# Constructions of open books and applications of convex surfaces in contact topology

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# Abstract

In the present thesis we introduce an extension of the contact connected sum, in the sense that we replace the tight 3-balls by standard neighbourhoods of Legendrian graphs $G \subset (S^3, \xi_{st})$. By the use of convex surface theory we show that there is a Weinstein cobordism from the original contact manifold to the result of the extended contact connected sum. We approach the analogue of this result in higher dimensions, using different methods, and present a generalised symplectic 1-handle which is used for the construction of exact symplectic cobordisms. Furthermore we describe compatible open books for the fibre connected sum along binding components of open books as well as for the fibre connected sum along multi-sections of open books. Given a Legendrian knot $L$ with standard neighbourhood $N$ in a closed contact 3-manifold $(M, \xi)$, the homotopy type of the contact structure $\xi|_{M \setminus N}$ on the knot complement depends on the rotation number of $L$. We give an alternative proof of this folklore theorem, as well as for a second folklore theorem that states, up to stabilisation, the classification of Legendrian knots is purely topological. Let $\zeta$ denote the standard contact structure on the 3-dimensional torus $T^3$. Denoting by $\Xi(T^3, \zeta)$ the connected component of $\zeta$ in the space of contact structures on $T^3$, we show that the fundamental group $\pi_1\big(\Xi(T^3, \zeta)\big)$ is isomorphic to $\mathbb{Z}$.

# Kurzzusammenfassung

In der vorliegenden Arbeit erweitern wir die Kontakt-verbundene Summe indem wir die straffen 3-Bälle durch Standardumgebungen von Legendre Graphen $G \subset (S^3, \xi_{st})$ ersetzen. Mit Hilfe von konvexen Flächen zeigen wir die Existenz eines Weinstein Kobordismuses zwischen der ursprünglichen Kontaktmannigfaltigkeit und dem Resultat der erweiterten verbundenen Summe. Mit anderen Methoden zeigen wir ein analoges Resultat in höheren Dimensionen und präsentieren einen verallgemeinerten symplektischen 1-Henkel mit dessen Hilfe wir exakte symplektische Kobordismen konstruieren. Darüber hinaus konstruieren wir sowohl kompatible offene Bücher für die Faser-verbundene Summe entlang von Bindungskomponenten offener Bücher, als auch kompatible offene Bücher für die Faser verbundene Summe entlang von mehrfachen Schnitten offener Bücher. Für einen gegebenen Legendre Knoten $L \subset (M, \xi)$ mit Standardumgebung $N$ in einer Kontaktmannigfaltigkeit $(M, \xi)$ hängt der Homotopietyp der Kontaktstruktur $\xi|_{M \setminus N}$ über dem Knotenkomplement von der Rotationszahl von $L$ ab. Wir geben einen alternativen Beweis sowohl für dieses als auch für ein weiteres Folkloretheorem, das besagt, dass die Klassifikation von Legendre Knoten, bis auf Stabilisierungen, rein topologischer Natur ist. Sei mit $\zeta$ die Standardkontaktstruktur auf dem 3-dimensionalen Torus $T^3$ bezeichnet. Ferner bezeichne $\Xi(T^3, \zeta)$ die Zusammenhangskomponente von $\zeta$ im Raum der Kontaktstrukturen auf $T^3$, dann zeigen wir, dass die Fundamentalgruppe $\pi_1\big(\Xi(T^3, \zeta)\big)$ isomorph zu $\mathbb{Z}$ ist.

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# Contents

# Introduction

Ever since Giroux [28] introduced the notion of convex surfaces their theory developed into a powerful concept frequently used in low-dimensional contact topology. Convex surfaces led to a complete classification of contact structures on some simple manifolds and, more recently, a rough classification on all closed manifolds [26,34,37]. The study of convex surfaces also initiated one of the most striking results of the last decade in contact topology. While it had been known for almost 50 years that open books carry a natural contact structure [47], at the beginning of the millennium it turned out that this was just one fragment of a much deeper correlation. As was observed by Giroux [25], contact structures in dimension 3 are of purely topological nature: he established a one-to-one correspondence between isotopy classes of contact structures and open book decompositions up to positive stabilisation. Giroux's correspondence turns out to be the starting point for various results such as a proof of Harer's Conjecture [30] or those of Eliashberg [7] and, independently, Etnyre [13] that symplectic fillings of contact 3-manifold can be capped off.

The present work begins with a preliminary chapter where we briefly introduce some of the standard tools surrounding convex surfaces and open book decompositions such as Giroux's Flexibility Theorem, edge rounding or the coherence of open books and standard handle decompositions. The results of this thesis are spread over the remaining chapters, which can be read, more or less, independently.

In the second chapter we introduce an extension of the contact connected sum, in the sense that we replace the tight 3-balls by standard neighbourhoods of Legendrian graphs $G \subset (S^3, \xi_{st})$. We use convex surface theory to show that there is a Weinstein cobordism from the original contact manifold to the result of the extended contact connected sum, cf. Theorem 11. Given a compact surface $\Sigma$ with non-empty boundary and two diffeomorphisms $f$ and $g$ of $\Sigma$ equal to the identity near the boundary, Baker–Etnyre–Van Horn-Morris [2] show the existence of a Stein cobordism whose negative end equals the disjoint union of the open books $(\Sigma, f)$ and $(\Sigma, g)$, and whose positive end induces $(\Sigma, f \circ g)$. Among other applications, we show how to deduce their result from our cobordism construction, cf. Corollary 13. We approach the analogue of this result in higher dimensions, using different methods, in Theorem 17 of Section 2.2. There we implicitly present a generalised symplectic 1-handle which is used for the construction of exact symplectic cobordisms, cf. Theorem 2.2.

Fibre connected sums recently drew some attention appearing in Wendl's [49] notion of planar torsion, an obstruction for strong fillability generalising overtwistedness and Giroux torsion. In essence, a contact manifold admits planar torsion if it can be written as the binding sum of a non-trivial number of open books, one of which has planar pages. In Chapter 3 we describe compatible open books for the fibre connected sum along binding components of open books, cf. Proposition 21, as well as for the fibre connected sum along multi-sections of open books, cf. Proposition 26. As an application, the first description provides a simple way of constructing open books compatible with all tight contact structures on $T^3$ and an open book supporting the result of performing a Lutz twist along a binding component of an open book.

Given a Legendrian knot $L$ with standard neighbourhood $N$ in a closed contact 3-manifold $(M, \xi)$, the homotopy type of the contact structure $\xi|_{M\setminus N}$ on the knot complement depends on the rotation number of $L$. This folklore result appeared in the literature, see for example [8, Section 4.1], though details of the argument have not appeared. Recently, details of the argument appeared in a preprint of Etnyre [12]. Alternative arguments were told to the author, in private communication, by Geiges. In Chapter 4 we present another alternative approach using the Pontryagin construction of maps to $S^2$, cf. Section 4.1. In Section 4.2 we approach another folklore theorem that states that, up to stabilisation, the classification of Legendrian knots is purely topological. For $(\mathbb{R}^3, \xi_{st})$ this theorem was proved by Fuchs and Tabachnikov [17, Theorem 4.4]. Recently Ding–Geiges gave detailed arguments for the general case [22]. Their proof is based on convex surface theory and a neighbourhood theorem for arbitrary knots in contact 3-manifolds. The proof presented in this thesis is also based on convex surface theory, however, without the need of a neighbourhood theorem.

The final chapter is concerned with the topology of the space of contact structures on the 3-dimensional torus $T^3$. To be more precise, let $\zeta$ denote the contact structure defined by the equation $\cos z\, dx - \sin z\, dy = 0$, where $(x, y, z)$ are coordinates on $T^3 \equiv \mathbb{R}^3/\mathbb{Z}^3$, and let $\Xi(T^3, \zeta)$ denote the connected component of $\zeta$ in the space of contact structures on $T^3$. We follow an outline of Geiges–Gonzalo [21] to show that the fundamental group $\pi_1\big(\Xi(T^3, \zeta)\big)$ based at $\zeta$ is isomorphic to $\mathbb{Z}$, cf. Theorem 33. The proof is based on convex surface theory and utilises ideas and results from [27] and [24].

# Chapter 1

# Preliminaries

We assume that the reader is familiar with the basic notions of contact topology. For an introduction to contact topology we point the reader to [19]. Most of the concepts used in the present work, such as convex surfaces or open book decompositions for example, are frequently used in the literature and can, more or less, be considered standard in the world of contact topology. For the sake of completeness however in the present chapter we will recall most of these concepts to reduce the need to consult external sources to a minimum. For a more detailed introduction to these concepts see [11] and [14].

We will denote by $\xi_{st}$ the standard contact structure on $S^3$. For convenience we will use the same notation for the standard contact structures on $S^1 \times S^2$, and $\#_g(S^1 \times S^2)$ in general, as well as for the contact structures on standard neighbourhoods of Legendrian knots. It will be clear from the situation which manifold the contact structure is referring to. We usually understand the unit circle $S^1$ as the quotient of the real numbers $\mathbb{R}$ by $\mathbb{Z}$ or sometimes with $2\pi\mathbb{Z}$ respectively. In the present work all manifolds will be oriented and, unless otherwise stated, compact. All contact structures will assumed to be cooriented and positive.

## 1.1 Convex surfaces in contact geometry

Throughout the whole section let $(M, \xi)$ denote a compact, oriented 3-manifold with contact structure $\xi$. A **convex surface** $\Sigma$ in $(M, \xi)$ is an oriented embedded surface with the property that there is a contact vector field $X$, i.e. a vector field whose flow preserves $\xi$, defined near and transverse to $\Sigma$. The contact vector field defines an $\mathbb{R}$-invariant neighbourhood $\Sigma \times \mathbb{R} \subset M$ of $\Sigma$, where $\Sigma = \Sigma \times \{0\}$. We call $\Sigma \times \mathbb{R}$ a **vertically invariant** neighbourhood of the convex surface $\Sigma$. In the present paper all convex surfaces will either be closed or compact with Legendrian boundary. The **dividing set** $\Gamma_\Sigma$ of a convex surface $\Sigma \subset (M, \xi)$ (corresponding to a given contact vector field $X$) is the set of points $p \in \Sigma$ where $X$ is tangent to $\xi$. One can show that $\Gamma$ is a collection of circles and, in case $\partial\Sigma$ is non-empty and Legendrian, properly embedded arcs. It turns out that all the crucial information about the contact structure in a neighbourhood of a convex surface is encoded in its dividing set, cf. Subsection 1.1.2 below.

### 1.1.1 Perturbation into a convex surface

A **Legendrian knot** $K \subset (M, \xi)$ is an embedded curve which is everywhere tangent to $\xi$. Suppose the Legendrian knot $K$ either appears as boundary component of a surface or is contained in the interior of a surface $\Sigma$. Then one defines the **twisting number** $\mathtt{tw}(K, \Sigma)$ to be the number of (right-handed) twists of $\xi$ along $K$, where we measure the twists with respect to the framing induced by the surface $\Sigma$. If $\Sigma$ is a Seifert surface of $K$, then $\mathtt{tw}(K, \Sigma)$ is the Thurston-Bennequin invariant $\mathtt{tb}(K)$. Suppose $K \subset \Sigma$ is a Legendrian knot lying on a convex surface $\Sigma$ with dividing set $\Gamma$. Suppose further $K$ and $\Gamma$ intersect transversely, then, according to [14], we have

$$\mathtt{tw}(K, \Sigma) = -\frac{1}{2}|K \cap \Gamma|, \tag{1.1}$$

where $|K \cap \Gamma|$ denotes the number of points in the set $K \cap \Gamma$.

Suppose $\Sigma$ is a convex surface with Legendrian boundary $\partial\Sigma$. For each component $L \subset \partial\Sigma$, identify a neighbourhood of $L$ with $S^1 \times \mathbb{R}^2$, where $L = S^1 \times \{0\}$. Then, with $S^1$-coordinates $\theta$ and Cartesian coordinates $(x, y)$ on $\mathbb{R}^2$, we say that the boundary of $\Sigma$ is in **standard form** if $\Sigma$ (over this neighbourhood) is given by $\{x = 0, y \leq 0\}$ and the contact structure

$$\cos n\theta \, dx - \sin n\theta \, dy = 0$$

provides a model for the above neighbourhood of $L$. Let $\Sigma'$ be a second convex surface such that $\partial\Sigma \subset \Sigma'$ (or $\partial\Sigma = \partial\Sigma'$ respectively) is Legendrian. For each component $L \subset \partial\Sigma$ we can identify a neighbourhood of $L$ with $S^1 \times \mathbb{R}^2$ as above. We say that $\Sigma$ and $\Sigma'$ are in **standard position** if $\Sigma$ is given by $\{x = 0, y \leq 0\}$ and $\Sigma'$ is given by $\{y = 0\}$ (or by $\{y = 0, x \geq 0\}$ respectively).

Giroux [28] proved that a closed oriented embedded surface $\Sigma$ can be deformed by a $C^\infty$-small isotopy so that the resulting embedded surface is convex. Honda [34] extends this result for embedded surfaces $\Sigma$ with non-empty boundary as follows.

**Lemma 1** (Honda [34])**.** *Let $\Sigma \subset M$ be a compact, oriented surface with Legendrian boundary satisfying $\mathtt{tw}(K, \Sigma) \leq 0$ for all boundary components $K \subset \partial\Sigma$. There exists a $C^0$-small perturbation near the boundary $\partial\Sigma$ that puts $\partial\Sigma$ into standard form (while fixing $\partial\Sigma$), and a $C^\infty$-small perturbation of the perturbed surface, which makes $\Sigma$ convex while fixing the standard neighbourhood of $\partial\Sigma$.* □

### 1.1.2 Flexibility of the characteristic foliation

Let $\mathfrak{F}$ be singular 1-dimensional foliation on a compact surface $\Sigma$. A collection $\Gamma$ of circles and properly embedded arcs is said to **divide** $\mathfrak{F}$ if the following conditions hold:

(i) $\Gamma$ is transverse to $\mathfrak{F}$.

(ii) There is an area form $\Omega$ on $\Sigma$ and a vector field $X$ defining $\mathfrak{F}$ such that $\mathcal{L}_X\Omega \neq 0$ on $\Sigma \setminus \Gamma$, and with $\Sigma_\pm := \{p \in \Sigma \mid \pm \operatorname{div}_\Omega(X) > 0\}$, so that $\Sigma \setminus \Gamma = \Sigma_+ \sqcup \Sigma_-$, the vector field $X$ points out of $\Sigma_+$ along $\Gamma$.

As its name suggests, the dividing set $\Gamma$ of a convex surface $\Sigma$ divides its characteristic foliation $\Sigma_\xi$. On the other hand, any surface $\Sigma$ whose characteristic foliation $\Sigma_\xi$ is divided by a collection $\Gamma$ of circles is convex (cf. [19, Theorem 4.8.5]).

The next theorem is referred to as **Flexibility Theorem** and is due to Giroux [28]. It asserts, in essence, that all the crucial information about the contact structure in a neighbourhood of a convex surface $\Sigma$ is encoded in the dividing set $\Gamma$.

**Theorem 2** (Giroux [28])**.** *Let $\Sigma$ be an oriented convex surface with (possibly empty) Legendrian boundary in a contact 3-manifold $(M, \xi)$. Let $\mathfrak{F}$ be a singular 1-dimensional foliation on $\Sigma$ divided by the dividing set $\Gamma$ of the characteristic foliation $\Sigma_\xi$, and let $X$ denote a contact vector field transverse to $\Sigma$. Then there is an isotopy $\psi_t\colon \Sigma \to M$, $t \in [0,1]$, of $\Sigma$ such that $\psi_0$ is the inclusion $\Sigma \subset M$, the characteristic foliation $\psi_1(\Sigma)_\xi$ coincides with $\psi_1(\mathfrak{F})$, and $\psi_t(\Sigma)$ is transverse to $X$ for all $t \in [0,1]$.* □

Let $M$ be a compact, oriented 3-manifold with non-empty boundary $\partial M$ which is equipped with a collection of circles $\Gamma \subset \partial M$. Let $\xi_0$ and $\xi_1$ be two contact structures on $M$ such that the characteristic foliations $\partial M_{\xi_0}$ and $\partial M_{\xi_1}$ are both divided by $\Gamma$.

**Definition 1.** We say $(M, \xi_0)$ and $(M, \xi_1)$ are contactomorphic **up to flexibility (of the characteristic foliation on the boundary)** if they are contactomorphic after a perturbation of the boundary in the sense of Theorem 2.

A properly embedded graph $G \subset \Sigma$ on a convex surface $\Sigma$ with Legendrian boundary is called **non-isolating** if $G$ and $\Gamma$ intersect transversely and every component $\Sigma \setminus G$ intersects $\Gamma$. We refer to the next theorem as **Legendrian realisation principle**.

**Theorem 3** (Honda [34])**.** *Let $G \subset \Sigma$ be a non-isolating graph on a convex surface $\Sigma \subset M$ with Legendrian boundary. Then there is an isotopy $\psi_t\colon \Sigma \to M$, $t \in [0,1]$, of $\Sigma$ such that $\psi_0$ is the inclusion $\Sigma \subset M$, the characteristic foliation $\psi_1(\Sigma)_\xi$ contains $\psi_1(G)$, and $\psi_t(\Sigma)$ is transverse to $X$ for all $t \in [0,1]$.* □

According to the preceding theorem any closed curve $C$ on a convex surface $\Sigma$ that intersects the dividing set transversely and non-trivially can be realised (in the sense of Theorem 3) as a Legendrian curve. This observation is originally due to Kanda [40] and the Legendrian realisation principle above is a generalisation of this result. Kanda actually observed the following.

**Corollary 4** (Kanda [40])**.** *Let $C$ be a closed curve on a convex surface $\Sigma$. Suppose $C$ intersects the dividing set transversely and non-trivially then $C$ can be realised (in the sense of Theorem 3) as a Legendrian curve. Moreover we can actually realise a characteristic foliation for which an annular collar neighbourhood of $C$ consists of a 1-parameter family of Legendrian ruling curves that are translates of $C$.*

Suppose $\Sigma$ is a closed surface. We call a contact structure on $\Sigma \times [-1, 1]$ **horizontally convex** if each level set $\Sigma^z = \Sigma \times \{z\}$, $z \in [-1, 1]$, is convex. The following lemma is a reformulation of the Uniqueness Lemma in [19].

**Lemma 5** (Uniqueness Lemma)**.** *Let $\Sigma$ be a compact surface with (possibly empty) boundary and $\mathfrak{F}_-, \mathfrak{F}_+$ two singular 1-dimensional foliations on $\Sigma$ which are divided by a collection $\Gamma$ of embedded circles and properly embedded arcs. Then there exists a unique (up to isotopy rel boundary) horizontally convex contact structure $\eta$ on $\Sigma \times [-1, 1]$ satisfying the boundary conditions imposed by $\mathfrak{F}_-$ and $\mathfrak{F}_+$.*

*Proof.* The existence of a horizontally convex contact structure $\eta$ on $\Sigma \times [-1, 1]$ satisfying the boundary conditions imposed by $\mathfrak{F}_-$ and $\mathfrak{F}_+$ follows from Giroux's Flexibility Theorem (cf. Theorem 2 above). The uniqueness is a consequence of the original Uniqueness Lemma (cf. [19, Lemma 4.9.2]) which states that two horizontally convex contact structures $\eta$ and $\eta'$ on $\Sigma \times [-1, 1]$ satisfying the boundary conditions imposed by $\mathfrak{F}_-$ and $\mathfrak{F}_+$ are in fact contact isotopic relative to the boundary. □

### 1.1.3 Gluing and decomposing along convex surfaces

#### Edge rounding

Given two convex surfaces $\Sigma$ and $\Sigma'$ such that $\partial\Sigma = \partial\Sigma'$ is Legendrian and such that they are in standard position (in the sense of Subsection 1.1.1). Then according to Honda [34] we can form a new convex surface $\Sigma''$ by **(convexly) rounding the edges** of $\Sigma \cup \Sigma'$ as follows: for each component $L \subset \partial\Sigma$ we choose a standard neighbourhood as in Subsection 1.1.1 above, a suitable $\delta > 0$, and replace $(\Sigma \cup \Sigma') \cap \{x^2 + y^2 \leq \delta^2\}$ by $\{(x-\delta)^2 + (y+\delta)^2 = \delta^2\} \cap \{y \leq 0, x \geq 0\}$. The result $\Sigma''$ is a convex surface whose dividing set matches the original one away from $(\Sigma \cup \Sigma') \cap \{x^2 + y^2 \leq \delta^2\}$ and over $\{(x-\delta)^2 + (y+\delta)^2 = \delta^2\} \cap \{y \leq 0, x \geq 0\}$ connects up as indicated in Figure 1.1.

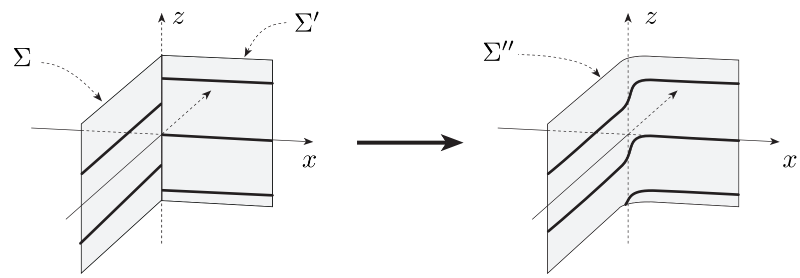


Figure 1.1: Rounding a corner between two convex surfaces.

### Gluing

Let $j_0, j_1 \colon \Sigma \hookrightarrow \partial M$ be two embeddings of a compact surface $\Sigma$ into the convex boundary of a contact 3-manifold $(M, \xi)$. Denote by $\Sigma_0$ and $\Sigma_1$ the embedded copies of $\Sigma$ corresponding to the embeddings $j_0$ and $j_1$ respectively. Suppose that the boundaries of $\Sigma_0$ and $\Sigma_1$ are Legendrian and in standard form. Let $\Gamma_0 \subset \Sigma_0$ and $\Gamma_1 \subset \Sigma_1$ denote the dividing sets induced by the dividing set on $\partial M$ and suppose that $j_0 \circ (j_1|_{\Sigma_1})^{-1}$ maps $\Gamma_1$ to $-\Gamma_0$, where the dividing curves are oriented as boundaries of the regions of positive divergence. Hence $(-\Sigma_0, -\Gamma_0)$ and $(\Sigma_1, \Gamma_1)$ are isomorphic as convex surfaces. Finally assume that $\Sigma_0$ and $\Sigma_1$ are disjoint from each other. By the Uniqueness Lemma there is a unique contact structure $\eta$ on $[0,1] \times \Sigma$ satisfying the boundary conditions induced by $(\Sigma_0)_\xi$ and $(\Sigma_1)_\xi$ and such that for each $t \in [0,1]$ the level set $\{t\} \times \Sigma$ is convex. We may now consider the contact manifold

$$(M', \xi') = (M, \xi) \cup_{\Sigma_0 \sqcup \Sigma_1} \big([0,1] \times \Sigma, \eta\big).$$

Observe that its boundary decomposes as $\big(\partial M \setminus (\Sigma_0 \sqcup \Sigma_1)\big) \cup \big([0,1] \times \partial\Sigma\big)$ and both parts are in standard position. Hence we can round the edges of $(M', \xi')$ and refer to the result as obtained by **gluing** $(M, \xi)$ along $\Sigma_0$ and $\Sigma_1$. As always, depending on the situation we will sometimes think of $(M', \xi')$ as manifold with edges, i.e. before we applied edge rounding. It will be clear from the situation which version of $(M', \xi')$ is considered.

### Decomposing

Let $(M, \xi)$ be a contact 3-manifold with (possibly empty) convex boundary $\partial M$ and let $\Sigma \subset (M, \xi)$ be a properly embedded, oriented convex surface with (possibly empty) boundary $\partial\Sigma$. Suppose $\Sigma$ and the convex boundary of $M$ are in standard position. Furthermore let $[-\varepsilon, \varepsilon] \times \Sigma$ denote a vertically invariant neighbourhood of $\Sigma$. We will refer to the new contact manifold

$$M(\Sigma) := (M, \xi) \setminus \big((-\varepsilon, \varepsilon) \times \Sigma\big)$$

(after convex edge rounding) as obtained by **decomposing** $M$ **along** $\Sigma$. Depending on the situation we will sometimes think of $M(\Sigma)$ as manifold with edges i.e. before we applied edge rounding. It will be clear from the situation which version of $M(\Sigma)$ is considered.

Observe that if we decompose a contact manifold along a convex surface $\Sigma$ as described above and reglue the resulting copies $\{\pm\varepsilon\} \times \Sigma$ we end up with the original contact manifold.

## 1.2 Contact surgery and symplectic cobordisms

### 1.2.1 Contact connected sum

Let $B_{st}$ denote a standard tight 3-ball, i.e. a 3-ball equipped with a tight contact structure and convex boundary. Let $(M, \xi)$ be a compact, oriented, not necessarily connected, contact 3-manifold. Suppose we are given a contact embedding $S^0 \times B_{st} \hookrightarrow (M, \xi)$, then one defines the **contact connected sum** $\#(M, \xi)$ as

$$\#(M, \xi) := \big((M, \xi) \setminus \operatorname{Int}(S^0 \times B_{st})\big) \cup_{S^0 \times S^2} (D^1 \times S^2, \eta),$$

where $\eta$ is the unique $D^1$-invariant contact structure – induced by the Uniqueness Lemma (cf. Lemma 5) – satisfying the relevant boundary conditions .

The contact connected sum does not depend on the choice of the embedding of $S^0 \times B_{st}$, since by the contact disc theorem (cf. [19, Theorem 2.6.7]) any two embeddings of a standard 3-ball are contact isotopic. The contact connected sum corresponds to the symplectic 1-handle attachment described by Eliashberg [9] and Weinstein [48], cf. also Section 1.2.3 below.

### 1.2.2 Contact $(\pm 1)$-surgery

Let $L \subset (M, \xi)$ be a Legendrian knot sitting in some contact 3-manifold. Identify a neighbourhood of $L$ with $S^1 \times \mathbb{R}^2$, where $L = S^1 \times \{0\}$. Then, with $S^1$-coordinate $\theta$ and Cartesian coordinates $(x, y)$ on $\mathbb{R}^2$, the contact structure

$$\cos\theta \, dx - \sin\theta \, dy = 0$$

provides a model for a neighbourhood of $L$. This is a consequence of a neighbourhood theorem for isotropic submanifolds (cf. [19, Theorem 2.5.8]). For $\varepsilon > 0$ we will refer to $\{x^2 + y^2 \leq \varepsilon\}$ as a **standard neighbourhood of** $L$. Since the radial vector field $x\,\partial_x + y\,\partial_y$ is a contact vector field, $\{x^2 + y^2 \leq \varepsilon\}$ has a convex boundary. The dividing set is given by two closed curves corresponding to the transverse push offs of $L$, namely $(\theta, x = \pm\varepsilon \sin\theta, y = \pm\varepsilon\cos\theta)$, $\theta \in S^1$.

Let $N \subset (M, \xi)$ denote a standard neighbourhood of $L$. Cut out the neighbourhood $N$ and glue back a solid torus $S^1 \times D^2$ by sending its meridian to $\lambda \pm \mu$, where $\mu$ is the meridian of $\partial N$, and $\lambda$ corresponds to the contact framing (the direction of the dividing curves on the boundary $\partial N$). By results of Giroux [29] and Honda [34] there is a unique contact structure on $S^1 \times D^2$ satisfying the relevant boundary conditions. Therefore the contact structure on $\xi|_{M \setminus N}$ uniquely extends to a contact structure $\xi_{\pm 1}(L)$ on the result $M_{\pm 1}(L)$ of $(\pm 1)$-surgery on $M$ along $L$.

**Definition 2.** The contact manifold $\big(M_{\pm 1}(L), \xi_{\pm 1}(L)\big)$, described above, is said to be obtained by **contact** $(\pm 1)$**-surgery on** $L$.

Contact $(-1)$-surgery corresponds to the symplectic 2-handle attachments described by Eliashberg [9] and Weinstein [48], cf. also Section 1.2.3 below. The following lemma, due to Ding–Geiges, shows that it is possible to cancel a contact $(-1)$-surgery by a certain contact $(+1)$-surgery and vice versa.

