Sequential Change-Point Detection for Diffusion Processes

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Zusammenfassung

In der vorliegenden Arbeit wird das Problem der sequentiellen Aufdeckung von Strukturbrüchen im Driftparameter von Diffusionsprozessen unter der Annahme betrachtet, dass die Prozesse stetig beobachtbar sind. Eine entsprechende Überwachungsprozedur wird vorgeschlagen und ihr asymptotisches Verhalten unter der Nullhypothese wie unter der Alternative untersucht. Das vorgeschlagene Überwachungsverfahren ähnelt der CUSUM Prozedur für Prozesse in diskreter Zeit.
Zur Konstruktion der Teststatistik wird die Einschrittmethode von Le Cam angewandt. Um Grenzwertsätze in der Strukturbruchanalyse zu beweisen, sind typischerweise starke Approximationen durch Gaußprozesse die wesentlichen Hilfsmittel. Zwei Hauptresultate der Arbeit sind die starken Invarianzprinzipien (mit Rate) für bestimmte stochastische Integrale und für den als Prozess betrachteten Schätzer. Auf Grundlage dieser Approximationen werden zwei Beweismethoden für die schwache Konvergenz der Teststatistik unter der Nullhypothese entwickelt. Weiterhin wird die asymptotische Normalität der Stoppzeit unter der Alternative bewiesen. Die Arbeit wird mit der Untersuchung einiger Beispiele von stochastischen Differentialgleichungen vervollständigt, welche mit der vorgestellten Methodik behandelt werden können.

Abstract

In this work the problem of sequential detection of changes in the drift parameter of diffusion processes is considered under the assumption that the processes can be observed continuously. A corresponding monitoring procedure is suggested and its asymptotic behaviour under the null hypothesis as well as under the alternative is investigated. The proposed procedure is similar to the CUSUM one for discrete-time processes.
For constructing the test statistic, the one-step method of Le Cam is applied. In order to prove limit theorems in change-point analysis, typically strong approximations by Gaussian processes are the key tools. Two main results of the thesis are the strong invariance principles (with rate) for certain stochastic integrals and for the estimator process. Based on these approximations, two methods of proof are developed for the weak convergence of the test statistic under the null hypothesis. Moreover, the asymptotic normality of the stopping time under the alternative is proven. The thesis is completed by studying some examples of stochastic differential equations which can be treated by the presented methodology.

Contents

Chapter 1

Introduction

1.1 Setting of the problem

In this dissertation we discuss a monitoring procedure to detect a change in the drift parameter of a diffusion process, which is observed sequentially. Throughout the entire work we assume that it is possible to collect data continuously. This assumption is adopted from the monographies of Kutoyants (2004), Prakasa Rao (1999a), and Prakasa Rao (1999b).

First, let a historical data set be available which is a realization of the unique solution $(X_s\colon 0 \le s \le m)$, $m \ge 0$, to the Ito stochastic equation

$$dX_s = b(\theta_0, X_s)ds + \sigma(X_s)dW_s, \quad X_0 \sim \mu(\theta_0), \quad 0 \le s \le m, \tag{1.1}$$

where θ_0 belongs to a compact interval $\Theta \subset \mathbb{R}$, $b\colon \Theta \times \mathbb{R} \to \mathbb{R}$ and $\sigma\colon \mathbb{R} \to \mathbb{R}$ denote the drift and the diffusion function, respectively, and where a unique stationary distribution $\mu(\theta_0)$ exists. The interval $[0, m]$ is called training period.

Immediately after the training period the sequential observation of a diffusion process begins which may have a structural break, i.e., the diffusion satisfies

$$X_{m+t} = \begin{cases} X_m + \int\limits_m^{m+t} b(\theta_0, X_s)ds + \int\limits_m^{m+t} \sigma(X_s)dW_s, & 0 < t \le t^*, \\ X_{m+t^*} + \int\limits_{m+t^*}^{m+t} b(\theta_1, X_s)ds + \int\limits_{m+t^*}^{m+t} \sigma(X_s)dW_s, & t > t^*, \end{cases} \tag{1.2}$$

if the change-point $t^* \in (0, \infty)$, and it satisfies

$$X_{m+t} = X_m + \int\limits_m^{m+t} b(\theta_0, X_s)ds + \int\limits_m^{m+t} \sigma(X_s)dW_s, \quad t \ge 0, \tag{1.3}$$

if $t^* = \infty$. In case of $t^* = \infty$, no structural break occurs. We allow the change-point $t^* = t^*(m)$ to depend on the length m of the training period, but for brevity we often write t^*.
Sequential observation, also called "online" observation, means that up to each time $t \geq 0$ a trajectory of $(X_{m+s} \colon 0 \leq s \leq t)$ has been observed.

In the described model the functions b and σ are supposed to be known, while t^* and the parameter values $\theta_0, \theta_1 \in \Theta$ are unknown. Throughout the thesis let

$$b \in C^{3,1}(\Theta \times \mathbb{R}), \quad \sigma \in C^1(\mathbb{R}), \quad \text{and} \quad \sigma^2 > 0. \tag{1.4}$$

Conventions:

a) We say that a real-valued function f defined on the closed set Θ is continuously differentiable if

$$f \in C^1(\mathring{\Theta}) \quad \text{and} \quad \dot{f} \in C(\Theta)$$

where $\mathring{\Theta}$ denotes the interior of Θ.

b) Set $\dot{b} = \partial_\theta b$, $b' = \partial_x b$ and $b(\theta)(x) = b(\theta, x)$.

Assumption (1.4) implies that, for any $\theta \in \Theta$, $b(\theta, \cdot)$ and σ satisfy the locally Lipschitz condition: for any $N > 0$ there exists a constant $L_N > 0$ such that

$$|b(\theta, y) - b(\theta, x)| + |\sigma(y) - \sigma(x)| \leq L_N |y - x| \quad \forall\, x, y \in [-N, N].$$

In order to guarantee that the solution of (1.1) does not explode, we can either assume the classical linear growth condition or, according to Durrett (1996) and Kutoyants (2004), the following modified condition: there exists some $K_\theta > 0$ such that

$$x\, b(\theta, x) + |\sigma(x)|^2 \leq K_\theta \left(1 + |x|^2\right) \quad \forall\, x \in \mathbb{R}. \tag{1.5}$$

The locally Lipschitz condition together with the linear growth condition or with condition (1.5) imply the existence of a unique Ito solution to equations (1.1) - (1.3).

Since the process X can be continuously sampled and the equation

$$\int_0^m \sigma^2(X_s) ds = \lim_{n \to \infty} \sum_{i=1}^{2^n} \left| X_{im/2^n} - X_{(i-1)m/2^n} \right|^2 \quad P\text{-a.s.} \quad \forall\, m \geq 0$$

holds, it is possible to determine σ^2 with arbitrary precision on the image $\{X_s(\omega)\colon s \le m\}, \omega \in \Omega,$ of the realization of X, at least from the theoretical point of view. Note that the image of the trajectory P-a.s. forms an interval. For the case of sequentially collected data it is necessary to assume that the functional form of σ^2 is determined if it is known on any interval, e.g. on $\{X_s(\omega)\colon s \le m\}$. Therefore, the previous assumption of a known diffusion function makes sense.

Now the question arises whether

$$H_0\colon\ t^* = \infty$$

or

$$H_1\colon\ t^* < \infty \quad \text{and} \quad \theta_0 \neq \theta_1$$

is true.
For testing these hypotheses sequentially, the decision rule is given by a stopping time. In change-point analysis and in the case of two-sided alternatives the stopping rules are usually of the form

$$\tau_m = \inf\big\{t > 0\colon |S_t^m| > c\big\}, \quad m > 0, \tag{1.6}$$

with a suitable statistic S_t^m and a well-chosen critical value $c > 0$. From results in Section 2.2 will follow that τ_m is really a stopping time with respect to any filtration of the process X. By means of τ_m the monitoring procedure is the following: at each time $t > 0$ one of the decisions

- if τ_m stops, accept H_1,
- if $\tau_m > t$, continue the observation,

has to be made based on the observation of the trajectories of $(X_{m+s}\colon 0 \le s \le t)$ and $(X_s\colon 0 \le s \le m)$. Note that in contrast to the case of a finite time horizon there is no explicit decision in favour of H_0. Thus, in applications the procedure can be stopped if there is no need for monitoring the diffusion process anymore.

By intuition, S_t^m should contain the difference between an estimator $\hat\theta_{m,t}$ of the drift parameter based on the online observed trajectory of $(X_{m+s}\colon 0 \le s \le t)$ and an estimator $\hat\theta_{0,m}$ constructed by the historical data set, which was observed during the training period $[0, m]$. Thus, we set

$$S_t^m = \frac{t}{g_m(t)}\big(\hat\theta_{m,t} - \hat\theta_{0,m}\big) \quad \forall\, m, t > 0. \tag{1.7}$$

Here, the weighting $t/g_m(t)$ and the concrete weighting function

$$g_m(t) = \frac{(m+t+1)^{1-\gamma}(t+1)^{\gamma}}{\sqrt{m}}, \quad t \geq 0, \quad \text{for} \quad 0 \leq \gamma < \frac{1}{2},$$

are taken from the literature about change-point analysis for discrete-time models where we have replaced the discrete time index $k \in \mathbb{N}$ by $t+1$. This weighting function was successfully applied, e.g., by Chu et al. (1996), Horváth et al. (2004), Aue (2004) and Aue and Horváth (2004).
In (1.7) we choose the same estimator as Lee et al. (2006). $\hat{\theta}_{m,t}$ as well as $\hat{\theta}_{0,m}$ is the one-step maximum likelihood estimator (one-step MLE or one-step estimator) introduced by Le Cam. Lee et al. (2006) considered the asymptotics under the null hypothesis of a statistic used for an a-posteriori test for parameter change in a continuously sampled diffusion process.
Briefly explained, the one-step estimator, e.g. $\hat{\theta}_{0,m}$, is the sum of a consistent starting estimator and a correction term such that $\hat{\theta}_{0,m}$ becomes asymptotically efficient as $m \to \infty$. Details can be read in Section 2.2.

1.2 Outline of the thesis

The critical value c in (1.6) is determined by the equation

$$\lim_{m\to\infty} P\{\tau_m < \infty\} = \alpha$$

where P is taken under the null hypothesis and α represents the prescribed level of the test. Thus, it is necessary to obtain under H_0 a known limit distribution of $\sup_{t>0} |S_t^m|$ as $m \to \infty$. In Chapter 3, Theorem 3.7, we will show that

$$\lim_{m\to\infty} \mathcal{L}\left(\sup_{t>0} |S_t^m|\right) = \mathcal{L}\left(\sup_{0<t\leq 1} \frac{|W_t|}{c'\, t^{\gamma}}\right) \tag{1.8}$$

is true under H_0. Here, c' represents some positive constant. Except for the factor, the limit distribution is the same as in Horváth et al. (2004) where some quantiles obtained by simulations are presented.

The convergence (1.8) will be proven in three steps.

Step 1. We adopt from Lee et al. (2006) the Taylor expansion of the correction term contained in the one-step estimator. Lee et al. (2006) showed that the term of order zero is responsible for the convergence in distribution. Following this idea, our term of order zero has the form

$$\frac{\psi_{m,t}(\theta_0)(X) - \frac{t}{m}\psi_{0,m}(\theta_0)(X)}{c' g_m(t)}, \quad m, t > 0, \tag{1.9}$$

where $c' > 0$. Under the null hypothesis, the random variables $\psi_{m,t}(\theta_0)(X)$ and $\psi_{0,m}(\theta_0)(X)$ denote stochastic integrals depending on $(X_{m+s}\colon 0 \le s \le t)$ and on $(X_s\colon 0 \le s \le m)$, respectively.
Expression (1.9) looks quite similar to the expression in Horváth et al. (2004) and Aue (2004) which is responsible for the asymptotics under H_0; this is

$$Q(m,k) = \frac{1}{g_m(k-1)}\left(\sum_{i=m+1}^{m+k} Y_i - \frac{k}{m}\sum_{i=1}^{m} Y_i\right), \quad k,m \in \mathbb{N},$$

where $(Y_i\colon i \in \mathbb{N})$ is a sequence of real-valued random variables. The first paper deals with the sequential detection of a structural change in a linear model, while in the second one the same problem is investigated for a location model with certain dependence of the errors. In order to derive the asymptotic behaviour of $\sup_k |Q(m,k)|$ as $m \to \infty$, the key tool in both papers is the following uniform weak invariance principle: there exists a family of Wiener processes $(W^{(m)}\colon m \in \mathbb{N})$ such that

$$\sup_{k\in\mathbb{N}} \frac{1}{k^\alpha}\left|\sum_{i=m+1}^{m+k} Y_i - c'W_k^{(m)}\right| = \mathcal{O}_P(1) \quad \text{as} \quad m \to \infty$$

for a positive constant c'.

Step 2. We follow Horváth et al. (2004) and Aue (2004) for proving the convergence of the term given in (1.9). For this purpose, the stochastic integrals involved in (1.9) should be approximated by Wiener processes with a suitable rate. We give a simple proof for the following strong invariance principle (see Theorem 2.3): If Y is a time-homogeneous, ergodic diffusion process and if $f\colon \mathbb{R} \to \mathbb{R}$ is a function satisfying suitable conditions, then one can find a Wiener process B and a constant $c' > 0$ such that

$$\int_0^t f(Y_s)dW_s - c'B_t = \mathcal{O}\left((t\log_2 t)^{1/4}(\log t)^{1/2}\right) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty. \quad (1.10)$$

Here, $\log$ denotes the natural logarithm and $\log_2 = \log \circ \log$. From (1.10) we deduce the uniform weak invariance principle: for any $\alpha > 1/4$ there exists a family of Wiener processes $(W^{(m)}\colon m \ge 0)$ such that

$$\sup_{t>0} \frac{1}{(t+1)^\alpha}\left|\int_m^{m+t} f(Y_s)dW_s - c'W_t^{(m)}\right| = \mathcal{O}_P(1) \quad \text{as} \quad m \to \infty.$$

Such a uniform approximation is necessary in sequential change-point analysis, at least for the asymptotics under the alternative.

Now we explain why it is not appropriate to use an already known result about strong approximations of stochastic integrals by Wiener processes. In literature one can find different papers about this topic, e.g. Ekushov (1984), Besdziek (1991), and Gerencsér (1991b).
Ekushov (1984) approximated general local martingales by Wiener processes P-a.s. This result contains two problems: first, the proofs are not available and second, if applied on stochastic integrals, the rate of approximation has the order of $(t \log\log t)^{1/2}$. In this case the approximation by Wiener processes does not yield the desired limit theorems of change-point analysis.
Theorems for the strong approximation of real semimartingales by processes with continuous trajectories and independent increments were derived by Besdziek (1991). He obtained an error term of the form $\mathcal{O}(t^{(1/2)-\delta})$ as $t \to \infty$ for some small $\delta > 0$. As mentioned above, this rate was sufficient for Horváth et al. (2004) and Aue (2004) to obtain the weak convergence of $\sup_k |Q(m,k)|$ essentially for all parameters $\gamma \in [0, 1/2)$. Unfortunately, in the proof of Theorem 3.7 we will see that in our setting $\gamma \le \delta$ should hold if we apply the approximation of Besdziek (1991). Therefore, an approximation error as in Besdziek (1991) would restrict the variability of the test and would furthermore reduce its performance because the delay time of the detection procedure becomes smaller if γ increases (see Remark 4.22 in this work). Besides, Besdziek (1991) applied a technique of reducing the problem to the strong approximation proven by Berkes and Philipp (1979). The most general result of this kind is a strong approximation of processes with cadlag trajectories (see Eberlein (1989)). However, the result of Berkes and Philipp (1979) cannot yield a better rate than the one in (1.10).
Trying to get a small order of the error, the result in Gerencsér (1991b) about the approximation of vector-valued stochasic integrals seems to be suitable because it has an error term of the form $\mathcal{O}_M(t^{2/5+\delta})$ for any $\delta > 0$. The Landau symbol $\mathcal{O}_M(1)$ stands for M-boundedness: a stochastic process $(Y_t\colon t \ge 0)$ is called M-bounded if and only if

$$\sup_{t\ge 0} \big\|Y_t\big\|_{L^p} < \infty \quad \forall\, p \ge 1.$$

It is unclear whether the existing methods of change-point analysis work in combination with the concept of M-boundedness. The second disadvantage of Gerencsér's theorem is that it cannot be applied under H_1 because it requires the constancy of the function $t \mapsto Ef^2(X_t)$, $t \ge 0$.

In conclusion, to our best knowledge the rate of approximation in (1.10) is the best one among all strong invariance principles for the class of continuous-time processes which includes stochastic integrals. This is not surprising because the stochastic integral as a process is not far away from a Wiener process.

Step 3. In the last step of the proof of (1.8) we have to show that all remainder terms in the mentioned Taylor expansion converge to zero. For this purpose, Lee et al. (2006) assume that there exists a starting estimator $\bar{\theta}_{0,t},\ t \geq 0$, based on the observation period $[0, t]$ which fulfils

$$\sup_{t>1} \varphi_t \left|\bar{\theta}_{0,t} - \theta_0\right| < \infty \quad P\text{-a.s.}$$

with some function $t \mapsto \varphi_t$ of a higher order than $t^{1/4}$ as $t \to \infty$.
In this doctoral thesis we prove under H_0 and suitable conditions that the estimator of the method of moments (EMM) $\hat{\theta}^0_{0,t},\ t \geq 0,$ satisfies

$$\sup_{t \geq e^e} \frac{\sqrt{t}}{\sqrt{\log_2 t}} \left|\hat{\theta}^0_{0,t} - \theta_0\right| < \infty \quad P\text{-a.s.}$$

(see Proposition 2.16). In addition, it will be shown that

$$\sup_{t \geq e^e} \frac{\sqrt{t}}{\sqrt{\log_2 t}} \left|\hat{\theta}^0_{m,t} - \theta_0\right| = \mathcal{O}_P(1) \quad \text{as} \quad m \to \infty \tag{1.11}$$

for the EMM $\hat{\theta}^0_{m,t}$ based on the observation of $(X_{m+s} \colon 0 \leq s \leq t)$. The possibility of cancelling the length m of the training period will simplify the proofs essentially (see, e.g. Lemmata 3.3, 3.4, or 3.5).

In Chapter 4 we discuss the asymptotic behaviour of the stopping time under the alternative. First, it turns out that our test has asymptotic power one, i.e.,

$$\lim_{m\to\infty} P\{\tau_m < \infty\} = 1.$$

Moreover, the stopping procedure asymptotically does not react too early such that, putting both results together, we obtain

$$\lim_{m\to\infty} P\{t^*(m) \leq \tau_m < \infty\} = 1$$

(see Theorem 4.1 and Proposition 4.10).
For proving the main result of the chapter, i.e. the asymptotic normality of the stopping time, we transfer the method developed by Aue and Horváth (2004) for location models to our diffusion model. It is necessary to prove that in a slightly modified Taylor expansion of the statistic S^m_t all expressions of the order greater than zero does not contribute to the convergence. On that account, we need a similar result as in (1.11) for the EMM under the alternative. This result is proven in Proposition 2.17.
It turns out that the asymptotic normality has the same form as the result of Aue and Horváth (2004), i.e.,

$$\lim_{m\to\infty} \mathcal{L}\left(\frac{\tau_m - a_m}{b_m}\right) = N(0,1) \tag{1.12}$$

for similar families of positive numbers $(a_m\colon m > 0)$, $(b_m\colon m > 0)$. The weak convergence in (1.12) is only true for an early change. We have the same condition for an early change as Aue and Horváth (2004):

$$t^*(m) = \mathcal{O}(m^\beta) \quad \text{as} \quad m \to \infty \quad \text{for some} \quad \beta < \left(\frac{1/2-\gamma}{1-\gamma}\right)^2 .$$

In Chapter 5 we study two examples of stochastic equations which satisfy all our assumptions for the presented results. These are the Ornstein-Uhlenbeck equation and

$$dX_t = (\theta - X_t)^3 dt + \sigma dW_t, \quad t \geq 0, \tag{1.13}$$

where θ belongs to a compact interval $[\alpha, \beta]$. Equation (1.13) represents some sort of a nonlinear location model because the mean of its stationary distribution equals θ.
In particular, we show that it is possible to check the technical integrability condition contained in the definition of the set $\mathcal{M}(b, \sigma)$ in Subsection 2.1.1.

Finally, the thesis ends with an alternative approach to change-point analysis for diffusion processes and with some perspectives for future research.
Under H_0, we derive from the strong invariance principle (1.10) the strong approximation

$$t\big(\hat\theta_{0,t} - \theta_0\big) - c' B_t = \mathcal{O}\left((t\log_2 t)^{1/4}(\log t)^{1/2}\right) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty \tag{1.14}$$

where $c' > 0$, B is some Wiener process and where the starting estimator for $\hat\theta_{0,t}$ is the EMM. This direct approximation of the estimator process $(t(\hat\theta_{0,t} - \theta_0)\colon t \geq 0)$ simplifies the proof for (1.8) essentially (see Section 6.2). The crucial point in the proof of (1.14) is the following strong approximation by a stochastic integral:

$$t\big(\hat\theta_{0,t} - \theta_0\big) - c' \int_0^t \frac{\dot b(\theta_0, X_s)}{\sigma(X_s)} dW(s) = \mathcal{O}\left(\log\log t\right) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty \tag{1.15}$$

for some $c' > 0$. The proof of (1.15) is given in Proposition 6.1. The idea for the statements (1.14) and (1.15) was borrowed from Gerencsér (1991a). Gerencsér (1991a) showed for the maximum-likelihood estimator $\bar\theta_{0,t}$ of a parameter in a continuous-time linear process that for some $c' > 0$ and for some Wiener process B

$$t\big(\hat\theta_{0,t} - \theta_0\big) - c' B_t = \mathcal{O}_M\left(t^{2/5+\delta}\right) \quad \text{as} \quad t \to \infty \quad \forall\, \delta > 0$$

where once again, $\mathcal{O}_M(1)$ denotes M-boundedness.

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Chapter 2

Preliminary results

2.1 Asymptotics of stochastic integrals

Let $b\colon \mathbb{R} \to \mathbb{R}$ be a drift function and $\sigma\colon \mathbb{R} \to \mathbb{R}$ a diffusion function with $\sigma^2 > 0$. In this section denote by Y the unique solution to the Ito stochastic equation

$$dY_s = b(Y_s)ds + \sigma(Y_s)dW_s, \quad Y_0 = \xi, \quad s \geq 0, \tag{2.1}$$

where the initial value ξ is independent of the Wiener process W.
Let us introduce

Assumption $\mathcal{RP}$. *The functions b and σ satisfy*

$$\int_0^r \exp\left(-2\int_0^y \frac{b(u)}{\sigma^2(u)}du\right) dy \longrightarrow \pm\infty \quad as \quad r \to \pm\infty$$

and

$$G := \int_{\mathbb{R}} \frac{1}{\sigma^2(x)} \exp\left(2\int_0^x \frac{b(u)}{\sigma^2(u)}du\right) dx < \infty.$$

$\mathcal{RP}$ is a sufficient condition for the process Y to be ergodic with the unique stationary distribution

$$d\mu(x) = \frac{1}{G\sigma^2(x)} \exp\left(2\int_0^x \frac{b(u)}{\sigma^2(u)}du\right) dx, \quad x \in \mathbb{R},$$

(see Kutoyants (2004), Section 1.2).

In this section we discuss some asymptotic properties of the stochastic integral

$$\int_{t_0}^{t_0+t} f(Y_s)dW_s \quad \text{as} \quad t \to \infty$$

for an arbitrary starting time $t_0 \geq 0$.
For any function $f \in L^2(\mu)$ with

$$E_\mu|f|^2 := \int_{\mathbb{R}} |f(x)|^2 d\mu(x) > 0$$

and

$$\lim_{t\to\infty} \int_{t_0}^{t_0+t} f^2(Y_s)ds = \infty \quad P\text{-a.s.} \tag{2.2}$$

one can apply a representation theorem for local martingales with continuous trajectories to the concrete process

$$M_t = \int_{t_0}^{t_0+t} f(Y_s)dW_s, \quad t \geq 0, \tag{2.3}$$

with the quadratic variation

$$\langle M\rangle_t = \int_{t_0}^{t_0+t} f^2(Y_s)ds, \quad t \geq 0. \tag{2.4}$$

Then one obtains that M is a time-changed Wiener process. In fact, for the family of stopping times

$$\tau(t) = \tau_{t_0}(t) = \inf\big\{s \geq 0\colon \langle M\rangle_s > t\big\}, \quad t \geq 0, \tag{2.5}$$

the process $W^{(t_0)}$ given by

$$W_t^{(t_0)} = \int_{t_0}^{t_0+\tau(t)} f(Y_s)dW_s, \quad t \geq 0, \tag{2.6}$$

is a Wiener process, and the equation

$$\int_{t_0}^{t_0+t} f(Y_s)dW_s = W^{(t_0)}_{\langle M\rangle_t} \quad \forall\, t \geq 0 \tag{2.7}$$

holds P-a.s. We only have to check that the quadratic variation $\langle M\rangle$ satisfies (2.2):
If $\mathcal{RP}$ is true, by a time substitution and the ergodic theorem we obtain

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t_0+t} f^2(Y_s)ds = \lim_{t\to\infty} \frac{1}{t} \int_0^t f^2(Y_{t_0+s})ds = E_\mu|f|^2 > 0 \quad P\text{-a.s.}$$

Hence, (2.2) follows.

For a proof of the representation theorem in the case of stochastic integrals confer Gihman and Skorohod (1972), Part I., §. 4. In the general case of local martingales a proof is presented, e.g. by Karatzas and Shreve (1998), Chapter 3, Theorem 4.6.
According to the remark after Theorem (4.9) of Durrett (1996), by means of (2.7) the asymptotic behaviour of stochastic integrals can be derived from the asymptotics of Wiener processes. We apply this principle in the following

Lemma 2.1. *Let* $t_0 \geq 0$, $f \in L^2(\mu)$ *and Assumption* $\mathcal{RP}$ *be true. If* Y *is a stationary process or if* $E_\mu |f|^2 > 0$, *then we have*

$$\limsup_{t\to\infty} \frac{\int_{t_0}^{t_0+t} f(Y_s)dW_s}{\sqrt{2t \log_2 t}} = \sqrt{E_\mu |f|^2} \quad P\text{-a.s.}$$

Proof. If Y is stationary and $E_\mu |f|^2 = 0$, we have $f = 0$ μ-a.s. Hence,

$$f(Y_s) = 0 \quad P\text{-a.s.} \quad \forall\, s \geq 0.$$

Since Y has continuous trajectories, it also holds that

$$f(Y_s) = 0 \quad \forall\, s \geq 0 \quad P\text{-a.s.}$$

Therefore, the stochastic integral of $f(Y)$ vanishes and the statement of the lemma is true.
Now we consider the case $E_\mu |f|^2 > 0$. Since (2.2) is valid, the representation in (2.7) results, and the law of the iterated logarithm (LIL) for Wiener processes yields

$$\limsup_{t\to\infty} \frac{\int_{t_0}^{t_0+t} f(Y_s)dW_s}{\sqrt{2\langle M\rangle_t \log_2 \langle M\rangle_t}} = 1 \quad P\text{-a.s.} \tag{2.8}$$

Moreover, we have

$$\lim_{t\to\infty} \frac{\log\langle M\rangle_t}{\log t} = \lim_{t\to\infty} \left(\frac{\log \frac{\langle M\rangle_t}{t}}{\log t} + 1 \right) = 1 \quad P\text{-a.s.}$$

because the ergodic theorem gives

$$\lim_{t\to\infty} \frac{\langle M\rangle_t}{t} = E_\mu |f|^2 > 0 \quad P\text{-a.s.} \tag{2.9}$$

Similarly, one obtains

$$\lim_{t\to\infty} \frac{\log_2\langle M\rangle_t}{\log_2 t} = \lim_{t\to\infty} \left(\frac{1}{\log_2 t} \log \frac{\log\langle M\rangle_t}{\log t} + 1 \right) = 1 \quad P\text{-a.s.} \tag{2.10}$$

Equations (2.9) and (2.10) imply

$$\lim_{t\to\infty} \frac{\langle M\rangle_t \log_2 \langle M\rangle_t}{t \log_2 t} = E_\mu |f|^2 \quad P\text{-a.s.} \tag{2.11}$$

Considerating the definition of the limes superior, by (2.8) and (2.11) we obtain the statement of the lemma.

□

2.1.1 Strong invariance principle

First, let us adopt some useful notations of Mandl (1968): using the function

$$B(x) = \int_0^x \frac{b(u)}{\sigma^2(u)} du, \quad x \in \mathbb{R},$$

define the measure p by $dp(x) = e^{-2B(x)}dx$, $x \in \mathbb{R}$. Then the stationary distribution μ of the process Y can be written in the compact form

$$d\mu(x) = \frac{e^{2B(x)}}{G\sigma^2(x)} dx, \quad x \in \mathbb{R}.$$

Now we introduce the set $\mathcal{M}(b,\sigma) \subset L^1(\mu)$ of functions g which satisfy

(i) $E_\mu g = 0$;

(ii) $\displaystyle\int_{\mathbb{R}} \left| g(y) \int_y^0 \int_{-\infty}^s g(z)\, d\mu(z)\, dp(s) \right| d\mu(y) < \infty.$

The dependence on the functions b and σ in the notation $\mathcal{M}(b,\sigma)$ shall indicate that properties (i) and (ii) are formulated by means of b and σ.

