peter and Bob agree to allocate the eleven tickets by $(4, 4, 3)$. But as mentioned in 2.1.2 this allocation treats Peter unfair since he could get his 4 tickets solo and by only investing 30 of his 35 Euros. But the model does restrict Peter to participate in the joint cinema ticket buy with his whole budget. Obviously it makes a difference if Peter is present with 30 or with 35 Euros.
This is the first point of criticism of the above mentioned models:

- Only presence of players in a coalition is modeled, not their *amount of presence*.

2.4.1 Multi-choice games

This problem was solved model-sided through the idea of so called *multi-choice games* introduced by Hsiao and Raghavan [49], [50]. We will briefly recall a model for multi-choice games, which was proposed by Grabisch and Lange [44] as an extension: one

assumes that each player $i \in N$ has certain *activity levels* which are linearly ordered $L_i := \{0, 1, ..., m_i\}$ where m_i is the maximal activity level of player i. A feasible coalition is a weighted incidence vector $S = (s_1, .., s_n) \in L_1 \times ... \times L_n$. The case $s_i = 0$ models the absence of player i in the coalition S and $s_i = k_i \in L_i$ the presence of player i in S with activity k_i. A cooperative multi-choice game on $\mathcal{S} := L_1 \times ..., \times L_n$ is a valuation $v : \mathcal{S} \to \mathbb{R}$ such that $v(0, ..., 0) = 0$.
This model of multi-choice games yields a special case of the model presented in Section 2.3: Set

$$V := \mathcal{S}, A_i := \{((s_1, ..., s_n), (t_1, ..., t_n)) | s_j = t_j \text{ for } j \neq i \text{ and } s_i < t_i\},$$
$$A := \bigcup_{i \in N} A_i \text{ and } s := (0, ..., 0).$$

Then $(N, V, A, (A_1, ..., A_n), s)$ is a game system in the sense of 2.3.
Back to the Cinema Example. In February Christy and Bob propose to allocate the eleven tickets in the same way, they did in January. But Peter disagrees. In January Peter payed 5 Euros too much in order to maximize the overall number of tickets they can buy together. He is not willing to do so again in February.
This leads to our second point of criticism:

- All set-theoretic models are static and do not take past events into account.

Assume further that Christy, Peter and Bob disagree about a solution to their allocation problem in February, and that Peter decides not to participate in the joint cinema ticket buy this month. The classical models can not model such situations. Further assume that Christy, Peter and Bob discuss about their February problem in the beginning of March and that they agree again. Peter joins the cinema ticket coalition again. Thus there is a cycle in the cooperative behavior of Christy, Peter and Bob, which leads us to the following point of criticism:

- All above mentioned models are acyclic and can not handle cyclic cooperative situations.

2.5 Coalition formation processes

In the first section, we had a look on cooperative games and associated allocation problems. We thought of situations in which a certain value was jointly generated by the players in N. By examples we pointed out, that for distributing this value among the players in a *fair* way, it seems eligible to take the *possibilities of cooperation* of the players into account. Another aspect of cooperative situations is the question, how players form coalitions and coalition structures in a given economic context.

Example 2.5.1 *Given a classical cooperative game (N, v), the formation of the grand coalition N takes place by players that consecutively join in any order. The process starts with the empty set and one by another players form larger and larger coalitions, until the grand coalition is build. With each step the value of the marginal contribution $v(S \cup i) - v(S)$ $(S \subseteq N, i \in N)$ of this step is generated.*

The study of such processes - so called *coalition formation processes* - was mainly started by the work of Thrall and Lucas [87]. They studied so called

Example 2.5.2 (Games in partition function form) *Given a set N of players denote by $\Pi(N)$ the set of all partitions of N. That is the set of all possibilities to distribute the players into pairwise disjoint coalitions. In terms of coalition formation processes a partition is also often called a coalition structure, and the contained coalitions are sometimes called blocks of the partition. The idea of Thrall and Lucas [87] was, that the value of a coalition should not depend only on the coalition itself, but on the coalition structure in which this coalition has formed. Therefore, define an embedded coalition as a pair, $(S, \pi) \in 2^N \times \Pi(N)$, such that $S \in \pi$. Denote by $\mathcal{C}(N)$ the set of all embedded coalitions.*
A game in partition function form is a pair (N, v) with a valuation function

$$v : \mathcal{C}(N) \to \mathbb{R}.$$

$\Pi(N)$ is partially ordered in a natural way induced by the partial order $\subseteq$ on 2^N: given two partitions π_1, π_2 we set $\pi_1 \leq \pi_2$ if for each $S \in \pi_1$ there exists $T \in \pi_2$ with $S \subseteq T$.

Aumann and Drèze [6] already marked, that the theory of cooperative games may be viewed as a general study of coalition formation. Traditionally one assumes that coalitional *moves* are monotonic; *i.e.*: the situation is acyclic, in order to guarantee the convergence of the coalition formation process after finitely many steps (*cf.* Funaki and Grabisch [43], Macho-Stadler *et al.* [63], Hajdukova [52]).

Faigle *et al.* [33] took a step beyond the acyclic point of view on coalition formation processes and proposed a model in which the acting agents could join and leave coalitions arbitrarily. They investigated coalition formation processes that not only arise from personal interests of the players, but also from the benefit of the *society*. We will discuss this model later in Chapter 10.
Later Faigle and Grabisch [31] investigated settings on the power-set 2^N of the player-set such that all transitions $S \to T$ $(S, T \subseteq N)$ are feasible moves.

3 The model

In the previous chapter we recalled various models for cooperative situations and discussed, in which situations these models are not adequate for many real situations. The two main points of criticism were:

- the statics and
- the acyclicity

of these models. In this chapter we propose a new model for cooperative situations, which can deal dynamic and cyclic settings. We will carefully differentiate between two perceptions: cooperative situations in which the underlying cooperation structure is externally given and those situations where it is unknown and could only be *observed* by an ongoing cooperative processes. Many of the used terms in the last sentence are not yet well defined, but in view of the previous chapter, one should have a rough idea of them.
In Section 3.1, we develop the abstract idea of *states*, state the *prediction problem* and finally we define what a *state space* is. After that we introduce the formalism to describe evolutionary processes in terms of *evolution operators*, which were already studied by Faigle and Schönhuth [38], in Section 3.2.
Section 3.3 makes the term *system* available to us as a tuple consisting of a state space and an evolution operator, which describes a certain evolution on this state space. Moreover, by defining the *transition graph* of a system, we emphasize the strong relation of our model to the theory of directed graphs. This finally leads us to the definition of *cooperation systems* in Section 3.5, where we will also see, that the models mentioned in the previous chapter yield special cases of our model.

3.1 Systems and states

For this whole chapter let $\mathbb{K}$ be a subfield of $\mathbb{C}$. Whenever a total ordering of $\mathbb{K}$ is needed, we implicitly assume $\mathbb{K} = \mathbb{R}$. In most applications $\mathbb{K} = \mathbb{R}$ will hold. Since most of our results even hold for arbitrary fields of characteristic 0, we state them for arbitrary subfields of $\mathbb{C}$ for the sake of mathematical generality. However, it is riskless if the reader assumes $\mathbb{K}$ to be the field of real numbers. We begin this section with an example which was motivated by Faigle [29]:

Example 3.1.1 *Assume there is an investor, who considers to bet a certain budget on a future event with $n \in \mathbb{N}$ possible different outcomes O_i ($i = 1, .., n$). The investor assigns weights p_i to the possible outcomes, in order to measure which outcome he expects to come true. In order to maximize his expected gain, he will bet each outcome O_i with the p_i-th part of his designed budget.*
At the time of his decision to bet the p_i-th part of his budget on outcome O_i, the investor is in an economic "state" described by the parameter vector $(p_1, ..., p_n)$.

In the first instance, the cooperative flavor of this example stays hidden. But abstractly thinking of the possible outcomes as *players*, which share the budget of the investor among them, the investors problem of allocating his budget to the outcomes, is also an allocation problem of players in a certain cooperative context.
Besides the allocation problem we are interested in another question:

3.1.1 The prediction problem

Assume a cooperative situation of n players is given, which can perform certain cooperative actions over time (think of "joining" or "leaving" certain coalitions, or "investing x Euros into a certain stock",...). Assume there is a time dependent payoff rule, $(\phi^t)_{t\geq 0}$ ($\phi^t \in \mathbb{K}^n$), that allocates certain gains to the players dependent on which actions they were able to perform. We refer to the question:

"How could the behavior of the players in performing actions be predicted?"

as the *prediction problem*.
We will see in Section 5.5 that there is a strong relation between the prediction problem and the allocation problem.

3.1.2 A first step to systems

We aim for an algebraic formalization of the above ideas into a general framework in which cooperative situations can be analyzed with respect to evolution of cooperation, formation of coalitions and solutions to the allocation- and the prediction problem. We will omit an abstract definition of a system at first and will instead define all needed components, in order to go step by step towards our general model.

We think of a certain situation that should be modeled, as a (finite) collection of certain observable *basic events*. We will also call these events *ground states*. The possibility that a basic event takes place or not can change over time. We assume that time steps are discrete and that the changes over time could be observed.

Think again of Example 3.1.1. A basic event that could be observed is, that the investor bets all his budget on O_i ($i \in N$). We denote the observation of the event "bet everything on O_i" by a tuple $(0, .., 0, 1, 0, ..., 0)$ with 1 in the i-th component. Just as well: if we have a finite collection V of basic events the observation "event x takes place" ($x \in V$) by $(0, .., 0, 1, 0, ..., 0)$ with one in the x-th component.

REMARK Note that we restrict ourselves to finite sets of basic events only for simplicity of the presentation. The basic model could be generalized to arbitrary discrete sets of ground states.

Superposition of ground states

Since the investor has the possibility to split up his budget among all possible events, the definition of ground states is not sufficient to model this situation. Also one could think of situations in which not only parts of an event take place, but it can take place more than once at a time (for instance think of a second investor that also bets his budget on the same events). In order to model those situations, we build formal linear combinations of the ground states in V (so called superpositions) with respect to a given field $\mathbb{K}$:

$$\sum_{x \in V} c_x x \qquad (c_x \in \mathbb{K}).$$

According to the setting that should be modeled, the observed formal linear combinations of ground states could vary extremely. For instance, a certain split up of the budget of the investor $(p_1, ..., p_n)$ is a convex combination of the ground states "bet everything on O_i". In quantum mechanics a possible event is a normalized superposition of basic events. The general case of a quantum state space will be treated later in Chapter 8. We are mainly interested in convex or generalized convex combinations of ground states, but also other reasonable sets of states are thinkable (*cf.* Chapter 11). We split up definitions and merge all states into a set and call this set the *state space*:

Classical Markovian state space

If all possible events are convex combinations of ground states as in Example 3.1.1, we merge all possible events in a set

$$\mathcal{V} := \{v | v = \sum_{x \in V} c_x x, \sum_{x \in V} c_x = 1, c_x \geq 0\}$$

and call $\mathcal{V}$ the *(classical) Markovian state space* with respect to V.

Generalized Markovian state space

If all possible events are generalized convex combinations of ground states, *i.e.*: if the occurring probabilities could be negative, we set

$$\mathcal{V} := \{v | v = \sum_{x \in V} c_x x, \sum_{x \in V} c_x = 1\}$$

and call $\mathcal{V}$ the *generalized Markovian state space* with respect to V.

Negative probabilities

In physics, negative probabilities were discussed by Dirac [24] and Feynman [41]. Recently Faigle and Schönhuth [39] developed a model for certain observable processes in which probabilities could also be negative. The term *probability* is plausible, even if probabilities could become negative or exceed 1: let $v \in \mathbb{K}^V$ s.t. $\mathbf{1}^T v = 1$. For a subset $S \subseteq V$, set $v(S) := \sum_{x \in S} v_x$. Then v obeys some of the familiar Kolmogorov axioms:

$$v(V) = 1, v(V \setminus S) = 1 - v(S), v(S \cup T) + v(S \cap T) = v(S) + v(T)$$

for all $S, T \subseteq V$. We will give no further interpretations of negative probabilities and we will just accept them as a feasible mathematical model.

If no further concretization is needed, we refer to $\mathcal{V}$ as the *state space* and will call the elements of $\mathcal{V}$ *states*. We define the *dimension* of a state space $\mathcal{V}$ to be the cardinality $|V| =: \dim(\mathcal{V})$ of the underlying set of ground states.

Dirac notation

We will often use the so called *Dirac notation*, which is widely used in quantum physics. We write a vector $v \in \mathbb{K}^V$ by $|v\rangle$ and its counterpart (its image under the natural isomorphism) in the dual space by $\langle v|$. We will note some advantages of this notation in the following lemma. Later we will use this lemma without further mentioning.

Lemma 3.1.1

(a) For all $v, w \in V : \langle v| |w\rangle = \langle v| w\rangle = \sum_{x\in V} \langle v|_x |w\rangle_x$.

(b) For all $v, w \in V : \langle v| w\rangle = 0, \langle v| v\rangle = 1$.

(c) $\sum_{x\in V} |x\rangle \langle x| = Id$

Proof. (a) is clear by inserting the definitions. (b) is clear by (a) since V is the standard orthonormal basis of $\mathbb{K}^V$. We prove (c) point-wise on the standard basis V. Let $|u\rangle \in V$. Then

$$\sum_{x\in V} |x\rangle \langle x| |u\rangle \overset{(b)}{=} |u\rangle \langle u| |u\rangle = |u\rangle .$$

Therefore (c) holds. ■

Example 3.1.2 *Consider again the investor of Example 3.1.1 that bets his budget on the* n *outcomes* O_i*. We model the* n *possible events "bet everything on outcome* i*" by defining a set of ground states* $V := \{|i\rangle \,|i = 1, .., n\}$*. The event that the investor bets the* p_i*-th part of his budget on outcome* O_i*, is then described by the state:*

$$|p\rangle := \sum_{i=1}^{n} p_i |i\rangle \qquad (p_i \geq 0, \sum_i p_i = 1).$$

Example 3.1.3 (Classical cooperative games) *Let* (N, v) *be a classical cooperative game. Assume that*

$$|v| := \sum_{S\subseteq N} v(S) \neq 0.$$

Define ground states by $V := \{|S\rangle \,|S \subseteq N\}$. *Then* (N, v) *gives rise to a state*

$$v' := \sum_{S \subseteq N} \frac{v(S)}{|v|} |S\rangle .$$

Conversely, each state is a cooperative game on N.

3.2 Evolution

This section wants to give a formal framework for describing evolutionary processes on Markovian (resp. generalized Markovian) state spaces. We will therefor characterize state preserving maps and give a definition of *evolution operators* which is due to Faigle and Schönhuth [38].

3.2.1 State preserving maps in the (generalized) Markovian case

In order to describe evolutionary processes, we are interested in linear operators $L : \mathbb{K}^V \to \mathbb{K}^V$, that leave the state space invariant, *i.e.*:

$$L(\mathcal{V}) \subseteq \mathcal{V}. \qquad (*)$$

Those maps induce transformations of states in a natural way, since V is a basis of $\mathbb{K}^V$ and (*) guarantees that $L\,|x\rangle \in \mathcal{V}$ for all $|x\rangle \in \mathcal{V}$.

Assume for the rest of this section: $\mathcal{V}$ is a Markovian or generalized Markovian state space.

We give an example of state preserving maps:

Example 3.2.1 (Stochastic matrices) *Assume* $\mathcal{V}$ *to be a (generalized) Markovian state space. Let* $(s_{ij}) =: S \in \mathbb{K}^{V \times V}$ *be a stochastic matrix. That is:*

$$\sum_{i \in V} s_{ij} = 1, \quad s_{ij} \geq 0, \textit{for all } j \in V.$$

And let $|p\rangle \in \mathcal{V}$. *Then*

$$\langle \mathbf{1}|\, S\,|p\rangle = \langle \mathbf{1}|\, p\rangle = 1.$$

And therefore $S\,|p\rangle \in \mathcal{V}$. *Note that the assumption* $s_{ij} \geq 0$ *was not needed if we assume a generalized Markovian state space. We therefore relax the assumptions and call* S

a generalized stochastic matrix if $\sum_{i \in V} s_{ij} = 1$*. Thus we have proved: (generalized) stochastic matrices leave* $\mathcal{V}$ *invariant if* $\mathcal{V}$ *is (generalized) Markovian. Note further that by dropping the non-negativity assumption, this statement also stays true if* $\mathbb{K} = \mathbb{C}$*.*

Denote by

$$\mathcal{O} := \mathcal{O}(V) := \{L \in \mathbb{K}^{V \times V} | L(\mathcal{V}) \subseteq \mathcal{V}\}$$

the set of all state preserving maps.

Lemma 3.2.1 $\mathcal{O}$ *is a convex set (if* $\mathbb{K} = \mathbb{R}$*) that is closed under matrix multiplication (also true if* $\mathbb{K} = \mathbb{C}$*).*

Proof. That invariance of $\mathcal{V}$ is preserved under matrix multiplication is clear. Let $\lambda \in [0, 1]$, $|p\rangle \in \mathcal{V}$ and $L, M \in \mathcal{O}$. Then

$$\langle \mathbf{1}| (\lambda L + (1 - \lambda)M) |p\rangle = \lambda \langle \mathbf{1}| L |p\rangle + (1 - \lambda) \langle \mathbf{1}| M |p\rangle = \lambda + 1 - \lambda = 1.$$

And the statement follows. ∎

We aim for a characterization of state preserving maps in the (generalized) Markovian case:

Theorem 3.2.1 *The state preserving maps in* $\mathcal{O}$ *are precisely the (generalized) stochastic matrices.*

Proof. That (generalized) stochastic matrices are state preserving, was already shown in Example 3.2.1. Let $M \in \mathcal{O}$. Then for all $|p\rangle \in \mathcal{V}$

$$\langle \mathbf{1}| M |p\rangle = 1 \tag{*}$$

holds, since $M |p\rangle \in \mathcal{V}$. Let $u \in V$. Then

$$\sum_{x \in V} m_{xu} = \sum_{x \in V} \langle x| M |u\rangle = (\sum_{x \in V} \langle x|) M |u\rangle = \langle \mathbf{1}| M |u\rangle \overset{(*)}{=} 1.$$

Hence M is a (generalized) stochastic matrix. ∎

3.2.2 Evolution operators

We aim for describing dynamic cooperative processes as sequences of states. Given a discrete time horizon $t = 0, 1, 2, ...$ and an observed sequence of states $(|p_t\rangle)_t$, we imagine that the states evolution, from time t to time $t+1$, can be described by a state preserving map L_t. We will call sequences of such maps, that describe an evolution, *evolution operators*. The following notion of *evolution operators* is already used in Schönhuth [81] and by Faigle and Schönhuth in [38]. The proposal of investigating evolution operators in cooperative situations is due to Faigle [29].

An *evolution operator* is a map

$$\Phi : \mathbb{K}^V \times \mathbb{N} \to \mathbb{K}^V$$

such that for all $t \geq 0$ the map

$$\Phi_t := \Phi(\cdot, t) = L_t$$

is state preserving (and therefore $\Phi_t \in \mathcal{O}$). We associate an evolution sequence with Φ and a given starting state $|p_0\rangle$:

$$|p_0\rangle := L_0 |p_0\rangle \quad \text{and} \quad |p_t\rangle := L_t |p_0\rangle .$$

Thus we made the assumption $L_0 = Id$.

REMARK Note that this notion of evolution operator does not restrict the sequences that can be described, since for any state $|p_t\rangle$, one could construct a matrix, such that $|p_t\rangle$ is the image of $|p_0\rangle$ under this matrix.

3.2.3 (Generalized) Markovian evolution

Given an evolution operator Φ, we call the induced evolution *(generalized) Markovian* if there is a (generalized) stochastic matrix $M \in \mathcal{O}$, such that for all $t \geq 0$

$$\Phi_t = M^t.$$

In this case we call M the *transition matrix* of Φ. Markovian evolution processes give rise to random walks in a natural way:

If the walk is in the ground state $|x\rangle \in V$, it moves to the ground state

$|y\rangle \in V$ with (possibly negative) probability m_{yx}.

Thus after t steps the walk, that started in a certain ground state $|s\rangle$, has evolved to the state

$$\Phi_t |s\rangle = \sum_{x \in V} p_x^t |x\rangle ,$$

where p_x^t is the (possibly negative) probability, that the walk is in x after t steps.
REMARK Later in Section 10.1 we will see a concrete example of an evolution operator, which is not induced by a single stochastic matrix, but by an inhomogeneous Markovian process. However, we will mainly consider Markovian evolution operators in this thesis.

3.3 Systems

The last section made the terms state space and evolution operator available to us. We define a *system* to be a tuple, (V, Φ), consisting of a set of ground states V and an evolution operator Φ. A system is called (generalized) Markovian if its evolution operator is (generalized) Markovian in the sense of Section 3.2.3.

3.3.1 Transition graphs of systems

Let (V, Φ) be a Markovian system with transition matrix $M = (m_{xy})$. Then Φ induces a directed graph on V *via* the arc set

$$A := \{xy \in V \times V | m_{yx} \neq 0\}.$$

The graph $G := G(\Phi) := (V, A)$ is called the transition graph of the system (or of the evolution operator Φ). Note that M is transposed in the definition of A. This is only for intuitive reasons, since by transposing M the row sums of M^T are equal to 1. Thus M^T_{xy} can really be interpreted as the probability for a transition from state x to state y. In our study of Markovian systems, transition graphs will play a central role. Note that the random walk induced by Φ equals a random walk on G with transition probabilities M^T_{xy}.

Example 3.3.1 *Let* $V = \{x, y, z\}$ *and* $M^T = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 2/3 & 1/3 & 0 \end{pmatrix}$. *The corresponding transition graph could be seen in Figure 3.1.*

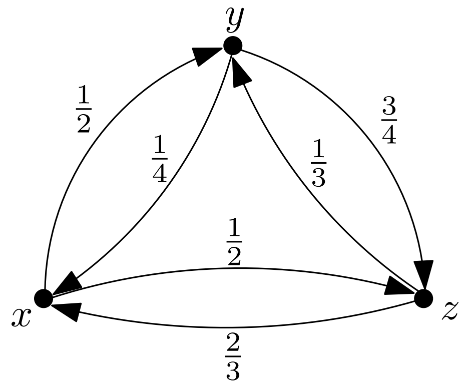

Figure 3.1:

The opposite is also true: given a directed Graph G with vertex set V and arc set A and given a random walk with transition matrix M on G, $(V, (M^{T^t})_{t\geq 0})$ is a Markovian system. We note this down in

Theorem 3.3.1 *(Generalized) Markovian systems and (generalized) random walks on directed graphs correspond.* ■

3.4 Graphs

While in the last section the idea of observing and describing the evolution of a process was ostensible, Theorem 3.3.1 gives rise to another perception, which reflects the spirit of the system-theoretic model presented in [37] (see also: Section 2.3) more. We head for this second perception in this section. Assume that, given a set of ground states V, the feasible transitions from one state to another are fixed in advance by the circumstances that should be modeled. Think for instance of cooperative games like the government formation game (*cf.* Example 2.1.1). Denote the set of feasible transitions by

$$A := \{xy \in V \times V | x \to y \text{ is feasible. }\}.$$

Again $G = (V, A)$ is a graph which is induced by a *feasibility structure*.
With this perception the whole language of directed graph theory is available to us. Therefore, we will sometimes call ground states vertices and feasible transitions arcs. We briefly recall the graph-theoretic terminology that is needed.

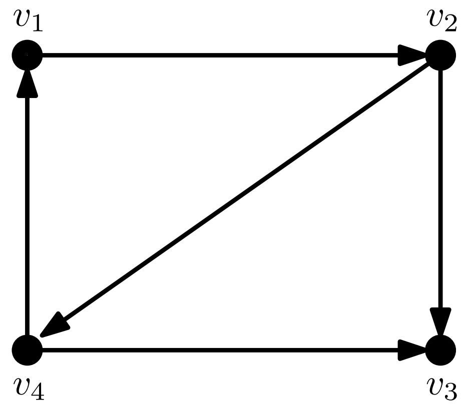

Figure 3.2:

By a *directed path* in G we understand - depending on what "parts" of it we want to emphasize - a tuple of adjacent vertices or a tuple of incident arcs. Whenever there could be any confusion on which of these tuples is meant, we will be concrete.
We assume that there is a given *starting* ground state $|s\rangle \in V$ in which the observation of certain processes on G starts. Without loss of generality we henceforth assume that G is *strongly s-connected*, that is: for all $x \in V$ exists a directed path from s to x. Actually, this is not a loss of generality: if G is not strongly s-connected, remove all vertices, that are not reachable by a directed path from s, and consider the smaller graph instead.
A graph, which is strongly connected for all of its vertices, is called *strongly connected*. Strong connectedness of a graph and strong connectedness relative to its vertices are terms that differ extremely. Consider the following example.

Example 3.4.1 *The graph in Figure 3.2 is strongly v-connected for each vertex v, except for v_3. But G is not strongly connected. Thus it is not sufficient for strong connectedness of a graph, that it is strongly connected for a certain subset of its vertices.*

Sources, sinks and flows

For $x \in V$ denote by $N^+(x)$ resp. $N^-(x)$ the set of successors resp. predecessors of x in the graph G. If x has no successors (*i.e.*: $N^+(x) = \emptyset$), it is called a *sink*, if it has no predecessors (*i.e.*: $N^-(x) = \emptyset$), it is called a *source*. It follows directly from the s-connectedness of G:

Lemma 3.4.1 *If G is strongly s-connected, either s is the unique source in G, or G has no sources.* ∎

A mapping $f : A \to \mathbb{R}$ is called a *flow* if for all $x \in V$ with $N^+(x) \neq \emptyset \neq N^-(x)$ the *flow conservation* holds. That is:

$$\sum_{u \in N^-(x)} f(ux) = \sum_{v \in N^+(x)} f(xv).$$

Lemma 3.4.2 *The set of all flows on a graph G is a $\mathbb{K}$-vector space.*

Proof. This is seen by direct calculation or by this argument: let $\overline{V}$ be the set of all *inner* vertices of G (*i.e.*: all vertices that are neither a source nor a sink) and consider the map

$$\Delta : \mathbb{K}^A \to \mathbb{K}^{\overline{V}}, f \mapsto (\sum_{v \in N^+(x)} f(xv) - \sum_{u \in N^-(x)} f(ux))_{x \in \overline{V}}.$$

Let $f, g \in \mathbb{K}^A$ and $c \in \mathbb{K}$. Then

$$\begin{aligned}
\Delta(cf + g) &= (\sum_{v \in N^+(x)} (cf + g)(xv) - \sum_{u \in N^-(x)} (cf + g)(ux))_{x \in \overline{V}} \\
&= c(\sum_{v \in N^+(x)} f(xv) - \sum_{u \in N^-(x)} f(ux))_{x \in \overline{V}} + (\sum_{v \in N^+(x)} g(xv) - \sum_{u \in N^-(x)} g(ux))_{x \in \overline{V}} \\
&= c\Delta(f) + \Delta(g)
\end{aligned}$$

holds. Hence Δ is a vector space homomorphism whose kernel equals the set of all flows on G. Hence the set of all flows is also a vector space. ∎

We will tacitly use throughout the whole thesis the following fact, which is well-known from graph theory:

Lemma 3.4.3 *Let $G = (V, A)$ be a graph and $B \in \mathbb{K}^{V \times V}$ a matrix with the property:*

$$b_{xy} = 0 \textit{ if } xy \notin A. \tag{*}$$

Let $t > 0$. If there is no path of length t from x to y, then

$$\left\langle x \middle| B^t y \right\rangle = 0.$$

Proof. We use induction on t with trivial beginning by (*). Thus assume $t > 1$ and that the statement is true for all $t' < t$. Let $x, y \in V$ and assume there is no path of length t from x to y. Thus there is also no path of length $t-1$ from x to u for all $u \in N^-(y)$. It holds:

$$\left\langle x \middle| B^t y \right\rangle = \sum_{u \in V} \left\langle x \middle| B^{t-1} u \right\rangle \left\langle u \middle| By \right\rangle .$$

Hence by induction $\left\langle x \middle| B^{t-1} u \right\rangle = 0$, for all $u \in N^-(y)$ and by (*) $0 = b_{uy} = \langle u | By \rangle$, for all $u \in V \setminus N^-(y)$. Therefore, $\left\langle x \middle| B^t y \right\rangle = 0$ holds. ∎

3.5 Cooperation systems

Since the cooperative flavor of the considered objects stepped a bit into the background in the last sections, we will dedicate our attention to it all the more now. The goal of this section is to show, how the different presented cooperative models fit into our model, and how our model is able to deal with the criticisms given in Section 2.4.

Let N be a finite set of players, V be finite set of ground states, $A \subseteq V \times V$ a set of feasible transitions and for $S \subseteq N$ let $A_S \subseteq A$ such that $\mathcal{A} := (A_S)_{S \subseteq N}$ is a partition of A. We think of a transition $x \to y \in A_S$ as an *action* that could be performed by the players in S in order to bring the system from state x to state y. Furthermore, assume that there is an emphasized starting state $s \in V$ such that (V, A) is strongly s-connected. As mentioned in 3.4 this is not any loss of generality. We call the tuple, $\Gamma := (N, V, A, \mathcal{A}, s)$, a *cooperation system*.

Times

We fix a discrete time-horizon by assuming that Γ is in the state s at time 0, in one of its neighbors at time 1, in one of their neighbors at time 2 and so on. For $t \geq 0$ we will denote by $\mathcal{P}_t$ the set of all directed paths of length t, starting in s and by $E_t \subseteq V$ the set of all endpoints of paths in $\mathcal{P}_t$. For technical simplification we assume that each sink in G is provided with a loop. Thus the system stays in a sink-state for all times if it is ever reached.

3.5.1 Cooperative games on cooperation systems

A cooperative game on Γ is a valuation function of the ground states

$$v : V \to \mathbb{R},$$

such that $v(s) = 0$. We denote the set of all cooperative games on Γ by

$$\mathcal{G}(\Gamma) := \{v : V \to \mathbb{R} | v(s) = 0\}.$$

If there is no confusion which cooperation system is meant, we omit the specification of Γ and write $\mathcal{G}$ instead of $\mathcal{G}(\Gamma)$. $\mathcal{G}$ is a $(|V| - 1)$-dimensional vector space. We will consecrate ourselves to it in Chapter 4.
We gathered all basic definitions needed. The next paragraphs will show that the mentioned classical models are special cases of cooperation systems.

Set-theoretic models

Let $\mathcal{F} \subseteq 2^N$ be a set of feasible coalitions s.t. $\emptyset \in \mathcal{F}$ and let $(\mathcal{F}, v)$ be a cooperative game in the sense of Section 2.2. By setting

$$V := \mathcal{F}$$

as set of ground states,

$$A := \{ST \in \mathcal{F} \times \mathcal{F} | S \subseteq T, \text{ for all } S \subseteq U \subseteq T : U = S \text{ or } U = T\}$$

as set of feasible transitions and by setting

$$A_S := \{UW \in A | W \setminus U = S\}$$

as action set of $S \subseteq N$ and finally $s := \emptyset$, one gets a cooperation system $\Gamma := (N, V, A, (A_S)_{S \subseteq N}, s)$. Then $(\mathcal{F}, v)$ is also a cooperative game on Γ. Note that it is possible that $A_S = \emptyset$. If this distracts the nature of the circumstances that should be modeled, there is technically no constraint in assuming A to be partitioned into non empty blocks.
By this construction, our model covers also the set-theoretic models mentioned in Section 2.2.3, especially closed set-systems w.r.t. a certain closure operator. In the case of the model of Faigle and Grabisch [31] in which all transitions $S \to T$ ($S, T \subseteq 2^N$) are feasible, we need to adjust the action set. For $S \subseteq N$ set

$$A_S := \{UW \in 2^N \times 2^N | U \Delta W = S\}.$$

Where Δ means the symmetric difference of two sets: $S \Delta T := (S \cup T) \setminus (S \cap T)$, $S, T \subseteq N$.

The first system-theoretic approach

The model of Faigle and the author [37] that was briefly introduced in 2.3, yields a special case of our model in the following sense: the actions mentioned there are governed by players in N, and therefore A was partitioned into n blocks A_i, $i \in N$. Simply by setting

$$A_{\{i\}} := A_i \text{ and } A_S := \emptyset \text{ for } |S| \neq 1$$

the game systems of Section 2.3 become cooperation systems in our sense. Therefore, all models that are based on greedoids, even the precedence approaches, as well as all mentioned models on combinatorial structures, yield special cases of cooperation systems.
As seen in Subsection 2.4.1, multi-choice games could be considered as games on an acyclic graph on certain lattice points in $\mathbb{N}^{|N|}$. Hence also multi-choice games yield a special case of cooperative games on cooperation systems.

Games in partition function form

Let N be a finite set of players and $\mathcal{C}(N)$ the set of all embedded coalitions of N (*cf.* Section 2.5). Set $V := \mathcal{C}(N)$ and define for $i \in N$:

$$\begin{aligned} A_i := \{((S,\pi_1),(S \cup i, \pi_2)) \in \mathcal{C}(N) \times \mathcal{C}(N)| \\ \forall S \neq X \in \pi_1, i \notin X : X \in \pi_2 \text{ and for } i \in X \in \pi_1 : X \setminus i \in \pi_2\} \end{aligned}$$

the action set of player i. Moreover, set $A := \bigcup_{i \in N} A_i$, $s := (\emptyset, (\emptyset, N))$ and $\mathcal{A} := (A_1, ..., A_n)$. Then $(\mathcal{C}(N), V, A, \mathcal{A}, s)$ is a cooperation system and cooperative games on $\mathcal{C}(N)$ are games on this cooperation system. Hence also games in partition function form yield a special case of our model.

3.5.2 Advantages of our model

Note that the acyclicity assumption, as well as the assumption that s is a source of the graph (V, A), is dropped in our model. Therefore cooperation systems are more general than all mentioned models of Chapter 2. Our model allows a much wider range of interpretations and concentrates on actions that could be performed by players, instead of the players themselves. Time-dependent circumstances can be modeled by cooperation systems and past events could be taken into account in valuating certain actions. Hence we came across all points of criticism, we pointed out in the previous chapter. Even processes of joining, leaving and rejoining a coalition at a later point are covered by cooperation systems.

Moreover, the strong relation of our model to graph theory enables a more abstract view on cooperative settings. This abstraction yields several tools known from graph theory (*cf.* Section 6.1), and transports them naturally to the theory of cooperative games.

The connection of Markovian systems and directed graphs yields another advantage: the allocation problem and the prediction problem are related to each other. We will see a direct connection between these two problems in Section 5.5.

4 Incidence algebras and the space of cooperative games

In this short chapter we recall a very fruitful concept of combinatorics: so called *incidence algebras*. We give the framework to apply this concept to our general model of cooperative games.
Section 4.1 briefly recalls basic terms of incidence algebras and gives well-known relations to classical cooperative games. Also two important bases of the vector space of all classical cooperative games are presented. In Section 4.2, we give a relation between acyclic graphs and certain partially ordered sets, which gave Faigle and the author [37] the opportunity to speak of the incidence algebra of an acyclic graph and to use the concept of incidence algebras in their model.
After that we generalize this idea to arbitrary graphs in Section 4.3 in a way, such that the classical theory of incidence algebras is applicable to our model and has similar relations to the vector space of cooperative games on cooperation systems and yields similar interpretations.

4.1 Classical cooperative games & incidence algebras

Rota [77] developed the theory of *incidence algebras*. Given a partial order $P := (X, \leq)$ of a set X and given a field $\mathbb{K}$, he studied the following set of functions:

$$\mathcal{I}_{\mathbb{K}}(P) := \{f : X \times X \to \mathbb{K} | f(x, y) = 0, \text{ if } x \not\leq y (x, y \in X)\}.$$

He showed that this set is a $\mathbb{K}$-vector space which becomes a $\mathbb{K}$-algebra *via* the multiplication given by:

$$(f * g)(x, y) := \sum_{x \leq z \leq y} f(x, z)g(z, y).$$

$\mathcal{I}_{\mathbb{K}}(P)$ is called the incidence algebra of P with respect to $\mathbb{K}$. We note two essential statements:

Lemma 4.1.1 ([3, p. 138 f.]) *Let $P = (X, \leq)$ be a partial order.*

(a) The map $\delta : X \times X \to \mathbb{K}$, $(x, y) \mapsto \begin{cases} 1 & , x = y \\ 0 & \textit{else}. \end{cases}$ is a (both-sided) neutral element of $(\mathcal{I}_{\mathbb{K}}(P), *)$.

(b) The map

$$\Lambda : \mathcal{I}_{\mathbb{K}}(P) \to \mathbb{K}^{X \times X}, f \mapsto (f(x, y))_{x,y \in X}$$

is a monomorphism from the incidence algebra to the algebra of upper triangular matrices over $\mathbb{K}$.

■

REMARK An important consequence of the lemma above is: Λ is an isomorphism into a sub-algebra of the upper triangular matrices over $\mathbb{K}$. Therefore the multiplication $*$ in $\mathcal{I}_{\mathbb{K}}(P)$ and the matrix multiplication are structurally the same. Thus one could identify these multiplications. By this, the structure of an incidence algebra equals the structure of a matrix algebra which is much more familiar (at least to the author).

Given a set N of players, it is well known that the set

$$\mathcal{G} := \{v : 2^N \to \mathbb{R} | v(\emptyset) = 0\}$$

is a $(2^{|N|}-1)$ dimensional $\mathbb{R}$-vector space - the space of all cooperative games on N. In order to study cooperative games, one is interested in certain bases of $\mathcal{G}$ to write an arbitrary cooperative game as a linear combination of simpler games. Note that 2^N is partially ordered by $\subseteq$. Therefore, the incidence algebra of $(2^N, \subseteq)$ is well defined.

4.1.1 Dirac games

For $\emptyset \neq S \subseteq N$ define

$$\delta_S(T) := \begin{cases} 1 & S = T \\ 0 & \text{else.} \end{cases} \quad (T \subseteq N).$$

These games are called *Dirac games*. An obvious basis of $\mathcal{G}$ is the set of all *Dirac games*:

$$\{\delta_S | S \in 2^N \setminus \{\emptyset\}\}$$

(by identifying this set with the standard basis of $\mathbb{R}^{2^N}$). The connection between cooperative games and the incidence algebra associated with $(2^N, \subseteq)$ is described by

Lemma 4.1.2 *The Dirac games correspond to the columns of δ if we interpret $\delta = (\delta(S,T))_{S,T \in 2^N}$ as a $2^N \times 2^N$ matrix.* ■

4.1.2 Unanimity games

Another set of well studied cooperative games are the so called *unanimity games*. For a coalition $S \subseteq N$ the *unanimity game with respect to S* is defined by:

$$\zeta_S(T) := \begin{cases} 1 & , S \subseteq T \\ 0 & \text{else.} \end{cases}$$

Theorem 4.1.1 *The set of unanimity games*

$$\{\zeta_S | S \in 2^N \setminus \{\emptyset\}\}$$

is a basis of $\mathcal{V}$.