**Lemma 6** (Cancellation Lemma, [19, Proposition 6.4.5])**.** *Let $(M', \xi')$ be the contact manifold obtained from $(M, \xi)$ by contact $(-1)$-surgery along a Legendrian knot $L$ and contact $(+1)$-surgery along a Legendrian push-off $L'$ of $L$. Then $(M', \xi')$ is contactomorphic to $(M, \xi)$.* □

### 1.2.3 Symplectic cobordisms and symplectic handles

Suppose we are given a symplectic $2n$-manifold $(X, \omega)$, oriented by the volume form $\omega^n$, such that the oriented boundary $\partial X$ decomposes as $\partial X = (-M_-) \sqcup M_+$, where $-M_-$ stands for $M_-$ with reversed orientation. Suppose further that in a neighbourhood of $\partial X$ there is a Liouville vector field $Y$ for $\omega$, transverse to the boundary and pointing outwards along $M_+$, inwards along $M_-$. The 1-form $\alpha = i_Y \omega$ restricts to $TM_\pm$ as a contact form defining cooriented contact structures $\xi_\pm$.

**Definition 3.** We will call $(X, \omega)$ a **(strong) symplectic cobordism** from $(M_-, \xi_-)$ to $(M_+, \xi_+)$, with **convex** boundary $M_+$ and **concave** boundary $M_-$. In case $(M_-, \xi_-)$ is empty $(X, \omega)$ is called a **(strong) symplectic filling** of $(M_+, \xi_+)$. If the Liouville vector field is defined not only in a neighbourhood of $\partial X$ but everywhere on $X$ we call the cobordism or the filling respectively **exact**.

A **Stein manifold** is an affine complex manifold, i.e. a complex manifold that admits a proper holomorphic embedding into $\mathbb{C}^N$ for some large integer $N$. By work of Grauert [32] a complex manifold $(X, J)$ is Stein if and only if it admits an exhausting plurisubharmonic function $\rho\colon X \to \mathbb{R}$. Eliashberg and Gromov's symplectic counterparts of Stein manifolds are *Weinstein manifolds.* A **Weinstein manifold** is a quadruple $(X, \omega, Z, \varphi)$, see [10], where $(X, \omega)$ is an exact symplectic manifold, $Z$ is a complete globally defined Liouville vector field, and $\varphi\colon X \to \mathbb{R}$ is an exhausting (i.e. proper and bounded below) Morse function for which $Z$ is gradient-like.

**Definition 4.** Suppose $(X, \omega)$ is an exact symplectic cobordism with boundary $\partial X = (-M_-) \sqcup M_+$ and with Liouville vector field $Z$. We call $(X, \omega)$ **Weinstein cobordism** if there exists a Morse function $\varphi\colon X \to \mathbb{R}$ which is constant on $M_-$ and on $M_+$, has no boundary critical points, and for which $Z$ is gradient-like.

Eliashberg [9] and Weinstein [48] show how to add symplectic $k$-handles to a convex boundary component $(M_+, \xi_+)$ of a symplectic $2n$-manifold $(X, \omega)$ provided $k \leq n$. In the following we will briefly sketch the construction of a **symplectic model $k$-handle** closely following the description in [19]: on $\mathbb{R}^{2n} = \mathbb{R}^k \times \mathbb{R}^{2n-k}$ with coordinates $(q_1, \dots, q_k)$ on the first factor, and with coordinates $(q_{k+1}, \dots, q_n, p_1, \dots, p_n)$ on the second factor, we have the standard symplectic form

$$\omega_0 = \sum_{j=1}^{n} dp_j \wedge dq_j.$$

A Liouville vector field $Y$ for $\omega_0$ is given by

$$Y := \sum_{j=1}^{k} (-q_j\, \partial_{q_j} + 2p_j\, \partial_{p_j}) + \frac{1}{2} \sum_{j=k+1}^{n} (q_j\, \partial_{q_j} + p_j\, \partial_{p_j}).$$

Notice that $Y$ is the gradient vector field, with respect to the standard Euclidian metric on $\mathbb{R}^{2n}$, of the function

$$g\colon\ (\boldsymbol{q}, \boldsymbol{p}) \mapsto \sum_{j=1}^{k} (-\frac{1}{2} q_j^2 + p_j^2) + \frac{1}{4} \sum_{j=k+1}^{n} (q_j^2 + p_j^2).$$

Suppose $S \subset (M_+, \xi_+)$ is an isotropic $(k-1)$-sphere with trivial conformal symplectic normal bundle. According to a neighbourhood theorem for isotropic submanifolds we can identify a neighbourhood of $S$ with an open neighbourhood $N_H \cong S^{k-1} \times \mathrm{Int}(D^{2n-k})$ in the hypersurface $g^{-1}(-1) \subset \mathbb{R}^{2n}$ of the $(k-1)$-sphere

$$S_H^{k-1} := \{\sum_{j=1}^{k} q_j^2 = 2, q_{k+1} = \ldots = q_n = p_1 = \ldots = p_n = 0\}.$$

The neighbourhood $N_H$ corresponds to the lower boundary of the **symplectic handle** $H$ defined as the locus of points $(\boldsymbol{q}, \boldsymbol{p}) \in (\mathbb{R}^{2n}, \omega_0)$ satisfying the inequality

$$-1 \leq g(\boldsymbol{q}, \boldsymbol{p}) \leq 1$$

and lying on a gradient flow line of $g$ through a point on $N_H$. Since the Liouville vector field $Y$ is transverse to the level sets of $g$ it induces a contact form on the upper boundary of $H$. For a detailed instruction how to attach such handles see [19].

**Remark.** The exact cobordism corresponding to the attachment of the symplectic model handle described above is actually a Weinstein cobordism.

## 1.3 Open books

Given a topological space $\Sigma$ and a homeomorphism $\phi\colon\ \Sigma \to \Sigma$, the **mapping torus** $\Sigma(\phi)$ is the quotient space obtained from $\Sigma \times [0, 1]$ by identifying $(x, 1)$

with $(\phi(x), 0)$ for each $x \in \Sigma$. Suppose $\Sigma$ is a smooth manifold with non-empty boundary $\partial\Sigma$ and $\phi$ a diffeomorphism equal to the identity near the boundary, then $\Sigma(\phi)$ is in a natural way a smooth manifold with boundary $\partial\Sigma \times S^1$, where we identify $S^1 \equiv \mathbb{R}/\mathbb{Z}$. The pair $(\Sigma, \phi)$ determines a closed manifold $M_{(\Sigma,\phi)}$ defined by

$$M_{(\Sigma,\phi)} := \Sigma(\phi) \cup_{\mathrm{id}} (\partial\Sigma \times D^2), \tag{1.2}$$

where we identify $\partial\Sigma(\phi) = \partial\Sigma \times S^1$ with $\partial(\partial\Sigma \times D^2)$ using the identity map. The pair $(\Sigma, \phi)$ is called an **abstract open book**. The hypersurface $\Sigma$ is referred to as the **page** of the open book and the diffeomorphism $\phi$ is referred to as the **monodromy** of the open book.

According to Alexander [1] every closed oriented 3-manifold $M$ there is an abstract open book $(\Sigma, \phi)$ such that $M$ is diffeomorphic to $M_{(\Sigma,\phi)}$. Write $B \subset M$ for the embedded link (or codimension 2 submanifold in general) $\partial\Sigma \times \{0\}$ under this diffeomorphism. Then we can define a fibration $\pi\colon\ M \setminus B \to S^1$ by

$$\left.\begin{array}{l} [x, \varphi] \\ [\theta, r\mathrm{e}^{i\pi\varphi}] \end{array}\right\} \mapsto [\varphi],$$

where we understand $M \setminus B$ as decomposed in (1.2) and $[x, \varphi] \in \Sigma(\phi)$ or $[\theta, r\mathrm{e}^{i\pi\varphi}] \in \partial\Sigma \times D^2 \subset \partial\Sigma \times \mathbb{C}$ respectively. This gives rise to the following definition.

**Definition 5.** An **open book decomposition** of an $n$-dimensional manifold $M$ is a pair $(B, \pi)$, where $B$ is a codimension 2 submanifold in $M$, called the **binding** of the open book and $\pi\colon\ M \setminus B \to S^1$ is a (smooth, locally trivial) fibration such that each fibre $\pi^{-1}(\varphi)$, $\varphi \in S^1$, corresponds to the interior of a compact hypersurface $\Sigma_\varphi \subset M$ with $\partial\Sigma_\varphi = B$. The hypersurfaces $\Sigma = \Sigma_\varphi$, $\varphi \in S^1$, are called the **pages** of the open book.

As we saw above an abstract open book $(\Sigma, \phi)$ naturally gives rise to an open book decomposition of the corresponding manifold $M_{(\Sigma,\phi)}$. On the other hand, an open book decomposition $(B, \pi)$ of some $n$-manifold $M$ defines an abstract open book as follows: identify a neighbourhood of $B$ with $B \times D^2$ such that $B = B \times \{0\}$ and such that the fibration on this neighbourhood is given by the angular coordinate, $\varphi$ say, on the $D^2$-factor. We can define a 1-form $\alpha$ on the complement $M \setminus (B \times D^2)$ by pulling back $d\varphi$ under the fibration $\pi$, where this time we understand $\varphi$ as the coordinate on the target space of $\pi$. The vector field $\partial\varphi$ on $\partial\big(M \setminus (B \times D^2)\big)$ extends to a nowhere vanishing vector field $X$ which we normalise by demanding it to satisfy $\alpha(X) = 1$. Let $\phi$ denote the time-1 map of the flow of $X$. Then the pair $(\Sigma, \phi)$, with $\Sigma = \overline{(\pi|_{M\setminus(B\times D^2)})^{-1}(0)}$, defines an abstract open book such that $M_{(\Sigma,\phi)}$ is diffeomorphic to $M$.

**Example 1.** Understand $S^3$ as the unit sphere in $\mathbb{C}^2$, i.e. as the subset of $\mathbb{C}^2$ given by

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2\colon\ |z_1|^2 + |z_2|^2 = 1\}.$$

We give three examples of open book decompositions of $S^3$:

(1) Set $B = \{(z_1, z_2) \in S^3\colon\ z_1 = 0\}$. Note that $B$ is an unknotted circle in $S^3$. Consider the fibration

$$\pi\colon\ S^3 \setminus B \to S^1 \subset \mathbb{C},\ (z_1, z_2) \mapsto \frac{z_1}{|z_1|}.$$

In polar coordinates this map is given by $(r_1 \mathrm{e}^{i\varphi_1}, r_2 \mathrm{e}^{i\varphi_2}) \mapsto \varphi_1$. Observe that $(B, \pi)$ defines an open book decomposition of $S^3$ with pages diffeomorphic to $D^2$ and trivial monodromy.

(2) Set $B_+ = \{(z_1, z_2) \in S^3\colon\ z_1 z_2 = 0\}$. Observe that $B_+$ describes the positive Hopf link. Consider the fibration

$$\pi_+\colon\ S^3 \setminus B_+ \to S^1 \subset \mathbb{C},\ (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}.$$

One can show that $(\pi_+, B_+)$ defines an open book decomposition of $S^3$ with annular pages and monodromy given by a left-handed Dehn twist along the core of the annulus.

(3) Set $B_- = \{(z_1, z_2) \in S^3\colon\ z_1 \overline{z_2} = 0\}$. Observe that $B_-$ describes the negative Hopf link. Consider the fibration

$$\pi_-\colon\ S^3 \setminus B_- \to S^1 \subset \mathbb{C},\ (z_1, z_2) \mapsto \frac{z_1 \overline{z_2}}{|z_1 \overline{z_2}|}.$$

One can show that $(\pi_+, B_+)$ defines an open book decomposition of $S^3$ with annular pages and monodromy given by a right-handed Dehn twist along the core of the annulus.

Let $\Sigma$ be a compact surface with non-empty boundary and $\phi\colon\ \Sigma \to \Sigma$ a diffeomorphism equal to the identity near $\partial\Sigma$. Suppose further we are given a properly embedded arc $a \subset \Sigma$. The **positive (negative) stabilisation** of the abstract open book $(\Sigma, \phi)$ is the abstract open book obtained by adding a 1-handle to the original page $\Sigma$ along the endpoints of $a$, and changing the monodromy by composing it with a right- (left-) handed Dehn twist along the simple closed curve obtained by the union of $a$ and the core of the 1-handle. The open books described in parts (2) and (3) of the preceding example are instances of a positive and negative stabilisation respectively of the open book described in the first part.

A positive and negative stabilisation respectively can be understood as a suitable connected sum with the open book described in part (2) or part (3) of the preceding example. This viewpoint draws more attention when we dive into the interplay of open books and contact structures, cf. Subsection 1.3.1 and 1.3.2 below. To be more precise: a positive stabilisation of an open book does not change the underlying contact structure, whereas a negative stabilisation turns it into an overtwisted one.

### 1.3.1 From open books to contact structures

Let $\Sigma$ be a compact surface with non-empty boundary and $\phi\colon\ \Sigma \to \Sigma$ be a diffeomorphism equal to the identity near $\partial\Sigma$. In [47] Thurston and Winkelnkemper describe the construction of a contact structure $\xi_{(\Sigma,\phi)}$ on the 3-manifold $M_{(\Sigma,\phi)}$ corresponding to the open book $(\Sigma, \phi)$. We briefly sketch this construction (for further details see [19] or [11]): let $\beta$ be a 1-form on $\Sigma$ satisfying

(i) $\beta = e^r\, d\theta$ on $[-\varepsilon, 0] \times \partial\Sigma \subset \Sigma$, and

(ii) $d\beta$ is a volume form on $\Sigma$.

One can show (cf. [19]) that the set of 1-forms on $\Sigma$ satisfying these properties is non-empty and convex. The 1-form on $\Sigma \times [0, 1]$ defined by

$$\varphi\, \beta + (1 - \varphi)\, \phi^* \beta$$

descends to a 1-form on the mapping torus $\Sigma(\phi)$. We continue to denote this 1-form by $\beta$. Note that the 1-form induced by $\beta$ on each fibre of $\Sigma(\phi)$ over any $\varphi \in S^1$ satisfies the properties (i) and (ii) given above. For some sufficiently large $C > 0$ the 1-form

$$\alpha = \beta + C\, d\varphi$$

defines a contact form on $\Sigma(\phi)$ (cf. [11]). It remains to extend this contact form over $\partial\Sigma \times D^2$. Write $M_{(\Sigma,\phi)}$ as the quotient space

$$\big(\Sigma(\phi) \cup (\partial\Sigma \times D^2_{1+\varepsilon})\big)/\sim,$$

where we identify $(r, \theta, \varphi) \in [-\varepsilon, 0] \times \partial\Sigma \times S^1$ with $(\theta, (1-r)\mathrm{e}^{i\varphi}) \in \partial\Sigma \times D^2_{1+\varepsilon} \subset \partial\Sigma \times \mathbb{C}$. Mimicking the Lutz twist, we make the ansatz $\alpha = h_1(r)\, d\theta + h_2(r)\, d\varphi$ on $\partial\Sigma \times D^2_{1+\varepsilon}$. The boundary conditions in the present situation may be taken to be

(i) $h_1(r) = 2 - r^2$ and $h_2(r) = r^2$ near $r = 0$,

(ii) $h_1(r) = e^{1-r}$ and $h_2(r) = C$ for $1 \le r \le 1 + \varepsilon$.

The contact condition $\alpha \wedge d\alpha \neq 0$ translates into a third condition

(iii) $\big(h_1(r), h_2(r)\big)$ is never parallel to $\big(h_1'(r), h_2'(r)\big)$ for $r \neq 0$.

It can be shown that such functions $h_1$ and $h_2$ respectively (satisfying properties (i),(ii) and (iii)) indeed exist (cf. [19]). The contact structure $\xi = \ker\alpha$ and the open book decomposition corresponding to $(\Sigma, \phi)$ are compatible in the following sense.

**Definition 6.** A positive contact structure $\xi = \ker\alpha$ is said to be **supported** by the open book decomposition $(B, \pi)$ of $M$, or $(B, \pi)$ is said to be **compatible** with $\xi$ respectively, if the 2-form $d\alpha$ induces a symplectic form on each page, defining its positive orientation, and the 1-form $\alpha$ induces a positive contact form on $B$.

In dimension equal to 3 it can be shown that any two contact structures supported by the same open book decomposition are in fact contact isotopic (cf. [11]). It can be shown that the open books described in parts (1) and (2) of Example 1 above support the standard contact structure $\xi_{st}$ on $S^3$, whereas part (3) supports the overtwisted contact structure $\xi_1$ which is obtained by a Lutz twist along a transverse unknot $U \subset (S^3, \xi_{st})$ with self-linking number $-1$.

Let $\xi$ be a contact structure on a 3-manifold which is supported by an open book decomposition $(B, \pi)$. Then $(B, \pi)$ induces a Heegaard decomposition of $M$ given by

$$M = \overline{\pi^{-1}\big([0, \tfrac{1}{2}]\big)} \cup_{\Sigma'} \overline{\pi^{-1}\big([\tfrac{1}{2}, 1]\big)}, \tag{1.3}$$

where $\Sigma' = \overline{\pi^{-1}(0)} \cup_B \overline{\pi^{-1}(\frac{1}{2})}$ is the union of two opposite pages. It is not hard to show (cf. [19, Example 4.8.4(4)]) that $\Sigma'$ is a convex surface with dividing set $B$. Note that $\overline{\pi^{-1}([0, \frac{1}{2}])}$ is homeomorphic to the quotient space of $\Sigma \times [0, \frac{1}{2}]$ by identifying $(x, t)$ with $(x, t')$ for all $x \in \partial\Sigma$ and $t, t' \in [0, \frac{1}{2}]$. In particular $\overline{\pi^{-1}([0, \frac{1}{2}])}$ is indeed a 1-handle body. It can be shown that the 1-handle body $\overline{\pi^{-1}([0, \frac{1}{2}])}$ is actually *standard* in the sense of Definition 7 below. The analogue statement holds for $\overline{\pi^{-1}\big([\frac{1}{2}, 1]\big)}$.

### 1.3.2 From contact structures to open books

There is a remarkably deep correlation between contact structures and open books. A theorem by Giroux and Mohsen states that any contact structure $\xi$ on a closed manifold $M$ of dimension at least equal to 3 admits a compatible open book $(B, \pi)$. Unfortunately, complete details of its proof, except for the 3-dimensional case, have not yet appeared, but see [25]. In this subsection we sketch the proof of the above statement (cf. Theorem 9) and the methods involved for the 3-dimensional case.

We will refer to a 3-ball equipped with a tight contact structure and convex boundary as a (3-dimensional) **standard contact 0-handle** and **standard contact 3-handle** respectively. According to the Uniqueness Lemma we can equip $D^1 \times D^2$ with a unique contact structure $\eta$ such that each level $\{z\} \times D^2$, $z \in [-1, 1]$, is a standard 2-disc, i.e a convex 2-disc whose dividing set is given by a single properly embedded arc. We refer to $(D^1 \times D^2, \eta)$ as a (3-dimensional) **standard contact 1-handle** and to its dual handle $(D^2 \times D^1, \eta)$ as a (3-dimensional) **standard contact 2-handle**.

**Definition 7.** Let $B^3_{st}$ be a standard contact 0-handle and let $H$ denote a solid handle body obtained by adding standard 1-handles to $B^3_{st}$. We call $H$ a **standard 1-handle body**.

Standard solid handle bodies play an important role in the correlation of contact structures and open books in dimension 3 as we will see in Proposition 8. Using analogue arguments as in the prime decomposition of tight contact manifolds (cf. [3]) one shows that standard 1-handle bodies are tight. Actually each standard 1-handle body can be embedded into $S^3$ equipped with its standard contact structure $\xi_{st}$. It turns out that standard 1-handle bodies carry a unique tight contact structure as the following lemma shows.

**Lemma 7.** *Let $H$ be a standard solid handle body with tight contact structure $\xi_0$. Suppose $\xi_1$ is another tight contact structure which induces the same characteristic foliation on $\partial H$. Then $\xi_0$ and $\xi_1$ are contact isotopic rel boundary.*

*Proof.* Let $g \in \mathbb{N}$ be the genus of $H$ and let $D_1, \ldots, D_g \subset H$ denote the meridional disc corresponding to the co-cores of the attached standard 1-handles. Perturb each disc $D_j$, $j = 1, \ldots, g$, with respect to the contact structure $\xi_i$, $i = 0, 1$, into a convex disc $D_j^i$. Note that we may assume that $\xi_0$ and $\xi_1$ already agree in a neighbourhood of the boundary of $H$. There is an isotopy $\psi_t : H \to H$, $t \in [0, 1]$, which fixes the boundary pointwise such that $\psi_1(D_j^0) = D_j^1$ and $\psi_1\big((D_j^0)_{\xi_0}\big) = (D_j^1)_{\xi_1}$. Therefore $\xi_t' = T\psi_t(\xi_0)$ defines an isotopy of contact structures such that $\xi_1$ and $\xi_1'$ agree over each 2-disc $D_j^1$, $j = 1, \ldots, g$ and over a neighbourhood of $\partial H$. Observe that $\xi_1$ and $\xi_1'$ define two tight contact structures on a 3-ball $B$ which is given by the complement of the 2-discs $D_j^1$, $j = 1, \ldots, g$ in $H$. In particular the characteristic foliations on the boundary $B$ induced by $\xi_1$ and $\xi_1'$ agree. By work of Eliashberg [5] it follows that these two contact structures are isotopic relative to $\partial B$. □

Observe that the convex boundary $\partial H$ of standard solid handle body $H$ of genus $g$ decomposes as

$$\partial H = (-\Sigma_-) \cup_\Gamma \Sigma_+, \tag{1.4}$$

where $\Gamma \subset \partial H$ is the dividing set and $\Sigma = \Sigma_\pm$ is the compact surface with boundary $\Gamma$ and of Euler characteristic $1 - g$. Let $G \subset \Sigma$ be the 1-skeleton of $\Sigma$ and note that $\Sigma \setminus G \cong (0, 1] \times \partial\Sigma$ is a collection of annuli with coordinates $(r, \theta)$. The singular 1-dimensional foliation generated by the vector field $Y = r^2\,\partial_r$ is divided by $\Gamma$ and can therefore be realised on $\Sigma$ after a $C^\infty$-small perturbation. There exists a neighbourhood $\Sigma \times \mathbb{R}$ of $\Sigma_+$ (which we understand as slightly pushed into the interior of $H$), with $\Sigma = \Sigma \times \{0\}$, such that the contact structure is given by $\beta + dt = 0$, where $t$ denotes the $\mathbb{R}$-coordinate and $\beta = \alpha|_{T\Sigma}$. Observe that $X = Y + t\,\partial_t$ is a contact vector field on $\Sigma \times \mathbb{R}$ which is transverse to the set

$$\{r^2 + t^2 \leq 1\} \subset \Sigma \times \mathbb{R}.$$

Note that the negative flow of the contact vector field $X$ defines a retraction of $\{r^2 + t^2 \leq 1\}$ onto $G$. We call this the **standard neighbourhood of** $G$. Furthermore $\{r^2 + t^2 \leq 1\}$ defines a standard 1-handle body which is contactomorphic (up to flexibility on the boundary) to $H$. We sometimes refer to $G$ as the **Legendrian core of** $H$.

**Remark.** Actually the contact vector field $X$, which is only defined over $\{r^2 + t^2 \leq 1\} \subset H$, can be extended over the rest of $H$ such that the negative flow induces a contact retraction of $H$ onto $G$. Hence all the information about $H$ is encoded in its Legendrian core $G$.

With this model 1-handle body in hand it can be shown that $H \setminus \Gamma$ admits a (smooth, locally trivial) fibration $\pi\colon\, \underline{H \setminus B} \to [-1,1]$ such that with respect to the above equation (1.4) we have $\pi^{-1}(\pm 1) = \pm\Sigma$. Moreover we can assume that the 2-form $d\alpha$ induces a symplectic form on each fibre $\pi^{-1}(\theta)$, $\theta \in [-1,1]$. We may summarise the above discussion by saying:

*"Standard 1-handle bodies are halves of open books."*

This (together with the discussion at the end of the preceding subsection) leads to the following.

**Proposition 8.** *A contact 3-manifold $(M,\xi)$ admits a compatible open book if and only if it admits a handle decomposition all of whose handles are standard contact handles.* □

We are now ready to prove the existence of a compatible open book for a given contact structure $\xi$ on a closed oriented 3-manifold $M$.

**Theorem 9** (Giroux [25])**.** *Any contact structure $\xi$ on a closed 3-manifold $M$ admits a compatible open book $(B,\pi)$.*

*Proof.* Let $\xi$ be a contact structure on a closed oriented 3-manifold $M$. According to Proposition 8 we have to define a handle decomposition of $M$ all of whose handles are standard contact handles.

Start with an arbitrary finite CW-decomposition of $M$. We can assume that the 1-skeleton is actually Legendrian and all the 3-cells are contained in Darboux balls (this can be achieved by refining the cell decomposition). Hence the 1-skeleton defines a Legendrian graph $G$. Each Legendrian graph $G$ admits a neighbourhood $H$ which defines a standard 1-handle body. We want to understand the 2-cells as attached to the boundary of $H$ (not to $G$ itself). Then after a perturbation of the attaching circles we can assume that the boundary of each 2-cell intersects the dividing set of $H$ nontrivially and transversely. So, after applying the Legendrian realisation principle, we may assume the attaching circles to be Legendrian. Suppose that all the 2-cells are convex. We can understand each 2-cell as the core of a 2-handle which is not necessarily standard yet. Let $D$ be one of those non-standard 2-cells. The dividing set $\Gamma \subset D$ is a collection of properly embedded arcs (since each 3-cell is contained in a Darboux ball there are no closed curves). There exists a properly embedded graph $X \subset D$ such that $X$ meets $\Gamma$ transversely and such that $D \setminus X$ defines a collection of 2-discs with dividing set a single properly embedded arc. We can now add standard 1-handles to $H$ along the components of $X$ and end up with a new standard 1-handle body $H'$. Suppose we performed this procedure for each non-standard 2-cell. Then the result is a standard handle decomposition as desired and we are done. □

**Example 2.** Suppose $\Sigma$ is a compact surface with a single boundary component $\Gamma = \partial\Sigma$. Let $\Sigma'$ be the closed surface obtained by gluing a copy of $\Sigma$ to its mirror along $\Gamma$, i.e we have

$$\Sigma' = \Sigma \cup_\Gamma \overline{\Sigma}.$$

By the Uniqueness Lemma there is a unique contact structure $\xi$ on $\Sigma' \times S^1$ such that each level set $\Sigma' \times \{\theta\}$, $\theta \in S^1$, is a convex surface with dividing curve $\Gamma \times \{\theta\}$. We will now construct a standard handle decomposition of $(\Sigma' \times S^1, \xi)$.

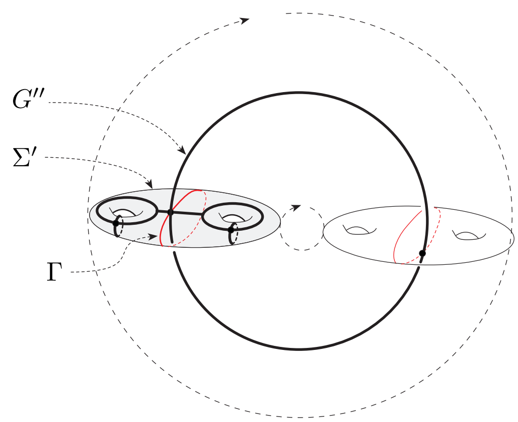


Figure 1.2: Schematic picture of the 1-skeleton $G'' \subset \Sigma' \times S^1$.

Let $G \subset \Sigma$ be the 1-skeleton of $\Sigma$ and $\overline{G} \subset \overline{\Sigma}$ its mirror in $\overline{\Sigma}$. Choose an arc $a \subset \Sigma'$ connecting $G$ and $\overline{G}$ which intersects the dividing set $\Gamma$ transversely and exactly once. Hence the union $G \cup a \cup \overline{G}$ defines a new graph $G' \subset \Sigma'$ which intersects the dividing set transversely and exactly once. Therefore the Legendrian realisation principle applies and we can actually assume $G'$ to be Legendrian. Let $x$ denote the unique intersection point of $G'$ with the dividing set and note that $\{x\} \times S^1 \subset \Sigma' \times S^1$ is Legendrian as well. Let $G'' \subset \Sigma' \times S^1$ be the Legendrian graph defined by the union of $G' \times \{0\}$ and $\{x\} \times S^1$ (cf. Figure 1.2) and let $H_0$ denote its standard neighbourhood. We will now show that the complement $(\Sigma' \times S^1, \xi) \setminus H_0$ of $H_0$ in $(\Sigma' \times S^1, \xi)$ also defines a standard 1-handle body: let $N(G') \subset \Sigma'$ be a neighbourhood of $G'$ in $\Sigma'$. Note that we can choose $N(G')$ such that its complement $D = \Sigma' \setminus N(G')$ defines a standard 2-disc. Let $D_x \subset N(G')$ denote a standard disc neighbourhood of the intersection point $x$. Observe that we have

$$H_0 \cong \big((\Sigma' \setminus D) \times [0, \pi]\big) \cup \big(D_x \times [\pi, 2\pi]\big).$$

Observe further that (since $(\Sigma' \setminus D) \cong (\Sigma' \setminus D_x)$ as convex surfaces) $H_0$ is also contactomorphic to

$$H_1 \cong \big((\Sigma' \setminus D_x) \times [\pi, 2\pi]\big) \cup \big(D \times [0, \pi]\big).$$

We end up with a standard handle decomposition $(\Sigma' \times S^1, \xi) = H_0 \cup H_1$.