Remark 2.2. Let $t_0 \geq 0$ and $g \in \mathcal{M}(b,\sigma)$. Then Assumption $\mathcal{RP}$ ensures the following LIL:

$$\limsup_{t\to\infty} \frac{\int_{t_0}^{t_0+t} g\big(Y_s\big) ds}{\sqrt{2t \log\log t}} = \sqrt{D} < \infty \quad P\text{-a.s.}$$

where D is a positive constant.

Proof. First, we show that Assumption $\mathcal{RP}$ implies

$$\int_0^r \int_0^s \frac{e^{2B(x)}}{\sigma^2(x)}\, dx\, dp(s) \longrightarrow \infty \quad \text{as} \quad r \to \pm\infty. \tag{2.12}$$

We restrict ourself to the case $r \to -\infty$ because the other case is similar and even easier.
There exists a real number $A > 0$ such that

$$\int_s^0 \frac{e^{2B(x)}}{\sigma^2(x)} dx \geq A \quad \forall\, s < -1$$

because the integrand is positive. Then consider for $r < -1$

$$\begin{aligned} \int_0^r \int_0^s \frac{e^{2B(x)}}{\sigma^2(x)}\, dx\, dp(s) &= \int_r^0 \int_s^0 \frac{e^{2B(x)}}{\sigma^2(x)}\, dx\, dp(s) \\ &\geq \int_{-1}^0 \int_s^0 \frac{e^{2B(x)}}{\sigma^2(x)}\, dx\, dp(s) + A \int_r^{-1} dp(s). \end{aligned}$$

Moreover, the first condition of Assumption $\mathcal{RP}$ states that

$$\lim_{r \to -\infty} \int_r^{-1} dp(s) = \infty. \tag{2.13}$$

Equation (2.13) implies (2.12).
Now, by (2.12) and a time substitution one can apply Theorem 10 of Mandl (1968), Chapter IV, in order to get the following LIL:

$$\limsup_{t \to \infty} \frac{\int_{t_0}^{t_0+t} g(Y_s) ds}{\sqrt{2t \log \log t}} = \sqrt{D} < \infty \quad P^x\text{-a.s.}$$

where $D > 0$ and P^x represents the probability measure on the underlying measurable space if the process Y starts at the arbitrary deterministic point $x \in \mathbb{R}$. Due to the Markov property of Y, we obtain the LIL for any initial distribution ν, too:

$$\begin{aligned} & P^\nu\left(\limsup_{t \to \infty} \frac{\int_{t_0}^{t_0+t} g(Y_s) ds}{\sqrt{2Dt \log_2 t}} = 1 \right) \\ &= \int_{\mathbb{R}} P^x \left(\limsup_{t \to \infty} \frac{\int_{t_0}^{t_0+t} g(Y_s) ds}{\sqrt{2Dt \log_2 t}} = 1 \right) d\nu(x) \\ &= 1 \end{aligned}$$

where P^ν denotes the probability measure if the initial distribution of Y is given by ν.

□

Now we give a simple proof for one of the key results of the thesis, i.e., a strong invariance principle with rate for certain stochastic integrals:

Theorem 2.3. *Let f be a function in $L^2(\mu)$ such that $E_\mu f^2 > 0$ and $(f^2 - E_\mu f^2) \in \mathcal{M}(b,\sigma)$. Then, under Assumption $\mathcal{RP}$ and for any time $t_0 \geq 0$, there exists a Wiener process $W^{(t_0)}$ such that P-a.s.*

$$\int\limits_{t_0}^{t_0+t} \frac{f(Y_s)}{\sqrt{E_\mu f^2}}\, dW_s - W_t^{(t_0)} = \mathcal{O}\left((t\log_2 t)^{1/4}(\log t)^{1/2}\right) \quad as \quad t \to \infty.$$

Proof. Define $a = \sqrt{E_\mu f^2}$. Replace the function f by f/a in the definition of the quadratic variation $\langle M\rangle$ (see (2.4)). Hence, there exists a Wiener process $W^{(t_0)}$ such that

$$\int\limits_{t_0}^{t_0+t} \frac{f(Y_s)}{a}\, dW_s = W^{(t_0)}_{\langle M\rangle_t} \quad \forall\, t \geq 0.$$

Applied on $g = (f/a)^2 - 1$, the LIL of Remark 2.2 implies that for almost every $\omega \in \Omega$ and for an arbitrary $\varepsilon > 0$ there exists some $T > 0$ such that

$$\frac{|\langle M\rangle_t(\omega) - t|}{\sqrt{t\log\log t}} \leq \sqrt{2}(\sqrt{D}+\varepsilon) =: c \quad \forall\, t > T. \tag{2.14}$$

By (2.14) and the estimate

$$\log_2 t - \tfrac{1}{2}\log_3 t \leq \log\sqrt{t} = \tfrac{1}{2}\log t \quad \text{for sufficiently large} \quad t$$

we get

$$\begin{aligned} \frac{\left|W^{(t_0)}_{\langle M\rangle_t} - W^{(t_0)}_t\right|}{(t\log_2 t)^{1/4}(\log t)^{1/2}} &\leq \frac{\sup\left\{\left|W^{(t_0)}_s - W^{(t_0)}_t\right| : |s-t| \leq c\sqrt{t\log_2 t}\right\}}{(t\log_2 t)^{1/4}(\log t)^{1/2}} \\ &\leq \frac{\sup\left\{\left|W^{(t_0)}_s - W^{(t_0)}_t\right| : |s-t| \leq c\sqrt{t\log_2 t}\right\}}{(t\log_2 t)^{1/4}(\log t - \frac{1}{2}(\log t + \log_3 t) + \log_2 t)^{1/2}} \end{aligned} \tag{2.15}$$

if t is large enough. Set for $t \geq e$

$$\begin{aligned} a_t &= \sqrt{t\log_2 t}, \quad & b_t &= \sqrt{a_t(\log t - \log a_t + \log_2 t)}, \\ \tilde{a}_t &= c a_t, & \tilde{b}_t &= \sqrt{\tilde{a}_t(\log t - \log\tilde{a}_t + \log_2 t)}. \end{aligned}$$

By Theorem 1.2.1, equation (1.2.3), of Csörgő and Révész (1981) we have

$$\limsup_{t\to\infty} \frac{\sup\left\{\left|W^{(t_0)}_s - W^{(t_0)}_t\right| : t \leq s \leq t + \tilde{a}_t\right\}}{\sqrt{\tilde{a}_t(\log t - \log\tilde{a}_t + \log_2 t)}} = \sqrt{2} \quad P\text{-a.s.}$$

Thereby, the convergence

$$\lim_{t\to\infty} \frac{\sqrt{\tilde{a}_t(\log t - \log \tilde{a}_t + \log_2 t)}}{\sqrt{a_t(\log t - \log a_t + \log_2 t)}} = \sqrt{c} \tag{2.16}$$

implies

$$\frac{\sup\left\{\left|W_s^{(t_0)} - W_t^{(t_0)}\right| : t \le s \le t + \tilde{a}_t\right\}}{\sqrt{a_t(\log t - \log a_t + \log_2 t)}} = \mathcal{O}(1) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty. \tag{2.17}$$

In order to obtain

$$\frac{\sup\left\{\left|W_s^{(t_0)} - W_t^{(t_0)}\right| : t - \tilde{a}_t \le s \le t\right\}}{\sqrt{a_t(\log t - \log a_t + \log_2 t)}} = \mathcal{O}(1) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty, \tag{2.18}$$

consider for sufficiently large t

$$\tilde{a}_t\big(\log t - \log \tilde{a}_t + \log_2 t\big) \ge \tilde{a}_t\big(\log(t - \tilde{a}_t) - \log \tilde{a}_t + \log_2(t - \tilde{a}_t)\big)$$

and combine (2.16) with (1.2.3) of Csörgő and Révész (1981). By (2.15), (2.17) and (2.18) the asymptotics

$$\frac{\left|W_{\langle M\rangle_t}^{(t_0)} - W_t^{(t_0)}\right|}{(t\log_2 t)^{1/4}(\log t)^{1/2}} = \mathcal{O}(1) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty$$

results.

□

Corollary 2.4. *Let* $\alpha > 1/4$. *Under the assumptions of Theorem 2.3 we have*

$$\lim_{m\to\infty} \sup_{t>0} \frac{1}{(m+t)^\alpha} \left| \int_0^{m+t} f(Y_s)dW_s - aW_{m+t}^{(0)} \right| = 0 \quad P\text{-a.s.}$$

where $a = \sqrt{E_\mu f^2}$.

Proof. By Theorem 2.3 for almost every $\omega \in \Omega$ and for given $\varepsilon > 0$ there exists some $T > 0$ such that

$$\frac{\left| \int_0^{m+t} f(Y_s)dW_s - aW_{m+t}^{(0)} \right|}{(m+t)^\alpha} < \varepsilon \quad \forall\, t > 0 \quad \forall\, m > T.$$

Thereby, we obtain

$$\sup_{t>0} \frac{\left| \int_0^{m+t} f(Y_s)dW_s - aW_{m+t}^{(0)} \right|}{(m+t)^\alpha} \le \varepsilon \quad \forall\, m > T.$$

□

2.1.2 Uniform weak invariance principle

Now, from the strong approximation in Theorem 2.3 we derive under H_1 a uniform weak invariance principle for the observed process X. For the definition of X see (1.2).
We start the subsection with a lemma which will be useful in the proof of the weak invariance principle as well as in later chapters:

Lemma 2.5. *Let* $f, F \in C(\mathbb{R})$ *and* $t_0 \geq 0$. *The maps* $\Phi_1, \Phi_2 \colon C[0,\infty) \to C[0,\infty)$ *given by*

$$\Phi_1(x)(t) = F\big(x(t_0+t)\big) - F\big(x(t_0)\big), \quad x \in C[0,\infty),\, t \geq 0,$$
$$\Phi_2(x)(t) = \int_{t_0}^{t_0+t} f\big(x(s)\big)\, ds, \quad x \in C[0,\infty),\; t \geq 0,$$

are continuous.

Proof. It is sufficient to prove the statement of the lemma for $t_0 = 0$ since the shift operator $T_{t_0} \colon C[0,\infty) \to C[0,\infty)$, $x \mapsto T_{t_0}x$ with

$$(T_{t_0}x)(t) = x(t_0+t), \quad t \geq 0,$$

is a continuous map. Note for this argument that by time substitution we have

$$\int_{t_0}^{t_0+t} f(x(s))ds = \int_0^t f(x(t_0+u))du \quad \forall\, x \in C[0,\infty),\, t \geq 0.$$

Let $x_0 \in C[0,\infty)$. Remember that the topology of the locally convex space $C[0,\infty)$ is given by the sequence of seminorms

$$\|x\|_N := \sup_{t\in[0,N]} |x(t)|, \quad N \in \mathbb{N},$$

in the way that for any x_0 the family of neighbourhoods

$$U_{1/N} = \big\{x \in C[0,\infty) \colon \|x - x_0\|_N < \tfrac{1}{N}\big\}, \quad N \in \mathbb{N},$$

forms a fundamental system of neighbourhoods of x_0, also called "base of neighbourhoods" or "local base". Thus, in order to prove continuity, for any $N' \in \mathbb{N}$ we have to find an $N \in \mathbb{N}$ such that for $x \in C[0,\infty)$:

$$\|x - x_0\|_N < \frac{1}{N} \quad \Longrightarrow \quad \|\Phi_i(x) - \Phi_i(x_0)\|_{N'} < \frac{1}{N'}, \quad i = 1, 2.$$

Set $\varepsilon = 1/(2N')$. For any N and for any x with $\|x - x_0\|_N < 1/N$ the image $x[0,N]$ is contained in the compact neighbourhood $\overline{V_1(x_0[0,N])}$ where

$$V_1\big(x_0[0,N]\big) = \big\{y \in \mathbb{R}\colon \operatorname{dist}\big(y, x_0[0,N]\big) < 1\big\}.$$

Since F is uniformly continuous on $\overline{V_1(x_0[0,N_1])}$ for some arbitrarily chosen integer $N_1 > N'$, for the chosen ε there exists an $N > N_1$ such that

$$\|F \circ x - F \circ x_0\|_{N'} < \varepsilon \quad \text{if} \quad \|x - x_0\|_N < 1/N.$$

Hence,

$$\begin{aligned} \|\Phi_1(x) - \Phi_1(x_0)\|_{N'} &\le 2\|F \circ x - F \circ x_0\|_{N'} \\ &< \frac{1}{N'} \end{aligned}$$

holds if $\|x - x_0\|_N < 1/N$.
Along the lines of the previous argumentation, it follows from the uniform continuity of f on compact sets that for $\varepsilon = (1/N')^2$ an $N > N'$ exists such that

$$\|x - x_0\|_N < \frac{1}{N} \quad \Longrightarrow \quad \|f \circ x - f \circ x_0\|_{N'} < \varepsilon.$$

Thereby, we obtain the desired estimate:

$$\begin{aligned} \left\| \int_0^{(\cdot)} f(x(s))ds - \int_0^{(\cdot)} f(x_0(s))ds \right\|_{N'} &\le \sup_{t\in[0,N']} t \sup_{s\in[0,t]} |f(x(s)) - f(x_0(s))| \\ &\le N' \, \|f \circ x - f \circ x_0\|_{N'} \\ &< \frac{1}{N'}. \end{aligned}$$

□

Theorem 2.6. *Let* $\alpha > 1/4$, $\theta_1 \neq \theta_0$ *and* $f \in L^2(\mu(\theta_1)) \cap C^1(\mathbb{R})$ *be a function such that* $E_{\mu(\theta_1)} f^2 > 0$, *and* $(f^2 - E_{\mu(\theta_1)} f^2) \in \mathcal{M}(b(\theta_1), \sigma)$. *Suppose that the functions* $b(\theta_1, \cdot)$, σ *satisfy Assumption* $\mathcal{RP}$.
If H_1 *holds, then there exists a family of Wiener processes* $(W^{(m+t^*)}\colon m \ge 0)$ *such that*

$$\sup_{t>0} \frac{1}{(t+1)^\alpha} \left| \frac{1}{\sqrt{E_{\mu(\theta_1)} f^2}} \int_{m+t^*}^{m+t^*+t} f(X_s)dW_s - W_t^{(m+t^*)} \right| \tag{2.19}$$

is a P*-a.s. finite random variable which distribution is independent of* $m \ge 0$.

Remark 2.7. We write in the denominator $(t+1)^\alpha$ instead of t^α in order to avoid complications in the neighbourhood of $t = 0$.

Proof. Set for brevity $t_0 = m + t^*$. By Theorem 2.3 the approximation

$$\int_{t_0}^{t_0+t} \frac{f(X_s)}{\sqrt{E_{\mu(\theta_1)} f^2}} dW_s - W_t^{(t_0)} = \mathcal{O}\left((t\log_2 t)^{1/4} (\log t)^{1/2}\right) \quad \text{as} \quad t \to \infty$$

holds P-a.s. Hence, for any m the expression in (2.19) belongs to $\mathbb{R}$ P-a.s. Set $g = f/(E_{\mu(\theta_1)} f^2)^{1/2}$ and denote by F a primitive of g/σ. It follows from the Ito formula that P-a.s.

$$\begin{aligned} \int_{t_0}^{t_0+t} g(X_s) dW_s &= \int_{t_0}^{t_0+t} \frac{g(X_s)}{\sigma(X_s)} \cdot \sigma(X_s) dW_s \\ &= F(X_{t_0+t}) - F(X_{t_0}) - \int_{t_0}^{t_0+t} F'(X_s) b(\theta_1, X_s) ds \\ &\quad - \frac{1}{2} \int_{t_0}^{t_0+t} F''(X_s) \sigma^2(X_s) ds, \quad \forall\, t \geq 0. \end{aligned} \tag{2.20}$$

Thus, by Lemma 2.5 the stochastic integral, interpreted as a process in $t \geq 0$, is a composition of the process $(X_{t_0+s} \colon s \geq 0)$ and a measurable map in $C[0, \infty)$. We would like to apply the Ito formula in the same way to the stochastic integral with random integration limit

$$W_t^{(t_0)} = \int_{t_0}^{t_0+\tau(t)} g(X_s) dW_s, \quad t \geq 0,$$

where we recall that

$$\tau(t) = \tau_{t_0}(t) = \inf\{s \geq 0 \colon \langle M \rangle_s > t\}$$

and

$$\langle M \rangle_s = \int_{t_0}^{t_0+s} g^2(X_u) du, \quad s \geq 0.$$

Toward this end, note that the strict stopping time $t_0 + \tau(t)$ is P-a.s. finite because we can repeat the computations from the introduction of the chapter in order to obtain by the ergodic theorem and $E_{\mu(\theta_1)} g^2 > 0$ that

$$\lim_{s \to \infty} \int_{t_0}^{t_0+s} g^2(X_u) du = \infty \quad P\text{-a.s.}$$

(see equation (2.2)).
Finally, we get

$$\int_{t_0}^{t_0+(\cdot)} f(X_s)dW_s - \int_{t_0}^{t_0+\tau(\cdot)} f(X_s)dW_s = \Phi(X_{t_0+s}\colon s \geq 0)$$

with a measurable map $\Phi\colon C[0,\infty) \to C[0,\infty)$. Therefore, since X is a Markov process, the distribution of expression (2.19) only depends on the transition probabilities and the initial distribution $\mathcal{L}(X_{t_0})$. However, the distribution of $X_{t_0} = X_{m+t^*}$ is independent of m because, by assumption, the process X runs stationarily up to the change-point $m + t^*$.

□

Remark 2.8. If we replace in Theorem 2.6 θ_1 by θ_0 and $m + t^*$ by m, the statement remains true under H_0. I.e., there exists a family of Wiener processes $(W^{(m)}\colon m \geq 0)$ such that

$$\sup_{t>0} \frac{1}{(t+1)^\alpha} \left| \frac{1}{\sqrt{E_{\mu(\theta_0)} f^2}} \int_m^{m+t} f(X_s)dW_s - W_t^{(m)} \right|$$

is a P-a.s. finite random variable which distribution is independent of $m \geq 0$.

2.2 Parameter estimation

Recall that $\Theta \subset \mathbb{R}$ is defined to be a compact interval. For any $\theta \in \Theta$ let $X(\theta)$ be the unique solution to the Ito stochastic equation

$$dX_s = b(\theta, X_s)ds + \sigma(X_s)dW_s, \quad X_0 \sim \mu(\theta), \quad 0 \leq s \leq m, \tag{2.21}$$

where a unique stationary distribution $\mu(\theta)$ exists and the initial value is independent of the Wiener process W. Denote by P_θ the induced measure of $X(\theta)$ on the path space $C[0,\infty)$ and by f_θ the density of the starting distribution $\mu(\theta)$. Under the defined statistical model we present two procedures for estimating the unknown parameter θ if a trajectory of $(X(\theta)\colon 0 \leq s \leq m)$ was observed. In the first subsection we will prove some important properties of the estimator of the method of moments (EMM). These properties will be needed for proving the limit theorems in Chapters 3 and 4.
The set of functions $g\colon \mathbb{R} \to \mathbb{R}$ which have at most polynomial growth will be denoted by

$$\mathcal{P} = \left\{ g\colon\ \exists\, K, p > 0 \quad \text{s.t.} \quad |g(x)| \leq K\,(1 + |x|^p) \quad \forall\, x \right\}$$

For a function $g\colon \Theta \times \mathbb{R} \to \mathbb{R}$ we say that $g(\theta, \cdot)$ belongs to $\mathcal{P}$ uniformly in θ if the constants K and p can be chosen to be independent of θ.
Let us adopt from Kutoyants (2004) a version of Khasminskii's condition which is uniform in θ:

Assumption $\mathcal{A}_0(\Theta)$. *The function* $1/\sigma$ *belongs to* $\mathcal{P}$ *and*

$$\limsup_{|x|\to\infty} \left(\sup_{\theta\in\Theta} \operatorname{sgn}(x) \frac{b(\theta, x)}{\sigma^2(x)} \right) < 0. \tag{2.22}$$

As mentioned by Kutoyants (2004), the condition of Khasminskii implies Assumption $\mathcal{RP}$ defined in the introduction of Section 2.1. Hence, for any $\theta \in \Theta$ the process $X(\theta)$ is ergodic.

Remark 2.9. Set

$$B_\theta(x) = \int_0^x \frac{b(\theta, y)}{\sigma^2(y)} dy, \quad x \in \mathbb{R}.$$

If $b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ and Assumption $\mathcal{A}_0(\Theta)$ is satisfied, then the exponential part e^{2B_θ} of the stationary density

$$f_\theta(x) = \frac{e^{2B_\theta(x)}}{G(\theta)\sigma^2(x)}, \quad x \in \mathbb{R},$$

has exponentially decreasing tails uniformly in θ.

Proof. The condition in (2.22) means that there exist real numbers $\kappa, R > 0$ such that

$$\sup_{\theta\in\Theta} \operatorname{sgn}(x) \frac{b(\theta, x)}{\sigma^2(x)} < -\kappa < 0 \quad \text{if} \quad |x| > R.$$

Hence, for $x > R$ we have

$$\begin{aligned} \int_0^x \frac{b(\theta, y)}{\sigma^2(y)} dy &\le \int_0^R \frac{b(\theta, y)}{\sigma^2(y)} dy - \kappa(x - R) \\ &\le c_1 - \kappa(x - R) \end{aligned} \tag{2.23}$$

for some positive constant c_1. For $x < -R$ a similar computation yields

$$\begin{aligned} \int_x^0 -\frac{b(\theta, y)}{\sigma^2(y)} dy &\le \int_{-R}^0 -\frac{b(\theta, y)}{\sigma^2(y)} dy + \kappa(x + R) \\ &\le c_2 + \kappa(x + R) \end{aligned} \tag{2.24}$$

for some positive constant c_2. Equations (2.23) and (2.24) imply the desired statement.

□

A very useful property of the normalization constant $G(\theta)$ will be proven in the following

Lemma 2.10. *Let* $1/\sigma \in \mathcal{P}$ *as well as* $b(\theta,\cdot) \in \mathcal{P}$ *uniformly in* θ. *Then we have* $\inf_{\theta\in\Theta} G(\theta) > 0$.

Proof. We have

$$\inf_\theta G(\theta) \geq \int_{\mathbb{R}} \frac{\inf_\theta e^{2B_\theta(x)}}{\sigma^2(x)} dx.$$

If $\inf_\theta G(\theta) = 0$, then

$$\lambda\Big\{x \in \mathbb{R}\colon \inf_\theta e^{2B_\theta(x)} \neq 0\Big\} = 0 \tag{2.25}$$

would be true where λ denotes the Lebesgue measure. By assumption there exists some positive polynomial p such that for any point $x > 0$

$$|2B_\theta(x)| \leq \int_0^x p(y)dy =: q(x) \quad \forall\, \theta \in \Theta$$

where q is a second positive polynomial. Hence, for any $x > 0$

$$e^{2B_\theta(x)} \geq e^{-q(x)} \quad \forall\, \theta \in \Theta.$$

However, this is a contradiction to (2.25).

□

2.2.1 Estimator of the method of moments

Consider the set $\mathcal{D}(\Theta)$ of functions $q\colon \mathbb{R} \to \mathbb{R}$ characterized by the following regularity conditions taken from Kutoyants (2004), Section 2.4: q belongs to $C(\Theta) \cap \mathcal{P}$ and the function $a\colon \Theta \to \mathbb{R}$, $a(\theta) = E_{\mu(\theta)}q$, is a continuously differentiable injection.
Recall from Section 1.1 that under $a \in C^1(\Theta)$ we understand that $a \in C^1(\mathring{\Theta}) \cap C(\Theta)$ and that $\partial_\theta a$ has a continuous extension on the closed set Θ.

Remark 2.11. If $b(\theta,\cdot), \partial_\theta b(\theta,\cdot) \in \mathcal{P}$ uniformly in θ, then, under Assumption $\mathcal{A}_0(\Theta)$, we have $a \in C^1(\Theta)$.

Proof. In the interior $\mathring{\Theta}$ the dominated convergence theorem can be applied on the function

$$a(\theta) = \int_{\mathbb{R}} q(x) \frac{e^{2B_\theta(x)}}{G(\theta)\sigma^2(x)} dx. \tag{2.26}$$

For this purpose, note that

$$\partial_\theta \frac{e^{2B_\theta(x)}}{G(\theta)} = \frac{[2G(\theta)\dot{B}_\theta(x) - \dot{G}(\theta)]e^{2B_\theta(x)}}{G^2(\theta)} \quad \forall\, \theta \in \Theta,\, x \in \mathbb{R} \tag{2.27}$$

and

$$G(\theta) = \int\limits_{\mathbb{R}} \frac{e^{2B_\theta(x)}}{\sigma^2(x)} dx.$$

In view of Lemma 2.10, it remains to show that the numerator in (2.27) is dominated uniformly in θ by an integrable function. The assumptions imply that $B_\theta,\, \partial_\theta B_\theta \in \mathcal{P}$ uniformly in θ. Considerating Remark 2.9 and the dominated convergence theorem, this yields $G \in C^1(\Theta)$. Finally, the boundedness of G and $\partial_\theta G$, the growth behaviour of $\partial_\theta B_\theta$ and Remark 2.9 imply the boundedness of the numerator in (2.27) by an integrable function uniformly in θ.

By the same conditions and the same arguments one obtains the continuity of a and $\partial_\theta a$ in the boundary points.

□

We introduce the notations $\hat{a}_{m,0} = q(X_m)$ and

$$\hat{a}_{m,t} = \frac{1}{t} \int\limits_m^{m+t} q(X_s) ds, \quad t > 0,$$

where X is the observed process which solves (2.21) with the true parameter value.

Choose some $q \in \mathcal{P}$ such that a is a continuous injection. The estimator of the method of moments (EMM) $\hat{\theta}^0_{m,t}$ based on the observation of a trajectory of $\big(X_s \colon m \le s \le m+t\big)$ is defined as the point in Θ where the function

$$\theta \longmapsto \left| a(\theta) - \frac{1}{t} \int\limits_m^{m+t} q(X_s) ds \right| = \left| a(\theta) - \hat{a}_{m,t} \right|, \quad \theta \in \Theta,$$

attains its minimum. E.g., if a is strictly increasing and $\Theta = [\alpha, \beta]$, $\alpha \le \beta$, then the EMM has the form

$$\hat{\theta}^0_{m,t} = \begin{cases} a^{-1}(\hat{a}_{m,t}), & \hat{a}_{m,t} \in a(\Theta), \\ \alpha, & \hat{a}_{m,t} < a(\alpha), \\ \beta, & \hat{a}_{m,t} > a(\beta), \end{cases} \qquad m,\, t \ge 0.$$

Remark 2.12. Under some additional conditions it was proven by Kutoyants (2004), Theorem 2.28, that for $m = 0$ the EMM is consistent and asymptotically normal as $t \to \infty$.

Lemma 2.13. *Let* $q \in \mathcal{P}$ *and the function* a *be injective and continuous. Then the function* $t \mapsto \hat{\theta}^0_{m,t}$, $t \geq 0$, *is* P*-a.s. continuous for any* $m \geq 0$.

Proof. Without loss of generality, let a be strictly increasing. Choose an arbitrary time $s > 0$. We have to distinguish between two cases:

a) First, suppose that $\hat{a}_{m,s} \in a(\mathring{\Theta})$. Then there exists $\hat{\theta}^0_{m,s} \in \mathring{\Theta}$ such that $a(\hat{\theta}^0_{m,s}) = \hat{a}_{m,s}$. Since the inverse function a^{-1} is continuous, the image $a(\mathring{\Theta})$ is open. Hence, by the continuity of the function $\hat{a}_{m,(\cdot)}$ there exists some $\delta > 0$ such that for all t with $|t - s| < \delta$ one has $\hat{a}_{m,t} \in a(\mathring{\Theta})$. Take $\hat{\theta}^0_{m,t} \in \mathring{\Theta}$ with $a(\hat{\theta}^0_{m,t}) = \hat{a}_{m,t}$. Once again, by the continuity of a^{-1} we can find for any $\varepsilon > 0$ an $\varepsilon' > 0$ with $(\hat{a}_{m,s} - \varepsilon', \hat{a}_{m,s} + \varepsilon') \in a(\mathring{\Theta})$ such that from

$$\left|\hat{a}_{m,t} - \hat{a}_{m,s}\right| = \left|a(\hat{\theta}^0_{m,t}) - a(\hat{\theta}^0_{m,s})\right| < \varepsilon'$$

we obtain $|\hat{\theta}^0_{m,t} - \hat{\theta}^0_{m,s}| < \varepsilon$. Finally, for ε' there exists some $\delta' > 0$ such that

$$\left|\hat{a}_{m,t} - \hat{a}_{m,s}\right| < \varepsilon' \quad \forall\, t \quad \text{with} \quad |t - s| < \delta'.$$

b) In case of $\hat{a}_{m,s} \notin a(\mathring{\Theta})$, we can suppose without loss of generality that $\hat{a}_{m,s} > a(\theta)\ \forall\, \theta \in \mathring{\Theta}$.