Proof. Again by interpreting $\zeta = (\zeta_S(T))_{S,T \subseteq N}$ as a matrix, and by ordering rows and columns of ζ, ζ becomes a triangular matrix with 1 on each diagonal entry. Therefore, ζ is invertible in $\mathcal{I}_{\mathbb{K}}(P)$, hence its columns are linearly independent. ■

Interpretation

The idea behind *unanimity* games ζ_S is the following. A coalition is *winning* (*i.e.*: is valued with 1) if and only if S is a subset of this coalition. Thus one could argue: if S is part of a coalition, it has the power to achieve unanimity of the players of this coalition.

4.2 Partial orders and acyclic graphs

An essential observation, which will lead us to our generalization, is the following connection between partial orders and acyclic graphs.

Theorem 4.2.1 *The directed graph of a Hasse diagram of a partial order is acyclic and each acyclic directed graph induces a certain partial order.*

Proof. Assume that the Hasse diagram of a partial order $P := (X, \leq)$ contains a directed cycle $x_0, x_1, ..., x_k = x_0$ with $x_i \neq x_j$ for $i \neq j$. Then $x_0 \leq x_1 \leq ... \leq x_k = x_0$ holds. By the antisymmetry of P it follows that $x_0 = x_i$ for all $i \in \{1, ..., k\}$. A contradiction. Thus the Hasse diagram of P is acyclic.
Let the other way around $G = (V, A)$ be an acyclic graph. For $x, y \in V$ define

$$x \leq y :\Leftrightarrow \text{There is a directed path from } x \text{ to } y.$$

This relation is apparently reflexive and transitive and by the acyclicity of G it is also antisymmetric. Therefore $(V, \leq)$ is a partial order. ■

This theorem gives us the opportunity to speak of the *incidence algebra of an acyclic graph* and to assign the whole theory developed in [77] to acyclic graphs. The transfer of the concepts of incidence algebras to cooperative games on acyclic graph structures was already done by Faigle and the author in [37]. We aim for dropping the acyclicity assumption and to generalize these ideas to arbitrary graphs.

4.3 Incidence algebra of a graph

As seen in the last section, the antisymmetry of a partial order is the essential property to prove the acyclicity of its directed Hasse diagram. In order to drop the acyclicity assumption and to develop a more general theory of incidence algebras of graphs, the approach to associate a preorder with a graph $G = (V, E)$, in analogy to the partial order defined in 4.2.1, seems fruitful:

$$x \leq y :\Leftrightarrow \text{There is a directed path from } x \text{ to } y.$$

By $\mathcal{I}_{\mathbb{K}}(G)$ we denote the incidence algebra induced by G. In fact, this approach was taken by several authors (mainly started by Belding in [10]), in order to generalize the ideas of the theory of incidence algebras to arbitrary preorders. We will make a different approach. But first of all, we give a reason, why this approach to incidence algebras of graphs is not suitable for the application of this theory to cooperative games in our model.
A basic observation already made by Belding in [10] is the following. Since the proof is embedded in a more complicated setting and is not direct in [10], we give a simple proof.

Theorem 4.3.1 *Given a graph $G = (V, A)$, ζ is invertible in $\mathcal{I}_{\mathbb{K}}(G)$ if and only if G is acyclic.*

Proof. If G is acyclic, the rows of ζ (interpreted as a matrix) could be arranged, such that ζ is upper triangular with each diagonal entry equal to 1. Thus ζ is invertible.
Assume the other way around, that G is not acyclic. Then there exists a directed cycle $x_0, ..., x_k$ in G with $x_{i-1}x_i \in A$, for $i = 1, .., k$. Thus $x_i \leq y$ if and only if $x_j \leq y$ for all $i, j \in 1, ..., k$ and $y \in V$. Hence the rows of ζ which correspond to x_i and x_j are equal, and therefore especially linearly dependent, thus ζ is not invertible. ∎

Recall that the rows of ζ yield the unanimity basis of the space of all cooperative games in the example of classical cooperation structures and therefore give rise to representations of arbitrary cooperative games in this basis. This is the main reason why the generalization of incidence algebras to preordered sets is not adequate from a cooperative game theoretic point of view: a simultaneous generalization of the idea of unanimity games and the ideas of incidence algebras is impossible *via* this approach.

4.3.1 Our approach to incidence algebras of graphs

Let $G = (V, A)$ be a graph. We define a relation on V *via*

$$x \leq y :\Leftrightarrow \begin{cases} x = y & \text{or} \\ \text{There exists a path from } x \text{ to } y\text{, but none from } y \text{ to } x. \end{cases}$$

This relation is an extension of the idea of the partial order induced in the acyclic case.

Lemma 4.3.1 *$\leq$ is a partial ordering of V. Moreover, if G is acyclic, it equals the partial order induced by G.*

Proof. Reflexivity holds by definition of $\leq$. Let $x, y, z \in V$. If $x \leq y$ and $y \leq x$, the conditions

There exists a path from x to y, but none from y to x.

and

There exists a path from y to x, but none from x to y.

are mutually exclusive. Therefore $x = y$ holds, and thus $\leq$ is antisymmetric. Assume $x \leq y \leq z$. If $x = y$ or $y = z$, $x \leq z$ follows directly. Therefore assume that $x \neq y$ and $y \neq z$. Thus there is a path from x to y and one from y to z, but none from y to x and none from z to y. Hence the concatenation of those paths yields a path from x to z. Assume now that there is a path from z to x. Then x, y and z are on a circle which contradicts the $\leq$ relation between them. Therefore, also $x \leq z$ holds. Hence transitivity for $\leq$ is proven.
Assume now that G is acyclic. The condition that there is no return path from y to x if $x \leq y$, is redundant by the acyclicity of G. Thus the partial order induced by an acyclic graph equals $\leq$. ∎

By the last lemma, there is a partial order of the vertices of an arbitrary graph such that it corresponds to the partial order induced by acyclic graphs. Therefore, we can speak of *the partial order induced by G*. If the context is clear, we will denote this order by $\leq$. Since $\leq$ is a partial order, the term *incidence algebra of a graph* is well defined for arbitrary directed graphs, and by that we mean the incidence algebra induced by $\leq$. Now we could assign the theory of incidence algebras to arbitrary graphs:

4.3.2 ζ-Functions of arbitrary graphs

The main advantage of this approach (from a cooperative game theoretic point of view) is, that the ζ-function is well defined for arbitrary graphs. Therefore also ζ-games are well defined in our model. The first question that comes into mind is:

“What is the interpretation of these games?”

Given a cooperation system $\Gamma := (N, V, A, \mathcal{A}, s)$ and a cooperative game v on Γ in the sense of 3.5, we consider the partial order $\leq$ induced by the transition graph (V, A). For any $x \in V \setminus \{s\}$ we set:

$$\zeta_x(y) := \zeta(x, y) := \begin{cases} 1 & , x \leq y \\ 0 & \text{else.} \end{cases}$$

Note that in fact ζ_x is a cooperative game in the sense of 3.5 (*i.e.*: is s-normalized): since (V, A) is strongly s connected, there is a path from s to x. Assume $\zeta_x(s) = 1$. Then there exists a path from x to s and none from s to x. Which contradicts the previous. Thus $\zeta_x(s) = 0$ for all $x \in V \setminus \{s\}$.

Lemma 4.3.2 *The set $\{\zeta_x | x \in V \setminus \{s\}\}$ is a basis of the vector space of all cooperative games on Γ. Thus for any cooperative game v, there exist $c_x(v) \in \mathbb{K}$ ($x \in V \setminus \{s\}$), such that*

$$v = \sum_{x \in V \setminus \{s\}} c_x(v) \zeta_x.$$

Proof. By Lemma 4.3.1, $\leq$ is a partial order and therefore ζ is invertible in $\mathcal{I}_{\mathbb{K}}(G)$. ∎

Interpretation

Given a ζ-game, ζ_x, the *unanimity interpretation* of classical cooperative games is preserved and generalized: a state $y \in V$ is winning (*i.e.*: $\zeta_x(y) = 1$) relative to x if and only if y is reachable from x and there is *unanimity* among the players that the system should not be taken back to x (there is no return path from y to x). The other way around, one could also argue that if there is a path from y back to x, there is no unanimity among the players, since bringing the system into the state y was not as good as it seemed at first. Hence the players took actions to return the system to a state, it previously was in.

REMARK Note that the ζ-function is strongly related to the structure of the graph: if G is strongly connected, there are no vertices, that are strictly smaller than any other vertex in the graph. Therefore $\zeta = \delta$ holds in the case of strong connectedness. On the other hand, we have already seen that if G is acyclic (which is the other extreme), $\leq$ (and therefore also ζ) equals its classical versions. Hence ζ reflects the “circular structure” of a graph in this sense.

4.4 Möbius-inversion

Let $P = (V, \leq)$ be a partially ordered set and let $f : V \to \mathbb{K}$. The map

$$\Sigma_f : V \to \mathbb{K}, x \mapsto \sum_{y \leq x} f(y)$$

is called the *sum-function of* f. Originally one was interested in the question: is it possible to give an inversion of the sum-function in a way, such that f could be calculated out of Σ_f?
We formulate this in the terms of a graph G and its associated partial order $\leq$. Therefore let $G = (V, A)$ be a directed graph and let $s \in V$ be a vertex, such that G is strongly s-connected. We define the sum-function of a vertex valuation $f : V \to \mathbb{K}$ to be

$$\Sigma_f : V \to \mathbb{K}, x \mapsto \sum_{y \leq x} f(y).$$

By Theorem 4.3.1 we know, that ζ is invertible in $\mathcal{I}_{\mathbb{K}}(G)$. We denote the inverse of ζ by μ, and call μ the Möbius function of G. Now we have collected all results and definitions to give an answer to the initial question:

Theorem 4.4.1 (Möbius-inversion) *Let $f, g : V \to K$. Then the following holds for all $x \in V$:*

$$g(x) = \sum_{y \leq x} f(y) \Leftrightarrow f(x) = \sum_{y \leq x} g(y)\mu(y, x),$$

for all $x \in V$.

Proof. A proof could be found in the book of Aigner [3, Theorem 4.18] ∎

REMARK Note that (given that μ is known) the Möbius-inversion gives rise to the concrete coefficients in the basis representation of cooperative games, in the ζ-basis in the sense of 4.3.2. Write $v = \sum_{x \in V} \Delta_x \zeta_x$ in its ζ-basis representation. Fix $u \in V$. Then $v(u) = \sum_{x \in V} \Delta_x \zeta_x(u) = \sum_{x \leq u} \Delta_x \zeta_x(u)$. Set $f(x) := \Delta_x \zeta_x(u)$ for all $x \in V$. Then $v(u) = \sum_{x \leq u} f(x)$. Hence by Möbius-inversion one gets: $f(u) = \sum_{y \leq u} v(y)\mu(y, u)$. On the other hand: $f(u) = \Delta_u \zeta_u(u) = \Delta_u$. Thus a concrete representation of Δ_u is found. The coefficients Δ_x are known in classical cooperative game theory as Harsanyi Dividends [48].

Positive decomposition of cooperative games

As known from the last section, the ζ games are a basis of the space of all cooperative games on Γ. By writing a game in this basis one gets a useful decomposition:

Lemma 4.4.1 *Let $v \in \mathcal{G}$. Then there are $v^+, v^- \in \mathcal{G}$ with the property $v^+, v^- \geq 0$ such that $v = v^+ - v^-$.*

Proof. Write $v = \sum_{x \in V} c_x \zeta_x$. And set

$$v^+ := \sum_{x \in V, c_x \geq 0} c_x \zeta_x \text{ and } v^- := \sum_{x \in V, c_x < 0} (-c_x) \zeta_x.$$

This yields the desired representation. ∎

This decomposition becomes useful, if one wants to argue why games are assumed to be non-negative in a certain setting.

5 Allocation mechanisms: Weber theory

After stating our general model for cooperative settings, the following question was left open in the previous chapters:

"How to solve the allocation problem for a cooperative game?"

In this chapter, we aim for a fractional answer to this question. Section 5.1 gives a brief overview of classical fairness concepts and restates famous characterization results of Weber [91] and Shapley [82]. We introduce the classical concept of *values* and generalize it to our model. Furthermore, we propose new *values* that arise naturally from our modeling and yield also new values in the classical model.
In Section 5.2, we generalize the classical theory of Weber to our model of cooperation systems and develop time-dependent fairness criteria. In the end of Section 5.2 and in Section 5.3 allocation rules, which satisfy those fairness criteria, are identified as being induced by random walks on transition graphs of generalized Markovian systems.
In Section 5.4, we give generalizations of the famous Shapley value and define the entropy of an allocation mechanism. This definition gives rise to two different perceptions of *symmetry*, which in turn give rise to two different notions of Shapley values (which coincide in the case of classical cooperation systems).
In Section 5.5, relations between a certain class of allocation mechanisms and evolution operators in the sense of Chapter 3 are pointed out. Basic convergence properties of these allocation mechanism are investigated in Section 5.6. Finally we give examples of classical values as special cases of allocation mechanisms in our model in Section 5.8. Also the famous Banzhaf value [8] is treated there.

5.1 Classical Weber theory

Given a classical cooperative game (N, v), the allocation problem (*cf.* Section 2.1.1) is: how should a jointly generated value be distributed among the players such that the different cooperation possibilities of the players are taken into account *fairly*? In other words: we search for a vector $x \in \mathbb{R}^N$, which allocates a certain fixed value to the players. Since one heads for general fairness concepts that yield a solution for every cooperative game (N, v), this search is equivalent to the retrieval for a general allocation rule that assigns to each valuation function v an allocation vector $\phi(v) \in \mathbb{R}^N$. Denote by $\mathcal{G}(N)$ the set of all classical cooperative games on N (and note that by image-wise addition resp. scalar multiplication, $\mathcal{G}(N)$ becomes a $2^{|N|} - 1$-dimensional vector space). Thus we are seeking for a map

$$\phi : \mathcal{G}(N) \to \mathbb{R}^N,$$

that enjoys certain fairness criteria. The proposed fairness criteria vary widely through the literature. Depending on the situation that should be valuated, and also depending on the subjective understanding of fairness of different authors, more or less countless fairness axioms were proposed in the past. We will present some of the famous classical axioms and will give a brief introduction to some characterization results. The goal of this section is to briefly present the main ideas of the theory of so called "random order values" developed by Weber [91] and to give some famous examples of allocation rules for classical cooperative games.

5.1.1 Classical fairness axioms

We fix a player set N and a solution function $\phi : \mathcal{G}(N) \to \mathbb{R}^N, v \mapsto \phi(v)$.

Efficiency

The first idea is that a solution to the allocation problem should really distribute a certain value. In classical cooperative game theory one assumes that this value equals the worth $v(N)$ of the grand coalition. Therefore, ϕ is called *efficient* if for all $v \in \mathcal{G}(N)$:

$$\sum_{i \in N} \phi(v)_i = v(N) \tag{EFF}$$

holds.

Additivity & linearity

Let $v, w \in \mathcal{G}(N)$ be two games. If one perceives a solution $\phi(v)$ to be fair for v and $\phi(w)$ to be fair for w, also the sum of the solutions should be perceived to be fair for the sum game $v + w$. ϕ is called *additive* if for all $v, w \in \mathcal{G}(N)$

$$\phi(v + w) = \phi(v) + \phi(w) \tag{ADD}$$

holds.
Extending this idea to multiples of a game, one assumes homogeneity for ϕ in addition. Hence ϕ is called linear if

$$\phi \text{ is a linear map.} \tag{LIN}$$

holds.

Marginal worth & null-player

Given a game $v \in \mathcal{G}(N)$, a player $i \in N$ and a coalition $S \subseteq N \setminus \{i\}$, which does not contain i, the *marginal worth* of player i relative to S and v is defined by

$$v(S \cup i) - v(S).$$

A player is called a *null player* for the game v if his marginal worth is zero for all $S \subseteq N \setminus \{i\}$. The idea behind the so called *Null-player Axiom* is the following: a null player does not contribute any worth to any coalition, therefore he should yield zero payoff.
Thus ϕ fulfills the *Null-player Axiom*, if for all $v \in \mathcal{G}(N)$, $i \in N$:

$$\text{If } i \text{ is a null player for } v\text{, then } \phi(v)_i = 0. \tag{NUL}$$

holds.

Symmetry

Another idea of fairness is *symmetry*. Essentially that means, if two players contribute the same marginal worth to all coalitions they can join, they should yield the same payoff.
ϕ is called *symmetric* if for all $i, j \in N$, $v \in \mathcal{G}(N)$:

$$\text{If } v(S \cup i) = v(S \cup j) \text{ for all } S \subseteq N \setminus \{i, j\}\text{, then } \phi(v)_i = \phi(v)_j. \tag{SYM}$$

holds.

Monotonicity

A player $i \in N$ is called *monotonic* for v if his marginal worth contribution to any coalition S is non-negative, *i.e.*:

$$v(S \cup i) \geq v(S), \text{ for all } S \subseteq N.$$

The solution ϕ is called *monotonic* if the payoff of monotonic players is non-negative. That is:

$$\phi(v)_i \geq 0 \text{ if } i \text{ is a monotonic player.} \tag{MON}$$

5.1.2 Characterization results

We will now give concrete examples of some famous classical solutions. After that, we will briefly restate a result of Weber [91] and introduce the so called "randomized values".

Shapley value

In 1953 Shapley proved the following theorem:

Theorem 5.1.1 ([82]) *There exists precisely one solution which suffices (ADD), (EFF), (SYM) and (NUL).* ■

This unique solution became famous under the name *Shapley value*. We denote the Shapley value by Φ^{Sh}. Furthermore, Shapley proved for all games (N, v) and all players $i \in N$, that:

$$\Phi^{Sh}(v)_i = \sum_{S \subseteq N} \frac{(|N| - |S|)!(|S| - 1)!}{|N|!}(v(S \cup i) - v(S))$$

holds.

Another representation of the Shapley value is achieved through so called *marginal vectors*. For each permutation $\pi = p_1 ... p_n \in Sym(N)$ define a tuple $S_i(\pi)$ as the ordered set of all players, which precede player i in π. The Shapley value receives the following form:

$$\Phi^{Sh}(v)_i = \frac{1}{|N|!} \sum_{\pi \in Sym(N)} v(S_i(\pi) \cup i) - v(S_i(\pi)).$$

How the different representations of the classical Shapley value are derived from each other, is not discussed here. We refer to the original work of Shapley [82] for further information on this topic in the classical model. We are interested in highlighting the different ideas, which are involved in the Shapley value. Thus we summarize: the Shapley value of player $i \in N$ is the mean marginal value over all permutations of players. One imagines that the grand coalition N is build by consecutively taken joins of players. The order in which this join process takes place is irrelevant for the Shapley value, since each permutation gets the same weight $\frac{1}{|N|!}$.

Weber values

In 1988 Weber [91] studied so called *random order values*. In view of the mean value characterization of the Shapley value (*cf.* 5.1.2), random order values generalize this aspect. A solution ϕ is called a *random order value* if there exists a probability distribution $(\lambda_\pi)_{\pi \in Sym(N)}$ on the set of all permutations of players such that

$$\phi(v)_i = \sum_{\pi \in Sym(N)} \lambda_\pi (v(S_i(\pi) \cup i) - v(S_i(\pi))$$

holds. Thus the Shapley value is a special random order value with $\lambda_\pi = \frac{1}{|N|!}$. Weber proved the following characterization of random order values:

Theorem 5.1.2 ([91]) *A solution ϕ is a random order value if and only if it suffices (LIN), (NUL), (EFF) and (MON).* ∎

Moreover, Weber proved another interesting statement:

Theorem 5.1.3 ([91]) *A solution ϕ is a random order value if and only if for all $S \subseteq N$ there exists a probability distribution $(q_i^S)_{S \subseteq N \setminus i}$, such that for all $i \in N$*

$$\phi(v)_i = \sum_{S \subseteq N, i \notin S} q_i^S (v(S \cup i) - v(S))$$

holds. ∎

We will go into detail, why this result is interesting to us. The set of coalitions 2^N, together with $\cap$ and $\cup$, is a lattice. Think of the Hasse diagram of this lattice as a directed graph (directed in an acyclic way from $\emptyset$ to N). Thus in this terms Theorem 5.1.3 says that each random order value describes a random walk on the Hasse diagram of 2^N with transition matrix $\pi_{S,S\cup i} = \frac{q_i^S}{\sum_{j \in N \setminus S} q_j^S}$. The beauty of this theorem is in connecting the theory of values with the theory of random walks on graphs.

5.2 Weber theory for cooperation systems

In this section we aim for a generalization of the solution concepts introduced in Section 5.1 to our general model of cooperation systems. For the rest of the chapter we fix a cooperation system $\Gamma := (N, V, A, \mathcal{A}, s)$ in the sense of Section 3.5. We denote by $\mathcal{G}$ the $|V| - 1$ dimensional vector space of all s-normalized games on Γ. Instead of valuating players depending on a given game v, we pursue the goal to valuate the single actions in A. Afterwards we deduce how to valuate the individual players in N. Therefore, for the moment it is enough to assume a directed graph (V, A) is given. One could also think of a System (V, Φ) and its transition graph (V, A) (*cf.* Chapter 3). The relation between Markovian systems and Weber's theory will be highlighted later in this chapter in Section 5.5.

5.2.1 Allocation mechanisms

We will constitute solutions as *allocation mechanisms* if they fulfill certain properties. Note that the term *allocation mechanism* is yet defined as an established term in the theory of mechanism design (*cf.* [71]). Since a solution is a mechanism to allocate a certain value to the players of a game, we think the term *allocation mechanism* is nevertheless accurate.
We define an *allocation rule* to be a map

$$\phi : \mathcal{G} \times \mathbb{N} \to \mathbb{K}^A, (v, t) \mapsto \phi^t(v),$$

which assigns to a game v at time t, a valuation of the actions, $\phi^t(v) \in \mathbb{K}^A$. We call an allocation rule ϕ an *allocation mechanism* if the following basic axioms are satisfied:

(A1) The value of an action $xy \in A$ at time t is independent of the values of the states in $V \setminus \{x, y\}$; *i.e.*: for all $v, v' \in \mathcal{G}$ with $v(x) = v'(x), v(y) = v'(y)$, $\phi^t(v)_{xy} = \phi^t(v')_{xy}$ holds.

(A2) An action, which could not be performed at time t, yields zero payoff; *i.e.*: for all $t > 0$, $xy \in A$ with $d(s, x) > t$ and all $v \in \mathcal{G}$: $\phi^t(v)_{xy} = 0$ holds.

(A3) The value of an action changes over time, only if this action could be performed (possibly anew); *i.e.*: $\phi^t(v)_{xy} = \phi^{t+1}(v)_{xy}$ if $x \notin E_t$.

Axioms (A1) and (A2) are quite intuitive. However, there may be situations in which (A3) is not plausible. We refer to Chapter 7 for a discussion of allocation rules which do not necessarily satisfy (A3).

5.2.2 The marginal operator

The map $\partial : \mathcal{G} \to \mathbb{K}^A, v \mapsto (v(y) - v(x))_{xy \in A}$ is called *marginal operator* and assigns to each action, $xy \in A$, its marginal worth w.r.t. v: $v(y) - v(x)$. ∂ will be a main tool in the understanding of allocation mechanisms. An important basic principle is the following lemma:

Lemma 5.2.1 *∂ is a $\mathbb{K}$-vector-space-monomorphism.*

Proof. Let $v, w \in \mathcal{G}$ and $c \in \mathbb{K}$. Then for all $xy \in A$:

$$\begin{aligned}\partial_{xy}(cv + w) &= (cv + w)(y) - (cv + w)(x) = cv(y) - cv(x) + w(y) - w(x) \\ &= c\partial_{xy}(v) + \partial_{xy}(w)\end{aligned}$$

holds. Thus ∂ is a homomorphism. Uniqueness is left to prove. Assume $\partial(v) = 0$ and let $x \in V$. Since G is strongly s-connected, there is a directed path $P = (s = x_0, x_1, ..., x_k = x)$ from s to x. By assumption, v is s-normalized and therefore $v(s) = 0$. But then also $v(x_1) = v(x_1) - v(x_0) = \partial_{x_0x_1}(v) = 0$. It follows by induction on k that $v(x_i) = 0$ for all $i \leq k$. Thus $v = 0$, hence ∂ is injective. ∎

5.2.3 Linearity

An allocation mechanism ϕ is called *linear* if

$$\text{For all } t \geq 0, \phi^t \text{ is a } \mathbb{K}\text{-linear map from } \mathcal{G} \text{ to } \mathbb{K}^A. \tag{LIN}$$

holds.

Theorem 5.2.1 *Let ϕ be a linear allocation mechanism. Then for all $t \geq 0$ there exists $\alpha^t \in \mathbb{R}^A$ such that*

$$\phi^t(v)_{xy} = \alpha^t_{xy}\partial_{xy}(v) = \alpha^t_{xy}(v(y) - v(x))$$

for all $v \in \mathcal{G}$.

Proof. Since ϕ^t is linear, the map

$$\phi^t_{xy} : \mathcal{G} \to \mathbb{K}, v \mapsto \phi^t(v)_{xy}$$

is also linear for all $xy \in A$. And since ∂ is a monomorphism, ϕ^t_{xy} could be considered as a linear map from $\partial(\mathcal{V})$ to $\mathbb{K}$. Thus there is $\beta \in \mathbb{K}^A$ with

$$\phi^t_{xy}(v) = \beta^T\partial(v) = \sum_{uw \in A} \beta_{uw}(v(w) - v(u)), \text{ for all } v \in \mathcal{G}.$$

On the other hand by (A1) the value of the action xy only depends on the values of x and y. Thus $\beta_{uv} = 0$ holds for all $uv \in A \setminus \{xy\}$. Therefore $\phi^t_{xy}(v) = \beta_{xy}(v(y) - v(x))$, as stated. ■

An interpretative plausible deduction of this theorem is the following:

Corollary 5.2.1 *Let ϕ be a linear allocation mechanism and let $v(x) = 0$ for all $x \in V$. Then for all $t > 0$ and all $xy \in A$, $\phi^t_{xy}(v) = 0$ holds.* ■

In other words: if no value is generated over time, also no value is allocates to the players. We are interested in describing linear allocation mechanisms *via* sequences $\alpha := (\alpha^t)_{t>0}$ in $\mathbb{K}^A$. The next lemma characterizes those sequences that yield a linear allocation mechanism.

Lemma 5.2.2 *Let $\alpha := (\alpha^t)_{t>0}$ be a sequence in $\mathbb{K}^A$. Then α induces a linear allocation mechanism via*

$$\phi^t(v)_{xy} := \alpha^t_{xy}(v(y) - v(x)) \qquad \text{, for all } xy \in A, t > 0, v \in \mathcal{G}$$

if and only if for all $xy \in A, t > 0$

$$\alpha^{t+1}_{xy} = \alpha^t_{xy} \text{ if } x \notin E_t. \tag{*}$$

holds.

Proof. By the last statements on linear allocation mechanisms it is clear, that if (*) holds, ϕ is an allocation mechanism, which is also linear. The other way around: if ϕ^t is a linear allocation mechanism, then by linearity and (A3) also (*) holds. ■

5.2.4 Non-negativity

We call an allocation mechanism ϕ non-negative (or monotone) if it satisfies the following axiom for all $v \in \mathcal{G}$:

$$\text{For all } xy \in A : v(y) - v(x) \geq 0 \Rightarrow \phi^t_{xy}(v) \geq 0, \text{ for all } t > 0. \tag{NN}$$

In words: if the marginal contribution of an action is non-negative, then also its value is.

REMARK In view of the classical monotonicity axiom (MON) from Section 5.1.1, (NN) is a direct generalization of it.
Let us give a characterization of non-negative allocation mechanisms, which will clarify, why we prefer the term "non-negative" instead of "monotone".

Lemma 5.2.3 *Let $\phi = \phi^\alpha$ be a linear allocation mechanism. Then ϕ is non-negative if and only if $\alpha^t_{xy} \geq 0$ for all $t \geq 0$ and $xy \in A$.*

Proof. Assume first that (NN) holds. Let $xy \in A$ and set

$$v(z) := \delta_y(z) = \begin{cases} 1 & \text{if } z = y \\ 0 & \text{else.} \end{cases}$$

Then $v(y) - v(x) = 1 > 0$ holds. Thus by (NN) also $\phi^t_{xy}(v) \geq 0$ holds. Combined with Theorem 5.2.1 we get

$$0 \leq \phi^t_{xy}(v) = \alpha^t_{xy}(v(y) - v(x)) = \alpha^t_{xy}. \tag{*}$$

Assume the other way around: $\alpha^t_{xy} \geq 0$ for all $t \geq 0$ and $xy \in A$. Given $v \in \mathcal{G}$ with $v(y) - v(x) \geq 0$ the statement follows also directly by (*). ∎

5.2.5 t-efficiency & efficiency

Let $t > 0$. Recall the definition of E_t from Chapter 3 as the set of endpoints of all paths of length exactly t, which start in s.
We call ϕ *t-efficient* if there is a vector $(\mu^t_e)_{e \in E_t}$ with $\sum_{e \in E_t} \mu^t_e = 1$ such that for all $v \in \mathcal{G}$:

$$\sum_{xy \in A} \phi^t(v)_{xy} = \sum_{e \in E_t} \mu^t_e v(e)$$

holds. The idea behind this axiom is the following: it is only known to us that a certain state in E_t is achieved after t time steps, but not which of them. In order to model this uncertainty, we assume a certain probability distribution μ is given and assume that the expected value relative to μ equals the generated value at time t. Therefore, this expected value should be allocated. This idea is not new and is strongly inspired by notions of efficiency presented in the models of Bilbao [13] and Faigle and Voss [37]

which are based on the same principle. Since t-efficiency is just a part of the desired efficiency concept, we will show later (*cf.* Ex. 5.2.2) that classical efficiency concepts are special cases of our.
t-efficiency of linear allocation mechanisms can be restated in terms of flows on the graph (V, A):

Theorem 5.2.2 *Let ϕ be a linear allocation mechanism. Then ϕ is t-efficient for $t > 0$ if and only if each of the following statements is true:*

(a) α^t fulfills the flow conservation property at all vertices $x \in V \setminus (E_t \cup s)$. That is:

$$\sum_{u \in N^-(x)} \alpha^t_{ux} = \sum_{w \in N^+(x)} \alpha^t_{xw}.$$

(b) It holds:

$$\sum_{e \in E_t} (\sum_{ue \in A} \alpha^t_{ue} - \sum_{ew \in A} \alpha^t_{ew}) = 1.$$

.

Proof. Let $t > 0$ and assume first that ϕ is t-efficient. Let $\delta_x : V \to \{0, 1\}$ with $\delta_x(y) = 1$ if and only if $x = y$. For $x \in V$ the following is true:

$$\sum_{uw \in A} \phi^t(\delta_x)_{uw} = \sum_{uw \in A} \alpha^t_{uw}(\delta_x(w) - \delta_x(u)) = \sum_{u \in N^-(x)} \alpha^t_{ux} - \sum_{w \in N^+(x)} \alpha^t_{xw}. \qquad (*)$$

Because of the t-efficiency of ϕ, there is a generalized probability distribution $(\mu^t_e)_{e \in E_t}$ on E_t, such that

$$\sum_{uw \in A} \phi^t(\delta_x)_{uw} = \sum_{e \in E_t} \mu^t_e \delta_x(e). \qquad (**)$$

If $x \notin E_t$, and therefore $\delta_x(e) = 0$ for all $e \in E_t$, (**) needs to equal zero. Together these two equations yield the flow conservation at state x. It remains to show (b):
Set $\delta := \sum_{e \in E_t} \delta_e$. Then the following holds:

$$\begin{aligned} 1 &= \sum_{e \in E_t} \mu^t_e \delta(e) = \sum_{uw \in A} \phi^t(\delta)_{uw} = \sum_{uw \in A} \alpha^t_{uw}(\delta(w) - \delta(u)) \\ &= \sum_{e \in E_t} (\sum_{ue \in A} \alpha^t_{ue} - \sum_{ew \in A} \alpha^t_{ew}). \end{aligned}$$

Assume the other way around: α^t fulfills the flow conservation on $V \setminus E_t$ and

$$\sum_{e \in E_t} (\sum_{ue \in A} \alpha^t_{ue} - \sum_{ew \in A} \alpha^t_{ew}) = 1$$

holds. Then

$$\sum_{e\in E_t}\sum_{uw\in A}\phi^t(\delta_e)_{uw}=\sum_{e\in E_t}\sum_{ue\in A}\alpha^t_{ue}-\sum_{ew\in A}\alpha^t_{ew}=1$$

holds. On the other hand, by the linearity of ϕ,

$$\phi^t(v)=\sum_{x\in V}v(x)\phi^t(\delta_x)$$

holds for all $v\in\mathcal{G}$. But then also

$$\begin{aligned}\sum_{uw\in A}\phi^t(v)_{uw} &= \sum_{uw\in A}\sum_{x\in V}v(x)\phi^t(\delta_x)_{uw}\\ &= \sum_{uw\in A}\sum_{x\in E_t}v(x)\phi^t(\delta_x)_{uw}\\ &= \sum_{e\in E_t}(\sum_{ue\in A}\phi^t(\delta_e)_{ue})v(e)\end{aligned}$$

is true for all $v\in\mathcal{G}$. The third equality holds since - as shown above - $\sum_{uw\in A}\phi^t(\delta_x)_{uw}$ equals the net-flow at vertex x, which is 0 by the flow conservation for all $x\in V\backslash(E_t\cup s)$. Moreover, we assumed $v(s)=0$. Therefore, only those summands yield a contribution to the sum, that correspond to E_t.
We proved above that $\mu^t_e:=\sum_{ue\in A}\phi^t(\delta_e)_{ue}$ add up to 1. Hence ϕ is t-efficient. ■

In the last proof we found a nice representation of the coefficients μ^t_e, which we want to write down in:

Corollary 5.2.2 *Let $\phi=\phi^\alpha$ be a t-efficient allocation mechanism which suffices (LIN) and (NN) and let $(\mu^t_e)_{e\in E_t}$ be the coefficient vector on E_t, that corresponds to the t-efficiency of ϕ. Then $\mu^t_e=\sum_{ue\in A}\phi^t_{ue}(\delta_e)=\sum_{x\in N^-(e)}\alpha^t_{xe}-\sum_{y\in N^+(e)}\alpha^t_{ey}$ holds.* ■

The last theorem characterized linear and t-efficient allocation mechanisms. The property of t-efficiency seems to be a very “local” axiom. One could think, that valuations of actions at different times are independent of each other. We will be disabused by

Corollary 5.2.3 *Let $\phi=\phi^\alpha$ be an allocation mechanism, which enjoys (LIN) and is $t+1$-efficient (for some $t>0$). Then for all $e\in E_t\setminus E_{t+1}$:*

$$\sum_{x\in N^-(e)}\alpha^t_{xe}=\sum_{y\in N^+(e)}\alpha^{t+1}_{ey}$$

holds.

Proof. Since $e \notin E_{t+1}$, it follows for all $x \in N^-(e)$: $x \notin E_t$. By axiom (A3) this means: $\alpha^t_{xe} = \alpha^{t+1}_{xe}$ for all $x \in N^-(e)$. On the other hand, we have $e \notin E_{t+1}$ and Theorem 5.2.2 yields the flow conservation of α^{t+1} at e, *i.e.*:

$$\sum_{x \in N^-(e)} \alpha^{t+1}_{xe} = \sum_{y \in N^+(e)} \alpha^{t+1}_{ey}.$$

Combining both statements yields the claimed. ∎

The restriction to endpoints in $E_t \setminus E_{t+1}$ in the last corollary seems a bit unnatural. Unfortunately, this assumption could not readily be dropped in order to extend this result. As the following example shows:

Example 5.2.1 *Let $G = (V, A)$ be the complete directed graph on V. Thus each vertex is an end vertex for all times t. Assuming t-efficiency for all times $t > 0$ does not yield the "time comprehensive flow conservation" of Corollary 5.2.3. This is because (A3) does not guarantee the conservation of the value of actions to certain times any longer. Hence the α-values could vary arbitrarily and are a priori not liable to any restrictions.*

t-efficiency is not efficient

One of the ideas behind efficiency axioms is: a jointly generated value is allocated to all players that participated in the generation of it. Since we want to allocate time-dependently, this idea should be extended to time steps. Consider the following example: assume there is only one *winning* state $e \in V$ and consider the game

$$\delta_e(y) = \begin{cases} 1 & e = y, \\ 0 & \text{else.} \end{cases}$$

Assume that $e \in E_t \cap E_{t+1}$ and that ϕ is $(t+1)$-efficient. We want to allocate the generated value μ^{t+1}_e among the actions in A. The essential actions that could take the game to the only winning state e at time $t+1$ are exactly the actions in:

$$\{xe \in A | x \in N^-(e) \cap E_t\}.$$

Since those actions could be performed to bring the system to the state e in time step $t \to t+1$ independently from all other actions and on their own, the owners of those actions could raise a plea if the whole generated value at time $t+1$ is not fully allocated to them. Therefore, one could argue that it is fair if they demand:

$$\mu_e^{t+1} = \sum_{x \in E_t \cap N^-(e)} \phi_{xe}^{t+1}(\delta_e) - \phi_{xe}^t(\delta_e),$$

which means, that the time-marginal value of those essential actions is the value that is generated at this time. On the other hand:

$$\mu_e^{t+1} = \sum_{xy \in A} \phi_{xy}^{t+1}(\delta_e)$$

holds by $(t+1)$-efficiency. Therefore, we call ϕ *time efficient* if for all $t > 0$ and all $e \in E_t$:

$$\sum_{xy \in A} \phi_{xy}^{t+1}(\delta_e) = \sum_{x \in E_t \cap N^-(e)} (\phi_{xe}^{t+1}(\delta_e) - \phi_{xe}^t(\delta_e)) \tag{TE}$$

holds.
In view of Example 5.2.1 and the restrictive assumption of Corollary 5.2.3, there is a nice characterization of (TE):

Lemma 5.2.4 *Let $\phi = \phi^\alpha$ be a linear allocation mechanism. Then ϕ is time efficient if and only if for all $t > 0$ and $e \in E_t$:*

$$\sum_{x \in N^-(e)} \alpha_{xe}^t = \sum_{y \in N^+(e)} \alpha_{ey}^{t+1}$$

holds.