**Remark.** The contact manifold $(\Sigma' \times S^1, \xi)$ described in the preceding example can be written as the zero-framed binding sum $(\Sigma, \mathrm{id}) \boxplus (\Sigma, \mathrm{id})$ of two copies of the open book $(\Sigma, \mathrm{id})$. The open book for $(\Sigma, \mathrm{id}) \boxplus (\Sigma, \mathrm{id})$ constructed in Chapter 3 agrees with the one constructed in the example above.

Of course the construction of compatible open books presented in the proof of Theorem 9 is far from being unique. However the correlation between contact structures and open books discovered by Giroux goes even deeper. As Giroux [25] discovered, there is in fact a one-to-one correspondence, which is referred to as **Giroux correspondence**, between contact structures and open books up to positive stabilisations. Unfortunately, complete details of the proof have not yet appeared.

From the construction of compatible open books given in the proof of Theorem 9 we almost immediately can deduce the following (cf. [11] for the first part and [15] for the second part).

**Proposition 10.** *(i) Suppose $L$ is a Legendrian knot (or link) in $(M, \xi)$ then there is an open book decomposition supporting $\xi$ such that $L$ sits on a page and the framing given by the page and by $\xi$ agree.*

*(ii) Suppose $K$ is a transverse knot (or link) in $(M, \xi)$ then there is an open book decomposition supporting $\xi$ such that $K$ is part of the binding.* □

### Stabilisations of standard handle decompositions

As it turned out standard handle decompositions and open book decompositions of contact manifolds describe equivalent structures. In the following we will describe the analogues of a positive and negative stabilisation of an abstract open book in the language of standard handle decompositions.

**Positive stabilisations.** Let $(M, \xi)$ be contact 3-manifold which admits a standard handle decomposition

$$(M, \xi) = H_0 \cup H_1.$$

We will now show how to add a cancelling pair of a standard 1- and 2-handle to the decomposition above. To simplify notation we understand $H_0$ and $H_1$ as manifolds with edges and identify $H_0 \equiv \Sigma \times [0, \frac{1}{2}]$ and $H_1 \equiv \Sigma \times [\frac{1}{2}, 1]$. Under this identification the dividing set $\Gamma \subset \partial H_0 = \partial H_1$ is identified with $\partial\Sigma \times \{\frac{1}{4}\}$ and $\partial\Sigma \times \{\frac{3}{4}\}$ respectively. Let $a \subset \Sigma$ be a properly embedded Legendrian arc. Add a standard 1-handle $h$, in form of a standard neighbourhood of $a \times \{\frac{3}{4}\}$, to $H_0$ and denote the result by $H_0'$. Observe that $a \times [\frac{1}{2}, \frac{3}{4}]$ descends to a 2-disc $D$ whose boundary $\alpha = \partial D$ intersects the dividing set $\Gamma' \subset H_0'$ exactly twice. Furthermore since $D$ is contained in $H_1$, and $H_1$ is tight, $D$ can be understood as core of a standard 2-handle $\mathbb{D}$. One can show that we have

$$H_0 \cup h \cup \mathbb{D} \cong H_0. \tag{1.5}$$

Actually the attachment of $h \cup \mathbb{D}$ corresponds to a trivial bypass attachment. In consequence $\mathbb{D}$ can be thought as a dual 1-handle attached to the complement of $H_0$ in $M$ and hence we end up with a **positively stabilised** standard handle decomposition

$$(M, \xi) = (H_0 \cup h) \cup (\mathbb{D} \cup H_1). \tag{1.6}$$

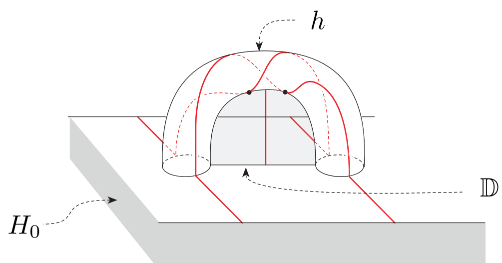


Figure 1.3: A positive stabilisation of a standard handle decomposition.

Let us take a look at what changed on the abstract level. Observe that, just as expected, the new page $\Sigma'$ is obtained by attaching a (2-dimensional) 1-handle to the original page $\Sigma$ along the endpoints of $a \subset \Sigma$. To investigate the change of the monodromy identify $\partial H_1'$ with $(-\Sigma') \cup \Sigma'$ and decompose the curve $\partial D \subset \partial H_1'$ into $a_0 \cup a_1$, where $a_0 = \Sigma' \cap \partial D$ and $a_1 = -\Sigma' \cap \partial D$. Note that $a_1$ can be understood as co-core of the attached (2-dimensional) 1-handle of $\Sigma'$ as well as the image of $a_0$ under the monodromy map of the abstract open book corresponding to decomposition (1.6). Referring to Figure 1.3, one concludes that $a_0$ and $a_1$ are related by the effect of a right-handed Dehn twist along the simple closed curve given by the union of $a$ and the core of the attached (2-dimensional) 1-handle of $\Sigma'$.

**Negative stabilisations.** Suppose we decomposed $(M, \xi)$ as in (1.6). Recall that the 1-handle $h$ did correspond to a standard neighbourhood of the Legendrian arc $a \times \{\frac{3}{4}\}$. With respect to the page $\Sigma_{\frac{3}{4}}$ the contact planes make a left-handed half-twist along $a \times \{\frac{3}{4}\}$, and so does the dividing set on the boundary of the 1-handle $h$. Let $B$ denote a little neighbourhood of $a \times [\frac{1}{4}, \frac{3}{4}] \subset M$ such that $\partial B$ is convex and $\xi$ restricted to $B$ is tight. From now on we understand all changes to take place inside of $B$. Replace the 1-handle $h$ by the new 1-handle $h_-$ indicated in Figure 1.4. Note that the dividing set on the boundary of the new 1-handle $h^-$ makes a right-handed half-twist. Although equation 1.5 does not hold anymore we still have

$$\partial(H_0 \cup h_- \cup \mathbb{D}) = \partial(H_0) \tag{1.7}$$

as convex surfaces. In fact this is all we need to understand $\mathbb{D}$ as dual 1-handle attached to the complement of $H_0 \cup h_- \cup \mathbb{D}$ in $M$ and hence we end up with a

**negatively stabilised** standard handle decomposition

$$(M, \xi_-) = (H_0 \cup h_-) \cup (\mathbb{D} \cup H_1). \tag{1.8}$$

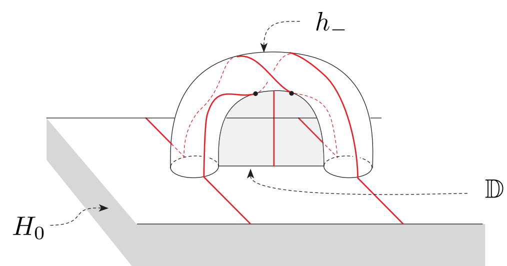


Figure 1.4: A negative stabilisation of a standard handle decomposition.

As indicated on the left side of the above equation the negatively stabilised handle decomposition does not support the original contact structure $\xi$ anymore. Though the new contact structure $\xi_-$ agrees with the original one outside of $B$. In particular $\partial B$ is also convex with respect to $\xi_-$ and its dividing set is standard, i.e. given by a single simple closed curve. Observe that (with respect to $\xi_-$) the the union of $a \times \{\frac{1}{4}\}$ and the core of $h_-$ defines a Legendrian unknot in $B$ which violates the Bennequin Inequality. Hence $\xi_-$ is overtwisted. It is not hard to show that $\xi_-$ is obtained from $\xi$ by a Lutz twist along a transverse unknot $U \subset (B, \xi)$ with self-linking number $-1$. Actually we have $(B, \xi_-) \cup B_{st} \cong (S^3, \xi_1)$ and hence we can conclude

$$(M, \xi_-) \cong (M, \xi) \# (S^3, \xi_1).$$

In analogy to the investigations of the effect of a positive stabilisation on the level of abstract open books we conclude that the new page of the abstract open book corresponding to the decomposition (1.8) is given by the attachment of a (2-dimensional) 1-handle to $\Sigma$ along the endpoints of $a$. Furthermore the monodromy changes by a right-handed Dehn twist along the closed curve given by the union of $a$ and the core of the attached 1-handle.

# Chapter 2

# An extension of the connected sum

In the present chapter we extend the operation of the contact connected sum, in the sense that we replace the tight 3-balls by standard 1-handle bodies which can be thought of as standard neighbourhoods of Legendrian graphs. We show that there is a Weinstein cobordism from the original manifold to the result of the extended contact connected sum, cf. Theorem 11.In Section 2.1 various applications of this result are presented. In Section 2.2 we step into arbitrary dimensions and approach the results from a symplectic perspective. There we implicitly present a generalised symplectic 1-handle which is used for the construction of exact symplectic cobordisms.

Unless otherwise stated let $(M, \xi)$ be a compact, oriented, not necessarily connected, contact 3-manifold with (possibly empty) convex boundary $\partial M$. Furthermore let $H$ be a standard 1-handle body of genus $g \in \mathbb{N}$.

**Definition 8.** Given a contact embedding $S^0 \times H \hookrightarrow (M, \xi)$, where $H$ is as above, we define the **extended connected sum** $\#_H(M, \xi)$ as

$$\#_H(M, \xi) := \big((M, \xi) \setminus \operatorname{Int}(S^0 \times H)\big) \cup_{S^0 \times \partial H} (D^1 \times \partial H, \eta),$$

where $\eta$ is the unique $D^1$-invariant contact structure induced by $(\partial H)_\xi$ – cf. the Uniqueness Lemma (Lemma 5).

In contrast to the ordinary contact connected sum (cf. Subsection 1.2.1), the extended contact connected sum does strongly depend on the choice of the embedding of $S^0 \times H$, since the topology of the manifold obviously depends on the topological knot type of the embedded core, the Legendrian graph $G$, of $H$, whereas the underlying contact structure depends on the embedded Legendrian graph $G$ up to Legendrian isotopy.

**Theorem 11.** *Let $(M, \xi)$ be a compact, oriented, not necessarily connected, contact 3-manifold and let $H$ be a standard 1-handle body. Given a contact-embedding $S^0 \times H \hookrightarrow (M, \xi)$ let $\#_H(M, \xi)$ denote the result of the extended* *connected sum on* $(M,\xi)$. *Then there is a Weinstein cobordism from* $(M,\xi)$ *to* $\#_H(M,\xi)$.

Before we dive into the proof of Theorem 11 we start by recalling some easy facts about standard 1-handle bodies and fix some notation: let $H$ be a standard 1-handle body of genus $g \in \mathbb{N}$ and let $\Gamma \subset \partial H$ denote the dividing set of its convex boundary. There are meridional 2-discs $D_1, \ldots, D_g \subset H$ such that each disc $D_i$ bounds a simple closed curve

$$\alpha_i = \partial D_i \tag{2.1}$$

on $\partial H$, which intersects the dividing set transversely and exactly twice. Hence the Legendrian Realisation Principle (see [34] or cf. Theorem 3) applies and we can assume the curves $\boldsymbol{\alpha}$ to be Legendrian. According to equation (1.1) on page 4 we compute $\mathtt{tw}(\alpha_i, \partial H) = -1$.

Let $\partial H = \{0\} \times \partial H$ denote the core of the generalised connecting tube $D^1 \times \partial H \subset \#_H(M,\xi)$, cf. Definition 8 above. Since $\partial H$ is convex we can identify a neighbourhood $N$ of $\partial H$ with

$$N = (\partial H \times \mathbb{R}, \eta) \subset \#_H(M,\xi),$$

where $\eta$ is the $\mathbb{R}$-invariant contact structure induced by the dividing set $\Gamma_H \subset \partial H$. Note that, in contrast to the whole meridional discs, cf. description (2.1) above, we still recover their boundaries, the curves $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_g\} \subset \partial H$. Let $\boldsymbol{\alpha} \times [-\varepsilon, \varepsilon] \subset \partial H$ denote a neighbourhood of the curves $\boldsymbol{\alpha}$ in $\partial H$ which consists of Legendrian translates of $\boldsymbol{\alpha}$. We set

$$\mathbb{A} = \boldsymbol{\alpha} \times [-\varepsilon, \varepsilon] \times [-1, 1] \subset N.$$

Note that the boundary of $\mathbb{A}$ decomposes as

$$\partial \mathbb{A} = \big(\boldsymbol{\alpha} \times [-\varepsilon, \varepsilon] \times \{-1\}\big) \cup \mathbb{A}^+ \cup \mathbb{A}^- \cup \big(\boldsymbol{\alpha} \times [-\varepsilon, \varepsilon] \times \{1\}\big),$$

where we use the shorthand notation $\mathbb{A}^\pm$ for the regions $\boldsymbol{\alpha} \times \{\pm\varepsilon\} \times [-1, 1]$. Another way to see $\mathbb{A}$ is as follows: note that $\alpha_i \times [-1, 1] \subset N$ defines a convex annulus with dividing set given by two vertical line segments, i.e. the dividing set is given by $\{x_i, y_i\} \times [-1, 1]$ for suitable points $x_i, y_i \in \alpha_i$. This gives rise to a family of convex annuli $\boldsymbol{\alpha} \times [-1, 1]$ for which we can think of its result after thickening as $\mathbb{A}$.

We will now consider the following submanifold $N_{\mathbb{A}}$ of $N$ given as

$$N_{\mathbb{A}} = \Big(N \setminus \big(\partial H \times (-1, 1)\big)\Big) \cup \mathbb{A}, \tag{2.2}$$

after convex edge rounding (cf. Subsection 1.1.3 for convex edge rounding or see [34]). This defines a contact submanifold $N_{\mathbb{A}} \subset N$ with convex boundary $\partial N_{\mathbb{A}}$. Observe that $\partial N_{\mathbb{A}}$ is a surface of genus $2g - 1$. A precise description of the boundary is given by

$$\partial N_{\mathbb{A}} = \Big(\big(\partial H \setminus \alpha \times (-\varepsilon, \varepsilon)\big) \times \{1\}\Big) \cup \mathbb{A}^+ \cup \mathbb{A}^- \cup \Big(\big(\overline{\partial H} \setminus \alpha \times (-\varepsilon, \varepsilon)\big) \times \{-1\}\Big). \tag{2.3}$$

Depending on the situation we will sometimes think of $N_{\mathbb{A}}$ as manifold with edges, i.e. before we applied the edge rounding. It will be clear from the situation which version of $N_{\mathbb{A}}$ is considered.

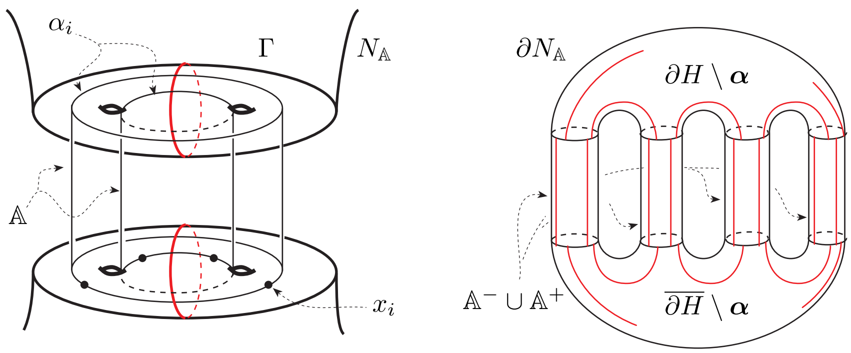


Figure 2.1: Schematic picture of $N_{\mathbb{A}}$ and $\partial N_{\mathbb{A}}$.

Let us introduce some further notation. Denoting by $D^2_{st}$ a convex disc with dividing set a single properly embedded arc, let

$$\mathbb{D} = D^2_{st} \times D^1$$

denote a 3-dimensional standard contact 2-handle. Assume we are given a family $\boldsymbol{\delta} = \{\delta_1, \ldots, \delta_k\} \subset \partial N_{\mathbb{A}}$ of Legendrian curves satisfying $\mathtt{tw}(\delta_i, \partial N_{\mathbb{A}}) = -1$. Then we define $N(\boldsymbol{\delta})$ to be the contact manifold obtained by attaching 3-dimensional standard contact 2-handles $\mathbb{D}_1, \ldots, \mathbb{D}_k$ along $\boldsymbol{\delta} = \{\delta_1, \ldots, \delta_k\} \subset \partial N_{\mathbb{A}}$, i.e. in symbols we have

$$N(\boldsymbol{\delta}) = N(\delta_1, \ldots, \delta_k) = N_{\mathbb{A}} \cup_{\boldsymbol{\delta}} (\mathbb{D}_1 \cup \ldots \cup \mathbb{D}_k).$$

In case the boundary of $\partial N(\boldsymbol{\delta})$ matches the boundary of another contact 3-manifold $(N', \xi')$ we define

$$N(\boldsymbol{\delta}; N') = N(\delta_1, \ldots, \delta_k; N') = N(\boldsymbol{\delta}) \cup_\partial (N', \xi').$$

From now on let $\boldsymbol{\gamma} = \{\gamma_1, ..., \gamma_{2g}\} \subset \partial N_{\mathbb{A}}$ denote the curves corresponding to the cores of the annuli $\mathbb{A}^+ \cup \mathbb{A}^-$ (cf. also Figure 2.2), to be precise, for $i = 1, \ldots, g$ we define

$$\gamma_{2i} = \alpha_i \times \{\varepsilon\} \times \{0\} \quad \text{and} \quad \gamma_{2i-1} = \alpha_i \times \{-\varepsilon\} \times \{0\}. \tag{2.4}$$

Note that $\gamma_{2i}$ and $\gamma_{2i-1}$ are Legendrian isotopic to $\alpha_i$ for each $i = 1, \ldots, g$ and we have $\mathtt{tw}(\gamma_{2i}, \partial N_{\mathbb{A}}) = \mathtt{tw}(\gamma_{2i-1}, \partial N_{\mathbb{A}}) = -1$.

Finally we define another family $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_{2g-1}\} \subset \partial N_{\mathbb{A}}$ of closed curves as follows. The $\boldsymbol{\alpha}$-curves cut $\Gamma_H$ into a family $\{b_1, ..., b_{2g}\} \subset \partial H$ of arcs with endpoints on $\boldsymbol{\alpha}$. Actually we assume the endpoints $\partial b_i$ of the arc $b_i$ to lie on $\boldsymbol{\alpha}\times\{\pm\varepsilon\}$. Recall that $\partial H$ decomposes as $\partial H = (-\Sigma)\cup_\Gamma \Sigma$ (cf. equation (1.4) on page 15) and push $\{b_1, ..., b_{2g}\}$ slightly into the $\Sigma$-part. Let $\boldsymbol{b} = \{b_1, ..., b_{2g-1}\}$ denote the first $2g-1$ of those arcs. We can think of $b_i \times [-1, 1]$ as a 2-disc sitting inside of $N$ and we denote its boundary by $\beta_i$, i.e. we set

$$\beta_i = \partial\big(b_i \times [-1, 1]\big) \subset N_{\mathbb{A}}. \tag{2.5}$$

Observe, by the choice of $\{b_1, ..., b_{2g-1}\} \subset \Sigma$, for each $i = 1, \ldots, 2g-1$, after rounding edges of $\partial N_{\mathbb{A}}$, $\beta_i$ descends to a simple closed curve with $\mathtt{tw}(\beta_i, \partial N_{\mathbb{A}}) = -1$ (cf. Figure 2.2).

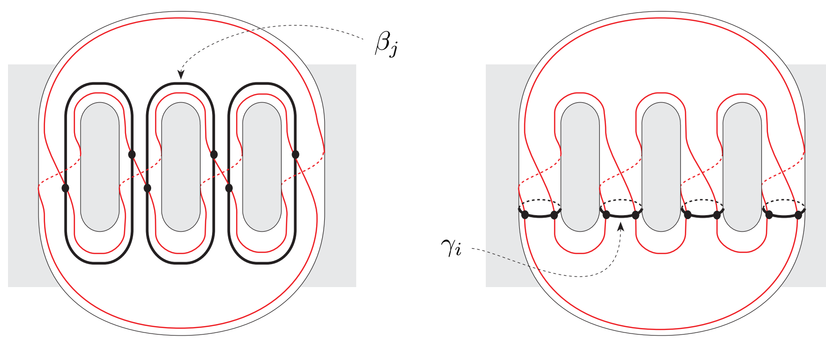


Figure 2.2: The attaching curves $\boldsymbol{\beta}, \boldsymbol{\gamma} \subset \partial N_{\mathbb{A}}$. Note that the edges of $\partial N_{\mathbb{A}}$ are considered to be rounded.

*Proof of Theorem 11.* The strategy of the proof is as follows: we will relate $\#_H(M, \xi)$ and the ordinary $g$-fold connected sum $\#_g(M, \xi)$ by a sequence of contact $(+1)$-surgeries. To do so we start by identifying $\#_H(M, \xi)$ with $\big(\#_H(M, \xi)\setminus N\big) \cup N(\boldsymbol{\beta}; B^3_{st})$. In a second step we will identify $\#_g(M, \xi)$ with $\big(\#_H(M, \xi) \setminus N\big) \cup N(\boldsymbol{\gamma}; B^3_{st} \cup B^3_{st})$. In the final step we relate $N(\boldsymbol{\beta}; B^3_{st})$ and $N(\boldsymbol{\gamma}; B^3_{st} \cup B^3_{st})$ by a sequence of contact $(+1)$-surgeries.

As a consequence of the Cancellation Lemma (cf. Lemma 6), $\#_H(M, \xi)$ can be understood as the result of a series of contact $(-1)$-surgeries on $\#_g(M, \xi)$. Finally the remark at the end of Subsection 1.2.3 (on page 10) and the fact that contact $(-1)$-surgery and the contact connected sum respectively correspond to the attachments of symplectic model 2- and 1-handles respectively, imply the existence of the desired Weinstein cobordism between $(M, \xi)$ and $\#_H(M, \xi)$.

**Step 1: Identify** $N(\boldsymbol{\beta}; B^3_{st})$**.** Actually we can think of $N(\boldsymbol{\beta}; B^3_{st})$ as embedded in $N$: let us recall the construction of $N(\boldsymbol{\beta}; B^3_{st})$. By construction $N_{\mathbb{A}}$ embeds into $N$ and for $i = 1, \ldots, 2g-1$ we can think of the standard 2-handle $\mathbb{D}_{\beta_i}$ as $\big(\beta_i \times [-\delta, \delta]\big) \times [-1, 1]$, cf. equation (2.5) above. Therefore we can think of $N(\boldsymbol{\beta})$ as embedded into $N$.

Finally by choice of the curves $\boldsymbol{\alpha}$ and the arcs $\boldsymbol{b}$ their complement $\partial H \setminus \big((\boldsymbol{\alpha}\times[-\varepsilon,\varepsilon])\cup(\boldsymbol{b}\times[-\delta,\delta])\big)$ defines a 2-disc $D$ with boundary parallel dividing set. Hence $N\setminus N(\boldsymbol{\beta}) = D\times[-1,1]$ defines a tight 3-ball and indeed we have

$$N(\boldsymbol{\beta}; B^3_{st}) = N.$$

**Step 2: Identify $N(\boldsymbol{\gamma}; B^3_{st}\cup B^3_{st})$.** We trivially decompose $(M,\xi)$ as $\big((M,\xi)\setminus \mathrm{Int}(H_0\cup H_1)\big)\cup(H_0\cup H_1)$. Observe that the first summand $(M,\xi)\setminus \mathrm{Int}(H_0\cup H_1)$ equals $\#_H(M,\xi)\setminus N$ and we can express the $g$-fold connected sum as

$$\#_g(M,\xi) = \big(\#_H(M,\xi)\setminus N\big)\cup\#_g(H_0\cup H_1).$$

It remains to show that $\#_g(H_0\cup H_1)$ equals $N(\boldsymbol{\gamma}; B^3_{st}\cup B^3_{st})$. Let us start with a description of $\#_g(H_0\cup H_1)$: recall that the $\boldsymbol{\alpha}$-curves correspond to the meridional discs of $H$. Therefore $(H_0\cup H_1)$ is given as

$$(H_0\cup H_1) = \Big(N\setminus\big(\partial H\times(-1,1)\big)\cup\big(\mathbb{D}_{\alpha_1^+}\cup\ldots\cup\mathbb{D}_{\alpha_g^+}\big)\cup\big(\mathbb{D}_{\alpha_1^-}\cup\ldots\cup\mathbb{D}_{\alpha_g^-}\big)\cup B^3_{st}\cup B^3_{st}\Big),$$

where we use the shorthand notation $\alpha_i^\pm = \alpha_i\times\{0\}\times\{\pm1\}$. For $i=1,\ldots,g$ choose tight 3-balls $B_i^+\cup B_i^-\subset\mathbb{D}_{\alpha_i^+}\cup\mathbb{D}_{\alpha_i^-}$ and let $\#_g(H_0\cup H_1)$ denote the $g$-fold connected sum where we identify $B_i^+$ with $B_i^-$.

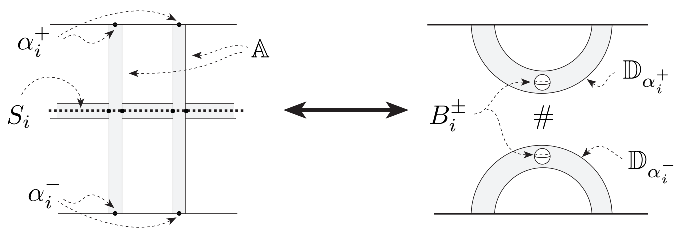


Figure 2.3: Relation between $N(\boldsymbol{\gamma})$ and $\#_g(H_0\cup H_1)$.