By the continuity of $\hat{a}_{m,(\cdot)}$ we can find an $\varepsilon' > 0$ such that either

$$\text{(i)} \quad \hat{a}_{m,t} > a(\theta) \ \ \forall\, \theta \in \mathring{\Theta} \qquad \text{or} \qquad \text{(ii)} \quad \hat{a}_{m,t} \in a(\mathring{\Theta})$$

holds if $|\hat{a}_{m,t} - \hat{a}_{m,s}| < \varepsilon'$ for some $t > 0$. I.e., the case $\hat{a}_{m,t} < a(\theta)\ \forall\, \theta \in \mathring{\Theta}$ can be excluded. If (i) is valid, we have $\hat{\theta}^0_{m,t} = \hat{\theta}^0_{m,s}$ anyway. In case (ii) and for any $\varepsilon > 0$ one has to choose ε' sufficiently small in order to get

$$\left|a(\hat{\theta}^0_{m,t}) - a(\hat{\theta}^0_{m,s})\right| < \varepsilon' \quad \Longrightarrow \quad \left|\hat{\theta}^0_{m,t} - \hat{\theta}^0_{m,s}\right| < \varepsilon.$$

Since a is strictly increasing and $\hat{\theta}^0_{m,s}$ is the right boundary point of Θ, for ε' there exists a $\delta > 0$ such that for t with $|t - s| < \delta$ the estimate

$$\left|a(\hat{\theta}^0_{m,t}) - a(\hat{\theta}^0_{m,s})\right| = \left|\hat{a}_{m,t} - a(\hat{\theta}^0_{m,s})\right| = a(\hat{\theta}^0_{m,s}) - \hat{a}_{m,t} \leq \hat{a}_{m,s} - \hat{a}_{m,t} < \varepsilon'$$

finally results.

□

Now it is advisable to be more precise with notations and to understand under

$$\hat\theta^0_{m,t}\colon C[0,\infty) \longrightarrow \mathbb{R}, \quad x \longmapsto \hat\theta^0_{m,t}(x), \quad m,t \geq 0,$$

the functional which constructs the estimator of the method of moments from the restriction $(x(s)\colon m \leq s \leq m+t)$ of a path $x \in C[0,\infty)$. Moreover, let us introduce for any $m \geq 0$ the map

$$\hat\theta^0_m\colon C[0,\infty) \longrightarrow C[0,\infty), \quad x \longmapsto \big(\hat\theta^0_{m,t}(x)\colon t \geq 0\big).$$

Lemma 2.14. *Let $q \in \mathcal{P}$ be continuous and the function a be injective and continuous. Then for any $m \geq 0$ the map $\hat\theta^0_m$ is continuous.*

Proof. Without loss of generality, assume a to be strictly increasing. Let $x_0 \in C[0,\infty)$ be an arbitrary point. In the course of this proof, let us simply denote the shifted functions $x(m+(\cdot))$, $x_0(m+(\cdot))$ by x, x_0, respectively. As explained in the proof of Lemma 2.5, the issue is to find for each $N' \in \mathbb{N}$ some $N \in \mathbb{N}$ such that

$$\|x - x_0\|_N < \frac{1}{N} \quad \Longrightarrow \quad \big\|\hat\theta^0_m(x) - \hat\theta^0_m(x_0)\big\|_{N'} < \frac{1}{N'}.$$

Due to the uniform continuity of a^{-1} on the compact set $a(\Theta)$, for given N' there exists an $\varepsilon > 0$ with the following implication:

$$\big\|a\big(\hat\theta^0_m(x)\big) - a\big(\hat\theta^0_m(x_0)\big)\big\|_{N'} < \varepsilon \quad \Longrightarrow \quad \big\|\hat\theta^0_m(x) - \hat\theta^0_m(x_0)\big\|_{N'} < \frac{1}{N'}. \tag{2.28}$$

In the next step we state that

$$\big|a\big(\hat\theta^0_{m,t}(x)\big) - a\big(\hat\theta^0_{m,t}(x_0)\big)\big| \leq \big|\hat a_{m,t}(x) - \hat a_{m,t}(x_0)\big| \quad \forall\, t \geq 0,\, x \in C[0,\infty). \tag{2.29}$$

In order to prove (2.29), we have to consider three cases:

(1) If $\hat a_{m,t}(x), \hat a_{m,t}(x_0) \in a(\Theta)$, then

$$\big|a\big(\hat\theta^0_{m,t}(x)\big) - a\big(\hat\theta^0_{m,t}(x_0)\big)\big| = \big|\hat a_{m,t}(x) - \hat a_{m,t}(x_0)\big|.$$

(2) In the case that exactly one of the numbers $\hat a_{m,t}(x)$, $\hat a_{m,t}(x_0)$ belongs to $a(\Theta)$, we suppose that $\hat a_{m,t}(x_0) \in a(\Theta)$ without loss of generality. Then

$$\big|a\big(\hat\theta^0_{m,t}(x)\big) - a\big(\hat\theta^0_{m,t}(x_0)\big)\big| = \big|a(\beta_*) - \hat a_{m,t}(x_0)\big|$$

for some point β_* on the boundary $\partial\Theta$.

(3) If $\hat a_{m,t}(x_0), \hat a_{m,t}(x) \notin a(\Theta)$, two further cases must be considered:

(i) $\hat{a}_{m,t}(x_0),\ \hat{a}_{m,t}(x) < a(\theta) \quad \forall\, \theta \in \Theta$ or
$\hat{a}_{m,t}(x_0),\ \hat{a}_{m,t}(x) > a(\theta) \quad \forall\, \theta \in \Theta$;

(ii) $\hat{a}_{m,t}(x_0) < a(\theta) < \hat{a}_{m,t}(x) \quad \forall\, \theta \in \Theta$ or vice versa.

If (i) is true, one obtains $a(\hat{\theta}^0_{m,t}(x)) = a(\hat{\theta}^0_{m,t}(x_0))$, while in case (ii) we have $|a(\hat{\theta}^0_{m,t}(x)) - a(\hat{\theta}^0_{m,t}(x_0))| = \lambda(a(\Theta))$ where λ denotes the one-dimensional Lebesgue measure.

Hence, for any of the cases (1), (2), or (3) we get (2.29).

Note in the following that for any $N \in \mathbb{N}$ the function q is uniformly continuous on the compact neighbourhood $\overline{V_1(x_0[0,N'])}$ of the image $x_0[0,N']$ where

$$V_1\big(x_0[0,N']\big) = \big\{y \in \mathbb{R} \colon \operatorname{dist}\big(y, x_0[0,N']\big) < 1\big\}.$$

Hence, for ε we can find some $N > N'$ such that

$$\sup_{t \le N} |x(t) - x_0(t)| < \frac{1}{N} \quad \Longrightarrow \quad \sup_{t \le N'} \big|q\big(x(t)\big) - q\big(x_0(t)\big)\big| < \varepsilon.$$

Finally,

$$\begin{aligned} \sup_{t \le N'} |\hat{a}_{m,t}(x) - \hat{a}_{m,t}(x_0)| &\le \sup_{t \le N'} \|q \circ x - q \circ x_0\|_t \\ &< \varepsilon. \end{aligned}$$

Inequality (2.29) yields

$$\big\|a\big(\hat{\theta}^0_m(x)\big) - a\big(\hat{\theta}^0_m(x_0)\big)\big\|_{N'} < \varepsilon$$

Taking (2.28) into consideration, the assertion of the lemma results.

□

Remark 2.15. Let $\theta_0 \in \Theta$ and $q \in \mathcal{D}(\Theta)$. Then there exists some $\kappa > 0$ such that for all $\nu > 0$

$$\inf_{|\theta - \theta_0| \ge \nu} |a(\theta) - a(\theta_0)| \ge \kappa\nu. \tag{2.30}$$

For proving this, by injectivity and continuity one can derive that for any $\nu > 0$ and any compact set $K \subset \Theta$

$$\inf_{\theta_* \in K} \inf_{|\theta - \theta_*| > \nu} |a(\theta) - a(\theta_*)| > 0. \tag{2.31}$$

By means of (2.31) the statement of the remark was proven by Kutoyants (2004) (see equation (2.111)) where (2.31) was applied without the infimum over compact sets but in the form

$$\inf_{|\theta - \theta_0| > \nu} |a(\theta) - a(\theta_0)| > 0 \quad \forall\, \nu > 0.$$

From now on, let X be the observed process defined in Section 1.1. The next two propositions contain the most important properties of the EMM for using it in change-point analysis: we will find, under H_0 as well as under H_1, an almost sure rate of convergence and will show that the length m of the training period has no influence on the distribution of $\hat{\theta}^0_m(X)$.

Proposition 2.16. *Let $q \in \mathcal{D}(\Theta)$ and $(q - a(\theta_0)) \in \mathcal{M}(b(\theta_0), \sigma)$ for the true value $\theta_0 \in \Theta$ in (1.3). Then, under H_0, we have*

$$\sup_{t \geq e^e} \frac{\sqrt{t}}{\sqrt{\log_2 t}} \left|\hat{\theta}^0_{m,t} - \theta_0\right| < \infty \quad P\text{-a.s.} \quad \forall\, m \geq 0 \tag{2.32}$$

and

$$\hat{\theta}^0_m(X) \overset{D}{=} \hat{\theta}^0_0(X) \quad \forall\, m > 0. \tag{2.33}$$

Proof. In order to prove (2.32), we follow the proof of the consistency of the EMM presented by Kutoyants (2004), Theorem 2.28. Define the function

$$\ell(t) = \frac{\sqrt{t}}{\sqrt{\log_2 t}}, \quad t \geq e^e.$$

Suppose that, with positive probability, for any real number $M > 0$ there exists some $t_0 \geq 0$ such that

$$\ell(t_0)\left|\hat{\theta}^0_{m,t_0} - \theta_0\right| \geq M.$$

Then, on the set

$$\left\{\forall\, M > 0 \ \exists\, t_0(M) \geq 0 \colon \ell(t_0)\left|\hat{\theta}^0_{m,t_0} - \theta_0\right| \geq M\right\}$$

the inequality

$$\inf_{|\theta-\theta_0|<M/\ell(t_0)} |\hat{a}_{m,t_0} - a(\theta)| > \inf_{|\theta-\theta_0|\geq M/\ell(t_0)} |\hat{a}_{m,t_0} - a(\theta)|$$

would be true. Hence, by the triangle inequality we get the following estimate:

$$\begin{aligned}
2|\hat{a}_{m,t_0} - a(\theta_0)| &> |\hat{a}_{m,t_0} - a(\theta_0)| + \inf_{|\theta-\theta_0|\geq M/\ell(t_0)} |\hat{a}_{m,t_0} - a(\theta)| \\
&\geq |\hat{a}_{m,t_0} - a(\theta_0)| \\
&\quad + \inf_{|\theta-\theta_0|\geq M/\ell(t_0)} \big(|a(\theta) - a(\theta_0)| - |\hat{a}_{m,t_0} - a(\theta_0)|\big) \\
&= \inf_{|\theta-\theta_0|\geq M/\ell(t_0)} |a(\theta) - a(\theta_0)|
\end{aligned}$$

Using (2.30), one obtains

$$P\left\{\forall M > 0 \;\exists\, t_0(M)\colon 2\ell(t_0)\left|\hat{a}_{m,t_0} - a(\theta_0)\right| > \kappa M\right\} > 0. \tag{2.34}$$

Now we proceed to derive a contradiction to (2.34). According to Remark 2.2, the LIL

$$\limsup_{t\to\infty} \frac{1}{\sqrt{2t\log_2 t}} \int\limits_m^{m+t} \big(q(X_s) - a(\theta_0)\big)ds = \sqrt{D} \quad P\text{-a.s.}$$

is true with a constant $D > 0$. Hence, the expression

$$\ell(t)\left|\hat{a}_{m,t} - a(\theta_0)\right| = \frac{1}{\sqrt{t\log_2 t}} \left| \int\limits_m^{m+t} \big(q(X_s) - a(\theta_0)\big)ds \right| \tag{2.35}$$

is P-a.s. bounded as $t \to \infty$. This statement is a contradiction to (2.34). Thus, (2.32) is proven.
For proving (2.33), note that by Lemma 2.14 the process

$$\hat{\theta}^0_m(X) = \hat{\theta}^0_m(X_{m+s}\colon s \geq 0)$$

is a composition of a Markov process and a measurable map for any $m \geq 0$. Moreover, the initial distribution $\mathcal{L}(X_m) = \mu(\theta_0)$ is independent of m. Since the distribution of any Markov process is determined by the transition probabilities and the initial distribution, the proof is complete.

□

Proposition 2.17. *Let $q \in \mathcal{D}(\Theta)$ and $(q - a(\theta_1)) \in \mathcal{M}(b(\theta_1), \sigma)$ for $\theta_1 \neq \theta_0$. Then, under H_1, we have*

$$\sup_{t \geq \max\{t^*, e^e\}} \frac{\sqrt{t}}{\sqrt{\log_2 t}} \left|\hat{\theta}^0_{m,t} - \theta_1\right| < \infty \quad P\text{-a.s.} \quad \forall\, m \geq 0 \tag{2.36}$$

and

$$\hat{\theta}^0_m(X) \stackrel{D}{=} \hat{\theta}^0_0(Y) \quad \forall\, m > 0 \tag{2.37}$$

where Y represents the unique solution to

$$Y_t = \begin{cases} Y_0 + \int\limits_0^t b(\theta_0, Y_s)ds + \int\limits_0^t \sigma(Y_s)dW_s, & 0 < t \leq t^*(m), \\ Y_{t^*} + \int\limits_{t^*}^t b(\theta_1, Y_s)ds + \int\limits_{t^*}^t \sigma(Y_s)dW_s, & t > t^*(m), \end{cases}$$

$$Y_0 \sim \mu(\theta_0).$$

Proof. Along the lines of the proof of Proposition 2.16, for proving (2.36), it is sufficient to show the P-a.s. boundedness of the function

$$[\max\{t^*, e^e\}, \infty) \longrightarrow [0, \infty), \quad t \longmapsto \frac{\sqrt{t}}{\sqrt{\log_2 t}} \left|\hat{a}_{m,t} - a(\theta_1)\right| \tag{2.38}$$

(see (2.34) and (2.35)). By definition we have for all $\forall\, t \geq \max\{t^*, e^e\}$

$$\begin{aligned}\frac{\sqrt{t}}{\sqrt{\log_2 t}} &\left|\hat{a}_{m,t} - a(\theta_1)\right| \\ &= \frac{\left|\int\limits_m^{m+t^*} [q(X_s) - a(\theta_1)]ds + \int\limits_{m+t^*}^{m+t} [q(X_s) - a(\theta_1)]ds\right|}{\sqrt{t \log_2 t}}.\end{aligned} \tag{2.39}$$

Applying the LIL presented in Remark 2.2 on the second integral in (2.39), it follows that

$$\sup_{t \geq \max\{t^*, e^e\}} \frac{\left|\int\limits_{m+t^*}^{m+t} [q(X_s) - a(\theta_1)]ds\right|}{\sqrt{t \log_2 t}}$$

is P-a.s. finite.
Equation (2.37) can be proven by almost the same arguments as in the proof of (2.33): We have only to note that X runs stationarily up to the change-point $m + t^*$. Hence, $\mathcal{L}(X_m) = \mu(\theta_0)$. This equality implies that the processes $X_{m+(\cdot)}$ and Y are equivalent.

□

2.2.2 One-step MLE

Denote by $\mathcal{B}_{m,t}$ the σ-Algebra on $C[0, \infty)$ generated by the restriction $(X^0_s\colon m \leq s \leq m + t)$ of the canonical process X^0. Let $\theta_* \in \Theta$ be an arbitrary value. From now on, we do not distinguish between the notations $b(\theta, x)$ and $b(\theta)(x)$ for $(\theta, x) \in \Theta \times \mathbb{R}$. We correspondingly treat all occuring derivatives of b.
According to Kutoyants (2004), Theorem 1.12, the log-likelihood ratio is given by

$$\begin{aligned}\log \frac{dP_\theta|_{\mathcal{B}_{m,t}}}{dP_{\theta_*}|_{\mathcal{B}_{m,t}}}(X) = \log \frac{f_\theta(X_m)}{f_{\theta_*}(X_m)} &+ \int\limits_m^{m+t} \left[\frac{[b(\theta) - b(\theta_*)]}{\sigma^2}\right] (X_s)\, dX_s \\ &- \int\limits_m^{m+t} \left[\frac{b(\theta)^2 - b(\theta_*)^2}{2\sigma^2}\right] (X_s)\, ds \quad \forall\, \theta \in \Theta\end{aligned}$$

where X denotes the process defined in Section 1.1 and f_θ denotes the density of the stationary distribution $\mu(\theta)$.
Define the function $\theta \mapsto \psi_{m,t}(\theta)(X)$, $\theta \in \Theta$, by

$$\psi_{m,t}(\theta)(X) = \partial_\theta \log \frac{dP_\theta|_{\mathcal{B}_{m,t}}}{dP_{\theta_*}|_{\mathcal{B}_{m,t}}}(X) - \frac{\partial_\theta f_\theta}{f_\theta}(X_m) \quad \text{for} \quad \theta \in \mathring{\Theta}$$

and by a continuous extension of the right hand side on the boundary $\partial\Theta$ if the derivatives and the corresponding extension exist (see Remarks 2.20 and 2.21). We often write shorter $\psi_{m,t}(\theta)$.
Let

$$I(\theta) = E_{\mu(\theta)} \left| \frac{\dot{b}(\theta, \cdot)}{\sigma} \right|^2, \quad \theta \in \Theta, \tag{2.40}$$

be the Fisher information which we would get if we observe a trajectory of the process $X(\theta)$ defined by (2.21) during a period of length $t = 1$.
Suppose that $I(\theta) > 0$ for all $\theta \in \Theta$.
Using the previous notations, we define the one-step maximum likelihood estimator (one-step estimator or one-step MLE) by

$$\hat{\theta}_{m,t} = \bar{\theta}_{m,t} + \frac{\psi_{m,t}(\bar{\theta}_{m,t})}{tI(\bar{\theta}_{m,t})}, \quad m \geq 0,\, t > 0, \tag{2.41}$$

where $\bar{\theta}_{0,t}$, $t \geq 0$, represents a consistent starting estimator.

Remark 2.18. Under suitable conditions the one-step MLE is consistent, asymptotically normal and asymptotically efficient in the sense of the Hajek-Le Cam bound. Essentially, the same conditions are used throughout this work.

Now it is necessary to derive from Assumption $\mathcal{A}_0(\Theta)$ the smoothness of the Fisher information I. Remember that we denoted by $G(\theta)$ the normalisation constant in the stationary distribution $\mu(\theta)$, i.e,

$$G(\theta) = \int_{\mathbb{R}} \frac{e^{2B_\theta(x)}}{\sigma^2(x)} dx$$

where

$$B_\theta(x) = \int_0^x \frac{b(\theta, y)}{\sigma^2(y)} dy, \quad x \in \mathbb{R}.$$

Lemma 2.19. *Let* $\mathcal{A}_0(\Theta)$ *be satisfied.*

a) If $b(\theta,\cdot),\ \partial_\theta b(\theta,\cdot)\in\mathcal{P}$ *uniformly in* θ, *then* $I\in C(\Theta)$.

b) If $b(\theta,\cdot),\ \partial_\theta b(\theta,\cdot),\ \partial_\theta^2 b(\theta,\cdot)\in\mathcal{P}$ *uniformly in* θ, *then* $I\in C^1(\Theta)$.

c) If $b(\theta,\cdot),\ \partial_\theta b(\theta,\cdot),\ \partial_\theta^2 b(\theta,\cdot)$ *and* $\partial_\theta^3 b(\theta,\cdot)$ *belong to* $\mathcal{P}$ *uniformly in* θ, *then* $I\in C^2(\Theta)$.

Proof. Consider

$$I(\theta)=\int_{\mathbb{R}}\left|\frac{\dot b(\theta,x)}{\sigma(x)}\right|^2\frac{e^{2B_\theta(x)}}{G(\theta)\sigma^2(x)}dx,\quad \theta\in\Theta. \tag{2.42}$$

Recall that by Remark 2.9 and Lemma 2.10 the function e^{2B_θ} has exponentially decreasing tails uniformly in θ and $\inf_{\theta\in\Theta}G(\theta)>0$. In combination with the growth conditions $1/\sigma\in\mathcal{P}$ and $\partial_\theta b(\theta,\cdot)\in\mathcal{P}$ uniformly in θ we obtain by the dominated convergence theorem that $I\in C(\Theta)$.
Moreover, if it is possible to interchange differentiation and integration, then

$$\dot I(\theta)=2\int_{\mathbb{R}}\left(\frac{\dot B_\theta\dot b(\theta)^2+\dot b(\theta)\ddot b(\theta)}{\sigma^2}-I(\theta)\dot B_\theta\right)d\mu(\theta)\quad\forall\,\theta\in\Theta \tag{2.43}$$

would be valid. However, by the differentiation lemma I belongs to $C^1(\mathring{\Theta})$ and fulfils (2.43) because $1/\sigma\in\mathcal{P},\ \partial_\theta b(\theta,\cdot),\ \partial_\theta^2 b(\theta,\cdot),\ \partial_\theta B_\theta\in\mathcal{P}$ uniformly in θ, and because I is bounded, $\inf_{\theta\in\Theta}G(\theta)>0$ as well as e^{2B_θ} has exponentially decreasing tails uniformly in θ. The same conditions imply that $\partial_\theta I\in C(\Theta)$.
Similarly, one can show that our assumptions yield $I\in C^2(\Theta)$ with

$$\begin{aligned}\ddot I(\theta)&=2\int_{\mathbb{R}}\frac{\ddot B_\theta\dot b(\theta)^2+2\dot B_\theta\dot b(\theta)\ddot b(\theta)+\ddot b(\theta)^2+\dot b(\theta)\partial_\theta^3 b(\theta)}{\sigma^2}d\mu(\theta)\\&+4\int_{\mathbb{R}}\frac{\dot B_\theta\dot b(\theta)^2+\dot b(\theta)\ddot b(\theta)}{\sigma^2}\big(\dot B_\theta-E_{\mu(\theta)}\dot B_\theta\big)d\mu(\theta)\\&-2I(\theta)\int_{\mathbb{R}}\big(\ddot B_\theta+2\dot B_\theta^2-2\dot B_\theta E_{\mu(\theta)}\dot B_\theta\big)d\mu(\theta)\\&-2\dot I(\theta)E_{\mu(\theta)}\dot B_\theta\quad\forall\,\theta\in\Theta.\end{aligned}$$

□

Remark 2.20. Let $m, t \geq 0$. Moreover, let H_0 and Assumption $\mathcal{A}_0(\Theta)$ be satisfied.

a) If $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot) \in \mathcal{P}$ uniformly in θ, then $\psi_{m,t}$ exists, belongs to $C(\Theta)$, and has the form

$$\psi_{m,t}(\theta) = \int_m^{m+t} \left[\frac{\dot b(\theta)}{\sigma}\right](X_s)\,dW_s + \int_m^{m+t} \left[\frac{[b(\theta_0) - b(\theta)]\dot b(\theta)}{\sigma^2}\right](X_s)\,ds \tag{2.44}$$

for all $\theta \in \Theta$.

b) If $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial_\theta^2\partial_x b(\theta,\cdot) \in \mathcal{P}$ uniformly in θ, then $\psi_{m,t} \in C^1(\Theta)$, and we have

$$\begin{aligned}\dot\psi_{m,t}(\theta) = &\int_m^{m+t} \left[\frac{\ddot b(\theta)}{\sigma}\right](X_s)\,dW_s + \int_m^{m+t} \left[\frac{[b(\theta_0) - b(\theta)]\ddot b(\theta)}{\sigma^2}\right](X_s)\,ds \\ &- \int_m^{m+t} \left[\frac{\dot b(\theta)}{\sigma}\right]^2 (X_s)\,ds\end{aligned} \tag{2.45}$$

for all $\theta \in \Theta$.

c) If $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial_\theta^2\partial_x b(\theta,\cdot)$, $\partial_\theta^3\partial_x b(\theta,\cdot) \in \mathcal{P}$ uniformly in θ, then $\psi_{m,t} \in C^2(\Theta)$, and we have

$$\begin{aligned}\ddot\psi_{m,t}(\theta) = &\int_m^{m+t} \left[\frac{\partial_\theta^3 b(\theta)}{\sigma}\right](X_s)\,dW_s - 3\int_m^{m+t} \left[\frac{\partial_\theta b(\theta)\cdot\partial_\theta^2 b(\theta)}{\sigma^2}\right](X_s)\,ds \\ &+ \int_m^{m+t} \left[\frac{[b(\theta_0) - b(\theta)]\partial_\theta^3 b(\theta)}{\sigma^2}\right](X_s)\,ds\end{aligned} \tag{2.46}$$

for all $\theta \in \Theta$.

Note that

$$\begin{aligned}\partial_\theta\partial_x b(\theta,\cdot) \in \mathcal{P} \text{ uniformly in } \theta \quad &\Longrightarrow \quad \partial_\theta b(\theta,\cdot) \in \mathcal{P} \text{ uniformly in } \theta, \\ \partial_\theta^2\partial_x b(\theta,\cdot) \in \mathcal{P} \text{ uniformly in } \theta \quad &\Longrightarrow \quad \partial_\theta^2 b(\theta,\cdot) \in \mathcal{P} \text{ uniformly in } \theta, \\ \partial_\theta^3\partial_x b(\theta,\cdot) \in \mathcal{P} \text{ uniformly in } \theta \quad &\Longrightarrow \quad \partial_\theta^3 b(\theta,\cdot) \in \mathcal{P} \text{ uniformly in } \theta.\end{aligned} \tag{2.47}$$

Proof. Considerating the process X under H_0 (see (1.3)), we get for the log-likelihood ratio

$$\log \frac{dP_\theta|_{\mathcal{B}_{m,t}}}{dP_{\theta_*}|_{\mathcal{B}_{m,t}}}(X) = \log \frac{f_\theta(X_m)}{f_{\theta_*}(X_m)} + \int\limits_m^{m+t} \left[\frac{b(\theta)-b(\theta_*)}{\sigma}\right](X_s)\,dW_s$$
$$+ \int\limits_m^{m+t} \left[\frac{[b(\theta)-b(\theta_*)]b(\theta_0)}{\sigma^2}\right](X_s)\,ds$$
$$- \frac{1}{2}\int\limits_m^{m+t} \left[\frac{b(\theta)^2-b(\theta_*)^2}{\sigma^2}\right](X_s)\,ds, \quad \forall\, \theta \in \Theta.$$

For any $\theta \in \Theta$ the functions

$$\frac{b(\theta,\cdot)-b(\theta_*,\cdot)}{\sigma^2}, \quad \frac{\partial_\theta b(\theta,\cdot)}{\sigma^2}, \quad \frac{\partial_\theta^2 b(\theta,\cdot)}{\sigma^2}, \quad \frac{\partial_\theta^3 b(\theta,\cdot)}{\sigma^2}$$

are continuous (see (1.4)) and thereby, they possess primitives which we denote by D_θ, F_θ, G_θ and H_θ, respectively.
By means of the Ito formula it follows that

$$\log \frac{dP_\theta|_{\mathcal{B}_{m,t}}}{dP_{\theta_*}|_{\mathcal{B}_{m,t}}}(X) = \int\limits_m^{m+t} [D'_\theta \sigma](X_s)dW_s + \int\limits_m^{m+t} [D'_\theta b(\theta_0)](X_s)ds$$
$$- \frac{1}{2}\int\limits_m^{m+t} \left[\frac{b(\theta)^2-b(\theta_*)^2}{\sigma^2}\right](X_s)\,ds + \log \frac{f_\theta(X_m)}{f_{\theta_*}(X_m)}$$
$$= D_\theta(X_{m+t}) - D_\theta(X_m) - \frac{1}{2}\int\limits_m^{m+t} [D''_\theta \sigma^2](X_s)ds \tag{2.48}$$
$$- \frac{1}{2}\int\limits_m^{m+t} \left[\frac{b(\theta)^2-b(\theta_*)^2}{\sigma^2}\right](X_s)\,ds + \log \frac{f_\theta(X_m)}{f_{\theta_*}(X_m)}.$$

The differentiability with respect to θ of the density

$$f_\theta(x) = \frac{\exp\big(2B_\theta(x)\big)}{G(\theta)\sigma^2(x)} = \frac{1}{G(\theta)\sigma^2(x)} \exp\left(2\int_0^x \frac{b(\theta,y)}{\sigma^2(y)}dy\right), \quad x \in \mathbb{R},$$

where

$$G(\theta) = \int\limits_{\mathbb{R}} \frac{1}{\sigma^2(x)} \exp\left(2\int_0^x \frac{b(\theta,y)}{\sigma^2(y)}dy\right)dx, \quad \theta \in \Theta,$$

can be checked by the dominated convergence theorem: since $\partial_\theta b(\theta)$ is dominated by a polynomial uniformly in θ, $\exp(2B_\theta)$ is differentiable and one can differentiate B_θ under the integral. The differentiability of G is obtained by Remark 2.9 and the additional growth condition $1/\sigma \in \mathcal{P}$ contained in Assumption $\mathcal{A}_0(\Theta)$.
Moreover, since $b(\theta,\cdot)$, $\partial_\theta b(\theta,\cdot)$, $\partial_\theta \partial_x b(\theta,\cdot) \in \mathcal{P}$ uniformly in θ, we can apply the dominated convergence theorem in order to differentiate under the integrals in (2.48). Note that

$$D_\theta(X_{m+t}) - D_\theta(X_m) = \int_{X_m}^{X_{m+t}} \left[\frac{b(\theta,y) - b(\theta_*,y)}{\sigma^2(y)}\right] dy, \quad \theta \in \Theta.$$

Hence, we obtain for any θ in the interior $\mathring{\Theta}$:

$$\begin{aligned} \psi_{m,t}(\theta) = F_\theta(X_{m+t}) - F_\theta(X_m) - \frac{1}{2}\int_m^{m+t} [F''_\theta \sigma^2](X_s) ds \\ - \int_m^{m+t} \left[\frac{\dot{b}(\theta) b(\theta)}{\sigma^2}\right](X_s)\, ds. \end{aligned} \tag{2.49}$$

By the Ito formula (2.44) results for any θ in the interior.
Similarly, one can compute by the dominated convergence theorem that

$$\begin{aligned} \dot{\psi}_{m,t}(\theta) &= G_\theta(X_{m+t}) - G_\theta(X_m) \\ &- \int_m^{m+t} \left[G'_\theta b(\theta) + F'_\theta \dot{b}(\theta) + \frac{1}{2} G''_\theta \sigma^2\right](X_s)\, ds, \quad \theta \in \mathring{\Theta}, \end{aligned} \tag{2.50}$$

and

$$\begin{aligned} \ddot{\psi}_{m,t}(\theta) &= H_\theta(X_{m+t}) - H_\theta(X_m) \\ &- \int_m^{m+t} \left[H'_\theta b(\theta) + 3G'_\theta \dot{b}(\theta) + \frac{1}{2} H''_\theta \sigma^2\right](X_s)\, ds, \quad \theta \in \mathring{\Theta}. \end{aligned} \tag{2.51}$$

Once again, the Ito formula yields (2.45) and (2.46) for any θ in the interior.