Proof. Consider again the game δ_e. Then

$$\sum_{xy \in A} \phi_{xy}^{t+1}(\delta_e) = \sum_{x \in N^-(e)} \phi_{xe}^{t+1}(\delta_e) - \sum_{y \in N^+(e)} \phi_{ey}^{t+1}(\delta_e)$$

holds. On the other hand

$$\begin{aligned}
\sum_{x \in N^-(e)} \phi_{xe}^{t+1}(\delta_e) &= \sum_{x \in N^-(e) \cap E_t} \phi_{xe}^{t+1}(\delta_e) + \sum_{x \in N^-(e) \setminus E_t} \phi_{xe}^{t+1}(\delta_e) \\
&= \sum_{x \in N^-(e) \cap E_t} \phi_{xe}^{t+1}(\delta_e) + \sum_{x \in N^-(e) \setminus E_t} \phi_{xe}^{t}(\delta_e) \\
&= \sum_{x \in N^-(e) \cap E_t} \phi_{xe}^{t+1}(\delta_e) + \sum_{x \in N^-(e)} \phi_{xe}^{t}(\delta_e) - \sum_{x \in N^-(e) \cap E_t} \phi_{xe}^{t}(\delta_e)
\end{aligned}$$

holds. Together these equations yield

$$\sum_{xy\in A}\phi_{xy}^{t+1}(\delta_e)=\sum_{x\in N^-(e)\cap E_t}\phi_{xe}^{t+1}(\delta_e)-\phi_{xe}^{t}(\delta_e)+\sum_{x\in N^-(e)}\phi_{xe}^{t}(\delta_e)+\sum_{y\in N^+(e)}\phi_{ey}^{t+1}(\delta_e). \quad (*)$$

Therefore, time efficiency is equivalent to

$$\sum_{x\in N^-(e)}\phi_{xe}^{t}(\delta_e)+\sum_{y\in N^+(e)}\phi_{ey}^{t+1}(\delta_e)=0,$$

which is equivalent to

$$\sum_{x\in N^-(e)}\alpha_{xe}^{t}-\sum_{y\in N^+(e)}\alpha_{ey}^{t+1}=0,$$

by (LIN) and Theorem 5.2.1, as stated in this lemma. ∎

REMARK Note that by the last lemma time efficiency is implied by t-efficiency if the underlying graph has the property that there is no vertex $e\in V$ with $e\in E_t\cap E_{t+1}$. Which is especially the case in the classical cooperation system. Hence in the classical model time efficiency is a redundant property.

Efficiency

In view of the last subsection we merge the two terms t-efficiency and time efficiency and call a value ϕ efficient if

$$\phi \text{ is } t\text{-efficient for all } t>0 \text{ and time efficient.} \qquad \text{(EFF)}$$

We consider the example of classical cooperative games and allocation mechanisms and show that our notion of efficiency agrees with the classical one:

Example 5.2.2 *Let (N,v) be a classical cooperative game on the cooperation system $\Gamma:=(2^N,A,(A_i)_{i\in N},\emptyset)$ (with $A_i:=\{UW\in A|W\setminus U=\{i\}\}$) and ϕ an efficient (in the sense of (EFF)) allocation mechanism. Consider the value ϕ^n ($n:=|N|$). Set*

$$\phi_i^n(v):=\sum_{xy\in A_i}\phi_{xy}^n(v).$$

Since $E_n=\{N\}$ holds in the classical cooperation system, (EFF) guarantees:

$$v(N)=\sum_{xy\in A}\phi_{xy}^n(v)=\sum_{i\in N}\sum_{xy\in A_i}\phi_{xy}^n(v)=\sum_{i\in N}\phi_i^n(v).$$

Therefore the value $v\mapsto(\phi_i^n(v))_{i\in N}$ is efficient in the classical sense.

Theorem 5.2.2 and Lemma 5.2.4 together give us a characterization of linear and efficient allocation mechanisms:

Corollary 5.2.4 *Let $\phi = \phi^\alpha$ be a linear allocation mechanism. Then ϕ is efficient if and only if each of the following statements is true for all $t > 0$:*

(a) α^t fulfills the flow conservation property at all vertices $x \in V \setminus (E_t \cup s)$. That is:

$$\sum_{u \in N^-(x)} \alpha^t_{ux} = \sum_{w \in N^+(x)} \alpha^t_{xw}.$$

(b) $\sum_{e \in E_t} (\sum_{ue \in A} \alpha^t_{ue} - \sum_{ew \in A} \alpha^t_{ew}) = 1$.

(c) $\sum_{x \in N^-(e)} \alpha^t_{xe} = \sum_{y \in N^+(e)} \alpha^{t+1}_{ey}$, for all $e \in E_t$.

■

In fact, these conditions have a much nicer interplay and we could give a better and more compact characterization of linear and efficient allocation mechanisms, which summarizes the results of this section:

Theorem 5.2.3 *Let $\phi = \phi^\alpha$ be a linear allocation mechanism. Then ϕ is efficient if and only if the following two conditions are true:*

(a) For all $t > 0$: $\sum_{x \in N^-(e)} \alpha^t_{xe} = \sum_{y \in N^+(e)} \alpha^{t+1}_{ey}$ for all $e \in E_t$.

(b) $\sum_{u \in N^+(s)} \alpha^1_{su} = 1$.

Proof. Set $\delta := \sum_{u \in N^+(s)} \delta_s$. By efficiency ϕ is especially 1-efficient. Hence

$$\sum_{u \in N^+(s)} \alpha^1_{su} = \sum_{xy \in A} \phi^1(\delta) = \sum_{e \in E_1} \mu^1_e \delta(e) = \sum_{e \in E_1} \mu^1_e = 1.$$

Thus, if ϕ is efficient, we already know that (a) and (b) are satisfied.
Assume the other way around that (a) and (b) hold. We have to show for all $t > 0$:

(i) $\sum_{u \in N^-(x)} \alpha^t_{ux} = \sum_{w \in N^+(x)} \alpha^t_{xw}$ for all $x \in V \setminus (E_t \cup s)$.

(ii) $\sum_{e \in E_t} \sum_{ue \in A} \alpha^t_{ue} - \sum_{ew \in A} \alpha^t_{ew} = 1$.

Then the last corollary yields the efficiency of ϕ.
(i): Let $x \in V \setminus (E_t \cup s)$. We use induction on t. At time $t = 1$ we have $\alpha^1_{xy} = 0$ if $x \neq s$. Thus the flow conservation for all $x \in V \setminus (E_1 \cup s)$ is clear. Let $t > 1$ and assume that the statement is true for all smaller t. Let $x \in V \setminus (E_t \cup s)$. If $x \notin E_{t-1}$, the following holds by induction:

$$\sum_{u \in N^-(x)} \alpha^{t-1}_{ux} = \sum_{w \in N^+(x)} \alpha^{t-1}_{xw} = \sum_{w \in N^+(x)} \alpha^{t}_{xw}.$$

Where the last equality holds by (A3). On the other hand, since $x \notin E_t$, no predecessor of x could be in E_{t-1}. Thus again by (A3):

$$\sum_{u \in N^-(x)} \alpha^{t-1}_{ux} = \sum_{u \in N^-(x)} \alpha^{t}_{ux}$$

holds and we are done. Assume that $x \in E_{t-1}$. Then by (a)

$$\sum_{u \in N^-(x)} \alpha^{t-1}_{ux} = \sum_{y \in N^+(x)} \alpha^{t}_{xy}$$

holds. Again $\sum_{u \in N^-(x)} \alpha^{t-1}_{ux} = \sum_{u \in N^-(x)} \alpha^{t}_{ux}$ by (A3). Since $x \notin E_t$, $u \notin E_{t-1}$ holds for all $u \in N^-(x)$.

(ii): By (a) we know that α is a flow over time. By (b) we know it sends out the value 1 at time 1 from s. Hence at time t the flow transports the value 1 to the endpoints in E_t and (ii) holds. Alternatively again one uses induction on t with trivial beginning and the fact that $E_t = \bigcup_{u \in E_{t-1}} N^+(u)$. Let $t > 1$ and assume (ii) to hold for all smaller t. Then

$$\begin{aligned}
1 &= \sum_{e \in E_{t-1}} \sum_{ue \in A} \alpha^{t-1}_{ue} - \sum_{ew \in A} \alpha^{t-1}_{ew} \\
&= \sum_{e \in E_{t-1}} \sum_{ew \in A} \alpha^{t}_{ew} - \sum_{ew \in A} \alpha^{t-1}_{ew} \\
&= \sum_{f \in E_t} \sum_{uf \in A} \alpha^{t}_{uf} - \sum_{uf \in A} \alpha^{t-1}_{uf} \\
&= \sum_{f \in E_t} \sum_{uf \in A} \alpha^{t}_{uf} - \sum_{fw \in A} \alpha^{t}_{fw}.
\end{aligned}$$

■

5.2.6 Ratio fairness

Recall the arguments of Section 5.2.5, which lead us to the notion of time efficiency:

$$\sum_{xy\in A} \phi_{xy}^{t+1}(\delta_e) = \sum_{x\in E_t\cap N^-(e)} \phi_{xe}^{t+1}(\delta_e) - \phi_{xe}^{t}(\delta_e) \tag{TE}$$

for all $t > 0$ and $e \in E_t$. In Lemma 5.2.4 we proved, that time efficiency is equivalent to

$$\sum_{x\in N^-(e)} \phi_{xe}^{t}(\delta_e) = -\sum_{y\in N^+(e)} \phi_{ey}^{t+1}(\delta_e). \tag{*}$$

The interpretation behind (*) is the following: those actions, which take the game from the only winning state e to a non-winning state $y \in N^+(e)$ at time $t+1$, are responsible for the loss of the value that was generated by the actions $xe(x \in N^-(e))$ at time t by taking the game to the winning state e. Right now, there is nothing said about, how these costs are allocated to the actions $ey(y \in N^+(e))$. Assume that this situation is cyclic, *i.e.*: there is a circle in the graph, such that the system is in state e at time $t' > t$ again. The abstract situation of possibilities and actions that could be performed has not changed. The only thing that probably changed is the value of $\sum_{x\in N^-(e)} \phi_{xe}^{t'}(\delta_e)$. Therefore, the owners of actions ey could argue, that at time $t'+1$ they take over liabilities only at the same ratio, they did in the past. We want to call this property *ratio fairness*. To be more formal:

Let ϕ be an allocation mechanism. ϕ is called *ratio fair* if for all $t' > t > 0$ and $e \in E_t \cap E_{t'}$ the following holds:

$$\phi_{ey}^{t+1}(\delta_e) \sum_{u\in N^-(e)} \phi_{ue}^{t'}(\delta_e) = \phi_{ey}^{t'+1}(\delta_e) \sum_{u\in N^-(e)} \phi_{ue}^{t}(\delta_e). \tag{RAT}$$

A reformulation of (RAT) in terms of linear values is

Lemma 5.2.5 *Let $\phi = \phi^\alpha$ a linear allocation mechanism. ϕ enjoys (RAT) if and only if*

$$\alpha_{xy}^{t+1} \sum_{ux\in N^-(x)} \alpha_{ux}^{t'} = \alpha_{xy}^{t'+1} \sum_{ux\in N^-(x)} \alpha_{ux}^{t} \textit{ for all } t, t' \textit{ with } x \in E_t \cap E_{t'}.$$

■

REMARK (A3) together with Lemma 5.2.5 implies

$$\alpha_{xy}^{t'} \sum_{ux \in N^-(x)} \alpha_{ux}^{t'} = \alpha_{xy}^{t'+1} \sum_{ux \in N^-(x)} \alpha_{ux}^{t'-1}$$

for linear, ratio fair allocation mechanism ϕ^α and all $t' > 0$ with $x \in E_{t'}$.

Lemma 5.2.6 *Let $\phi = \phi^\alpha$ be an allocation mechanism that satisfies (LIN), (EFF) and (RAT). Then for all $xy \in A$ and all $t > 0$ with $d(s,x) < t$ and $\alpha_{xy}^t \neq 0$:*

$$\alpha_{xy}^{t} \sum_{w \in N^+(x)} \alpha_{xw}^{t+1} = \alpha_{xy}^{t+1} \sum_{w \in N^+(x)} \alpha_{xw}^{t}$$

holds.

Proof. If $x \notin E_t$, all considered α_{xy} values are constant in the time step $t \to t+1$ by Axiom (A3). Hence the claim follows trivially. Therefore, assume $x \in E_t$. By (EFF) and Theorem 5.2.3

$$\sum_{w \in N^+(x)} \alpha_{xw}^{t+1} = \sum_{u \in N^-(x)} \alpha_{ux}^{t}$$

holds. Moreover, by (RAT) and Lemma 5.2.5 for all $xy \in A$:

$$\alpha_{xy}^{t} \sum_{u \in N^-(x)} \alpha_{ux}^{t} = \alpha_{xy}^{t+1} \sum_{u \in N^-(x)} \alpha_{ux}^{t-1}$$

holds. Thus

$$\alpha_{xy}^{t} \sum_{w \in N^+(x)} \alpha_{xw}^{t+1} = \alpha_{xy}^{t+1} \sum_{u \in N^-(x)} \alpha_{ux}^{t-1}$$

holds. If $x \notin E_{t-1}$,

$$\sum_{u \in N^-(x)} \alpha_{ux}^{t-1} = \sum_{w \in N^+(x)} \alpha_{xw}^{t-1} = \sum_{w \in N^+(x)} \alpha_{xw}^{t}$$

and the claim follows. Hence we may assume that $x \in E_{t-1}$. It follows by (EFF):

$$\sum_{u \in N^-(x)} \alpha_{ux}^{t-1} = \sum_{w \in N^+(x)} \alpha_{xw}^{t}.$$

∎

Now we have gathered all preliminaries to generalize Weber's theory of random values and to discover linear, efficient and ratio fair allocation mechanisms as random walks on the graph (V, A). Given $\phi = \phi^\alpha$ which suffices (LIN), (EFF) and (RAT) we define a random walk on (V, A) *via*:

(0.) Start in s and move with (possibly negative) probability α^1_{sx} along the arc sx.

(1.) If the random walk arrived in $x \in V$ at time $t > 0$, move with (possibly negative) probability $\pi_{xy} := \frac{\alpha^t_{xy}}{\sum_{w \in N^+(x)} \alpha^t_{xw}}$ along the arc xy.

REMARK This random walk is well defined since ϕ is efficient and therefore α^1_{sx} $(x \in N^+(s))$ actually is a (generalized) probability distribution. If the walk arrives in $x \in V$ with nonzero probability $\sum_{u \in N^-(x)} \alpha^{t-1}_{ux} \neq 0$, by Lemma 5.2.5 also $\sum_{w \in N^+(x)} \alpha^t_{xw} \neq 0$ and therefore the term

$$\pi^t_{xy} := \frac{\alpha^t_{xy}}{\sum_{w \in N^+(x)} \alpha^t_{xw}}$$

is well defined. Given that the walk arrives in x also at another time t', Lemma 5.2.6 guarantees:

$$\pi^t_{xy} = \pi^{t'}_{xy}.$$

Hence the time-independence of π in the definition of the walk is justified.

By setting $\pi_{xy} := 0$ for $xy \notin A$, this random walk gives rise to a transition matrix $(\pi_{xy})_{xy \in V \times V}$ with row-sums equal to 1. The lemma above gives us another nice characterization of (NN) and another reason for preferring the term "non-negative" instead of "monotone":

Corollary 5.2.5 *Let $\phi = \phi^\alpha$ be a linear, efficient and ratio fair allocation mechanism with corresponding transition matrix $\Pi := (\pi_{xy})_{xy \in V \times V}$. Then ϕ suffices (NN) if and only if Π is a classical Markovian matrix.*

Proof. Assume first that ϕ is non-negative. Let $xy \in A$ and $t > 0$ s.t. $x \in E_{t-1}$. Then $\pi_{xy} = \frac{\alpha^t_{xy}}{\sum_{w \in N^+(x)} \alpha^t_{xw}} \geq 0$. Hence Π is a classical Markovian matrix. Assume the other way around: for all $xy \in A$ and all $t > 0$ s.t. $x \in E_{t-1}$:

$$0 \leq \frac{\alpha^t_{xy}}{\sum_{w \in N^+(x)} \alpha^t_{xw}} \tag{*}$$

holds. Assume by an inductive argument, that $\alpha^{t-1}_{ab} \geq 0$ for all $ab \in A$. By the efficiency of ϕ and induction we have

$$\sum_{w \in N^+(x)} \alpha^t_{xw} = \sum_{u \in N^-(x)} \alpha^{t-1}_{ux} \geq 0.$$

Hence multiplying (*) by $\sum_{w\in N^+(x)} \alpha^t_{xw}$ yields $\alpha^t_{xy} \geq 0$. ■

5.3 Allocation mechanisms induced by random walks

In the last section we proved that each allocation mechanism, which enjoys the properties (LIN), (EFF) and (RAT), induces a (generalized) random walk on the underlying graph. In this section we want to consider the contrary question:

Does a random walk on the graph induce an allocation mechanism?

Let $\Pi = (\pi_{xy})_{xy\in V\times V}$ be a generalized stochastic matrix, which is compatible with the graph G, *i.e.*:

(1) $\sum_{w\in V} \pi_{uw} = 1$ for all $u \in V$ and

(2) $uw \notin A \Leftrightarrow \pi_{uw} = 0$.

We define recursively for all $t > 0$ and $u, w \in V$: $p^0(u) := 0$, $p^0(s) = 1$ and

$$p^t(u) := \begin{cases} p^{t-1}(u) & \text{if } u \notin E_t \\ \sum_{xu\in A} p^{t-1}(x)\pi_{xu} & \text{if } u \in E_t. \end{cases}$$

Moreover, define for all $t \geq 0$:

$$\alpha^{t+1}_{uw}(\Pi) := \alpha^{t+1}_{uw} := p^t(u)\pi_{uw}.$$

Lemma 5.3.1 *For all $xy \in A, t > 0$ $\alpha^{t+1}_{xy} = \alpha^t_{xy}$ holds if $x \notin E_t$, i.e.: α induces a linear allocation mechanism by Lemma 5.2.2.*

Proof. Since $x \notin E_t$, $p^t(x) = p^{t-1}(x)$. Hence $\alpha^{t+1}_{xy} = p^t(x)\pi_{xy} = p^{t-1}(x)\pi_{xy}\alpha^t_{xy}$. ■

Lemma 5.3.2 *Let Π be a generalized stochastic matrix. The linear allocation mechanism $\phi := \phi^\Pi := \phi^\alpha$ induced by Π satisfies (EFF) and (RAT).*

Proof. Let $t > 0$ and $x \in E_t$. Then

$$\sum_{u \in N^-(x)} \alpha^t_{ux} = \sum_{u \in N^-(x)} p^{t-1}(u)\pi_{ux}. \tag{*}$$

This equals

$$p^t(x) = p^t(x) \sum_{w \in N^+(x)} \pi_{xw} = \sum_{w \in N^+(x)} \alpha^{t+1}_{xw}.$$

Moreover, $\sum_{u \in N^+(s)} \alpha^1_{su} = \sum_{u \in N^+(s)} p^0(s)\pi_{su} = 1$ holds.
This completes the proof of (EFF) holding for ϕ by the characterization given in Theorem 5.2.3. It remains to prove ratio fairness for ϕ. Therefor let $x \in E_t \cap E_{t'}$. Then

$$\begin{aligned}
\alpha^{t+1}_{xy} \sum_{u \in N^-(x)} \alpha^{t'}_{ux} &= p^t(x)\pi_{xy} \sum_{u \in N^-(x)} p^{t'-1}(u)\pi_{ux} \\
&= \sum_{u \in N^-(x)} p^{t-1}(u)\pi_{ux}\pi_{xy}p^{t'}(x) \\
&= \sum_{u \in N^-(x)} \alpha^t_{ux}\alpha^{t'+1}_{xy}
\end{aligned}$$

holds which is exactly the condition of (RAT). ∎

To subsume the results of the last two subsection, we formulate:

Theorem 5.3.1 *Allocation mechanisms, which satisfy (LIN), (EFF) and (RAT), correspond to generalized stochastic matrices over V; i.e.: to generalized random walks on G.* ∎

Moreover, as seen in the definition of the random walk associated to an allocation mechanism, these results characterize non-negativity in a way, which is another argument for calling the axiom (NN) non-negativity and not monotonicity:

Theorem 5.3.2 *Let $\phi = \phi^\Pi$ be an allocation mechanism that enjoys (LIN), (EFF) and (RAT). Then ϕ suffices (NN), if and only if Π is a classical stochastic matrix; i.e.: $\pi_{xy} \geq 0$ for all $xy \in A$.* ∎

Thus we achieved a generalization of Weber's result (Theorem 5.1.3) and characterized allocation mechanisms, which enjoy certain fairness axioms, as being induced by random walks. We will merge those allocation mechanism into a class and will call an allocation mechanism *randomized* (RAN) if it enjoys (LIN), (EFF) and (RAT).

5.3.1 Action sequences & path probabilities

As presented in Section 5.1.2, Weber proved a bit more than a characterization of random walks. He characterized *random order values* which evaluate players with respect to random permutations of the player set. His result was restated in Theorem 5.1.2. We aim for a generalization of the idea behind random order values.
In terms of our model, an order of the players (*i.e.* a permutation of N) in the classical case, is a path of length $|N|$ from $\emptyset$ to N. Therefore we consider action sequences $x_1y_1, ..., x_ky_k$, such that $y_i = x_{i+1}$ and $x_1 = s$. In terms of graph theory those sequences of actions are precisely the paths that start in s. Set

$$\mathcal{P} := \{P | P \text{ is a path of finite length in } G \text{, that starts in } s.\}$$

and for all $t > 0$:

$$\mathcal{P}_t := \{P \in \mathcal{P} | \, |P| = t\}. \qquad \text{(where } |P| \text{ is the length of } P\text{)}$$

Randomized allocation mechanisms induced by random orders

Let $t_{\max}$ be the cover time of G with respect to s (*i.e.*: the first time such that each arc was at least once visitable). And let $0 < \pi := (\pi_{\max}(P))_{P \in \mathcal{P}_{t_{\max}}}$ be a probability distribution on $\mathcal{P}_{t_{\max}}$. Set

$$\beta_{xy} := \sum_{P \in \mathcal{P}_{t_{\max}}, xy \in P} \pi_{\max}(P)$$

for all $xy \in A$ (that is: the probability of xy being part of a randomly chosen path of length $t_{\max}$). In analogy to the last section one can associate a random walk with β by setting transition probabilities to:

$$\pi_{xy} := \frac{\beta_{xy}}{\sum_{z \in N^+(x)} \beta_{xz}}.$$

And in turn, by the last section of this chapter: this random walk induces a randomized allocation mechanism ϕ.

Assume the other way around that ϕ is a randomized allocation mechanism. As seen in Subsection 5.2.6, ϕ induces a random walk with transition matrix $(\pi_{xy})_{xy\in A}$. Let $P = x_0, ..., x_{t_{\max}} \in \mathcal{P}_{t_{\max}}$. By setting

$$\pi(P) := \prod_{i=1}^{t_{\max}} \pi_{x_{i-1}x_i},$$

π induces a probability distribution on $\mathcal{P}_{t_{\max}}$. Hence we find:

Theorem 5.3.3 *Each random order induces a randomized allocation mechanism and each randomized allocation mechanism induces a random order.* ■

Hence the strong relation between *random orders* and randomized allocation mechanisms known from the classical result of Weber is preserved. This relation will give us the opportunity to introduce the *entropy* of a non-negative randomized allocation mechanism in Section 5.4.2 which in turn yields another notion of symmetry and a measure for impartiality.

5.4 Characterization results

In the first section of this chapter we presented some classical allocation mechanisms, which received much attention in the theory of cooperative games. In this section we want to generalize the main ideas behind those allocation mechanism, that made them that famous and aim for analogous characterization results in our model.

5.4.1 Shapley values

In the beginning of this chapter we restated the famous theorem of Shapley which characterized the allocation mechanism that got his name:

Theorem 5.4.1 ([82]) *Given a classical cooperation structure on a finite player set N there exists exactly one allocation mechanism which enjoys the properties (ADD), (NUL), (EFF) and (SYM).* ■

Compared with the results of Weber (*cf.* Section 5), it seems that the characterizing axiom is (SYM).

The symmetry axiom

In order to generalize the idea behind (SYM) to our model, let us restate (SYM) and recapitulate the idea behind it in the case of a classical cooperation structure:

$$v(S \cup i) = v(S \cup j) \text{ for all } S \subseteq N \setminus \{i, j\} \Rightarrow \phi_i(v) = \phi_j(v). \text{ (for all } v \in \mathcal{G}) \qquad \text{(SYM)}$$

In words the symmetry axiom means the following: if two players generate the same marginal contributions to any coalition they can join, they only differ in their name and should be treated equally in the payoff process.
Adapted to the general case, this idea could be generalized as follows: if two actions that could bring the system out of state x to states y and z yield the same marginal contribution, they should be treated equally. Thus we call an allocation mechanism ϕ *symmetric*, if

$$\text{For all } xy, xz \in A : v(y) = v(z) \Rightarrow \phi^t_{xy}(v) = \phi^t_{xz}(v) \text{ for all } t > 0. \qquad \text{(SYM')}$$

One directly observes:

Lemma 5.4.1 *Let Γ be the classical cooperation system and let ϕ be an allocation mechanism that enjoys (SYM'). Then $v \to (\sum_{xy \in A_i} \phi^n_{xy}(v))_{i \in N}$ satisfies (SYM).* ■

Since (SYM') implies (SYM) in the classical case, it is riskless to omit the prime, and we will do so in the following.

A first Shapley value

Theorem 5.4.2 *There is precisely one randomized allocation mechanism Φ which is symmetric.*

Proof. Consider the transition matrix $(\pi_{xy})_{xy \in V \times V}$ given by

$$\pi_{xy} := \begin{cases} \frac{1}{deg^+(x)} & \text{if } xy \in A, \\ 0 & \text{else.} \end{cases}$$

Since the associated linear allocation mechanism ϕ^α (*cf.* Section 5.3) is induced by

$$\alpha^t_{xy} := p^{t-1}(x)\pi_{xy},$$

and the first factor only depends on x, ϕ^α is symmetric. Assume the other way around: there is another symmetric allocation mechanism which is induced by

$$\beta^t_{xy} = \tilde{p}^{t-1}(x)\tilde{\pi}_{xy}.$$

Since ϕ^β is symmetric,

$$\beta^t_{xy} = \beta^t_{xz}$$

holds. Thus $\tilde{\pi}_{xy} = \tilde{\pi}_{xz}$. Hence $\alpha = \beta$. ∎

This theorem gives a generalization of the classical Shapley value. Therefore, we call Φ the *symmetric Shapley allocation mechanism*.

REMARK As seen in the proof of Theorem 5.4.2, symmetry implies non-negativity for Φ since the (a priori generalized) probabilities π_{xy} ($y \in N^+(x)$) need to sum up to 1 and $\pi_{xy} = \pi_{xz}$ for all $z \in N^+(x)$.

5.4.2 Entropy - another notion of symmetry

Recall the representation of the classical Shapley value given in Section 5.1.2 as the allocation mechanism, which is induced by the uniform distribution on the set of permutations of the players:

$$\Phi^{Sh}(v)_i = \frac{1}{|N|!} \sum_{\sigma \in Sym(N)} v(S_i(\sigma) \cup i) - v(S_i(\sigma)),$$

where $S_i(\sigma)$ is the ordered set of all players, which precede player i in σ. This reflects another idea of symmetry in the Shapley value: all rankings of the players should be treated equally and the payoff of a player should only depend on his marginal contributions to coalitions he could join, not on the order in which this joining process takes place. Given a classical probability distribution π on the set of all permutations $Sym(N)$, one could measure by the so called *entropy* (known from information theory [51]), how symmetric a value induced by π is. Also Faigle and the author [37] proposed the entropy of an allocation mechanism as a measure of fairness. Define the *entropy* of π *via*:

$$H(\pi) := -\sum_{\sigma \in S_N} \pi_\sigma \log \pi_\sigma.$$

It is well known (*cf.* [51]), that the entropy of classical probability distributions is maximized by the uniform distribution. This fact yields another characterization of the classical Shapley value:

Theorem 5.4.3 ([37]) *There exists precisely one random order value (in the classical sense of Weber), that maximizes the entropy. This maximizer is the classical Shapley value.* ■

This is our starting point for generalizing this notion of symmetry. In view of Section 5.3.1 we define the *entropy* of a non-negative randomized allocation mechanism ϕ with underlying probability distribution $\pi_{\max}$ on $\mathcal{P}_{t_{\max}}$, as

$$H(\phi) := H(\pi_{t_{\max}}) := -\sum_{P \in \mathcal{P}_t} \pi_{t_{\max} P} \log(\pi_{t_{\max} P}).$$

Theorem 5.4.4 *There is exactly one non-negative randomized allocation mechanism Ψ with maximal entropy.*

Proof. This is direct, since if two random walks induce the same distribution on $\mathcal{P}_{t_{\max}}$ they equal each other. ■

Since Ψ is also a generalization of the classical Shapley value (and in the classical case $\Psi = \Phi$ holds) we call Ψ the *entropy-symmetric* Shapley value.
There is a large class of graphs on which these two concepts of symmetry coincide. We will note this down in the following

Lemma 5.4.2 *If G is a graph with the property*

$$\textit{For all } t > 0, \quad x, y \in E_t : d^+(x) = d^+(y). \tag{R}$$

Then $\Psi = \Phi$ holds.

Proof. Let $t := t_{\max}$ and $P = p_0, ..., p_t$ and $Q = q_0, ..., q_t$ be paths of length t with $p_0 = s = q_0$. The induced path probability of Φ is given by:

$$\pi(P) = \prod_{i=0}^{t-1} \pi_{p_i p_{i+1}} = \prod_{i=0}^{t-1} \frac{1}{d^+(p_i)}$$

Since $p_i, q_i \in E_i$, by (R) it holds:

$$\pi(P) = \pi(Q).$$

Hence Φ induces the uniform distribution on $\mathcal{P}_{t_{\max}}$. Therefore $\Phi = \Psi$ holds. ■

5.5 Markovian systems and Weber theory

In Theorem 3.3.1 connected generalized Markovian systems and generalized random walks on the associated transition graphs. This connection gives us the "switch" between two points of view on cooperative situations:

(I) Given a cooperative situation that could be modeled by states, actions and transition graphs, the Weber theory presented in this chapter could be used to propose solutions to the Allocation Problem. As long as all participating players agree with the fairness axioms of randomized allocation mechanisms, each allocation that suffice (RAN) would yield a fair solution to the Allocation Problem.

(II) Given a cooperative situation, assume there is a certain randomized allocation mechanism, which is the fixed payoff rule for this situation. In view of the Prediction Problem (see 3.1.1) one could argue: the evolution operator of the associated Markovian system solves the Prediction Problem, since it describes the random walk induced by the fixed randomized allocation mechanism.

This observation shows that the Allocation Problem and the Prediction Problem are related to each other and that solving one of these problems leads to a solution of the other.

5.5.1 Allocation mechanisms induced by inhomogeneous random walks

So far, we studied allocation mechanism that are induced by Markovian evolution operators. A natural generalization is the following: let (V, Φ) be a system with Markovian state space $\mathcal{V}$ and let $\Phi = (M_t)_{t \geq 0}$ be a sequence of $V \times V$-matrices, such that

(a) $M_0 = Id$ and

(b) M_t is stochastic for $t > 0$.

Assume that there is $s \in V$, such that for all $x \in V \setminus s$ there is a sequence $s = x_0, x_1, ..., x_k = x$ with $(M_i)_{x_{i-1}x_i} \neq 0$ for all i. Then Φ also describes a random walk on V *via*:

(0) Start in s and move with probability $(M_1)_{sx}$ to $x \in V \setminus s$.

(1) If the walk arrives in $x \in V$ at time $t > 0$, move to $y \in V$ with probability $(M_{t+1})_{xy}$.

By adapting the ideas developed in this chapter, one could define a linear allocation mechanism relative to Φ by:

$$\alpha_{xy}^{t+1} := \begin{cases} p^t(x)(M_{t+1})_{xy} & \text{if } x \in E_t \\ \alpha_{xy}^t & \text{else.} \end{cases}$$

Where p^t is recursively defined by: $p^0(u) := 0$ for $u \in V \setminus s$, $p^0(s) = 1$ and

$$p^t(u) := \begin{cases} p^{t-1}(u) & \text{if } u \notin E_t \\ \sum_{xu \in A} p^{t-1}(x)(M_t)_{xu} & \text{if } u \in E_t. \end{cases}$$

In analogy to the results of Section 5.3 one gets:

Corollary 5.5.1 *ϕ^α is a linear allocation mechanism which is t-efficient for all $t > 0$.* ∎

Later in Section 10.1 we will see a concrete example of an evolution operator which is defined by an inhomogeneous random walk. But our focus still lies on homogeneous processes.

5.6 Weber allocation in the long run

Until now we developed a theory of randomized allocation mechanisms that reflect fair allocations over time in some sense. Whenever certain time-dependent objects are considered, it is quite natural to ask

- How does an allocation mechanism evolve over time?
- Does $(\phi^t)_{t>0}$ converge?
- And if so, what does a limit look like?

In this last part of this chapter we want to give answers to these questions. To do so, we need to recall some basic facts about random walks.

5.6.1 Basic facts about random walks

In order to give all tools needed for answering the above stated question but not losing our focus on our main tasks of this thesis, this section is kept very brief and restates

well-known facts about random walks. For further and much more detailed reading on this topic we refer the reader to an introductory book on probability theory [55]. Deeper matter and a beautiful presentation of the whole theory of Markov chains and mixing times could be found in [62]. Especially the theory of random walks with continuous time horizon is treated there in detail.
All considered random walks in this thesis stray on finite directed graphs with a discrete time horizon. All results of this section are well-known in one or another form and could be found in a more general or slightly different version in [62].
Assume for the rest of this section that $G = (V, A, s)$ is a graph which is strongly s connected and that $\Pi := (\pi_{xy})_{xy \in A}$ is a classical stochastic matrix, which is compatible with G, *i.e.*:

$$\pi_{xy} = 0 \Leftrightarrow xy \notin A, \tag{*}$$

and for all $x \in V$

$$\sum_{y \in V} \pi_{xy} = 1, \quad \pi_{xy} \geq 0.$$

Let us start by some definitions: Π is called *irreducible* if the probability of getting to any vertex y, starting in any other node x, is positive. In other words: if every two vertices $x, y \in V$ are connected by a directed path, whose arcs have all a positive transition probability. By (*) this is equivalent to G being strongly connected.
The *period* of a vertex $x \in V$, $per(x)$, is the greatest common divisor of the length of all directed circles, which contain x and have a positive probability. The first observation is the following:

Lemma 5.6.1 ([62]) *If Π is irreducible, all vertices in V have the same period.* ∎

Π is called *aperiodic* if all vertices in V have period 1.
A probability distribution

$$p \in \Delta(V) := \{x \in \mathbb{R}^V | x_i \geq 0, \sum_{i \in V} x_i = 1\}$$

is called *stationary* with respect to Π if

$$p\Pi = p$$

holds. A stationary distribution p therefore is an eigenvector to the eigenvalue 1 (or: a fixed point of Π).

Lemma 5.6.2 ([62]) *For every stochastic matrix Π there exists a stationary distribution.* ■

Given a starting distribution s, p is called a *limiting distribution* if the limit

$$\lim_{t\to\infty} s\Pi^t$$

exists and equals p.

Lemma 5.6.3 *If p is a limiting distribution for Π, then p is stationary.*

Proof. It is $p\Pi = (\lim_{t\to\infty} s\Pi^t)\Pi = \lim_{t\to\infty} s\Pi^{t+1} = \lim_{t\to\infty} s\Pi^t = p$. ■

We are interested in the question: under which circumstances does a unique stationary distribution exist. Obviously this is exactly then the case if the dimension of the eigenspace to the eigenvalue 1 is 1. Because in a one dimensional space there is exactly one non-negative vector with component sum equal to 1. Therefore we could find in every one dimensional subspace of the eigenspace to the eigenvalues 1, a stationary distribution.
We want to use a theorem from linear algebra: the theorem of Perron and Frobenius, which was first proven in the case of positive matrices in [74] by Perron and was generalized to non-negative ones later by Frobenius in [42]. Before we are able to state this theorem, we need a further definition:
A matrix $A \in \mathbb{R}^{n\times n}$ is called *reducible* if it is conjugated to a matrix in block-diagonal form. If a matrix is not reducible, it is called *irreducible*. To avoid confusion, let us first show, that this notion of irreducibility does not conflict the notion of irreducibility of transition matrices:

Lemma 5.6.4 *Stochastic matrices which underlying random walk is irreducible, are irreducible matrices.*

Proof. Assume the random walk associated to Π is irreducible, but Π is not. Then there exists a block-diagonal form of Π and w.l.o.g. we assume Π to be in this form. It is known from linear algebra that powers of block-diagonal matrices are block-diagonal. Let $x, y \in V$ such that the xy-entry of Π is in the 0-block of Π. Then also $\Pi^k_{xy} = 0$ holds for all $k > 0$. But that means: there is no $k > 0$ such that x and y are connected by a path of length k in G. therefore Π could not be irreducible. ■

We only need a part of the statement of the Perron-Frobenius Theorem. Thus we only restate a part of it:

Theorem 5.6.1 (Theorem of Perron-Frobenius) *Let $0 \leq A \in \mathbb{R}^{n \times n}$ a non-negative irreducible matrix and λ an eigenvalue with maximal absolute value. Then the following holds:*

(a) The eigenspace of λ is of dimension 1.

(b) There is a (left-)eigenvector v of A to the eigenvalue λ with $0 < v$.

Proof. The complete and a bit technical proof could be found in [54, p. 356 ff.]. ■

In order to apply the theorem of Perron-Frobenius to stochastic matrices, we need:

Lemma 5.6.5 ([62]) *For all eigenvalues λ of Π, $|\lambda| \leq 1$ holds.* ■

Combining 5.6.2, 5.6.5 and 5.6.1 we get:

Lemma 5.6.6 *Let Π be irreducible. Then there exists a unique stationary distribution p of Π.* ■

As mentioned in the beginning of this section, we are interested in the existence of limiting distributions.