Now consider the contact manifold $N(\boldsymbol{\gamma})$ and recall that it is given by attaching standard 2-handles $\mathbb{D}_{\gamma_1},\ldots,\mathbb{D}_{\gamma_{2g}}$ to $N_{\mathbb{A}}$ along $\boldsymbol{\gamma}$. Recall further that the family of closed curves $\boldsymbol{\gamma}$ corresponds to the cores of the annuli $\mathbb{A}^+\cup\mathbb{A}^-$ in the description (2.3) of $\partial N_{\mathbb{A}}$ above. Moreover, for $i=1,\ldots,g$, the consecutive curves $\gamma_{2i-1}$ and $\gamma_{2i}$ lie on the opposite boundary regions of $\alpha_i\times[-\varepsilon,\varepsilon]\times[-1,1]$. Hence the attachment of a pair of 2-cells $\mathbb{D}_{\gamma_{2i-1}}$ and $\mathbb{D}_{\gamma_{2i}}$ gives birth to a tight 2-sphere $S_i^2\subset N(\boldsymbol{\gamma})$ as result of gluing together the core discs of $\mathbb{D}_{\gamma_{2i-1}}$ and $\mathbb{D}_{\gamma_{2i}}$. We will now show that these 2-spheres correspond to the connecting tubes of $\#_g(H_0\cup H_1)$. Just cut $N(\boldsymbol{\gamma})$ along the $S_i^2$ and fill in the resulting boundary with tight 3-balls $B_i^+$ and $B_i^-$. This affects $N(\boldsymbol{\gamma})$ as follows. For $i=1,\ldots,g$ we have

$$\Big(\big((\alpha_i\times[-\varepsilon,\varepsilon]\times[-1,1])\cup\mathbb{D}_{\gamma_{2i}}\cup\mathbb{D}_{\gamma_{2i-1}}\big)\setminus S_i^2\Big)\cup B_i^+\cup B_i^- = \mathbb{D}_{\alpha_i^+}\cup\mathbb{D}_{\alpha_i^-}. \quad (2.6)$$

Now, for $i = 1, \ldots, g$ build the connected sum along the pair $B_i^+ \cup B_i^- \subset \mathbb{D}_{\alpha_i^+} \cup \mathbb{D}_{\alpha_i^-}$ of tight 3-balls. Comparing this with the description of $\#_g(H_0 \cup H_1)$ at the beginning of this discussion we conclude

$$N(\boldsymbol{\gamma}; B^3_{st} \cup B^3_{st}) = \#_g(H_0 \cup H_1).$$

**Final step: Transform $N(\boldsymbol{\beta}; B^3_{st})$ into $N(\boldsymbol{\gamma}; B^3_{st} \cup B^3_{st})$.** Relabel the $\boldsymbol{\beta}$- and $\boldsymbol{\gamma}$-curves such that for $k = 1, \ldots, 2g-1$ we have (cf. figure 2.2)

$$\#(\beta_k \cap \gamma_k) = \#(\beta_k \cap \gamma_{k+1}) = 1. \tag{2.7}$$

Furthermore we can assume that these are the only intersection points between the curves. Let us introduce some further notation. Set

$$N(k) = N(\gamma_1, \ldots, \gamma_{k-1}, \beta_k, \ldots, \beta_{2g-1})$$

and for compatible boundary conditions, in the same fashion as above, we set

$$N(k; N') = N(k) \cup N'.$$

We first want to show that we can get from $N(k; B^3_{st})$ to $N(k+1; B^3_{st})$ by a contact $(+1)$-surgery on $\gamma_k$: recall (cf. equation (2.7) above) that we chose $\gamma_k$ such that $\#(\gamma_k \cap \beta_k) = 1$. Therefore we can understand $\mathbb{D}_{\beta_k}$ as meridional disc for $\gamma_k$. Hence $B^3_{st} \cup \mathbb{D}_{\beta_k} \subset N(k; B^3_{st})$ provides a neighbourhood of $\gamma_k$ (actually it provides a neighbourhood of a nearby Legendrian isotopic copy of $\gamma_k$). A priori we cannot ensure that this is a standard Legendrian neighbourhood for $\gamma_k$. But for topological surgery this neighbourhood works just fine. Since $\mathtt{tw}(\gamma_k, \partial N_{\mathbb{A}}) = -1$ contact $(+1)$-surgery corresponds to topological zero-surgery where the framing is measured with respect to $\partial N_{\mathbb{A}}$, that is, we cut out the neighbourhood $B^3_{st} \cup \mathbb{D}_{\beta_k}$ of $\gamma_k$ and glue back in a 2-cell along $\gamma_k$ followed by a 3-ball. Topologically this gives $N(k+1; B^3_{st})$.

Let us take a closer look at the contact situation. Let $\nu\gamma_k$ be a standard Legendrian neighbourhood of $\gamma_k$. Hence denoting the result of contact $(+1)$-surgery along $\gamma_k$ by $N(k; B^3_{st})_{+1}(\gamma_k)$ we have

$$N(k; B^3_{st})_{+1}(\gamma_k) \quad = \quad \big(N(k; B^3_{st}) \setminus \nu\gamma_k\big) \cup_{\gamma_k} \mathbb{D}_{\gamma_k} \cup (B^3, \xi_{st}).$$

Given this description, we conclude that $N(k+1)$ embeds into $N(k; B^3_{st})_{+1}(\gamma_k)$ and we can investigate its complement. Observe that $N(k; B^3_{st})_{+1}(\gamma_k) \setminus N(k+1)$ is topologically a 3-ball. Assume for a moment that the contact structure on $N(k; B^3_{st})_{+1}(\gamma_k)$ is tight (this will be shown in Lemma 12 below), then this 3-ball is tight and hence we have

$$N(k; B^3_{st})_{+1}(\gamma_k) \cong N(k+1; B^3_{st})$$

So far we achieved the following. By a sequence of contact $(+1)$-surgeries we can transform $N(\beta_1, \ldots, \beta_{2g'-1}; B^3_{st})$ into $N(\gamma_1, \ldots, \gamma_{2g'-1}; B^3_{st})$. Consider the tight 3-ball $B^3_{st}$ from the latter manifold. Note that $\gamma_{2g}$ is a simple closed curve on the boundary of $B^3_{st}$ with $\mathtt{tb}(\gamma_{2g}, \partial B^3_{st}) = -1$. Therefore $\gamma_{2g}$ bounds a standard 2-disc $D \subset B^3_{st}$. Thickening up $D$ we can interpret it as a standard 2-handle $\mathbb{D}_{\gamma_{2g}}$. In the light of this interpretation the transformation into $N(\boldsymbol{\gamma}; B^3_{st} \cup B^3_{st})$ is finished and we are done. □

**Lemma 12.** *The result* $N(k; B^3_{st})_{+1}(\gamma_k)$ *of contact* $(+1)$*-surgery at each step* $k = 1, ..., 2g-1$ *in the above construction is tight.*

*Proof.* Recall that $\partial H$ was originally the boundary of some solid handle body $H$, which can be understood as standard neighbourhood of a Legendrian graph $G \subset (S^3, \xi_{st})$. This provides an embedding of $N \hookrightarrow (S^3, \xi_{st})$. Note that for each $i = 1, \dots, 2g$ there is a $j = 1, \dots, g$ such that $\gamma_i$ is Legendrian isotopic to $\alpha_j$ (cf. equation 2.4), which itself is a Legendrian unknot with $\mathtt{tb}(\alpha_j) = -1$. Furthermore the link $\gamma \subset S^3$ is trivial and contact $(+1)$-surgery on $\gamma$ understood as sitting in $(S^3, \xi_{st})$ yields $\#_{2g}(S^1 \times S^2, \xi_{st})$. In particular $N(k; B^3_{st})_{+1}(\gamma_k)$ embeds into $\#_k(S^1 \times S^2, \xi_{st})$ and hence is tight. □

**Remark.** In the final step of the proof of Theorem 11 we showed that contact $(+1)$-surgery on $\gamma_k$ gets us from $N(k; B^3_{st})$ to $N(k+1; B^3_{st})$. Since topologically there is no problem in going the other direction by using a zero-surgery on $\beta_k$, it suggests itself to ask whether the same holds in the contact category, i.e. does contact $(+1)$-surgery on $\beta_k$ get us back from $N(k+1; B^3_{st})$ to $N(k; B^3_{st})$?

The answer is no. Consider the embedding $N(\beta; B^3_{st}) \hookrightarrow (S^3, \xi_{st})$ from Lemma 12 above, then $\gamma_k \cup \beta_k$ gives the Hopf link. Recall that we have $\mathtt{tb}(\gamma_k) = \mathtt{tb}(\beta_k) = -1$. One computes the $d^3$-invariant (see [23] for a formula) of the result of contact $(+1)$-surgery on the Hopf link to be $\frac{3}{2}$ and concludes that it corresponds to performing a Lutz-twist along a transverse unkot $K$ with $\mathtt{sl}(K) = -1$. Alternatively, if one is familiar with open books, one recognises the open book with annular pages and monodromy a single negative Dehn twist, which is known to be compatible with the above contact structure. In particular $N(k+1; B^3_{st})_{+1}(\beta_k)$ is overtwisted and differs from $N(k; B^3_{st})$.

## 2.1 Applications

### 2.1.1 Monodromies and their concatenation

Let us apply Theorem 11 to the following particular situation. Given an open book $(\Sigma, g)$ we can think of the corresponding manifold $M_{(\Sigma,g)}$ as

$$M_{(\Sigma,g)} = \big(\Sigma \times [0,1]\big)/_\sim,$$

where we identify $(x,1) \sim (g(x), 0)$ for all $x \in \Sigma$ and $(x,t) \sim (x,s)$ for all $x \in \partial\Sigma$ and $s, t \in [0,1]$. Observe that

$$H = \Sigma \times [\tfrac{1}{2} + \varepsilon, \tfrac{1}{2} - \varepsilon]$$

descends to a standard solid handle body in $M_{(\Sigma,g)}$. Now assume we are given a second open book $(\Sigma, h)$ having the same page $\Sigma$ with some monodromy $h$, possibly different from the former one. Considering the corresponding standard handle body $H$, as above, but this time in $M_{(\Sigma,h)}$, we obtain an embedding

$$S^0 \times H \hookrightarrow M_{(\Sigma,g)} \sqcup M_{(\Sigma,h)}.$$

By a slightly different interpretation of the definition, the result of the extended sum $M_{(\Sigma,g)} \#_H M_{(\Sigma,h)}$ is given as $\big(M_{(\Sigma,g)} \sqcup M_{(\Sigma,h)}\big) \setminus \mathrm{Int}(S^0 \times H)$, where we identify $\Sigma \times \{\frac{1}{2} \pm \varepsilon\}$ with $-\big(\Sigma \times \{\frac{1}{2} \mp \varepsilon\}\big)$. It is not hard to see that $M_{(\Sigma,g)} \#_H M_{(\Sigma,h)}$ can be identified as $M_{(\Sigma,h\circ g)}$, cf also Figure 2.4. Thus we recover the following result first observed by Baker–Etnyre–Van Horn-Morris [2, Theorem 1.3]. Note that we approach this result in higher dimensions, using different methods, in Theorem 17 of section 2.2 below .

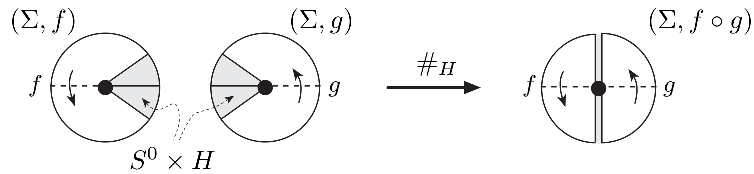


Figure 2.4: Schematic picture of summing two open books along thickened pages. The grey area in the left part corresponds to two embedded copies of $H = \Sigma \times [\frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon]$.

**Corollary 13.** *Assume we are given two open books* $(\Sigma, g)$ *and* $(\Sigma, h)$ *having the same page* $\Sigma$. *Then there is a Weinstein cobordism between the corresponding contact manifolds* $(M_{(\Sigma,g)}, \xi_{(\Sigma,g)}) \sqcup (M_{(\Sigma,h)}, \xi_{\Sigma,h)})$ *and* $(M_{(\Sigma,h\circ g)}, \xi_{(\Sigma,h\circ g)})$. □

Strengthening the conditions in the corollary above a little bit, namely supposing that the initial open books are Stein fillable, we can alternatively proof a slightly different version of it utilising the Giroux correspondence [25] of open books and contact structures. Of course Corollary 13 covers the statement of the corollary below – even a stronger version holding for all kinds of fillability.

**Corollary 14.** *Assume we are given two open books* $(\Sigma, g)$ *and* $(\Sigma, h)$ *having the same page* $\Sigma$. *Furthermore assume that the corresponding contact manifolds* $(M_{(\Sigma,g)}, \xi_{(\Sigma,g)})$ *and* $(M_{(\Sigma,h)}, \xi_{\Sigma,h)})$ *are Stein fillable. Then so is the contact manifold* $(M_{(\Sigma,h\circ g)}, \xi_{(\Sigma,h\circ g)})$.

*Proof.* Since $(M_{(\Sigma,g)}, \xi_{(\Sigma,g)})$ is Stein fillable, the open book $(\Sigma, g)$ is stably equivalent to an open book $(\Sigma', g')$ such that $g'$ factorises as a product of positive Dehn twists, cf. [25]. Now consider $(\Sigma, h)$ or the manifold $(M_{(\Sigma,h)}, \xi_{(\Sigma,h)})$ respectively, which is again Stein fillable by assumption. Start adding 1-handles to $(\Sigma, h)$ consecutively till we end up with $(\Sigma', h)$. On the level of abstract open books, adding 1-handles to a page corresponds to performing a contact connected sum on the corresponding contact manifold. As already mentioned the contact connected sum corresponds to adding a symplectic 1-handle on the level of cobordisms and is known to preserve all kinds of fillability (see[48] and [9]). So does contact $(-1)$-surgery (see[48] and [9]) and hence, since $g'$ factorises as a product of positive Dehn twists, $(\Sigma', h \circ g')$ is still Stein fillable. By the choice of $\Sigma'$ and $g'$, at the beginning of the proof, $(\Sigma', h \circ g')$ destabilises to $(\Sigma, h \circ g)$ and we are done. □

### 2.1.2 On fibrations over the circle

**Symmetric open books**

Let $\Sigma$ be a compact surface with non-empty boundary and $f\colon \Sigma \to \Sigma$ a diffeomorphism, equal to the identity near $\partial\Sigma$. Recall that for such data $\Sigma(f)$ denotes the mapping torus, that is, the quotient space obtained from $\Sigma \times [0,1]$ by identifying $(x,1)$ with $(f(x),0)$ for each $x \in \Sigma$. Furthermore we denote by $\bar{f}\colon \bar{\Sigma} \to \bar{\Sigma}$ the induced diffeomorphism on the mirror of $\Sigma$.

**Definition 9.** Given two diffeomorphisms $f, g\colon \Sigma \to \Sigma$, equal to the identity near $\partial\Sigma$, we define a surface bundle $\Sigma(f,g) \to S^1$ as follows. Take the mapping tori $\Sigma(f)$ and $\Sigma(g)$ and glue them together along their boundary $\partial\Sigma \times S^1$ using the orientation reversing diffeomorphism that reverses the orientation on the $S^1$-factor, i.e.

$$\Sigma(f,g) = \big(\Sigma(f) \cup \Sigma(g)\big)/_{\sim},$$

where we identify $(x,\theta)$ with $(x,\theta)$ for all $(x,\theta) \in \partial\Sigma \times S^1$. The result $\Sigma(f,g)$ yields a surface bundle over the circle with fibre $\Sigma \cup_{\partial\Sigma} \bar{\Sigma}$ and monodromy given by $f \cup \bar{g}$. Furthermore $\Sigma(f,g)$ carries a natural contact structure $\xi_{(\Sigma,f,g)}$ such that each fibre $\Sigma \cup_{\partial\Sigma} \bar{\Sigma}$ is convex with dividing set $\Gamma$ given by $\partial\Sigma$. Since the dividing set divides the surface $\Sigma \cup_{\partial\Sigma} \bar{\Sigma}$ into two parts of equal genus, we will refer to $\Sigma(f,g)$ as **balanced**.

**Remark.** The surface bundle $\Sigma(f,g)$ can be understood as $(\Sigma,f) \boxplus (\Sigma,g)$, the binding sum of the open books $(\Sigma,f)$ and $(\Sigma,g)$, cf. Definition 13. That is why in the literature $\Sigma(f,g)$ is sometimes referred to as **symmetric open book**.

In [49] Wendl introduces the notion of *planar torsion* for a contact manifold, an obstruction to strong fillability generalising overtwistedness and Giroux torsion. In essence, a contact manifold admits *planar torsion* if it can be written as the binding sum of a non-trivial number of open books, one of which has planar pages. However he excludes the class of symmetric open books from his definition. It is stated, without a proof though, that some of them occur as boundary of a Lefschetz fibration and thus admit a strong filling. Along these lines and with Theorem 11 in hand we can proof the following.

**Corollary 15.** *Let $\Sigma$ be a compact surface with non-empty boundary and $f\colon \Sigma \to \Sigma$ a diffeomorphism of $\Sigma$, equal to the identity near $\partial\Sigma$. Then the symmetric open book $\Sigma(f,f)$ with its natural contact structure admits a Weinstein filling.*

*Proof.* Let $k \in \mathbb{N}$ be the Euler characteristic of $\Sigma$. Then $(\Sigma, \mathrm{id})$, the open book with page $\Sigma$ and trivial monodromy, is compatible with the $k$-fold self-connected

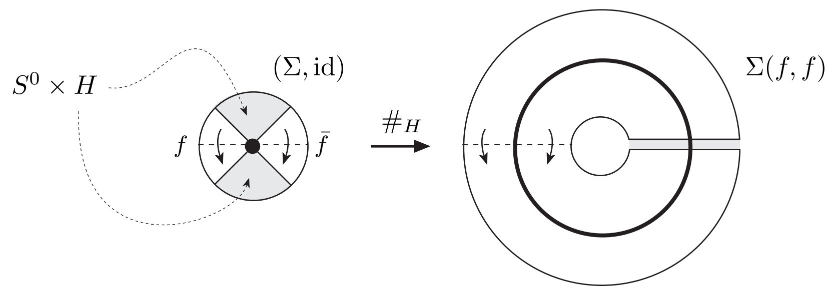


Figure 2.5: Schematic picture of the proof of Corollary 15.

sum of $S^1 \times S^2$ with its standard contact structure $\xi_{st}$. In particular $(\Sigma, \mathrm{id})$ admits a Weinstein filling. Recall that $M_{(\Sigma,\mathrm{id})}$ is given by $(\Sigma \times S^1) \cup_{\mathrm{id}} (\partial\Sigma \times D^2)$. Observe that, for any diffeomorphism $f$ of $\Sigma$, equal to the identity near $\partial\Sigma$, we can describe $\Sigma \times S^1$ as

$$\Big(\big(\Sigma \times [0,1]\big) \cup \big(\Sigma \times [2,3]\big)\Big)/_{\sim},$$

where we identify $(x, 3)$ with $(f(x), 0)$ and $(f(x), 1))$ with $(x, 2))$ for all $x \in \Sigma$. Now two embeddings of $H = \Sigma \times [-\varepsilon, \varepsilon]$ are given by $\Sigma \times [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ and $\Sigma \times [\frac{5}{2} - \varepsilon, \frac{5}{2} + \varepsilon]$. The result $\#_H M_{(\Sigma,\mathrm{id})}$ is easily identified as $\Sigma(f, f)$ (cf. also Figure 2.5). □

### An operation on trivial fibrations

Let $\Sigma$ be a compact, oriented surface with (possibly empty) boundary $\partial\Sigma$. Furthermore let $\Gamma \subset \Sigma$ be a collection of oriented, properly embedded arcs and circles. In case $\Sigma$ has non-empty boundary we assume for each component $K \subset \partial K$ the number of intersection points $\Gamma \cap K$ to be non-zero and even.

**Definition 10.** We will refer to $(\Sigma, \Gamma)$ as an **abstract convex surface** if there is a choice of orientations on the regions $\Sigma \setminus \Gamma$ which is coherent with the orientation of $\Gamma$. We will denote by $\Sigma^+$ and $\Sigma^-$ the collection of positive respectively negative oriented regions of $\Sigma \setminus \Gamma$.

Let $\phi$ denote a monodromy map of $\Sigma$ that restricts to the identity in a neighbourhood $\mathcal{N}(\Gamma) \subset \Sigma$ of the abstract dividing set $\Gamma$. Write $\pi$ for the projection from the mapping torus $\Sigma(\phi) = (\Sigma, \Gamma)(\phi)$ to the circle. Note that there is a natural contact structure $\xi_\Gamma$, such that for each $\theta \in S^1$ the fibre $\pi^{-1}(\theta) \cong \Sigma$ is a convex surface with dividing set $\Gamma$ and the boundary $\partial\Sigma(\phi)$ is a collection of convex tori $\partial\Sigma \times S^1$ with dividing curves of slope $\infty$.

Given two embedded standard two discs $D_0, D_1 \subset \mathcal{N}(\Gamma)$, that is, for $i = 0, 1$ the intersection $D_i \cap \Gamma$ contains a single properly embedded arc, the connected sum along $D_0$ and $D_1$ again gives rise to an abstract convex surface $(\Sigma', \Gamma')$. Furthermore $D_i \times S^1$ defines a standard neighbourhood of the Legendrian knot

$\{p_i\} \times S^1$ embedded in $\big(\Sigma(\phi), \xi_\Gamma\big)$, where $p_i$ is a point on $D_i \cap \Gamma$ and $i = 0, 1$. Performing the extended connected sum along $D_0 \times S^1$ and $D_1 \times S^1$ yields $\big(\Sigma'(\phi'), \xi_{\Gamma'}\big)$, where the monodromy $\phi'$ is the map that equals $\phi$ over $\Sigma \backslash (D_0 \cup D_1)$ and restricts to the identity elsewhere. Using Theorem 11 we conclude the following.

**Corollary 16.** *Assume we are in the situation described above. Then there is a Weinstein cobordism from $\big(\Sigma(\phi), \xi_\Gamma\big)$ to $\big(\Sigma'(\phi'), \xi_{\Gamma'}\big)$.* □

Let us put the corollary into a different light. Assume that the monodromy $\phi$ is trivial. Furthermore assume that $\Gamma \subset \Sigma$ isolates a component $R \subset \Sigma \setminus \Gamma$ from the boundary $\partial\Sigma$, i.e. we have $R \cap \partial\Sigma = \emptyset$. In [35] Honda–Kazez–Matić show that in presence of such an isolating region $R$ the contact invariant of $\big(\Sigma(\mathrm{id}), \xi_\Gamma\big)$ (in the sutured Floer homology) over $\mathbb{Z}_2$-coefficients vanishes (Massot extends this in [42] for $\mathbb{Z}$-coefficients). With the help of Corollary 16 we can reduce this statement to the case where $R$ has just a single boundary component as follows: assume that the number of boundary components $\#(\partial R)$ is bigger than 1. Now choose the standard 2-discs $D_0, D_1 \subset \Sigma$ from the construction above to lie in neighbourhoods of different components of $\partial R$. Then $R$ descends to an isolated region $R'$ in $\Sigma' \setminus \Gamma'$ whose number of boundary components decreased by 1 (though $R'$ differs from $R$ by an attached 1-handle), cf. also Figure 2.6. Since Corollary 16 provides a Stein cobordism from $\big(\Sigma(\mathrm{id}), \xi_\Gamma\big)$ to $\big(\Sigma'(\mathrm{id}), \xi_{\Gamma'}\big)$ the vanishing of the contact invariant of $\Sigma'(\mathrm{id})$ implies the vanishing of the contact invariant of $\Sigma(\mathrm{id})$ (see [38] and [36]).

We can reduce the above statement even further. We observed above that we can assume the isolated region $R$ to have a single boundary component. Now take a curve $\alpha \subset \Sigma \setminus R$ in the region adjacent to $R$, parallel to $\partial R$ and such that it intersects $\Gamma$ exactly twice and away from $\partial R \subset \Gamma$. This curve $\alpha$ separates $\Sigma$ into two regions. Let $(\hat{R}, \hat{\Gamma})$ denote the region containing $R$. The embedding $(\hat{R}, \hat{\Gamma}) \hookrightarrow (\Sigma, \Gamma)$ gives rise to a contact embedding $\big(\hat{R}(\mathrm{id}), \xi_{\hat{\Gamma}}\big) \hookrightarrow \big(\Sigma(\mathrm{id}), \xi_\Gamma\big)$ of the corresponding contact manifolds. Such an embedding in turn implies that the vanishing of the contact invariant of $\big(\hat{R}(\mathrm{id}), \xi_{\hat{\Gamma}}\big)$ is passed on to the contact invariant of $\big(\Sigma(\mathrm{id}), \xi_\Gamma\big)$ (this is a consequence of the Gluing-Theorem in [35]).

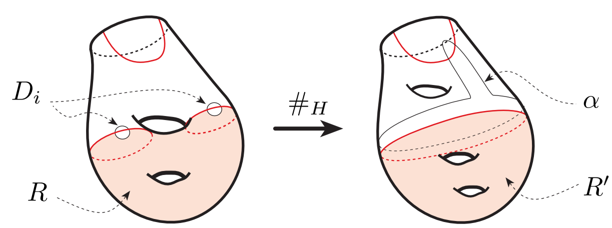


Figure 2.6: Decreasing the number of boundary components of an isolating region $R$.

Finally let us take a closer look on the abstract convex surface $(\hat{R}, \hat{\Gamma})$ and its corresponding contact structure $\xi_{\hat{\Gamma}}$. First let us shorten the notation a bit and denote by $(M,\xi) = (M_{(\hat{R},\mathrm{id})}, \xi_{(\hat{R},\mathrm{id})})$ the contact manifold corresponding to $(\hat{R},\mathrm{id})$ understood as an abstract open book. Let $K \subset (M,\xi)$ denote the unique binding component. Performing a Lutz-twist along $K \subset (M,\xi)$ yields an overtwisted contact manifold $(M',\xi')$ for which we denote by $K'$ the core of the Lutz-tube (actually the Lutz-twist does not change the topology of the underlying manifold but we write $M'$ anyway). The convex blow up of $K' \subset (M',\xi')$ can now easily be identified as $(\hat{R},\hat{\Gamma})(\mathrm{id})$ (cf. Definition 12 on page 41 below if the notion *convex blow up* is not familiar). Furthermore it is well known that the *hat*-version of the knot Floer homology of $K'$ can be naturally identified with the sutured Floer homology of $(\hat{R},\hat{\Gamma})(\mathrm{id})$ (cf. [39]). In particular the natural identification is compatible with their respective contact invariants. Thus the statement at the beginning of the discussion becomes a statement on the transverse invariant of a fibered knot after performing a Lutz-twist and raises the following question:

**Question 1.** Let $K \subset (M,\xi)$ be a transverse knot whose complement fibres as pages of an open book supporting the contact structure $\xi$. Let $\xi'$ denote the result of performing a Lutz-twist along $K$ and let $K'$ denote the transverse knot corresponding to the core of the twist. Does the transverse invariant of $K'$ vanish?

### 2.1.3 Normal sum along Legendrian knots

The operation on trivial surface bundles in Section 2.1.2 can be understood as a normal sum along Legendrian knots. In the present section we choose $H$ to be the standard neighbourhood of a Legendrian knot. Let $L_0, L_1 \subset (M,\xi)$ denote two Legendrian knots in some contact 3-manifold $(M,\xi)$. Choosing standard Legendrian neighbourhoods $\mathcal{N}(L_0)$ and $\mathcal{N}(L_1)$ respectively defines an embedding

$$S^0 \times H \hookrightarrow (M,\xi).$$

Let us denote the result of performing the contact connected sum along $\mathcal{N}(L_0)$ and $\mathcal{N}(L_1)$ by $M(L_0,L_1)$, i.e. we have

$$M(L_0,L_1) = \#_H(M,\xi).$$

Now let $L_2, ..., L_n$ be Legendrian knots in the same knot type as $L_1$ and with the same classical invariants. Then by normal summing along $L = L_0$ and $L_1, \dots, L_n$ we obtain contact structures $\xi_1, \dots, \xi_n$ on the same manifold. It may be worth to explore the differences between these contact structures.

**Remark.** The normal sum along Legendrian knots described in this subsection can be generalised to a normal sum along isotropic spheres in higher dimensional contact manifolds. It will be the result of a connected sum followed by a contact surgery, hence fillability is also preserved.

## 2.2 Generalisation to higher dimensions

Let $\Sigma$ denote a compact, $2n$-dimensional manifold admitting an exact symplectic form $\omega = d\beta$ and let $Y$ denote the Liouville vector field defined by $i_Y\omega = \beta$. Assume that $Y$ is transverse to $\partial\Sigma$, pointing outwards. Note that these properties are precisely the ones requested for $\Sigma$ to be a page of an abstract open book in the contact setting. The main result of the present section is the following.

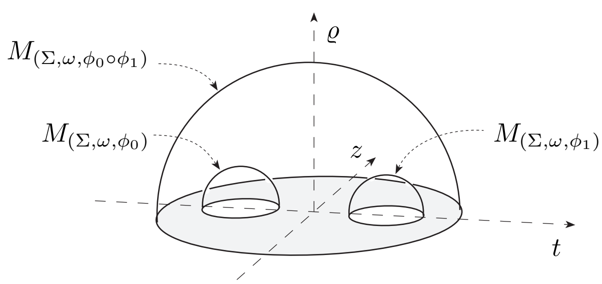


Figure 2.7: Schematic picture of the symplectic cobordism constructed in Theorem 17.

**Theorem 17.** *Given two symplectomorphisms $\phi_0$ and $\phi_1$ of $(\Sigma, \omega)$, equal to the identity near the boundary $\partial\Sigma$, there is an exact symplectic cobordism from the disjoint union $M_{(\Sigma,\omega,\phi_0)} \sqcup M_{(\Sigma,\omega,\phi_1)}$ to $M_{(\Sigma,\omega,\phi_0\circ\phi_1)}$.*

*Proof.* Let $(r, x)$ denote coordinates on a collar neighbourhood $(-\varepsilon, 0] \times \partial\Sigma$ induced by the negative flow corresponding to the Liouville vector field $Y$. Let $\varrho\colon \Sigma \to [0, \infty]$ be a $C^\infty$-function on $\Sigma$ satisfying the following properties:

- $\varrho \equiv 0$ over $\Sigma \setminus \big((-\varepsilon, 0] \times \partial\Sigma\big)$,
- $\varrho \equiv \infty$ over $\partial\Sigma$,
- $\frac{\partial\varrho}{\partial_r} > 0$ and $\frac{\partial\varrho}{\partial_x} \equiv 0$ over $\big((-\varepsilon, 0] \times \partial\Sigma\big)$ with coordinates $(r, x)$.