Along the previous lines, the dominated convergence theorem implies that $\psi_{m,t}$ and its first and second derivative are continuous in the boundary points of Θ.

□

Remark 2.21. Let $m \geq 0$, $t > t^*$. Moreover, let H_1 and Assumption $\mathcal{A}_0(\Theta)$ be satisfied.

a) If $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ, then $\psi_{m,t}$ exists, belongs to $C(\Theta)$, and has the form

$$\begin{aligned}\psi_{m,t}(\theta) = \int_m^{m+t} \frac{\dot{b}(\theta, X_s)}{\sigma(X_s)} dW_s + \int_m^{m+t^*} \left[\frac{[b(\theta_0) - b(\theta)]\dot{b}(\theta)}{\sigma^2}\right](X_s)\, ds \\ + \int_{m+t^*}^{m+t} \left[\frac{[b(\theta_1) - b(\theta)]\dot{b}(\theta)}{\sigma^2}\right](X_s)\, ds\end{aligned} \tag{2.52}$$

for all $\theta \in \Theta$.

b) If $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot)$, $\partial_\theta^2 \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ, then $\psi_{m,t} \in C^1(\Theta)$, and we have

$$\begin{aligned}\dot{\psi}_{m,t}(\theta) = \int_m^{m+t} \frac{\ddot{b}(\theta, X_s)}{\sigma(X_s)} dW_s + \int_m^{m+t^*} \left[\frac{[b(\theta_0) - b(\theta)]\ddot{b}(\theta)}{\sigma^2}\right](X_s)\, ds \\ + \int_{m+t^*}^{m+t} \left[\frac{[b(\theta_1) - b(\theta)]\ddot{b}(\theta)}{\sigma^2}\right](X_s)\, ds - \int_m^{m+t} \left[\frac{\dot{b}(\theta)}{\sigma}\right]^2 (X_s)\, ds\end{aligned} \tag{2.53}$$

for all $\theta \in \Theta$.

c) If $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot)$, $\partial_\theta^2 \partial_x b(\theta, \cdot)$, $\partial_\theta^3 \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ, then $\psi_{m,t} \in C^2(\Theta)$, and we have

$$\begin{aligned}\ddot{\psi}_{m,t}(\theta) = \int_m^{m+t} \left[\frac{\partial_\theta^3 b(\theta)}{\sigma}\right](X_s)\, dW_s - 3 \int_m^{m+t} \left[\frac{\partial_\theta b(\theta) \partial_\theta^2 b(\theta)}{\sigma^2}\right](X_s)\, ds \\ + \int_m^{m+t^*} \left[\frac{[b(\theta_0) - b(\theta)]\partial_\theta^3 b(\theta)}{\sigma^2}\right](X_s)\, ds \\ + \int_{m+t^*}^{m+t} \left[\frac{[b(\theta_1) - b(\theta)]\partial_\theta^3 b(\theta)}{\sigma^2}\right](X_s)\, ds\end{aligned} \tag{2.54}$$

for all $\theta \in \Theta$.

Proof. We proceed as in Remark 2.20. Once again, for any $\theta \in \Theta$ let D_θ, F_θ, G_θ and H_θ be primitives of the functions

$$\frac{b(\theta,\cdot)-b(\theta_*,\cdot)}{\sigma^2}, \quad \frac{\partial_\theta b(\theta,\cdot)}{\sigma^2}, \quad \frac{\partial_\theta^2 b(\theta,\cdot)}{\sigma^2}, \quad \frac{\partial_\theta^3 b(\theta,\cdot)}{\sigma^2},$$

respectively. Under the alternative the log-likelihood has the form

$$\begin{aligned}\log \frac{dP_\theta|_{\mathcal{B}_{m,t}}}{dP_{\theta_*}|_{\mathcal{B}_{m,t}}}(X) = &\int_m^{m+t} \left[\frac{b(\theta)-b(\theta_*)}{\sigma}\right](X_s)\,dW_s \\ &+ \int_m^{m+t^*} \left[\frac{\big(b(\theta)-b(\theta_*)\big)b(\theta_0)}{\sigma^2}\right](X_s)\,ds \\ &+ \int_{m+t^*}^{m+t} \left[\frac{\big(b(\theta)-b(\theta_*)\big)b(\theta_1)}{\sigma^2}\right](X_s)\,ds \\ &- \frac{1}{2}\int_m^{m+t} \left[\frac{b(\theta)^2-b(\theta_*)^2}{\sigma^2}\right](X_s)\,ds + \log\frac{f_\theta(X_m)}{f_{\theta_*}(X_m)}\end{aligned}$$

(see (1.2)). Applying the Ito formula in the same way as in (2.48), we obtain the same formula as under H_0:

$$\begin{aligned}\log \frac{dP_\theta|_{\mathcal{B}_{m,t}}}{dP_{\theta_*}|_{\mathcal{B}_{m,t}}}(X) = &D_\theta(X_{m+t}) - D_\theta(X_m) - \frac{1}{2}\int_m^{m+t} [D_\theta''\sigma^2](X_s)ds \\ &- \frac{1}{2}\int_m^{m+t} \left[\frac{b(\theta)^2-b(\theta_*)^2}{\sigma^2}\right](X_s)\,ds + \log\frac{f_\theta(X_m)}{f_{\theta_*}(X_m)}.\end{aligned}$$

By differentiation we get for $\psi_{m,t}$ and its first and second derivative the same formulas as in (2.49) - (2.51). The continuity in the boundary points of Θ follows in a similar way. Applying the Ito formula three times, equations (2.52) - (2.54) result.

□

For later use it is convenient to know the facts proven in the following three lemmata.

Lemma 2.22. *Let* $g \in C^1(\mathbb{R})$. *Assume that* Y *is the solution to* (2.1) *and the diffusion function* σ *belongs to* $C^1(\mathbb{R})$.

(i) There exists a measurable map $\Phi_1 \colon C[0,\infty) \to C[0,\infty)$ *such that*

$$\int_{t_0}^{t_0+t} g(Y_s)dW_s = \Phi_1\big(Y_{t_0+(\cdot)}\big)(t) \quad \forall\, t_0,\, t \geq 0.$$

(ii) Let X *be the process defined in Section 1.1. Then, under* H_1, *there exists a measurable map* $\Phi_2 \colon C[0,\infty) \to C[0,\infty)$ *such that*

$$\int_{m}^{m+t} g(X_s)dW_s = \Phi_2\big(X_{m+(\cdot)}\big)(t) \quad \forall\, m,\, t \geq 0.$$

Proof. Denote by F a primitive of g/σ. It follows from the Ito formula that

$$\begin{aligned}\int_{t_0}^{t_0+t} g(Y_s)dW_s &= \int_{t_0}^{t_0+t} \frac{g}{\sigma}(Y_s)\cdot\sigma(Y_s)dW_s \\ &= F(Y_{t_0+t}) - F(Y_{t_0}) - \int_{t_0}^{t_0+t} (F'b)(Y_s)ds \\ &\quad - \frac{1}{2}\int_{t_0}^{t_0+t} (F''\sigma^2)(Y_s)ds \quad \forall\, t \geq 0 \quad P\text{-a.s.}\end{aligned} \tag{2.55}$$

Hence, the stochastic integral can be written as the composition of $Y_{t_0+(\cdot)}$ and the map Φ_1 given by

$$\Phi_1(x)(t) = F\big(x(t)\big) - F\big(x(0)\big) - \int_0^t (F'b)\big(x(s)\big)ds - \int_0^t \frac{(F''\sigma^2)\big(x(s)\big)}{2}ds$$

for $x \in C[0,\infty)$, $t \geq 0$. According to Lemma 2.5, Φ_1 is continuous and thereby, measurable. This proves (i).

For proving (ii), we can argue in exactly the same way if we replace (2.55) by

$$\int_m^{m+t} g(X_s)dW_s = F(X_{m+t}) - F(X_m) - \int_m^{m+(t\wedge t^*)} \big(F'b(\theta_0)\big)(X_s)ds$$
$$- \int_{m+t^*}^{m+(t\vee t^*)} \big(F'b(\theta_1)\big)(X_s)ds$$
$$- \int_m^{m+t} \frac{(F''\sigma^2)(X_s)}{2}ds \quad \forall\, t \geq 0 \quad P\text{-a.s.}$$

and Φ_1 by the map Φ_2,

$$\Phi_2(x)(t) = F\big(x(t)\big) - F\big(x(0)\big) - \int_0^{t\wedge t^*} \big(F'b(\theta_0)\big)\big(x(s)\big)ds$$
$$- \int_{t^*}^{t\vee t^*} \big(F'b(\theta_1)\big)\big(x(s)\big)ds - \int_0^t \frac{(F''\sigma^2)\big(x(s)\big)}{2}ds$$

for $x \in C[0,\infty)$, $t \geq 0$. Here, we use the notations $t \wedge t^* = \min\{t, t^*\}$ and $t \vee t^* = \max\{t, t^*\}$.

□

Lemma 2.23. *Let the functions* $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial_\theta^2\partial_x b(\theta,\cdot)$ *belong to* $\mathcal{P}$ *uniformly in* θ. *Then, under* H_0 *and Assumption* $\mathcal{A}_0(\Theta)$, *we have*

(i) $\big(\psi_{m,t}(\theta)(X) \colon t \geq 0\big) \overset{D}{=} \big(\psi_{0,t}(\theta)(X) \colon t \geq 0\big) \quad \forall\, m > 0,\, \theta \in \Theta,$

(ii) $\big(\dot\psi_{m,t}(\theta)(X) \colon t \geq 0\big) \overset{D}{=} \big(\dot\psi_{0,t}(\theta)(X) \colon t \geq 0\big) \quad \forall\, m > 0,\, \theta \in \Theta.$

Proof. In view of formulas (2.44) and (2.45), by Lemma 2.22.(i) and Lemma 2.5 the processes $\psi_{m,(\cdot)}(\theta)(X)$ and $\partial_\theta\psi_{m,(\cdot)}(\theta)(X)$ are compositions of the process $X_{m+(\cdot)}$ and measurable maps in $C[0,\infty)$.
Now we can argue in the same way as in the proof of (2.33): since the initial distribution $\mathcal{L}(X_m) = \mu(\theta_0)$ is independent of m and the distribution of any Markov process is determined by the transition probabilities and the initial distribution, the processes $X_{m+(\cdot)}$ and X are equivalent. This completes the proof.

□

Lemma 2.24. *Let the functions* $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial^2_\theta\partial_x b(\theta,\cdot)$ *belong to* $\mathcal{P}$ *uniformly in* θ. *Then, under* H_1 *and Assumption* $\mathcal{A}_0(\Theta)$, *we have*

(i) $\big(\psi_{m,t}(\theta)(X)\colon t\ge 0\big) \overset{D}{=} \big(\psi_{0,t}(\theta)(Y)\colon t\ge 0\big) \quad \forall\ m>0,\ \theta\in\Theta,$

(ii) $\big(\dot\psi_{m,t}(\theta)(X)\colon t\ge 0\big) \overset{D}{=} \big(\dot\psi_{0,t}(\theta)(Y)\colon t\ge 0\big) \quad \forall\ m>0,\ \theta\in\Theta$

where Y *denotes the unique solution to*

$$Y_t = \begin{cases} Y_0 + \int_0^t b(\theta_0, Y_s)ds + \int_0^t \sigma(Y_s)dW_s, & 0<t\le t^*(m),\\ Y_{t^*} + \int_{t^*}^t b(\theta_1, Y_s)ds + \int_{t^*}^t \sigma(Y_s)dW_s, & t>t^*(m),\end{cases} \tag{2.56}$$

$$Y_0 \sim \mu(\theta_0).$$

Proof. In view of formulas (2.52) and (2.53), by Lemma 2.22.(ii) and Lemma 2.5 the processes $\psi_{m,(\cdot)}(\theta)(X)$ and $\partial_\theta\psi_{m,(\cdot)}(\theta)(X)$ are compositions of the process $X_{m+(\cdot)}$ and measurable maps in $C[0,\infty)$.
The same argument as in the proof of Lemma 2.23 yields the assertion of the Lemma because the process $(X_s\colon 0\le s\le m+t^*)$ is stationary.

□

Chapter 3

Asymptotics under the hypothesis

3.1 Test statistic

Recall that the question is to decide sequentially between the hypotheses

$$H_0\colon\ t^* = \infty,\ \text{ i.e. no change}$$

and

$$H_1\colon\ t^* < \infty \quad \text{and} \quad \theta_0 \neq \theta_1$$

for the model defined in Section 1.1.
In this chapter we will determine under the hypothesis the limit distribution of $\sup_{t>0} |S_t^m|$ as $m \to \infty$ (see Theorem 3.7 and its Corollary 3.8). Therefore, the critical value $c > 0$ of the test procedure given by (1.6) can be chosen according to

$$\alpha = \lim_{m\to\infty} P\{\tau_m < \infty\} = \lim_{m\to\infty} P\Big\{ \sup_{t>0} |S_t^m| > c \Big\}$$

where $\alpha \in (0,1)$ denotes the prescribed level of the test.

Remember our test statistic introduced in Section 1.1:

$$\begin{aligned} S_t^m &= \frac{t}{g_m(t)} (\hat{\theta}_{m,t} - \hat{\theta}_{0,m}) \\ &= \frac{t}{g_m(t)} \left[\hat{\theta}^0_{m,t} + \frac{1}{t} \frac{\psi_{m,t}(\hat{\theta}^0_{m,t})}{I(\hat{\theta}^0_{m,t})} - \left(\hat{\theta}^0_{0,m} + \frac{1}{m} \frac{\psi_{0,m}(\hat{\theta}^0_{0,m})}{I(\hat{\theta}^0_{0,m})} \right) \right], \quad m,t > 0, \end{aligned}$$

where it must be supposed that $I > 0$. Moreover, assume that $I, \psi_{m,t} \in C^2(\Theta)$ for all $m, t \geq 0$. Under these conditions, one obtains by Taylor expansion of the functions $\psi_{m,t}/I$ and $\psi_{0,m}/I$ around θ_0 and some rearrangement

$$\begin{aligned} S_t^m = & \frac{\psi_{m,t}(\theta_0) - \frac{t}{m}\psi_{0,m}(\theta_0)}{I(\theta_0) g_m(t)} \\ & + \frac{t}{g_m(t)} \left(R_1 - R_2 + R_3 - R_4\right), \quad m, t > 0, \end{aligned} \tag{3.1}$$

where

$$R_1 = \hat{\theta}^0_{m,t} - \theta_0 + \frac{\dot{\psi}_{m,t}(\theta_0)}{tI(\theta_0)}\big(\hat{\theta}^0_{m,t} - \theta_0\big)$$

$$R_2 = \frac{\dot{I}(\theta_0)\psi_{m,t}(\theta_0)}{tI^2(\theta_0)}\big(\hat{\theta}^0_{m,t} - \theta_0\big),$$

$$R_3 = \partial^2_\theta\Big(\frac{\psi_{m,t}}{I}\Big)(\tilde{\theta}_{m,t})\frac{\big(\hat{\theta}^0_{m,t} - \theta_0\big)^2}{t},$$

and

$$\begin{aligned} R_4 = & (\hat{\theta}^0_{0,m} - \theta_0) + \frac{\dot{\psi}_{0,m}(\theta_0)}{mI(\theta_0)}\big(\hat{\theta}^0_{0,m} - \theta_0\big) - \frac{\dot{I}(\theta_0)\psi_{0,m}(\theta_0)}{mI^2(\theta_0)}\big(\hat{\theta}^0_{0,m} - \theta_0\big) \\ & + \partial^2_\theta\Big(\frac{\psi_{0,m}}{I}\Big)(\tilde{\theta}_{0,m})\frac{\big(\hat{\theta}^0_{0,m} - \theta_0\big)^2}{m}. \end{aligned} \tag{3.2}$$

Here, we have $|\tilde{\theta}_{0,m} - \theta_0| \leq |\hat{\theta}^0_{0,m} - \theta_0|$ and $|\tilde{\theta}_{m,t} - \theta_0| \leq |\hat{\theta}^0_{m,t} - \theta_0|$.
The first summand in (3.1) will turn out to be the one which converges in distribution, while the expressions $R_1, \ldots, R_4$ converge to zero as $m \to \infty$.

3.2 Preliminary results

We proceed to approximate the first term on the right hand side of (3.1) by a functional of Wiener processes. For this purpose, let us introduce the following assumption of a strong invariance principle with rate:

Assumption 3.1. For any $\theta_0 \in \Theta$ there exists a Wiener process B and a number $\alpha < 1/2$ such that under H_0

$$\psi_{0,t}(\theta_0) - \sqrt{I(\theta_0)}B_t = o(t^\alpha) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty.$$

Lemma 3.2. *Let Assumptions 3.1 and $\mathcal{A}_0(\Theta)$ be satisfied. Suppose that $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)\in\mathcal{P}$ uniformly in θ.*
If $0\le\gamma\le 1/2-\alpha$ and H_0 holds, then there exists for any $m>0$ a Wiener process $B^{(m)}$ such that $B^{(m)}$ is independent of $(B_s\colon 0\le s\le m)$ and

$$\lim_{m\to\infty}\sup_{t>0}\frac{\left|\psi_{m,t}(\theta_0)-\sqrt{I(\theta_0)}B_t^{(m)}+\frac{t}{m}\big(\sqrt{I(\theta_0)}B_m-\psi_{0,m}(\theta_0)\big)\right|}{g_m(t)}=0\quad P\text{-a.s.}$$

Remember that γ is the parameter involved in the weighting function

$$g_m(t)=\frac{(m+t+1)^{1-\gamma}(t+1)^\gamma}{\sqrt{m}},\qquad t\ge 0.$$

Proof. Under H_0 we have by Remark 2.20

$$\psi_{m,t}(\theta_0)=\int_m^{m+t}\frac{\dot b(\theta_0,X_s)}{\sigma(X_s)}dW_s\quad\forall\ m,\,t\ge 0.$$

Set $a=\sqrt{I(\theta_0)}$ and for any $m>0$ $B^{(m)}=(B_{m+t}-B_m\colon t\ge 0)$. By the triangle inequality it follows that

$$\begin{aligned}
&\sup_{t>0}\frac{\left|\psi_{m,t}(\theta_0)-aB_t^{(m)}+\frac{t}{m}\big(aB_m-\psi_{0,m}(\theta_0)\big)\right|}{g_m(t)}\\
&\quad\le\sup_{t>0}\frac{|\psi_{0,m+t}(\theta_0)-aB_{m+t}-\psi_{0,m}(\theta_0)+aB_m|}{(m+t+1)^\alpha}\sup_{t>0}\frac{(m+t+1)^\alpha}{g_m(t)}\\
&\qquad+\frac{|aB_m-\psi_{0,m}(\theta_0)|}{m^\alpha}\sup_{t>0}\frac{t}{m^{1-\alpha}g_m(t)}.
\end{aligned}$$

Along the lines of the proof of Corollary 2.4, the invariance principle of Assumption 3.1 implies

$$\lim_{m\to\infty}\sup_{t>0}\frac{1}{(m+t)^\alpha}\,|\psi_{0,m+t}(\theta_0)-aB_{m+t}|=0\quad P\text{-a.s.}$$

Hence, we get

$$\begin{aligned}
&\sup_{t>0}\frac{\left|\psi_{m,t}(\theta_0)-aB_t^{(m)}+\frac{t}{m}\big(aB_m-\psi_{0,m}(\theta_0)\big)\right|}{g_m(t)}\\
&\quad\le o(1)\sup_{t>0}\frac{\sqrt{m}}{(m+t+1)^{1-\gamma-\alpha}(t+1)^\gamma}\\
&\qquad+o(1)\sup_{t>0}\frac{t}{m^{1/2-\alpha}(m+t+1)^{1-\gamma}(t+1)^\gamma}
\end{aligned}$$

where the Laundau symbols represent terms which have the corresponding asymptotic behaviour P-a.s. as $m \to \infty$. The fact that the family

$$\sup_{t>0} \frac{\sqrt{m}}{(m+t+1)^{1-\gamma-\alpha}(t+1)^{\gamma}} = \frac{\sqrt{m}}{(m+1)^{1-\gamma-\alpha}}, \quad m>0,$$

is bounded completes the proof.

□

Remember the notation $a(\theta) = E_{\mu(\theta)}q,\ \theta \in \Theta$, from Subsection 2.2.1.

Lemma 3.3. *Let Assumption $\mathcal{A}_0(\Theta)$ be satisfied. Let $q \in \mathcal{D}(\Theta)$ as well as $(q - a(\theta_0))$, $(|\partial_\theta b(\theta_0,\cdot)/\sigma|^2 - I(\theta_0)) \in \mathcal{M}(b(\theta_0,\cdot),\sigma)$ for $\theta_0 \in \Theta$.*
Moreover, suppose that $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial_\theta^2\partial_x b(\theta,\cdot) \in \mathcal{P}$ uniformly in θ.
Then, under H_0, we have

$$\sup_{t>0} \frac{\left|\hat{\theta}^0_{m,t} - \theta_0\right|}{g_m(t)} \left|tI(\theta_0) + \dot{\psi}_{m,t}(\theta_0)\right| = o_P(1) \quad \textit{as} \quad m \to \infty.$$

Proof. Remember that the distributions of the processes $(\hat{\theta}^0_{m,t} \colon t \geq 0)$ and $(\partial_\theta \psi_{m,t}(\theta_0) \colon t \geq 0)$ are independent of $m \geq 0$, see Proposition 2.16 and Lemma 2.23. Hence, it is sufficient to consider the asymptotics of

$$\sup_{t>0} \frac{\left|\hat{\theta}^0_{0,t} - \theta_0\right|}{g_m(t)} \left|tI(\theta_0) + \dot{\psi}_{0,t}(\theta_0)\right|$$

as $m \to \infty$.
By formula (2.45) one has

$$tI(\theta_0) + \dot{\psi}_{0,t}(\theta_0) = \int_0^t \frac{\ddot{b}(\theta_0,X_s)}{\sigma(X_s)} dW_s - \int_0^t \left(\left| \frac{\dot{b}(\theta_0,X_s)}{\sigma(X_s)} \right|^2 - I(\theta_0) \right) ds \tag{3.3}$$

for all $t \geq 0$.
The LILs of Lemma 2.1 and Remark 2.2 yield that both integrals, the stochastic as well as the ordinary one, are P-a.s. of the order $\mathcal{O}((t \log_2 t)^{1/2})$ as $t \to \infty$.
Choose some $\alpha < 1/2$ and a small $\delta \geq 0$ with $1/2 - \alpha - \gamma < \delta < 1/2 - \gamma$.
Using temporarily the notation

$$M_t = \int_0^t \frac{\ddot{b}(\theta_0,X_s)}{\sigma(X_s)} dW_s - \int_0^t \left(\left| \frac{\dot{b}(\theta_0,X_s)}{\sigma(X_s)} \right|^2 - I(\theta_0) \right) ds, \quad t \geq 0, \tag{3.4}$$

we consider the estimate

$$\begin{aligned} \sup_{t>0} \frac{\left|\hat{\theta}^0_{0,t} - \theta_0\right|}{g_m(t)} |M_t| \leq \sup_{t>0} (t+1)^{\alpha} \left|\hat{\theta}^0_{0,t} - \theta_0\right| \cdot \sup_{t>0} \frac{|M_t|}{(t+1)^{\alpha+\gamma+\delta}} \\ \cdot \sup_{t>0} \frac{\sqrt{m}(t+1)^{\delta}}{(m+t+1)^{1-\gamma}}. \end{aligned} \tag{3.5}$$

In view of Proposition 2.16, the first supremum on the right hand side of (3.5) is P-a.s. bounded. As already explained, the second supremum is bounded, too. Note that the function

$$x \longmapsto \frac{x^\delta}{(m+x)^{1-\gamma}}, \quad x > 1, \quad \text{for} \quad 0 < \delta < 1-\gamma,$$

achieves its unique maximum at the point

$$x_0 = \frac{\delta m}{(1-\gamma-\delta)}. \tag{3.6}$$

Hence, we have for any $\delta \in [0, 1/2-\gamma)$

$$\lim_{m\to\infty} \sup_{t>0} \frac{\sqrt{m}(t+1)^\delta}{(m+t+1)^{1-\gamma}} = 0. \tag{3.7}$$

□

Lemma 3.4. *Let Assumption $\mathcal{A}_0(\Theta)$ be satisfied. Let $q \in \mathcal{D}(\Theta)$ as well as $(q - a(\theta_0)) \in \mathcal{M}(b(\theta_0, \cdot), \sigma)$ for $\theta_0 \in \Theta$. Moreover, suppose that $b(\theta, \cdot)$, $\partial_\theta^2 b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ.*
Then, under H_0, we have

$$\sup_{t>0} \frac{|\psi_{m,t}(\theta_0)|}{g_m(t)} \left|\hat{\theta}^0_{m,t} - \theta_0\right| = o_P(1) \quad \text{as} \quad m \to \infty.$$

Proof. Exactly the same argumentation as in the proof of Lemma 3.3 works in this case, too. We only have to replace the process $(M_t \colon t \geq 0)$ in (3.4) by

$$M_t = \psi_{0,t}(\theta_0) = \int_0^t \frac{\dot{b}(\theta_0, X_s)}{\sigma(X_s)} dW_s, \quad t \geq 0.$$

□

Lemma 3.5. *Let Assumption $\mathcal{A}_0(\Theta)$ and $I > 0$ be satisfied. Suppose that $q \in \mathcal{D}(\Theta)$ as well as $(q - a(\theta_0)) \in \mathcal{M}(b(\theta_0, \cdot), \sigma)$ for $\theta_0 \in \Theta$. In addition, suppose that $\sigma' \in \mathcal{P}$ and $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot)$, $\partial_\theta^2 \partial_x b(\theta, \cdot)$, $\partial_\theta^3 \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ.*
Then, under H_0, we have

$$\sup_{t>0} \partial_\theta^2 \left(\frac{\psi_{m,t}}{I}\right)(\tilde{\theta}_{m,t}) \frac{(\hat{\theta}^0_{m,t} - \theta_0)^2}{g_m(t)} = o_P(1) \quad \text{as} \quad m \to \infty.$$

Proof. The functions $\partial_\theta b(\theta,\cdot)/\sigma^2$, $\partial_\theta^2 b(\theta,\cdot)/\sigma^2$, $\partial_\theta^3 b(\theta,\cdot)/\sigma^2$ are continuous and thereby, they possess primitives which we denote by F_θ, G_θ and H_θ. Recall that we consider the notations $b(\theta,x)$ and $b(\theta)(x)$ for $(\theta,x)\in\Theta\times\mathbb{R}$ to be equivalent.
Equations (2.49) - (2.51) state that

$$\begin{aligned}\psi_{m,t}(\theta) &= F_\theta(X_{m+t}) - F_\theta(X_m)\\ &\quad - \int_m^{m+t}\Big[F_\theta' b(\theta) + \frac{1}{2}F_\theta''\sigma^2\Big](X_s)\,ds,\\ \dot\psi_{m,t}(\theta) &= G_\theta(X_{m+t}) - G_\theta(X_m)\\ &\quad - \int_m^{m+t}\Big[G_\theta' b(\theta) + F_\theta'\dot b(\theta) + \frac{1}{2}G_\theta''\sigma^2\Big](X_s)\,ds,\end{aligned}$$

and

$$\begin{aligned}\ddot\psi_{m,t}(\theta) &= H_\theta(X_{m+t}) - H_\theta(X_m)\\ &\quad - \int_m^{m+t}\Big[H_\theta' b(\theta) + 3G_\theta'\dot b(\theta) + \frac{1}{2}H_\theta''\sigma^2\Big](X_s)\,ds.\end{aligned}$$

By the assumptions each integrand as well as each function F_θ, G_θ, H_θ is dominated by a positive polynomial uniformly in θ. Remember (1.5) which is a linear growth condition for the diffusion function σ. Hence, there exist positive polynomials Q_1 and Q_2 such that for all $\theta\in\Theta$ the estimate

$$\begin{aligned}&|\psi_{m,t}(\theta)| + |\dot\psi_{m,t}(\theta)| + |\ddot\psi_{m,t}(\theta)|\\ &\qquad\le Q_1(X_{m+t}) + Q_1(X_m) + \int_m^{m+t} Q_2(X_s)ds\\ &\qquad= 2Q_1(X_m) + \int_m^{m+t} Q_1'(X_s)\sigma(X_s)dW_s\\ &\qquad\quad + \int_m^{m+t}\Big[Q_1' b(\theta_0) + \frac{1}{2}Q_1''\sigma^2 + Q_2\Big](X_s)\,ds\end{aligned} \tag{3.8}$$

The last equality was obtained by the Ito formula.
Since the Fisher information I fulfils $\inf_{\theta\in\Theta} I(\theta)>0$ and belongs to $C^2(\Theta)$ (see Lemma 2.19), the functions $1/I$, $\partial_\theta(1/I)$ and $\partial_\theta^2(1/I)$ are bounded.