Lemma 5.6.7 ([62]) *Let Π be aperiodic and irreducible. Then the limit*

$$p := \lim_{t \to \infty} sP^t$$

exists for each starting distribution s. Furthermore, p is the only stationary distribution of Π and each row of $\lim_{t \to \infty} \Pi^t$ equals p. ■

5.6.2 Convergence results

Theorem 5.6.2 *Let G be strongly connected. If the underlying random walk of the non-negative randomized allocation mechanism $\phi = \phi^\alpha$ has a limiting distribution $(\rho_u)_{u\in V}$, then the sequence (α^t) converges for $t \to \infty$. And for all $xy \in A$ it is $\lim_{t\to\infty} \alpha^t_{xy} = \rho_x \pi_{xy}$, where π_{xy} is the transition probability of $x \to y$.*

Proof. Let Π be the transition matrix of the random walk associated with ϕ. It holds:

$$\lim_{t\to\infty} \alpha^t_{uw} = \lim_{t\to\infty} p^{t-1}(u)\pi_{uw}. \tag{*}$$

By the theorem of Perron-Frobenius ρ is the unique stationary distribution. Furthermore it is: $\lim_{t\to\infty} p^t = \rho$. Hence (*) equals:

$$\lim_{t\to\infty} \alpha^t_{uw} = \rho_u \pi_{uw}.$$

In particular: α^t converges for $t \to \infty$. ∎

Moreover, we can achieve a nice structural property of the limit of $(\alpha^t)_{t>0}$, if it exists:

Theorem 5.6.3 *Let G be strongly connected. If the sequence (α^t) converges for $t \to \infty$ to α^∞, then α^∞ is a circulation.*

Proof. Let $u \in V$. Then the following holds:

$$\begin{aligned}
\sum_{xu\in A} \alpha^\infty_{xu} &= \sum_{xu\in A} \lim_{t\to\infty} \alpha^t_{xu} = \lim_{t\to\infty} \sum_{xu\in A} p^{t-1}(x)\pi_{xu} \\
&= \lim_{t\to\infty} p^t(u) = \lim_{t\to\infty} \sum_{uw\in A} \pi_{uw} p^t(u) \\
&= \lim_{t\to\infty} \sum_{uw\in A} \alpha^{t+1}_{uw} = \sum_{uw\in A} \alpha^\infty_{uw}
\end{aligned}$$

Thus α^∞ is a circulation on G. ∎

The convergence of random walks gives us a solution to another practical problem, which we want to describe in the following example:

Example 5.6.1 *Assume there are two possible starting states s and $\tilde{s}$ in a cooperative situation. Moreover, assume that the participants already agreed to take pay offs* *according to a certain randomized allocation mechanism ϕ, and assume that there is disunity among two parties if the process should start in s or in $\tilde{s}$. The convergence results for allocation mechanism above then counter: if ϕ has a limiting distribution it is regardless in which of those two states the situation starts. The situation will converge to the same distribution in the long run.*

5.6.3 Examples of convergent allocation mechanisms

In this section we want to briefly describe the convergence behavior of the Shapley values Φ and Ψ defined in Section 5.4 by some examples. Recall that Φ is the allocation mechanism induced by the transition probabilities $\pi_{xy} := \frac{1}{d^+(x)}$.

Example 5.6.2 *Let $G = K_V$ the complete directed graph on V. The unique stationary distribution of Φ is given by $\pi(x) = \frac{1}{|V|}$ $(x \in V)$. This holds in a much more general context. Namely for regular graphs (i.e.: graphs in which $d^+(x) = d^-(x) =: d(x)$ holds for all $x \in V$ and $d(x) = d(y)$ for all $x, y \in V$). Note that such graphs satisfy condition (R) of Lemma 5.4.2. Therefore $\Psi = \Phi$ holds.*
The value that is allocated to an action $xy \in A$ by the Shapley value in the limit is:

$$\Phi_{xy}^{\infty}(v) = \alpha_{xy}^{\infty}(v(y) - v(x)) = \frac{1}{|V|}\pi_{xy}(v(y) - v(x)) = \frac{1}{|V|\,d(x)}(v(y) - v(x)).$$

Since $d(x)$ is independent of x in these graphs, we also get $\alpha_{xy}^{\infty} = \alpha_{uw}^{\infty}$ for all $xy, uw \in A$.

Example 5.6.3 *Let G be an undirected graph. We identify its edges $\{x, y\}$ with the directed arcs $(x, y), (y, x)$. By this we define a directed graph and call it also G. Furthermore, let Π be the underlying transition matrix of Φ*
It is a direct calculation to see that the limiting distribution of Π is given by

$$\pi(x) = \frac{d(x)}{\sum_{u \in N^+(x)} d(u)}.$$

This holds for more general graphs, namely for Eulerian graphs (i.e.: $d^+(x) = d^-(x)$ for all $x \in V$).
We get:

$$\begin{aligned}\Phi_{xy}(v) &= \alpha_{xy}^{\infty}(v(y) - v(x)) = \frac{d(x)}{\sum_{u \in N^+(x)} d(u)}\pi_{xy}(v(y) - v(x)) \\ &= \frac{1}{\sum_{u \in N^+(x)} d(u)}(v(y) - v(x)).\end{aligned}$$

Example 5.6.4 *Let G be acyclic. If d is the length of a maximal path in G which starts in s, then a random walk on G stops after d steps by acyclicity. In other words: $\alpha^t_{xy} = \alpha^d_{xy}$ for all $t > d$ and $xy \in A$. More precisely: set $t_x := \max\{j|x \in E_j\}$. Then $\alpha^t_{xy} = \alpha^{t_x+1}_{xy}$ for all $t > t_x$ and $y \in N^+(x)$. Hence each randomized allocation mechanism on G converges. Moreover: its limit equals the allocation mechanisms presented in the acyclic model in [37].*

5.7 Weber-allocation with coalitions

Till now, we considered very general situations and aimed for allocating certain values to all actions. Now we will have a more detailed view on situations, in which players, or whole coalitions of players need to be treated fair in a certain context.
For the rest of this section let $\Gamma := (N, V, A, \mathcal{A}, s)$ be a cooperation system in the sense of 3.5. Thus the action set is partitioned into blocks A_S ($S \subseteq N$) and each block indicates, which actions are governed by which coalition. We want to highlight the special case: $A_S = \emptyset$, if $|S| \neq 1$. This leads to a cooperative situation, in which only single players are allowed to perform actions. In the acyclic case this situation was investigated by Faigle and the author in [37]. We will treat the general case here. In a first step we define the value of $S \subseteq N$ relative to a given randomized allocation mechanism ϕ and a cooperative game v at time t to be:

$$\phi^t_S(v) := \sum_{xy \in A_S} \phi^t_{xy}(v).$$

In other words: S should get all value, that was generated by actions governed by S. This does not fully solve the allocation problem for the players in N. One question remains open: how should S distribute the value $\phi^t_S(v)$ among the players in S?
One could easily argue: it should be distributed equally among the players in S, since if any player $i \in S$ refuses his cooperation to take the action $x \to y \in A_S$, $S \setminus \{i\}$ is not able to take the action at all. However, one easily could think of situations in which the power ratios between the players in S are not equal by any external circumstances. To be more general and to capture also those situations, we make a further assumption:
If the players in a coalition S agree to perform an action $x \to y \in A_S$ there should be an agreement amongst the players in S from the outset, how worthy each single player is for S. If there is no such agreement, S is not stable enough to really take the action xy in common. We will model this agreement by a weightvector $(p^S_i)_{i\in S}$, $p^S_i \geq 0$, $\sum_{i\in S} p^S_i = 1$.

The value of player i then is given by his proportional value of the actions in which his presence was involved:

$$\phi_i^t(v) := \sum_{S \ni i, S \subseteq N} p_i^S \phi_S^t(v)$$

and we call $\phi_i^t(v)$ the *individual value* of player i w.r.t. v at time t.

5.8 Examples of classical values

In this section we want to demonstrate the power of the developed concept of allocation mechanisms, by giving various examples of classical values which arise as special cases of allocation mechanisms in our general model.
Let $\lambda \in \mathbb{R}^N_{>0}$. And set

$$\lambda_{xy} := \lambda_i \text{ if } xy \in A_i.$$

Then λ gives rise to a random walk on G by defining the following transition probabilities:

$$\pi_{xy} := \frac{\lambda_{xy}}{\sum_{z \in N^+(x)} \lambda_{xz}}.$$

Denote by ϕ^λ the allocation mechanism induced by the transition matrix

$$\Pi(\lambda) := (\pi_{xy})_{xy \in V \times V}$$

and call it λ-value.

Example 5.8.1 *Let $\lambda := (1, ..., 1)^T$. Then for all $xy \in A$*

$$\pi_{xy} = \frac{1}{d^+(x)}$$

holds. Hence ϕ^λ equals the symmetric Shapley value. Together with Lemma 5.4.2, ϕ^λ also equals the entropy-symmetric Shapley value if for all $t > 0$ and all $x, \tilde{x} \in E_t$: $d^+(x) = d^+(\tilde{x})$ holds.

Example 5.8.2 *In the acyclic case Faigle and the author [37] defined an analogue to λ-values. If k is the length of a largest source-sink path in G, then $(\phi^\lambda)^k$ equals the λ-value defined there. There it is shown that certain concepts as weighted Shapley values (cf. [83]) and, as a generalization of them, also weighted values of Kalai and Samet [57] (see also: Chun [21]) yield special cases of λ-values in the acyclic case. Hence they are also special cases of λ-values in our general context.*

Hence λ-values are a special class of randomized allocation mechanism which generalizes many classical value concepts and could be easily defined in our model.
There is a rich literature on generalizations of the classical value introduced by Shapley [82]. Faigle and Kern [34] introduced a Shapley value for games on precedence structures. They proposed a random order value in the sense of Section 5.3.1 by investigating the uniform distribution on all maximal paths in the precedence structure (they call such paths *rankings* of players). Hence their Shapley value yields a special case of the entropy-symmetric Shapley value in the special case of cooperation systems induced by precedence structures.
For instance Derks and Peters [23] investigate a model with restricted coalition structure, which is restricted by a certain set-function $\rho : 2^N \to 2^N$, such that the image of ρ is the set of feasible coalitions $\mathcal{F}$. Essentially they define a cooperative game on the classical cooperation structure *via* $\overline{v}(S) := v(\rho(S))$ and define the Shapley value of the game v on the restricted system to be the Shapley value of $\overline{v}$ on the classical cooperation system. Hence in this sense their Shapley value is just a special case of the classical one and in turn a special case of our Shapley values.

5.8.1 Example: the Banzhaf value

Another very famous value is the so called Banzhaf value introduced by Banzhaf [8] for voting situations and later generalized by Owen [72] to general cooperative games. The original idea of Banzhaf was the following: given a finite player set N and a monotone simple game $v : 2^N \to \{0,1\}$ the value of player i should depend on the number of coalitions which become "winning" if i joins them. This means coalitions $S \subseteq N \setminus i$, such that $v(S) = 0$ and $v(S \cup i) = 1$. Denote the number of such coalitions by η_i. Banzhaf proposed that the power of i in such a game should be measured *via* its proportional possibilities to "make" coalitions winning:

$$B_i(v) := \frac{\eta_i}{\sum_{j \in N} \eta_j}.$$

B_i is called the *Banzhaf voting index*. Owen [72] observed that

$$\eta_i = \sum_{S \subseteq N \setminus i} v(S \cup i) - v(S),$$

since $v(S \cup i) - v(S) = \begin{cases} 1 & \text{if } v(S) = 0 \text{ and } v(S \cup i) = 1 \\ 0 & \text{else wise.} \end{cases}$
This expression only depends on marginal values which gives a direct generalization to arbitrary classical cooperative games v. By definition of $B_i(v)$ only ratios of the η_i

matter. Thus one can multiply the vector $(\eta_i)_{i\in N}$ with a constant c without changing $B_i(v)$ in the case of a monotone simple game. For $c \in \mathbb{R}$ and a classical cooperative game (N, v) define the vector

$$\psi_i^c(v) := c \cdot \sum_{S\subseteq N\setminus i} v(S\cup i) - v(S).$$

Then ψ^c is a classical allocation mechanism and for all c and each monotone simple game v the proportional contribution $\frac{\psi_i^c(v)}{\sum_{j\in N}\psi_j^c(v)}$ equals the Banzhaf voting index of player i.
Owen proposed to take $c := \frac{1}{2^{n-1}}$ such that

$$\psi_i^c(v) = \frac{1}{2^{n-1}} \sum_{S\subseteq N\setminus i} v(S\cup i) - v(S)$$

becomes an expected marginal worth. Due to this proposal the allocation mechanism $\beta := \psi^{1/2^{n-1}}$ became famous under the name *Banzhaf value*. We want to define a linear value which coincides with the classical Banzhaf value in the classical case. Therefor we rewrite β in terms of the ζ-basis (*cf.* Section 4.4):

$$\beta_i(\zeta_S) = \begin{cases} \frac{1}{2^{|S\setminus i|}} & \text{if } i \in T \\ 0 & \text{else.} \end{cases}$$

A Banzhaf value in our model

For $t > 0$ set $A(t) := \{xy \in A | d(s,x) < t-1\}$. We define a linear and non-negative (*cf.* Theorems 5.2.2 & 5.2.3) allocation mechanism $\phi = \phi^\alpha$ by setting $\alpha_{xy}^1 := 0$, if $x \neq s$. For $t \geq 0$ and $xy \in A$ define:

$$\alpha_{xy}^{t+1} := \begin{cases} \frac{1}{|A_i \cap A(t+1)|} & \text{if } x \in E_t \text{ and } xy \in A_i. \\ \alpha_{xy}^t & \text{else.} \end{cases}$$

This expression is well-defined, since $xy \in A_i \cap A(t+1)$. Assume that Γ is the classical cooperation system on 2^N and let $S \in V$. Then for all $i \in S$

$$\begin{aligned} \phi_i^n(\zeta_S) &= \sum_{xy\in A_i} \phi_{xy}^n(\zeta_S) = \sum_{xy\in A_i} \alpha_{xy}^n(\zeta_S(y) - \zeta_S(x)) \\ &= \alpha_{S\setminus i,S}^n = \alpha_{S\setminus i,S}^{|S|} \\ &= \frac{1}{|A_i \cap A(|S|)|}. \end{aligned}$$

holds. If $i \notin S$, $\sum_{xy \in A_i} \phi^n_{xy}(\zeta_S) = 0$. Hence ϕ^n equals β on a basis of $\mathcal{G}$. Therefore ϕ^n is the Banzhaf value for classical cooperative games.
Also the Banzhaf value of Bilbao *et al.* [17] is covered by this generalization, since it arises in an analogous way in the special case of convex geometries.

5.8.2 The model of Faigle & Grabisch

In [31] Faigle and Grabisch propose a model for Markovian values for coalition formation processes. They consider a set $N = \{1, ..., n\}$ of players and call a sequence $\emptyset = S_0 S_1 S_2 ...$ of coalitions $S_i \subseteq N$ a *scenario* of a coalition formation process. The transition from a coalition S to a coalition T at time t is modeled by transition probabilities $\pi^t_{S,T}$, in order to account for incomplete information. Even if their general model considers time-dependent transition probabilities, they restrict to the time-independent case. Then the probability of a certain scenario $\mathcal{S} := S_0 S_1 S_2 ... S_k$ is given by

$$\prod_{i=1}^{k} \pi_{S_{i-1} S_i}.$$

This model is easily seen as a special case of our model: set $V := 2^N$, $A_S := \{UW \in 2^N \times 2^N | W \Delta U = S\}$ $(S \subseteq N)$, $s = \emptyset$. Then $(N, V, \bigcup A_S, \mathcal{A}, s)$ is a cooperation system and the transition probabilities given above yield random walks on its transition graph. For a given scenario $\mathcal{S} = S_0 ... S_t$ Faigle and Grabisch define the *scenario value* of $\mathcal{S}$ w.r.t. a game $v : V \to \mathbb{R}$ to be a vector $\phi^{\mathcal{S}}(v) \in \mathbb{R}^N$. For a given length $t > 0$ they define

$$\phi^t(v) = \sum_{\mathcal{S}, |\mathcal{S}| \leq t} \pi_{\mathcal{S}} \phi^{\mathcal{S}}(v).$$

For a scenario $\mathcal{S} = S_0 ... S_t$ and $i \in N$ they consider a special value and call it Shapley scenario value:

$$\phi^{\mathcal{S}}_i := \sum_{k | i \in S_k \Delta S_{k+1}} \frac{1}{S_k \Delta S_{k+1}} (v(S_{k+1}) - v(S_k)).$$

Hence Faigle and Grabisch consider the case in which each coalition agrees to divide all jointly generated values equally among its players (*cf.* individual values of Section 5.7). Thus the Shapley scenario value yields a special payoff rule according to the randomized allocation mechanism induced by the initially given transition probabilities $\pi_{S,T}$.

6 Tensor products

Now we will turn to the question:

How could two previously modeled settings be jointly modeled?

In Section 6.1, we give a general framework for jointly modeling two settings on different state spaces of the same type. Formally we build tensor products of states and argue, why this construction yields a *good* framework for our task.
Section 6.2 is dedicated to evolution operators of concatenated systems and their decompositions. We find a nice relation to Cartesian products of graphs in Subsection 6.2.2. Finally we have a look at tensor products of cooperative games in our model in Section 6.2.3 and give an example of a voting situation which is a tensor product of two independent voting situations.
We study randomized allocation mechanisms of concatenated systems in Section 6.3. On our way, we give a construction to define a random walk on the Cartesian product of two graphs out of random walks on its factors. This will give us the opportunity to assign to each pair of randomized allocation mechanisms on two arbitrary cooperation systems a randomized allocation mechanism on the product of these systems.

6.1 Concatenation of state spaces

Think of two situations that were previously modeled by ground states V and W and the state spaces $\mathcal{V}$ and $\mathcal{W}$ (of same type: Markovian or generalized Markovian). Assume one wants to model these two situations in a joint model. This is possible by the following definition. Set

$$\mathcal{V} \otimes \mathcal{W}$$

as the state space (of the same type as $\mathcal{V}$ and $\mathcal{W}$) induced by the ground states in $V \times W$ of pairs of ground states in V and W. Thus states $|p\rangle \in \mathcal{V} \otimes \mathcal{W}$ are of the form

$$|p\rangle = \sum_{(x,y) \in V \times W} c_{x,y} \, |x, y\rangle$$

with restriction to the scalars $c_{x,y}$ as needed by the type of state space.
Obviously by construction

$$\dim(\mathcal{V} \otimes \mathcal{W}) = \dim(\mathcal{V}) \cdot \dim(\mathcal{W})$$

holds.

Example 6.1.1 (Bicooperative games) *In [16] Bilbao et al. introduced so called bi-cooperative games. The idea behind these games is the following: assume a voting situation in which n players can vote for a certain circumstance, against it or be absent. All possible coalitions are pairs of disjoint subsets of the player set $N := \{1, ..., n\}$:*

$$3^N := \{(S, T) | S, T \subseteq N, S \cap T = \emptyset\}.$$

A bicooperative game is a function

$$b : 3^N \to \mathbb{R}.$$

This model could easily be extended to coalitions as arbitrary pairs of subsets of N by thinking of the players in $S \cap T$ for a coalition $(S, T) \in 2^N \times 2^N$ as double-minded. By setting $b(S, T) := b(S, T \setminus S)$ we extend a classical bicooperative game to the set of all pairs of subsets of players without losing any of the interpretation given by the model of bicooperative games. Thus one could interpret bicooperative games as games on the concatenated state space $2^N \otimes 2^N$.

Example 6.1.2 *Let (N, v) and (M, w) be classical cooperative games. By interpreting v, w as vectors, one can define the tensor product of v and w. This gives a natural definition of the tensor product of classical cooperative games:*

$$(N, v) \otimes (M, w) := (N \otimes M, v \otimes w),$$

where $(v \otimes w)(S, T) = v(S)w(T)$ for $S \subseteq N$, $T \subseteq M$.

Tensor products of states

For $|v\rangle \in \mathcal{V}$ and $|w\rangle \in \mathcal{W}$ with ground state representations

$$|v\rangle = \sum_{x \in V} c_x |x\rangle \text{ and } |w\rangle = \sum_{y \in W} d_y |y\rangle ,$$

we set in analogy to the concatenation of state spaces:

$$|v\rangle \otimes |w\rangle := \sum_{(x,y) \in V \times W} (c_x \cdot d_y) |x, y\rangle$$

and call $|v\rangle \otimes |w\rangle$ the *tensor product* of $|v\rangle$ and $|w\rangle$. By this definition one can identify:

$$|x\rangle \otimes |y\rangle = |x, y\rangle$$

for ground states $x \in V, y \in W$.
Note that indeed $|v\rangle \otimes |w\rangle$ is a state (of the same type as $|v\rangle$ and $|w\rangle$) since:

$$\sum_{xy \in V \times W} c_x \cdot d_y = \sum_{x \in V} c_x (\sum_{y \in W} d_y) = 1$$

in the Markovian and generalized Markovian case (and also non-negativity is preserved in the Markovian case).
Not all states $|z\rangle \in \mathcal{V} \otimes \mathcal{W}$ allow a representation as a tensor product of states in $\mathcal{V}$ and $\mathcal{W}$ as the following example shows:

Example 6.1.3 *Let* $V := \{x, y\}$ *and* $W := \{a, b\}$. *Set*

$$|z\rangle := \frac{1}{5} |x, a\rangle + \frac{3}{5} |x, b\rangle + \frac{1}{4} |y, a\rangle + \frac{1}{4} |y, b\rangle .$$

Then $|z\rangle$ *is irreducible and could not be expressed as a direct tensor product of states in* $\mathcal{V}$ *and* $\mathcal{W}$.

Without further mentioning we will use the following fact known from linear algebra.

Lemma 6.1.1 $\otimes$ *is bilinear and associative.* ■

6.2 Evolution operators of concatenated state spaces

Let $\mathcal{V}, \mathcal{W}$ be (generalized) Markovian state spaces, Φ an evolution operator on $\mathcal{V}$ and Ψ one on $\mathcal{W}$. Since we introduced in Section 6.1 the concept of concatenating state spaces, a natural question after defining evolution operators is

Is there a relation between evolution operators on $\mathcal{V} \otimes \mathcal{W}$ and those on $\mathcal{V}$ and $\mathcal{W}$?

The answer to this question is positive. But first recall the definition of the tensor product (sometimes also called Kronecker-product) of two matrices $A \in \mathbb{K}^{n\times n}, B \in \mathbb{K}^{m\times m}$:

$$(A \otimes B)_{(i,j),(k,l)} := a_{ij} b_{kl}$$

and the Kronecker-sum:

$$A \oplus B := A \otimes Id_m + Id_n \otimes B$$

Lemma 6.2.1 *Define* $(\Phi \boxdot \Psi)^t := (\frac{1}{2}(M(\Phi) \oplus M(\Psi)))^t$. *Then* $\Phi \boxdot \Psi$ *is an evolution operator on* $\mathcal{V} \otimes \mathcal{W}$.

Proof. We have to prove that for all $t > 0$ $(\Phi \boxdot \Psi)^t$ is state preserving. Thus it is enough to show that $\frac{1}{2}(M(\Phi) \oplus M(\Psi))$ is state preserving. Since the tensor products of the ground states yield a basis of $\mathbb{K}^{V\times W}$, each state $|p\rangle \in \mathcal{V} \otimes \mathcal{W}$ has a unique representation

$$|p\rangle = \sum_{x,y \in V\times W} c_{xy} |x, y\rangle .$$

By definition $M(\Phi) \boxdot M(\Psi)$ is a linear operator. Thus it is enough to prove the state preservation on the basis $V \times W$. Therefor let $x \in V$ and $y \in W$. Then

$$\begin{aligned}(\Phi \boxdot \Psi)^1 |x, y\rangle &= \frac{1}{2}(M(\Phi) \oplus M(\Psi)) |x, y\rangle \\ &= \frac{1}{2}(M(\Phi) |x\rangle \otimes Id |y\rangle + Id |x\rangle \otimes M(\Psi) |y\rangle)\end{aligned}$$

holds. Since $M(\Phi)$ and $M(\Psi)$ are state preserving by assumption, this term is a convex combination of two states and thus a generalized Markovian state itself. Furthermore,

$$(\Phi \boxdot \Psi)^0 = (\frac{1}{2}(M(\Phi) \oplus M(\Psi)))^0 = Id.$$

Hence $\Phi \boxdot \Psi$ is indeed an evolution operator on $\mathcal{V} \otimes \mathcal{W}$. ∎

REMARK Note that the last lemma in truth yields much more than only one evolution operator on $\mathcal{V} \otimes \mathcal{W}$ constructed from Ψ and Φ: the factor $\frac{1}{2}$ in the definition of $\boxdot$ was arbitrary. Any convex combination of the summands of the Kronecker-sum of $M(\Phi)$ and $M(\Psi)$ gives rise to an evolution operator on $\mathcal{V} \otimes \mathcal{W}$, since - as seen in the proof of the previous lemma - this is the only important property of this factor in this proof. However, we decided to consider evolution operators with the factor $\frac{1}{2}$ for an interpretative reason, which we dwell on in the end of this chapter. For other interpretations or application we had not thought of, also other weightings of the Kronecker-sum could yield *nice* evolution operators on the concatenated space.

Tensor products of systems

Given two generalized Markovian systems $\mathcal{S} = (V_1, \Phi)$ and $\mathcal{T} = (V_2, \Psi)$ we define the *tensor product* of $\mathcal{S}$ and $\mathcal{T}$ to be

$$\mathcal{S} \otimes \mathcal{T} := (V_1 \otimes V_2, \Phi \boxdot \Psi),$$

As proved in Lemma 6.2.1 this definition is well defined and $\Phi \boxdot \Psi$ is an evolution operator on $V_1 \otimes V_2$.

6.2.1 Products of graphs

Given two graphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$ the *product* of G_1 and G_2 is defined by the graph:

$$G_1 \otimes G_2 := (V_1 \times V_2, A_1 \otimes A_2),$$

where

$$A_1 \otimes A_2 := \{((u, u'), (v, v')) \in A_1 \times A_2 | u = v \text{ and } u'v' \in A_2 \text{ or } u' = v' \text{ and } uv \in A_1\}.$$

This notion of the product (also called *Cartesian product*) of two graphs was first defined and studied by Russell in [79].

Example 6.2.1 (Tensor products of graphs of set-systems) *Consider the set systems $\{\emptyset, \{i\}\}$ and $\{\emptyset, \{j\}, \{k\}, \{j, k\}\}$ and the underlying directed graphs, G_i and $G_{j,k}$, of the Hasse diagrams of the $\subseteq$-partial orders.*
As seen in Figure 6.1, the Cartesian product of G_i with $G_{j,k}$ is isomorphic to the Graph $G_{i,j,k}$ of the set system of a power set of 3 elements.

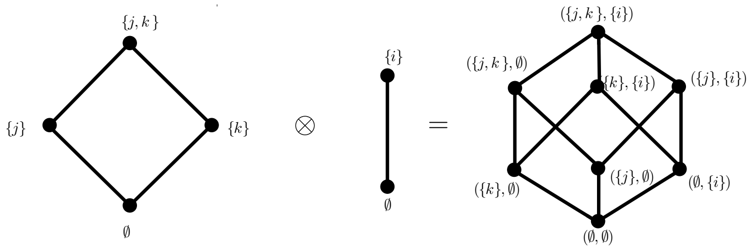

Figure 6.1:

We will show that this definition agrees with our notion of tensor products of systems. Therefore, it is safe to equally speak of tensor products of systems and products of the underlying transition graphs.

Lemma 6.2.2 *Let $\mathcal{S} = (V, \Phi)$ and $\mathcal{T} = (W, \Psi)$ be two (generalized) Markovian systems and $G_{\mathcal{S}}$ resp. $G_{\mathcal{T}}$ the underlying transition graphs. Then the following holds:*

$$G_{\mathcal{S}\otimes\mathcal{T}} = G_{\mathcal{S}} \otimes G_{\mathcal{T}}.$$

Proof. Observe first, that $V \times W$ is the vertex set of $G_{\mathcal{S}\otimes\mathcal{T}}$. For any pair of vertices $((u,u'),(v,v'))$ the following holds:

$$\begin{aligned} M(\Phi \boxdot \Psi)_{((u,u'),(v,v'))} &= (\frac{1}{2}(M(\Phi) \otimes M(\Psi)))_{((u,u'),(v,v'))} \\ &= \frac{1}{2}(M(\Phi) \otimes Id)_{((u,u'),(v,v'))} + \frac{1}{2}(Id \otimes M(\Psi))_{((u,u'),(v,v'))} \\ &= \frac{1}{2}M(\Phi)_{(u,v)} Id_{(u',v')} + \frac{1}{2} Id_{(u,v)} M(\Psi)_{(u',v')} \end{aligned}$$

By definition $((u,u'),(v,v'))$ is an arc of $G_{\mathcal{S}} \otimes G_{\mathcal{T}}$ if (w.l.o.g. by symmetry) $u = v$ and $u'v' \in A_2$. Hence $M(\Psi)_{(u',v')} \neq 0$. But this is exactly the case if

$$0 \neq \frac{1}{2}M(\Psi)_{(u',v')} = \frac{1}{2}M(\Phi)_{(u,u)} Id_{(u',v')} + \frac{1}{2} Id_{(u,u)} M(\Psi)_{(u',v')} = M(\Phi \boxdot \Psi)_{((u,u'),(u,v'))}.$$

Hence the arc sets of $G_{\mathcal{S}\otimes\mathcal{T}}$ and $G_{\mathcal{S}} \otimes G_{\mathcal{T}}$ are equal. ∎

A graph (resp. a system) is called *reducible* if it allows a representation as a tensor product, all of whose factors are strictly smaller (*i.e.* with strictly smaller vertex set) as the graph itself. And we call a graph (resp. system) irreducible if it is not reducible. In more formal words: A graph is reducible if and only if it is isomorphic to a tensor product of smaller graphs. Note that the product of graphs is associative (up to isomorphisms). An obvious observation is the following:

Lemma 6.2.3 *Let $G = (V, A)$ be a graph. Then there exists $k \in \mathbb{N}$ and irreducible graphs $G_1, ..., G_k$ ($G_i = (V_i, A_i)$) with*

$$G = \bigotimes_{i=1}^{k} G_i.$$

Proof. We use induction on $|V|$. If G itself is irreducible there is nothing to proof. Thus assume G to be reducible and that $G = G_1 \otimes G_2$ with $|V_i| < |V|$ ($i = 1, 2$). By induction G_1 and G_2 both admit a presentation as in the statement of this lemma. Hence also G does. ∎

A natural questions that arises is: is this presentation unique? Sabidussi [80] and independently of him Vizing [89] proved:

Theorem 6.2.1 *Let G be a finite graph. Then there exists a factorization of G into irreducible factors and this factorization is unique up to isomorphisms.* ∎

Moreover, Feigenbaum *et al.* [40] and Winkler [92] gave algorithms for finding this unique factorization in polynomial time.

6.2.2 Factorizations of systems

In a natural generalization of the term "reducible graph" we call a generalized Markovian system (V, Φ) reducible if there are nontrivial generalized Markovian systems $(V_1, \Phi_1), (V_2, \Phi_2)$ such that

$$(V, \Phi) = (V_1 \times V_2, \Phi_1 \boxdot \Phi_2).$$

By the same inductive argument as in Lemma 6.2.3 we find

Lemma 6.2.4 *Each system is $\otimes$-decomposable and has a factorization as tensor product of finitely irreducible systems.* ∎

Note that $\boxdot$ is not associative. Hence such a factorization could not be unique in general. A trivial observation is: since $|V| = |V_1|\,|V_2|$, (V, Φ) is irreducible if $|V|$ is prime.

6.2.3 Tensor products of cooperative games

In the previous sections we introduced the tensor product of game systems and evolution operators. We want to direct our attention more to the idea of building products of games. Products of classical cooperative games were already considered by Shapley [84] for the case of *simple games* (that is a classical monotone game (N, v) s.t. $v(S) \in \{0, 1\}$ for all $S \subseteq N$) under some restrictive assumptions. Owen [73] and Megiddo [64] generalized this concept to non-negative and monotone games. But in these works products were build such that the product of two games could only be defined relative to a third game - the so called *quotient game*. However, the idea of building products of games is not new. The mentioned papers essentially investigate decomposition theorems. In contrast to that, we are interested in applying Chapter 5 and in finding solutions for products of games.
Let us motivate the wish of building tensor-products of games by an example:

Example 6.2.2 (Decomposition of Elections) *A political party is electing its candidate for chancellor for the coming German "Bundestag". Assume the elections for the Bundestag are done and the coalition formation process is in progress. The question of how much power each party has in the coalition poker, was already discussed in Example 2.1.1. For party internal decisions it could be important, to analyze how much power a certain wing of the party has. Thus one could ask: how much influence has a certain party wing on the cast of the new chancellor of Germany?*
Obviously the answer to this question depends on two things: first of all, it depends on the influence of this wing in its own party, and second it depends on the influence of the party in the coalition poker. This could be modeled by the following classical cooperative game: let N be the set of all members of the party conference and M be the set of all parties taking part in the coalition poker. Define for $S = S_N \cup S_M \subseteq 2^{N \cup M}$

$$v(S) := \begin{cases} 1 & \textit{if } S_M \textit{ is up to governance, and } S_N \textit{ has a majority in its party.} \\ 0 & \textit{else.} \end{cases}$$

This game pictures all possible situations in which the party is able to govern Germany, and takes into account, how and by whom a certain candidate for chancellor could be chosen. A nearby question is: is v related to the coalition poker game and the underlying candidate election? The answer is: yes, it is!

Define by

$$w(S_N) := \begin{cases} 1 & \textit{if } S_N \textit{ has a majority at the party conference.} \\ 0 & \textit{else} \end{cases}$$

and by

$$u(S_M) := \begin{cases} 1 & \textit{if } S_M \textit{ is up to governance.} \\ 0 & \textit{else.} \end{cases}$$

Consider the tensor product $w \otimes u$ as defined in Example 6.1.2:

$$w \otimes u(S_N, S_M) = w(S_N)u(S_M).$$

Thus v decomposes into w and u; i.e.: $v = w \otimes u$.

This more or less simple example gives rise to a new question:

- If a game is decomposable, is there a relation between solutions to the factors of the game and solutions to itself?

We consider this question in the next section. We will give compositions of a wide range of solution concepts to factors of decomposable games there.
In general, the question, if a certain vector is decomposable by tensors, is very difficult. Even if the underlying system is decomposable, the games on this system need not to be:
Let Γ be a cooperation system and $G = G(\Gamma)$ its transition graph. Note that if G is irreducible with respect to $\otimes$ there could not exist any $\otimes$-decomposition of a game on G. Thus whenever asking for an irreducible decomposition of a game, it is necessary to ask for an irreducible decomposition of the underlying graph (or system) first. Thus assume that $G = G_1 \otimes G_2$ for strictly smaller graphs G_1 and G_2 and let $v \in \mathcal{G}$. $\otimes$-decomposability of a game could be characterized directly by the definition of the tensor product:

Lemma 6.2.5 *v is $\otimes$-decomposable by the decomposition $G = G_1 \otimes G_2$ if and only if there are games $v_i \in \mathcal{G}_i$ on G_i ($i = 1, 2$) such that for all $(x_1, x_2) \in V$:*

$$v(x_1, x_2) = v_1(x_1)v_2(x_2)$$

holds. In particular: the decomposability of G does not imply the decomposability of all games on G in general. ■

We will give a complete application of the theory developed in Chapter 5 to tensor-decomposable systems in the next section:

6.3 Allocation mechanisms of tensor-decomposable games

For the rest of this section let $G_1 = (V_1, A_1), G_2 = (V_2, A_2)$ be transition graphs of cooperation systems Γ_1 and Γ_2 with starting vertices s_1 resp. s_2 and Φ_1 resp. Φ_2 linear allocation mechanisms. Recall the definition of the Cartesian product $G := G_1 \otimes G_2 = (V_1 \times V_2, A)$ of the graphs G_1 and G_2 and the fact, that the adjacency matrix of G is the Kronecker-sum of the adjacency matrices of the G_i's mentioned in Section 6.2.1.

6.3.1 The randomized approach

Let Φ_1 (resp. Φ_2) be an allocation mechanism on G_1 (resp. G_2) which suffices (RAN). And let P_1 (resp. P_2) be the associated transition matrix. Assume that (RAN) is a commonly accepted fairness concept of players of games on G_1 and G_2. Is there a randomized allocation mechanism on G which is related to Φ_1 and Φ_2? The answer to this question is already known to us: in Section 5.5 we pointed out the relation between evolution operators of generalized Markovian systems and randomized allocation mechanisms. Lemma 6.2.2 yielded a construction for evolution operators on the tensor product of two systems, out of the evolution operators of its factors. We concatenated the underlying transition matrices *via*

$$P := P_1 \boxdot P_2 := \frac{1}{2}(P_1 \otimes Id + Id \otimes P_2)$$

and showed in Lemma 6.2.2 that P is a generalized stochastic matrix over $V_1 \times V_2$. Thus by Theorem 5.3.1 P induces a randomized allocation mechanism $\Phi := \Phi(P)$ on G.

REMARK The factor $\frac{1}{2}$ in the definition of P has an interpretation in terms of random walks: imagine a random walker on G walks according to P. If he reaches the vertex $(x_1, x_2) \in V_1 \times V_2$, at first he flips a coin and by that he decides to take the next step in the walk either in the graph G_1 or in the graph G_2. Assume w.l.o.g. he chose G_1. After that he performs the next step of the walk only on arcs, which are induced by G_1, *i.e.*: arcs of the form $(x_1, x_2)(y_1, x_2) \in A$ with $x_1 y_1 \in A_1$, with probability $(P_1)_{x_1 y_1}$. This process is described by the definition of $\boxdot$.

Entropy

Recall the definition of the entropy of a randomized allocation mechanism of Section 5.4.2 as a measure of symmetry and impartiality. Can we say something about the entropy of Φ depending on the entropies of Φ_1 and Φ_2?

Theorem 6.3.1 *Let Φ_1, Φ_2 be as above. Then $H(\Phi) \geq H(\Phi_1) + H(\Phi_2)$ holds.*

Before we are able to prove this theorem, we need more information about covering paths of length t_{max} (where again t_{max} is the time at which each arc of G is at least once visitable). For the rest of this subsection let $t := t_{max}(G)$, $t_i := t_{max}(G_i)$ $(i = 1, 2)$, P_1, P_2 the transition matrices of Φ_1 resp. Φ_2 and P the transition matrix of Φ. For any path $Q \in \mathcal{P}(G_i)$ we write $P_i(Q)$ for the probability of a random walker that starts in s_i to take the path Q w.r.t. P_i (*i.e.*: the product over the transition probabilities of the arcs of Q).