Note that over the collar neighbourhood $(-\varepsilon, 0] \times \partial\Sigma$ the vector field $Y$ is gradient-like for $\varrho$. Consider the space $\Sigma \times \mathbb{R}^2$ with coordinates $(p; z, t)$. This space is symplectic with symplectic form

$$\Omega = \omega + dz \wedge dt.$$

Consider the vector field $Z_0$ on $\Sigma \times \mathbb{R}^2$ defined by

$$Z_0 = Y + X,$$

where $X = \big(1 - f'(t)\big)z\,\partial_z + f(t)\,\partial_t$ and $f : \mathbb{R} \to \mathbb{R}$ is the function satisfying the following properties:

- $f(\pm\sqrt{2}) = f(0) = 0$,
- $|f'(t)| < 1$ for each $t \in \mathbb{R}$ and
- $f'$ has exactly two zeros $t_\pm$ and they satisfy $0 < \pm t_\pm < \sqrt{2}$.

An easy computation shows that $X$ is a Liouville vector field on $(\mathbb{R}^2, dz \wedge dt)$ for any function $f$. Hence $Z_0$ defines a Liouville vector field on $\big(\Sigma \times \mathbb{R}^2, \Omega\big)$.

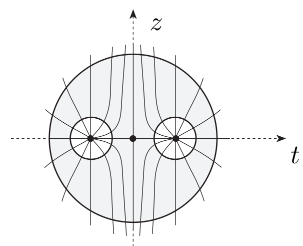


Figure 2.8: Flow lines of the Liouville vector field $X$.

We are now ready to define the desired symplectic cobordism $W$. Let $P$ denote the subset of $\Sigma \times \mathbb{R}^2$ defined by

$$P := \big\{(p; z, t)\colon\ \varrho \leq 0,\ z^2 + t^2 \leq C \text{ and } (t \pm \sqrt{2})^2 + z^2 \geq 1\big\},$$

where $C \in \mathbb{R}$ is some constant satisfying $C > (\sqrt{2} + 1)^2$. We will now cut $P$ along $\{z = 0\}$ and then reglue with respect to $\phi_0$ and $\phi_1$ as follows. Set $P_\pm := P \cap \{\pm z \geq 0\}$ and $P_0 = P \cap \{z = 0\}$. Obviously $P_0$ can be understood as part of the boundary of $P_+$ as well as of $P_-$. Now consider

$$P(\phi_0, \phi_1) := (P_+ \sqcup P_-)/_{\sim_\Phi},$$

where we identify with respect to the map $\Phi\colon P_0 \to P_0$ (understanding the domain of definition of $\Phi$ as part of the boundary of $P_+$ and the target space as part of $P_-$) given by

$$\Phi(p; 0, t) := \begin{cases} (\phi_0(p); 0, t) & , \text{ for } t \leq \sqrt{2} - 1, \\ (\phi_1^{-1}(p); 0, t) & , \text{ for } t \geq \sqrt{2} + 1, \\ (p; 0, t) & , \text{ for } |t| \leq \sqrt{2} - 1. \end{cases}$$

Note that, since $\phi_0$ and $\phi_1$ are symplectomorphisms of $(\Sigma, \omega)$ and $\Phi$ keeps the $t$-coordinates fixed $\Omega$ descends to a symplectic form on $P(\phi_0, \phi_1)$ which we will continue to denote by $\Omega$. We are now going to define a Liouville vector field $Z_1$ on $P(\phi_0, \phi_1)$. Let $g, h\colon [-\varepsilon, 0] \to \mathbb{R}$ be the functions satisfying the following properties:

- $g(z) = 0$, for $z \in [-\varepsilon, 0]$ near $-\varepsilon$,

- $g(z) = 1$, for $z \in [-\varepsilon, 0]$ near 0,
- $g'(z) \geq 0$, for each $z \in [-\varepsilon, 0]$,
- $h(z) = g(-\varepsilon - z)$, for each $z \in [-\varepsilon, 0]$,
- $g(z) + h(z) = 1$, for each $z \in [-\varepsilon, 0]$.

Observe that for these functions we have $dh = -dg$. The symplectomorphisms $\phi_0$ and $\phi_1$ can be chosen to be exact (cf. [19]), i.e. for $i = 0, 1$ the equation $\phi_i^* \beta - \beta = d\varphi_i$ defines a function $\varphi_i$ on $\Sigma$, unique up to adding a constant. By the compactness of $\Sigma$ we may assume that $\varphi_0$ takes only negative values whereas $\varphi_1$ takes only positive values — this will be needed to ensure that $Z_1$ will be transverse to $\partial W$. To avoid confusing indices we will write

$$\Phi^* \beta - \beta = d\varphi$$

to summarise these facts. Over $P_-$ we define $Z_1$ to be given as

$$Z_1 = \Big( g(z)\, (T\Phi^{-1})(Y) + h(z)\, Y \Big) + X + g'(z) \varphi(p)\, \partial_t .$$

To show that $Z_1$ is indeed a Liouville vector field we have to take a look at the Lie derivative of $\Omega$ along $Z_1$. With the help of the Cartan formula we compute

$$\begin{aligned}
\mathcal{L}_{Z_1} \Omega &= d\big( g\, \Phi^* \beta + h\, \beta \big) + dz \wedge dt - d\big( g' \varphi\, dz \big) \\
&= \big( dg \wedge (\Phi^* \beta) + dh \wedge \beta + g\, (\Phi^* \omega) + h\, \omega \big) + dz \wedge dt - g'\, d\varphi \wedge dz \\
&= \big( g'\, dz \wedge (\Phi^* \beta) - g'\, dz \wedge \beta + (g + h)\, \omega \big) + dz \wedge dt - g'\, d\varphi \wedge dz \\
&= \big( g'\, dz \wedge d\varphi + \omega \big) + dz \wedge dt - g'\, d\varphi \wedge dz \\
&= \omega + dz \wedge dt \\
&= \Omega .
\end{aligned}$$

Observe that we can extend $Z_1$ over $P_+$ by $Z_0$. In particular $Z_1$ descends to a vector field on $P(\phi_0, \phi_1)$. Set

$$W' := \big\{ (p, z, t) \colon\ \varrho^2 + z^2 + t^2 \leq C \ \text{ and } \ \varrho^2 + z^2 + (t \pm \sqrt{2})^2 \geq 1 \big\}$$

and note that we have $P \subset W'$. Finally we define the symplectic cobordism $W$ by

$$W := (W' \setminus P) \cup P(\phi_0, \phi_1).$$

The boundary of $W$ decomposes as $\partial W = \partial_- W \sqcup \partial_+ W$, where we have

$$\partial_- W = \{ \varrho^2 + z^2 + (t \pm \sqrt{2})^2 = 1 \} \ \text{ and } \ \partial_+ W = \{ \varrho^2 + z^2 + t^2 = C \}.$$

We do not have to worry about the well-definedness of the function $\varrho$ on $P(\phi_0, \phi_1) \subset W$ since $\phi_0$ and $\phi_1$ can be assumed to equal the identity over $(-\varepsilon, 0] \times \partial\Sigma$, which is the only region where $\varrho$ is non-trivial. Observe that the Liouville vector field $Z_1$ is transverse to $\partial W$ pointing inwards along $\partial_- W$ and outwards along $\partial_+ W$. Finally observe that we indeed have $\partial_- W = M_{(\Sigma, \omega, \phi_0)} \sqcup M_{(\Sigma, \omega, \phi_1)}$ and $\partial_+ W = M_{(\Sigma, \omega, \phi_0 \circ \phi_1)}$, which completes the proof. □

**Remark.** The cobordism $W$ constructed in the proof of Theorem 17 can be thought of as the result of attaching a generalised symplectic 1-handle of the form $D^1 \times (\Sigma \times D^1)$ to the symplectization of $M_{(\Sigma,\omega,\phi_0)} \sqcup M_{(\Sigma,\omega,\phi_1)}$.

### 2.2.1 Symplectic fibrations over the circle

Let $(M,\xi)$ be a contact 3-manifold and $\omega$ a closed 2-form on $M$ such that $\omega|_\xi > 0$. Suppose that we are given an open book decomposition of $M$ with binding $B$. Let $M'$ denote the result of Morse surgery along $B$ with the natural zero-framing induced by the pages of the open book. In particular $M'$ is fibered over the circle, where the fibre is the closed surface obtained by capping off the boundary components of the page. Denoting by $W$ the induced cobordism, one of the main results (Theorem 1.1) in [7] states that there is a symplectic form $\Omega$ on $W$ such that $\Omega|_M = \omega$ and $\Omega$ is positive on fibres of the fibration $M' \to S^1$. We now want to utilise the language established in the proof of Theorem 17 to sketch how we can extend the above result in [7] to higher dimensions.

Let $(M,\xi)$ be a closed, oriented, $(2n+1)$-dimensional contact manifold supported by an open book with page $(\Sigma,\omega)$ and monodromy $\phi$. Suppose that $(\Sigma,\omega)$ symplectically embeds into a second $2n$-dimensional (not necessarily closed) symplectic manifold $(\Sigma',\omega')$, i.e.

$$(\Sigma,\omega) \subset (\Sigma',\omega')$$

(For $n=2$ we could, for example, choose $\Sigma'$ to be the closed surface obtained by capping off the boundary components of $\Sigma$). Let $M'$ be the symplectic fibration over the circle with fibre $(\Sigma',\omega')$ and monodromy equal to $\phi$ over $\Sigma \subset \Sigma'$ and equal to the identity elsewhere.

**Corollary 18.** *There is a cobordism $W$ with $\partial W = (-M) \sqcup M'$ and a symplectic form $\Omega$ on $W$ for which $-M$ is a concave boundary component and $\Omega$ induces $\omega'$ on the fibres of the fibration $M' \to S^1$.*

*Proof.* Let $\varrho\colon \Sigma \to [0,\infty]$ be a $C^\infty$-function on $\Sigma$ as in the proof of Theorem 17. We can extend this function by $\infty$ over the rest of $\Sigma'$. Analogous to the proof of Theorem 17 we consider the symplectic space $\Sigma' \times \mathbb{R}^2$ with symplectic form $\Omega = \omega' + dz \wedge dt$. Over $\Sigma \times \mathbb{R}^2 \subset \Sigma' \times \mathbb{R}^2$ we define the Liouville vector field $Z_0 = Y + (z\,\partial_z + t\,\partial_t)$. Let $A$ denote the subset of $\Sigma' \times \mathbb{R}^2$ defined by

$$A := \big\{(p,z,t)\colon\ \varrho \le 0,\ z^2+t^2 \ge 1 \text{ and } z^2+t^2 \le 2\big\}.$$

In analogy of the definition of $P(\phi_0,\phi_1)$ in the proof of Theorem 17 we define $A(\phi)$. Set

$$W' := \big\{(p,z,t)\colon\ \varrho^2+z^2+t^2 \ge 1 \ \text{ and }\ z^2+t^2 \le 2\big\}$$

and note that we have $A \subset W'$. Finally we define the symplectic cobordism $W$ by

$$W := (W' \setminus A) \cup A(\phi).$$

Observe that $\Omega$ descends to a symplectic form on $W$. Furthermore we indeed have $\partial W = (-M) \cup M'$. With $g, h, \varphi$ as in the proof of Theorem 17 we define a Liouville vector field $Z_1$ on $W_{\varrho \leq 0}$ by

$$Z_1 = \big(g(z)\,(T\Phi^{-1})(Y) + h(z)\,Y\big) + (z\,\partial_z + t\,\partial_t) + g'(z)\varphi(p)\,\partial_t.$$

This vector field is transverse to the lower boundary $\partial_- W = M(\Sigma, \omega, \phi)$ pointing inwards. Finally observe that $\Omega$ induces $\omega'$ on the fibres of the fibration $M' \to S^1$. $\square$

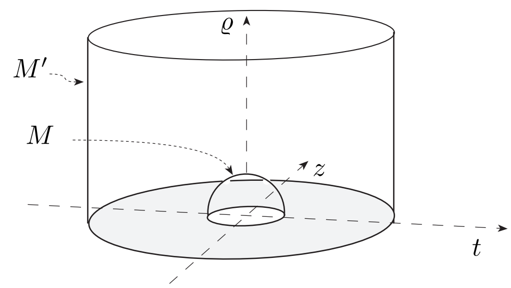


Figure 2.9: Schematic picture of the symplectic cobordism constructed in Corollary 18.

# Chapter 3

# On fibre sums in contact geometry

In the present chapter we describe compatible open books for the fibre connected sum along binding components of open books and for the fibre connected sum along multi-sections of open books. As an application the first description provides a simple way of constructing open books compatible with all tight contact structures on $T^3$ and an open book supporting the result of performing a Lutz twist along a binding component.

We assume that the reader is familiar with the basic notions of braid theory. For a brief introduction we point the reader to [45]. Given a simple closed curve $\alpha$ on some surface $\Sigma$ we denote by $\tau_\alpha$ and $(\tau_\alpha)^{-1}$ respectively the right- and left-handed Dehn twist along $\alpha$. When we deal with the concatenation of Dehn twists we sometimes omit the concatenation symbol "$\circ$" to simplify the notation. In this fashion it makes sense to consider $n$th-powers $(\tau_\alpha)^n$ of Dehn twists, where the zero power $(\tau_\alpha)^0$ is defined to be the identity map.

## 3.1 The fibre connected sum

Let us recall the definition of the fibre connected sum. In this section we closely follow the construction given in [19, Section 7.4]. Let $M, M'$ be two closed, oriented (not necessarily connected) manifolds with $\dim M' < \dim M$. Suppose we are given two embeddings $j_0, j_1 \colon M' \hookrightarrow M$ with disjoint images. Further assume that the normal bundles $N_i$ of the submanifolds $M'_i = j_i(M') \subset M$, $i = 0, 1$, are isomorphic under a fibre-orientation-reversing bundle isomorphism $\Psi$ covering $j_1 \circ j_0^{-1}|_{M'_0}$. Choose a bundle metric on $N_0$, and give $N_1$ the induced metric that turns $\Psi$ into a bundle isometry. Write $\|v\|$ for the corresponding norm of an element $v \in N_i$. Identify $N_i$ with an open tubular neighbourhood of $M'_i$ in $M$, with $N_0, N_1 \subset M$ still disjoint. For any interval $I \subset \mathbb{R}_{\geq 0}$, denote by $N_i^I$ the subset given by $\{v \in N_i : \|v\| \in I\}$.

**Definition 11.** The **fibre connected sum** $\#_\Psi M$ is the closed oriented manifold given as the quotient space

$$\#_\Psi M = \big(M \setminus (N_0^{[0,\varepsilon/2]} \cup N_1^{[0,\varepsilon/2]})/ \sim_\Psi \big),$$

where the identification is given by

$$N_0^{(\varepsilon/2,\sqrt{3}\varepsilon/2)} \ni v \ \overset{\sim}{\leftrightarrow}\ \frac{\sqrt{\varepsilon^2 - \|v\|^2}}{\|v\|} \cdot \Psi(v) \in N_1^{(\varepsilon/2,\sqrt{3}\varepsilon/2)}.$$

Under suitable assumptions, one can form the fibre connected sum of a contact manifold $(M, \xi)$ along contact submanifolds.

**Theorem 19** (Geiges, [19, Theorem 7.4.3])**.** *Let $(M, \xi)$ and $(M', \xi')$ be contact manifolds of dimension $\dim M' = \dim M - 2$, where the contact structures $\xi, \xi'$ are assumed to be cooriented. Let $j_0, j_1\colon (M', \xi') \hookrightarrow (M, \xi)$ be disjoint contact embeddings that respect the coorientations, and such there exists a fibre-orientation-reversing bundle isomorphism $\Psi\colon N_0 \to N_1$ of the normal bundles of $M'_0 = j_0(M')$ and $M'_1 = j_1(M')$. Then the fibre connected sum $\#_\Psi M$ admits a contact structure that coincides with $\xi$ outside tubular neighbourhoods of $M'_0$ and $M'_1$. We will denote the manifold $\#_\Psi M$ endowed with this contact structure by $\#_\Psi(M, \xi)$.* □

In the present thesis we are only interested in fibre sums of codimension 2 submanifolds in contact 3-manifolds. The following subsection gives an explicit description of the construction for the low dimensional case.

### 3.1.1 Alternative description of the fibre connected sum

Let $K \subset (M, \xi)$ be a positively transverse knot in a contact 3-manifold. We may identify a neighbourhood of $K$ with an $\varepsilon$-neighbourhood $N_\varepsilon \subset S^1 \times \mathbb{R}^2$, where $K = S^1 \times \{0\}$. Then, with $S^1$-coordinate $\theta$, polar coordinates $(r, \varphi)$ on $\mathbb{R}^2$, and for a suitable $\varepsilon > 0$, the contact structure

$$d\theta + r^2\, d\varphi = 0$$

provides a model for the above neighbourhood of $K$. Let $M_0 = M \setminus N_\delta$ denote the complement of a $\delta$-neighbourhood $N_\delta \subset N_\varepsilon$, with $0 < \delta < \varepsilon$. Replace the contact structure $\xi$ over $N_\varepsilon \setminus N_\delta$ by the kernel of the contact 1-form $d\theta + f(r)\, d\varphi$, where $f : [\delta, \infty] \to \mathbb{R}$ is a function that equals $r^2$ away from $\delta$, satisfies $f' > 0$, $f'(\delta) = 1$ and $f(\delta) = 0$.

For $s > 0$ let $\xi_s$ denote the contact structure on $[0, s] \times T^2$ given by the kernel of the 1-form $\cos r\, d\theta + \sin r\, d\varphi$. Observe that the function $f$ above is chosen in a way such that we can extend $M_0$ by attaching $([0, s] \times T^2, \xi_s)$. Denote the result of this attachment by $M_s$, i.e. in symbols we have

$$M_s = M_0 \cup ([0, s] \times T^2, \xi_s).$$

Note that for $s \in \mathbb{Q}$ the boundary of $M_s$ is a linearly foliated torus of slope $s$. Furthermore $M_s$ can be understood as sitting in $M_\infty$. Hence the boundary can be perturbed into a convex torus with dividing set $\Gamma_s$ given by two closed curves of slope $s$. Let $\tilde{M}_s$ denote the contact manifold with convex boundary obtained by this perturbation. We end up with a contact manifold with convex boundary which we denote by $\tilde{M}_s$.

**Definition 12.** Suppose we are in the situation described above. For $s = 0$ we will refer to $M_0$ and $\tilde{M}_0$ as obtained by **blowing up** $K$ and **convexly blowing up** $K$ respectively. The inverse operation of (convexly) blowing up will be referred to as **collapsing**.

In the 3-dimensional setting, a pair of codimension 2 submanifolds matching the conditions in Theorem 19 above is just a pair of positive transverse knots $K_0$ and $K_1$. For cohomological reasons the normal bundles $N_0$ and $N_1$ are trivial and fixing a fibre-orientation-reversing bundle isomorphism $\Psi\colon N_0 \to N_1$ between them corresponds to a choice of framings for the knots $K_0$ and $K_1$. If we blow up a transverse knot $K$ (in the sense of Definition 12) which admits some framing we obtain a natural identification of the boundary torus associated to $K$ with $\mathbb{R}^2/\mathbb{Z}^2$ by sending the meridian to the $x$-axis and the framing-direction to the $y$-axis.

We can now describe the fibre connected sum along transverse knots as the following two-step process: start by blowing up the framed knots and finish by identifying the boundary tori with respect to the gluing map sending $(x, y)$ to $(-x, y)$.

## 3.2 An open book supporting the binding sum

In the previous section we introduced the fibre connected sum along codimension 2 contact submanifolds. In this section we proceed by considering two special cases, the fibre connected sum along binding components of open books and the fibre connected sum along sections of open books. Throughout the whole section let $(M, \xi)$ be a closed, not necessarily connected, contact 3-manifold supported by an open book $(\Sigma, \phi)$. Let $B \subset M$ denote the embedded binding of the open book.

Suppose we have chosen the transverse knots $K_0$ and $K_1$ to be components of the binding of the open book decomposition $(\Sigma, \phi)$. Since the pages induce a natural framing for $K_0$ and $K_1$ respectively we can think of it as the zero-framing and hence can measure all other trivialisations relative to it. Note that performing the fibre connected sum with framings $m_1, m_2 \in \mathbb{Z}$ equals the the result of performing the fibre connected sum with framings $\widetilde{m}_1 = m_1 + m_2$ and $\widetilde{m}_2 = 0$. So in the following we just fix one framing assuming the other one to be zero.

**Definition 13.** The result of performing the fibre connected sum along two binding components $K_0$ and $K_0$ with framing $m \in \mathbb{Z}$ will be referred to as **binding sum along** $K_0$ **and** $K_1$ and will be denoted by $\boxplus_m(\Sigma, \phi)$.

Now suppose $K_0$ and $K_1$ are positively transverse knots intersecting every page transversely and exactly once. We will refer to such a knot as a **section of the open book**, since it induces a section of the fibration $M \setminus B \to S^1$. Again understand these knots as endowed with a framing. By nature of the sections we can embed the normal bundle $N_i$, $i = 0, 1$, such that each fibre corresponds to a disc neighbourhood $D_i$ of the intersection point $\{p_i\} = K_i \cap \Sigma$. We will see that fibre connected sums of this kind are nicely adapted to the underlying open book decomposition.

The new fibre is obtained by replacing $D_0 \cup D_1$ by $[-1, 1] \times S^1$. However, the change of monodromy is less obvious. To see how the monodromy changes, consider a vector field transverse to the fibres in $M$ with $K_0$ and $K_1$ as closed orbits such that the return map $h$ on a fibre $\Sigma$ fixes a disc neighbourhood $D_i$ of each $\Sigma \cap K_i$ and such that closed orbits close to $K_0$ and $K_1$ represent the trivialisations of the sections. The new monodromy is equal to $h$ on $\Sigma \setminus (D_0 \cup D_1)$ and the identity on $[-1, 1] \times S^1$.

For our purposes it will be sufficient just to consider **trivial sections**, that is, sections corresponding to a single fix point $p \in \Sigma$ of the monodromy of a given abstract open book $(\Sigma, \phi)$. In this case we obtain natural trivialisations of the normal bundles given by a parallel copy of the knot corresponding to a nearby point. Furthermore we can assume the given monodromy $\phi$ to be the identity on $D_0 \cup D_1$. So by the observations above, the new monodromy will be given by $\phi$ on $\Sigma \setminus (D_0 \cup D_1)$ and the identity on $[0, 1] \times S^1$.

**Lemma 20.** *The contact manifold resulting from the (contact) fibre connected sum along a section of an open book is compatible with the corresponding open book. The analogous statement holds for multi-sections of open books as defined in Section 3.3.*

*Proof.* We can assume, by applying an isotopy of $\xi$ together with a perturbation of the fibration $\pi : M \setminus B \to S^1$ corresponding to $(\Sigma, \phi)$, that the intersections of $K_i$ with the pages $\Sigma_\theta$ correspond to singularities of the characteristic foliation $(\Sigma_\theta)_\xi$: according to [11, Lemma 3.5] the contact structure can be isotoped such that it becomes arbitrarily close (as an oriented plane field) to the pages, outside any open neighbourhood of the binding, and without loosing the compatibility with the open book. By a little perturbation of $\pi$ inside a small neighbourhood of $K$ we can assume the contact planes, over $K$, to be tangent to the pages.

If we perform the fibre connected sum, the resulting contact structure and open book, $\xi'$ and $(\Sigma', \phi')$, are related as follows. As observed in the above discussion the new fibre $\Sigma'$ is obtained by replacing $D_0 \cup D_1 \subset \Sigma$ by $[-1, 1] \times S^1$. Since the origin $0 \in D_i$ corresponds to a singularity of the characteristic foliation $(D_i)_\xi$ it gives rise to a closed leaf in the characteristic foliation $([-1, 1] \times S^1)_{\xi'}$ corresponding to the core $\{0\} \times S^1$ of the annulus. Outside this curve the characteristic foliation agrees with the foliations on $D_i \setminus \{0\}$.

Thus the new contact structure $\xi'$ can be isotoped to be arbitrarily close (as oriented plane fields), on compact subsets of the pages, to the tangent planes to the pages of the open book in such a way, that after some point in the isotopy the contact planes are transverse to $B'$ and transverse to the pages of the open book in a fixed neighbourhood of $B'$ (because this holds for the original open book $(\Sigma, \phi)$). Hence, again according to [11, Lemma 3.5], the contact structure $\xi'$ is supported by the open book $(\Sigma', \phi')$. □

**Remark.** An open book decomposition can be understood as the boundary of an *achiral Lefschetz fibration*. In a similar fashion the fibre sum along sections corresponds to the boundary of a *broken achiral Lefschetz fibration*, see [18] for reference.

Let $K \subset \partial\Sigma$ denote a boundary component of the page $\Sigma$ provided with some framing $m \in \mathbb{Z}$. We will refer to $(K, m)$ as **admitting a navel** if the monodromy near the boundary component is given by $\tau_\alpha \tau_\beta^{-1} \tau_\gamma^{m-1}$, where the curves $\alpha, \beta, \gamma \subset \Sigma$ are given as in Figure 3.1. The transverse knot $K' = (K', 0)$ indicated by the black dot in Figure 3.1 will be referred to as **core of the navel** corresponding to $(K, m)$. The framings are understood as measured with respect to their respective natural zero-framings as explained above. Observe that we can change every boundary component into a navel, since the monodromy $\tau_\alpha \tau_\beta^{-1} \tau_\gamma^{m-1}$ is isotopic to the identity. In the following proposition we will express the binding sum as the fibre sum along the core of its corresponding navel.

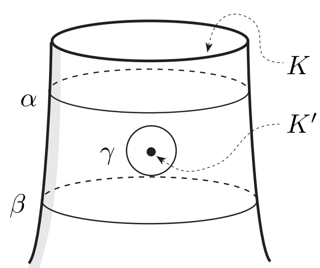


Figure 3.1: Binding component admitting a navel.

**Proposition 21.** *Let $K \subset \partial\Sigma$ be a binding component provided with a framing $m \in \mathbb{Z}$. Then the framed knot $(K, m)$ is transversely isotopic to the corresponding core $(K', 0)$ of its navel. In consequence the result of performing the binding sum with framing $m \in \mathbb{Z}$ along two binding components $K_0, K_1 \subset \Sigma$ corresponds to the fibre sum along the cores $K_0', K_1'$ of their corresponding navels (cf. also Figure 3.2).*

*Proof.* Choose coordinates $(\theta, r, \varphi)$ on a neighbourhood $N \equiv S^1 \times D^2$ of the binding component $K \subset \partial\Sigma$, where $(r, \varphi)$ denote polar coordinates on $D^2$ and $K$ is identified with $S^1 \times \{0\} \subset S^1 \times D^2$. The contact structure over this neighbourhood can be assumed to be given by the kernel $\xi_0$ of the contact 1-form

$$d\theta + r^2 \, d\varphi.$$

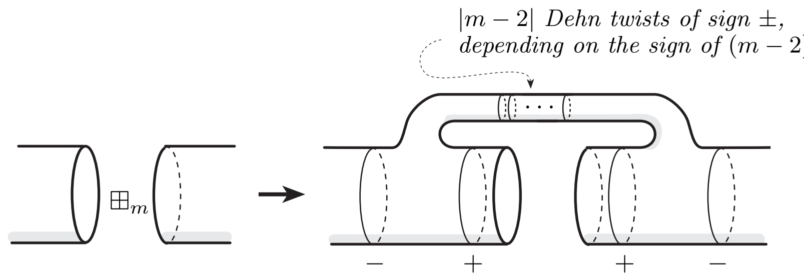


Figure 3.2: An open book supporting the binding sum.

Since $K$ is a binding component of an open book, which implies that the contact planes can be isotoped to be arbitrarily close to the pages away from a little neighbourhood of the binding [11, Lemma 3.5], we can assume the 2-disc $D^2$ to be of radius 2 (or even bigger). Secondly, we can assume that over this neighbourhood the pages are given by the pre images of the projection on the angular-coordinate $\varphi$, i.e. the closure of every page can be described as $A_\varphi = S^1 \times [2, 0] \times \{\varphi\}$ for some appropriate $\varphi \in S^1$.