Hence, there exists a constant $K > 0$ such that

$$\left|\partial_\theta^2\left(\frac{\psi_{m,t}}{I}\right)(\tilde{\theta}_{m,t})\right| \leq \sup_\Theta \max\left\{\left|\partial_\theta^2\left(\frac{1}{I}\right)(\theta)\right|, \left|2\partial_\theta\left(\frac{1}{I}\right)(\theta)\right|, \frac{1}{I(\theta)}\right\} \cdot$$
$$\cdot \sup_\Theta\left(|\psi_{m,t}(\theta)| + \left|\dot{\psi}_{m,t}(\theta)\right| + \left|\ddot{\psi}_{m,t}(\theta)\right|\right)$$
$$\leq K \sup_\Theta\left(|\psi_{m,t}(\theta)| + \left|\dot{\psi}_{m,t}(\theta)\right| + \left|\ddot{\psi}_{m,t}(\theta)\right|\right) \quad \forall\, t > 0.$$

We conclude that

$$\left|\partial_\theta^2\left(\frac{\psi_{m,t}}{I}\right)(\tilde{\theta}_{m,t})\right| \frac{\left|\hat{\theta}^0_{m,t} - \theta_1\right|^2}{g_m(t)} \leq K \frac{\left|\hat{\theta}^0_{m,t} - \theta_1\right|^2}{g_m(t)} Z_t^{(m)}, \tag{3.9}$$

where for any $m \geq 0$

$$Z_t^{(m)} = 2Q_1(X_m) + \int_m^{m+t} Q_1'(X_s)\sigma(X_s)dW_s$$
$$+ \int_m^{m+t} \left[Q_1' b(\theta_0) + \frac{1}{2}Q_1''\sigma^2 + Q_2\right](X_s)\, ds, \quad t \geq 0.$$

Choose some $\alpha \in (1/4, 1/2)$ and set $\delta = \max\{0, 1-\gamma-2\alpha\}$. By Proposition 2.16 we have

$$\sup_{t>0}(t+1)^{2\alpha}\left|\hat{\theta}^0_{m,t} - \theta_0\right|^2 = \mathcal{O}_P(1) \quad \text{as} \quad m \to \infty. \tag{3.10}$$

Moreover, note that by Lemma 2.22.(i) the process $Z^{(m)}$ is the composition of $X_{m+(\cdot)}$ and a measurable map in $C[0,\infty)$. Then, since the processes X and $X_{m+(\cdot)}$ are equivalent, the equality in distribution

$$Z^{(m)} \stackrel{D}{=} Z^{(0)} \quad \forall\, m > 0 \tag{3.11}$$

results.
Recall that by Remark 2.9 all polynomials belong to $L^1(\mu(\theta_0))$. Hence, the ergodic theorem and Lemma 2.1 imply

$$\sup_{t>0} \frac{\left|Z_t^{(m)}\right|}{(t+1)^{2\alpha+\delta+\gamma}} = \mathcal{O}_P(1) \quad \text{as} \quad m \to \infty \tag{3.12}$$

Combining (3.10), (3.12), and (3.7), we obtain

$$\sup_{t>0}\left|Z_t^{(m)}\right| \frac{\left(\hat{\theta}^0_{m,t} - \theta_0\right)^2}{g_m(t)} = o_P(1) \quad \text{as} \quad m \to \infty.$$

□

Lemma 3.6. *Let Assumption $\mathcal{A}_0(\Theta)$ and $I > 0$ be satisfied. Suppose that $q \in \mathcal{D}(\Theta)$ and $(q - a(\theta_0))$, $(|\partial_\theta b(\theta_0,\cdot)/\sigma|^2 - I(\theta_0)) \in \mathcal{M}(b(\theta_0,\cdot),\sigma)$ for $\theta_0 \in \Theta$.*
In addition, suppose that $\sigma' \in \mathcal{P}$ as well as $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial_\theta^2\partial_x b(\theta,\cdot)$, $\partial_\theta^3\partial_x b(\theta,\cdot) \in \mathcal{P}$ uniformly in θ.
Then, under H_0,

$$\sup_{t>0} \frac{t}{g_m(t)} \bigg| \big(\hat\theta^0_{0,m} - \theta_0\big)\Big(1 + \frac{\dot\psi_{0,m}(\theta_0)}{mI(\theta_0)} - \frac{\dot I(\theta_0)\psi_{0,m}(\theta_0)}{mI^2(\theta_0)}\Big) + \partial_\theta^2\Big(\frac{\psi_{0,m}}{I}\Big)(\tilde\theta_{0,m})\frac{\big(\hat\theta^0_{0,m} - \theta_0\big)^2}{m}\bigg| = o_P(1) \quad as \quad m \to \infty.$$

Proof. According to (3.9), the third remainder

$$R_3 = \partial_\theta^2\Big(\frac{\psi_{m,t}}{I}\Big)(\tilde\theta_{m,t})\frac{\big(\hat\theta^0_{m,t} - \theta_0\big)^2}{t}, \quad t > 0,$$

in the Taylor expansion of (3.1) can be bounded by

$$K\big(\hat\theta^0_{m,t} - \theta_1\big)^2 \frac{Z_t^{(m)}}{t}, \quad t > 0,$$

for some $K > 0$.
Hence, by the proofs of (2.33) and of Lemma 2.23, there exists for any $m \geq 0$ a measurable map

$$R_m \colon C[0,\infty) \longrightarrow C[0,\infty), \quad x \longmapsto R_m(x),$$

such that, using the notations of (3.1),

$$|R_1 + R_2 + R_3| \leq R_m(X)(t) \quad P\text{-a.s.}, \qquad |R_4| \leq R_0(X)(m) \quad P\text{-a.s.}$$

where, for $x \in C[0,\infty)$, $R_m(x)(t)$ represents the value of the function $R_m(x) \in C[0,\infty)$ in $t \geq 0$. From (2.33), Lemma 2.23 and (3.11) it follows that

$$\sup_{t>0} \frac{t}{g_m(t)} R_m(X)(t) \overset{D}{=} \sup_{t>0} \frac{t}{g_m(t)} R_0(X)(t) \quad \forall\, m > 0.$$

According to the proofs of Lemmata 3.3, 3.4, and 3.5, we have

$$\sup_{t>0} \frac{t}{g_m(t)} R_0(X)(t) = o_P(1) \quad \text{as} \quad m \to \infty.$$

Consider the following lower bound:

$$\begin{aligned}\sup_{t>0} \frac{t}{g_m(t)} R_0(X)(t) &\geq \frac{m}{g_m(m)} R_0(X)(m) \\ &= \left(\frac{m}{2m+1}\right)^{1-\gamma} \left(\frac{m}{m+1}\right)^{\gamma} \sqrt{m} R_0(X)(m).\end{aligned}$$

Hence, $\sqrt{m}R_0(X)(m)$ tends to zero in probability as $m \to \infty$. Finally, the inequality

$$\sup_{t>0} \frac{t}{g_m(t)} = \sup_{t>0} \frac{t\sqrt{m}}{(m+t+1)^{1-\gamma}(t+1)^{\gamma}} \leq \sqrt{m} \quad \forall\, m \geq 0$$

yields

$$\sup_{t>0} \frac{t}{g_m(t)} R_0(X)(m) = o_P(1) \quad \text{as} \quad m \to \infty.$$

□

3.3 Main result

Theorem 3.7. *Let Assumptions 3.1 and* $\mathcal{A}_0(\Theta)$ *be satisfied. Let* $I > 0$, $q \in \mathcal{D}(\Theta)$ *and* $(|\partial_\theta b(\theta_0, \cdot)/\sigma|^2 - I(\theta_0))$, $(q - a(\theta_0)) \in \mathcal{M}(b(\theta_0, \cdot), \sigma)$ *for* $\theta_0 \in \Theta$.
In addition, suppose that $\sigma' \in \mathcal{P}$ *as well as* $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot)$, $\partial_\theta^2 \partial_x b(\theta, \cdot)$, $\partial_\theta^3 \partial_x b(\theta, \cdot) \in \mathcal{P}$ *uniformly in* θ.
If $0 \leq \gamma \leq 1/2 - \alpha$ *and* H_0 *holds, then we have*

$$\lim_{m\to\infty} \mathcal{L}\left(\sup_{t>0} |S_t^m|\right) = \mathcal{L}\left(\sup_{0<t\leq 1} \frac{|W_t|}{\sqrt{I(\theta_0)}t^{\gamma}}\right)$$

where W *represents a Wiener process.*

Proof. In view of Lemmata 3.3, 3.4, and 3.5, it is sufficient to show

$$\lim_{m\to\infty} \mathcal{L}\left(\sup_{t>0} \frac{1}{g_m(t)} \left|\frac{\psi_{m,t}(\theta_0)}{I(\theta_0)} - \frac{t}{m}\frac{\psi_{0,m}(\theta_0)}{I(\theta_0)}\right|\right) = \mathcal{L}\left(\sup_{0<t\leq 1} \frac{|W_t|}{\sqrt{I(\theta_0)}t^{\gamma}}\right).$$

It was proven in Lemma 3.2 that the assumption of the strong invariance principle yields

$$\sup_{t>0} \frac{1}{g_m(t)} \left|\frac{\psi_{m,t}(\theta_0)}{I(\theta_0)} - \frac{B_t^m}{\sqrt{I(\theta_0)}} + \frac{t}{m}\left(\frac{B_m}{\sqrt{I(\theta_0)}} - \frac{\psi_{0,m}(\theta_0)}{I(\theta_0)}\right)\right| = o(1) \qquad (3.13)$$

P-a.s. as $m \to \infty$ where for any $m > 0$ $B^{(m)}$ and B represent two independent Wiener processes. Similar to Horváth et al. (2004), one can compute that

$$\lim_{m\to\infty} \mathcal{L}\left(\sup_{t>0} \frac{|B_t^{(m)} - \frac{t}{m}B_m|}{g_m(t)}\right) = \mathcal{L}\left(\sup_{0<t\leq 1} \frac{|W_t|}{t^\gamma}\right) \tag{3.14}$$

with a new Wiener process W in the following way:
Note that the weighting function can be written as

$$g_m(t) = \sqrt{m}\left(1 + \frac{t+1}{m}\right)\left(\frac{t+1}{m+t+1}\right)^\gamma, \quad t \geq 0,\, m > 0.$$

Due to the independence of $B^{(m)}$ and B, we can replace $B^{(m)}$ for any $m > 0$ by an arbitrary Wiener process W and apply the property of rescaling in order to obtain

$$\begin{aligned}
\sup_{t>0} \frac{\left|B_t^{(m)} - \frac{t}{m}B_m\right|}{g_m(t)} &\stackrel{D}{=} \sup_{t>0} \frac{\left|W_t - \frac{t}{m}B_m\right|}{g_m(t)} \\
&\stackrel{D}{=} \sup_{t>0} \frac{\left|W_{t/m} - \frac{t}{m}B_1\right|}{\left(1 + \frac{t+1}{m}\right)\left(\frac{t+1}{m+t+1}\right)^\gamma}.
\end{aligned}$$

Then the substitution $t = ms$ yields

$$\begin{aligned}
\sup_{t>0} \frac{\left|B_t^{(m)} - \frac{t}{m}B_m\right|}{g_m(t)} &\stackrel{D}{=} \sup_{s>0} \frac{|W_s - sB_1|}{\left(1 + s + \frac{1}{m}\right)}\left(1 + \frac{m}{1+ms}\right)^\gamma \\
&\stackrel{m\to\infty}{\longrightarrow} \sup_{s>0} \frac{|W_s - sB_1|}{(1+s)}\left(\frac{1+s}{s}\right)^\gamma
\end{aligned} \tag{3.15}$$

where the convergence holds pathwise. According to equation (5.15) of Horváth et al. (2004), the limit random variable in (3.15) has the same distribution as

$$\sup_{0<t\leq 1} \frac{|W_t|}{t^\gamma}.$$

Finally, since for all functions $a, b \colon (0,\infty) \to \mathbb{R}$ the inequality

$$\left|\sup_{t>0} |a_t| - \sup_{t>0} |b_t|\right| \leq \sup_{t>0} |a_t - b_t| \tag{3.16}$$

holds, Slutsky's lemma can be applied in order to obtain the statement of the theorem from (3.13) and (3.14).

□

Theorem 3.7 is still useless for determining the critical value c of the test procedure (see (1.6)) because the unknown parameter θ_0 is involved in the limit distribution. Therefore, we shall estimate θ_0 from the training period $[0, m]$. For this purpose, consider the modified statistic

$$\widehat{S_t^m} = \frac{\sqrt{I(\hat{\theta}_{0,m}) \cdot t}}{g_m(t)} \big(\hat{\theta}_{m,t} - \hat{\theta}_{0,m}\big), \quad m, t > 0. \tag{3.17}$$

Corollary 3.8. *Under the assumptions of Theorem 3.7 the convergence*

$$\lim_{m\to\infty} \mathcal{L}\left(\sup_{t>0} |\widehat{S_t^m}|\right) = \mathcal{L}\left(\sup_{0<t\le 1} \frac{|W_t|}{t^\gamma}\right)$$

holds.

Proof. Since the Fisher information I is continuous (see Lemma 2.19), Slutsky's lemma implies the desired statement.

□

Remark 3.9. We proved in Theorem 2.3 a strong invariance principle of the form assumed in Theorem 3.7 and Corollary 3.8. The approximation error is $o(t^\alpha)$ for an arbitrary $\alpha > 1/4$. Thus, the presented test procedure works as long as $0 \le \gamma < 1/4$ is chosen in the weighting function g_m.

Remark 3.10. The problem of a one-sided alternative can be treated along the lines of the previous discussion. In this case we would like to test sequentially

$H_0\colon\ t^* = \infty$

versus

$H_1\colon\ t^* < \infty \quad \text{and} \quad \theta_1 > \theta_0.$

Toward this end, we modify the stopping rule by cancelling the absolute value:

$$\tau_m = \inf\left\{t > 0\colon \widehat{S_t^m} > c\right\}, \quad m > 0,$$

where $\widehat{S_t^m}$ is the same statistic as for the two-sided alternative. The method used in the proofs of Theorem 3.7 and Corollary 3.8 can be applied once again. We only have to replace the inequality in (3.16) by

$$\left|\sup_{t>0} a_t - \sup_{t>0} b_t\right| \le \sup_{t>0} |a_t - b_t|$$

for functions $a, b\colon (0,\infty) \to \mathbb{R}$.
Then, under the assumptions of Theorem 3.7, the convergence

$$\lim_{m\to\infty} \mathcal{L}\left(\sup_{t>0} \widehat{S_t^m}\right) = \mathcal{L}\left(\sup_{0<t\le 1} \frac{W_t}{t^\gamma}\right)$$

holds.

Chapter 4

Asymptotics under the alternative

4.1 Asymptotic power

In this section we prove that the procedure given by (1.6) has asymptotic power one. In sequential change-point analysis the asymptotic power of a test with infinite time horizon is defined by

$$\lim_{m\to\infty} P\{\tau_m < \infty\} \tag{4.1}$$

where P represents the measure under H_1.
Now we study a Taylor expansion of the statistic S_t^m although the mean value theorem would be sufficient in this section. However, in order to simplify the proofs, we want to use the same expansion for the asymptotic normality of the stopping time (see Section 4.3).
Consider

$$\begin{aligned} S_t^m = \frac{t}{g_m(t)} \Bigg[&(\hat{\theta}^0_{m,t} - \theta_1) + \frac{\psi_{m,t}(\hat{\theta}^0_{m,t})}{tI(\hat{\theta}^0_{m,t})} + (\theta_1 - \theta_0) \\ &- \left((\hat{\theta}^0_{0,m} - \theta_0) + \frac{\psi_{0,m}(\hat{\theta}^0_{0,m})}{mI(\hat{\theta}^0_{0,m})} \right) \Bigg] \end{aligned}$$

and expand $\psi_{m,t}/I$ around θ_1 and $\psi_{0,m}/I$ around θ_0. We have to assume that $I,\, \psi_{m,t} \in C^2(\Theta)$ and $I > 0$. One obtains

$$\begin{aligned} S_t^m = &\frac{(\theta_1 - \theta_0)t}{g_m(t)} + \frac{\psi_{m,t}(\theta_1)}{g_m(t)I(\theta_1)} - \frac{t}{g_m(t)} \frac{\psi_{0,m}(\theta_0)}{mI(\theta_0)} \\ &+ \frac{t}{g_m(t)} (R_1 - R_2 + R_3 - R_4) \end{aligned} \tag{4.2}$$

where the terms $R_1, \dots, R_4$ are similar to the terms in (3.1), i.e.,

$$\begin{aligned} R_1 &= \big(\hat{\theta}^0_{m,t} - \theta_1\big) + \frac{\dot{\psi}_{m,t}(\theta_1)}{tI(\theta_1)}\big(\hat{\theta}^0_{m,t} - \theta_1\big), \\ R_2 &= \frac{\dot{I}(\theta_1)\psi_{m,t}(\theta_1)}{tI^2(\theta_1)}\big(\hat{\theta}^0_{m,t} - \theta_1\big), \\ R_3 &= \partial^2_\theta\Big(\frac{\psi_{m,t}}{I}\Big)(\tilde{\theta}_{m,t})\frac{\big(\hat{\theta}^0_{m,t} - \theta_1\big)^2}{t}, \end{aligned}$$

and

$$\begin{aligned} R_4 = (\hat{\theta}^0_{0,m} - \theta_0) &+ \frac{\dot{\psi}_{0,m}(\theta_0)}{mI(\theta_0)}\big(\hat{\theta}^0_{0,m} - \theta_0\big) - \frac{\dot{I}(\theta_0)\psi_{0,m}(\theta_0)}{mI^2(\theta_0)}(\hat{\theta}^0_{0,m} - \theta_0) \\ &+ \partial^2_\theta\Big(\frac{\psi_{0,m}}{I}\Big)(\tilde{\theta}_{0,m})\frac{\big(\hat{\theta}^0_{0,m} - \theta_0\big)^2}{m}. \end{aligned}$$

Here, we have $\big|\tilde{\theta}_{0,m} - \theta_0\big| \le \big|\hat{\theta}^0_{0,m} - \theta_0\big|$ and $\big|\tilde{\theta}_{m,t} - \theta_1\big| \le \big|\hat{\theta}^0_{m,t} - \theta_1\big|$.

Theorem 4.1. *Assume that either*

(i) $t^*(m) = \mathcal{O}(1)$ *as* $m \to \infty$ *or*

(ii) $\lim_{m\to\infty} t^*(m) = \infty$ *and* $t^*(m) = \mathcal{O}(m^{3/2})$ *as* $m \to \infty$.

Let Assumption $\mathcal{A}_0(\Theta)$, $I > 0$ *and*

$$E_{\mu(\theta_1)}\left|\frac{\ddot{b}(\theta_1, \cdot)}{\sigma}\right|^2 > 0 \quad \textit{for} \quad \theta_1 \neq \theta_0$$

be satisfied.
Suppose that $q \in \mathcal{D}(\Theta)$, $(|\partial_\theta b(\theta_0, \cdot)/\sigma|^2 - I(\theta_0)), (q - a(\theta_0)) \in \mathcal{M}(b(\theta_0, \cdot), \sigma)$ *for* $\theta_0 \in \Theta$ *and* $(|\partial_\theta b(\theta_1, \cdot)/\sigma|^2 - I(\theta_1)), (q - a(\theta_1)) \in \mathcal{M}(b(\theta_1, \cdot), \sigma)$ *for* $\theta_1 \neq \theta_0$.
In addition, suppose that $\sigma' \in \mathcal{P}$ *as well as* $b(\theta, \cdot)$, $\partial_\theta\partial_x b(\theta, \cdot)$, $\partial^2_\theta\partial_x b(\theta, \cdot)$, $\partial^3_\theta\partial_x b(\theta, \cdot) \in \mathcal{P}$ *uniformly in* θ.
Then, under the alternative, we have

$$\lim_{m\to\infty} P\big\{\tau_m < \infty\big\} = 1.$$

Proof. We observe that

$$\big\{\tau_m < \infty\big\} = \Big\{\sup_{t>0} |S^m_t| > c\Big\} \quad \forall\, m \ge 0.$$

Set $t_m = m^2 + t^*(m),\ m \geq 0,$ and

$$R_m = S^m_{t_m} - \frac{(\theta_1 - \theta_0)t_m}{g_m(t_m)}, \quad m \geq 0.$$

Since

$$\left\{\sup_{t>0} |S^m_t| > c\right\} \supset \left\{|S^m_{t_m}| > c\right\} \supset \left\{\tfrac{|\theta_1-\theta_0|t_m}{g_m(t_m)} - |-R_m| > c\right\} \quad \forall\, m \geq 0,$$

it is sufficient to show that the deterministic drift term $(\theta_1 - \theta_0)t_m/g_m(t_m)$ is unbounded as $m \to \infty$ and that $R_m = \mathcal{O}_P(1)$ as $m \to \infty$. Note that

$$\frac{t_m}{g_m(t_m)} = \frac{t_m\sqrt{m}}{(m + t_m + 1)^{1-\gamma}(t_m + 1)^{\gamma}} \geq \frac{t_m\sqrt{m}}{m + t_m + 1} \quad \forall\, m \geq 0.$$

Hence, the drift term crosses the critical value as $m \to \infty$. We show in Lemmata 4.2 - 4.7 that $(R_m \colon m \geq 0)$ is bounded in probability. □

Lemma 4.2. *Assume that either*

(i) $t^*(m) = \mathcal{O}(1)$ *as* $m \to \infty$ *or*

(ii) $\lim_{m\to\infty} t^*(m) = \infty$ *and* $t^*(m) = \mathcal{O}(m^{3/2})$ *as* $m \to \infty$.

Moreover, let Assumption $\mathcal{A}_0(\Theta)$ *be satisfied. In addition, suppose that* $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot) \in \mathcal{P}$ *uniformly in* θ *and that*

$$I(\theta_1) = E_{\mu(\theta_1)}\left|\frac{\dot{b}(\theta_1,\cdot)}{\sigma}\right|^2 > 0 \quad \text{for} \quad \theta_1 \neq \theta_0. \tag{4.3}$$

Then, under H_1, *we have*

$$\frac{|\psi_{m,t_m}(\theta_1)|}{g_m(t_m)} = \mathcal{O}_P(1) \quad \text{as} \quad m \to \infty.$$

Proof. By Remark 2.21 we have

$$\psi_{m,t_m}(\theta_1) = \int_{m+t^*}^{m+t_m} \frac{\dot{b}(\theta_1, X_s)}{\sigma(X_s)}dW_s + \int_m^{m+t^*} \frac{\dot{b}(\theta_1, X_s)}{\sigma(X_s)}dW_s + \int_m^{m+t^*} \left[\frac{[b(\theta_0) - b(\theta_1)]\dot{b}(\theta_1)}{\sigma^2}\right](X_s)\, ds.$$

Remember that by Lemma 2.24 $\psi_{m,t_m}(\theta_1)$ has the same distribution as

$$\begin{aligned}\psi_{0,t_m}(\theta_1) = &\int_{t^*}^{t_m} \frac{\dot b(\theta_1, Y_s)}{\sigma(Y_s)} dW_s + \int_0^{t^*} \frac{\dot b(\theta_1, Y_s)}{\sigma(Y_s)} dW_s \\ &+ \int_0^{t^*} \left[\frac{[b(\theta_0) - b(\theta_1)]\dot b(\theta_1)}{\sigma^2}\right](Y_s)\, ds\end{aligned} \tag{4.4}$$

where the process Y is the solution of (2.56).
For studying the asymptotic behaviour of the first stochastic integral in (4.4), define the process

$$M_s^* = \int_0^s \frac{\dot b\big(\theta_1, X(\theta_1)_u\big)}{\sigma(X(\theta_1)_u)} dW_u, \quad s \geq 0,$$

where $X(\theta_1)$ represents the solution of the homogeneous equation (2.21) with $\theta = \theta_1$ but with initial condition $\mu(\theta_0) = \mathcal{L}(Y_{t^*})$. According to Lemma 2.22.(i), the random variable $M_{m^2}^*$ represents the composition of the process $X(\theta_1)$ and a measurable functional defined in $C[0,\infty)$. Since, in addition, the process $(Y_{t^*+s} \colon s \geq 0)$ is equivalent to $X(\theta_1)$, it follows that

$$M_{m^2}^* \stackrel{D}{=} \int_{t^*}^{t_m} \frac{\dot b(\theta_1, Y_u)}{\sigma(Y_u)} dW_u. \tag{4.5}$$

By Lemma 2.1 one obtains for any $\delta \in (0, 1/4]$

$$\frac{\sqrt{m}\,\big|M_{m^2}^*\big|}{(m + t_m + 1)^{1-\gamma}(t_m + 1)^\gamma} \leq \frac{\big|M_{m^2}^*\big|}{(m^2)^{1/2+\delta}} \cdot \frac{\sqrt{m}}{(m^2)^{1/2-\delta}} = o(1) \tag{4.6}$$

P-a.s. as $m \to \infty$.
In order to investigate the integrals up to the change-point in (4.4), define the process

$$M_t = \int_0^t \frac{\dot b(\theta_1, Y_s)}{\sigma(Y_s)} dW_s + \int_0^t \left[\frac{\dot b(\theta_1)[b(\theta_0) - b(\theta_1)]}{\sigma^2}\right](Y_s)\, ds, \quad t \geq 0.$$

The ergodic theorem and Lemma 2.1 yield

$$\lim_{m\to\infty} \frac{M_{t^*(m)}}{t^*(m)} = E_{\mu(\theta_0)}\left(\frac{\dot b(\theta_1, \cdot)[b(\theta_0, \cdot) - b(\theta_1, \cdot)]}{\sigma^2}\right) \quad P\text{-a.s.}$$

Hence,

$$\frac{\sqrt{m}\,|M_{t^*}|}{(m+t_m+1)^{1-\gamma}(t_m+1)^\gamma} \leq \frac{\sqrt{m}\cdot t^*}{(m^2+t^*)}\frac{|M_{t^*}|}{t^*} \\ = \mathcal{O}(1) \quad P\text{-a.s.} \quad \text{as} \quad m\to\infty \tag{4.7}$$

according to the growth condition of t^*.
Combining (4.5) - (4.7), we obtain the statement of the lemma.