Lemma 6.3.1 *(a) The map $\Delta : \mathcal{P} \to \mathcal{P}(G_1) \times \mathcal{P}(G_2), Q \mapsto \Delta(Q) := (\Delta(Q)_1, \Delta(Q)_2)$*[1]*, where*

$$\Delta(Q)_i := \{(x^i_j, y^i_j) \in A_i | \text{there is } z^k_j \in V_k (k \neq i) \text{ with } (x^i_j, z^k_j), (y^i_j, z^k_j) \in Q\},$$

is surjective and for $Q \in \mathcal{P}(G)$:

$$\left|\{\tilde{Q} \in \mathcal{P}(G) | \Delta(Q) = \Delta(\tilde{Q})\}\right| = \binom{|Q|}{|\Delta(Q)_1|} = \binom{|Q|}{|\Delta(Q)_2|}.$$

(b) $t = t_1 + t_2$

(c) Let $Q \in \mathcal{P}_t(G)$. Then $P(Q) = \frac{1}{2^{|Q|}} P_1(\Delta(Q)_1) P_2(\Delta(Q)_2)$ holds.

Proof. (a): The surjectivity of Δ holds by concatenating paths in $\mathcal{P}(G_1)$ and $\mathcal{P}(G_2)$ to a path in $\mathcal{P}(G)$ and by identifying an arc in A_i with an accordant arc in A. To see the second statement of (a) we give a counting argument: any path $Q \in \mathcal{P}$ with predefined image $(P_1, P_2) = \Delta(Q)$ has $|\Delta(Q)_1|$ arcs "in" A_1, which could be in any position on Q. Thus we have to choose $|\Delta(Q)_1|$ positions for arcs of $\Delta(Q)_1$ on Q and for that, there are $\binom{|Q|}{|\Delta(Q)_1|}$ possibilities (and symmetrically also $\binom{|Q|}{|\Delta(Q)_2|}$ possibilities for fixing positions of arcs of $\Delta(Q)_2$).

(b): Let $(x_1, x_2) \in E_t(G)$. Since x_i is reachable by a path of length at most t_i in G_i, we have $t \leq t_1 + t_2$. On the other hand, if $d(s_i, x_i) = t_i$, it follows that $d(s_1, s_2; x_1, x_2) = t_1 + t_2$. Hence $t \geq t_1 + t_2$.

(c): Follows directly by the definition of $P := \frac{1}{2}(P_1 \otimes Id + Id \otimes P_2)$. ∎

[1] Δ splits a path $P \in \mathcal{P}$ into its unique *component paths* in G_1 and G_2.

Now we turn back to Theorem 6.3.1 and can give a proof:
Proof of Theorem 6.3.1.

$$
\begin{aligned}
&H(\Phi_1)+H(\Phi_2)\\
&=-\sum_{Q_1\in\mathcal{P}_{t_1}(G_1)} P_1(Q_1)\log(P_1(Q_1)) - \sum_{Q_2\in\mathcal{P}_{t_2}(G_2)} P_2(Q_2)\log(P_2(Q_2))\\
&\leq -\sum_{(Q_1,Q_2)\in\mathcal{P}_{t_1}(G_1)\times\mathcal{P}_{t_2}(G_2)} P_1(Q_1)\log(P_1(Q_1)) + P_2(Q_2)\log(P_2(Q_2))\\
&\leq -\sum_{(Q_1,Q_2)\in\mathcal{P}_{t_1}(G_1)\times\mathcal{P}_{t_2}(G_2)} P_1(Q_1)P_2(Q_2)\log(P_1(Q_1)) + P_1(Q_1)P_2(Q_2)\log(P_2(Q_2))\\
&\leq -\sum_{(Q_1,Q_2)\in\mathcal{P}_{t_1}(G_1)\times\mathcal{P}_{t_2}(G_2)} \frac{1}{2^{|Q_1|+|Q_2|}}P_1(Q_1)P_2(Q_2)(\log(P_1(Q_1)) + \log(P_2(Q_2)))\\
&\leq -\sum_{(Q_1,Q_2)\in\mathcal{P}_{t_1}(G_1)\times\mathcal{P}_{t_2}(G_2)} \frac{1}{2^{|Q_1|+|Q_2|}}P_1(Q_1)P_2(Q_2)\log(\frac{1}{2^{|Q_1|+|Q_2|}}P_1(Q_1)P_2(Q_2))\\
&\leq -\sum_{Q\in\mathcal{P}_t(G)} \frac{P_1(\Delta(Q)_1)P_2(\Delta(Q)_2)}{2^{|\Delta(Q)_1|+|\Delta(Q)_2|}}\log(\frac{P_1(\Delta(Q)_1)P_2(\Delta(Q)_2)}{2^{|\Delta(Q)_1|+|\Delta(Q)_2|}})\\
&=-\sum_{Q\in\mathcal{P}_t(G)} P(Q)\log(P(Q))\\
&=H(\Phi).
\end{aligned}
$$

Where the first inequality holds, since $-P_i(Q_i)\log(P_i(Q_i)) \geq 0$. Since $P_1(Q_1), P_2(Q_2) \geq P_1(Q_1)P_2(Q_2)$, $-P_1(Q_1)\log(P_1(Q_1))+P_2(Q_2)\log(P_2(Q_2)) \leq -P_1(Q_1)P_2(Q_2)\log(P_1(Q_1))-P_1(Q_1)P_2(Q_2)\log(P_2(Q_2))$ holds for all $(Q_1,Q_2)\in\mathcal{P}_{t_1}(G_1)\times\mathcal{P}_{t_2}(G_2)$ which yields the second inequality. An anologous argument for the factor $\frac{1}{2^{|Q_1|+|Q_2|}}$ and the monotonicity of the logarithm yield inequalities number three and four. By the last lemma Δ is surjective on $\mathcal{P}_{t_1}(G_1)\times\mathcal{P}_{t_2}(G_2)$. Hence the last inequality holds. ∎

Thus if it is an agreement (among all parties participating in the cooperative situation), that the entropy of a randomized allocation mechanism is a measure for fairness in a sense of impartiality, Theorem 6.3.1 yields the following interpretation: by concatenating allocation mechanisms, no impartiality is lost.
One could intuitively hope, that the entropy-symmetric Shapley value of Section 5.4.2 of the product graph G is the product of the entropy-symmetric Shapley values on its factors. But in most cases it is not. There are structural conditions needed in order to get the equality hoped for:

Corollary 6.3.1 *Let Ψ_1, Ψ_2, Ψ be the entropy-symmetric Shapley value on G_1, G_2 and $G := G_1 \otimes G_2$ respectively and t as above. Then $\Psi = \Psi_1 \boxdot \Psi_2$ holds if and only if*

$$|\mathcal{P}_t(G)| = 2^t \, |\mathcal{P}_k(G_1)| \, |\mathcal{P}_{t-k}(G_2)|$$

is true for all $k \leq t$.

Proof. Assume that $\Psi = \Psi_1 \boxdot \Psi_2$. Thus any path $Q \in \mathcal{P}_t(G)$ is chosen with equal probability $\frac{1}{|\mathcal{P}_t(G)|}$ with respect to the random walk P associated with $\Psi_1 \boxdot \Psi_2$. On the other hand, by Lemma 6.3.1 (c) we have: $P(Q) = \frac{1}{2^{|Q|}} P_1(\Delta(Q)_1) P_2(\Delta(Q)_2)$ for all $Q \in \mathcal{P}_t(G)$. Since Q is arbitrary, it follows that $|\mathcal{P}_t(G)| = 2^t \, |\mathcal{P}_k(G_1)| \, |\mathcal{P}_{t-k}(G_2)|$ for all $k \leq t$. All made conclusion were equivalence transformations and the statement is proven. ∎

6.3.2 The linear approach

Let $\Phi_1 = \Phi^\alpha$ and $\Phi_2 = \Phi^\beta$ be randomized allocation mechanisms on G_1 resp. G_2. We now turn to the questions: is it possible to give a sequence $(\gamma^t) \in (\mathbb{K}^A)^\infty$, depending on α and β, such that Φ^γ is a linear allocation mechanism on G? And if so, does Φ^γ equal $\Phi_1 \boxdot \Phi_2$?

Recall that we want to evaluate possibilities of taking actions. Since for any two paths of length t and t' in G_1 and G_2 there are $\binom{t+t'}{t}$ paths in G corresponding to them, the number of possibilities of being in a certain ground state in G is much bigger than it is in the factor graphs of G. In particular, the possibilities of taking the action $(x_1, x_2)(y_1, x_2)$ in G are more numerous than performing the action $x_1 y_1$ in G_1 because also paths in G_2 give rise to new paths in G which end in $(x_1, x_2)(y_1, x_2)$. This new possibilities for cooperation should be taken into account, if $(x_1, x_2)(y_1, x_2)$ is valuated.

We will give a possibility for defining γ in dependency of α and β first and argue afterwards why this possibility is the right choice. Note that α^t and β^t are completely known for every time t. For $(x_1, x_2)(y_1, y_2) \in A$ and $t > 0$ we set

$$\gamma^{t+1}_{(x_1,x_2)(y_1,y_2)} := \begin{cases} \gamma^t_{(x_1,x_2)(y_1,y_2)} & \text{if } (x_1, x_2) \notin E_t(G). \\ \frac{1}{2^{t+1}} \sum_{k=0}^t \binom{t}{k} \alpha^{k+1}_{x_1 y_1} \sum_{u_2 \in N^-(x_2)} \beta^{t-k}_{u_2 x_2} & \text{if } (x_1, x_2) \in E_t(G) \\ & \text{and } x_2 = y_2. \\ \frac{1}{2^{t+1}} \sum_{k=0}^t \binom{t}{k} \sum_{u_1 \in N^-(x_1)} \alpha^{k+1}_{u_1 x_1} \beta^{t-k}_{x_2 y_2} & \text{if } (x_1, x_2) \in E_t(G) \\ & \text{and } x_1 = y_1. \end{cases}$$

By Lemma 5.2.2, Φ^γ is a linear allocation mechanism. This construction is quite technical and a bit intransparent. An argument for defining γ in the way we did, is the following theorem:

Theorem 6.3.2 *Let Φ_1, Φ_2 and α, β, γ as above. Furthermore, let $\Phi := \Phi_1 \boxdot \Phi_2$. Then Φ^γ is a randomized allocation mechanism and equals Φ.*

Proof. It is sufficient to show that Φ equals Φ^γ, since by Section 6.3.1 Φ is a randomized allocation mechanism. By Theorem 5.2.1 Φ is uniquely determined by its sequence $(\kappa^t)_{t>0}$ of coefficient vectors. Thus we will show that $\gamma^t = \kappa^t$ for all $t > 0$ to prove this theorem. Section 5.3, tells us how κ^t is computed concretely. Therefor let $((x_1, x_2), (y_1, x_2)) \in A$ (w.l.o.g. we again only consider the case in which $x_2 = y_2$) and $t > 0$. Denote again by P, P_1, P_2 the associated transition matrices of Φ, Φ_1, Φ_2. Then

$$\kappa^{t+1}_{(x_1,x_2),(y_1,x_2)} = \left\langle s_1, s_2 \middle| P^t x_1, x_2 \right\rangle \langle x_1, x_2 | \, P y_1, x_2 \rangle \text{ if } (x_1, x_2) \in E_t(G)$$

and $\kappa^t_{(x_1,x_2),(y_1,x_2)} = \kappa^{t+1}_{(x_1,x_2),(y_1,x_2)}$ else wise. Since

$$\begin{aligned}
&\kappa^{t+1}_{(x_1,x_2),(y_1,x_2)} \\
&= \left\langle s_1, s_2 \middle| P^t x_1, x_2 \right\rangle \langle x_1, x_2 | \, P y_1, x_2 \rangle \\
&= \left\langle s_1, s_2 \middle| (\frac{1}{2}(P_1 \otimes Id + Id \otimes P_2)^t x_1, x_2 \right\rangle \left\langle x_1 \middle| (\frac{1}{2}(P_1 \otimes Id + Id \otimes P_2) y_1, x_2 \right\rangle
\end{aligned}$$

Note that $P_1 \otimes Id$ and $Id \otimes P_2$ commute (via matrix multiplication). Hence by the general ring-theoretic Binomial Theorem one has:

$$\begin{aligned}
(\frac{1}{2}(P_1 \otimes Id + Id \otimes P_2))^t &= \frac{1}{2^t} \sum_{k=0}^{t} \binom{t}{k} (P_1 \otimes Id)^k (Id \otimes P_2)^{t-k} \\
&= \frac{1}{2^t} \sum_{k=0}^{t} \binom{t}{k} (P_1^k \otimes Id)(Id \otimes P_2^{t-k}) \\
&= \frac{1}{2^t} \sum_{k=0}^{t} \binom{t}{k} (P_1^k \otimes P_2^{t-k})
\end{aligned}$$

The $(s_1, s_2)(x_1, x_2)$ component of this matrix equals

$$\frac{1}{2^t} \sum_{k=0}^{t} \binom{t}{k} \left\langle s_1 \middle| P_1^k x_1 \right\rangle \left\langle s_2 \middle| P_2^{t-k} x_2 \right\rangle .$$

All together we have:

$$
\begin{aligned}
&\kappa^{t+1}_{(x_1,x_2),(y_1,x_2)} \\
&= \frac{1}{2^t} \sum_{k=0}^{t} \binom{t}{k} \left\langle s_1 \middle| P_1^k x_1 \right\rangle \left\langle s_2 \middle| P_2^{t-k} x_2 \right\rangle \langle x_1, x_2 | P y_1, x_2 \rangle \\
&= \frac{1}{2^t} \sum_{k=0}^{t} \binom{t}{k} \left\langle s_1 \middle| P_1^k x_1 \right\rangle \left\langle s_2 \middle| P_2^{t-k} x_2 \right\rangle \frac{1}{2} \langle x_1 | P y_1 \rangle \\
&= \frac{1}{2^{t+1}} \sum_{k=0}^{t} \binom{t}{k} \alpha^{k+1}_{x_1 y_1} \left\langle s_2 \middle| P_2^{t-k} x_2 \right\rangle \\
&= \frac{1}{2^{t+1}} \sum_{k=0}^{t} \binom{t}{k} \alpha^{k+1}_{x_1 y_1} \left\langle s_2 \middle| P_2^{t-k} x_2 \right\rangle \sum_{w_2 \in N^+(x_2)} \langle x_2 | P_2 w_2 \rangle \\
&= \frac{1}{2^{t+1}} \sum_{k=0}^{t} \binom{t}{k} \alpha^{k+1}_{x_1 y_1} \sum_{w_2 \in N^+(x_2)} \beta^{t-k+1}_{x_2 w_2} \\
&= \frac{1}{2^{t+1}} \sum_{k=0}^{t} \binom{t}{k} \alpha^{k+1}_{x_1 y_1} \sum_{u_2 \in N^-(x_2)} \beta^{t-k}_{u_2 x_2}
\end{aligned}
$$

Tacitly we used the facts that $\sum_{w_2 \in N^+(x_2)} \langle x_2 | P_2 w_2 \rangle = 1$ since P_2 is a generalized stochastic matrix and $\sum_{w_2 \in N^+(x_2)} \beta^{t-k+1}_{x_2 w_2} = \sum_{u_2 \in N^-(x_2)} \beta^{t-k}_{u_2 x_2}$ since Φ_2 is time-efficient. All in all we get $\kappa^{t+1} = \gamma^{t+1}$ and with that the statement of the theorem is proven. ■

This theorem gives to each pair of randomized allocation mechanisms on factor systems a randomized allocation mechanism on their product. Hence it is quite easy to construct solutions to allocation problems on products of cooperations systems out of already *known* solutions.

7 More on allocation mechanisms

Efficiency is one of the most powerful axioms in our characterization of randomized allocation mechanisms (*cf.* Chapter 5). Section 7.1 extends our study of allocation mechanisms in another direction, by dropping the assumption (EFF). We characterize allocation mechanisms, which satisfy (LIN), (RAT) and (SYM) and give a generalization of the idea behind a theorem of Dubey, Neyman and Weber [26]. By giving a generalization of the classic dummy player axiom, we achieve a generalized representation of classical semi-values in our model.

We want to discuss another point of view. Assume that ϕ is an allocation mechanism. Our previous discussion of allocation mechanisms leads to a problem in certain cooperative settings: one could argue that the value of an action $xy \in A$ at time $t > 0$, should not only depend on the value that is created by xy at time t, but it should be a value that takes the "past" of the valuations of this arc more into account.

In Section 7.2 we introduce *Cesàro values*, which are a concept for fair time-dependent valuations of actions that heed the past in another way than allocation mechanisms do. The concept of Cesàro values is new to the theory of cooperative games. Already Faigle and Grabisch [31] mentioned an approach for Cesàro values in their model (*cf.* Section 5.8.2) but, however, did not study them.

Subsection 7.2.2 first gives some basic convergence results of Cesàro values and states the famous *Ergodic Theorem*. In the end of this section we will give a convergence result, which is mainly based on this theorem and states that many Cesàro values of linear and ratio fair allocation mechanism converge. We will see that, under some assumption on the graph, each Cesàro value of a non-negative randomized allocation mechanism converges. Subsequently we will discuss the few restricting assumptions of this convergence result in detail.

7.1 Semi-value theory

In the classical theory of cooperative games allocation mechanisms which are not efficient received some attention ([26], [25], [78]). Since the symmetry axiom (SYM) is a quite natural fairness requirement and since Shapley's Theorem [82] stated that there is exactly one symmetric randomized allocation mechanism, one needed to drop certain axioms in order to study symmetric allocation mechanisms. This was done by dropping the efficiency axiom. In this chapter we want to characterize some classes of allocation mechanisms which are not efficient. Dubey, Neyman and Weber [26] gave a characterization of symmetric, linear, monotone values which are not necessarily efficient but fulfill the so called *Dummy Axiom* (DA). In words (DA) means that, if the marginal contribution of a player is constant, his value should equal this constant. To be formal: let (N, v) be a classical cooperative game and $i \in N$. A classical allocation mechanism ϕ suffices the dummy player axiom if and only if

$$\text{If } v(S \cup i) - v(S) = v(i) \text{ for all } S \subseteq N \setminus \{i\}, \text{ then } \phi_i(v) = v(i). \tag{DA}$$

holds. Dubey, Neyman and Weber proved:

Theorem 7.1.1 ([26]) *Let (N, v) be a classical cooperative game and ϕ an allocation mechanism on (N, v). Then ϕ fulfills (LIN), (SYM), (NN) and (DA) if and only if there is a vector $p = (p_1, ..., p_n)$ with $p_i \geq 0$ and $\sum_{k=1}^{n-1} \binom{n-1}{k} p_k = 1$ such that*

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} p_{|S|}(v(S \cup i) - v(S)).$$

We aim for a generalization of the approach which lead to the definition of semi-values. On our way to it we will prove characterization results for classes of allocation mechanisms. Finally we will find a new representation of *semi-allocation mechanisms* in our model, which is also new to the classical model. We will vary from the classic definition of semi-values and will omit the dummy axiom. But in the end of this section we will give a generalization of the dummy axiom as well and will also give a representation of semi-allocation mechanisms that satisfy the generalized dummy axiom which coincides with the representation of Theorem 7.1.1 in the case of a classical cooperation system.

Let V be a finite set of ground states and $A \subseteq V \times V$ a set of feasible actions. We begin with a characterization of linear and ratio fair allocation mechanisms:

Theorem 7.1.2 *The allocation mechanism $\phi = \phi^\alpha$ enjoys the axioms (LIN) and (RAT) if and only if there exists a matrix $(b_{xy})_{xy\in V\times V} =: B \in \mathbb{K}^{V\times V}$ with the property $b_{xy} = 0$ if $xy \notin A$, such that for all $t \geq 0$, $xy \in A$ the following holds:*

$$\alpha_{xy}^{t+1} = \begin{cases} \left\langle s \middle| B^t x \right\rangle b_{xy} & \text{if } x \in E_t \\ \alpha_{xy}^t & \text{else.} \end{cases}$$

Proof. Assume first that $\phi = \phi^B$ is induced by a matrix B. Note that indeed ϕ^B is a linear allocation mechanism by Theorem 5.2.2. It remains to prove ratio fairness of ϕ. Therefor let $x \in E_t \cap E_{t'}$ ($t' > t$) and $y \in N^+(x)$. Then the following holds:

$$\begin{aligned}
\alpha_{xy}^{t+1} \sum_{u\in N^-(x)} \alpha_{ux}^{t'} &= \left\langle s \middle| B^t x \right\rangle b_{xy} \sum_{u\in N^-(x)} \left\langle s \middle| B^{t'-1} u \right\rangle b_{ux} \\
&= \left\langle s \middle| B^t x \right\rangle \langle x| By\rangle \sum_{u\in N^-(x)} \left\langle s \middle| B^{t'-1} u \right\rangle \langle u| Bx\rangle \\
&= \left\langle s \middle| B^t x \right\rangle \langle x| By\rangle \left\langle sB^{t'-1} \middle| \sum_{u\in N^-(x)} |u\rangle \langle u| Bx \right\rangle \\
&= \left\langle s \middle| B^t x \right\rangle \langle x| By\rangle \left\langle sB^{t'-1} \middle| \sum_{u\in V} |u\rangle \langle u| Bx \right\rangle \\
&= \left\langle s \middle| B^t x \right\rangle \langle x| By\rangle \left\langle sB^{t'-1} \middle| |Bx\rangle \right. \\
&= \left\langle s \middle| B^t x \right\rangle \langle x| By\rangle \left\langle s \middle| B^{t'} x \right\rangle \\
&= \left\langle s \middle| B^t x \right\rangle \alpha_{xy}^{t'+1} \\
&= \left\langle sB^{t-1} \middle| \sum_{u\in N^-(x)} |u\rangle \langle u| |Bx\rangle \right. \alpha_{xy}^{t'+1} \\
&= \sum_{u\in N^-(x)} \left\langle s \middle| B^{t-1} u \right\rangle \langle u| Bx\rangle \alpha_{xy}^{t'+1} \\
&= \sum_{u\in N^-(x)} \alpha_{ux}^t \alpha_{xy}^{t'+1}.
\end{aligned}$$

Thus ϕ satisfies (RAT). Assume the other way around that ϕ is linear and ratio efficient. For all $u \in N^+(x)$ set: $b_{su} := \alpha_{su}^1$. Let $xy \in A$. If there is a time $t > 0$ with $x \in E_t$ and $\sum_{u\in N^-(x)} \alpha_{ux}^t \neq 0$, define

$$b_{xy} := \frac{\alpha_{xy}^{t+1}}{\sum_{u\in N^-(x)} \alpha_{ux}^t} \text{ and } b_{xy} := 0 \text{ else wise.} \tag{*}$$

If $uw \notin A$, we set $b_{uw} := 0$. By (RAT) the definition of b_{xy} is well defined and independent of t. Set $B := (b_{xy})_{xy \in V \times V}$. It remains to prove

$$\alpha_{xy}^{t+1} = \left\langle s \middle| B^t x \right\rangle b_{xy} \text{ f.a. } t > 0,\, x \in E_t \text{ and } y \in N^+(x).$$

Therefor let $t \geq 0$, $x \in E_t$ and $y \in N^+(x)$. If $t = 0$, $x = s$ and $\left\langle s \middle| B^0 s \right\rangle b_{sy} = b_{sy} = \alpha_{sy}^1$ holds. Thus let $t > 0$ and the statement be true for all smaller t. As above we have:

$$\left\langle s \middle| B^t x \right\rangle b_{xy} = b_{xy} \sum_{u \in N^-(x)} \left\langle s \middle| B^{t-1} u \right\rangle \langle u | Bx \rangle .$$

By induction on t and the definition of B this equals:

$$b_{xy} \sum_{u \in N^-(x)} \alpha_{ux}^t = \alpha_{xy}^{t+1}.$$

Which completes the proof. ∎

REMARK We know from the last chapter, that B equals the transition matrix of the associated random walk if ϕ is efficient in addition to the assumptions of this theorem. Moreover, this theorem associates to arbitrary matrices a certain allocation mechanism which is linear and ratio fair.
A direct consequence of Theorem 7.1.2 and the representation of the α-values is the following:

Corollary 7.1.1 *Let $\phi = \phi^B$ be a linear and ratio efficient allocation mechanism.*

(a) ϕ is symmetric if and only if $b_{xy} = b_{xy'}$ for all $x \in V$ and $y, y' \in N^+(x)$.

(b) ϕ is non-negative if and only if $B \geq 0$.

Proof. (a) is clear by the representation of the α-values of Theorem 7.1. So we prove (b): assume first that ϕ is non-negative. Thus $\alpha^t \geq 0$ for all $t > 0$. For $x \in E_t$ we have as above:

$$\alpha_{xy}^{t+1} = b_{xy} \sum_{u \in N^-(x)} \alpha_{ux}^t. \tag{*}$$

Since $\alpha \geq 0$, also $b_{xy} \geq 0$ has to hold. Assume the other way around, that $B \geq 0$ holds. Let $t = 1$ then $\alpha_{sy}^1 = b_{sy} \geq 0$ for all $y \in N^+(x)$. For $t > 1$ (*) yields inductively $\alpha_{xy}^t \geq 0$ for all $xy \in A$. ∎

This corollary together with Theorem 7.1 yields a complete characterization of linear, ratio fair and symmetric allocation mechanisms. Hence we generalized the idea of dropping the efficiency assumption in order to study symmetric allocation mechanisms.

Theorem 7.1.3 *Let Φ be an allocation mechanism. Φ suffices (LIN), (SYM) and (RAT) if and only if there exists a vector $b \in \mathbb{R}^V$ such that (with $B := (b_{xy})_{xy \in A}$ $b_{xy} = b(x)$ if $xy \in A$ and 0 else wise) $\Phi = \Phi^B$ holds. Moreover: the set of all linear, symmetric and ratio fair allocation mechanisms is a vector space and the map $b \mapsto \Phi^b$ is a vector space isomorphism onto it. The dimension of this vector space equals $|V|$.*

Proof. By Theorem 7.1 there is a matrix B such that $\Phi = \Phi^B$. Since Φ is symmetric Corollary 7.1.1 (a) yields $b_{xy} = b_{xy'}$ for all $x \in V$ and all $y, y' \in N^+(x)$. Which proves the first statement of this theorem. That the map $b \mapsto \Phi^b$ is linear and injective is a direct consequence of the construction. This implies that the dimension of the space of all linear, symmetric and ratio fair allocation mechanisms equals $|V|$. ■

These results give us the possibility to characterize the (symmetric) Shapley value on another way:

Corollary 7.1.2 *Let $\Phi = \Phi^b$ be a linear, symmetric, non-negative and ratio fair allocation mechanism. Then Φ is efficient if and only if Φ equals the symmetric Shapley value. Moreover, this is the case if and only if $b(x) = \frac{1}{|N^+(x)|}$ for all $x \in V$.* ■

7.1.1 The Dummy axiom

In the beginning of this section we gave some characterization results of linear, ratio fair and symmetric allocation values. We still aim for a characterization and a definition of semi-values in our model. Thus we need to give a generalization of (DA) to our model. Since (DA) strongly depends on players and less on actions taken by them, we will deviate from our present basic idea of taking actions as main objects of study and will bring players more into the foreground. Therefore let $(N, V, A, \mathcal{A}, s)$ be a cooperation system in the sense of Section 3.5 and ϕ an allocation mechanism. For ease of notation we assume that arcs are governed by players; *i.e.*:

$$A = \bigcup_{i \in N} A_i,$$

but also the general case in which each coalition $S \subseteq N$ governs certain arcs is coverable by the following.
Let $t := t_{\max}$ be the cover time of G. We say that ϕ suffices the dummy player axiom if for all $v \in \mathcal{G}$ and all $i \in N$:

$$\text{If there exists } c \in \mathbb{R}, \text{ such that } v(y) - v(x) = c \text{ f.a. } xy \in A_i, \text{ then } \phi_i^t(v) = c. \tag{DA'}$$

holds. The idea behind this axiom is the following: if all actions of a certain player yield a fixed marginal contribution, this player should only be awarded this direct worth, he produces.
First of all, this general version of the dummy axiom agrees with the classical dummy player axiom in the classic case:

Lemma 7.1.1 *Let (N, v) be a classical cooperative game and ϕ be an allocation mechanism on the classical cooperation system. Then ϕ^n suffices (DA') if and only if ϕ suffices (DA).*

Proof. If (DA') holds, also the dummy property at time n is fulfilled. But n is the cover time of the classical cooperation graph. Thus given a classical dummy player i, all marginal contributions of this player are constant. Moreover, this constant needs to equal $v(i)$ since $v(i) = v(i) - 0 = v(i) - v(\emptyset)$. Hence (DA) holds. The same argument in the backward order yields the other implication. ∎

Since (DA') and (DA) are equivalent in the classic case, we omit the prime in the following. There is a nice characterization of allocation mechanisms which fulfill (LIN) and (DA):

Theorem 7.1.4 *Let $\Phi = \Phi^\alpha$ be a linear allocation mechanism. Then Φ suffices the dummy axiom, if and only if*

$$\sum_{xy \in A_i} \alpha_{xy}^t = 1$$

for $t = t_{\max}$.

Proof. Let v be a game on G such that player i is a dummy player (*i.e.*: has constant marginal contributions c_i). Then

$$\Phi_i^t(v) = \sum_{xy \in A_i} \Phi_{xy}^t(v) = \sum_{xy \in A_i} \alpha_{xy}^t(v(y) - v(x)) = \sum_{xy \in A_i} \alpha_{xy}^t c_i.$$

Thus (DA) holds, if and only if

$$c_i = \Phi_i^t(v) = \sum_{xy \in A_i} \alpha_{xy}^t c_i$$

which is equivalent to:

$$\sum_{xy \in A_i} \alpha_{xy}^t = 1.$$

■

This implies a generalized version of the representation of individual values Dubey, Neyman and Weber 7.1.1 gave:

Corollary 7.1.3 *Let $\Phi = \Phi^\alpha$ be an allocation mechanism that suffices (LIN), (RAT), (DA) and (SYM) and let $t := t_{\max}$. Then there are coefficients $\alpha^t(x)$ such that*

$$1 = \sum_{x \in V} \alpha^t(x) \frac{1}{n} \left| N^+(x) \right|$$

and

$$\phi_i^t(v) = \sum_{xy \in A_i} \alpha^t(x)(v(y) - v(x)).$$

Proof. By the last theorem we have $1 = \sum_{xy \in A_i} \alpha_{xy}^t$. Therefore $n = \sum_{xy \in A} \alpha_{xy}^t$. By Lemma 7.1.1 (SYM) is equivalent to $\alpha_{xy}^t = \alpha_{x\overline{y}}^t$ for all $x \in V$ and all $y, \overline{y} \in N^+(x)$. Thus $\alpha^t(x) := \alpha_{xy}^t$ only depends on x. Then

$$n = \sum_{xy \in A} \alpha_{xy}^t = \sum_{x \in V} \alpha^t(x) \left| N^+(x) \right|. \qquad (*)$$

Dividing by n yields the desired statement. ■

REMARK For classical cooperative games and a coalition $S \subseteq N$ there are $|N \setminus S|$ out-neighbors of S in the transition graph. On the other hand, to a fixed cardinality $s \in \{1, ..., n\}$ there are $\binom{n}{s}$ coalitions of this cardinality. By the symmetry of the classical cooperation graph one could prove that $\alpha^t(S) = \alpha^t(T)$ if $|S| = |T|$. And by that one gets:

$$\begin{aligned} 1 &= \sum_{x \in V} \alpha^t(x) \frac{\left|N^+(x)\right|}{n} = \sum_{S \in 2^N} \alpha^t(|S|) \frac{\left|N^+(S)\right|}{n} \\ &= \sum_{|S|=1}^{n} \alpha^t(|S|) \binom{n}{s} \frac{n-|S|}{n} \\ &= \sum_{|S|=1}^{n} \alpha^t(|S|) \binom{n-1}{s}. \end{aligned}$$

Hence this corollary indeed implies the value representation of Dubey, Neyman and Weber.

7.2 Cesàro values

We start by giving an example:

Example 7.2.1 *Assume a cooperative setting, such that player i governs the only out-arc, sa, of s. Hence at time $t = 0$ player 1 governs the only possibility for the whole cooperative process to take place. Furthermore, assume that there is a possibility to bring the system back to s at time $t' > 0$ with probability strictly smaller than 1. Thus at time $t' + 1$, the value of the action sa has decreased compared to its value at time t'. Thus player i could raise the following plea: if it is not guaranteed that his engagement to start the whole process is esteemed more in the future, he will deny his cooperation.*

We come across this plea by taking an average. Therefor we consider a new valuation of actions to a given allocation mechanism ϕ:

$$\overline{\phi}^t_{xy} := \frac{1}{t} \sum_{i=1}^{t} \phi^i_{xy}. \quad (t > 0, xy \in A).$$

$\overline{\phi}$ allocates to every action xy at time t its so called *Cesàro average* of the previously generated values of this action. Because of its relation to the Cesàro average we will call this valuation the *Cesàro value* of ϕ. Building Cesàro averages of allocation mechanisms was already softly foretelled by Faigle and Grabisch [31] (*cf.* the model in Section 5.8.2). However, the possibility to build these averages in an allocation context was just mentioned there. Cesàro values are a good argument against the above stated plea, only if certain fairness criteria are passed over to them. Thus immediately two questions arise:

- Is the Cesàro value of a randomized allocation mechanism at least as fair as the original value is?
- When do Cesàro values converge?

In the rest of this section we aim for answers to these questions.

7.2.1 Fairness of Cesàro values

As seen before, $\overline{\phi}$ does not need to suffice (A3) and therefore it is no allocation mechanism in the sense of the definition given in 5.2.1.

Lemma 7.2.1 *Let ϕ be an allocation mechanism.*

(a) $\overline{\phi}$ satisfies (A1) and (A2).

(b) If $\phi = \phi^\alpha$ suffices (LIN), also $\overline{\phi}$ does. Moreover,

$$\overline{\phi}^t_{xy}(v) = \frac{1}{t}\sum_{i=1}^{t} \alpha^i_{xy}(v(y) - v(x))$$

.

(c) If ϕ is non-negative, also $\overline{\phi}$ is.

(d) If ϕ is symmetric, then also $\overline{\phi}$ is.

Proof.

(a) Let $v, v' \in \mathcal{G}$, $xy \in A$ with $v(x) = v'(x), v(y) = v'(y)$ and $t > 0$. Then

$$\overline{\phi}^t_{xy}(v) = \frac{1}{t}\sum_{i=1}^{t} \phi^i_{xy}(v) = \frac{1}{t}\sum_{i=1}^{t} \phi^i_{xy}(v') = \overline{\phi}^t_{xy}(v')$$

and thus (A1) holds. In the same way (A2) follows directly for $\overline{\phi}$ by (A2) holding for ϕ.

(b) Assume ϕ^t is a linear map for all $t > 0$. Then also $\frac{1}{t}\sum_{i=1}^{t} \phi^i$ is a linear map. Let $t > 0$, $xy \in A$ and $v \in \mathcal{G}$. Then

$$\overline{\phi}^t_{xy}(v) = \frac{1}{t}\sum_{i=1}^{t} \phi^i_{xy}(v) = \frac{1}{t}\sum_{i=1}^{t} \alpha^i_{xy}(v(y) - v(x)).$$

(c) Let $v \in \mathcal{G}$ and $xy \in A$ with $v(x) \leq v(y)$. Then for all $t > 0$, $\phi^t_{xy}(v) \geq 0$ holds. Hence $\frac{1}{t}\sum_{i=1}^{t} \phi^i_{xy}(v) \geq 0$ which proves the non-negativity of $\overline{\phi}$.

(d) Let $v \in \mathcal{G}$ and $xy, xz \in A$ with $v(z) = v(y)$. Since ϕ is symmetric, we have $\phi^t_{xy} = \phi^t_{xz}$ for all $t > 0$. Hence

$$\overline{\phi}^t_{xy} = \frac{1}{t}\sum_{i=1}^{t} \phi^t_{xy} = \frac{1}{t}\sum_{i=1}^{t} \phi^t_{xz} = \overline{\phi}^t_{xz}.$$

Thus $\overline{\phi}$ is symmetric. ■

7.2.2 Convergence of Cesàro values

The limit theorem of Cauchy, well known in calculus states:

Lemma 7.2.2 (Theorem of Cauchy) *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ which converges to $a \in \mathbb{R}$ and define $c_n := \frac{1}{n}\sum_{i=1}^{n} a_n$. Then also $(c_n)_{n\in\mathbb{N}}$ converges and its limit equals a. In this case the sequence $(a_n)_{n\in\mathbb{N}}$ is called Cesàro-summable.* ■

Note that the converse is not true. A Cesàro-summable sequence must not converge, as shown by the following example:

Example 7.2.2 *Let $a_n := (-1)^n$. The sequence $(a_n)_{n\in\mathbb{N}}$ diverges. But the Cesàro-averages*

$$c_n := \frac{1}{n}\sum_{i=1}^{n}(-1)^n$$

converge to 0. Hence $(a_n)_{n\in\mathbb{N}}$ is not Cesàro-summable.

The limit theorem of Cauchy directly gives us an answer to one of our stated questions:

Corollary 7.2.1 *If ϕ converges to ϕ^∞ for $t \to \infty$, then also $\overline{\phi}$ converges and*

$$\lim_{t\to\infty} \overline{\phi^t} = \phi^\infty$$

holds. ■

Thus in the long run it does not matter, if we allocate by ϕ or $\overline{\phi}$. One can say even more on the convergence of Cesàro values of randomized allocation mechanism by applying the so called *Ergodic Theorem*:

Theorem 7.2.1 ([54, Satz 3.7.]) *Let $\|\cdot\|$ be an algebra norm on $\mathbb{K}^{n\times n}$ and let $A \in \mathbb{K}^{n\times n}$ with $\|A\| \leq 1$. Then the limit*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} A^i$$

exists. ■

We will prove the following theorem:

Theorem 7.2.2 *Let G be strongly connected and let $\phi = \phi^B$ be a linear and ratio-fair allocation mechanism. Let $\|\cdot\|$ be an algebra norm on $\mathbb{K}^{V\times V}$. If $\|B\| \leq 1$, then*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} \phi^t$$

exists. Hence the Cesàro value $\overline{\phi}$ associated to ϕ converges.