We will now apply the first part $\tau_\alpha \tau_\beta^{-1}$ of the monodromy of the navel (the twists $\tau_\gamma^{m-1}$ just take care of the framings, but we will come to that later). Let $S^1 \times (\delta, 2] \times S^1$ denote the complement of a $\delta$-neighbourhood $N^{[0,\delta)}$ of $K$ in $S^1 \times D^2$ for some small $\delta > 0$. Consider the map $\phi\colon\ S^1 \times [\delta, 2] \to S^1 \times [\delta, 2]$ defined by

$$\phi(\theta, r) := (\theta +\, h(r), r),$$

where $h : [\delta, 2] \to [0, 1]$ is the function satisfying the following properties:

- $h(r) = 0$ for $r$ near $\delta$ and near 2,
- $h(r) = 1$ on an interval containing $\frac{2+\delta}{2}$,
- $h'(r) \geq 0$ for $r < \frac{2+\delta}{2}$ and
- $h'(r) \leq 0$ for $r > \frac{2+\delta}{2}$.

Note that with respect to the identification $S^1 \equiv \mathbb{R}/\mathbb{Z}$ the map $\phi$ is indeed well-defined. Observe that $\phi$ equals $\tau_\alpha \tau_\beta^{-1}$ and is isotopic to the identity. Consider the corresponding mapping torus $A(\phi)$, that is

$$A(\phi) = \big(S^1 \times [2, \delta] \times [0, 1]\big)/\sim_\phi,$$

where we identify $(\theta + h(r), r, 1)$ with $(\theta + h(r), r, 0)$ for each $(\theta, r) \in S^1 \times [2, \delta]$. Following the construction of Thurston-Winkelnkemper [47] we can endow $A(\phi)$ with the contact structure $\xi_1$ given by the kernel of the contact 1-form

$$\big((1 - \varphi)\, d\theta + \varphi\, \phi^* d\theta\big) + r^2\, d\varphi$$

(actually this defines a contact structure on $S^1 \times [2, \delta] \times [0, 1]$ that descends to a contact structure on $A(\phi)$). Observe that $(A(\phi), \xi_1)$ and $(S^1 \times [\delta, 2] \times S^1, \xi_0)$ are contactomorphic under a contactomorphism keeping little neighbourhoods of the boundary fixed.

Since we have $\phi \simeq \mathrm{id}$, the complement $S^1 \times [2, \delta] \times S^1$ of the $\delta$-neighbourhood $N^{[0,\delta)}$ of $K$ in $S^1 \times D^2$ can be identified with $A(\phi)$. The space $A(\phi)$ is foliated by tori $T_r$ of the form

$$T_r = \big(S^1 \times \{r\} \times [0, 1]\big)/_{\sim_\phi}, \tag{3.1}$$

where we identify $(\theta, r, 1)$ with $(\theta + h(r), r, 0)$ for each $\theta \in S^1$. We can also understand these tori as the quotient of $\mathbb{R}^2$ and the lattice spanned by $(1, 0)$ and $(h(r), 1)$ (in the same manner as we understand $S^1 \times S^1$ as $\mathbb{R}^2/\mathbb{Z}^2$). The characteristic foliation $(T_r)_\xi$ of each torus is given by linear curves of slope $s(T_r) = -\frac{1}{r^2}$, where we measure the slope with respect to the identification as above. Hence any closed curve $c$ on $T_r$ describes a transverse knot as long as the slope of $\dot{c}$ does not equal $-\frac{1}{r^2}$. Now let $K^+$ be the positive, transverse push-off of $K$, i.e. $K^+$ is a linear curve on $T_\delta$ of slope $+1$.

We are now going to define an isotopy $K_s^+$ of transverse knots connecting $K^+$ with the core of the navel $K'$. For $s \in [\delta, \frac{2+\delta}{2}]$ let $c_s : [0, 1] \to [0, 1] \times [0, 1]$ denote the family of embedded curves with the following properties:

- $c_s(0) = (h(s), 0)$,
- $c_s(1) = (1, 1)$,
- $\dot{c}_s \geq 0$ and
- $\dot{c}_s(0) = \dot{c}_s(1) = \pm\infty$.

Observe that each of the curves $c_s$ gives rise to a closed curve $K_s^+$ on $T_s$ (cf. Equation (3.1) above). In particular these knots are transverse, since we have $\dot{c}_s \geq 0 > s(T_s)$. Furthermore we have $K_\delta^+ = K^+$ and $K^+_{\frac{2+\delta}{2}} = K'$.

Let us see what happens to the framing of the knots. Assume that the initial framing of $K$ was $m$. Observe that the framing of the transverse push-off $K^+$ with respect to $T_1$ is given by $m-1$. The isotopy of knots $K_s^+$ does not change the framing at all. Hence at this point we constructed a transverse isotopy connecting $(K, m)$ and $(K', m-1)$. Since the slope of $K'$ equals $\infty$ we can apply the twists $\tau_\gamma^{m-1}$ around $K'$ such that we end up with a zero-framed knot $K'$ and we are done. □

**Example.** Consider two copies of the open book $(D^2, \mathrm{id})$ supporting $S^3$ with the standard contact structure $\xi_{st}$. It is easy to see that the result of the fibre connected sum along the only binding components yields $S^1 \times S^2$ with its standard contact structure. Using the description of a compatible open book for the binding sum in Proposition 21 we obtain the standard open book description of $S^1 \times S^2$ given by an annulus with trivial monodromy.

Before we present some applications of the description given in Proposition 21 we explain a few useful operations to change the appearance of an open book. The following trick is due to Goodman [31]: let $S_3$ denote a 3-holed sphere decorated with curves $\alpha, \beta, \delta \subset S_3$ as given in the left part of Figure 3.3. Suppose one can embed $S_3$ in $\Sigma$ such that the boundary component parallel to the $\delta$-curve corresponds to a binding component of $(\Sigma, \phi)$ and the monodromy restricted to $S_3$ is given by $\tau_\delta \tau_\alpha^{-1} \tau_\beta^{-1}$. Positively stabilising $(\Sigma, \phi)$ as in Figure 3.3 turns $S_3$ into a 4-holed sphere $S_4$ with monodromy $\tau_\delta \tau_\alpha^{-1} \tau_\beta^{-1} \tau_\sigma$. Applying the lantern-relation (sufficiently adjusted) turns the monodromy on $S_4 \subset \Sigma$ into $\tau_{\delta_1} \tau_{\delta_2} \tau_\gamma^{-1}$, where $\delta_1, \delta_2, \gamma \subset S_4$ are the curves given in the right part of Figure 3.3.

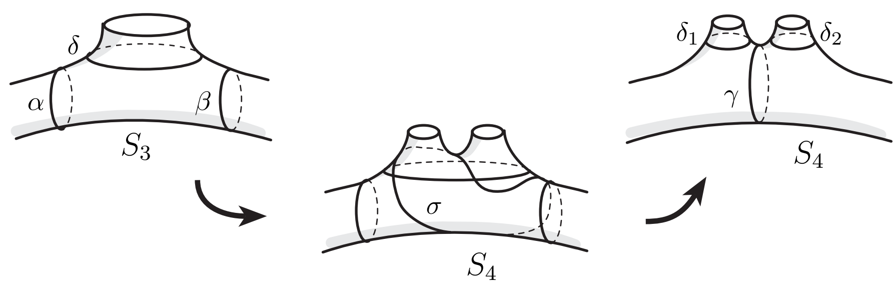


Figure 3.3: A trick to change the appearance of an open book.

**Lemma 22.** *Suppose that on an abstract page of an open book $(\Sigma, \phi)$ you see the constellation as given in*

*(i) the left part of Figure 3.3,*

*(ii) the top left part of Figure 3.4, or*

*(iii) the bottom left part of Figure 3.4.*

*Then, without changing the topology nor the induced contact structure on the corresponding ambient space $M_{(\Sigma,\phi)}$ of the open book, we can replace this part by the constellation as given in*

*(i) the right part of Figure 3.3,*

*(ii) the top right part of Figure 3.4, or*

*(iii) the bottom right part of Figure 3.4.*

*Proof.* The first part is is due to Goodman [31] and is just what we considered in the discussion above. For the remaining parts we only give proofs by picture, cf. Figure 3.4. □

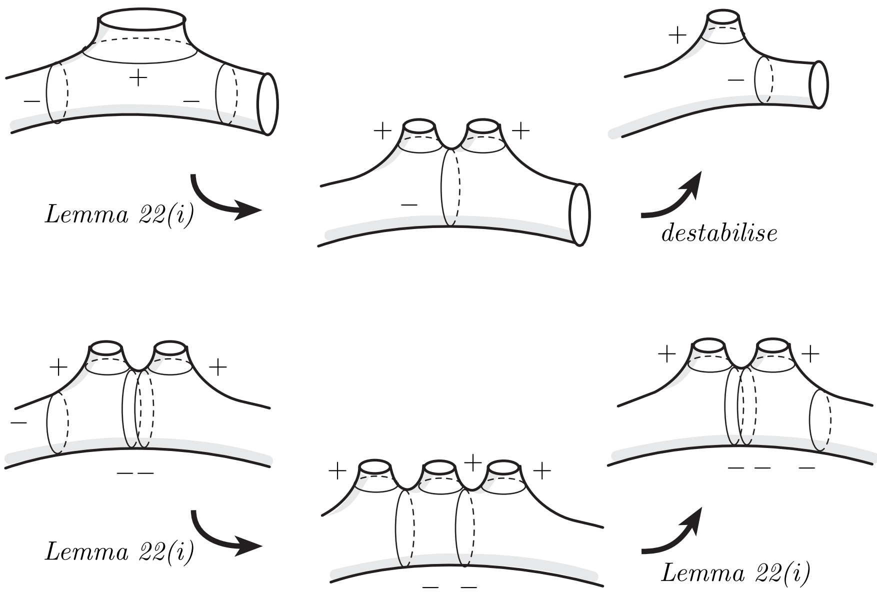


Figure 3.4: Two applications of the trick given in Figure 3.3.

### 3.2.1 Applications

**Tight contact structures on $T^3$**

Let $(\theta_1, \theta_2, \varphi)$ denote coordinates on the 3-dimensional torus $T^3$ and consider the tight contact structure $\xi_n$ given by the kernel of the contact form $\cos(n\varphi)\, d\theta_1 + \sin(n\varphi)\, d\theta_2$. The contact structures $\xi_n$ provide a complete list of tight contact structures on $T^3$ (cf. [40])and can also be described in the following way. Take $2n$ copies of the open book $(S^1 \times [0,1], \mathrm{id})$, which is an open book compatible with the standard contact structure on $S^1 \times S^2$, and then perform the $2n$-fold binding sum in the obvious way. Now using Proposition 21 we are able to translate the above construction of $(T^3, \xi_n)$ into compatible open books. These open books for $(T^3, \xi_n)$ were first computed by Van Horn-Morris [46] using different methods than the ones presented here.

**Full Lutz twist along binding component**

Consecutively performing the binding sum with two copies of the open book $(S^1 \times [0,1], \mathrm{id})$ has the effect of a full Lutz twist along the binding component. Again using Proposition 21 we are able to compute a compatible open book. With the help of Lemma 22 one can show that this open book is stably equivalent to the compatible open book computed in [44]. Obviously we can compute the effect of a regular Lutz twist in the same fashion.

**Recovering Giroux torsion**

Let $(M,\xi)$ be a contact-3-manifold with non-zero Giroux torsion, i.e. if we choose $(\theta_1,\theta_2)$ to be coordinates on $T^2$ there exists an embedding of the contact manifold

$$\Big(T^2\times[0,2\pi],\xi_{2\pi}=\ker\big(\cos(t)\,d\theta_{2\pi}+\sin(t)\,d\theta_2\big)\Big)$$

into $(M,\xi)$. So far, there was no way to recover Giroux-torsion in the language of open books. We approach this question by computing a certain compatible open book for $(M,\xi)$.

Consider the complement $(M,\xi)\setminus\big(T^2\times[0,2\pi],\xi_1\big)$ of the Giroux domain in $(M,\xi)$. The boundary of $(M,\xi)\setminus\big(T^2\times[0,2\pi],\xi_1\big)$ consists of two pre-Lagrangian tori which are foliated by an $S^1$-family of closed curves. Collapsing these tori, in the sense of Definition 12, gives rise to a new closed contact manifold $(M',\xi')$ with two distinguished transverse knots $K_0$ and $K_1$. In this particular case we decorate these knots with the framing corresponding to the $\theta_1$-coordinate. Let $(\Sigma',\phi')$ denote a compatible open book decomposition of $(M',\xi')$ such that $K_0$ and $K_1$ are part of the binding $\partial\Sigma'$. Let $m,n\in\mathbb{Z}$ be the above framings (induced by $\theta_1$) expressed with respect to the page-framing induced by the open book $(\Sigma',\phi')$.

Observe that if we take two copies of the open book $(S^1\times[0,1],\mathrm{id})$, perform the binding sum along $S^1\times\{0\}$ in each copy of $(S^1\times[0,1],\mathrm{id})$ and blow up the two remaining components corresponding to $S^1\times\{1\}$ in each copy we end up with $\big(T^2\times[0,2\pi],\xi_{2\pi}\big)$. Hence performing the 2-fold binding sum of $(\Sigma',\phi')$ with $(S^1\times[0,1],\mathrm{id})\boxplus(S^1\times[0,1],\mathrm{id})$ along $K_0$ and the first copy of $S^1\times\{1\}$ and along $K_1$ and the second copy of $S^1\times\{1\}$ actually gives us a description of $(M,\xi)$ which, using Proposition 21, may be translated into an open book.

**Surface bundles with invariant dividing set**

Suppose we are given a closed (abstract) convex surface $(\Sigma,\Gamma)$ and a diffeomorphism $\phi$ of $\Sigma$ that restricts to the identity in a neighbourhood $N(\Gamma)\subset\Sigma$ of the (abstract) dividing set $\Gamma$. Note that $(\Sigma^+,\phi|_{\Sigma^+})$ and $(\Sigma^-,\phi|_{\Sigma^-})$ are both open books with binding $\Gamma$. Observe that the surface bundle $(\Sigma,\Gamma)(\phi)$ described in Subsection 2.1.2 can be understood as (zero-framed) binding sum of $(\Sigma^+,\phi_{\Sigma^+})$ and $(\Sigma^-,\phi_{\Sigma^-})$. Hence we can use Proposition 21 to compute compatible open books for $(\Sigma,\Gamma)(\phi)$.

## 3.3 Fibre connected sum along multi-sections

In this section we try to approach the following question. Assume we are given two knots $K_0$ and $K_1$ in the 3-dimensional sphere $S^3$ which are braided over the unkot $U\subset S^3$. Furthermore we assume the knots to have the same braid index, $n\in\mathbb{N}$ say. Now recall the standard open book description of $S^3$ with binding the unknot and pages diffeomorphic to the 2-disc and note that each of the knots $K_0$ and $K_1$ provide an $n$**-fold section of of the open book**, i.e. each of the knots intersects every page transversely and exactly $n$ times. A representation of the knots $K_0$ and $K_1$ as braids endows them with a natural framing given by the the blackboard framing. Taking two copies of $(D^2, \mathrm{id})$ we can perform the fibre connected sum along $K_0$ and $K_1$ and ask for a description of the resulting open book

$$(\Sigma, \phi) := (D^2, \mathrm{id}) \#_{K_0,K_1} (D^2, \mathrm{id}). \tag{3.2}$$

Obviously the page $\Sigma$ will be the $n$-fold connected sum of the two original pages. However it is not clear what the monodromy $\phi$ looks like. This question will be settled in the following two subsections.

### 3.3.1 Monodromy corresponding to a pair of crossings

Before we dive into the description of $(\Sigma, \phi)$ we first set up some notation and define a relative version of the fibre connected sum. For a, not necessarily connected, manifold $M$ with non-empty boundary $\partial M$ and two collections of properly embedded, oriented, framed arcs $\boldsymbol{a} = \{a_1, \dots, a_k\}$ and $\boldsymbol{a}' = \{a'_1, \dots, a'_k\}$ with neighbourhoods $N_{\boldsymbol{a}}$ and $N_{\boldsymbol{a}'}$ we denote by $\#_{\boldsymbol{a}} M$ the manifold

$$\#_{\boldsymbol{a}} M := \big(M \setminus (N_{\boldsymbol{a}} \cup N_{\boldsymbol{a}'})\big)/_{\partial N_{\boldsymbol{a}} \sim \partial N_{\boldsymbol{a}'}},$$

where we identify as follows: for $i = 1, \dots, k$ the framing together with the orientation induce identifications of both components $N_{a_i} \subset N_{\boldsymbol{a}}$ and $N_{a'_i} \subset N_{\boldsymbol{a}'}$ with $[0,1] \times D^2$. Now we identify $(t, \theta) \in [0,1] \times \partial D^2 \subset \partial N_{a_i}$ with $(t, -\theta) \in [0,1] \times \partial D^2 \subset \partial N_{a'_i}$.

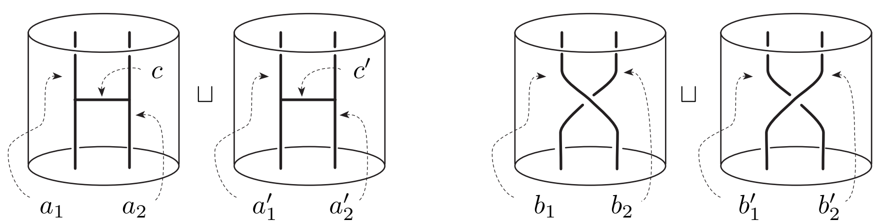


Figure 3.5: The arcs $\boldsymbol{a}, \boldsymbol{a}', \boldsymbol{b}, \boldsymbol{b}', c, c' \subset \big(D \times [0,1]\big) \sqcup \big(D' \times [0,1]\big)$.

From now on let $M$ be the disjoint union of $D \times [0,1]$ and $D' \times [0,1]$, in symbols

$$M = \big(D \times [0,1]\big) \sqcup \big(D' \times [0,1]\big),$$

where $D$ and $D'$ respectively denote a copy of the 2-disc $D^2$. Consider the two sets of properly embedded, framed (by the blackboard-framing) arcs $\boldsymbol{a}, \boldsymbol{a}', \boldsymbol{b}, \boldsymbol{b}' \subset M$ indicated in Figure 3.5. Understand these arcs as oriented upwards and consider the corresponding manifolds $\#_{\boldsymbol{a}} M$ and $\#_{\boldsymbol{b}} M$. Furthermore let $K, K' \subset$

$\#_{\boldsymbol{a}}M$ denote the framed knots indicated in Figure 3.6. The framings are measured with respect to $\Sigma' \subset \#_{\boldsymbol{a}}M$, the genus-1 surface with two boundary components obtained by the 2-fold connected sum of $D$ and $D'$. Observe that $\#_{\boldsymbol{a}}M$ is naturally diffeomorphic to $\Sigma' \times [0,1]$.

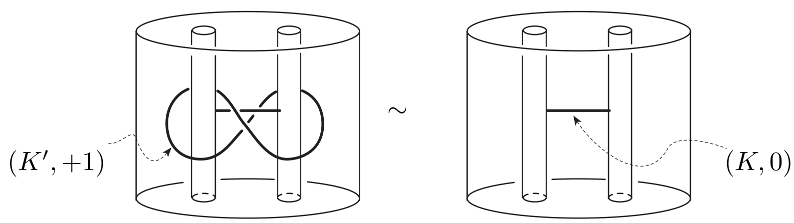


Figure 3.6: The knots $K$ and $K'$ sitting in $\#_{\boldsymbol{a}}M$. Framings are measured with respect to $\Sigma' \subset \#_{\boldsymbol{a}}M$.

**Lemma 23.** *Denote by $(\#_{\boldsymbol{a}}M)(K,K')$ the result of surgery along $K_0, K_1$ with respect to their framings (cf. Figure 3.6). Then we have*

$$\#_{\boldsymbol{b}}M \cong (\#_{\boldsymbol{a}}M)(K,K').$$

*Proof.* Let us give an explicit description of $M = \big(D \times [0,1]\big) \sqcup \big(D' \times [0,1]\big)$ embedded in $\mathbb{R}^3$ with coordinates $(x,y,z)$: understand $D$ and $D'$ as unit 2-discs in the $xy$-plane centred at the points $(0,2)$ and $(0,-2)$. Identify $\boldsymbol{a}, \boldsymbol{a}'$ with $\{(\pm\frac{1}{2}, 2)\} \times [0,1]$ and $\{(\pm\frac{1}{2}, -2)\} \times [0,1]$. Denote by $c, c'$ the arcs given as $[-\frac{1}{2}, \frac{1}{2}] \times \{\pm 2\} \times \{\frac{1}{2}\}$. Then $\#_{\boldsymbol{a}}M$ is given as the quotient

$$\Big(\big(D \times [0,1]\big) \setminus N_{\boldsymbol{a}}\Big) \cup \Big(\big(D' \times [0,1]\big) \setminus N_{\boldsymbol{a}'}\Big) / \sim_{\partial N_{\boldsymbol{a}}},$$

where we identify points $(x,y,z) \in \partial N_{\boldsymbol{a}}$ with their mirror image $(x,y,-z) \in \partial N_{\boldsymbol{a}'}$. Note that the pair of arcs $c, c'$ descends to the closed curve $K \subset \#_{\boldsymbol{a}}M$.

Denote by $H \subset M$ a neighbourhood of the graph $(a_1 \cup a_2) \cup c$. Choose $H'$ to be the reflection of $H$ with respect to the $xz$-plane and note that $H'$ provides a neighbourhood of the graph $(a_1' \cup a_2') \cup c'$. Observe that the result $(\#_{\boldsymbol{a}}M)(K)$ of zero-surgery along $K$ is given by

$$\Big(\big(D \times [0,1]\big) \setminus H\Big) \cup \Big(\big(D' \times [0,1]\big) \setminus H'\Big) / \sim_{\partial H},$$

where we identify a point $(x,y,z) \in \partial H$ with its mirror $(x,-y,z) \in \partial H'$.

Let us now perform the surgery along $K'$. Isotope $K'$ such that it lies on $\partial H$ sitting inside of $\#_{\boldsymbol{a}}M$. Note that the framing of $K'$ and the framing induced by $\partial H$ agree. Let $\nu K' = (-\varepsilon, \varepsilon) \times S^1$ denote a small open neighbourhood of

$K'$ in $\partial H$. Remove a neighbourhood $N_{K'} \subset (\#_{\boldsymbol{a}} M)(K)$ of $K'$ and observe that topologically the complement of $N_{K'}$ is given by

$$\Big(\big(D \times [0,1]\big) \setminus H\Big) \cup \Big(\big(D' \times [0,1]\big) \setminus H'\Big) / \sim_{\partial H \setminus \nu K'}, \qquad (3.3)$$

where we just identify points $(x, y, z) \in \partial H \setminus \nu K'$ with their mirror $(x, -y, z) \in \partial H'$. We will glue back the surgery torus $S^1 \times D^2$ in two steps. Take $[0, \pi] \times D^2 \subset S^1 \times D^2$ (where we identify $S^1 \equiv \mathbb{R}/2\pi\mathbb{Z}$) and attach it along $[0, \pi] \times \partial D^2$ to the closure of the neighbourhood $\nu K' \subset \partial H$, which we identify with $[-\varepsilon, \varepsilon] \times S^1$. Simultaneously attach $[\pi, 2\pi] \times D^2$ along $[\pi, 2\pi] \times \partial D^2$ to the mirror image of $\nu K'$ on $\partial H'$. We can actually picture this to be done inside of $H$ and $H'$ respectively. Observe that the two pieces $\big([0, \pi] \times D^2\big)$, $\big([\pi, 2\pi] \times D^2\big)$, attached to the complement described in description (3.3) above, really descend to a solid torus. Moreover observe that we can understand the boundary of $N_{\boldsymbol{b}}$ as decomposes as $(\partial H \setminus \nu K') \cup (\{0, \pi\} \times D^2)$. Hence gluing back the surgery torus to the space given in (3.3) gives

$$\Big(\big(D \times [0,1]\big) \setminus N_{\boldsymbol{b}}\Big) \cup \Big(\big(D' \times [0,1]\big) \setminus N_{\boldsymbol{b}'}\Big) / \sim_{\partial N_{\boldsymbol{b}}},$$

which describes $\#_{\boldsymbol{b}} M$. This completes the proof. □

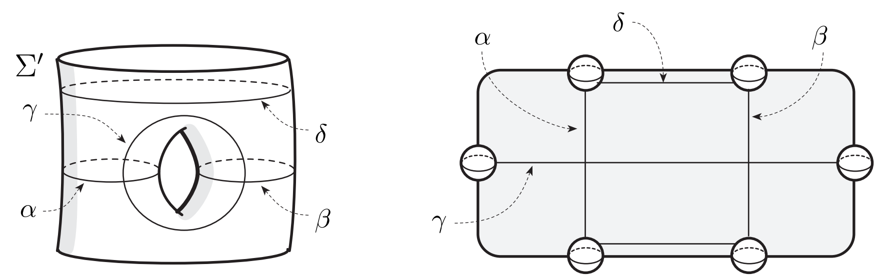


Figure 3.7: Two identifications of $\Sigma'$ with the curves $\alpha, \beta, \gamma, \delta$ used in Lemma 24.

Recall that $\Sigma'$ denotes the genus-1 surface with two boundary components understood as obtained by the 2-fold connected sum of $D$ and $D'$. Recall further $\#_{\boldsymbol{a}} M$ is naturally isomorphic to $\Sigma' \times [0, 1]$. We would now like to express the surgery along $K, K' \subset \#_{\boldsymbol{a}} M$ in the above description of $\#_{\boldsymbol{b}} M$ as a sequence of $\pm 1$-surgeries along certain curves on $\Sigma'$, where the framing is measured with respect to $\Sigma'$.

**Lemma 24.** *Let $\alpha, \beta, \gamma, \delta \subset \Sigma'$ denote the curves described in Figure 3.7. Then setting $\psi = (\tau_\alpha \tau_\beta \tau_\gamma)^2 (\tau_\delta)^{-1}$ we have*

$$(\#_{\boldsymbol{a}} M)(K, K') = \Big(\big(\Sigma' \times [0, 1]\big) \cup \big(\Sigma' \times [2, 3]\big)\Big) / \sim_\psi,$$

*where we identify $(p, 1)$ with $(\psi(p), 2)$ for each $p \in \Sigma'$.*

*Proof.* Recall that in Lemma 23 we identified $(\#_{\boldsymbol{a}}M)(K,K')$ with $\#_{\boldsymbol{b}}M$. Observe that the latter space admits the structure of a $\Sigma'$-fibration which is induced by the projection on the unit interval $[0,1]$. The map $\psi$ is actually a factorisation of the monodromy of this fibration into Dehn twists. A little caution is needed: unfortunately we are actually computing the inverse of $\psi$, since in our computations we push arcs from the top to the bottom, not the other way round.

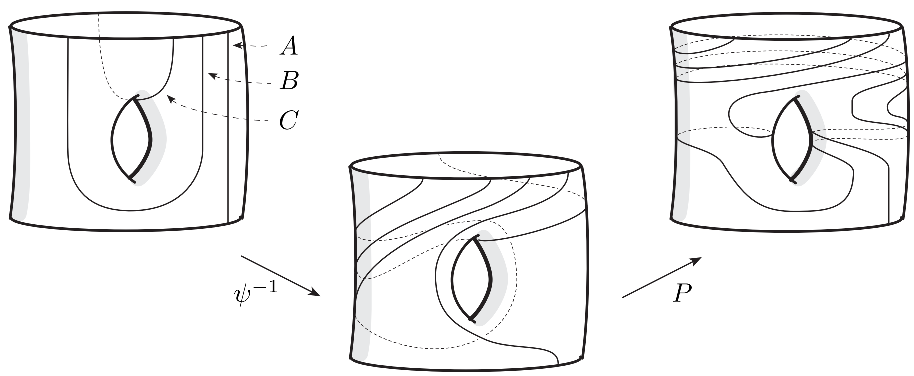


Figure 3.8: A cut system for $\Sigma'$ and its image under $P \circ \psi^{-1}$.