□

Lemma 4.3. *Let Assumption* $\mathcal{A}_0(\Theta)$ *be satisfied. Moreover, suppose that* $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)\in\mathcal{P}$ *uniformly in* θ *and that*

$$I(\theta_0) = E_{\mu(\theta_0)}\left|\frac{\dot{b}(\theta_0,\cdot)}{\sigma}\right|^2 > 0, \quad \theta_0\in\Theta. \tag{4.8}$$

Then, under H_1, *we have*

$$\frac{t_m}{g_m(t_m)}\cdot\frac{\psi_{0,m}(\theta_0)}{m} = \mathcal{O}_P(1) \quad \textit{as} \quad m\to\infty.$$

Proof. By (4.8) the central limit theorem for stochastic integrals can be applied (see Kutoyants (2004), Theorem 1.19):

$$\frac{\psi_{0,m}(\theta_0)}{\sqrt{m}} = \mathcal{O}_P(1) \quad \text{as} \quad m\to\infty.$$

Hence,

$$\frac{\sqrt{m}t_m}{(m+t_m+1)^{1-\gamma}(t_m+1)^\gamma}\cdot\frac{\psi_{0,m}(\theta_0)}{m} = \mathcal{O}_P(1) \quad \text{as} \quad m\to\infty.$$

□

Lemma 4.4. *Let either* t^* *be bounded or* $\lim_{m\to\infty} t^*(m)=\infty$. *Let Assumption* $\mathcal{A}_0(\Theta)$ *and*

$$E_{\mu(\theta_1)}\left|\frac{\ddot{b}(\theta_1,\cdot)}{\sigma}\right|^2 > 0, \quad \theta_1\neq\theta_0, \tag{4.9}$$

be satisfied. Suppose that $q\in\mathcal{D}(\Theta)$ *as well as* $(|\partial_\theta b(\theta_1,\cdot)/\sigma|^2 - I(\theta_1))$, $(q-a(\theta_1))\in\mathcal{M}(b(\theta_1,\cdot),\sigma)$ *for* $\theta_1\neq\theta_0$.
In addition, suppose that $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial_\theta^2\partial_x b(\theta,\cdot)\in\mathcal{P}$ *uniformly in* θ.
Then, under the alternative, we have

$$\frac{\left|t_m I(\theta_1)+\dot{\psi}_{0,t_m}(\theta_1)\right|}{g_m(t_m)}\left|\hat{\theta}^0_{0,t_m}-\theta_1\right| = \mathcal{O}_P(1) \quad \textit{as} \quad m\to\infty.$$

Proof. First, by means of Proposition 2.17 and Lemma 2.24 we obtain that $\hat{\theta}^0_{m,t_m}$ and $\partial_\theta \psi_{m,t_m}(\theta_1)$ have the same distribution as $\hat{\theta}^0_{0,t_m}$ and $\partial_\theta \psi_{0,t_m}(\theta_1)$, respectively. Therefore, it is sufficient to discuss the asymptotic behaviour of

$$\frac{\big|t_m I(\theta_1) + \dot{\psi}_{0,t_m}(\theta_1)\big|}{g_m(t_m)} \big|\hat{\theta}^0_{0,t_m} - \theta_1\big|$$

as $m \to \infty$. By Remark 2.21 one can see that

$$tI(\theta_1) + \dot{\psi}_{0,t}(\theta_1) = M^*_t + Z_{t^*} \quad \forall\, t \geq t^*$$

where, using the process Y defined in (2.56),

$$M^*_t = \int_{t^*}^{t} \frac{\ddot{b}(\theta_1, Y_s)}{\sigma(Y_s)} dW_s + \int_{t^*}^{t} \left(I(\theta_1) - \left| \frac{\dot{b}(\theta_1, Y_s)}{\sigma(Y_s)} \right|^2 \right) ds, \quad t \geq t^*,$$

$$Z_{t^*} = \int_0^{t^*} \frac{\ddot{b}(\theta_1, Y_s)}{\sigma(Y_s)} dW_s + \int_0^{t^*} \left(I(\theta_1) + \left[\frac{\ddot{b}(\theta_1)[b(\theta_0) - b(\theta_1)] - |\dot{b}(\theta_1)|^2}{\sigma^2} \right] (Y_s) \right) ds.$$

We can argue as in the explanation of (4.5) in order to obtain

$$\big(M^*_{t^*+s} \colon s \geq 0 \big) \overset{D}{=} \big(M_s \colon s \geq 0 \big) \tag{4.10}$$

with

$$M_s = \int_0^{s} \frac{\ddot{b}(\theta_1, X(\theta_1)_u)}{\sigma(X(\theta_1)_u)} dW_u + \int_0^{s} \left(I(\theta_1) - \left| \frac{\dot{b}(\theta_1, X(\theta_1)_u)}{\sigma(X(\theta_1)_u)} \right|^2 \right) du, \quad s \geq 0,$$

where $X(\theta_1)$ is the solution of (2.21) starting with the distribution $\mu(\theta_0)$. From the laws of the iterated logarithm given in Lemma 2.1 and Remark 2.2 follows that P-a.s.

$$\frac{\sqrt{m}\,|M_{m^2}|}{(m + t_m + 1)^{1-\gamma}(t_m + 1)^{\gamma}} \leq \frac{\sqrt{m}}{t_m^{1/4}} \cdot \frac{|M_{m^2}|}{t_m^{3/4}} = o(1) \quad \text{as} \quad m \to \infty \tag{4.11}$$

where we use $t_m = m^2 + t^*$. Choose some $\alpha \in (1/4, 1/2)$. We remark that Proposition 2.17 yields

$$(t_m + 1)^{\alpha} \big|\hat{\theta}^0_{0,t_m} - \theta_1\big| = \mathcal{O}(1) \quad P\text{-a.s.} \quad \text{as} \quad m \to \infty. \tag{4.12}$$

Moreover, by Lemma 2.1 and the ergodic theorem we have

$$\frac{Z_{t^*}}{g_m(t_m)(t_m+1)^\alpha} \leq \frac{Z_{t^*}}{t^*} \cdot \frac{\sqrt{m}}{t_m{}^{1/4}} \cdot \frac{t^*}{t_m^{3/4+\alpha}} = o(1) \tag{4.13}$$

P-a.s. as $m \to \infty$.
Finally, combining (4.10) - (4.13), we get

$$\frac{|M^*_{t_m} + Z_{t^*}|}{g_m(t_m)} \big|\hat{\theta}^0_{0,t_m} - \theta_1\big| = o_P(1) \quad \text{as} \quad m \to \infty.$$

□

Lemma 4.5. *Assume that either*

(i) $t^*(m) = \mathcal{O}(1)$ *as* $m \to \infty$ *or*

(ii) $\lim_{m\to\infty} t^*(m) = \infty$ *and* $t^*(m) = \mathcal{O}(m^{3/2})$ *as* $m \to \infty$.

Moreover, let Assumption $\mathcal{A}_0(\Theta)$ *be satisfied. In addition, suppose that* $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot) \in \mathcal{P}$ *uniformly in* θ *and that*

$$I(\theta_1) = E_{\mu(\theta_1)}\left|\frac{\dot{b}(\theta_1,\cdot)}{\sigma}\right|^2 > 0 \quad \textit{for} \quad \theta_1 \neq \theta_0.$$

Then, under H_1, *we have*

$$\frac{|\psi_{m,t_m}(\theta_1)|}{g_m(t_m)} \big|\hat{\theta}^0_{m,t_m} - \theta_1\big| = \mathcal{O}_P(1) \quad \textit{as} \quad m \to \infty.$$

Proof. The assertion directly follows from Lemma 4.2 because the parameter set Θ is bounded.

□

Lemma 4.6. *Let either* t^* *be bounded or* $\lim_{m\to\infty} t^*(m) = \infty$. *Let Assumption* $\mathcal{A}_0(\Theta)$ *and* $I > 0$ *be satisfied.*
Suppose that $q \in \mathcal{D}(\Theta)$ *and* $(q - a(\theta_1)) \in \mathcal{M}(b(\theta_1,\cdot),\sigma)$ *for* $\theta_1 \neq \theta_0$. *In addition, suppose that* $\sigma' \in \mathcal{P}$ *as well as* $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial^2_\theta\partial_x b(\theta,\cdot)$, $\partial^3_\theta\partial_x b(\theta,\cdot) \in \mathcal{P}$ *uniformly in* θ.
Then, under the alternative, we have

$$\partial^2_\theta\Big(\frac{\psi_{m,t_m}}{I}\Big)(\tilde{\theta}_{m,t_m})\frac{\big(\hat{\theta}^0_{m,t_m} - \theta_1\big)^2}{g_m(t_m)} = \mathcal{O}_P(1) \quad \textit{as} \quad m \to \infty.$$

Recall that $t_m = m^2 + t^*(m)$.

Proof. Similar to the proof of Lemma 3.5, let $F_\theta,\ G_\theta,\ H_\theta$ be primitives of $\partial_\theta b(\theta,\cdot)/\sigma^2,\ \partial_\theta^2 b(\theta,\cdot)/\sigma^2$ and $\partial_\theta^3 b(\theta,\cdot)/\sigma^2$. The local integrability of the last three functions is guaranteed by their continuity. For any $m,t \geq 0$ the Ito formula and Remark 2.21 imply for the function $\theta \mapsto \psi_{m,t}(\theta)$ and its derivatives

$$\psi_{m,t}(\theta) = F_\theta(X_{m+t}) - F_\theta(X_m) - \int_m^{m+t} \left[\frac{\dot b(\theta) b(\theta)}{\sigma^2} + \frac{F_\theta'' \sigma^2}{2}\right](X_s)\,ds,$$

$$\dot\psi_{m,t}(\theta) = G_\theta(X_{m+t}) - G_\theta(X_m) - \int_m^{m+t} \left[\frac{b(\theta)\ddot b(\theta) + \dot b(\theta)^2}{\sigma^2} + \frac{G_\theta'' \sigma^2}{2}\right](X_s)\,ds,$$

and

$$\begin{aligned}\ddot\psi_{m,t}(\theta) = {} & H_\theta(X_{m+t}) - H_\theta(X_m) \\ & - \int_m^{m+t} \left[\frac{3\partial_\theta b(\theta)\partial_\theta^2 b(\theta) + b(\theta)\partial_\theta^3 b(\theta)}{\sigma^2} + \frac{H_\theta'' \sigma^2}{2}\right](X_s)\,ds.\end{aligned}$$

By our assumptions each integrand as well as each function $F_\theta,\ G_\theta,\ H_\theta$ is dominated by a positive polynomial uniformly in θ. Hence, there exists positive polynomials Q_1 and Q_2 such that for all $\theta \in \Theta$ the inequality

$$\begin{aligned}|\psi_{m,t}(\theta)| & + |\dot\psi_{m,t}(\theta)| + |\ddot\psi_{m,t}(\theta)| \\ & \leq Q_1(X_{m+t}) + Q_1(X_m) + \int_m^{m+t} Q_2(X_s)ds \\ & = 2Q_1(X_m) + \int_m^{m+t} Q_1'(X_s)\sigma(X_s)dW_s + \int_m^{m+t^*} Q_1'(X_s) b(\theta_0, X_s)ds \\ & \quad + \int_{m+t^*}^{m+t} Q_1'(X_s) b(\theta_1, X_s)ds + \int_m^{m+t} \left[\frac{1}{2}Q_1''\sigma^2 + Q_2\right](X_s)\,ds\end{aligned}$$

holds. The last equality was obtained by the Ito formula. Taking into account that, combining $\inf_{\theta\in\Theta} I(\theta) > 0$ and Lemma 2.19, there exists a constant $K > 0$ such that

$$\begin{aligned}\left|\partial_\theta^2\Big(\frac{\psi_{m,t}}{I}\Big)(\tilde\theta_{m,t})\right| & \leq \sup_\Theta \max\left\{\left|\partial_\theta^2\Big(\frac{1}{I}\Big)(\theta)\right|, \left|2\partial_\theta\Big(\frac{1}{I}\Big)(\theta)\right|, \frac{1}{I(\theta)}\right\} \cdot \\ & \quad \cdot \sup_\Theta\big(|\psi_{m,t}(\theta)| + |\dot\psi_{m,t}(\theta)| + |\ddot\psi_{m,t}(\theta)|\big) \\ & \leq K \sup_\Theta\big(|\psi_{m,t}(\theta)| + |\dot\psi_{m,t}(\theta)| + |\ddot\psi_{m,t}(\theta)|\big) \quad \forall\ t > 0,\end{aligned}$$

we conclude that

$$\left|\partial_\theta^2\Big(\frac{\psi_{m,t_m}}{I}\Big)(\tilde\theta_{m,t_m})\right|\frac{\left|\hat\theta^0_{m,t_m}-\theta_1\right|^2}{g_m(t_m)}\le K\frac{\left|\hat\theta^0_{m,t_m}-\theta_1\right|^2}{g_m(t_m)}\Big(Z^{(m)}_{t^*}+A^{(m+t^*)}_{m^2}\Big)\tag{4.14}$$

where

$$\begin{aligned}Z^{(m)}_{t^*}&=2Q_1(X_m)+\int\limits_m^{m+t^*}Q_1'(X_s)\sigma(X_s)dW_s\\&\quad+\int\limits_m^{m+t^*}\Big[Q_1'b(\theta_0)+\frac12Q_1''\sigma^2+Q_2\Big](X_s)\,ds,\\A^{(m+t^*)}_s&=\int\limits_{m+t^*}^{m+t^*+s}Q_1'(X_u)\sigma(X_u)dW_u\\&\quad+\int\limits_{m+t^*}^{m+t^*+s}\Big[Q_1'b(\theta_1)+\frac12Q_1''\sigma^2+Q_2\Big](X_u)\,du,\quad s\ge0.\end{aligned}$$

Let $\alpha\in(1/8,1/2)$. By Proposition 2.17 we have

$$(t_m+1)^{2\alpha}\left|\hat\theta^0_{m,t_m}-\theta_1\right|^2=\mathcal{O}_P(1)\quad\text{as}\quad m\to\infty.\tag{4.15}$$

Note that by Lemmata 2.5, 2.22.(i) and by the argumentation used in the proof of (2.37) the equalities in distribution

$$\begin{aligned}Z^{(m)}_{t^*}&\overset{D}{=}Z^{(0)}_{t^*}\quad\forall\,m>0,\\(A^{(m+t^*)}_s\colon s\ge0)&\overset{D}{=}(A^{(0)}_s\colon s\ge0)\quad\forall\,m>0\end{aligned}\tag{4.16}$$

result where

$$\begin{aligned}A^{(0)}_s&=\int\limits_0^s Q_1'(X(\theta_1)_u)\sigma(X(\theta_1)_u)dW_u\\&\quad+\int\limits_0^s\Big[Q_1'b(\theta_1)+\frac12Q_1''\sigma^2+Q_2\Big](X(\theta_1)_u)\,du\end{aligned}$$

and where $X(\theta_1)$ is the solution of (2.21) starting with the distribution $\mu(\theta_0)=\mathcal{L}(X_{m+t^*})$.

Since, by assumption, the functions $Q_1'\sigma$, $Q_1'b(\theta_0,\cdot)$, $Q_1''\sigma^2$, and Q_2 are $\mu(\theta_0)$-integrable, Lemma 2.1 and the ergodic theorem imply

$$\begin{aligned}\frac{Z_{t^*}^{(0)}}{(t_m+1)^{2\alpha}g_m(t_m)} &= \frac{\sqrt{m}}{(m+t_m+1)^{1/4}}\frac{Z_{t^*}^{(0)}}{(m+t_m+1)^{3/4-\gamma}(t_m+1)^{2\alpha+\gamma}} \\ &= o(1) \quad P\text{-a.s.} \quad \text{as} \quad m\to\infty.\end{aligned} \tag{4.17}$$

Recall for the last computation that $t_m = m^2 + t^*(m)$.
In order to apply Lemma 2.1 on the stochastic integral contained in the process $A^{(0)}$, we have to choose a polynomial Q_1 such that $\mu(\theta_1)\{Q_1' \neq 0\} > 0$. Then $E_{\mu(\theta_1)}\left|Q_1'\sigma\right|^2 > 0$ and, applying Lemma 2.1 and the ergodic theorem, one obtains

$$\begin{aligned}\frac{A_{m^2}^{(0)}}{(t_m+1)^{2\alpha}g_m(t_m)} &= \frac{\sqrt{m}}{(m+t_m+1)^{1/4}}\frac{A_{m^2}^{(0)}}{(m+t_m+1)^{3/4-\gamma}(t_m+1)^{2\alpha+\gamma}} \\ &= o(1) \quad P\text{-a.s.} \quad \text{as} \quad m\to\infty.\end{aligned} \tag{4.18}$$

Finally, equations (4.14) - (4.18) yield the statement of the lemma.

□

Lemma 4.7. *Let Assumption $\mathcal{A}_0(\Theta)$ and $I>0$ be satisfied. Suppose that $q\in\mathcal{D}(\Theta)$ and $(q-a(\theta_0))$, $(|\partial_\theta b(\theta_0,\cdot)/\sigma|^2 - I(\theta_0)) \in \mathcal{M}(b(\theta_0,\cdot),\sigma)$ for $\theta_0\in\Theta$.*
In addition, suppose that $\sigma'\in\mathcal{P}$ as well as $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial_\theta^2\partial_x b(\theta,\cdot)$, $\partial_\theta^3\partial_x b(\theta,\cdot)\in\mathcal{P}$ uniformly in θ.
Then, under the alternative, we have

$$\begin{aligned}\frac{t_m}{g_m(t_m)}\Bigg|&(\hat\theta_{0,m}^0-\theta_0)\left(1+\frac{\dot\psi_{0,m}(\theta_0)}{mI(\theta_0)}-\frac{\dot I(\theta_0)\psi_{0,m}(\theta_0)}{mI^2(\theta_0)}\right) \\ &+\partial_\theta^2\Big(\frac{\psi_{0,m}}{I}\Big)(\tilde\theta_{0,m})\frac{\big(\hat\theta_{0,m}^0-\theta_0\big)^2}{m}\Bigg| \\ &= o_P(1)\end{aligned} \tag{4.19}$$

as $m\to\infty$.

Proof. For any $m>0$ the expression on the left hand side of (4.19) has under the alternative the same distribution as under the hypothesis. Since

$$\frac{t_m}{g_m(t_m)} \le \sup_{t>0}\frac{t}{g_m(t)} \quad \forall\, m>0,$$

the application of Lemma 3.6 completes the proof.

□

4.2 Probability for non-detection

Under the alternative and for any $m > 0$ the probability for non-detection is given by

$$P\big(\{\tau_m < t^*\} \cup \{\tau_m = \infty\}\big). \tag{4.20}$$

We want this probability to become arbitrarily small if m is large enough. Since we proved in Theorem 4.1 that the test has asymptotic power one, it remains to show that the procedure asymptotically does not stop before the structural break occurs.

The following two lemmata are preliminary results.

Lemma 4.8. *Let some* $\alpha \in (1/2 - \gamma, 1 - \gamma)$ *exist such that*

$$t^*(m) = o(m^\beta) \quad \textit{as} \quad m \to \infty \quad \textit{for} \quad \beta = \frac{1/2 - \gamma}{\alpha}. \tag{4.21}$$

Remember that γ *is the parameter involved in the weighting function* g_m*. Let Assumption* $\mathcal{A}_0(\Theta)$ *be satisfied. Suppose that* $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot) \in \mathcal{P}$ *uniformly in* θ*.*
Then, under H_1*, we have*

$$\sup_{0<t<t^*(m)} \left| \frac{\psi_{m,t}(\theta_0)}{g_m(t)} \right| = o_P(1) \quad \textit{as} \quad m \to \infty.$$

Proof. As in previous proofs, it is possible to simplify the problem by means of

$$\sup_{0<t<t^*} \left| \frac{\psi_{m,t}(\theta_0)}{g_m(t)} \right| \stackrel{D}{=} \sup_{0<t<t^*} \left| \frac{\psi_{0,t}(\theta_0)}{g_m(t)} \right|.$$

Take some $\alpha \in (1/2 - \gamma, 1 - \gamma)$ such that (4.21) holds. Then we have

$$\sup_{0<t<t^*} \left| \frac{\psi_{0,t}(\theta_0)}{g_m(t)} \right| \le \sqrt{m} \sup_{0<t<t^*} \frac{|\psi_{0,t}(\theta_0)|}{(t+1)^{\gamma+\alpha}} \sup_{0<t<t^*} \frac{(t+1)^\alpha}{(m+t+1)^{1-\gamma}}. \tag{4.22}$$

In view of equation (2.44), $\psi_{0,t}(\theta_0)$ is given by a stochastic integral. Apply Lemma 2.1 in order to see that the first supremum on the right hand side of (4.22) is P-a.s. bounded as $m \to \infty$.
Moreover, note that the function

$$x \longmapsto \frac{x^\alpha}{(m+x)^{1-\gamma}}, \quad x \ge 1, \tag{4.23}$$

is strictly increasing up to the point

$$x_0 = \frac{\alpha m}{(1 - \gamma - \alpha)}$$

Since $t^*(m) \leq x_0$ for large m, it follows that

$$\sup_{0<t<t^*} \frac{(t+1)^\alpha}{(m+t+1)^{1-\gamma}} = \frac{(t^*+1)^\alpha}{(m+t^*+1)^{1-\gamma}}.$$

In conclusion, we obtain

$$\sup_{0<t<t^*} \left| \frac{\psi_{0,t}(\theta_0)}{g_m(t)} \right| \leq \mathcal{O}(1) \left(\frac{(t^*+1)}{m^{(1/2-\gamma)/\alpha}} \right)^\alpha = o(1) \quad P\text{-a.s.} \quad \text{as} \quad m \to \infty.$$

□

Lemma 4.9. *Let some $\beta < 1$ exist such that $t^*(m) = \mathcal{O}(m^\beta)$ as $m \to \infty$. Let Assumption $\mathcal{A}_0(\Theta)$ be satisfied. Suppose that $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ.*
Then we have

$$\sup_{0<t<t^*} \left| \frac{t\psi_{0,m}(\theta_0)}{m g_m(t)} \right| = o(1) \quad P\text{-a.s.} \quad \text{as} \quad m \to \infty.$$

Proof. Lemma 2.1 implies

$$\frac{\psi_{0,m}(\theta_0)}{m^{1/2+\delta}} = o(1) \quad P\text{-a.s.} \quad \text{as} \quad m \to \infty$$

for any $\delta > 0$. If $\delta = (1-\beta)(1-\gamma)$, then

$$\sup_{0<t<t^*} \frac{m^\delta (t+1)}{(m+t+1)^{1-\gamma}(t+1)^\gamma} = \frac{m^\delta (t^*+1)^{1-\gamma}}{(m+t^*+1)^{1-\gamma}} = \mathcal{O}(1)$$

P-a.s. as $m \to \infty$.
Hence, we obtain

$$\sup_{0<t<t^*} \left| \frac{t\psi_{0,m}(\theta_0)}{m g_m(t)} \right| \leq \left| \frac{\psi_{0,m}(\theta_0)}{m^{1/2+\delta}} \right| \sup_{0<t<t^*} \frac{m^\delta (t+1)}{(m+t+1)^{1-\gamma}(t+1)^\gamma} = o(1)$$

P-a.s. as $m \to \infty$.

□

Proposition 4.10. *Let some $\alpha \in (1/2-\gamma, 1-\gamma)$ exist such that*

$$t^*(m) = o(m^\beta) \quad \textit{as} \quad m \to \infty \quad \textit{for} \quad \beta = \frac{1/2-\gamma}{\alpha}. \tag{4.24}$$

Let Assumption $\mathcal{A}_0(\Theta)$ and $I > 0$ be satisfied. Suppose that $q \in \mathcal{D}(\Theta)$ and $(q - a(\theta_0))$, $(|\partial_\theta b(\theta_0, \cdot)/\sigma|^2 - I(\theta_0)) \in \mathcal{M}(b(\theta_0, \cdot), \sigma)$ for $\theta_0 \in \Theta$.
In addition, suppose that $\sigma' \in \mathcal{P}$ as well as $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot)$, $\partial_\theta^2 \partial_x b(\theta, \cdot)$, $\partial_\theta^3 \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ.
Then, under the alternative, we have

$$\lim_{m\to\infty} P\{\tau_m < t^*(m)\} = 0.$$

Proof. Denote the weighted remainders in (3.1) by $R(m,t)$, i.e.,

$$R(m,t) = \frac{t}{g_m(t)}(R_1 - R_2 + R_3 - R_4).$$

By

$$\{\tau_m < t^*\} = \left\{ \sup_{0<t<t^*} |S_t^m| > c \right\}$$

and Lemmata 4.8 and 4.9 it remains to prove

$$\lim_{m\to\infty} P\left(\sup_{0<t<t^*} |R(m,t)| > c \right) = 0.$$

Applying the asymptotic results obtained under H_0 (see Lemmata 3.3, 3.4, 3.5, and 3.6), we obtain

$$\begin{aligned} P_{H_1}\left(\sup_{0<t<t^*} |R(m,t)| > c \right) &= P_{H_0}\left(\sup_{0<t<t^*} |R(m,t)| > c \right) \\ &\le P_{H_0}\left(\sup_{0<t<\infty} |R(m,t)| > c \right) \\ &= o(1) \quad \text{as} \quad m\to\infty \end{aligned}$$

where P_{H_0}, P_{H_1} represent the probability measures under H_0 and under H_1, respectively.

□

4.3 Asymptotic normality of the stopping time

Now we start to prove the key result of this chapter: the existence of two families $(a_m\colon m \ge 0)$, $(b_m\colon m\ge 0)$ such that

$$\frac{\tau_m - a_m}{b_m} \xrightarrow{D} N(0,1) \quad \text{as} \quad m\to\infty.$$

Toward this end, we will follow the procedure developed by Aue and Horváth (2004). Define in a similar way

$$a_m = \left(\frac{cm^{1/2-\gamma}}{|\theta_1 - \theta_0|} \right)^{1/(1-\gamma)}, \quad b_m = \frac{\sqrt{a_m}}{(1-\gamma)|\theta_1-\theta_0|\sqrt{I(\theta_1)}}, \quad m \ge 0,$$

where c is the critical value of the test.

In the course of this section we will assume the following uniform weak invariance principle:

Assumption 4.11. For any $\theta_1 \neq \theta_0$ there exists a family of Wiener processes $(W^{(m)}: m \geq 0)$ and a number $\alpha < 1/2$ such that, under H_1,

$$\sup_{t>0} \frac{1}{(t+1)^{\alpha}} \left| \int_{m+t^*}^{m+t^*+t} \frac{\dot{b}(\theta_1, X_s)}{\sigma(X_s)} dW_s - \sqrt{I(\theta_1)} W_t^{(m)} \right| = \mathcal{O}_P(1) \tag{4.25}$$

as $m \to \infty$.

Remark 4.12. We already know from Theorem 2.6 that, under suitable conditions, (4.25) is true for any $\alpha > 1/4$. E.g., the assumptions of Theorem 2.6 are satisfied if $(|\partial_\theta b(\theta_1, \cdot)/\sigma|^2 - I(\theta_1)) \in \mathcal{M}(b(\theta_1, \cdot), \sigma)$, $I(\theta_1) > 0$, and if Assumption $\mathcal{A}_0(\Theta)$ is valid.

Along the lines of the proof of Theorem 1.1 of Aue and Horváth (2004), it is sufficient to show that

$$\lim_{m\to\infty} P\{\tau_m \geq N(m,x)\} = \Phi(x), \quad x \in \mathbb{R}, \tag{4.26}$$

where Φ denotes the standard normal distribution function and

$$N(m,x)^{1-\gamma} = a_m^{1-\gamma} - \frac{x}{\sqrt{I(\theta_1)}} \frac{a_m^{1/2-\gamma}}{|\theta_1 - \theta_0|}.$$

The differences to Aue and Horváth (2004) are some constants in the definition of a_m, b_m and $N(m,x)$ and the fact that we have continuous time parameters t, $m > 0$.
We often will suppress the dependence on m and x of the function $N(m,x)$ and simply will write N. Note the growth behaviour of N:

$$N = \mathcal{O}(m^\beta) \quad \text{as} \quad m \to \infty \quad \text{with} \quad \beta = \frac{1/2-\gamma}{1-\gamma}. \tag{4.27}$$

Proposition 4.10 implies

$$\begin{aligned}
\lim_{m\to\infty} P\{\tau_m \geq N\} &= \lim_{m\to\infty} P\Big\{ \sup_{0<t<t^*} |S_t^m| \leq c, \sup_{t^*\leq t<N} |S_t^m| \leq c \Big\} \\
&= \lim_{m\to\infty} \Big(P\Big\{ \sup_{t^*\leq t<N} |S_t^m| \leq c \Big\} \\
&\qquad - P\Big\{ \sup_{0<t<t^*} |S_t^m| > c, \sup_{t^*\leq t<N} |S_t^m| \leq c \Big\} \Big) \\
&= \lim_{m\to\infty} P\Big\{ \sup_{t^*\leq t<N} |S_t^m| \leq c \Big\}.
\end{aligned} \tag{4.28}$$

4.3.1 Preliminary results

In order to compute the limit on the right hand side of (4.28), the first step is the approximation of the test statistic by Wiener processes with drift:

Proposition 4.13. *Assume that either*

(i) $t^*(m) = \mathcal{O}(1)$ *as* $m \to \infty$ *or*

(ii) $\lim_{m\to\infty} t^*(m) = \infty$ *and* $t^* = o(m^\beta)$ *as* $m \to \infty$ *for* $\beta = \frac{(1/2-\gamma)^2}{(1-\gamma)^2}$.