In order to prove this theorem, we need some lemmata. The first one is more or less mathematical folklore and is not due to a single author. Since all citable resources the author found leave this statement as an exercise to the reader, we will give a short proof here:

Lemma 7.2.3 *Let $l_1, ..., l_m \in \mathbb{N}$ such that $gcd(l_1, ..., l_m) = 1$. Then there exists $F \in \mathbb{N}$, such that for all $n' \geq F$ there are $c_1, ..., c_m \in \mathbb{N}$ with $n' = \sum_{i=1}^m c_i l_i$.*

Proof. Either by the fact that $\mathbb{Z}$ is a principal ideal ring or by the Euclidean algorithm one sees that, $gcd(l_1, ..., l_m)\mathbb{Z} = \sum_{i=1}^m l_i\mathbb{Z}$. Hence $1 = \sum_{i=1}^m a_i l_i$ for some $a_i \in \mathbb{Z}$. W.l.o.g. assume that $a_i < 0$ for $i = 1, ..., k$ and $a_i \geq 0$ for $i = k+1, ..., m$. Set

$$n := -(l_1 - 1) \sum_{i=1}^{k} a_i l_i.$$

Then n is a non-negative linear combination of the l_i. Moreover, it is

$$n+1 = n+\sum_{i=1}^{m} a_i l_i = -(l_1-1)\sum_{i=1}^{k} a_i l_i+\sum_{i=1}^{k} a_i l_i+\sum_{i=k+1}^{m} a_i l_i = (2-l_i)\sum_{i=1}^{k} a_i l_i+\sum_{i=k+1}^{m} a_i l.$$

Hence also $n+1$ is a non-negative linear combination of the l_i. This procedure could be iterated (l_1-1) times and we get a representation of $n+l_1-1$. This completes the proof, since with n, also $n+l_1$ is a non-negative linear combination of the l_i and with $n+1$, also $n+l_1+1$ is one too. By induction each integer, which is greater or equal to n is a non-negative linear combination of the l_i. ∎

Corollary 7.2.2 *Let $l_1, ..., l_m \in \mathbb{N}$. Then there exists $N \geq 0$, such that:*

$$\{N + l * gcd(l_1, ..., l_m) | l \in \mathbb{N}\} = \{\sum_{i=1}^{m} c_i l_i \geq N | c_i \in \mathbb{N}\}.$$

Proof. Let $g := gcd(l_1, ..., l_m)$. Then $\frac{l_1}{g}, ..., \frac{l_m}{g}$ are relatively prime. Hence Lemma 7.2.3 guarantees the existence of $F := F(\frac{l_1}{g}, ..., \frac{l_m}{g}) \in \mathbb{N}$ such that F and all $n > F$ are a non-negative linear combinations of the $\frac{l_i}{g}$. That is:

$$\{F + l | l \in \mathbb{N}\} = \{\sum_{i=1}^{m} c_i \frac{l_i}{g} \geq F | c_i \in \mathbb{N}\}.$$

Multiplying this by g yields:

$$\{F * g + l * g | l \in \mathbb{N}\} = \{\sum_{i=1}^{m} c_i l_i \geq F * g | c_i \in \mathbb{N}\}.$$

Hence by setting $N := F * g$ the statement is proven. ∎

REMARK The problem of finding the largest natural number which is not representable as a non-negative linear combination of relatively prime $l_1, ..., l_m \in \mathbb{N}$ is known as the coin-exchange problem of Frobenius (*cf.* [9]). A closed form of this number is known for $m = 1, 2, 3$. No closed-form solution is known for $m \geq 4$. For arbitrary m and arbitrary l_i the problem of finding the Frobenius number of the l_i is known to be NP-hard [75]. Note that we are not interested in a concrete computation. We only need the existence of this number in order to identify certain periodicities in our setting. These periodicities will yield the desired convergence result.

Lemma 7.2.4 *Let $G = (V, A)$ be a strongly connected directed graph. Then for all $x \in V$ the following statement is true: there are $N(x), k(x) \in \mathbb{N}$ such that $x \in E_{N(x)}$ and*

$$\{n \geq N(x) | x \in E_n\} = \{N(x) + l * k(x) | l \in \mathbb{N}\}.$$

Proof. Let $l_1, ..., l_m$ be the lengths of circles through x such that x occurs exactly twice (*i.e.*: as start- and end-vertex of a circle). Let $n \in \mathbb{N}$ such that $x \in E_n$. Then there exist a path $P = (s = x_0, ..., x_n = x)$ from s to t with $|P| = n$. Hence there is a minimal k such that $x_k = x$. Thus P could be divided into the path $x_0...x_k$ and some combination of circles through x for which reason $n = k + \sum_{i=1}^{m} c_i l_i$ has to hold for some $c_i \in \mathbb{N}$. The other way around: each such combination yields a path such that x is its end-vertex. Set $g(x) := gcd(l_1, ..., l_m)$. By Corollary 7.2.2 there exists $\tilde{N}(x)$ such that

$$\{\tilde{N}(x) + l * g(x) | l \in \mathbb{N}\} = \{\sum_{i=1}^{m} c_i l_i \geq \tilde{N}(x) | c_i \in \mathbb{N}\}.$$

Let $k_1, ..., k_t$ be the lengths of all smooth paths from s to x and assume that k_1 is maximal beneath these. We have

$$\{n \in \mathbb{N} | x \in E_n\} = \{k_1, ..., k_t\} + \{\sum_{i=1}^{m} c_i l_i | c_i \in \mathbb{N}\}.$$

Since G is strongly connected, there is a smooth path Q from x to s. Let $|Q| = q$. Hence there are circles of length $q + k_i$ through x for all $i = 1, ..., t$. Set $N(x) := k_1 + \tilde{N}(x)$. hence we find:

$$\begin{aligned}\{n \geq N(x) | x \in E_n\} &= \{k_1, ..., k_t\} + \{\sum_{i=1}^{m} c_i l_i \geq \tilde{N(x)} | c_i \in \mathbb{N}\} \\ &= \{k_1, ..., k_t\} + \{\tilde{N}(x) + l * g(x) | l \in \mathbb{N}\}.\end{aligned}$$

W.l.o.g. assume that $q + k_i = l_i$. Hence by dividing by $g(x)$, all k_i have the same remainder. Thus

$$\{k_1, ..., k_t\} + \{\tilde{N}(x) + l * g(x) | l \in \mathbb{N}\} = \{k_t + \tilde{N}(x) + l * g(x) | l \in \mathbb{N}\} = \{N(x) + l * g(x) | l \in \mathbb{N}\}$$

holds. ∎

Lemma 7.2.5 *Let G be strongly connected and let $P \in \mathbb{K}^{V \times V}$ be a matrix which is compatible with G such that $\|P\| \leq 1$ for some algebra norm on $\mathbb{K}^{V \times V}$. Let $t > 0$ and define for $0 < i < t$: $f(i, x) := \max\{j \leq i | x \in E_j\}$ and $s(i, t, x) := |\{j \leq t - 1 | f(j, x) = i\}|$. Then for all $x \in V$ the following limit exists:*

$$\lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} s(i, t, x) P^i.$$

Proof. Let $x \in V$, let $N(x)$ and $k(x)$ be as in Lemma 7.2.4 and let $t > N(x)$. Then

$$\frac{1}{t}\sum_{i=0}^{t-1} s(i,t,x)P^i = \frac{1}{t}\sum_{i=0}^{N(x)-1} s(i,t,x)P^i + \frac{1}{t}\sum_{i=N(x)}^{t-1} s(i,t,x)P^i$$

holds. For all $i \leq N(x)$ it is $s(i,t,x) \leq N(x)$. Hence $\frac{1}{t}\sum_{i=0}^{N(x)-1} s(i,t,x)P^i \to 0$ for $t \to \infty$. Divide $t-1-N(x)$ by $k(x)$ with remainder. Then there are a_t, b_t s.t. $t-1-N(x) = a_t * k(x) + b_t$ and $b_t < k(x)$. Hence we find

$$\begin{aligned}\frac{1}{t}\sum_{i=N(x)}^{t-1} s(i,t,x)P^i &= \frac{1}{t}\sum_{i=N(x)}^{N(x)+a_t*k(x)+b_t} s(i,t,x)P^i \\ =&\frac{1}{t}\sum_{i=N(x)}^{N(x)+a_t*k(x)} s(i,t,x)P^i + \frac{1}{t}\sum_{i=N(x)+a_t*k(x)}^{N(x)+a_t*k(x)+b_t} s(i,t,x)P^i.\end{aligned}$$

For $N(x) + a_t * k(x) < i \leq N(x) + a_t * k(x) + b_t$, $x \notin E_i$ holds. Thus: $s(i,t,x) = 0$. Hence the second sum equals:

$$\begin{aligned}\frac{1}{t}\sum_{i=N(x)+a_t*k(x)}^{N(x)+a_t*k(x)+b_t} s(i,t,x)P^i &= \frac{s(N(x)+a_t*k(x),t,x)}{t}P^{N(x)+a_t*k(x)} \\ &= \frac{b_t}{t}P^{N(x)+a_t*k(x)}.\end{aligned}$$

Since b_t is bounded by $k(x)$, this term converges to 0 for $t \to \infty$:

$$\left\|\frac{b_t}{t}P^{N(x)+a_t*k(x)}\right\| \leq \frac{k(x)}{t}\left\|P\right\|^{N(x)+a_t*k(x)} = \frac{k(x)}{t} * 1 \to 0$$

All in all we have until now:

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=0}^{t-1} s(i,t,x)P^i = \lim_{t\to\infty}\frac{1}{t}\sum_{i=N(x)}^{N(x)+a_t*k(x)} s(i,t,x)P^i.$$

Since $s(i,t,x) = 0$ for $x \notin E_i$, it holds:

$$\frac{1}{t}\sum_{i=N(x)}^{N(x)+a_t*k(x)} s(i,t,x)P^i = \frac{1}{t}\sum_{i=N(x),x\in E_i}^{N(x)+a_t*k(x)} s(i,t,x)P^i.$$

Those indices are well-known by Lemma 7.2.4:

$$= \frac{1}{t}\sum_{i=0}^{a_t} s(N(x)+i*k(x),t,x)P^{N(x)+i*k(x)}$$

Moreover it is: $s(N(x) + i * k(x), t, x) = k(x)$. Hence:

$$= P^{N(x)} \frac{1}{t} \sum_{i=1}^{a_t} k(x)(P^{k(x)})^i = P^{N(x)} \frac{k(x)}{t} \sum_{i=0}^{a_t} (P^{k(x)})^i.$$

Since $\|P\| \leq 1$, also $\left\|P^{k(x)}\right\| \leq 1$. Since $a_t \to \infty$ for $t \to \infty$, Theorem 7.2.1 implies:

$$\lim_{a_t \to \infty} \frac{1}{a_t} \sum_{i=0}^{a_t} (P^{k(x)})^i$$

exists. We will show that $\frac{k(x)}{t} \sum_{i=0}^{a_t} (P^{k(x)})^i$ converges to $\lim_{t\to\infty} \frac{1}{a_t} \sum_{i=0}^{a_t} (P^{k(x)})^i$. It suffices to show that $\left|\frac{k(x)}{t} - \frac{1}{a_t}\right| \to 0$ for $t \to \infty$. The following holds.

$$\left|\frac{k(x)}{t} - \frac{1}{a_t}\right| = \left|\frac{a_t k(x) - t}{a_t * t}\right|$$

On the other hand it is $t = a_t * k(x) + b_t + N(x) + 1$. Hence:

$$= \left|\frac{a_t k(x) - a_t k(x) - b_t - N(x) - 1}{a_t * t}\right| = \left|\frac{-b_t - N(x) - 1}{a_t * t}\right|.$$

This expression converges to 0 for $t \to \infty$, since b_t is bounded by $k(x)$. ∎

Now we are able to prove the desired convergence:

Proof of Theorem 7.2.2. Let $xy \in A$. And $\phi = \phi^B$ a linear and ratio fair allocation mechanism s.t. $\|B\| \leq 1$. We have to prove: $\frac{1}{t} \sum_{i=1}^{t} \alpha_{xy}^i$ converges for $t \to \infty$. The following holds:

$$\begin{aligned}
\frac{1}{t} \sum_{i=1}^{t} \alpha_{xy}^i &= \frac{1}{t} \sum_{i=0}^{t-1} \alpha_{xy}^{i+1} \\
&\overset{(A3)}{=} \frac{1}{t} \sum_{i=0}^{t-1} \alpha_{xy}^{f(i,x)+1} \\
&= \frac{1}{t} \sum_{i=0}^{t-1} \left\langle s \middle| \, B^{f(i,x)} x \right\rangle \langle x | \, By \rangle \\
&= \langle s | \, (\frac{1}{t} \sum_{i=0}^{t-1} B^{f(i,x)}) \, |x\rangle \, \langle x | \, By \rangle \\
&= \langle s | \, (\frac{1}{t} \sum_{i=0}^{t-1} s(i,t,x) B^i) \, |x\rangle \, \langle x | \, By \rangle \, .
\end{aligned}$$

By Lemma 7.2.5 the limit $\lim_{t\to\infty}\frac{1}{t}\sum_{i=0}^{t-1}s(i,t,x)B^i$ exists. Hence also

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t}\alpha_{xy}^i$$

exists and it holds:

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t}\alpha_{xy}^i = \langle s|\,(\lim_{t\to\infty}\frac{1}{t}\sum_{i=0}^{t-1}s(i,t,x)B^i)\,|x\rangle\,\langle x|\,By\rangle\,.$$

■

Corollary 7.2.3 *Let G be strongly connected and let ϕ be a non-negative randomized allocation mechanism. Then $\lim_{t\to\infty}\overline{\phi}^t$ converges.*

Proof. Choose the row-sum norm on $\mathbb{K}^{V\times V}$:

$$\|B\| := \max\{\sum_{y\in V}|b_{xy}|\,|x\in V\} \qquad (B\in\mathbb{K}^{V\times V}).$$

Let P be the transition matrix associated to ϕ. Since P is a stochastic matrix, it is $\|P\|=1$. Hence by Theorem 7.2.2 the Cesàro value induced by ϕ converges. ■

REMARK The assumption of G being strongly connected is more than really needed to achieve the convergence of Cesàro values. Note that Lemma 7.2.4 is the only part of the whole proof in which the strong connectedness was essential. We use the same notations as in this lemma. In the proof of Lemma 7.2.4 we proved a stronger statement, namely:

$$\{n\geq\max\{k_1,...,k_t\}+\tilde{N}(x)|x\in E_n\}=\{k_1,...,k_t\}+\{\tilde{N}(x)+l*g(x)|l\in\mathbb{N}\}.$$

By the strong connectedness of G we argued that k_i is congruent k_j modulo $g(x)$ and hence

$$\{n\geq N(x)|x\in E_n\}=\{N(x)+l*g(x)|l\in\mathbb{N}\}.$$

Hence the statement remains true if we have a graph with the property that k_i is congruent k_j modulo $g(x)$ for all i,j.
If G is not necessarily strongly connected, let $s=y_1,...,y_k=x$ be all vertices on all

smooth paths from x to s and for $i = 1, ..., k$ let $l_1^i, ..., l_{m(i)}^i$ be the length of all circles through y_i s.t. y_i occurs exactly twice on each of them. Hence for any s-x path in G there is a unique smooth s-x path Q, s.t. $V(Q) \subseteq V(P)$ and $|P| = |Q| + \sum_{i=1}^{k} \sum_{j=1}^{m(k)} c_j^i l_j^i$ for some $c_j^i \in \mathbb{N}$. Hence again by Lemma 7.2.2 there is a number $F(x)$

$$\begin{aligned} &\{n \geq \max\{k_1, ..., k_t\} + F(x) | x \in E_n\} \qquad (*) \\ =&\{k_1, ..., k_t\} + \{F(x) + l * gcd(l_j^i | i = 1, ..., k; j = 1, ..., m(i)) | l \in \mathbb{N}\}. \end{aligned}$$

In order to achieve a convergence result as above, one needs some kind of periodicity. In Lemma 7.2.5 the periodicity was needed in order to reduce the convergence of $\frac{1}{t}\sum_{i=0}^{t-1} s(i,t,x)P^i$ for $t \to \infty$ to the known convergence of the Cesàro mean of a fixed power $P^{k(x)}$ of P. Hence it seems natural to use (*) to build certain "periodic blocks". $\{n \geq \max\{k_1, ..., k_t\} + F(x) | x \in E_n\}$ is the union of the sets $\{k_p + F(x) + l * gcd(l_j^i | i = 1, ..., k; j = 1, ..., m(i)) | l \in \mathbb{N}\}$, but in most cases these sets do not build a partition. Hence it is not obvious that
$\frac{1}{t}\sum_{i=0}^{t-1} s(i,t,x)P^i$ could be split up into periodic parts. Moreover, the terms $s(i,t,x)$ need not to equal $k(x)$ any longer (for some range of i), hence even if a split-up into periodic parts is possible, one needs to determine the terms $s(i,t,x)$ in another way. The author believes that convergence of Cesàro values holds for arbitrary s-connected graphs and that the strong- connectedness assumption is dispensable.

8 The quantum case - new allocation mechanisms

As seen in Chapter 5, Markovian cooperation systems are a quite powerful tool in modeling cooperative settings and in giving answers to the allocation problem and the prediction problem. A main advantage of these systems is the simple representation of the evolution process in terms of the evolution operator by the powers of a single matrix.
Nevertheless, there are situations which could not be modeled as Markovian systems. In this chapter we take a step further and describe transitions from one state to another from a quantum mechanical viewpoint.
In Section 8.1, beside some basic notations, we give an example of an evolution process, which is not Markovian and aim for a compact description of it in the rest of this chapter.
Section 8.2 gives a brief overview of models for quantum random walks on graphs. After that we apply a model of Faigle and Schönhuth [38] for quantum random walks on graphs to our model. By that we are able to associate to each quantum random walk an allocation mechanism in Section 8.3 and call it *quantum allocation mechanism*. These allocation mechanisms are seen to be linear and efficient in the end of the section. Finally we apply a convergence theorem of Faigle and Schönhuth [38] to the Cesàro values of these allocation mechanism in Section 8.3.1 and achieve a convergence result similar to Theorem 7.2.2. Hence quantum allocation mechanisms are a class of non-randomized allocation mechanisms whose Cesàro values converge.

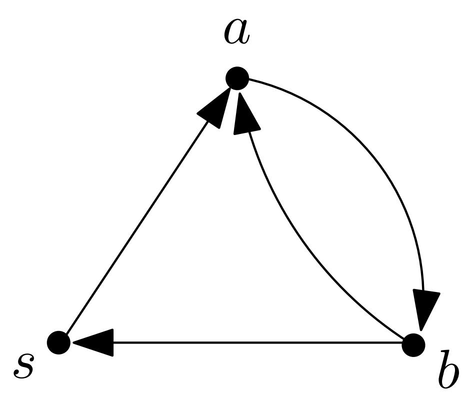

Figure 8.1:

8.1 Preliminaries & notations

We start by an example:

Example 8.1.1 *Let $V = \{s, a, b\}$ and $A = \{sa, ab, ba, bs\}$. Hence the graph $G = (V, A)$ equals the graph in Figure 8.1. Assume in time-steps $t = 0, ..., 6$ the following probability distributions $(p^t(s), p^t(a), p^t(b))$ have been observed:*

$$t = 0 : (1, 0, 0), t = 1 : (0, 1, 0), t = 2 : (\frac{1}{2}, 0, \frac{1}{2}), t = 3 : (0, 1, 0),$$
$$t = 4 : (\frac{1}{4}, 0, \frac{3}{4}), t = 5 : (0, 1, 0), t = 6 : (\frac{1}{8}, 0, \frac{7}{8}).$$

Obviously there is no way to describe this process by a classical random walk on the graph $G = (V, A)$.

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. Since all studied dimensions in this thesis are finite, we assume $\mathcal{H}$ also to be finite dimensional, even if general quantum theory is formulated for the infinite dimensional case. Fix a basis $V \subseteq \mathcal{H}$. A *wave function* is a state $|\psi\rangle \in \mathcal{H}$ with length 1, *i.e.*:

$$\langle\psi|\ \psi\rangle = \sum_{x \in V} |\psi_x|^2 = \|\psi\| = 1.$$

Thus each wave function describes a probability distribution on the basis V by interpreting the squared absolute component values as probabilities for the system *to be in state $x \in V$*. We are interested in linear transformations of $\mathcal{H}$ which preserve the wave function property; *i.e.*: we are interested in matrices $U \in \mathbb{C}^{V \times V}$, such that for any wave function ψ, also $U|\psi\rangle$ is a wave function. These matrices are known from linear algebra and are called *unitary*. We recall a standard characterization of unitary matrices in the following lemma. A proof is given in [54].

Lemma 8.1.1 *Let $U \in \mathbb{C}^{V \times V}$ be an invertible matrix. The following statements are equivalent:*

(a) U *is unitary.*

(b) $\|Uv\| = \|v\|$ *for all* $v \in \mathcal{H}$.

(c) $\langle Uv|\, Uw\rangle = \langle v|\, w\rangle$ *for all* $v, w \in \mathcal{H}$.

(d) $U^{-1} = U^*$.

(e) The columns of U *are an orthonormal basis of* $\mathcal{H}$.

■

Any unitary matrix U gives rise to an evolution process of ψ *via*

$$\left|\psi^t\right\rangle := U^t \left|\psi\right\rangle$$

for all $t \geq 0$.

8.1.1 Self-adjoint matrices

Given a wave function $|\psi\rangle$ one could identify it with a self-adjoint matrix (*i.e.* a matrix Q with the property $Q^* = Q$) via:

$$|\psi\rangle \leftrightarrow |\psi\rangle \langle\psi| =: Q_\psi.$$

With this definition one could describe the evolution of ψ also through its representation as self-adjoint matrix:

$$Q_{\psi^t} = \left|U^t\psi\right\rangle \left\langle U^t\psi\right| = U^t \left|\psi\right\rangle \left\langle\psi\right| {U^t}^* = U^t Q_\psi {U^t}^*.$$

In order to define certain mappings, we abstract from wave functions to self-adjoint matrices of trace 1. Denote by $\mathcal{S} := \mathcal{S}_\mathcal{H}$ the set of self-adjoint matrices over $\mathcal{H}$ with trace 1 (which is closed under convex combination by the linearity of the trace function). Later in this chapter we will describe *quantum random walks* by linear operations on $\mathcal{S}$.

Known from linear algebra an element $Q \in \mathcal{S}$ has only real eigenvalues and there is an orthonormal basis of eigenvectors. Hence if $c_1, ..., c_n \in \mathbb{R}$ are the (not necessarily different) eigenvalues of Q and $\psi_1, ..., \psi_n$ is an associated orthonormal basis of eigenvectors, we have:

$$Q = \sum_i c_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \text{ and } trQ = \sum_i c_i.$$

8.2 Quantum random walks

We want to generalize the concept of random walks on transition graphs to that of quantum random walks. Quantum random walks received some attention in the past. The wish to consider quantum walks is essentially based on the following: the model of classical random walks was able to speed up certain algorithms in graph theory by randomization. By development of the theory of a quantum computer one had the wish to get an analogue of random walks, which could also be understood by a quantum computer. The hope is that the theory of quantum random walks is as rich of results as it was the theory of classical random walks in the classical case. However, algorithmic aspects of quantum random walks are not treated here. We are interested in processes that are induced by quantum random walks. More on the algorithmic aspects of quantum random walks could be found in the work of Ambainis [4].

8.2.1 A very brief history of quantum random walks on graphs

There are several approaches to quantum random walks in the literature. First of all one should mention the model of Aharonov *et al.* [1]: first they considered d-regular graphs. Laxly spoken, they imagined a random walker that tosses a d-sided coin at each crossroad. This coin toss was modeled quantum-mechanically: by considering to each vertex v of the graph d ground states $\left|v, 1\right\rangle, ..., \left|v, d\right\rangle$, they studied unitary evolutions on the Hilbert space spanned by all these vectors. (To be honest: they investigated unitary transformations on the Hilbert space that is the tensor product of $\mathbb{C}^V$ and a certain d-dimensional *coin space*).

On a first view there was a problem: with this model only d-regular graphs were quantum-random-walkable. The idea of the coin toss was adapted later (*cf.* [60]) and one studied more complicated spaces by giving each node in the graph its own coin space in dependency of its neighborhood structure. An alternative (but equivalent) trick was the following: one takes the maximal degree of the graph and gave any node

x artificial and independent slopes $\{x, x\}$ since the degree of x is also maximal. Thus it was possible to model quantum walks on arbitrary graphs.
Faigle and Schönhuth [38] generalized the view on quantum walks on arbitrary graphs in a general model of so called *Quantum Markov Chains*. In the special case of graphs, this model yields a nicer representation of quantum walks and does not need the idea of many coin tosses: by forgetting coin tosses and tensor products with coin spaces completely, they studied unitary evolutions on the Hilbert space $\mathbb{C}^A$. We will view on quantum random walks in the same spirit.

8.2.2 Quantum random walks in our model

According to Faigle and Schönhuth, a quantum walk is a triple $(G = (V, A), U, \psi)$ consisting of a directed graph G, a unitary matrix $U \in \mathbb{C}^{A \times A}$ and an initial wave function $\psi \in \mathbb{C}^A$. By describing the evolution of the wave function as above: $\left|\psi^t\right\rangle := U^t \left|\psi\right\rangle$ for $t \geq 0$, one can give the probability of the walk that starts in ψ and evolves according to U being in node $v \in V$ at time t via:

$$p_t(v \mid \psi) := \sum_{x \in V} \left| \left\langle vx \middle| \psi^t \right\rangle \right|^2 . \tag{*}$$

Again we identified the arc $vx \in A$ with its associated element in the standard orthonormal basis. Thus one could think of U describing a walk on the arcs of G. We make the following assumption to U in order to get a much more interpretative view on quantum random walks:

$$U_{(xw),(yz)} = 0 \text{ if } z \neq x.$$

In words: we assume U^* to be compatible with the arc graph of G, since the $(yz), (xw)$ entry of U^* is zero, if the arcs (yz) and (xw) are not adjacent in the arc graph of G. The first question that comes into mind is

"Do unitary matrices of that kind exist for every graph G?"

The answer is: no. There are some previous works that give detailed answers to this question. For instance Montanaro [67] gave the answer that *essentially* there is a unitary matrix with this property if and only if the underlying graph is strongly connected. But we do not aim for a detailed answer to this question here. We just make the assumption: let G be a graph such that there is a unitary matrix U that has the desired property.

Equation (*) could be expressed in a much more elegant way: define for all $v \in V$

$$P_v : \mathbb{C}^A \to \mathbb{C}^A, (\beta_{xy})_{xy \in A} \mapsto \sum_{vy \in A} \beta_{vy} |vy\rangle$$

the projection onto the neighborhood of the node v. A first observation is:

Lemma 8.2.1

(a) $\sum_{x \in V} P_x = Id$

(b) For all $\psi \in \mathbb{C}^A$*:* $\sum_{x \in V} |P_x \psi\rangle \langle P_x \psi| = |\psi\rangle \langle \psi|$.

Proof. Let $\psi \in \mathbb{C}^A$. (a): Then

$$\sum_{x \in V} P_x |\psi\rangle = \sum_{x \in V} (\sum_{xy \in A} \psi_{xy} |xy\rangle) = \sum_{xy \in A} \psi_{xy} |xy\rangle = |\psi\rangle$$

holds.
(b): The following is true:

$$\begin{aligned} \sum_{x \in V} |P_x \psi\rangle \langle P_x \psi| &= \sum_{x \in V} (\sum_{y \in N^+(x)} \psi_{xy} |xy\rangle)(\sum_{z \in N^+(x)} \psi_{xz} \langle xz|) \\ &= \sum_{xy \in A} \sum_{xz \in A} \psi_{xy} \psi_{xz} |xy\rangle \langle xz| = |\psi\rangle \langle \psi| . \end{aligned}$$

∎

Moreover, define an operator on the set of all self-adjoint matrices of trace one over $\mathbb{C}^A$ ($\mathcal{S} := \mathcal{S}(\mathbb{C}^A)$):

$$T_v(Q) := P_v U Q (P_v U)^* \in \mathcal{S}.$$

Then for all $t \geq 0$, all wave functions $|\psi\rangle$ and all $v \in V$ one has

$$\begin{aligned} p_t(v \mid \psi) &= \sum_{x \in V} \left| \left\langle vx \middle| \psi^t \right\rangle \right|^2 = \left\langle P_v \psi^t \middle| P_v \psi^t \right\rangle \\ &= tr P_v \psi^t (P_v \psi^t)^* = tr T_v (U^{t-1} Q_\psi U^{t-1^*}) \end{aligned}$$

Thus one gets the probability of seeing a certain node at time t, by applying U^t, projecting and then renormalizing - which is a quantum theoretic measurement. Note that in the following we will make use of the well known basic facts of tr (tr is a linear map and $trAB = trBA$ for all quadratic matrices A, B) without further mentioning.

Example 8.2.1 *Recall the graph and the process described in Example 8.1.1. This process could be described as the quantum random walk induced by the matrix:*

$$U := \begin{pmatrix} 0 & 0 & 0 & i \\ \frac{1}{\sqrt{2}} & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix}$$

(which is in fact unitary, since it columns are an orthonormal basis) and by the wave function $|\psi\rangle := (1, 0, 0, 0)^T$. It is a direct calculation to check, that the vectors $p_t(\cdot \mid \psi)$ equal the distribution vectors given in Example 8.1.1 for $t = 0, ..., 6$ and that U satisfies the property: $U_{(xw)(yz)} = 0$ if $z \neq x$ for all $xw, yz \in A$.

The operators T_v yield a nice property, which will demonstrate the strong relation of a quantum walk to the graph itself:

Lemma 8.2.2 *Let $x, y \in V$. If $xy \notin A$, then $T_y T_x = 0$.*

Proof. Let $Q = |\psi\rangle \langle\psi| \in \mathcal{S}$. Then

$$T_y T_x Q = P_y U P_x U |\psi\rangle \langle\psi| (P_y U P_x U)^*.$$

We will prove that $P_y U P_x U |\psi\rangle = 0$, hence also $T_y T_x Q = 0$. Observe first that

$$P_x U |\psi\rangle \in span\{|xw\rangle \mid xw \in A\}.$$

Thus

$$U P_x U |\psi\rangle \in span\{U |xw\rangle \mid xw \in A\} = span\{(u_{(ab),(xw)})_{ab \in A} \mid xw \in A\}.$$

By projecting to the neighborhood of y we get:

$$P_y U P_v U |\psi\rangle \in span\{\sum_{yz \in A} u_{(yz),(xw)} |yz\rangle \mid xw \in A\}.$$

But $u_{(yz),(xw)} \neq 0$ only if $w = y$. Since $xy \notin A$ we have:

$$span\{\sum_{yz \in A} u_{(yz),(xw)} |yz\rangle \mid xw \in A\} = 0.$$

■

According to Faigle and Schönhuth we will call $M := \sum_{v \in V} T_v$ the evolution operator of the quantum walk (G, U, ψ). In fact, it is an evolution operator in the sense of Chapter 3 by interpreting it as an operator on the *state space* $\mathcal{S}$. The following Lemma gives an argument, why this term is reasonable:

Lemma 8.2.3 *Let $t \geq 0$, $x \in V$ and $|\psi\rangle$ a wave function. Then*

(a) $M^t = \sum_{(v_1,...,v_t)\in V^t} T_{v_t} T_{v_{t-1}} ... T_{v_1}$.

(b) $Q_{\psi^t} = M^t Q_\psi$

(c) $trMQ = trQ$ *for all* $Q \in \mathcal{S}$.

Proof. (a): We use induction on t with trivial beginning. Let $t > 1$ and the statement be true for all smaller t. Then:

$$M^t = MM^{t-1} = (\sum_{v\in V} T_v) \sum_{(v_1,...,v_{t-1})\in V^t} T_{v_{t-1}} ... T_{v_1} = \sum_{(v_1,...,v_t)\in V^t} T_{v_t} T_{v_{t-1}} ... T_{v_1}.$$

(b): Again we use induction on t. For $t = 1$ we have:

$$MQ_\psi = \sum_{x\in V} T_x Q_\psi = \sum_{x\in V} P_x U Q_\psi U^* P_x^* = UQ_\psi U^* = Q_{\psi^1}.$$

Where the last equations holds since the sum over all projections is the identity matrix. For $t > 1$ we have

$$M^t Q_\psi = M(M^{t-1} Q_\psi) = MQ_{\psi^{t-1}} = UQ_{\psi^{t-1}} U^* = Q_{\psi^t}$$

by induction and the same argument as above.
(c): Since conjugation with U is trace-preserving and

$$trMQ = tr \sum_{x\in V} T_x Q = tr \sum_{x\in V} P_x UQU^* P_x^*,$$

it is enough to show that

$$tr \sum_{x\in V} P_x Q P_x^* = trQ \text{ for all } Q \in \mathcal{S}.$$

But since Q is self-adjoint there is a spectral representation of Q as $Q = \sum_i c_i |\psi_i\rangle \langle\psi_i|$ - where c_i are the eigenvalues of Q and $(\psi_i)_{i=1,...,|A|}$ is an orthonormal basis of eigenvectors of Q - we know the desired equality from Lemma 8.2.1 (b) and the fact that Q has trace 1. ∎

REMARK Indeed M is an evolution operator in the sense of Section 3: by statement (c) of the last lemma, M (and each of its powers) leaves the state space $\mathcal{S}$ invariant. Moreover, since M is a linear operator it is $M^0 = Id$ by definition.

8.3 Allocation mechanisms induced by quantum random walks

We use the same notations as before. Let $\Gamma := (N, V, A, \mathcal{A}, s)$ be a cooperation system. Let $((V, A), U, \psi)$ be a quantum random walk on the transition graph G of Γ. Since we want the walk to "start in s", we make an additional assumption on the wave function ψ:

$$\langle P_s\psi|\ P_s\psi\rangle = 1. \tag{s}$$

In terms of the probabilities we introduced in the last section, this reads as: the probability to be in s at time $t = 0$ is 1.
We aim to define a linear allocation mechanism with respect to this quantum walk; *i.e.*: we want to define certain α-values. Recall Section 5.3.1. There we identified the α-value of an arc $xy \in A$ at time t ($x \in E_t$) to be the probability, that a walk which starts in s, is in x at time t and moves to y at time $t + 1$. We want to copy this idea and try to carry it over to the concept of quantum random walks. Therefore define inductively:

$$\alpha^t_{xy} := trT_yT_xM^{t-1}Q_\psi \text{ if } x \in E_{t-1}$$

and

$$\alpha^t_{xy} := \alpha^{t-1}_{xy} \text{ else.}$$

Thus by definition and Lemma 5.2.2 we have

Lemma 8.3.1 *$\phi = \phi^\alpha$ is a linear allocation mechanism.* ■

Our goal is to prove that ϕ^α is efficient.

Lemma 8.3.2 *Let α be as above and $t > 0$. Then .*

(a) $\sum_{x\in N^-(e)} \alpha^t_{xe} = \sum_{y\in N^+(e)} \alpha^{t+1}_{ey}$ *for all* $e \in E_t$.

(b) $\sum_{u\in N^+(s)} \alpha^1_{su} = 1.$

holds.

Proof. (a): Let $x \in E_t$. Since M is trace preserving and $T_b T_a = 0$ if $ab \notin A$, we have:

$$\begin{aligned}
\sum_{u \in N^-(x)} \alpha^t_{ux} &= \sum_{u \in N^-(x)} trT_xT_uM^{t-1}Q_\psi = trT_x(\sum_{u \in N^-(x)} T_u)M^{t-1}Q_\psi \\
&= trT_x(\sum_{u \in V} T_u)M^{t-1}Q_\psi = trT_xM^tQ_\psi \\
&= trMT_xM^tQ_\psi = tr(\sum_{y \in V} T_y)T_xM^tQ_\psi \\
&= tr(\sum_{y \in N^+(x)} T_y)T_xM^tQ_\psi = \sum_{y \in N^+(x)} trT_yT_xM^tQ_\psi \\
&= \sum_{y \in N^+(x)} \alpha^{t+1}_{xy}.
\end{aligned}$$

(b): Recall assumption (s) made on ψ and that M is trace preserving. By (s) we have

$$1 = \langle P_s\psi |\, P_s\psi\rangle = trT_sQ_\psi.$$

On the other hand, we have:

$$\begin{aligned}
\sum_{u \in N^+(s)} \alpha^1_{su} &= \sum_{u \in N^+(s)} trT_uT_sM^0Q_\psi = tr \sum_{u \in N^+(s)} T_uT_sQ_\psi \\
&= tr(\sum_{u \in V} T_u)T_sQ_\psi = trT_sQ_\psi \\
&= 1.
\end{aligned}$$

■

This lemma together with Theorem 5.2.3 yields:

Theorem 8.3.1 *Let $((V, A), U, \psi)$ be a quantum random walk such that ψ suffices (s). And let ϕ be the linear allocation mechanism associated with this walk. Then ϕ is efficient.* ■

Due to this theorem we will call an allocation mechanism, which is induced by a quantum random walk, a *quantum allocation mechanism*.

8.3.1 Convergence of quantum allocation mechanisms

We use all objects and notations from the last section without introducing them again. The convergence of quantum allocation mechanisms is strongly related to the convergence of quantum random walks. In a more general context Faigle and Schönhuth [38] (and similarly Schönhuth [81]) proved the following theorem, which we will restate to our restricted setting:

Theorem 8.3.2 ([38, Theorem 1]) *Let $|\psi\rangle \in \mathbb{C}^A$ be a wave function and let $U \in \mathbb{C}^{A\times A}$ be an unitary matrix. Furthermore, let $M := \sum_{x\in V} T_x$ be the evolution operator associated with U. Then the limit*

$$\overline{Q} := \lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} M^i(Q_\psi)$$

exists and is an eigenvector of M to the eigenvalue 1. Moreover, for each path $P = x_1...x_k$ in G also the limit

$$T_P(\overline{Q}) = \lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} T_P M^i(Q_\psi)$$

exists, where $T_P := T_{x_k}...T_{x_1}$. ■

We will use this theorem to prove

Theorem 8.3.3 *Let $G = (V, A)$ be a transition graph of a cooperation system Γ and assume that G is strongly connected. Moreover, let $(G, U, |\psi\rangle)$ be a quantum random walk on G and $\phi = \phi^\alpha$ be the quantum allocation mechanism relative to this walk. Then the associated Cesàro value (cf. Section 7.2) $\overline{\phi}$ converges, i.e.:*

$$\lim_{t\to\infty} \overline{\alpha}^t_{xy}$$

exists for all $xy \in A$.