Let $A, B, C \subset \Sigma'$ denote the cut system given in the left part of Figure 3.8. The images of this cut system under $\psi^{-1}$ are given in the middle part of Figure 3.8. The images were computed as follows: recall that $\#_{\boldsymbol{a}}M$ did correspond to $\Sigma' \times [0,1]$ which we understand as obtained by thicken up the shaded area in Figure 3.7 (or Figure 3.9 respectively). A description of the knots $K, K'$ with respect to this perspective is given in Figure 3.9. Understand the result of surgery $(\#_{\boldsymbol{a}}M)(K,K')$ on $K, K'$ as embedded in the Kirby diagram given in Figure 3.9. We can now recover the cut system, chosen above, in the Kirby diagram and manipulate it using Kirby calculus. The actual computations are given in Figures A.1, A.2 and A.3 on p. 69ff in Appendix A.

We could now just compare these images with the ones under the inverse of $(\tau_\alpha\tau_\beta\tau_\gamma)^2(\tau_\delta)^{-1}$ and conclude that both agree up to isotopy, showing that $\psi = (\tau_\alpha\tau_\beta\tau_\gamma)^2(\tau_\delta)^{-1}$. However the usual way to compute the factorisation of

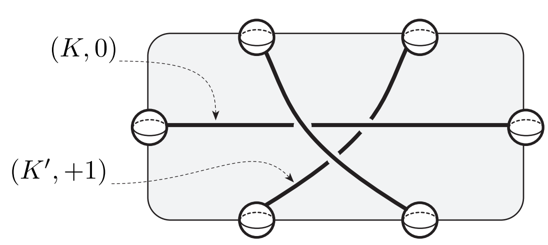


Figure 3.9: A Kirby diagram showing $K, K'$ and $\Sigma'$.

a monodromy map of a compact surface is by by reducing it to the case of a self-diffeomorphism of a disc with punctures, cf. [45]. Referring to Figure A.4 (see Appendix A on p. 72) the image of $\alpha$ under $\psi^{-1}$ is given as $\beta$. Set

$$P = \tau_\gamma\, \tau_\beta\, \tau_\alpha\, \tau_\gamma$$

and note that $P$ maps $\beta$ to $\alpha$. Therefore $P \circ \psi^{-1}$ fixes the curve $\alpha$ and hence can now be interpreted as a self-diffeomorphism of the 3-fold punctured disc $D_3$ obtained by cutting $\Sigma'$ along $\alpha$. Note that $A, B \subset \Sigma'$ descends to a cut system of $D_3 = \Sigma' \setminus \alpha$. Therefore all the data of $P \circ \psi^{-1}$ is encoded in the images of $A, B \subset \Sigma'$. The images of $A, B \subset \Sigma'$ under $P \circ \psi^{-1}$ are given in the right part of Figure 3.8 (cf. Figure A.5 and Figure A.6 in Appendix A for the actual computations). We conclude that we have

$$P \circ \psi^{-1} = \tau_\alpha^{-1}\, \tau_\beta^{-1}\, \tau_\delta.$$

Therefore $\psi^{-1}$ is given by $(\tau_\alpha^{-1}\, \tau_\beta^{-1}\, \tau_\delta) \circ P^{-1}$, which, computing the inverse, is exactly what we intended to show. □

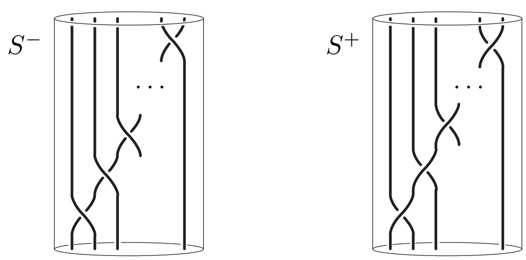


Figure 3.10: Description of the standard braids.

With this in hand we are able to compute the monodromy $\phi$ for the case that $K_0$ and $K_1$ are chosen among the standard braids $S^+, S^-$ given in Figure 3.10.

**Corollary 25.** *Let $\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_{n-1}, \delta^{(\prime)}_{1,2}, \ldots, \delta^{(\prime)}_{n-1,n} \subset \Sigma$ denote the curves indicated in Figure 3.11. Then we have*

*(i)* $\phi = \prod_{i=1}^{n-1} \tau_{\gamma_i}\, \tau_{\alpha_i}\, \tau_{\alpha_{i+1}}\, \tau_{\gamma_i}$, *for the pair of knots* $(K_0, K_1) = (S^+, S^+)$,

*(ii)* $\phi = \prod_{i=1}^{n-1} (\tau_{\gamma_i}\, \tau_{\alpha_i}\, \tau_{\alpha_{i+1}})^2\, (\tau_{\delta_{i,i+1}})^{-1}$, *for the pair of knots* $(K_0, K_1) = (S^-, S^+)$ *and*

*(iii)* $\phi = \prod_{i=1}^{n-1} \tau_{\alpha_i}\, \tau_{\alpha_{i+1}}\, (\tau_{\gamma_i}\, \tau_{\alpha_i}\, \tau_{\alpha_{i+1}})^2\, (\tau_{\delta_{i,i+1}})^{-1}\, (\tau_{\delta'_{i,i+1}})^{-1}$, *for the pair of knots* $(K_0, K_1) = (S^-, S^-)$.

*Proof.* We start by proving the second part of the statement. Let $n \in \mathbb{N}$ be the braid index of $K_0$ and $K_1$ respectively. Consider two copies $L_0, L_1$ of the

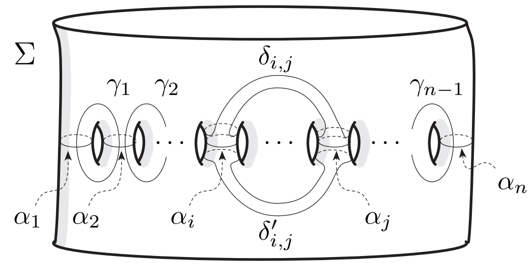


Figure 3.11: Definition of the curves used in Corollary 25 and Proposition 26.

trivial braid of $n$-strands describing an $n$-component unlink and perform a fibre connected sum for each pair of unknots. By applying Lemma 23 exactly $n-1$ times we can turn this $n$-fold fibre connected sum into the fibre connected sum along $S^+$ and $S^-$. Keeping track of the change of monodromy completes the proof of the second part.

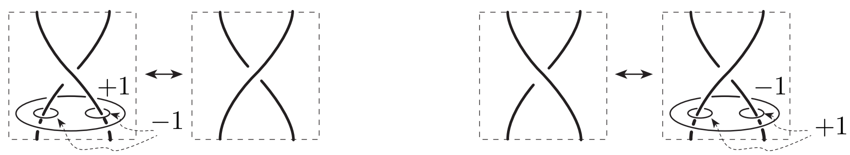


Figure 3.12: Using a Rolfsen twist we can switch between positive and negative crossings.

In Lemma 23 we are considering the result of fibre summing a negative crossing (the arcs $a_1, a_2$) with a positive one (the arcs $b_1, b_2$). Perform a surgery as indicated in the left part of Figure 3.12 (or right part of Figure 3.12 respectively) and observe that we turned the negative (or positive respectively) crossing into a positive (or negative respectively) crossing. This actually is nothing but a certain Rolfsen twist. However by performing one of these surgeries we can always set up the situation for which Lemma 23 applies. Translating the surgery into the language of Dehn twists sets the way to compute the monodromy for the remaining cases and we are done. □

### 3.3.2 Final computation of the monodromy

We almost have everything in place to compute the monodromy map $\phi$ of the fibre sum along multi-sections (see description (3.2) at the beginning of the section). The last ingredient is the following normal form for a braided knot $K$. Let $B$ be a braid representation of $K \subset S^3$ with braid index $n \in \mathbb{N}$ and let $S = S^+$ be the positive standard braid indicated in the right part of Figure 3.10. By an isotopy of $K$ we may assume that the permutation induced by $B$ is given by $(n\ 1\ \ldots\ n-1)$. Therefore $P = B * S^{-1}$ describes a pure braid for which we obviously have $B = P * S$. Here "$*$" denotes the composition of braids in the braid group.

According to [45] one can assign a pure braid to a diffeomorphism of the $n$-fold punctured disc, equal to the identity near the boundary, and vice versa. Let $\phi_K$ denote the map corresponding to the pure braid $P$ (which itself, by the consideration above, is induced by $K$ ). Note that the map $\phi_K$ encodes all information about $K$.

Let us return to the open book description $(\Sigma, \phi)$ of the fibre sum along $K_0$ and $K_1$ (cf. (3.2) above). Denote by $\phi_{K_0}$ and $\phi_{K_1}$ the maps associated to the knots $K_0$ and $K_1$ as described above. These maps trivially extend to $\Sigma$. Together with the first part of Corollary 25 we finally obtain the following description of $\phi$.

**Proposition 26.** *The monodromy $\phi$ of the open book described in (3.2) is given by*

$$\phi = \Big( \prod_{i=1}^{n-1} \tau_{\gamma_i}\, \tau_{\alpha_i}\, \tau_{\alpha_{i+1}}\, \tau_{\gamma_i} \Big) \circ \phi_{K_0} \circ \phi_{K_1},$$

*where $\phi_{K_0}$, $\phi_{K_1}$ are as described above and $\alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_{n-1} \subset \Sigma$ denote the curves indicated in Figure 3.11.* □

# Chapter 4

# Legendrian knots and their complements

## 4.1 Plane fields on knot complements

Given a Legendrian knot $L$ with standard neighbourhood $N$ in a closed contact 3-manifold $(M, \xi)$, the homotopy type of the contact structure $\xi|_{M \setminus N}$ on the knot complement depends on the rotation number of $L$. Here the homotopy is assumed to be stationary over the boundary of $M \setminus N$. This folklore result appeared in the literature, see for example [8, Section 4.1], though details of the argument have not appeared. Recently details of the argument appeared in a preprint of Etnyre [12]. Alternative arguments were told to the author, in private communication, by Geiges. In the present section we present another alternative approach using the Pontryagin construction of maps to $S^2$. In Section 4.1.1 we outline how this fact is used in [8] to coarsely (i.e. up to a global coorientation preserving contact diffeomorphism) classify loose Legendrian knots by their classical invariants.

Let $L \subset M$ be a homologically trivial knot with Seifert surface $\Sigma$, i.e $L = \partial\Sigma$. Let $\xi_0, \xi_1 \subset TM$ be two contact structures, such that $L$ is Legendrian with respect to both of the contact structures and such that $\xi_0|_L$ and $\xi_1|_L$ agree. Assuming that the contact structures $\xi_0$ and $\xi_1$ are homotopic as plane fields we ask whether we can find a homotopy stationary on $L$. If the Legendrian knot $L$ has the same rotation number with respect to both of the contact structures, the question can be positively answered.

**Lemma 27.** *Suppose we are in the situation described above. Then the restrictions $\xi_0|_\Sigma$ and $\xi_1|_\Sigma$ are homotopic as plane fields relative to the boundary $L = \partial\Sigma$.*

*Proof.* Choose a Riemannian metric on $M$ and a trivialisation of the tangent bundle $TM$. Then, for $i = 0, 1$ the 2-plane field $\xi_i|_\Sigma$ can be described in terms of its corresponding Gauß map $f_i : \Sigma \to S^2$, assigning to each $x \in M$ the positive normal vector to $\xi_i|_\Sigma(x)$. Since $\xi_0|_\Sigma$ and $\xi_1|_\Sigma$ agree over $L$ so do their corresponding Gauß maps and we obtain a map

$$f : \Sigma \cup_L \bar{\Sigma} \to S^2$$

by "gluing" together $f_0$ and $f_1$. In the same fashion we obtain a plane bundle $\xi$ over $\Sigma \cup_L \bar{\Sigma}$. Actually we obtain a trivial bundle — this is where the rotation numbers come into play. To see this choose a nowhere vanishing section $s$ in $\xi_0|_L = \xi_1|_L$ making $-\mathtt{rot}(L, [\Sigma])$ twists relative to the positive tangent vector of $L$ as we go around $L$. By definition of the rotation number, this section extends to nowhere vanishing sections $s_0, s_1$ in $\xi_0|_\Sigma$ and $\xi_1|_\Sigma$. In turn these two sections yield a nowhere vanishing section in $\xi$. Therefore $\xi$ is a trivial bundle.

We now want to relate the degree $\mathtt{deg} f$ of the map $f$ to the Euler class $\mathrm{e}(\xi)$ of the plane bundle $\xi$. Consider the tangent bundle $TS^2$ over the 2-sphere. Observe that with respect to $f$ this bundle pulls back to $\xi$, i.e.

$$\xi = f^* TS^2.$$

Therefore we have $\mathrm{e}(\xi) = f^*\mathrm{e}(TS^2)$. Recalling the characterisation of the mapping degree we now are able to compute

$$\begin{aligned} \mathtt{deg} f \cdot \int_{S^2} \mathrm{e}(TS^2) &= \int_{\Sigma \cup_L \bar{\Sigma}} f^*\mathrm{e}(TS^2) \\ &= \int_{\Sigma \cup_L \bar{\Sigma}} \mathrm{e}(\xi) \\ &= 0. \end{aligned}$$

Hence the degree $\mathtt{deg} f$ must be zero since $\int_{S^2} \mathrm{e}(TS^2)$ is not.

Now we are ready to construct the desired homotopy. Choose a regular value $y \in S^2 \setminus f(L)$ of $f$ in the complement of the image of $L$. In particular we can choose $y$ to be a regular value for both of the maps $f_0$ and $f_1$. The Pontryagin manifolds $f_0^{-1}(y)$ and $f_1^{-1}(y)$ of $f_0$ and $f_1$ respectively are a finite collection of signed points on $\Sigma$ and the mapping degree $\mathtt{deg} f$ corresponds to the signed count of these points. Since $\mathtt{deg} f$ is zero there exists a framed cobordism $X \subset \Sigma \times [0,1]$ between the Pontryagin manifolds $f_0^{-1}(y)$ and $f_1^{-1}(y)$ of $f_0$ respectively $f_1$. The Pontryagin construction then yields a map

$$F : \Sigma \times [0,1] \to S^2,$$

which induces the Pontryagin manifolds given above at the ends $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$, see [43]. Note that this is not a homotopy between $f_0$ and $f_1$ so far, $F$ just induces the same Pontryagin manifolds as $f_0$ and $f_1$ on the ends. The construction yields a map $F$ which takes a constant value say $-y \in S^2$ in a neighbourhood $\mathcal{N}$ of the boundary $\partial\Sigma \times [0,1]$. Since $S^2$ is simply connected we can contract the closed curve given by $f_0|_L = f_1|_L$ to the point $-y \in S^2$ — note that this can be done in the complement of $y \in S^2$. Let $\gamma_t : L \to S^2$ denote this contraction. This allows us to change the map $F$ into a map which is stationary and equal to $f_0|_L = f_1|_L$ over $L$ as follows. Just turn $F|_{\mathcal{N}}$, by redefining it, into $F|_{\mathcal{N}}(x,s,t) = \gamma_{(1-s)}(x)$, where we identify $\mathcal{N} \equiv \partial\Sigma\times[0,1)\times[0,1]$. This does not affect the feature concerning the Pontryagin manifolds, since the contraction is applied in the complement of $y$. Now the induced maps $F_0$ and $F_1$ on the ends of $\Sigma\times[0,1]$ are homotopic to the maps $f_0$ and $f_1$ respectively, via a homotopy fixed on $L$, see [43]. □

**Remark.** Lemma 27 still holds without the condition on the contact structures $\xi_0$ and $\xi_1$ to be homotopic as plane fields.

**Proposition 28.** *Suppose we are in the situation as in Lemma 27. Then the contact structures $\xi_0$ and $\xi_1$ are homotopic as plane fields relative to $L$.*

*Proof.* Again we understand tangent plane fields in terms of their corresponding Gauß maps and denote by $f_0$ and $f_1$ the Gauß maps of $\xi_0$ and $\xi_1$ respectively. Choose a regular value $y\in S^2\setminus f_0(L)$ of both $f_0$ and $f_1$ in the complement of the image of $L$ and denote by $X\subset M\times[0,1]$ the framed cobordism between the corresponding Pontryagin manifolds $f_0^{-1}(y)$ and $f_1^{-1}(y)$. The existence of the framed cobordism follows from the fact that $\xi_0$ and $\xi_1$ (hence $f_0$ and $f_1$) were supposed to be homotopic as plane fields. Perturb $X$ such that it becomes transverse to the submanifold $Y = L\times[0,1]$. Since both manifolds are of dimension two and compact, the transverse intersection $X \pitchfork Y$ is of dimension zero, and hence a finite collection of points. Actually we end up with a signed collection of finite points by comparing the orientation of $X$ followed by the orientation of $Y$ for each intersection point $p\in X\pitchfork Y$ with the orientation of the ambient space $M\times[0,1]$. Note that this is the same as comparing the orientation of the framing of $X$ and the orientation of $Y$. Choose a normal neighbourhood $N_Y\equiv Y\times D^2$ of $Y$ in $M\times I$, such that for every point $p\in X\pitchfork Y$ the fibre $\{p\}\times D^2$ yields a neighbourhood of $p$ in $X$.

Let us assume that the signed count of the intersection points $X\pitchfork Y$ is zero. Choose a pair $p,p'\in X\pitchfork Y$ of intersection points of opposite sign and let $\gamma\subset Y$ be an injective arc connecting them. For technical reasons we will parametrise $\gamma$ by $[-1,1]$ with coordinate $t$. Cut out the neighbourhoods $\{p\}\times D^2$ and $\{p'\}\times D^2$ described above and connect them up by a tube given by $\mathrm{Im}\gamma\times\partial D^2$. After rounding edges we will end up with another cobordism $X'$ between the given Pontryagin manifolds, not yet framed over $\mathrm{Im}\gamma\times\partial D^2$ and having higher genus. Moreover $X'$ is disjoint from $Y$. Observe that the framing of $X$ extends to a framing of $X'$: write $\{\partial_x,\partial_y\}$ for a positive oriented basis of $T_{(x,y)}D^2$ and let $n$ be a vector field over $\mathrm{Im}\gamma$ such that $\{\dot\gamma, n\}$ defines a positive oriented basis of $T_{\gamma(t)}Y$. Then, since $p$ and $p'$ are of opposite signs, the framing of $X$ at $p$ can be assumed to be given by $\{\dot\gamma,n\}$ and by $\{-\dot\gamma,n\}$ at $p'$. We can further assume that the framing is constant over the neighbourhoods $\{p\}\times D^2$ and $\{p'\}\times D^2$ of $p$ and $p'$ respectively. A framing on $\mathrm{Im}\gamma\times D^2$ that extends the given one on $X\setminus\big(\{p,p'\}\times D^2\big)$ is given by

$$\{t\,(-\dot\gamma)+(1-|t|)\,(x\,\partial_x+y\,\partial_y),\ n\},$$

where $|t|$ should actually be read as smooth approximation of the absolute value. We end up with a new framed cobordism $X'$ which induces the given Pontryagin manifolds $f_0^{-1}(y)$ and $f_1^{-1}(y)$ on $M \times \{0,1\}$ and which is disjoint from $Y$. Hence the Pontryagin construction (see [43]) translates $X'$ into the desired homotopy between $f_0$ and $f_1$ and we are done.

All that is left to do is to investigate the signed count of the intersection points $X \pitchfork Y$. Consider the submanifold $W = \Sigma \times [0,1]$ with boundary

$$\partial W = \Sigma \times \{0\} \ \cup_L \ L \times [0,1] \ \cup_L \ \Sigma \times \{1\}.$$

Recall that the signed count of intersection points of $\partial W$ and $X$ does only depend on the homology classes corresponding to $\partial W$ and $X$. Therefore it is zero, since the former set obviously bounds $W$ and hence its corresponding homology class is zero. Lemma 27 tells us that the same holds for the signed count over intersection points on $\Sigma \times \{0\} \cup \Sigma \times \{1\}$. Therefore summing over intersection points on the middle part $Y = L \times [0,1]$ is again zero. This completes the proof. □

**Remark.** Given a homotopy of plane fields between $\xi_0$ and $\xi_1$ Proposition 28 yields a homotopy rel $L$ which matches the given one outside an arbitrarily small neighbourhood of $L$.

### 4.1.1 On the coarse classification of loose knots

Let $(M, \xi)$ be an overtwisted contact 3-manifold. A Legendrian knot $L \subset (M, \xi)$ is called **loose** if the restriction of the contact structure on the knot complement $M \setminus L$ is still overtwisted. Now suppose we are given two loose Legendrian knots $L_0$ and $L_1$. The knots are called **coarsely equivalent** if there is a coorientation preserving contactomorphism of $(M, \xi)$ which sends $L_0$ to $L_1$.

In [8] Eliashberg and Fraser prove that the coarse classification of loose Legendrian knots is of pure homotopical nature:

**Proposition 29** (Eliashberg–Fraser, [8, Proposition 4.3])**.** *Let $L_0, L_1 \subset (M, \xi)$ be two Legendrian knots. Suppose that there exists a diffeomorphism $f\colon M \to M$ which sends $L_0$ to $L_1$, the plane fields $\xi_0 = \xi$ and $\xi_1 = (Tf)(\xi)$ coincide over $L_1$ and such that they are homotopic on $M \setminus L_1$ relative to the boundary. Then $L_0$ and $L_1$ are coarsely equivalent.*

*Proof.* By a neighbourhood theorem for isotropic submanifolds (cf. [19, Theorem 2.5.8]) we can actually assume that the diffeomorphism $f$ sends $\xi_0$ to $\xi_1$ on a neighbourhood $N_0 \supset L_0$ and $N_1 = f(N_0) \supset L_1$. Then the contact structures $\xi_0$ and $\xi_1$ coincide on the boundary of $N_1$ and are homotopic as plane fields via a homotopy fixed on $N_1$. Hence, according to the classification of overtwisted contact structures in [6] there exists an isotopy $h_t\colon M \to M$, $t \in [0,1]$, which is fixed on $N_1$ and such that $h_0 = \mathrm{id}$ and $(Th_1)(\xi_0) = \xi_1$. □

Now suppose we are given two topologically isotopic, null-homologous and loose Legendrian knots $L_0$ and $L_1$ in an overtwisted contact manifold $(M, \xi)$.

Suppose further that they have the same values of tb and rot. By the isotopy extension theorem we can assume the isotopy between $L_0$ and $L_1$ to be covered by an ambient isotopy $f_t\colon M \to M$, $t \in [0,1]$, such that we have $f_0 = \mathrm{id}$ and $f_1(L_0) = L_1$. Since the Thurston–Bennequin invariants of $L_0$ and $L_1$ agree we can actually assume (cf. [19, Theorem 2.5.8]) that the diffeomorphism $f_1$ is a contactomorphism over neighbourhoods $N_0 \supset L_0$ and $N_1 = f_1(N_0) \supset L_1$. To match the conditions in Proposition 29 it remains to show that $\xi$ and $(Tf_1)(\xi)$ are homotopic as plane fields via a homotopy that fixes $N_1$. But this is exactly the content of Proposition 28. Hence we obtain the following.

**Corollary 30** (Eliashberg–Fraser, [8, Corollary 4.4]). *Two topologically isotopic, null-homologous and loose Legendrian knots in an overtwisted contact manifold $(M,\xi)$ are coarsely equivalent if and only if they have the same values of* tb *and* rot.

One should note that the statement given in [8] is actually formulated for topologically trivial knots. However the above statement is a straightforward generalisation of the original one.

## 4.2 On isotopies of knots in contact 3-manifolds

In the present section we give an alternative proof of a folklore theorem that says, up to stabilisation, the classification of Legendrian knots is purely topological. For $(\mathbb{R}^3, \xi_{st})$ this theorem was proved by Fuchs and Tabachnikov [17, Theorem 4.4]. Recently Ding–Geiges gave detailed arguments for the general case [22]. Their proof is based on convex surface theory and a neighbourhood theorem for arbitrary knots in contact 3-manifolds. The proof presented in this thesis is also based on convex surface theory, however, without the need of a neighbourhood theorem.

**Theorem 31** (Ding–Geiges, [22]). *If two oriented Legendrian knots $L_0$ and $L_1$ in a 3-dimensional contact manifold $(M,\xi)$ are topologically isotopic, one can find Legendrian isotopic stabilisations $S_+^{m_0} S_-^{n_0} L_0$ and $S_+^{m_1} S_-^{n_1} L_1$.*

**Remark.** Since the result of stabilising a Legendrian knot is unique up to Legendrian isotopy Theorem 31 can be reformulated as follows: one can relate $L_0$ and $L_1$ by a sequence of stabilisations and destabilisations.

**Lemma 32.** *Let $L_0$ and $L_1$ be two Legendrian knots bounding an embedded convex annulus $A$ inside an arbitrarily contact 3-manifold $(M,\xi)$ (no assumption is made on the contact structure $\xi$ being tight or overtwisted). Then one can find Legendrian isotopic stabilisations $S_+^{m_0} S_-^{n_0} L_0$ and $S_+^{m_1} S_-^{n_1} L_1$.*

*Proof.* Observe that the statement is equivalent to the existence of a third knot which destabilises to $L_0$ as well as to $L_1$. Let $\Gamma \subset A$ denote the dividing set on the convex annulus $A$ which we understand as identified with $[-1,1] \times S^1$. We are allowed to isotope the dividing curves $\Gamma$ inside $A$ as long as we assure that $\partial\Gamma \subset \partial A$. This can be understood as corresponding to a different choice of coordinates on $A$.

Identifying $S^1$ with $\mathbb{R}/2\pi\mathbb{Z}$ there exists an isotopy of $\Gamma$ such that $\Gamma_0$ is contained in $[-1,1] \times (0,\pi)$ and $\Gamma_1$ is contained in $[-1,1] \times (\pi, 2\pi)$, where we denote by $\Gamma_0$ and $\Gamma_1$ the components of $\Gamma$ whose endpoints either both lie on $L_0$ or both on $L_1$ respectively. Note that at this point we do not worry about circles, closed curves or components running from one boundary to the other. We can further isotope such that $\Gamma_0 \cap \big([-1,0] \times S^1\big)$ just consists of arcs isotopic to $[-1,0] \times \{*\}$ and $\Gamma_1 \cap \big([0,1] \times S^1\big)$ just consists of arcs isotopic to $[0,1] \times \{*\}$. By a final isotopy we achieve that all circles intersect the core of the annulus $\{0\} \times S^1$ transversely and exactly twice.

Since $\Gamma$ intersects the core $\{0\} \times S^1$ of $A$ multiple times the Legendrian realisation principle [34, Theorem 3.7] applies and we can (after a small perturbation of $A$) assume $\{0\} \times S^1$ to be Legendrian. Observe that the intersection $\Gamma \cap \big([-1,0] \times S^1\big)$ contains only components running form one boundary to the other or components whose endpoints both lie on $\{0\} \times S^1$. Hence $\{0\} \times S^1$ destabilises to $L_0$. Analogous one shows that $\{0\} \times S^1$ destabilises to $L_1$. □

*Alternative proof of Theorem 31.* Let $\psi\colon [0,1]\times S^1 \to M$ denote the topological isotopy between $L_0$ and $L_1$. By applying an isotopy discretisation we can assume that there is a subdivision $0 = t_0 < t_1 < \ldots < t_n = 1$ of the unit interval such that for $i = 0, \ldots, n-1$ the knots $L_{t_i}$ and $L_{t_{i+1}}$ bound an embedded annulus $A_i$. Since every knot can be $C^0$-approximated by a Legendrian knot we can further assume that each of the knots $L_{t_i}$ is actually Legendrian. Finally after sufficiently many stabilisations the twisting of $L_{t_i}$ measured with respect to $A_i$ is negative and $A_i$ can be perturbed into a convex surface. Hence Lemma 32 applies and we are done. □

# Chapter 5

# The fundamental group of $\Xi(T^3, \zeta)$

Let $T^3$ denote the 3-dimensional torus, which we understand as the quotient of $\mathbb{R}^3$, equipped with coordinates $(x, y, z)$, and the integer lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$. Let $\zeta$ denote the contact structure defined by the kernel of the contact 1-form $\alpha = \cos z\, dx - \sin z\, dy$. In the present chapter we will prove the following.

**Theorem 33.** *Let $\Xi(T^3, \zeta)$ denote the connected component of $\zeta$ in the space of contact structures on $T^3$. Then the fundamental group $\pi_1\big(\Xi(T^3, \zeta)\big)$ with base point $\zeta$ is isomorphic to $\mathbb{Z}$.*

An outline of the proof of Theorem 33 is presented in [21] and was told to the author in private conversation by Geiges. The proof relies on the fact that $\pi_1\big(\Xi(T^3, \zeta)\big)$ fits into an exact sequence of homotopy groups (see sequence (5.1) below). One part of this sequence has already been studied by Geiges–Gonzalo in [20]. They show that $\pi_1\big(\Xi(T^3, \zeta)\big)$ contains an infinite cyclic subgroup (see Proposition 37 below), providing one half of the proof. The remaining half of the proof is based on a result of Giroux [27, Théorème 4]. However Giroux's paper [27] has to be read with a certain amount of caution. Proposition 10 and the proofs of the main results (though not the results as such), including [27, Théorème 4], are incorrect. The contribution of the author to the proof of Theorem 33 is presented in Proposition 39 below. It may be viewed as a replacement for Giroux's stronger result [27, Téorème 4].