Moreover, let Assumptions 4.11 and $\mathcal{A}_0(\Theta)$ *be satisfied and let* $I > 0$ *as well as*

$$E_{\mu(\theta_1)}\left|\frac{\ddot{b}(\theta_1,\cdot)}{\sigma}\right|^2 > 0 \quad \textit{for} \quad \theta_1 \neq \theta_0.$$

Suppose that $q \in \mathcal{D}(\Theta)$, $(|\partial_\theta b(\theta_0,\cdot)/\sigma|^2 - I(\theta_0))$, $(q - a(\theta_0)) \in \mathcal{M}(b(\theta_0,\cdot),\sigma)$ *for* $\theta_0 \in \Theta$ *as well as* $(|\partial_\theta b(\theta_1,\cdot)/\sigma|^2 - I(\theta_1))$, $(q - a(\theta_1)) \in \mathcal{M}(b(\theta_1,\cdot),\sigma)$ *for* $\theta_1 \neq \theta_0$.
In addition, suppose that $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial_\theta^2\partial_x b(\theta,\cdot)$, $\partial_\theta^3\partial_x b(\theta,\cdot) \in \mathcal{P}$ *uniformly in* θ.
Then, under the alternative, we have

$$\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^*\le t<N} \frac{\left|S_t^m g_m(t) - \frac{W_t^{(m)}}{\sqrt{I(\theta_1)}} - (\theta_1-\theta_0)t\right|}{g_m(t)} = o_P(1) \quad \textit{as} \quad m \to \infty.$$

Before starting with the proof, remember the Taylor expansion in (4.2):

$$\begin{aligned} S_t^m = {} & \frac{(\theta_1-\theta_0)t}{g_m(t)} + \frac{\psi_{m,t}(\theta_1)}{g_m(t)I(\theta_1)} - \frac{t}{g_m(t)}\frac{\psi_{0,m}(\theta_0)}{mI(\theta_0)} \\ & + \frac{t}{g_m(t)}\left(R_1 - R_2 + R_3 - R_4\right). \end{aligned}$$

Proposition 4.13 is proven by means of Lemmata 4.14 - 4.19.

Lemma 4.14. *Assume that either*

(i) $t^*(m) = \mathcal{O}(1)$ *as* $m \to \infty$ *or*

(ii) $\lim_{m\to\infty} t^*(m) = \infty$ *and* $t^* = o(m^\beta)$ *as* $m \to \infty$ *for* $\beta = \frac{(1/2-\gamma)^2}{(1-\gamma)^2}$.

Moreover, let Assumptions 4.11 and $\mathcal{A}_0(\Theta)$ *be satisfied. Suppose that* $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot) \in \mathcal{P}$ *uniformly in* θ.
Then, under H_1, *we have*

$$\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^*\le t<N} \frac{\left|\psi_{m,t}(\theta_1) - \sqrt{I(\theta_1)}W_t^{(m)}\right|}{g_m(t)} = o_P(1) \quad \textit{as} \quad m \to \infty.$$

Proof. Due to the formula

$$\begin{aligned}\psi_{m,t}(\theta_1) = &\int_{m+t^*}^{m+t} \frac{\dot{b}(\theta_1, X_s)}{\sigma(X_s)} dW_s + \int_{m}^{m+t^*} \frac{\dot{b}(\theta_1, X_s)}{\sigma(X_s)} dW_s \\ &+ \int_{m}^{m+t^*} \left[\frac{[b(\theta_0) - b(\theta_1)]\dot{b}(\theta_1)}{\sigma^2}\right](X_s)\, ds,\end{aligned} \tag{4.29}$$

only the first stochastic integral will be approximated by Wiener processes. Choose some $\alpha' \in (\gamma, 1/2)$ which fulfils Assumption 4.11. Then we have for large m

$$\begin{aligned}\frac{m^{1-\gamma}}{N^{1/2-\gamma}} \sup_{t^* \le t < N} \frac{(t+1)^{\alpha'}}{(m+t+1)^{1-\gamma}(t+1)^{\gamma}} &= \frac{m^{1-\gamma}(N+1)^{\alpha'-\gamma}}{N^{1/2-\gamma}(m+N+1)^{1-\gamma}} \\ &= o(1) \quad \text{as} \quad m \to \infty\end{aligned} \tag{4.30}$$

(confer the statement for the function defined in (4.23)).
Assumption 4.11 and (4.30) imply

$$\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^* \le t < N} \frac{\left| \int_{m+t^*}^{m+t} \frac{\dot{b}(\theta_1, X_s)}{\sigma(X_s)} dW_s - \sqrt{I(\theta_1)} W_t^{(m)} \right|}{g_m(t)} = o_P(1) \quad \text{as} \quad m \to \infty.$$

For investigating the integrals up to the change-point in (4.29), define the process

$$M_t = \int_0^t \frac{\dot{b}\big(\theta_1, X(\theta_0)_s\big)}{\sigma(X(\theta_0)_s)} dW_s + \int_0^t \left[\frac{[b(\theta_0) - b(\theta_1)]\dot{b}(\theta_1)}{\sigma^2}\right](X(\theta_0)_s)\, ds, \quad t \ge 0,$$

where we denoted by $X(\theta_0)$ the unique solution of (2.21) with $\theta = \theta_0$. By Lemmata 2.5 and 2.22.(i) M can be written as the composition of $X(\theta_0)$ and a measurable map in $C[0, \infty)$. From the equivalence of the Markov processes

$$(X(\theta_0)_s \colon 0 \le s \le t^*), \quad (X_{m+s} \colon 0 \le s \le t^*),$$

the equality in distribution

$$M_{t^*} \stackrel{D}{=} \int_m^{m+t^*} \frac{\dot{b}(\theta_1, X_s)}{\sigma(X_s)} dW_s + \int_m^{m+t^*} \left[\frac{[b(\theta_0) - b(\theta_1)]\dot{b}(\theta_1)}{\sigma^2}\right](X_s)\, ds \tag{4.31}$$

follows. The ergodic theorem and Lemma 2.1 yield

$$\lim_{m \to \infty} \frac{M_{t^*(m)}}{t^*(m)} = E_{\mu(\theta_0)} \left(\frac{\dot{b}(\theta_1, \cdot)[b(\theta_0, \cdot) - b(\theta_1, \cdot)]}{\sigma^2} \right) \quad P\text{-a.s.}$$

Hence, according to the growth of t^* and N (see (4.27)), we obtain

$$\begin{aligned}\Big(\frac{m}{N}\Big)^{1/2-\gamma} & \sup_{t^*\le t<N} \frac{\sqrt{m}\,|M_{t^*}|}{(m+t+1)^{1-\gamma}(t+1)^{\gamma}} \\ &= \frac{(t^*+1)^{1-\gamma}}{N^{1/2-\gamma}} \frac{m^{1-\gamma}|M_{t^*}|}{(m+t^*+1)^{1-\gamma}(t^*+1)} \\ &= o(1) \quad P\text{-a.s.} \quad \text{as} \quad m\to\infty.\end{aligned} \tag{4.32}$$

By (4.31) and (4.32) the proof is complete.

□

Lemma 4.15. *Assume that either*

(i) $t^*(m) = \mathcal{O}(1)$ *as* $m\to\infty$ *or*

(ii) $\lim_{m\to\infty} t^* = \infty$ *and* $t^* = o(m^{\beta})$ *as* $m\to\infty$ *for* $\beta = \frac{(1/2-\gamma)^2}{(1-\gamma)(1-\gamma-\alpha)}$ *with some* $\alpha < 1/2$.

Moreover, let Assumption $\mathcal{A}_0(\Theta)$ *and*

$$E_{\mu(\theta_1)}\left|\frac{\ddot{b}(\theta_1,\cdot)}{\sigma}\right|^2 > 0 \quad \text{for} \quad \theta_1 \ne \theta_0$$

be satisfied. Suppose that $q\in\mathcal{D}(\Theta)$ *as well as* $(|\partial_\theta b(\theta_1,\cdot)/\sigma|^2 - I(\theta_1))$, $(q-a(\theta_1)) \in \mathcal{M}(b(\theta_1,\cdot),\sigma)$ *for* $\theta_1\ne\theta_0$.
In addition, suppose that $b(\theta,\cdot)$, $\partial_\theta\partial_x b(\theta,\cdot)$, $\partial_\theta^2\partial_x b(\theta,\cdot)\in\mathcal{P}$ *uniformly in* θ.
Then, under the alternative, we have

$$\Big(\frac{m}{N}\Big)^{1/2-\gamma} \sup_{t^*\le t<N} \frac{1}{g_m(t)}\left|\big(tI(\theta_1)+\dot{\psi}_{m,t}(\theta_1)\big)\big(\hat{\theta}^0_{m,t}-\theta_1\big)\right| = o_P(1)$$

as $m\to\infty$.

Proof. First, by Proposition 2.17 and Lemma 2.24 we know that the distributions of the processes $(\hat{\theta}^0_{m,t}\colon t\ge 0)$, $(\partial_\theta\psi_{m,t}\colon t\ge 0)$ are independent of the starting point m. Therefore, it is sufficient to discuss the asymptotic behaviour of

$$\Big(\frac{m}{N}\Big)^{1/2-\gamma} \sup_{t^*\le t<N} \frac{1}{g_m(t)}\left|\big(tI(\theta_1)+\dot{\psi}_{0,t}(\theta_1)\big)\big(\hat{\theta}^0_{0,t}-\theta_1\big)\right|(Y)$$

as $m\to\infty$ where Y is the unique solution to

$$Y_t = \begin{cases} Y_0 + \int_0^t b(\theta_0,Y_s)ds + \int_0^t \sigma(Y_s)dW_s, & 0<t\le t^*(m), \\ Y_{t^*} + \int_{t^*}^t b(\theta_1,Y_s)ds + \int_{t^*}^t \sigma(Y_s)dW_s, & t>t^*(m), \end{cases} \tag{4.33}$$

$Y_0 \sim \mu(\theta_0)$.

By Remark 2.21 one can see that

$$tI(\theta_1) + \dot{\psi}_{0,t}(\theta_1) = M_t^* + Z_{t^*} \quad \forall\, t \geq t^*$$

where according to the proof of Lemma 4.4,

$$\begin{aligned} Z_{t^*} = & \int_0^{t^*} \frac{\ddot{b}(\theta_1, Y_s)}{\sigma(Y_s)} dW_s \\ & + \int_0^{t^*} \left(I(\theta_1) + \left[\frac{\ddot{b}(\theta_1)[b(\theta_0) - b(\theta_1)] - |\dot{b}(\theta_1)|^2}{\sigma^2} \right](Y_s) \right) ds \end{aligned}$$

and

$$\big(M^*_{t^*+s} \colon s \geq 0\big) \overset{D}{=} \big(M_s \colon s \geq 0\big) \tag{4.34}$$

with

$$M_s = \int_0^s \frac{\ddot{b}(\theta_1, X(\theta_1)_u)}{\sigma(X(\theta_1)_u)} dW_u + \int_0^s \left(I(\theta_1) - \left| \frac{\dot{b}(\theta_1, X(\theta_1)_u)}{\sigma(X(\theta_1)_u)} \right|^2 \right) du.$$

$X(\theta_1)$ represents the solution of (2.21), but starting with the distribution $\mu(\theta_0) = \mathcal{L}(Y_{t^*})$.
Choose real numbers $\alpha' \in (0, 1/2)$ and $\delta \geq 0$ with $1/2 - \alpha' - \gamma < \delta < 1/2 - \gamma$. Applying Lemma 2.1 and Remark 2.2 on the process M and considerating (4.34), it follows that

$$\sup_{t^* \leq t < N} \frac{|M_t^*|}{(t+1)^{\alpha' + \gamma + \delta}} = \mathcal{O}_P(1) \quad \text{as} \quad m \to \infty \tag{4.35}$$

where we have used the fact that P-a.s. boundedness implies boundedness in probability. In view of the increasing property of the function defined in (4.23), the equality

$$\sup_{t^* \leq t < N} \frac{(t+1)^\kappa}{(m+t+1)^{1-\gamma}} = \frac{(N+1)^\kappa}{(m+N+1)^{1-\gamma}} \tag{4.36}$$

holds for any $\kappa \in (0, 1-\gamma)$ if m is large enough. Thereby, the choice of δ implies

$$\lim_{m \to \infty} \left(\frac{m}{N} \right)^{1/2 - \gamma} \sqrt{m} \sup_{t^* \leq t < N} \frac{(t+1)^\delta}{(m+t+1)^{1-\gamma}} = 0. \tag{4.37}$$

Applying Proposition 2.17, we obtain

$$\sup_{t > 0} (t+1)^{\alpha'} \left| \hat{\theta}^0_{0,t} - \theta_1 \right| < \infty \quad P\text{-a.s.} \tag{4.38}$$

Finally, by (4.35), (4.37), and (4.38) the asymptotics

$$\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^* \le t < N} \sqrt{m} \frac{|M_t^*| \left|\hat{\theta}_{0,t}^0 - \theta_1\right|}{(m+t+1)^{1-\gamma}(t+1)^\gamma} = o_P(1) \quad \text{as} \quad m \to \infty$$

results.
In the last step we discuss the asymptotic behaviour of the family $\big(Z_{t^*(m)} : m \ge 0\big)$. Let $\alpha \in (0, 1/2)$ be chosen according to the growth condition in (ii). Proposition 2.17 implies

$$\sup_{t^* \le t < N} \frac{|Z_{t^*}|}{g_m(t)} \left|\hat{\theta}_{0,t}^0 - \theta_1\right| \le \mathcal{O}(1)\, |Z_{t^*}| \sup_{t^* \le t < N} \frac{1}{g_m(t)(t+1)^\alpha} \tag{4.39}$$

where the Landau symbol is understood for $m \to \infty$ and P-a.s.
Note that

$$\sup_{t^* \le t < N} \frac{1}{g_m(t)(t+1)^\alpha} = \frac{\sqrt{m}}{(m+t^*+1)^{1-\gamma}(t^*+1)^{\alpha+\gamma}}. \tag{4.40}$$

By the growth of N and t^*, the ergodic theorem, and Lemma 2.1 we have

$$\lim_{m \to \infty} \frac{Z_{t^*}}{N^{1/2-\gamma}(t^*+1)^{\alpha+\gamma}} = 0 \quad P\text{-a.s.} \tag{4.41}$$

Putting together (4.39) - (4.41), the desired convergence follows:

$$\lim_{m \to \infty} \left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^* \le t < N} \frac{|Z_{t^*}|}{g_m(t)} \left|\hat{\theta}_{0,t}^0 - \theta_1\right| = 0 \quad P\text{-a.s.}$$

□

Lemma 4.16. *Assume that either*

(i) $t^*(m) = \mathcal{O}(1)$ *as* $m \to \infty$ *or*

(ii) $\lim_{m\to\infty} t^* = \infty$ *and* $t^* = o(m^\beta)$ *as* $m \to \infty$ *for* $\beta = \frac{(1/2-\gamma)^2}{(1-\gamma)(1-\gamma-\alpha)}$ *with some* $\alpha < 1/2$.

Moreover, let Assumption $\mathcal{A}_0(\Theta)$ *and*

$$I(\theta_1) = E_{\mu(\theta_1)} \left| \frac{\dot{b}(\theta_1, \cdot)}{\sigma} \right|^2 > 0 \quad \text{for} \quad \theta_1 \ne \theta_0$$

be satisfied. Suppose that $q \in \mathcal{D}(\Theta)$ *and* $(q - a(\theta_1)) \in \mathcal{M}(b(\theta_1, \cdot), \sigma)$ *for* $\theta_1 \ne \theta_0$. *In addition, suppose that* $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot) \in \mathcal{P}$ *uniformly in* θ. *Then, under the alternative, we have*

$$\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^* \le t < N} \frac{|\psi_{m,t}(\theta_1)|}{g_m(t)} \left|\hat{\theta}_{m,t}^0 - \theta_1\right| = o_P(1) \quad \text{as} \quad m \to \infty.$$

Proof. Along the lines of many previous proofs, e.g. the proof of Lemma 4.15, we simplify the problem by using

$$\sup_{t^*\le t<N} \frac{|\psi_{m,t}(\theta_1)|}{g_m(t)} \left|\hat{\theta}^0_{m,t} - \theta_1\right| (X) \stackrel{D}{=} \sup_{t^*\le t<N} \frac{|\psi_{0,t}(\theta_1)|}{g_m(t)} \left|\hat{\theta}^0_{0,t} - \theta_1\right| (Y)$$

where Y is given by (4.33). Remark 2.21 yields

$$\psi_{0,t}(\theta_1)(Y) = M^*_t + Z_{t^*}, \quad t \ge t^*,$$

where

$$M^*_t = \int_{t^*}^{t} \frac{\dot{b}(\theta_1, Y_s)}{\sigma(Y_s)} dW_s, \quad t \ge t^*, \tag{4.42}$$

$$Z_{t^*} = \int_0^{t^*} \frac{\dot{b}(\theta_1, Y_s)}{\sigma(Y_s)} dW_s + \int_0^{t^*} \left[\frac{\dot{b}(\theta_1)[b(\theta_0) - b(\theta_1)]}{\sigma^2}\right](Y_s)\, ds.$$

We can replace the process M^* of the proof of Lemma 4.15 by the process defined in (4.42) because the new M^* has the asymptotic behaviour (4.35), too. In the same way, the new family $(Z_{t^*(m)} : m \ge 0)$ fulfils (4.41). Therefore, by the argumentation in the proof of Lemma 4.15 the statement of Lemma 4.16 results.

□

Lemma 4.17. *Assume that either*

(i) $t^*(m) = \mathcal{O}(1)$ *as* $m \to \infty$ *or*

(ii) $\lim_{m\to\infty} t^*(m) = \infty$ *and* $t^*(m)^{1-2\alpha-\gamma} = o(m^\beta)$ *as* $m \to \infty$ *for some* $\alpha < 1/2$ *and* $\beta = (1/2-\gamma)^2/(1-\gamma)$.

Let Assumption $\mathcal{A}_0(\Theta)$ *and* $I > 0$ *be satisfied. Suppose that* $q \in \mathcal{D}(\Theta)$ *and* $(q - a(\theta_1)) \in \mathcal{M}(b(\theta_1, \cdot), \sigma)$ *for* $\theta_1 \ne \theta_0$.
In addition, suppose that $\sigma' \in \mathcal{P}$ *as well as* $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot)$, $\partial^2_\theta \partial_x b(\theta, \cdot)$, $\partial^3_\theta \partial_x b(\theta, \cdot) \in \mathcal{P}$ *uniformly in* θ.
Then, under H_1, *we have*

$$\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^*\le t<N} \partial^2_\theta \left(\frac{\psi_{m,t}}{I}\right)(\tilde{\theta}_{m,t}) \frac{(\hat{\theta}^0_{m,t} - \theta_1)^2}{g_m(t)} = o_P(1) \quad as \quad m \to \infty.$$

Proof. Along the lines of the proof of (4.14), one obtains

$$\begin{aligned}
&\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^*\le t<N} \left|\partial^2_\theta \left(\frac{\psi_{m,t}}{I}\right)(\tilde{\theta}_{m,t})\right| \frac{\left|\hat{\theta}^0_{m,t} - \theta_1\right|^2}{g_m(t)} \\
&\quad \le K \left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^*\le t<N} \frac{\left|\hat{\theta}^0_{m,t} - \theta_1\right|^2}{g_m(t)} \left(Z^{(m)}_{t^*} + A^{(m+t^*)}_{t-t^*}\right)
\end{aligned} \tag{4.43}$$

where $K > 0$ is some constant and where

$$Z_{t^*}^{(m)} \overset{D}{=} Z_{t^*}^{(0)} \quad \forall\, m > 0,$$
$$(A_s^{(m+t^*)} : s \geq 0) \overset{D}{=} (A_s^{(0)} : s \geq 0) \quad \forall\, m > 0 \tag{4.44}$$

with

$$Z_{t^*}^{(0)} = 2Q(X_0) + \int_0^{t^*} Q'(X_s)\sigma(X_s)dW_s + \int_0^{t^*} \Big[Q'b(\theta_0) + \frac{1}{2}Q''\sigma^2 + \tilde{Q}\Big](X_s)\, ds,$$

$$A_s^{(0)} = \int_0^s Q'(X(\theta_1)_u)\sigma(X(\theta_1)_u)dW_u + \int_0^s \Big[Q'b(\theta_1) + \frac{1}{2}Q''\sigma^2 + \tilde{Q}\Big](X(\theta_1)_u)\, du, \quad s \geq 0.$$

Here, $X(\theta_1)$ is chosen to start with the distribution $\mu(\theta_0) = \mathcal{L}(X_{m+t^*})$. Since the functions $Q'\sigma$, $Q'b(\theta_0)$, $Q''\sigma^2$ and $\tilde{Q}$ are $\mu(\theta_0)$-integrable, the ergodic theorem, Lemma 2.1 and the growth behaviour of t^* imply

$$\lim_{m\to\infty} \frac{Z_{t^*}^{(0)}}{N^{1/2-\gamma}(t^*+1)^{2\alpha+\gamma}} = 0 \quad P\text{-a.s.} \tag{4.45}$$

where $\alpha \in (0, 1/2)$ is some number satisfying (ii). By Proposition 2.17 we have

$$\sup_{t^* \leq t < N} (t+1)^{2\alpha} \big|\hat{\theta}_{m,t}^0 - \theta_1\big|^2 = \mathcal{O}_P(1) \quad \text{as} \quad m \to \infty. \tag{4.46}$$

From (4.45) and (4.46) the asymptotic behaviour

$$\Big(\frac{m}{N}\Big)^{1/2-\gamma} \sup_{t^* \leq t < N} \big|\hat{\theta}_{m,t}^0 - \theta_1\big|^2 \frac{Z_{t^*}^{(m)}}{g_m(t)} = o_P(1) \quad \text{as} \quad m \to \infty$$

results.
For discussing the expression in (4.43) containing the process $A^{(m+t^*)}$, take new real numbers $\alpha \in (1/4, 1/2)$ and $\delta = \max\{0, 1-\gamma-2\alpha\}$. Moreover, choose the polynomial Q such that $\mu(\theta_1)\{Q' \neq 0\} > 0$. Remember that P-a.s. boundedness implies boundedness in probability. Then we have

$$\sup_{t^* \leq t < N} \frac{\Big|A_{t-t^*}^{(m+t^*)}\Big|}{(t+1)^{2\alpha+\gamma+\delta}} = \mathcal{O}_P(1) \quad \text{as} \quad m \to \infty \tag{4.47}$$

due to (4.44), the ergodic theorem, and Lemma 2.1.
Moreover, from (4.36) it follows for $\delta = \max\{0, 1-\gamma-2\alpha\}$ that

$$\sup_{t^* \le t < N} \frac{m^{1-\gamma}(t+1)^{\delta}}{(m+t+1)^{1-\gamma}} \le (N+1)^{\delta}.$$

Finally, applying (4.46) and (4.47) with the chosen α, we obtain

$$\begin{aligned} \left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^* \le t < N} \left|\hat{\theta}^0_{m,t} - \theta_1\right|^2 \frac{\left|A^{(m+t^*)}_{t-t^*}\right|}{g_m(t)} &= \sup_{t^* \le t < N} \frac{m^{1-\gamma}\left|A^{(m+t^*)}_{t-t^*}\right| \left|\hat{\theta}^0_{m,t} - \theta_1\right|^2}{N^{1/2-\gamma}(m+t+1)^{1-\gamma}(t+1)^{\gamma}} \\ &\le \mathcal{O}_P(1) \frac{(N+1)^{\delta}}{N^{1/2-\gamma}} \\ &= o_P(1) \quad \text{as} \quad m \to \infty \end{aligned}$$

because $\delta < 1/2 - \gamma$.

□

Lemma 4.18. *Let Assumption $\mathcal{A}_0(\Theta)$ be satisfied. Moreover, let $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ.*
Then, under H_1, we have

$$\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{t^* \le t < N} \frac{t}{g_m(t)} \cdot \frac{\psi_{0,m}(\theta_0)}{m} = o(1) \quad P\text{-}a.s. \quad as \quad m \to \infty.$$

Proof. For any real number $\delta > 0$ one obtains by Lemma 2.1 that

$$\frac{\psi_{0,m}(\theta_0)}{m^{1/2+\delta}} = o(1) \quad P\text{-a.s.} \quad \text{as} \quad m \to \infty. \tag{4.48}$$

Choose $\delta \in (0, 1/(4-4\gamma))$. Moreover, observe that

$$\sup_{t^* \le t < N} \frac{t}{(m+t+1)^{1-\gamma}(t+1)^{\gamma}} \le \frac{(N+1)^{1-\gamma}}{(m+N+1)^{1-\gamma}}. \tag{4.49}$$

Then

$$\lim_{m \to \infty} \frac{m^{1/2+\delta}(N+1)^{1-\gamma}}{N^{1/2-\gamma} m^{\gamma}(m+N+1)^{1-\gamma}} = 0 \tag{4.50}$$

holds by the asymptotic behaviour of N (see (4.27)). By (4.48) - (4.50) the proof is complete.

□

Lemma 4.19. *Let $\mathcal{A}_0(\Theta)$ and $I > 0$ be satisfied. Suppose that $q \in \mathcal{D}(\Theta)$ and $(|\partial_\theta b(\theta_0, \cdot)/\sigma|^2 - I(\theta_0))$, $(q - a(\theta_0)) \in \mathcal{M}(b(\theta_0, \cdot), \sigma)$ for $\theta_0 \in \Theta$.*
In addition, suppose that $\sigma' \in \mathcal{P}$ as well as $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot)$, $\partial_\theta^2 \partial_x b(\theta, \cdot)$, $\partial_\theta^3 \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ.
Then, under the alternative, we have

$$\begin{aligned} \Big(\frac{m}{N}\Big)^{1/2-\gamma} \sup_{t^* \le t < N} \frac{t}{g_m(t)} \Bigg[\big(\hat{\theta}^0_{0,m} - \theta_0\big) \Bigg(1 + \frac{\dot{\psi}_{0,m}(\theta_0)}{mI(\theta_0)} - \frac{\dot{I}(\theta_0)\psi_{0,m}(\theta_0)}{mI^2(\theta_0)}\Bigg) \\ + \partial_\theta^2 \Big(\frac{\psi_{0,m}}{I}\Big)(\tilde{\theta}_{0,m}) \frac{\big(\hat{\theta}^0_{0,m} - \theta_0\big)^2}{m} \Bigg] \\ = o_P(1) \quad as \quad m \to \infty. \end{aligned} \tag{4.51}$$

Proof. For any $m \ge 0$ the expression on the left hand side of (4.51) has the same distribution under the alternative as under the null hypothesis. One can check that

$$\Big(\frac{m}{N}\Big)^{1/2-\gamma} \sup_{t^* \le t < N} \frac{t}{g_m(t)} \le \sup_{t > 0} \frac{t}{g_m(t)} \quad \text{for sufficiently large} \quad m > 0$$

(see (4.49)). Hence, Lemma 3.6 yields the statement of the lemma. □

Thereby, the proof of Proposition 4.13 is complete.

The way is now paved for

Proposition 4.20. *Assume that either*

(i) $t^*(m) = \mathcal{O}(1)$ *as* $m \to \infty$ *or*

(ii) $\lim_{m\to\infty} t^*(m) = \infty$ *and* $t^* = o(m^\beta)$ *as* $m \to \infty$ *for* $\beta = \frac{(1/2-\gamma)^2}{(1-\gamma)^2}$.

Moreover, let Assumptions 4.11 and $\mathcal{A}_0(\Theta)$ be satisfied and let $I > 0$ as well as

$$E_{\mu(\theta_1)} \left| \frac{\ddot{b}(\theta_1, \cdot)}{\sigma} \right|^2 > 0 \quad for \quad \theta_1 \ne \theta_0.$$

Suppose that $q \in \mathcal{D}(\Theta)$, $(|\partial_\theta b(\theta_0, \cdot)/\sigma|^2 - I(\theta_0))$, $(q - a(\theta_0)) \in \mathcal{M}(b(\theta_0, \cdot), \sigma)$ for $\theta_0 \in \Theta$ as well as $(|\partial_\theta b(\theta_1, \cdot)/\sigma|^2 - I(\theta_1))$, $(q - a(\theta_1)) \in \mathcal{M}(b(\theta_1, \cdot), \sigma)$ for $\theta_1 \ne \theta_0$.
In addition, suppose that $\sigma' \in \mathcal{P}$ as well as $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot)$, $\partial_\theta^2 \partial_x b(\theta, \cdot)$, $\partial_\theta^3 \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ.
Then, under the alternative, we have

$$\lim_{m\to\infty} P\{ \sup_{t^* \le t < N} |S_t^m| \le c\} = \Phi(x) \quad \forall\, x \in \mathbb{R}$$

where Φ denotes the standard normal distribution function.