In order to prove Theorem 8.3.3 we need some further lemmata:

Lemma 8.3.3 *Let $(G, U, |\psi\rangle)$ be a quantum random walk on G with associated evolution operator M and let $k \in \mathbb{N}$. Then $(G, U^k, |\psi\rangle)$ is a quantum random walk with evolution operator M^k.*

Proof. Since U is unitary, also U^k is unitary. Hence $(G, U^k, |\psi\rangle)$ is indeed a quantum random walk. For $t > 0$ we have $(M^k)^t(Q) = U^{kt}Q(U^{kt})^* = (U^k)^t Q((U^k)^*)^t$. Thus M^k equals the evolution operator of this walk. ■

Lemma 8.3.4 *Let $(G, U, |\psi\rangle)$ be a quantum random walk, set $Q := Q_\psi$ and let $t > 0$. Define: $f(i, x) := \max\{j \leq i | x \in E_j\}$ and $s(i, t, x) := |\{j \leq t-1 | f(j, x) = i\}|$ for all $0 < i < t$. Then for all $x \in V$ the following limit exists:*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} s(i, t, x) M^i Q.$$

Proof. The idea of the proof is essentially the same as of Lemma 7.2.5. Let $x \in V$, let $N(x)$ and $k(x)$ be as in Lemma 7.2.4 and let $t > N(x)$. Then

$$\frac{1}{t} \sum_{i=0}^{t-1} s(i, t, x) M^i Q = \frac{1}{t} \sum_{i=0}^{N(x)-1} s(i, t, x) M^i Q + \frac{1}{t} \sum_{i=N(x)}^{t-1} s(i, t, x) M^i Q$$

holds. For all $i \leq N(x)$ it is $s(i, t, x) \leq N(x)$. Hence $\frac{1}{t} \sum_{i=0}^{N(x)-1} s(i, t, x) M^i Q \to 0$ for $t \to \infty$. Divide $t - 1 - N(x)$ by $k(x)$ with remainder. Then there are a_t, b_t s.t. $t - 1 - N(x) = a_t * k(x) + b_t$ and $b_t < k(x)$. Hence we find

$$\begin{aligned}
\frac{1}{t} \sum_{i=N(x)}^{t-1} s(i, t, x) M^i Q &= \frac{1}{t} \sum_{i=N(x)}^{N(x)+a_t*k(x)+b_t} s(i, t, x) M^i Q \\
&= \frac{1}{t} \sum_{i=N(x)}^{N(x)+a_t*k(x)} s(i, t, x) M^i Q + \frac{1}{t} \sum_{i=N(x)+a_t*k(x)}^{N(x)+a_t*k(x)+b_t} s(i, t, x) M^i Q.
\end{aligned}$$

For $N(x) + a_t * k(x) < i \leq N(x) + a_t * k(x) + b_t$, $x \notin E_i$ holds. Thus: $s(i, t, x) = 0$. Hence the second sum equals:

$$\begin{aligned}
\frac{1}{t} \sum_{i=N(x)+a_t*k(x)}^{N(x)+a_t*k(x)+b_t} s(i, t, x) M^i Q &= \frac{s(N(x) + a_t * k(x), t, x)}{t} M^{N(x)+a_t*k(x)} Q \\
&= \frac{b_t}{t} M^{N(x)+a_t*k(x)} Q.
\end{aligned}$$

Since b_t is bounded by $k(x)$, this term converges to 0 for $t \to \infty$:

$$\left\| \frac{b_t}{t} M^{N(x)+a_t*k(x)} Q \right\| \leq \frac{k(x)}{t} \left\| M^{N(x)+a_t*k(x) Q} \right\| = \frac{k(x)}{t} * 1 \to 0$$

All in all, we have until now:

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=0}^{t-1} s(i, t, x) M^i Q = \lim_{t\to\infty} \frac{1}{t} \sum_{i=N(x)}^{N(x)+a_t*k(x)} s(i, t, x) M^i Q.$$

Since $s(i,t,x) = 0$ for $x \notin E_i$, it holds:

$$\frac{1}{t} \sum_{i=N(x)}^{N(x)+a_t*k(x)} s(i,t,x) M^i Q = \frac{1}{t} \sum_{i=N(x),x\in E_i}^{N(x)+a_t*k(x)} s(i,t,x) M^i Q$$

Those indices are well-known by Lemma 7.2.4:

$$= \frac{1}{t} \sum_{i=0}^{a_t} s(N(x)+i*k(x),t,x) M^{N(x)+i*k(x)Q}$$

Moreover it is: $s(N(x)+i*k(x),t,x) = k(x)$. Hence:

$$= M^{N(x)} \frac{1}{t} \sum_{i=1}^{a_t} k(x)(M^{k(x)})^i Q = M^{N(x)} \frac{k(x)}{t} \sum_{i=0}^{a_t} (M^{k(x)})^i Q.$$

By Lemma 8.3.3 $M^{k(x)}$ is the evolution operator of $(G, U^{k(x)}, |\psi\rangle)$. Since $a_t \to \infty$ for $t \to \infty$, Theorem 8.3.2 implies:

$$\lim_{a_t \to \infty} \frac{1}{a_t} \sum_{i=0}^{a_t} (M^{k(x)})^i Q$$

exists. We will show that $\frac{k(x)}{t} \sum_{i=0}^{a_t} (M^{k(x)})^i Q$ converges to $\lim_{t\to\infty} \frac{1}{a_t} \sum_{i=0}^{a_t} (M^{k(x)})^i Q$. Therefor it suffices to show that $\left| \frac{k(x)}{t} - \frac{1}{a_t} \right| \to 0$ for $t \to \infty$. The following holds.

$$\left| \frac{k(x)}{t} - \frac{1}{a_t} \right| = \left| \frac{a_t k(x) - t}{a_t * t} \right|$$

On the other hand it is $t = a_t * k(x) + b_t + N(x) + 1$. Hence:

$$= \left| \frac{a_t k(x) - a_t k(x) - b_t - N(x) - 1}{a_t * t} \right| = \left| \frac{-b_t - N(x) - 1}{a_t * t} \right|.$$

This expression converges to 0 for $t \to \infty$, since b_t is bounded by $k(x)$. ∎

These two lemmata allow us to prove the desired convergence of Cesàro values induced by quantum random walks:

Proof of Theorem 8.3.3. Let $xy \in A$. And ϕ be a quantum allocation mechanism induced by the unitary matrix $U \in \mathbb{C}^{A\times A}$. We have to prove: $\frac{1}{t}\sum_{i=1}^{t} \alpha_{xy}^{i}$ converges for $t \to \infty$. The following holds:

$$\begin{aligned}
\frac{1}{t}\sum_{i=1}^{t} \alpha_{xy}^{i} = \frac{1}{t}\sum_{i=0}^{t-1} \alpha_{xy}^{i+1} \\
=&\frac{1}{t}\sum_{i=0}^{t-1} \alpha_{xy}^{f(i,x)+1} \\
=&\frac{1}{t}\sum_{i=0}^{t-1} tr(TyTxM^{f}(i,x)Q_{\psi}) \\
=&tr(T_yT_x\frac{1}{t}\sum_{i=0}^{t-1} M^{f(i,x)}Q_{\psi}) \\
=&tr(T_yT_x\frac{1}{t}\sum_{i=0}^{t-1} s(i,t,x)M^{i}Q_{\psi}).
\end{aligned}$$

By Lemma 8.3.4 the limit $\lim_{t\to\infty}\frac{1}{t}\sum_{i=0}^{t-1} s(i,t,x)M^{i}Q$ exists. Hence also

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t} \alpha_{xy}^{i}$$

exists and it holds:

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{t} \alpha_{xy}^{i} = tr(T_yT_x\lim_{t\to\infty}\frac{1}{t}\sum_{i=0}^{t-1} s(i,t,x)M^{i}Q_{\psi}).$$

■

9 Cores of cooperative games

In this chapter we want to generalize another notion of fairness from the classical models of cooperative games. The idea of the so called *core* (introduced by Gillies [47]) is the following: whenever players participate in the process of building a coalition, their payoff should at least be the value of this coalition. Else wise these players would be treated unfair, since the value they create by building this certain coalition, is not allocated to them.

Section 9.1 gives the classical model of the core and some concepts related to it. In Section 9.2, we generalize the classical core in two different ways. Each of them takes on a different idea of fairness of the classical core. We give a generalization of a famous fact and prove, that both cores in our general setting are subsets of the *Weber-set*. The proof is mainly based on a greedy-type algorithm, which is given in Subsection 9.2.2. Finally, we give examples of classical core concepts in Section 9.3 and show that they are special cases of our notions of core.

9.1 Classical core concept

Let (N, v) be a classical cooperative game with $v : 2^N \to \mathbb{R}_+$ and $x \in \mathbb{R}_+^N$. Then x is called fair in the above sense if

$$x(S) := \sum_{i \in S} x_i \geq v(S) \qquad \text{for all } S \subseteq N.$$

By merging all these vectors into a set, one defines the so called *open core*:

$$\text{core}^o(v) := \{x \in \mathbb{R}_+^N | x(S) \geq v(S)\}$$

which is by definition a polyhedron. The assumption that x has to be non-negative is often made by the following argument: if a player gets a negative payoff he would simply deny his participation in the game. Alternatively one could argue by the positive decomposition of cooperative games by means of their ζ-representation (*cf.* Section 4.4). Note that the open core of a game is non-empty since the vector with components all equal to $\max_{S \subseteq N} v(S)$ is obviously in the open core.
By additionally requiring the core vectors to be efficient, the *classical core* is defined which is due to Gillies [47]:

$$\text{core}(v) := \{x \in \mathbb{R}_+^N | x(S) \geq v(S), x(N) = v(N)\}.$$

This restriction seems natural, but yields a new problem: the nice property of the open core of being non-empty is lost by passing into the closed core. Thus one has a simple concept of fairness for allocations, but it is open if such allocations exist.
We want to point out two essential ideas of fairness of the classical core. The obvious idea is mentioned above: all players that could build a certain coalition should be rewarded by a core allocation. The second idea is the following: all players, which could deny their cooperation to build a certain coalition $S \subseteq N$ are *essential* for building S and should be rewarded. Even if these two points of view are different, they coincide in the classical case. We want to generalize both ideas of fairness.

9.1.1 Marginal-worth vectors and the Weber-set

Given a permutation $\pi = (p_1, ..., p_n) \in Sym(N)$ one defines the so called *marginal-worth vector* in $\mathbb{R}^N$ relative to it: fix $i \in N$ and let $j \in N$ such that $p_j = i$. Define:

$$h^\pi(v)_i := v(p_1, ..., p_j) - v(p_1, ..., p_{j-1}).$$

The convex hull of all these marginal-worth vectors is called the Weber-set of v, due to Weber [91], who first investigated this set:

$$\mathcal{W}(v) := \text{conv}\{h^\pi | \pi \in Sym(N)\}.$$

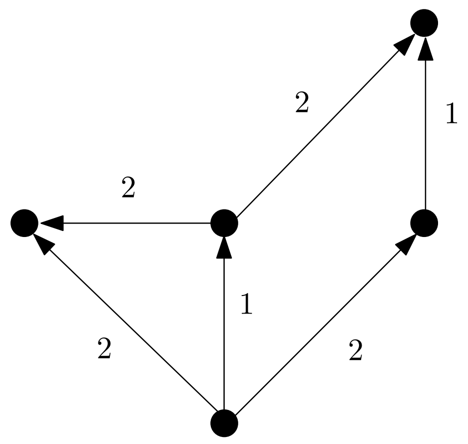

Figure 9.1:

9.2 A core concept for the acyclic model

Let Γ be a cooperation system. In this section we define a set-valued solution concept which is a generalization of the classical core to our model. Therefor we restrict ourselves to the acyclic case; *i.e.*: we assume:

$$\text{The transition graph of } \Gamma \text{ is acyclic.} \tag{A}$$

Moreover, we need another structural assumption on Γ. The so called *single action property*, which was already defined and used in [37] by Faigle and the author:

$$|P \cap A_i| \leq 1 \text{ for all } P \in \mathcal{P} \text{ and } i \in N. \tag{SA}$$

REMARK Conditions (A) and (SA) seem technical (and they are!), but note that all models that are based on greedoids (*cf.* Chapter 2) in fact enjoy (A) and (SA). One could think that (A) and (SA) are that restrictive, that Γ is also *just* a set system. But it is not as the following example shows:

Example 9.2.1 *Let $N := \{1, 2\}$ and consider the graph in Figure 9.1 which reflects actions that are governed by player 1 resp. player 2. Obviously this graph is no subgraph of the graph induced by the power-set lattice of N, but (A) and (SA) are satisfied.*

In analogy to our notion of efficiency, we call a vector $z \in \mathbb{R}^N$ efficient for the game $v \in \mathcal{G}$ if there is a probability distribution on the sinks of G, s.t. z allocates the expected v-value w.r.t. to this probability distribution. Denote by $\mathcal{P}(x)$ the set of all paths that start in s and end in x and set:

$$\text{core}(v) := \{z \in \mathbb{R}^N_+ | z(P) \geq v(x), \forall x \in V, P \in \mathcal{P}(x), z \text{ is efficient for } v\}.$$

Lemma 9.2.1 core(v) *is a polyhedron.*

Proof. Let $E := \{e_1, ..., e_k\}$ be the set of sinks of G and consider the following linear inequality system:

$$\begin{aligned} z(P) &\geq v(x) \text{ for all } x \in V, P \in \mathcal{P}(x) \\ (1, ..., 1, -v(e_1), .., -v(e_k)) \begin{pmatrix} z \\ \mu \end{pmatrix} &= 0 \\ 1^T \mu &= 1 \\ \mu &\geq 0 \end{aligned}$$

As known from the theory of polyhedra, also the projection of this polyhedron onto its z-components is a polyhedron (*cf.* [36, Theorem 2.5]). And this projection equals core(v). ■

Example 9.2.2 (Classical Core) *Let (N, v) be a classical cooperative game and let*

$$\Gamma := (2^N, \{(S, T) | \, |T \setminus S| = 1\}, \mathcal{A}, \emptyset)$$

be the classical cooperation system on N. Let $x \in \mathbb{R}^N$ be a classical core vector; i.e.: $v(S) \leq x(S)$ for all $S \subseteq N$ and $v(N) = x(N)$ holds. Then any coalition S is the endpoint of paths P of length $|S|$ from $\emptyset$ to S such that $A_i \cap P \neq \emptyset$ if and only if $i \in S$. Hence for such a path P, $x(P) = x(S) \geq v(S)$ holds.
The efficiency condition of the core in our model simplifies to:

$$x(N) = \sum_{e \in E} \mu_e v(e) = \mu_N v(N) = v(N),$$

since N is the only sink. Thus any classical core vector is a core vector in our sense and vice versa.

9.2.1 Marginal-worth vectors & the Weber-set

For a fixed game $v \in \mathcal{G}$ and a path $P \in \mathcal{P}$, in analogy to the classical case, we define the *marginal-worth vector* of P with respect to v via:

$$h_i^P := h_i^P(v) := \sum_{xy \in A_i \cap P} v(y) - v(x) \text{ for all } i \in N. \tag{*}$$

This gives a definition of marginal-worth vectors which is independent of (SA). However, by (SA) Equation (*) reduces to: $h_i^P = v(y) - v(x)$, if $\{xy\} = A_i \cap P$ and $h_i^P = 0$, if $A_i \cap P = \emptyset$.
By taking the convex hull of all marginal worth vectors, we define the *Weber-set of* v:

$$\mathcal{W}(v) := \text{conv}\{h^P | P \in \mathcal{P}\}.$$

9.2.2 A greedy-type algorithm

In this subsection we will give an algorithm, which will be the main tool in our analysis of relations between the core and the Weber-set of a game.
For $x \in V$ denote by

$$\gamma(x) := \{j \in N | A_i \cap N^-(x) \neq \emptyset\}$$

the set of *essential* players, which own actions that lead to x.

The algorithm

Input: $c \in \mathbb{R}_+^N$, $e \in E$

(0) Initialize:
$P \leftarrow [\,]; y(x) \leftarrow 0$ for all $x \in V; X \leftarrow e$.

(1) If $X = s$, output (P, y) and stop.

(2) If $X \neq s$, choose $i \in \gamma(X)$ with minimal c_i and update:
$y(X) \leftarrow c_i; c_j \leftarrow [c_j - c_i]$for all $j \in \gamma(X)$ with $\{ab \in A_i | a \leq X\} \neq \emptyset$; $P \leftarrow XP$.

(3) Choose $u \in \{w \in V | wX \in A_i\}$ uniformly distributed and update:
$X \leftarrow u$.

(4) Go to (1).

Lemma 9.2.2 *Let (P, y) be the output of the algorithm with respect to the input $e \in E$ and $c \in \mathbb{R}_+^N$. Then the following statements are true:*

(a) $y \geq 0$ and $y(x) = 0$, if $x \notin P$.

(b) For all $i \in N$:

$$\sum_{x \in V; i \in \gamma(x)} y(x) \leq c_i.$$

Moreover: equality holds if $P \cap A_i \neq \emptyset$.

Proof. Since only the y-values of vertices of P are updated by the algorithm, (a) follows immediately by the update rule of it. By (a) we have:

$$\sum_{x \in V; i \in \gamma(x)} y(x) = \sum_{x \in P; i \in \gamma(x)} y(x).$$

The weight c_i of player i is changed if and only if $i \in \gamma(X)$ for the current X in the algorithm step. Moreover, the weights are never increased by this change. If (and only if) an action of a player is chosen at state $x \in P$, $y(x)$ is set to the current weight of this player. By (SA) this happens at most once.
Let $i \in N$ and write $P = (s = x_0, x_1, ..., x_k = e)$. W.l.o.g. assume that $x_j x_{j-1} \in A_j$ and that $i \in \gamma(x_j)$ for $j \in \{1, ..., l\}$. Denote by c_i^j the current weight of i at state x_j. Then

$$c_i^l = c_i - y(x_l)$$

and inductively:

$$c_i^1 = c_i - y(x_l) - ... - y(x_1).$$

Since $c_i^1 \geq 0$ by the update rule, we have:

$$c_i = c_i^1 + \sum_{i=1}^{l} y(x_j) \geq \sum_{i=1}^{l} y(x_j).$$

Assume that $A_i \cap P \neq \emptyset$. Then i governs an arc on P and by the above notation, this arc needs to be $x_1 x_0$. But then the updated weight of i at x_1 is 0. It follows:

$$c_i = c_i^1 + \sum_{i=1}^{l} y(x_j) = \sum_{i=1}^{l} y(x_j).$$

■

Lemma 9.2.3 *Let (P, y) be the output of the algorithm with respect to the input (e, c) and let $h := h^P$ be the marginal-worth vector associated with P. Then*

$$\sum_{i \in N} c_i h_i = \sum_{x \in P} v(x) y(x) = \sum_{x \in V} v(x) y(x)$$

holds.

Proof. Again let $P = (s = x_0, x_1, ..., x_k = e)$ such that $x_i x_{i-1} \in A_i$. Note that $h_i = 0$ if $A_i \cap P = \emptyset$ and that $y(x) = 0$ if $x \notin P$. Hence

$$\sum_{i \in N} c_i h_i = \sum_{i \in N, A_i \cap P \neq \emptyset} c_i h_i = \sum_{x_i \in P} c_i h_i = \sum_{x_i \in P} (c_i - c_{i-1}) v(x_i). \tag{*}$$

On the other hand, $c_i = \sum_{x \in P, i \in \gamma(x)} y(x)$. Hence (*) equals

$$\sum_{x_i \in P} \Big(\sum_{x \in P, i \in \gamma(x)} y(x) - \sum_{\tilde{x} \in P, i+1 \in \gamma(\tilde{x})} y(\tilde{x}) \Big) v(x_i) = \sum_{x_i \in P} y(x_i) v(x_i),$$

which proves the desired equality. ∎

9.2.3 Relations of the core and the Weber-set

The following theorem was proven in the classical case by Weber [91] and was later simplified by Derks [22] by an argument of separating hyperplanes. We will use a similar argument here.

Theorem 9.2.1 *Let* $v \in \mathcal{G}$. *Then* $\text{core}(v) \subseteq \mathcal{W}(v)$.

Proof. Assume that the theorem is false and aim for a contradiction. Then there exists $z \in \text{core}(v) \setminus \mathcal{W}(v)$. Since $\mathcal{W}$ is a closed convex set, there exists a separating hyperplane that separates z from $\mathcal{W}(v)$. Hence there is $c \in \mathbb{R}^N$ with

$$\sum_{j \in N} c_j z_j < \sum_{j \in N} c_j w_j \text{ for all } w \in \mathcal{W}(v).$$

Since z is an efficient allocation for v, there exists a probability distribution $(\mu_e)_{e \in E}$ on E such that $z(N) = \sum_{e \in E} \mu_e v(e)$. In particular, the above inequality holds for all vectors h of the following kind: for all $e \in E$ let $P_e \in \mathcal{P}$ be a path that ends in e and set $h := \sum_{e \in E} \mu_e h^{P_e}$. Hence we have

$$(*) \qquad \sum_{j \in N} c_j z_j < \sum_{j \in N} c_j h_j \text{ for all } h \text{ of the above type.}$$

Moreover, we have

$$0 \leq z(N) = \sum_{e \in E} \mu_e v(e) = \sum_{e \in E} \mu_e h^{P_e}(N) = h(N)$$

and hence we can assume $c \geq 0$ without violating inequality (*). Run the greedy-type algorithm from the last subsection with respect to c for all $e \in E$ and let $(y_e, P_e)_{e\in E}$ be the sequence of its outputs. Set $h := \sum_{e\in E} \mu_e h^{P_e}$. By the bilinearity of the inner product, it is $c^T h = \sum_{e\in E} \mu_e c^T h^{P_e}$.
For $x \in P_e$ let $P_e(x)$ be the unique subpath of P_e which ends in x. From Lemmata 9.2.2 and 9.2.3 together with the fact that $z \in \text{core}(v)$ and $c, y, z \geq 0$ we get:

$$\begin{aligned}
c^T h^{P_e} &= \sum_{x\in V} v(x) y_e(x) = \sum_{x\in P_e} v(x) y_e(x) \\
&\leq \sum_{x\in P_e} z(P_e(x)) y_e(x) \leq \sum_{x\in V} z(P_e(x)) y_e(x) && (9.1)\\
&\leq \sum_{j\in N}\Big(\sum_{P_e\cap\gamma(x)\ni j} y_e(x)\Big) z_j = \sum_{j\in N}\Big(\sum_{x\in V, j\in\gamma(x)} y_e(x)\Big) z_j && (9.2)\\
&\leq \sum_{j\in N} c_j z_j = c^T z.
\end{aligned}$$

Summand-wise inserted in $c^T h$, the last inequality yields:

$$c^T h = \sum_{e\in E} \mu_e c^T h^{P_e} \leq \sum_{e\in E} \mu_e c^T z = c^T z.$$

A contradiction which proves the theorem. ■

Theorem 9.2.2 *Observe the following optimization problem (P):*

$$\begin{aligned}
\min \quad & c^T z \quad s.t. \\
z(P) &\geq v(x) \textit{ for all } x \in V, P \in \mathcal{P}(x) \\
(1, ..., 1, -v(e_1), .., -v(e_k)) \begin{pmatrix} z \\ \mu \end{pmatrix} &= 0 \\
1^T \mu &= 1 \\
z &\geq 0
\end{aligned}$$

The following statements are equivalent:

(i) For all $c \in \mathbb{R}^N_+$ and all $e \in E$ the associated marginal vector h^P of the output path P of the greedy algorithm solves this optimization problem optimally.

(ii) $\mathcal{W}(v) \subseteq \text{core}(v)$

Proof. Assume (i) holds. Let $P = x_0x_1...x_k \in \mathcal{P}$ and $e \in E$ the sink such that $P \in \mathcal{P}_e$. W.l.o.g. assume that $x_{i-1}x_i \in A_i$. Choose

$$0 \leq c_k < c_{k-1} < ... < c_2 < c_1 < c_{k+1} < ... < c_n.$$

Because of the random component of the greedy algorithm in the last step, the probability that the greedy algorithm with respect to (c, e) picks P is not zero. Hence after finitely many tries the greedy algorithm picks P and by (i) h^P is an optimal solution of (P). In fact, h^P is a feasible solution and therefore $h^P \in \text{core}(v)$.

Assume the other way around that (ii) holds. Let $c \in \mathbb{R}^N_+$. Since $\mathcal{W}(v) \subseteq \text{core}(v)$ the associated marginal vector h^P of the output path of the greedy algorithm is in the core. It remains to show the optimality of h^P for (P).
Assume that there is $z \in \text{core}(v)$ with

$$c^T h^P > c^T z.$$

In analogy to the proof of of the last theorem one gets a contradiction. ∎

9.2.4 Another core concept

The idea, that owners of arcs of paths, which bring the system to a certain state, should be treated fair, seems initial. But there is another idea of fairness included in the classical core, which differs from it. The system will be in state x if and only if the owners of direct actions that could bring the system to x agree to do so. Therefore, those players should be treated fair. In terms of the last section, the players in $\gamma(x)$ should gain at least the worth of x. This yields another notion of the core:

$$\text{core}^\sharp(v) := \{z \in \mathbb{R}^N_+ | z(\gamma(x)) \geq v(x); \forall x \in V, z \text{ is efficient for } v\}.$$

Note that Lemmata 9.2.2 and 9.2.3 only depend on the algorithm given in Section 9.2.2, but not on the definition of the core. Recall the proofs of Theorems 9.2.1 and 9.2.2 and observe that the only parts in which the definition of the core was essential are equations (9.1) and (9.2), and the definition of the linear program (P). We will be able to adapt these results to core$^\sharp$.

Theorem 9.2.3 *Let* $v \in \mathcal{G}$. *Then* $\text{core}^\sharp(v) \subseteq \mathcal{W}(v)$ *holds.*

Proof. As mentioned above, it is enough to show that the inequalities (9.1) and (9.2) also hold for $z \in \text{core}^\sharp(v)$. For z, c, h as in the proof of Theorem 9.2.1 with $z \in \text{core}^\sharp$ instead of $z \in \text{core}$, the following holds:

$$\begin{aligned}
c^T h^{P_e} &= \sum_{x \in V} v(x) y_e(x) = \sum_{x \in P_e} v(x) y_e(x) \\
&\leq \sum_{x \in P_e} z(\gamma(x)) y_e(x) \leq \sum_{x \in V} z(\gamma(x)) y_e(x) \\
&\leq \sum_{j \in N} \Big(\sum_{x \in P_e; j \in \gamma(x)} y_e(x) \Big) z_j = \sum_{j \in N} \Big(\sum_{\gamma(x) \ni j} y_e(x) \Big) z_j \\
&\leq \sum_{j \in N} c_j z_j = c^T z.
\end{aligned}$$

Hence by exactly the same arguments as in Theorem 9.2.1 we find: $\text{core}^\sharp(v) \subseteq \mathcal{W}(v)$. ■

Theorem 9.2.4 *Observe the following optimization problem (Q):*

$$\begin{aligned}
\min \quad c^T z \quad & s.t. \\
z(\gamma(x)) &\geq v(x) \textit{ for all } x \in V, \\
(1, ..., 1, -v(e_1), .., -v(e_k)) \begin{pmatrix} z \\ \mu \end{pmatrix} &= 0 \\
1^T \mu &= 1 \\
z &\geq 0
\end{aligned}$$

The following statements are equivalent:

(i) For all $c \in \mathbb{R}^N_+$ and all $e \in E$ the associated marginal vector h^P of the output path P of the greedy algorithm solves this optimization problem optimally.

(ii) $\mathcal{W}(v) \subseteq \text{core}^\sharp(v)$

Proof. The proof reads word to word from the proof of Theorem 9.2.2. ■

Thus one has two concepts of fairness and it is a matter of taste which one should be used.

9.3 Generalizations of the classical core to other models

In this section we want to give examples of core-generalizations made in the literature and we briefly show how they fit into our core concepts.

Example 9.3.1 (The first system-theoretic approach) *Faigle and the author [37] already investigated a core concept in the acyclic case. However, they did not give a concept for arbitrary acyclic graphs and only had a look onto the restricted case of graphs that arise as selection structures (cf. Section 2.2.2).*
Let N be a finite set, $\sigma : 2^N \to 2^N$ a selector and $\mathcal{S}$ the set of all selections induced by σ. For $i \in N$ define $A_i := \{(S, S \cup i) | S \in \mathcal{S}, i \in \sigma(S)\}$ and $A := \bigcup_{i \in N} A_i$. Let $\Gamma(\sigma) := (N, \mathcal{S}, A, (A_1, ..., A_n), \emptyset)$ the cooperation system induced by σ. Faigle and the author chose the second idea behind the classical core and defined for a selection $S \subseteq N$ by

$$\gamma(S) := \{i \in S | S \setminus i \in \mathcal{S}\}$$

the set of essential players for S. Relative to that they developed the following terminology of an open core of a game $v : \mathcal{S} \to \mathbb{R}$:

$$\text{core}^o_{FV}(v) := \{z \in \mathbb{R}^N_+ | z(\gamma(S)) \geq v(S), \textit{ for all } S \in \mathcal{S}\}.$$

By setting $v^* := \min\{z(N) | z \in \text{core}^o_{FV}(v)\}$ and restricting the open core to

$$\text{core}_{FV}(v) := \{z \in \text{core}^o_{FV}(v) | z(N) = v^*\},$$

they defined a core that is always non-empty. This notion of essential players agrees with our notion of $\gamma(x)$ ($x \in V$) in our model in the special case of selection structures. If one assumes $\text{core}^\sharp(v) \neq \emptyset$ we directly see $\text{core}_{FV}(v) \subseteq \text{core}^\sharp(v)$, hence this core is a special case of $\text{core}^\sharp$.

Example 9.3.2 (Cores on set-systems) *Let $(N, \mathcal{F}, v)$ be a cooperative game on a set-system $\mathcal{F} \subseteq 2^N$ with $\emptyset, N \in \mathcal{F}$. A natural definition of a core is the following*

$$\text{core}_{\mathcal{F}}(v) := \{z \in \mathbb{R}^N_+ | z(F) \geq v(F), \textit{ for all } F \in \mathcal{F}, z(N) = v(N)\}.$$

For instance Faigle [28], Faigle, Grabisch and Heyne [31] or Grabisch and Xie [45] investigate core concepts of this type. Define

$$A_S := \{(F, G) \in \mathcal{F} \times \mathcal{F} | G \setminus F = S\} \textit{ and } N(\mathcal{F}) := \{S \subseteq 2^N | A_S \neq \emptyset\}.$$

We define a cooperation system relative to $\mathcal{F}$ with abstract player set $N(\mathcal{F})$ via:

$$\Gamma(\mathcal{F}) := (N(\mathcal{F}), \mathcal{F}, A := \bigcup_{S \in N(\mathcal{F})} A_S, (A_S)_{S \in N(\mathcal{F})}, \emptyset\}$$

and view v as a cooperative game on $\Gamma(\mathcal{F})$. Then

$$\text{core}(v) = \{z \in \mathbb{R}_+^{N(\mathcal{F})} | z(P_F) \geq v(F), \textit{ for all } P_F \in \mathcal{P}(F), z(N(\mathcal{F})) = z(N)\}.$$

Note that the abstract players which govern arcs on a path that starts in $\emptyset$ and ends in $F \in \mathcal{F}$, build a partition of F. Given a vector $z \in \text{core}_{\mathcal{F}}(v)$ and a coalition $S \in N(\mathcal{F})$, define

$$\overline{z}_S := z(S).$$

Let $P_F \in \mathcal{P}(F)$ be a path from $\emptyset$ to F which partitions F by $(F_1, ..., F_k)$. Then

$$\overline{z}(P_f) = \sum_{i=1}^{k} \overline{z}_{F_i} = \sum_{i=1}^{k} z(F_i) = z(F) \geq v(F)$$

holds, since $z \in \text{core}_{\mathcal{F}}(v)$. Hence each allocation in $\text{core}_{\mathcal{F}}(v)$ induces a core allocation in $\text{core}(v)$ and therefore $\text{core}_{\mathcal{F}}(v)$ could be seen as a special case of $\text{core}(v)$.

10 Non-cooperative cooperative settings

This chapter wants to build a bridge between some tasks of non-cooperative and cooperative game theory. We will not prove any new theorems here. Instead one should view this chapter as an outlook and inspiration for further research.
Section 10.1 recalls the model of Faigle *et al.* [33] for coalition formation in societies in order to highlight an interesting idea included. This example will motivate us to define general cooperation systems for non-cooperative games in extensive form in Section 10.2. We will identify non-cooperative games as tuples of special cooperative games on cooperation systems. This gives a general framework to model *social welfare functions* as cooperative games. Applying any of the allocation mechanisms introduced in the previous chapters of this thesis to these games, one gets the answer to the question: how should a certain social welfare be allocated to society?

Moreover, we generalize the concept of *potential functions* to our general model of non-cooperative games and can adopt certain classical results that yield a relation between games with a potential function and the existence of pure equilibrium points.
Section 10.2.3 is attended to the special case of 2-player non-cooperative games and gives a relation between strong equilibrium points and certain vectors of the open core of a game in our model. Moreover, we expose a relation of *mixed strategies* and random walks on a tensor product of certain graphs.
The last section briefly introduces the classical model of social welfare and social choice, which was first proposed by Arrow [5]. After stating the model, we are able to associate to any allocation mechanism a *social welfare function*. This yields a connection between fairness criteria of social welfare functions and fairness criteria of allocation mechanisms.

10.1 Coalition formation in societies

As mentioned in Section 2.5, the model of Faigle *et al.* [33] yields a generalized view on coalition formation processes. We will go into detail and present the model here in order to point out an idea, which is included.

Let N be a finite set of players. A non-empty family $x \subseteq 2^N$ of coalitions is called a *coalition structure*. A *coalition system* $\mathcal{X}$ is a set of coalition structures. We imagine that N and its subsets are part of a *society* σ, which can allow or deny certain *moves* $x \to y$, $x, y \in \mathcal{X}$. Given a public benefit function $v : \mathcal{X} \to \mathbb{R}$, we assume that σ is interested in maximizing $v(x)$, $x \in \mathcal{X}$ and bases its decision of allowing or denying a move $x \to y$ on the marginal benefit $v(y) - v(x)$ by the following axioms:

(S_1) If $v(y) - v(x) \geq 0$, then σ allows the move $x \to y$.

(S_2) If $v(y) - v(x) < 0$, then $x \to y$ will be accepted by σ with a certain probability $a(v(x), v(y)) > 0$.

Furthermore assume there is a stochastic transition matrix $M \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}}$, which measures a certain weight of a move $x \to y$ relative to the other moves $x \to z$. If $m_{xy} = 0$, the move $x \to y$ is called *infeasible*.

A coalition formation process in the sense of [33] is a sequence $(x_t)_{t>0}$ in $\mathcal{X}$, such that $x_i \to x_{i+1}$ is a feasible move with respect to M. By taking into account the regulatory possibilities of σ, $a(v(x), v(y))$ ($x, y \in \mathcal{X}$), one gets the combined move matrix

$$a_{xy} = \begin{cases} m_{xy} & \text{if } v(y) > v(x) \text{ and } y \neq x. \\ m_{xy} a(v(x), v(y)) & \text{if } v(y) < v(x). \\ 1 - \sum_{z \neq x} a_{xz} & \text{if } y = x. \end{cases} \tag{*}$$

In [33] one of the questions that were considered is: is there a regulation scheme $a(v(x), v(y))$, such that σ regulates the coalition formation process in a way s.t. the expected social benefit is close to its maximum? Under mild assumptions (as reversibility and weak symmetry of M) a result of Faigle and Kern [35] was used to prove that a Metropolis regulation ($a_\theta(v(x), v(y)) := e^{\frac{v(x)-v(y)}{\theta}}$, $\theta > 0$; *cf.* [65]) converges to an optimum if $\theta \to 0$ is lowered *slowly enough*. These mild assumptions are:

(a) The transition graph induced by M is strongly connected.

(b) M is weakly symmetric; *i.e.*: $m_{xy} > 0 \Rightarrow m_{yx} > 0$ holds for all $x, y \in \mathcal{X}$.

10.1.1 A special case of a system

We use the same notations as in the last section. Set $V := \mathcal{X}$ and define by $\mathcal{V}$ the Markovian state space associated to V. Assume there is a lowering procedure $\theta_t \to 0$ for $t \to \infty$. Define $\Phi_0^\theta := Id$ and $\Phi_t^\theta := (a_{xy}^{\theta_t})_{xy \in V \times V}$, where a^{θ_t} is the regulated move matrix relative to θ_t from Equation (*). Then the pair (V, Φ) yields a special case of a (inhomogeneous) Markovian system, since Φ_t^θ is indeed a stochastic matrix for each $t > 0$.

As seen in Subsection 5.5.1, Φ^θ induces a linear and t-efficient (f.a. $t > 0$) allocation mechanism and due to its relation to the Metropolis algorithm, we call it the *Metropolis allocation mechanism relative to* θ. In view of the prediction problem and Section 5.5, together with the convergence result of Faigle and Kern and its interpretation above, one can argue that: for a suited lowering procedure θ the coalition formation behavior of a society converges to an optimum by allocating payoffs due to the Metropolis allocation mechanism.

The motivating idea of this example

Typically the behavior of people in a society is egoistic; *i.e.*: each player wants to maximize his own profit and does not care about the social benefit of his actions. The move matrix M from above could be interpreted as the probabilities, induced by individual incentives to take certain actions. The individual payoff functions of certain actors (and with that the egoistic reasons for the certain probabilities in M) are hidden in this model, but nevertheless it is possible to make statements on the evolution of the system and hence also on a social benefit function v expected value.

10.2 Non-cooperative games

10.2.1 Extensive-form games

Let $N = \{1, .., n\}$ be a finite set of players. Assume that each player has a finite set of strategies S_i. Denote by $\mathcal{S} := \times_{i \in N} S_i$ the Cartesian product of all strategy sets of all players. Moreover, assume that there are functions $u_i : \mathcal{S} \to \mathbb{R}$ which measure the profit of player i by $u_i(s)$ for each $s \in \mathcal{S}$. We call the tuple $\Theta := (N, (S_i)_{i \in N})$ a *strategy system* and the tuple $(N, (S_i)_{i \in N}, \mathcal{S}, (u_i)_{i \in N})$ is called a *non-cooperative game in extensive-form* (on Θ).