## 5.1 Contactomorphisms of the solid torus

Let $(M, \xi)$ denote a compact contact 3-manifold with (possibly empty) convex boundary. Let $\Sigma \subset M$ be an embedded convex surface with dividing set $\Gamma \subset \Sigma$ and (possibly empty) boundary $\partial\Sigma$.

**Definition 14.** We say that two embedded copies $\Sigma_0$ and $\Sigma_1$ of the surface $\Sigma$

are **parallel** if the following two conditions hold:

- $\Sigma_0 \cap \Sigma_1 = \partial\Sigma$;
- $\Sigma_0 \cup \Sigma_1$ bounds a domain in $M$ diffeomorphic to the product $\Sigma \times [0, 1]$.

Furthermore we will call the surfaces **Γ-parallel**, if in addition we have

- $\Sigma_0$ and $\Sigma_1$ are both convex with dividing set $\Gamma$.

We say that two surfaces $\Sigma_0$ and $\Sigma_1$ are **Γ-connected**, if there is path of convex surfaces connecting them. We will call two embeddings *parallel* or *connected* respectively if their corresponding images are. Furthermore we say that an isotopy $(\psi_t)_{t\in[0,1]}$ of embeddings of $\Sigma$ has **Γ-convex ends**, if the corresponding surfaces $\psi_0(\Sigma)$ and $\psi_1(\Sigma)$ both are convex with dividing set $\Gamma$.

**Lemma 34** (Isotopy discretisation)**.** *Given an isotopy of $\Sigma \subset (M, \xi)$ with $\Gamma$-convex ends there is another isotopy $(\psi_t)_{t\in[0,1]}$ with the same endpoints and a subdivision $s_0 = 0 < s_1 < \ldots < s_k = 1$ of the unit interval such that $\psi$ restricts to an embedding of $\Sigma \times [s_i, s_{i+1}]$ for each $i = 0, \ldots, k-1$. Furthermore we can assume that each $\Sigma_i = \psi_{s_i}(\Sigma)$ is convex. The surfaces $\Sigma_i$ will be referred to as **vertices** of the isotopy $\psi$.* □

*Proof.* Let $(\Sigma_t)_t \in [0, 1]$ be family of embedded copies of $\Sigma$ such that $\Sigma_0$ and $\Sigma_1$ are convex. For each $t \in [0, 1]$ there is a neighbourhood $N(\Sigma_t) = \Sigma_t \times [-\delta_t, \delta_t]$ of $\Sigma_t$ and an $\varepsilon_t > 0$ such that $\Sigma_{t+s}$ is contained in $N(\Sigma_t)$ for each $-\varepsilon_t \leq s \leq \varepsilon_t$. Let $s_0 = 0 < s_1 < \ldots < s_k = 1$ be a subdivision of the unit interval with $s_{i+1} - s_i < \varepsilon_*$, where $\varepsilon_*$ denotes the minimum of $\{\varepsilon_t \colon t \in [0, 1]\}$. Now replace the given isotopy between $\Sigma_{s_i}$ and $\Sigma_{s_{i+1}}$, $i = 0, \ldots, k$, by an isotopy which

(i) first connects $\Sigma_{s_i} \subset N(\Sigma_{s_i})$ with $\Sigma_{s_i} \times \{\delta_{s_i}\}$,

(ii) then connects $\Sigma_{s_i} \times \{\delta_{s_i}\}$ with $\Sigma_{s_i} \times \{-\delta_{s_i}\}$,

(iii) and finally $\Sigma_{s_i} \times \{-\delta_{s_i}\}$ with $\Sigma_{s_{i+1}}$.

Note that the isotopies in (i),(ii) and (iii) may be chosen to be parallel. All that is left to do is to perturb $\Sigma_{s_i} \times \{\pm\delta_{s_i}\}$ into a convex surfaces. This completes the proof. □

One should note that the Isotopy Discretisation can actually be chosen such that the contact structure on each block $\Sigma \times [s_i, s_{i+1}]$ is either vertically invariant or corresponds to a single non-trivial bypass attachment [33]. Our formulation of the Isotopy Discretisation is much weaker than the one described in [33]. However it is strong enough for the purposes of this work.

**Lemma 35.** *Let $\Sigma \subset (M, \xi)$ be a properly embedded convex surface and $\phi$ be a contactomorphism. Furthermore assume that the image $\Sigma' = \phi(\Sigma)$ of $\Sigma$ under $\phi$ is $\Gamma$-connected to $\Sigma$. Then $\phi$ is contact isotopic to a contactomorphism that keeps $\Sigma$ fixed.* □

*Proof.* Without any loss of generality we can assume that $\Sigma$ and $\Sigma'$ are connected by a discrete isotopy as in Lemma 34. We may further assume that, since $\Sigma$ and $\Sigma'$ are $\Gamma$-connected, consecutive vertices $\Sigma_i$ and $\Sigma_{i+1}$, $i = 0, \dots, k-1$, are $\Gamma$-connected as well. Finally by the Flexibility Theorem we can assume that the characteristic foliation $(\Sigma_i)_\xi$ on each vertex $\Sigma_i$ agrees with the characteristic foliation $(\Sigma_{i+1})_\xi$ of its successor $\Sigma_{i+1}$. By the Uniqueness Lemma the contact structure on the embedded copy of $\Sigma \times [s_i, s_{i+1}]$ can be assumed to be $[s_i, s_{i+1}]$-invariant. In particular the characteristic foliation on $\Sigma \times \{t\}$, $t \in [s_i, s_{i+1}]$, stays fixed. We end up with an isotopy $\psi_t : \Sigma \hookrightarrow M$, $t \in [0,1]$ connecting $\Sigma$ and $\Sigma'$ such that $\psi_t$ induces the same characteristic foliation on $\Sigma$. By [19, Theorem 2.6.13] there is a compactly supported contact isotopy $\phi_t : M \to M$ with $\phi_t \circ \psi_0 = \psi_t$. Hence $(\phi_1)^{-1} \circ \phi$ defines a contactomorphism satisfying

$$\big((\phi_1)^{-1} \circ \phi\big)(\Sigma) = \psi_0(\Sigma) = \Sigma.$$

This completes the proof. □

Every diffeomorphism of the solid torus that fixes the boundary point wise is isotopic to the identity. The following corollary states that the same holds for contactomorphisms of standard neighbourhoods of Legendrian knots.

**Corollary 36.** *Let $\xi_{\mathrm{st}}$ be the standard tight contact structure on the solid torus $D^2 \times S^1$ with convex boundary and two dividing curves of slope $-1$. Every contactomorphism $\phi$ of $(D^2 \times S^1, \xi_{\mathrm{st}})$ that fixes the boundary point wise is contact isotopic to the identity.*

*Proof.* Let $D \subset (D^2 \times S^1, \xi_{\mathrm{st}})$ be a convex meridional disc and let $D' = \phi(D)$ denote its image under some contactomorphism $\phi$ which fixes the boundary point wise. The two discs $D$ and $D'$ are clearly isotopic and by Lemma 34 we find an isotopy $\psi$ with convex vertices. Since $\xi_{\mathrm{st}}$ is tight and the Thurston-Bennequin invariant of the boundaries $\partial D_i$ of the vertices is $\mathtt{tb}(\partial D_i) = -1$, the dividing set is given by a single properly embedded arc. Furthermore $D_i$ and $D_{i+1}$ bound a tight 3-ball (with edges admittedly), which implies that they are $\Gamma$-connected. So by Lemma 35 we can assume that $\phi$ fixes a convex disc. Therefore $\phi$ restricts to a contactomorphism on the complement of the disc, which is a tight 3-ball. By work of Eliashberg [4] it is isomorphic to the identity and we are done. □

**Remark.** In the proof of Corollary 36 we used that the space of tight contact structures on the 3-ball (with fixed boundary foliation) is contractible. This result is due to Eliashberg [4]. However one should note that details of the proof were only published on the level of $\pi_0$.

## 5.2 Computation of the fundamental group

For a closed contact 3-manifold $(M, \xi_0)$ we denote by $\mathrm{Diff}_0(M)$ the identity component of the diffeomorphism group of $M$ and by $\mathrm{Cont}_0(M, \xi_0)$ the subgroup of $\mathrm{Diff}_0(M)$ built up of contactomorphisms of $(M, \xi_0)$. Furthermore let

$\Xi_0(M)$ denote the space of contact structures on $M$ isotopic to $\xi_0$. Consider the following map

$$\begin{aligned}\sigma\colon \mathrm{Diff}_0(M) &\longrightarrow \Xi_0(M)\\ \phi &\longmapsto T\phi(\xi_0)\end{aligned}$$

sending an element $\phi \in \mathrm{Diff}_0(M)$ to the contact structure $\phi_*\xi_0$ obtained by pushing forward $\xi_0$. By Gray stability this map is a surjection. Moreover, it is a well-known (folklore) result that this defines a Serre fibration with fibre equal to $\mathrm{Cont}_0(M,\xi_0)$. Write $i\colon \mathrm{Cont}_0(M,\xi_0) \to \mathrm{Diff}_0(M)$ for the obvious inclusion, then there is an exact sequence of homotopy groups:

$$\cdots \to \pi_k\big(\mathrm{Diff}_0\big) \stackrel{\sigma_\#}{\to} \pi_k\big(\Xi_0\big) \stackrel{\Delta_\#}{\to} \pi_{k-1}\big(\mathrm{Cont}_0\big) \stackrel{i_\#}{\to} \pi_{k-1}\big(\mathrm{Diff}_0\big) \to \dots. \tag{5.1}$$

From now on choose $M$ to be the 3-dimensional torus $T^3$ which we understand as $\mathbb{R}^3/2\pi\mathbb{Z}^3$. For $t \in \mathbb{R}$ let $\zeta_t$ denote the kernel of the contact 1-form

$$\cos(z+t)\,dx - \sin(z+t)\,dy.$$

In the following choose $\xi_0$ to be $\zeta = \zeta_0$. Since we are interested in $\pi_1(\Xi_0)$, we will have to investigate the above fragment (5.1) of the sequence of homotopy groups for $k = 1$. One part of this sequence has already been studied by Geiges–Gonzalo in [20]:

**Proposition 37** (Geiges–Gonzalo [20])**.** *The fundamental group $\pi_1\big(\Xi(T^3,\zeta)\big)$ based at $\zeta$ contains an infinite cyclic subgroup, generated by the loop $\{\zeta_t\colon 0 \le t \le 1\}$.*

For the sake of completeness we briefly sketch the proof of the above statement. We start with the following observation.

**Lemma 38.** *Let $\Sigma_0$ and $\Sigma_1$ be two parallel convex surfaces sitting in some contact 3-manifold $(M,\xi)$. Suppose $\Sigma_0$ and $\Sigma_1$ are connected by a path $(\Sigma_t)_{t\in[0,1]}$ of convex surfaces isotopic, in the space of embeddings $\Sigma \hookrightarrow M$, to the path induced by the embedded product $\Sigma \times [0,1]$ which is bounded by $\Sigma_0$ and $\Sigma_1$. If $\Sigma_0$ is separating, the contact structure $\xi|_{\Sigma\times[0,1]}$ can be isotoped such that it becomes vertically invariant.*

*Proof.* Let $\xi'$ denote the contact structure which equals a one-sided convex neighbourhood of $\Sigma_0$ over $\Sigma \times [0,1]$ and coincides with $\xi$ elsewhere. We are going to connect $\xi$ and $\xi'$ by a path of contact structures $\zeta_t$ and apply Gray stability to show that $\xi$ equals $\xi'$. Suppose the manifold can be decomposed along $\Sigma_t$ into two pieces, i.e. we have

$$M = M_t^- \cup_{\Sigma_t} M_t^+.$$

Note that for each $t \in [0,1]$ the piece $M_t^-$ is diffeomorphic to $M_0^-$ and let $f_t$ denote the diffeomorphism $f_t\colon M_0^- \to M_t^-$. We can endow $M_t^-$ with the contact structure $(f_t)_*(\xi|_{M_t^-})$ obtained by pushing forward $\xi|_{M_t^-}$ using the diffeomorphism $f_t$. On the other hand we endow $M_t^+$ with the contact structure $\xi_{M_t^+}$ induced by the inclusion $M_t^+ \hookrightarrow M$. We end up with a family of contact structures $\zeta_t$ on $M$ by setting

$$\zeta_t|_{M_t^-} = (f_t)_*(M_t^-) \quad \text{and} \quad \zeta_t|_{M_t^+} = \xi_{M_t^-}.$$

By Gray stability $(M, \zeta_0)$ and $(M, \zeta_1)$ are contactomorphic. Observe that we have $\zeta_0 = \xi$ and $\zeta_1 = \xi'$. □

**Remark.** If one drops the assumption on the surface $\Sigma_t$ in Lemma 38, to be separating, the statement does not hold anymore. A counterexample can be constructed choosing $\Sigma_0$ to be a vertical torus in $(T^3, \zeta)$ with non-minimal dividing set.

*Proof of Proposition 37.* It is a classical result that $\pi_1(\mathrm{Diff}_0) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$, where the $\mathbb{Z}$-factors are generated by the shifts along the $x$-, $y$- and $z$-axis respectively. Consider the following fragment of the homotopy sequence (5.1):

$$\pi_1\big(\mathrm{Cont}_0\big) \overset{i_\#}{\to} \pi_1\big(\mathrm{Diff}_0\big) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \overset{\sigma_\#}{\to} \pi_1\big(\Xi_0\big) \to \dots.$$

Since the contact structure $\zeta$ is invariant under shifts along the $x$- and $y$-axis the first two $\mathbb{Z}$-factors of $\pi_1(\mathrm{Diff}_0)$ lie in the kernel of $\sigma_\#$. To show that $\pi_1\big(\Xi(T^3, \zeta)\big)$ contains an infinite cyclic subgroup is suffices to show that $\sigma_\#$ restricted to the remaining $\mathbb{Z}$-factor is injective. Let $f_\theta^z\colon (x, y, z) \mapsto (x, y, z + \theta)$, $\theta \in S^1$, denote the $S^1$-family of shifts along the $z$-axis generating the third $\mathbb{Z}$-factor of $\pi_1(\mathrm{Diff}_0) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. We have to show that $[(f_\theta^z)_{\theta\in S^1}]$ does not lie in the kernel of $\sigma_\#$. Suppose it does. Then by the exactness of the homotopy sequence (5.1) we can assume that $[(f_\theta^z)_{\theta\in S^1}]$ is represented by a family $(\phi_\theta)_{\theta\in S^1}$ of contactomorphisms. Let $\varphi_n\colon (x, y, z) \mapsto (x, y, n \cdot z)$ denote the $n$-fold cover covering the $z$-axis and let $\zeta^n$ denote the contact structure defined by the pulling back a contact form for $\zeta$. Let $T \subset T^3$ be a 2-dimensional convex torus which is isotopic to the coset of the $xy$-plane in $T^3 \equiv \mathbb{R}^3/\mathbb{Z}^3$ and whose dividing set consists of two simple closed curves. Note that $T_\theta = \phi_\theta$, $\theta \in [0, 1]$, defines a path of convex surfaces in $(T^3, \zeta)$. For sufficiently large $n$ we can lift $(T_\theta)_{\in[0,1]}$ to a path of convex surfaces $(\widetilde{T}_\theta)_{\in[0,1]}$ in $(T^3, \zeta^{2n})$ such that $\widetilde{T}_0 = T^2 \times \{0\}$, $\widetilde{T}_1 = T^2 \times \{1\}$ and such that $\widetilde{T}_\theta$ is disjoint from $T^2 \times \{n\}$ for each $\theta \in [0, 1]$. We could cut $(T^3, \zeta^{2n})$ along $T^2 \times \{n\}$ and obtain a contact manifold (with boundary) for which $\widetilde{T}_0$ is separating. Then Lemma 38 implies that the block $\widetilde{T} \times [0, 1]$ that bounds $\widetilde{T}_0$ and $\widetilde{T}_1$ corresponds to a vertically invariant neighbourhood of $\widetilde{T}_0$. But this would induce a contactomorphism between $(T^3, \zeta^n)$ and $(T^3, \zeta^{n-1})$ which contradicts Kanda's classification of tight contact structures on the 3-torus [40]. Hence $[(f_\theta^z)_{\theta\in S^1}]$ does not lie in the kernel of $\sigma_\#$ showing that $(\sigma_\#)|_{\{0\}\oplus\{0\}\oplus\mathbb{Z}}$ is injective. This completes the proof. □

As mentioned at the beginning of the present chapter Proposition 37 provides the first half of the proof of Theorem 33. The main ingredient for the second half is content of the following statement.

**Proposition 39.** *Every contactomorphism of $(T^3, \zeta)$ that is isotopic to the identity is also isotopic to the identity via contactomorphisms.*

*Proof.* Suppose $\phi$ is a contactomorphism of $(T^3, \zeta)$ that is isotopic to the identity. Let $T$ be vertical convex torus with minimal dividing set, i.e given by two closed curves of slope zero. Let $T'$ denote its image under $\phi$. Since $\phi$ is isotopic to the identity the tori $T$ and $T'$ are smoothly isotopic. In [24, Lemma 6.5] Ghiggini shows that such tori are in fact contact isotopic. Hence by Lemma 35 we can assume that $\phi$ fixes the torus $T$. Now let $S$ be another vertical convex torus with minimal dividing set intersecting $T$ in a single vertical Legendrian curve with maximal value of `tb`. Applying [24, Lemma 6.5] once again we can assume that $\phi$ also fixes $S$. Therefore we can understand $\phi$ as a contactomorphism on the complement of the tori $T$ and $S$ fixing the boundary. The complement is a solid torus matching the conditions in Corollary 36. This completes the proof. $\square$

Now we have everything in place to prove Theorem 33.

*Proof of Theorem 33.* Note that putting Proposition 39 in different words, it states that the map $i_\#\colon \pi_0(\mathrm{Cont}_0) \to \pi_0(\mathrm{Diff}_0)$ induced by the natural inclusion is injective. In consequence $\pi_0(\mathrm{Cont}_0)$ is trivial, since $\pi_0(\mathrm{Diff}_0)$ is by trivial reasons, and hence $\sigma_\#$ is surjective (by exactness of the sequence). In conclusion the infinite cyclic subgroup in Proposition 37 is all there is, that is we have $\pi_1\big(\Xi_0\big) \cong \mathbb{Z}$. $\square$

**Remark.** It is possible to reduce the proof of Proposition 39 to the statement that two vertical Legendrian curves in $(T^3, \zeta)$ with maximal values of `tb` are Legendrian isotopic.

# Appendix A

# Monodromy computations

Below we give the monodromy computations used in the proof of Lemma 24 on page 51:

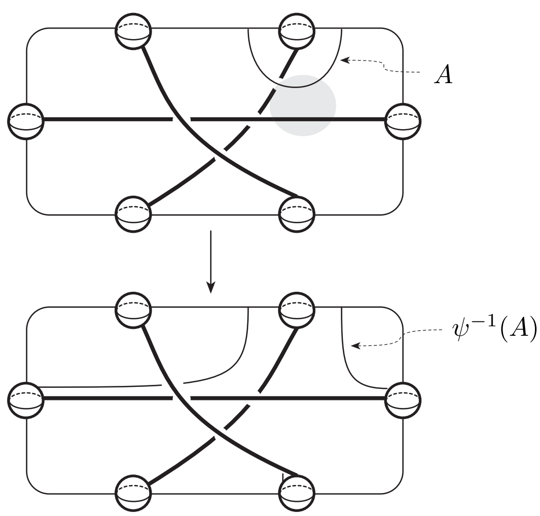


Figure A.1: The image of $A$ under $\psi^{-1}$. The shaded area indicates that a handle slide is performed.

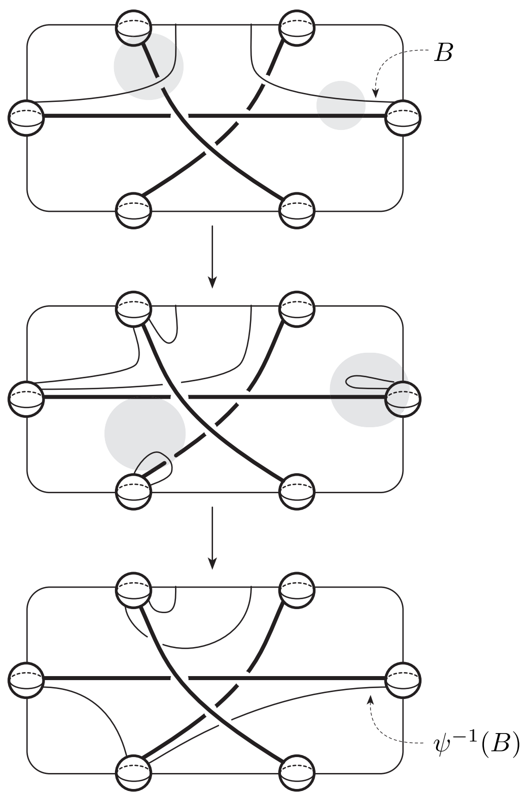


Figure A.2: The image of $B$ under $\psi^{-1}$. The shaded areas indicate that an isotopy or a handle slide is performed.

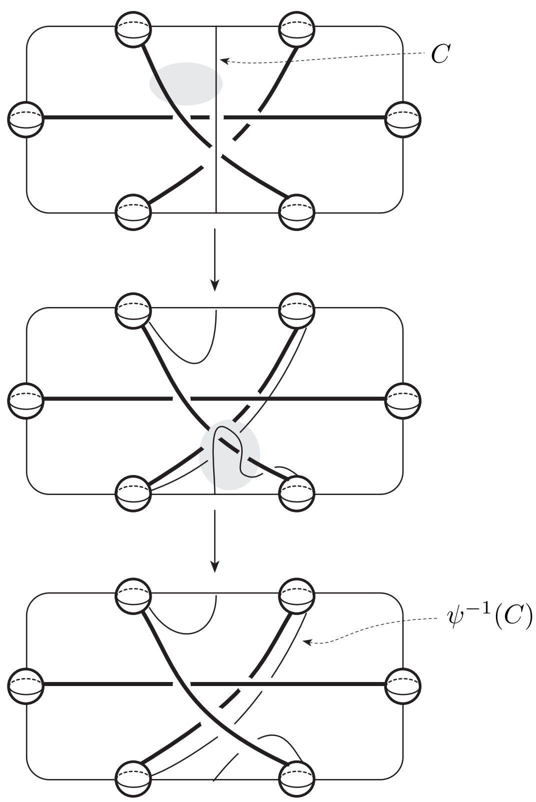


Figure A.3: The image of $C$ under $\psi^{-1}$. The shaded areas indicate that an isotopy or a handle slide is performed.

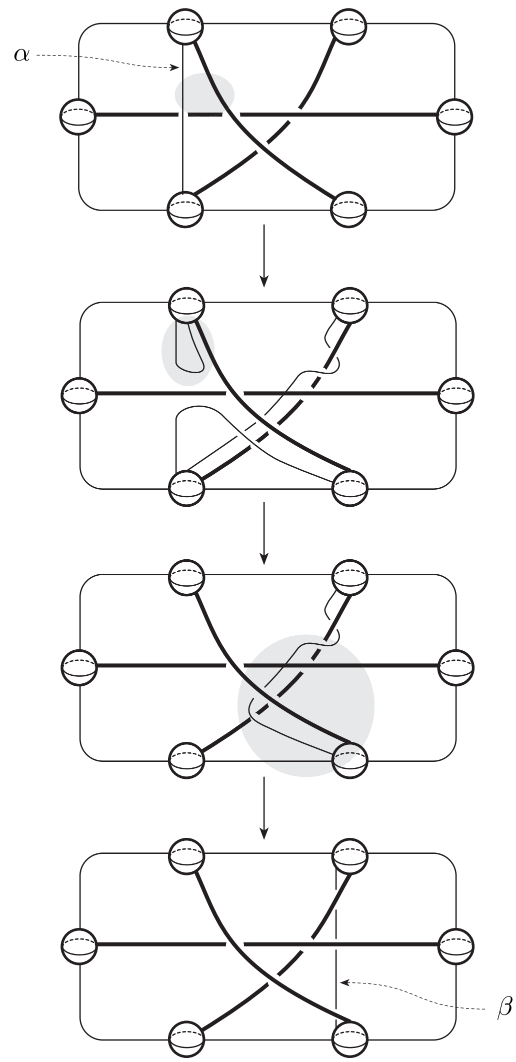


Figure A.4: Computation of the image of $\alpha$ under $\psi^{-1}$. The shaded areas indicate that an isotopy or a handle slide is performed.

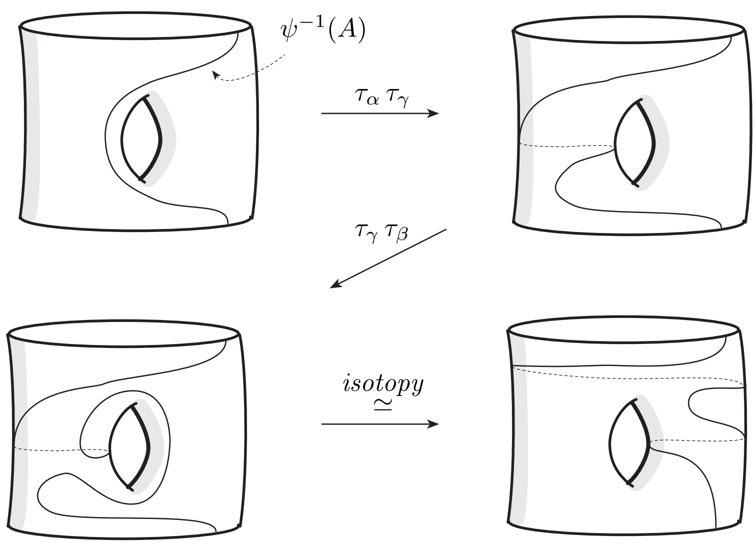


Figure A.5: Computation of the image of $\psi^{-1}(A)$ under $P = \tau_\gamma\,\tau_\beta\,\tau_\alpha\,\tau_\gamma$.

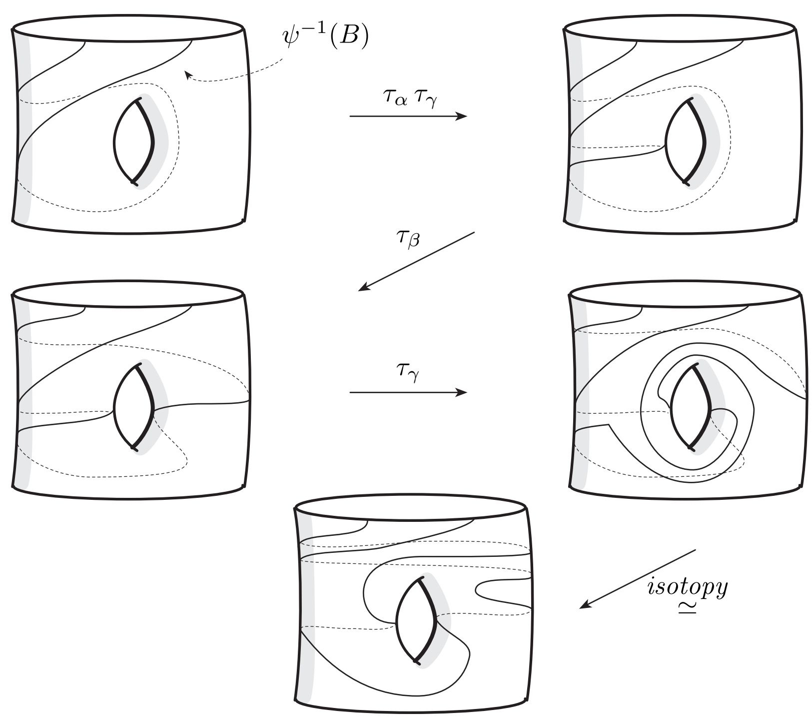


Figure A.6: Computation of the image of $\psi^{-1}(B)$ under $P = \tau_\gamma\,\tau_\beta\,\tau_\alpha\,\tau_\gamma$.

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Köln, den 22. März 2012

Mirko Klukas