Equilibria

In part of non-cooperative game theory one is interested in so called *equilibria*. A "state" $s = (s_1, ..., s_n) \in \mathcal{S}$ is called a (Nash-)equilibrium if for all $i \in N$ there is no appeal to vary from s; *i.e.*:

$$u_i(s_1, ..., s_n) \geq u_i(s_1, ..., s_{i-1}, \overline{s_i}, s_{i+1}, ..., s_n)$$

holds. By randomizing the choices of strategies of players and extending the u_i linearly to convex combination of strategies, one gets the term *(Nash-)equilibrium in mixed strategies* (named due to the famous existence result of Nash [69]). Formally: for $i \in N$ let $\pi_i \in \Delta(S_i)$ be a probability distribution. Define

$$u_i(\pi_1, ..., \pi_n) := \sum_{(s_1,...,s_n)\in\mathcal{S}} \pi_1(s_1)...\pi_n(s_n)u_i(s_1, ..., s_n).$$

Then $(\pi_i)_{i\in N}$ is called an *equilibrium in mixed strategies* if for all $i \in N$ and all $\overline{\pi}_i \in \Delta(S_i)$:

$$u_i(\pi_1, ..., \pi_n) \geq u_i(\pi_1, ..., \pi_{i-1}, \overline{\pi_i}, \pi_{i+1}, ..., \pi_n)$$

holds. Aumann [6] strengthened this notion. Due to him, an equilibrium is called strong if for all $S \subseteq N$, $(\overline{\pi}_i)_{i\in S}$

$$u_i(\pi_1, ..., \pi_n) \geq u_i((\pi_i)_{i\in N\setminus S}, (\overline{\pi}_i)_{i\in S})$$

holds.
Two games on Θ, $(u_i)_{i\in N}$ and $(\overline{u}_i)_{i\in N}$, are called *strategically equivalent* (*cf.* [90, p.4 ff.]) if there is $0 < k \in \mathbb{R}$ and $c \in \mathbb{R}^n$ such that for all $i \in N$ and $s \in S$

$$u_i(s) = k\overline{u}_i(s) + c_i.$$

Lemma 10.2.1 ([90]) *Strategic equivalence is an equivalence relation. Moreover, strategically equivalent games have the same equilibria.* ∎

REMARK Note that this lemma directly extends to equilibria in mixed strategies and also to strong equilibria.

10.2.2 Non-cooperative games as games on cooperation structures

Assume $\Theta := (N, (S_i)_{i\in N})$ is a strategy system and $(u_i)_{i\in N}$ is a non-cooperative game on Θ. Choose $s \in S$. By setting $c_i := u_i(s)$ and defining

$$\overline{u}_i(x) := u_i(x) - c_i, \text{ for all } i \in N,$$

we get a strategically equivalent game on Θ, which is s-normalized; *i.e.*: $\overline{u}_i(s) = 0$. Hence by Lemma 10.2.1 we may without loss of generality assume, that $(u_i)_{i\in N}$ is s-normalized for any predefined and fixed strategy state s.
Set $V := \mathcal{S}$ and for each $i \in N$:

$$A_i := \{(s,t) \in V \times V | s_i \neq t_i \text{ and } s_j = t_j \text{ for } j \neq i\}.$$

As before we set $\mathcal{A} := (A_1, ..., A_n)$ and $A := \bigcup_{i\in N} A_i$. Then $\Gamma := \Gamma(\Theta) := (N, V, A, \mathcal{A}, s)$ is a cooperation system. By fixing s, we tacitly assume in the following that non-cooperative games on Θ are s-normalized.
In view of Chapter 6.1, the following observation is interesting:

Theorem 10.2.1 *Let $(u_i)_{i\in N}$ a non-cooperative game on a strategy system Θ. Furthermore, let $G = G(\Gamma(\Theta)) = (V, A)$ be the transition graph associated with Θ as above. Then $G = \times_{i\in N} G_i$ is the Cartesian product of graphs G_i, where G_i is the complete graph on S_i without loops. Assume that u_i is independent of the choices of other players, then u_i is a function of S_i, hence $\prod_{i\in N} u_i = u_1 \otimes u_2 ... \otimes u_n$.*

Proof. This is direct by the definition of G and the definition of the Cartesian product of graphs (*cf.* Chapter 6). ∎

REMARK Obviously, $d^+(x) = d^+(y) = \sum_{i\in N}(|S_i| - 1)$ for all $x, y \in V$ by construction. Hence these graphs yield a whole class of examples, in which the symmetric and the entropy-symmetric Shapley value equal each other (*cf.* Lemma 5.4.2). Moreover, G is strongly connected and to each arc xy, also its reverse arc yx is an arc in G. Hence: given a Markovian matrix M with the property $M_{xy} > 0$ for all $xy \in A$, this matrix fulfills the weak symmetry and strong connectedness assumptions made for the convergence results of Faigle *et al.* [33] mentioned in the first section of this chapter.

As seen above:

Lemma 10.2.2 *Let $(u_i)_{i\in N}$ be a game on Θ. Then for all $i \in N$ u_i is a cooperative game on $\Gamma(\Theta)$. Hence also $u := \sum_{i\in N} u_i$ is a cooperative game on $\Gamma(\Theta)$. And conversely: each set $(v_i)_{i\in N}$ of cooperative games on $\Gamma(\Theta)$ is a non-cooperative game on Θ.* ∎

This lemma yields a cooperative interpretation of non-cooperative games in extensive-form: we associated to each non-cooperative game, a cooperative game u, such that u reflects the jointly generated payoff of all players. Hence a nearby question is: how should this jointly generated payoff be distributed? The whole theory of allocation mechanisms developed in this thesis makes a proposal.
The social benefit function u gives a characterization of equilibrium states in terms of marginal contributions:

Lemma 10.2.3 *Let $(u_i)_{i\in N}$ be a game on Θ which is s-normalized and such that $u_i \geq 0$ holds. Let $u := \sum_{i\in N} u_i$ be the associated cooperative game on $\Gamma(\Theta)$. Then $x \in V$ is an equilibrium state if and only if $u(y) - u(x) \leq 0$ for all $y \in N^+(x)$.*

Proof. If x is an equilibrium state, $u_i(x) \geq u_i(y)$ for all $y \in N^+(x)$ and all $i \in N$. Hence also $\sum_{i\in N} u_i(x) \geq \sum_{i\in N} u_i(y)$. By the assumption $u_i \geq 0$, also the reversal is true. ∎

REMARK Note that the "if"-part of the lemma above is also true, if the assumption $u_i \geq 0$ is dropped. Hence an equilibrium state is a local maximum of the social benefit function u.

Example 10.2.1 *If one restricts the arc-set of the graph $G := G(\Gamma(\Theta))$ with respect to a given non-cooperative game $(u_i)_{i\in N}$ in the following way:*

$$A_u := \{(s,t) \in A_i | u_i(s) < u_i(t), i \in N\},$$

and if this graph (V, A_u) is acyclic, its sinks are exactly the equilibrium states.

Example 10.2.2 (Potential games) *Due to Monderer and Shapley [66] a potential game is a non-cooperative game $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ such that there exists a potential function $P : \mathcal{S} \to \mathbb{R}$ with the property:*

$$\begin{aligned} &P(s_1, ..., s_{i-1}, s_i, s_{i+1}, ...s_n) - P(s_1, ..., s_{i-1}, \overline{s}_i, s_{i+1}, ...s_n) \\ =&u_i(s_1, ..., s_{i-1}, s_i, s_{i+1}, ...s_n) - u_i(s_1, ..., s_{i-1}, \overline{s}_i, s_{i+1}, ...s_n) \end{aligned}$$

for all $i \in N$, $s_j \in S_j$ and $s_i, \overline{s}_i \in S_i$. Shapley and Monderer proved that each potential game has an equilibrium point (namely each strategy profile whose potential equals $\max_{(s_1,...,s_n)\in\mathcal{S}} P(s_1, ..., s_n)$). In view of Example 10.2.1, this is not as surprising as it seems at first: consider the non-cooperative game $(P)_{i\in N}$ with pay-off functions equal to P. The sinks of the graph in Example 10.2.1 are exactly the maxima of P.

Due to the result of Monderer and Shapley, it is a natural wish to search for potential functions instead of equilibrium points. However, Rosenthal [76] gave a concrete potential function for a class of games called *congestion games*, without knowing the abstract connection between potential games and equilibrium points. Hence the idea of investigating potential functions is based on his work.

10.2.3 Non-cooperative games with two agents

In this section we restrict ourselves to the case $N = \{1, 2\}$. Let V_1, V_2 be finite strategy sets of player 1 resp. 2 and $u_1, u_2 : V := V_1 \times V_2 \to \mathbb{R}$ be utility functions such that u_i is $s := (s_1, s_2)$-normalized for a certain node $s \in V$. Denote by G_i the complete directed graph on V_i without loops. Let $\pi_i \in \Delta(V_i)$ be a mixed strategy of player $i = 1, 2$, such that $p_i(s_i) = 0$. For each vertex $(x, y) \in V_i \times V_i$ define $\pi^i_{xy} := \pi_i(y)\pi_i(x)$. Hence π_i induces a random walk on G_i with transition matrix P_i. By Section 6.2.1

$$P := \frac{1}{2}(P_1 \otimes Id + Id \otimes P_2)$$

is a transition matrix of a random walk on G. Hence we find:

Lemma 10.2.4 *Each pair of s-normalized mixed strategies, induces a randomized allocation mechanism on G.* ■

Proposition 10.2.1 *Let (π_1, π_2) be a strong equilibrium that is s-normalized and $\phi = \phi^\alpha$ the induced randomized allocation mechanism. Set $z := z(i) := (\phi^2_1(u_i), \phi^2_2(u_i))$. Then z is a 2-efficient allocation for u_i and for all $(x, y) \in V$:*

$$z(\gamma(x, y)) \geq \frac{1}{2}u_i(x, y) \textit{ for } i = 1, 2$$

holds.

Proof. Note first that by the 2-efficiency of ϕ, z is efficient. Hence there is a probability distribution $(\mu_e)_{e\in E_2}$ (which is independent of the games u_i), s.t.

$$z_1 + z_2 = \sum_{(xy)(wz)\in A} \phi^2_{(xy)(wz)}(u) = \sum_{e\in E_2} \mu_e u_i(e).$$

Recall that μ_e equals the probability of the underlying random walk to be in node e at time 2. For all $(x, y) \in E_2 \setminus (s_1, s_2)$:

$$\mu_{(x,y)} = \frac{1}{2}\pi_1(x)\pi_2(y)$$

holds. Combining these two equations yields:

$$z_1 + z_2 = \sum_{x\in V_1\setminus\{s_1\}, y\in V_2\setminus\{s_2\}} \frac{1}{2}\pi_1(x)\pi_2(y)u_i(x, y).$$

$\pi_i(s_i) = 0$ holds by assumption, hence:

$$\sum_{(x,y)\in V} \frac{1}{2}\pi_1(x)\pi_2(y)u_i(x, y) = \frac{1}{2}u_i(\pi_1, \pi_2) \geq u_i(a, b)$$

for all $(a, b) \in V$. The last inequality holds, since (π_1, π_2) is a strong equilibrium in mixed strategies. We just proved $z_1 + z_2 \geq \frac{1}{2}u_i(x, y)$ for $(x, y) \in V$, but desired the inequality $z(\gamma(x, y)) \geq \frac{1}{2}u_i(x, y)$. By the structure of G it holds $\gamma(x, y) = \{1, 2\}$ for all $(x, y) \in V$. Hence we achieved the desired inequality. ∎

REMARK Recall the definition of core$^\sharp$ from Section 9. The last proposition states that the vector $2z$ fulfills all inequalities of the core of the games u_i, except the efficiency condition. Since we restricted our analysis of the core to the acyclic case, no *core* for games on arbitrary graphs is yet defined. One could easily think of a core of a game v on a game system $\Gamma = (N, V, A, \mathcal{A}, s)$:

$$\text{core}(v) := \{z \in \mathbb{R}^N | z(\gamma(x)) \geq v(x) \text{ for all } x \in V\}.$$

Hence Proposition 10.2.1 yields a core vector in this sense.

10.2.4 Generalized non-cooperative games

We generalize the view on non-cooperative games to arbitrary graphs and aim define non-cooperative games on cooperation systems. Let $\Gamma := (N, V, A, \mathcal{A}, s)$ be a cooperation system s.t. $\mathcal{A} = (A_1, ..., A_n)$ and let $(u_i)_{i\in N}$ be a tuple of cooperative games on Γ. We call the pair $(\Gamma, (u_i)_{i\in N})$ a non-cooperative game on Γ. The terms equilibrium and potential function can be defined analogously *via*:

- $x \in V$ is an equilibrium if for all $y \in N^+(x)$ s.t. $xy \in A_i$: $u_i(x) \geq u_i(y)$ holds.
- $P : V \to \mathbb{R}$ is a potential function if for all $i \in N$ and all $xy \in A_i$: $P(y) - P(x) = u_i(y) - u_i(x)$ holds.

Hence in the special case of a cooperation system induced by a strategy system, these definitions coincide with the classical ones. By abstracting from sets of strategies to arbitrary sets of actions of the players, one could cover a much wider range of non-cooperative situations as in the extensive form. In terms of potential functions, even the argument for proving the theorem of Monderer and Shapley stays the same.

Proposition 10.2.2 ([66]) *Let $(u_i)_{i\in N}$ be a non-cooperative game on a cooperation system Γ. If there is a potential function P for $(u_i)_{i\in N}$, then there is an equilibrium point.*

Proof. Let $x \in V$ with $P(x)$ maximal. And let $y \in N^+(x)$ s.t. $xy \in A_i$. Then $u_i(y) - u_i(x) = P(y) - P(x) \leq 0$ holds by maximality of $P(x)$. Hence $u_i(y) \leq u_i(x)$. Thus x is an equilibrium point. ∎

Monderer and Shapley also gave a characterization of potential games which directly extends to our model. Their proof does not depend on the fact, that a non-cooperative game in extensive form is considered and reads one to one if one substitutes the classical notions with our notions. Hence we do not repeat their proof here and refer instead to it:

Theorem 10.2.2 ([66]) *The non-cooperative game $(u_i)_{i\in N}$ admits a potential function if and only if for all circles C in G*

$$\sum_{xy\in C} u_{i(xy)}(y) - u_{i(xy)}(x) = 0$$

holds. Where $i(xy) \in N$ s.t. $xy \in A_{i(xy)}$. ∎

Expected social welfare - a special case of the prediction problem

In the beginning of this chapter we proposed to assume $u := \sum_{i\in N} u_i$ to be the function which measures the social benefit. This function is known as the *utilitarian welfare function*. Also other measures are thinkable (and were already thought of [68]) depending on the concrete application. For instance, taking the maximum utility of the players,

the minimum, the product over all utility functions or the mean utility of the players are famous social welfare functions. The question, which social welfare function should be used in which situation, is a wide area of research. An overview of these (and other) topics can be found in the book of Moulin [68]. Mainly these topics were started by the work of Arrow [5] in which also the famous *Impossibility Theorem* was proven by him. This whole area of research was named *welfare economics*.
Besides the question, which social welfare function should be used to measure social welfare in a certain context, also the allocation problem of allocating the social benefit to certain groups of players is of interest. For instance, Hougaard and Østerdal [53] recently proposed to allocate social welfare by allocation procedures of cooperative game theory, by assuming a certain classical cooperative game is given that reflects certain societal claims. They used the concept of the core to prove certain impossibility theorems. Thus the idea of allocating a social benefit is not new to welfare economics. However, our model gives the opportunity to give a direct connection to cooperative allocation mechanisms and to directly give a cooperative setting, which suits the generation process of *any* social welfare.

Let $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a non-cooperative game and let Γ be the associated cooperation system, with underlying transition graph $G = (V, A)$. Moreover, let $u := \sum_{i\in N} u_i$ be the utilitarian social welfare function on V. Assume that there is an agreement (or a dictation) to allocate social welfare according to a certain randomized allocation mechanism ϕ. Since ϕ is efficient and $E_t = V$ for $t > 1$, there are $\pi^t(x) \in \mathbb{R}$ such that $\sum_{x\in V} \pi^t(x) = 1$ for all $t > 1$ and

$$\sum_{x\in V} \pi^t(x)u(x) = \sum_{xy\in A} \phi^t_{xy}(u).$$

Hence the expected social welfare could be expressed in terms of efficient allocation mechanisms. As seen in the chapters before (*cf.* Section 5.5), equivalently one could also give a concrete evolution operator in order to model the expected behavior of the players.

Example 10.2.3 (Maximizing social welfare at time t**)** *Assume a society consisting of players* $N = \{1, ..., n\}$ *wants not to maximize a social welfare function* u *depending on individual utility functions* u_i *(*$i \in N$*) overall, but at a given time step* t*. For instance: a company or a governmental department is called more to account for its financial situation during a legal year, as it is at the end of it. Hence one could be interested in maximizing* u *at a given time* t*. Again let* V *be the underlying state space and* s *an emphasized starting state. The society aims for being in a state* $e \in \arg\max\{u(x)|x \in E_t\}$

at time step t. In order to impulse certain actions of the players that lead to a desired state at time t, one could give a random walk with transition matrix P which starts in s such that

$$\sum_{e\in\arg\max\{u(x)|x\in E_t\}} \left\langle s\right| P^t e\rangle = 1. \qquad (*)$$

Each transition matrix P which satisfies (*) yields an evolution process that evolves s to a state of maximal social welfare at time t. Moreover, the randomized allocation mechanism induced by P yields a distribution of the worthiness of the individual actions for the generation of the social welfare at time t.

Example 10.2.4 (Multi-agent resource allocation) *We want to turn our attention to a related example. So called multi-agent resource allocation problems. The recent survey on multi-agent resource allocation of Chevaleyre et. al. [12] gives a nice overview over the whole topic. However, we will just give a brief idea of the modeling of such situations as cooperation systems here: assume there is a set of agents $N = \{1, ..., n\}$ and a set $R = \{r_1, ..., r_k\}$ of resources (e.g. electricity, money, man-power, time ... .). Moreover, assume that each agent $i \in N$ has a utility function $\overline{u}_i : 2^R \to \mathbb{R}$; i.e.: each agent measures a certain value to each package of resources. Finding a partition of R into n components, such that this partition is fair in some sense, is an allocation problem in this setting. Set*

$$V := \{(R_1, ..., R_n)|R_i \subseteq R, (R_i)_{i\in\mathbb{N}} \text{ is a partition of } R\}$$

and define an action-set of player $i \in N$ by

$$\begin{aligned} A_i := \quad & \{(S_1, ..., S_n)(R_1, ..., R_n)|\text{there is } j \in N \text{ and } r \in S_j, \text{ s.t.} \\ & S_i \cup \{r\} = R_i, S_j \setminus \{r\} = R_j, R_k = S_k \text{ for } k \in N \setminus \{i, j\}\}. \end{aligned}$$

We extend the functions $\overline{u_i}$ and define:

$$u_i : V \to \mathbb{R}, u_i(R_1, ..., R_n) := \overline{u}_i(R_i).$$

This yields a cooperative situation in the sense above and gives another view on resource allocation problems: agents trade resources until a certain state $v \in V$ is reached, in which nobody has incentives to trade any further. A main question in multi-agent resource allocation is: what is a good measure for the value of a certain partition and which partition of the resources is optimal for society? Thus the task is again to optimize a certain social welfare function.

Assume a certain social welfare function u (depending on the u_i) is given and its fairness is commonly accepted (or dictated). In view of Section 10.1, one could view resource allocation as a random process on the graph (V, A), hence a state in which u is maximized; *i.e.* a global social optimum is reached, seems to be a reasonable state for a "good" resource allocation.

Example 10.2.5 (Allocating a surplus - a connection to social welfare)
Assume a company has investors $N = \{1, ..., n\}$ and investor i has the possibility to invest a maximum of m_i Euros. Hence there are the following investment states the company is confronted with:

$$V := \{0, 1, ..., m_1\} \times ... \times \{0, 1, ..., m_n\}.$$

Assume that investments are not symmetric; i.e.: investor i could probably invest his money bounded on a certain purpose. Hence the state $(1, 2, 3)$ probably reflects another investment situation for the company as the state $(3, 2, 1)$ does. Moreover, assume that each investor can withdraw his investment after a certain time period or invest more money. The company predicts its gains depending on the certain states and in turn the payoffs of the individual investors in advance. Hence investor i is assured a payoff of $u_i(x)$ if the investment state $x \in V$ comes true. However, because of external unpredictable circumstances, the profit $c(x)$ of the company is strictly greater than the predicted profit $c_{pred}(x)$. Hence the company got a surplus $u(x) = c(x) - c_{pred}(x) - \sum_{i \in N} u_i(x)$ depending on the state $x \in V$ which was unexpected. The company wants to share this unexpected surplus with its investors in order to keep the investors willing to do further investments.
Thus the company has a social welfare allocation problem and could use the allocation mechanisms presented in the previous chapters to allocate its surplus to the investors.

10.3 Social welfare

The following setting is essentially due to Arrow [5]. Let V be a finite set of *alternatives*, $N = \{1, ..., n\}$ a finite set of players and for $i \in N$ let $\prec_i$ be a linear ordering of V which reflects the preferences of player i over the alternatives in V. Moreover, assume that there is $s \in V$ such that s is a joint minimum of $\prec_i$. To each $\prec_i$ we associate a utility function $u_i : V \to \mathbb{R}_{\geq 0}$, such that $u_i(s) = 0$, and for all $x, y \in V$: $u_i(x) \leq u_i(y)$ holds if $x \prec_i y$. A *social welfare function* is a function

$$W : (\mathbb{R}^V)^n \to \mathbb{R}^V,$$

which maps any tuple of utility functions to a mutual *social* utility function. One task of welfare economics is to define social welfare functions that fulfill certain fairness or rationality criteria.
REMARK Note that in welfare economics the domain of W is often assumed to be a subset of $(\mathbb{R}^V)^n$ and that players preferences are also given relative to these subsets (*cf.* [68]). By assuming W to be defined on $(\mathbb{R}^V)^n$, we implicitly assume the so called *unrestricted domain* axiom. However, we will not go into these details and for our purposes it is enough to define welfare functions on $(\mathbb{R}^V)^n$ instead.

10.3.1 Examples of fairness criteria

A social welfare function W satisfies *non-dictatorship* (ND) if there is no $i \in N$ such that for all $(u_j)_{j\in N} \in (\mathbb{R}^V)^n$ and all $x, y \in V$

$$u_i(x) \leq u_i(y) \Rightarrow W(u_1, ..., u_n)(x) \leq W(u_1, ..., u_n)(y)$$

holds; *i.e.*: the social preferences are not dictated by a single players preferences.

W is *independent of irrelevant alternatives* (IIA) if for all $x, y \in V$ and all $(u_j)_{j\in N}$, $(\overline{u}_j)_{j\in N} \in (\mathbb{R}^V)^n$

$$(\text{for all } i \in N : u_i(x) < u_i(y) \Leftrightarrow \overline{u}_i(x) < \overline{u}_i(y)) \Rightarrow W((u_j)_{j\in N})(x) < W((\overline{u}_j)_{j\in N})(y),$$

holds.

W satisfies *unanimity* (U) if for all $x, y \in V$:

$$(\forall i, j : u_i(x) \leq u_i(y) \Leftrightarrow u_j(x) \leq u_j(y)) \Rightarrow W((u_j)_{j\in N})(x) \leq W((u_j)_{j\in N})(y)$$

holds. Arrow proved in [5]:

Theorem 10.3.1 (Impossibility Theorem of Arrow) *If W fulfills (U) and (IIA), then it does not fulfill (ND). Hence there is a dictator.* ■

W satisfies anonymity if for each permutation $\sigma \in Sym(N)$, all $x, y \in V$ and all $(u_i)_{i\in N} \in (\mathbb{R}^V)^n$ the following holds:

$$W((u_j)_{j\in N})(x) < W((u_j)_{j\in N})(y) \Leftrightarrow W((u_\sigma(j))_{j\in N})(x) < W((u_\sigma(j))_{j\in N})(y).$$

10.3.2 Welfare functions induced by allocation mechanisms

By investigating non-negative utility functions instead of linear orderings, we abstracted from the classical model of Arrow. In order to apply our theory of cooperative situations to social welfare functions, we will additionally consider a graph on V. We will restrict ourselves to the case where $G = (V, A)$ is the complete directed graph on V without loops; *i.e.*: $A := V \times V \setminus \{(x,x)|x \in V\}$. If it is necessary for a concrete application to restrict this graph any further, there should be no formal reason not to do so, as long as G is s-connected.
Let $\phi : \mathcal{G} \to \mathbb{R}^A$ be an allocation mechanism. We propose a social welfare function associated to ϕ and define it point-wise on V. Therefor let $x \in V$ and $(u_i)_{i \in N}$ a tuple of utility functions as above. Moreover, let $t := |V|$. Define

$$W^{\phi}(u_1, ..., u_n)(x) := \frac{1}{N} \sum_{i \in N} \sum_{u \in N^-(x)} \phi^t_{ux}(u_i).$$

Thus $W^{\phi}(u_1, ..., u_n)(x)$ measures the value of the in-arcs of x according to ϕ. For further cooperative considerations it could be useful if $W^{\phi}(u_1, ..., u_n)$ again is a cooperative game (*i.e.*: is s-normalized). One gets this property by simply assuming s to be a source in G. But we will not do so. We conclude this chapter by giving two properties of W^{ϕ}:

Proposition 10.3.1 *W^{ϕ} satisfies anonymity for all allocation mechanisms ϕ. Moreover: if ϕ is linear, W^{ϕ} permits no dictators.*

Proof. Anonymity is clear by definition of W^{ϕ}, since permuting the summands does not change the sum. Let ϕ be linear. Assume there is a dictator $i \in N$. Consider the utility function $u_i(x) := 1$ for all $x \in V \setminus s$. Hence i is indifferent between all alternatives. Let $x, y \in V$. Since i is a dictator, it follows: $W^{\phi}(u_1, ..., u_n)(x) = W^{\phi}(u_1, ..., u_n)(y)$ for all utility functions u_j $(j \neq i)$. Obviously this yields a contradiction by the definition of W^{ϕ}. ∎

11 Open problems & perspective

In this thesis we presented a general model for settings, in which agents can cooperate. We gave a general algebraic framework for stating allocation and prediction problems of cooperative settings, in which the modeling of non-acyclic situations is possible. Even non-static, time-discrete cooperative circumstances could as well as static ones be represented by our model. The model puts the actions of players in the center of interest, instead of the players themselves.

Initially we aimed for a general model for cooperative games that covers time-dependent and dynamic cooperative settings. In Chapter 3 and 4 we achieved parts of this goal and transported some seminal questions of cooperative game theory to a much more general framework. Chapter 5 gave concrete and time-dependent solutions to the allocation problem. We developed solutions to the allocation problem in terms of randomized allocation mechanisms and were able to characterize these allocation mechanisms by certain fairness criteria. We generalized Weber's classical value theory and, moreover, we extended it by investigating generalized stochastic matrices. This yields a new class of fair allocation mechanisms, even in the classical case.
We uncovered a relation between tensor products of cooperation systems and Cartesian products of the underlying transition graphs in Chapter 6. Moreover, we proposed a composition of randomized allocation mechanisms on different systems, in order to construct a fair allocation mechanism on their tensor product.

We studied allocation mechanisms that are not necessarily efficient in Chapter 7 and gave several characterizations by means of various fairness axioms. After that we introduced the Cesàro value of an allocation mechanism, which gives another notion of fairness over time by taking certain Cesàro means. Even if arbitrary allocation mechanisms need not to converge, we achieved the convergence of a very large class of Cesàro values (*cf.* Theorem 7.2.2).

In Chapter 8, we showed, how a general model of quantum random walks on graphs could be used to propose a whole new class of linear and efficient, but not necessarily ratio fair, allocation mechanism: quantum allocation mechanisms. On the basis of the concepts developed in Chapter 7 we proved the convergence of all Cesàro values of quantum random walks.

We proposed two different ideas for cores in our general framework in Chapter 9 and proved their inclusion in the Weber-set. After that we had a look on non-cooperative games and could identify non-cooperative games in extensive form as a special case of games on cooperation systems. By that we jointly modeled cooperative and non-cooperative games. We also pointed out relations to welfare economics and gave a first idea of a social welfare function that is induced by a linear allocation mechanism.

We will state open problems and questions for future research in the spirit of the following quotation:

"As long as there is something left to do, we have not done anything."

-unknown author.[1]

Non-Markovian evolution

In view of Chapter 7 and especially of Theorems 7.1.2 & 7.2.2 plenty of linear and ratio fair allocation mechanisms with convergent Cesàro values are imaginable. For instance one could study allocation mechanisms $\phi = \phi^O$ induced by orthogonal matrices $O \in \mathbb{R}^{V \times V}$. Since O has euclidean norm equal to 1, the Cesaro value of ϕ converges. Orthogonal matrices preserve the lengths of vectors. Hence O gives also rise to an *orthogonal* evolution operator on the *orthogonal state space* $\mathcal{V} := \{v \in \mathbb{R}^V \mid \|v\| = 1\}$. These states could also be interpreted in a quantum mechanical spirit as probabilities, since $1 = \|v\| = (\sum_{x \in V} |v_x|^2)^{\frac{1}{2}}$. Are there concrete examples of cooperative processes, which are describable by orthogonal matrices? Is there a characterization of orthogonal allocation mechanisms in terms of fairness axioms? Even other state-spaces and evolution operators are thinkable by investigating another norm than the euclidean one. In view of Chapter 8, it is desirable to characterize quantum allocation mechanisms by means of fairness criteria. To prove that quantum allocation mechanisms enjoy certain properties is more or less easy by the concrete representation of them. But identifying an allocation mechanism to be induced by a quantum random walk, seems to be a much harder task.

[1] even if two websites indicate that Bertrand Russell is the originator of this quotation, we failed to prove this claim. Hence we refer to the author of this statement as *unknown*.

Convergence & Mixing times

As mentioned in the last remark of Section 7.2.2 the assumption of G being strongly connected in order to achieve convergence of Cesàro values is unsatisfying. Is this assumption avoidable or could it be weakened?

Except for Example 5.6.1 we neither gave a fairness interpretation of convergence of an allocation mechanism, nor of its speed of convergence. Is a convergent allocation mechanism in some sense fairer than a divergent one? Is it fairer, if it converges fast or if it converges slow? Chung [20] gave a model for measuring speed of convergence of random walks on directed graphs by means of inequalities relying on certain eigenvalues of a matrix, associated to the transition matrix of the considered random walks. Is there a cooperative interpretation of those eigenvalues and the induced inequalities?

Compromise values

Let $\Gamma := (N, V, A, \mathcal{A}, s)$ be a cooperation system s.t. $\mathcal{A} = (A_1, ..., A_{|N|})$. Assume there are reasonable time-dependent upper and lower bounds l^t and o^t such that it is an agreement that a solution ϕ^t should suffice:

$$l^t_{xy}(v) \leq \phi^t_{xy}(v) \leq u^t_{xy}(v), \tag{*}$$

for all $xy \in A$, $t > 0$ and $v \in \mathcal{G}$. Moreover, assume that at each time t the value $P^t(v) \in \mathbb{R}$ should be allocated (think for instance of the expected value of the t-endpoints relative to a given probability distribution). Then there exists $\lambda^t(v) \in \mathbb{R}$ such that $\lambda^t(v)(\sum_{xy \in A} l^t_{xy}(v)) + (1 - \lambda^t(v))(\sum_{xy \in A} u^t_{xy}(v)) = P^t(v)$. If l and u admit a $\lambda^t \in \mathbb{R}$ such that it is independent of v, we say that l and u admit a *t-compromise* and call

$$\phi^t = \lambda l^t + (1 - \lambda) u^t$$

the *t-compromise value* of u and l. The idea of compromise values was first developed in the context of bargaining problems by Kalai and Smorodinsky [58], later this idea was adapted for the case of non-transferable utility games [18] and for classical cooperative games [88] by Tijs *et al.* The above construction gives a first idea of a generalization of compromise values in the general case of cooperation systems. How could fairness ideas of concrete compromise values be generalized to cooperation systems? Is there a characterization of upper and lower bounds u and l such that they admit a compromise?

Non-cooperative settings

Chapter 10 gave a first idea of connecting cooperative allocation mechanisms with the construction of social welfare functions. Also other *natural* welfare functions could be associated with an allocation mechanism ϕ instead of the proposed one. For instance, one could measure the value of the out-arcs of a vertex instead of the in-arcs, or one could take the average value of the in- and the out-arcs.
However, this relation opens the door to a wide array of questions: is a certain welfare function associated to ϕ more reasonable than another? Is there a relation between fairness criteria for ϕ and W^{ϕ}? Do randomized allocation mechanisms ϕ yield a randomized interpretation in terms of social welfare? Could cooperative game theory learn something from impossibility theorems as Theorem 10.3.1 in this context?

Core & Weber-set

In Chapter 9 we only gave core concepts for games on acyclic cooperation systems. Thus natural questions are: are there extensions of these concepts to arbitrary graphs? What is a *good* notion for a time-dependent core concept?

In classical cooperative game theory the fact that the Weber-set of a game is a subset of the core of the game if and only if the game is supermodular, is very famous. Essentially this result is due to Edmonds [27], who studied certain polyhedra and submodular functions in a non-game-theoretic context. In the case of general cooperation systems: is there a nice property of characteristic functions $v \in \mathcal{G}$ which characterizes the above mentioned inclusion?

Invariants & continuous times

Attention: this paragraph uses some terms of physics which are not explained any further. In physics a theorem of Noether became very famous. We will give just the informal idea of its statement here: *"To every system symmetry there is an invariant."*. This theorem is the mathematical reason for several conservation-laws in physics (*e.g. energy conservation*, *impulse conservation*,...). Mathematically it means: whenever a *Hamilton operator* commutes with a matrix a certain *measurement* (*i.e.* expected value) is conserved over time. Faigle [29] suggested to adapt this idea and to study operators which commute with the evolution operator of a system (*i.e.* with its defining matrices). A main problem is to prove a discrete version of a theorem like the *Noether Theorem*. Since times are continuous in quantum mechanics one could consider the derivative of a wave function in time. We investigated discrete time steps, hence differential calculus w.r.t. to time is not available to us. It seems natural to avoid this problem by investigating difference equations instead of differential equations, in order to achieve a discrete Noether Theorem. How could this be done? And if it can be done, what is a cooperative interpretation of the conserved quantities? Besides invariants: are there any concrete applications of cooperative game theory in the sense of our model to physics?

Another aspect is interesting about continuous times: is there a generalization of our model to the time-continuous case? And if so, is this generalization compatible with the relation between discrete time Markov chains and continuous time Markov chains? Could the whole theory of transition graphs be embedded into a setting of so called *continuous graphs*?

This list of topics and question is by no means complete but it reflects and highlights the open problems that are perceived as interesting by the author.

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List of symbols

We list certain symbols that appear regularly in this thesis. In order to keep this list short, let V be a finite set, $G = (V, A)$ be a directed graph on V and $N = \{1, ..., n\}$ a finite set of players. Moreover, let $x \in V$ and $xy \in A$.

$N^+(x), N^-(x), d^+(x), d^-(x)$	the out- resp. in-neighbors of x and the out- resp. in-degree of x.
s	an emphasized start vertex in V
t	a discrete time $t = 0, 1, 2, ...$
2^N	the power-set of N
$gcd(l_1, ..., l_m)$	the greatest common divisor of natural numbers $l_1, ..., l_m$
$\mathbb{K}$	a subfield of $\mathbb{C}$
$z(S)$	sum over the S-components of a vector $z \in \mathbb{K}^N$
$Sym(N)$	the symmetric group of N
∂	the marginal operator
$\mathcal{P}_t$	set of all paths of length t starting in s
$\mathcal{P}(x)$	set of all paths that end in a fixed vertex x
$\mathcal{P}$	set of all paths starting in s
E_t	set of endpoints of all paths of length t
A_S	set of arcs, governed by a coalition $S \subseteq N$
A_i	set of arcs governed by $\{i\} \subseteq N$
$A(t), A_i(t)$	set of arcs of all paths of length t resp. $A(t) \cap A_i$
$\mathcal{A}$	a partition of A in blocks A_S ($S \subseteq N$)
$\mathcal{V}$	a state space
Γ	a cooperation system
$\mathcal{G}, \mathcal{G}(\Gamma)$	the space of all cooperative games on Γ
(V, Φ)	a system on V with evolution operator on Φ
$h^P(v), h^P$	the marginal-worth vector relative to a path $P \in \mathcal{P}$ and $v \in \mathcal{G}$

ϕ	an allocation mechanism
ϕ^t	the allocation vector in $\mathbb{K}^A$ induced by ϕ at time t
$\phi^t_{xy}(v)$	the payoff of $xy \in A$ at time t relative to ϕ and $v \in \mathcal{G}$
$\overline{\phi}$	the Cesàro value of an allocation mechanism ϕ
$\gamma(x)$	the set of essential players relative to a vertex $x \in V$
$core(v)$	the path-core relative to $v \in \mathcal{G}$
$core^\sharp(v)$	the essential player core relative to $v \in \mathcal{G}$
$\mathcal{W}(v)$	the Weber-set with respect to $v \in \mathcal{G}$
$G \otimes H$	the Cartesian product of two graphs G and H
$A \oplus B$	the Kronecker-sum of two quadratic matrices A, B
$\Phi \boxdot \Psi$	the concatenated evolution operator of evolution operators Φ and Ψ
$\mathcal{V} \otimes \mathcal{W}$	the tensor product of systems $\mathcal{V}$ and $\mathcal{W}$
$H(\phi)$	entropy of a randomized allocation mechanism ϕ
$\mathcal{I}(G)$	incidence algebra of the graph G
$\mathcal{S} := \mathcal{S}(\mathbb{K}^A)$	the space of all positive self-adjoint $(\lvert A\rvert \times \lvert A\rvert)$-matrices
$\lvert\psi\rangle$, $\lvert\psi^t\rangle$	a wave function, resp. an evolved wave-function w.r.t. an unitary matrix
$p_t(x\lvert\psi)$	probability to be in state $x \in V$ at time t w.r.t. a quantum walk that started in $\lvert\psi\rangle$
P_x	the projection operator onto the neighborhood of a vertex $x \in V$

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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbstständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit - einschließlich Tabellen, Karten und Abbildungen -, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie - abgesehen von unten angegebenen Teilpublikationen - noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Prof. Dr. Ulrich Faigle betreut worden.

Jan Voss