Proof. Aue and Horváth (2004) had the idea that by Proposition 4.13 it is sufficient to prove

$$\lim_{m\to\infty} P\left\{\left(\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{[t^*,N)} \frac{\left|W_t^{(m)}+\Delta t\right|}{g_m(t)} - \Delta\sqrt{N}\right) \le x_m\right\} = \Phi(x) \quad (4.52)$$

where $\Delta = \sqrt{I(\theta_1)}(\theta_1-\theta_0)$ and the family $(x_m \colon m>0)$,

$$x_m = \sqrt{I(\theta_1)}\left(\frac{m}{N}\right)^{1/2-\gamma}\left(c - \frac{(\theta_1-\theta_0)N}{m^{1/2-\gamma}N^\gamma}\right),$$

converges towards x as $m\to\infty$. Remember that N is a function of (m,x). We have suppressed the time variable under the supremum for typographic reasons and will suppress it in the course of this proof.
Without loss of generality, assume $\theta_1>\theta_0$. This assumption is allowed because the stopping rule in (1.6) is based on the absolute value of the test statistic. Hence, for proving Proposition 4.20, the considerations about S^m can be replaced by considerations about $-S^m$.

Now we follow the method of proof developed by Aue and Horváth (2004). For the reader's convenience we recall the program of approximations of Aue and Horváth (2004), Lemma 3.4, and explain the modifications which are necessary in our context.

Since the distribution of $W^{(m)}$ is independent of m, we can simply use an arbitrary Wiener process B instead of $W^{(m)}$. Moreover, note that we have

$$\sup_{[t^*,N)} \frac{|B_t+\Delta t|}{g_m(t)} = \sup_{[t^*,N]} \frac{|B_t+\Delta t|}{g_m(t)}$$

due to the continuity of the trajectories of Wiener processes.

1. One should change the weighting function:

$$\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{[t^*,N]}\left|\frac{B_t+\Delta t}{g_m(t)} - \frac{B_t+\Delta t}{m^{1/2-\gamma}(t+1)^\gamma}\right| = o_P(1) \quad (4.53)$$

as $m\to\infty$.

Proof. Along the lines of the proof of (3.18) by Aue and Horváth (2004), one obtains

$$\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{[t^*,N]}\left|\frac{B_t}{g_m(t)} - \frac{B_t}{m^{1/2-\gamma}(t+1)^\gamma}\right| = o_P(1) \quad \text{as} \quad m\to\infty.$$

Second, consider for any $m > 0$

$$\sup_{[t^*,N]} \left| \frac{t}{g_m(t)} - \frac{t}{m^{1/2-\gamma}(t+1)^\gamma} \right| = \sup_{[t^*,N]} \frac{t}{m^{1/2-\gamma}(t+1)^\gamma} \cdot \\ \cdot \left| 1 - \left(\frac{m}{m+t+1} \right)^{1-\gamma} \right| . \tag{4.54}$$

Note that by the mean value theorem there exists for any $m > 0,\, t \geq t^*$ a number $\xi_{m,t}$ with $m/(m+t+1) < \xi_{m,t} < 1$ such that

$$\begin{aligned} 1^{1-\gamma} - \left(\frac{m}{m+t+1} \right)^{1-\gamma} &= \frac{1-\gamma}{\xi_{m,t}^\gamma} \frac{t+1}{m+t+1} \\ &\leq (1-\gamma) \left(\frac{m+N+1}{m} \right)^\gamma \frac{t+1}{m+t+1} \end{aligned} \tag{4.55}$$

for all $t \in [t^*, N]$. Then we have

$$\begin{aligned} &\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{[t^*,N]} \frac{(t+1)^{1-\gamma}}{m^{1/2-\gamma}} \left(\frac{m+N+1}{m} \right)^\gamma \frac{t+1}{m+t+1} \\ &= \frac{(N+1)^{2-\gamma}}{N^{1/2-\gamma} m^\gamma (m+N+1)^{1-\gamma}} \\ &= o(1) \quad \text{as} \quad m \to \infty \end{aligned} \tag{4.56}$$

because $N(m) = \mathcal{O}(m^\beta)$ for $\beta = (1/2-\gamma)/(1-\gamma)$.

Equations (4.54) - (4.56) yield (4.53).

□

2. In order to get a functional corresponding to the one in the discrete-time case of Aue and Horváth (2004), consider for any $\delta \in (0,1)$ the following approximation:

$$\sup_{[(1-\delta)N,N]} \left| \frac{B_t + \Delta t}{N^{1/2-\gamma}(t+1)^\gamma} - \frac{B_t + \Delta t}{N^{1/2-\gamma} t^\gamma} \right| = o_P(1) \quad \text{as} \quad m \to \infty.$$

Proof. Similar to the proof of 1., using the mean value theorem, one obtains for any $t \in [(1-\delta)N, N]$ the existence of a number ξ_t where $t/(t+1) < \xi_t < 1$ such that

$$1^\gamma - \left(\frac{t}{t+1} \right)^\gamma = \frac{\gamma}{\xi_t^{1-\gamma}} \frac{1}{t+1} \leq \frac{\gamma}{t^{1-\gamma}(t+1)^\gamma} \tag{4.57}$$

Hence, there exists some constant $K > 0$ such that

$$\sup_{[(1-\delta)N,N]} \left(1 - \left(\frac{t}{t+1}\right)^{\gamma}\right) \leq \frac{K}{N^{1-\gamma}\big(1+(1-\delta)N\big)^{\gamma}} \tag{4.58}$$

According to equations (3.16) and (3.17) of Aue and Horváth (2004), we have

$$\sup_{[(1-\delta)N,N]} \frac{|B_t|}{N^{1/2-\gamma}t^{\gamma}} \leq \sup_{(0,N]} \frac{|B_t|}{N^{1/2-\gamma}t^{\gamma}} \overset{D}{=} \sup_{(0,1]} \frac{|B_t|}{t^{\gamma}}. \tag{4.59}$$

Considering (4.57), it follows that

$$\begin{aligned} t\left(\frac{1}{t^{\gamma}} - \frac{1}{(t+1)^{\gamma}}\right) &= t^{1-\gamma}\left(1 - \left(\frac{t}{t+1}\right)^{\gamma}\right) \\ &\leq \frac{\gamma}{(t+1)^{\gamma}} \quad \forall\, t \in [(1-\delta)N, N] \end{aligned}$$

and thereby, that

$$\sup_{[(1-\delta)N,N]} \left(\frac{t}{N^{1/2-\gamma}t^{\gamma}} - \frac{t}{N^{1/2-\gamma}(t+1)^{\gamma}}\right) \leq \frac{\gamma}{N^{1/2-\gamma}\big(1+(1-\delta)N\big)^{\gamma}}. \tag{4.60}$$

The statement in 2. follows by (4.58) - (4.60).

□

For proving the convergence in (4.52), we proceed to bound the corresponding distribution function from above and below. For this purpose, consider

$$\begin{aligned} &P\left\{\left(\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{[t^*,N]} \frac{|B_t + \Delta t|}{g_m(t)} - \Delta\sqrt{N}\right) \leq x_m\right\} \\ &\quad\leq P\left\{\left(\left(\frac{m}{N}\right)^{1/2-\gamma} \frac{|B_N + \Delta N|}{g_m(N)} - \Delta\sqrt{N}\right) \leq x_m\right\}. \end{aligned} \tag{4.61}$$

In view of the approximations in 1. and 2., for any $\varepsilon > 0$ and any $\varepsilon' > 0$ we have

$$\begin{aligned} &P\left\{\left(\left(\frac{m}{N}\right)^{1/2-\gamma} \frac{|B_N + \Delta N|}{g_m(N)} - \Delta\sqrt{N}\right) \leq x_m\right\} \\ &\quad\leq P\left\{\left(\left(\frac{m}{N}\right)^{1/2-\gamma} \frac{|B_N + \Delta N|}{m^{1/2-\gamma}N^{\gamma}} - \Delta\sqrt{N}\right) \leq x_m + \varepsilon\right\} + \varepsilon' \end{aligned} \tag{4.62}$$

for sufficiently large m.

3. Using similar considerations as for equation (3.20) of Aue and Horváth (2004), the absolute value can be cancelled: i.e.,

$$\lim_{m\to\infty} P\left\{\frac{|B_N+\Delta N|}{m^{1/2-\gamma}N^\gamma} = \frac{B_N+\Delta N}{m^{1/2-\gamma}N^\gamma}\right\} = 1$$

   and even

$$\lim_{m\to\infty} P\left\{\sup_{[(1-\delta)N,N]} \frac{|B_t+\Delta t|}{m^{1/2-\gamma}t^\gamma} = \sup_{[(1-\delta)N,N]} \frac{B_t+\Delta t}{m^{1/2-\gamma}t^\gamma}\right\} = 1$$

   for any $\delta \in (0,1)$.

Combining (4.62) and part 3., we get for large m

$$\begin{aligned} &P\left\{\left(\left(\frac{m}{N}\right)^{1/2-\gamma}\frac{|B_N+\Delta N|}{g_m(N)} - \Delta\sqrt{N}\right) \le x_m\right\} \\ &\le P\left\{\left(\left(\frac{m}{N}\right)^{1/2-\gamma}\frac{B_N+\Delta N}{m^{1/2-\gamma}N^\gamma} - \Delta\sqrt{N}\right) \le x_m+\varepsilon\right\} + 2\varepsilon' \qquad (4.63) \\ &= \Phi(x_m+\varepsilon) + 2\varepsilon'. \end{aligned}$$

Second, we need three further approximations for bounding the left hand side in (4.61) from below.

4. Similar to Aue and Horváth (2004), the supremum is attained in the neighbourhood of the right boundary, i.e., for any $\delta \in (0,1)$

$$\lim_{m\to\infty} P\left\{\sup_{[t^*,N]} \frac{|B_t+\Delta t|}{m^{1/2-\gamma}(t+1)^\gamma} = \sup_{[(1-\delta)N,N]} \frac{|B_t+\Delta t|}{m^{1/2-\gamma}(t+1)^\gamma}\right\} = 1.$$

   Proof. For the reader's convenience we explain the necessary modifications caused by using $(t+1)$ instead of $k \in \mathbb{N}$ in the weighting function. It must be shown that

$$\lim_{m\to\infty} P\left\{\sup_{[t^*,(1-\delta)N]} \frac{|B_t+\Delta t|}{m^{1/2-\gamma}(t+1)^\gamma} > \sup_{[(1-\delta)N,N]} \frac{|B_t+\Delta t|}{m^{1/2-\gamma}(t+1)^\gamma}\right\} = 0. \qquad (4.64)$$

   The supremum over $[(1-\delta)N, N]$ is bounded from below by

$$\frac{\Delta N}{m^{1/2-\gamma}(N+1)^\gamma} - \frac{|B_N|}{m^{1/2-\gamma}(N+1)^\gamma}.$$

Moreover, we have

$$\sup_{[t^*,(1-\delta)N]} \frac{t}{(t+1)^\gamma} = \frac{(1-\delta)N}{\big((1-\delta)N+1\big)^\gamma} \quad \text{for large} \quad m.$$

Then the probability in (4.64) is dominated by

$$P\Bigg\{ \frac{m^{1/2-\gamma}(N+1)^\gamma}{\Delta N} \Bigg[\sup_{[t^*,(1-\delta)N]} \frac{|B_t|}{m^{1/2-\gamma}(t+1)^\gamma} + \frac{\Delta(1-\delta)N}{m^{1/2-\gamma}\big((1-\delta)N+1\big)^\gamma} + \frac{|B_N|}{m^{1/2-\gamma}(N+1)^\gamma} \Bigg] > 1 \Bigg\}. \tag{4.65}$$

By the inequality

$$\frac{(N+1)^\gamma}{N} \sup_{[t^*,(1-\delta)N]} \frac{|B_t|}{(t+1)^\gamma} \le \frac{(N+1)^\gamma}{N^{1/2+\gamma}} \sup_{(0,(1-\delta)N]} \frac{|B_t|}{N^{1/2}(t/N)^\gamma} \tag{4.66}$$

and the scale transformation of Wiener processes the expressions which contain a Wiener process tend to zero P-a.s. as $m \to \infty$. Hence, since $0 < \delta < 1$, the probability in (4.65) tends to zero.

□

By the approximations presented in 1. and 4. we get for any $\varepsilon, \varepsilon' > 0$ and for sufficiently large m

$$P\Bigg\{ \Bigg(\Big(\frac{m}{N}\Big)^{1/2-\gamma} \sup_{[t^*,N]} \frac{|B_t+\Delta t|}{g_m(t)} - \Delta\sqrt{N} \Bigg) \le x_m \Bigg\} \ge P\Bigg\{ \Bigg(\sup_{[(1-\delta)N,N]} \frac{|B_t+\Delta t|}{N^{1/2-\gamma}(t+1)^\gamma} - \Delta\sqrt{N} \Bigg) \le x_m - \varepsilon \Bigg\} - 2\varepsilon'$$

Applying 2. and 3., one obtains for large m

$$P\Bigg\{ \Bigg(\Big(\frac{m}{N}\Big)^{1/2-\gamma} \sup_{[t^*,N]} \frac{|B_t+\Delta t|}{g_m(t)} - \Delta\sqrt{N} \Bigg) \le x_m \Bigg\} \ge P\Bigg\{ \Bigg(\sup_{[(1-\delta)N,N]} \frac{B_t+\Delta t}{N^{1/2-\gamma}t^\gamma} - \Delta\sqrt{N} \Bigg) \le x_m - 2\varepsilon \Bigg\} - 4\varepsilon' \tag{4.67}$$

5. The same argumentation as for equations (3.22) and (3.23) of Aue and Horváth (2004) yields that, independently of m,

$$\sup_{[(1-\delta)N,N]} \frac{B_t + \Delta t - \big(B_N + \Delta t\big)}{N^{1/2-\gamma} t^{\gamma}} = o_P(1) \quad \text{as} \quad \delta \to 0.$$

6. In addition, we have

$$\sup_{[(1-\delta)N,N]} \left(\frac{B_N + \Delta t}{N^{1/2-\gamma} t^{\gamma}}\right) - \frac{B_N + \Delta N}{N^{1/2-\gamma} N^{\gamma}} = o_P(1) \quad \text{as} \quad \delta \to 0$$

independently of m.

Since this quite easy statement is not mentioned by Aue and Horváth (2004), we give a proof for it:

Proof. First, compute for any $m > 0$

$$\begin{aligned} \sup_{[(1-\delta)N,N]} \frac{B_N}{N^{1/2-\gamma} t^{\gamma}} - \frac{B_N}{N^{1/2-\gamma} N^{\gamma}} &= \frac{B_N}{\sqrt{N}} \left(\frac{1}{(1-\delta)^{\gamma}} - 1\right) \\ &\overset{D}{=} B_1 \left(\frac{1}{(1-\delta)^{\gamma}} - 1\right). \end{aligned} \tag{4.68}$$

Second, we have

$$\sup_{[(1-\delta)N,N]} \frac{\Delta t}{N^{1/2-\gamma} t^{\gamma}} - \frac{\Delta N}{N^{1/2-\gamma} N^{\gamma}} = 0 \quad \forall m > 0. \tag{4.69}$$

□

From the approximations in 5. and 6. follows that for sufficiently small δ and independently of m

$$\begin{aligned} &P\left\{\left(\sup_{[(1-\delta)N,N]} \frac{B_t + \Delta t}{N^{1/2-\gamma} t^{\gamma}} - \Delta\sqrt{N}\right) \le x_m - 2\varepsilon\right\} \\ &\quad \ge P\left\{\left(\frac{B_N + \Delta N}{\sqrt{N}} - \Delta\sqrt{N}\right) \le x_m - 3\varepsilon\right\} - \varepsilon' \end{aligned}$$

Finally, considering (4.67), we obtain for the lower bound and for any ε, $\varepsilon' > 0$

$$\begin{aligned} &P\left\{\left(\left(\frac{m}{N}\right)^{1/2-\gamma} \sup_{[t^*,N]} \frac{|B_t + \Delta t|}{g_m(t)} - \Delta\sqrt{N}\right) \le x_m\right\} \\ &\quad \ge \Phi(x_m - 3\varepsilon) - 5\varepsilon' \end{aligned} \tag{4.70}$$

if m is sufficiently large.
By the fact that $x_m \to x$ as $m \to \infty$ and by (4.61), (4.63), and (4.70) we have

$$\begin{aligned}\Phi(x-3\varepsilon)-5\varepsilon' &\leq \liminf_{m\to\infty} P\left\{\left(\left(\frac{m}{N}\right)^{1/2-\gamma}\sup_{[t^*,N]}\frac{|B_t+\Delta t|}{g_m(t)}-\Delta\sqrt{N}\right)\leq x_m\right\}\\ &\leq \limsup_{m\to\infty} P\left\{\left(\left(\frac{m}{N}\right)^{1/2-\gamma}\sup_{[t^*,N]}\frac{|B_t+\Delta t|}{g_m(t)}-\Delta\sqrt{N}\right)\leq x_m\right\}\\ &\leq \Phi(x+\varepsilon)+2\varepsilon'\end{aligned}$$

Since ε and ε' can be chosen arbitrarily small, (4.52) follows.

□

4.3.2 Main result

Theorem 4.21. *Under the assumptions of Proposition 4.20 we have*

$$\lim_{m\to\infty}\mathcal{L}\Big(\frac{\tau_m-a_m}{b_m}\Big)=N(0,1)$$

where

$$a_m=\left(\frac{cm^{1/2-\gamma}}{|\theta_1-\theta_0|}\right)^{1/(1-\gamma)}, \quad b_m=\frac{\sqrt{a_m}}{(1-\gamma)|\theta_1-\theta_0|\sqrt{I(\theta_1)}}, \quad m\geq 0,$$

and where c represents the critical value involved in the stopping rule τ_m.

Proof. As mentioned above, the theorem results from (4.26), (4.28), and Proposition 4.20.

□

Remark 4.22. Theorem 4.21 has two important consequences:

(i) We obtain

$$\frac{\tau_m}{a_m}\xrightarrow{P}1 \quad \text{as} \quad m\to\infty \tag{4.71}$$

where $\xrightarrow{P}$ stands for convergence in probability.

In order to see this, write

$$\frac{\tau_m}{a_m}=\frac{b_m}{a_m}\left(\frac{\tau_m-a_m}{b_m}+\frac{a_m}{b_m}\right),$$

note that $b_m=o(a_m)$ as $m\to\infty$, and use Slutsky's lemma.

Since $t^*(m) = o(a_m)$ as $m \to \infty$, (4.71) implies

$$\frac{\tau_m - t^*(m)}{a_m} \xrightarrow{P} 1 \quad \text{as} \quad m \to \infty.$$

Hence, for a large training period $[0, m]$ the delay time $\tau_m - t^*$ of the detection can be estimated by a_m.

(ii) By (4.71) we obtain the result of Theorem 4.1 under somewhat stronger conditions, i.e.,

$$\lim_{m\to\infty} P\{\tau_m < \infty\} = 1$$

under H_1.

This equality results from

$$\lim_{m\to\infty} P\left\{\frac{\tau_m}{a_m} > 1 + \varepsilon\right\} = 0 \quad \forall\, \varepsilon > 0.$$

Remark 4.23. As in Remark 3.10, we consider the problem of the one-sided alternative at the end of the chapter. Recall that we want to study a sequential test for

$$H_0\colon\ t^* = \infty$$

versus

$$H_1\colon\ t^* < \infty \quad \text{and} \quad \theta_1 > \theta_0.$$

The stopping time τ_m, $m > 0$, is defined as in Remark 3.10. Along the lines of the proofs in this chapter, but in many cases by easier arguments, we obtain the asymptotic power one of the test and the asymptotic normality of the stopping time.

Chapter 5

Examples

This chapter is devoted to some examples of stochastic equations. We will focus on checking the technical integrability condition contained in the definition of the set $\mathcal{M}(b,\sigma)$ in Subsection 2.1.1 because the other assumptions do not cause problems.

5.1 Ornstein-Uhlenbeck process

Let X be a solution of the Ornstein-Uhlenbeck equation

$$dX_t = -\theta X_t dt + \sigma dW_t, \quad t \geq 0, \tag{5.1}$$

where $\sigma > 0$ and the parameter θ is an element of the compact interval $\Theta = [\alpha, \beta] \subset (0, \infty)$. The Ornstein-Uhlenbeck process X is ergodic, and its stationary distribution is given by $\mu(\theta) = N(0, \sigma^2/(2\theta))$. The measure p defined in Subsection 2.1.1 has the form

$$dp(x) = \exp\left(\frac{\theta x^2}{\sigma^2}\right) dx, \quad x \in \mathbb{R}.$$

As indicated in Kutoyants (2004), Section 2.4.2, for constructing the EMM in the Ornstein-Uhlenbeck case it is suitable to choose the function $q(z) = z^2,\ z \in \mathbb{R}$.
The functions

$$\left(\left|\frac{\dot{b}(\theta,\cdot)}{\sigma}\right|^2 - E_{\mu(\theta)}\left|\frac{\dot{b}(\theta,\cdot)}{\sigma}\right|^2\right), \quad \big(q - E_{\mu(\theta)} q\big),$$

where $b(\theta, x) = -\theta x,\ x \in \mathbb{R},$ belong to the class of functions

$$f(z) = c\left(z^2 - \tau^2\right), \quad c > 0,$$

where $\tau^2 = \sigma^2/(2\theta)$.
It will be sufficient to check whether $f(z) = z^2 - \tau^2$, $z \in \mathbb{R}$, belongs to $\mathcal{M}(b(\theta,\cdot),\sigma)$ which essentially means that

$$\int_{\mathbb{R}} \left| (y^2 - \tau^2) \int_y^0 \int_{-\infty}^s (z^2 - \tau^2)\, d\mu(\theta)(z)\, dp(s) \right| d\mu(\theta)(y) < \infty. \tag{5.2}$$

Toward this end, denote the density of the normal distribution $\mu(\theta)$ by φ_{0,τ^2}. The integration by substitution leads to

$$\begin{aligned} \int_{-\infty}^s (z^2 - \tau^2)\varphi_{0,\tau^2}(z)dz &= \tau^2 \int_{-\infty}^s \left[\left(\frac{z}{\tau}\right)^2 - 1 \right] \varphi_{0,1}(z/\tau)\frac{dz}{\tau} \\ &= \tau^2 \int_{-\infty}^{s/\tau} \left[x^2\varphi_{0,1}(x) - \varphi_{0,1}(x) \right] dx. \end{aligned}$$

The equation

$$x^2\varphi_{0,1}(x) = \varphi''_{0,1}(x) + \varphi_{0,1}(x) \quad \forall\, x \in \mathbb{R}$$

implies

$$\begin{aligned} \int_{-\infty}^s (z^2 - \tau^2)\varphi_{0,\tau^2}(z)dz &= (\tau^2)\varphi'_{0,1}(s/\tau) \\ &= -\tau s\varphi_{0,1}(s/\tau). \end{aligned}$$

Hence, the second integral in (5.2)

$$-\tau \int_y^0 s \cdot \varphi_{0,1}(s/\tau)dp(s) = \tau \int_0^y sds$$

is equal to $Q(y) = \tau y^2/2$. Since $f \cdot Q$ is again a polynomial, (5.2) follows.

5.2 Nonlinear location model

Let the process X be a solution of

$$dX_t = (\theta - X_t)^3 dt + \sigma dW_t, \quad t \geq 0, \quad \theta \in [\alpha, \beta]. \tag{5.3}$$

The stationary distribution is given by $d\mu(\theta)(x) = \varphi_\theta(x)dx$ where

$$\varphi_\theta(x) = \frac{1}{a\sqrt{\sigma}} \exp\left(-\frac{(x-\theta)^4}{2\sigma^2} \right), \quad x \in \mathbb{R},$$

and where the constant $a > 0$ represents the normalization factor. Observe that the drift coefficient $b(\theta, x) = (\theta - x)^3$, $x \in \mathbb{R}$, satisfies the modified growth condition

$$x\, b(\theta, x) \leq K_\theta(1 + |x|^2) \quad \forall\, x \in \mathbb{R} \quad \text{for some} \quad K_\theta > 0$$

presented in (1.5).

Remark 5.1. We have chosen the name "location model" for the stochastic equation (5.3) because one has

$$EX_t = \int_{\mathbb{R}} x\mu(\theta)(dx) = \theta \quad \forall\, t \geq 0. \tag{5.4}$$

The last equality in (5.4) follows by means of the Gamma function.

The measure p defined in Subsection 2.1.1 has the form

$$dp(x) = a' \exp\left(\frac{(x-\theta)^4}{2\sigma^2}\right) dx, \quad x \in \mathbb{R},$$

with some constant $a' > 0$.
In order to avoid too many cases, we suppose that $\sigma = 1$ and $\alpha, \beta > 0$. As a preliminary result, a kind of Feller inequality has to be proven for the distribution function F_0 of $\mu(0)$. The elementary proof is similar to the one of the Feller inequality (see Feller (1968), Chapter 7, Lemma 2).

Lemma 5.2. *For* $x > 0$ *the inequalities*

$$\text{(i)} \qquad 1 - F_0(x) < \frac{\varphi_0(x)}{2x^3},$$

$$\text{(ii)} \qquad \varphi_0(x)\left(\frac{1}{2x^3} - \frac{3}{4x^7}\right) < 1 - F_0(x)$$

hold.

Proof. Note in the following that

$$\varphi_0'(x) = -2x^3\varphi_0(x) \quad \forall\, x \in \mathbb{R}.$$

(i) The first inequality results from the following computation for any $x > 0$:

$$\begin{aligned} 1 - F_0(x) &= \int_x^\infty \varphi_0(y)dy \\ &< \int_x^\infty \varphi_0(y)\left(1 + \frac{3}{2y^4}\right) dy \\ &= \frac{\varphi_0(x)}{2x^3}. \end{aligned}$$

(ii) Consider

$$\varphi_0(x)\left(\frac{1}{2x^3}-\frac{3}{4x^7}\right)=\int\limits_x^\infty \varphi_0(y)\left(1-\frac{21}{4y^8}\right)dy.$$

Hence,

$$\varphi_0(x)\left(\frac{1}{2x^3}-\frac{3}{4x^7}\right)<1-F_0\left(x\right).$$

□

Now we are able to show that the function

$$\begin{aligned} f(z) &= \left(\Big|\frac{\dot b(\theta,z)}{\sigma}\Big|^2 - E_{\mu(\theta)}\Big|\frac{\dot b(\theta,\cdot)}{\sigma}\Big|^2\right) \\ &= 9(z-\theta)^4-\tau^2, \quad z\in\mathbb{R}, \end{aligned}$$

where $\tau^2 = E_{\mu(\theta)}|\partial_\theta b(\theta,\cdot)|^2$, is an element of $\mathcal{M}(b(\theta,\cdot),\sigma)$.

Lemma 5.3. *Let $\sigma=1$ and $0<\alpha\le\beta$. Then for any $\theta\in[\alpha,\beta]$ we have*

$$\int\limits_{\mathbb{R}}\left|f(y)\int\limits_y^0\int\limits_{-\infty}^s f(z)\varphi_\theta(z)dz\,dp(s)\right|\varphi_\theta(y)dy<\infty. \tag{5.5}$$

Proof. Let the centred interval $[-K,K]$, $K>0$, be large enough such that for $|x|>K$

$$9x^4-\tau^2>0, \quad 4x^6-6x^2>0,$$

and

$$9x^4<4x^6-6x^2 \tag{5.6}$$

hold. Consider an arbitrary number $s>K+\theta$ and note that

$$\tau^2=\int\limits_{\mathbb{R}} 9x^4\varphi_0(x)dx.$$

We have

$$\begin{aligned} \left|\int\limits_{-\infty}^{s-\theta}\left(9x^4-\tau^2\right)\varphi_0(x)dx\right| &= \left(\int\limits_{-\infty}^{s-\theta}9x^4\varphi_0(x)dx-\tau^2F_0(s-\theta)\right)^+ \\ &+\left(\int\limits_{-\infty}^{s-\theta}9x^4\varphi_0(x)dx-\tau^2F_0(s-\theta)\right)^-. \end{aligned} \tag{5.7}$$

If
$$\int_{-\infty}^{s-\theta} 9x^4\varphi_0(x)dx \geq \tau^2 F_0(s-\theta),$$
then we get by Lemma 5.2
$$\begin{aligned}\int_{-\infty}^{s-\theta} \big(9x^4-\tau^2\big)\varphi_0(x)dx &\leq \tau^2-\tau^2F_0(s-\theta)\\ &< \tau^2\frac{\varphi_0(s-\theta)}{2(s-\theta)^3}.\end{aligned} \tag{5.8}$$
In case of
$$\int_{-\infty}^{s-\theta} 9x^4\varphi_0(x)dx < \tau^2 F(s-\theta),$$
it follows that
$$\begin{aligned}\int_{-\infty}^{s-\theta} \big(\tau^2-9x^4\big)\varphi_0(x)dx &\leq \tau^2-\int_{-\infty}^{s-\theta} 9x^4\varphi_0(x)dx\\ &= \int_{s-\theta}^{\infty} 9x^4\varphi_0(x)dx.\end{aligned}$$
Hence, by (5.6) and the formulas
$$\varphi_0''(x)=(4x^6-6x^2)\varphi_0(x), \quad -\varphi_0'(x)=2x^3\varphi_0(x) \quad \forall\, x\in\mathbb{R} \tag{5.9}$$
one obtains
$$\begin{aligned}\int_{-\infty}^{s-\theta} \big(\tau^2-9x^4\big)\varphi_0(x)dx &\leq \int_{s-\theta}^{\infty} \big(4x^6-6x^2\big)\varphi_0(x)dx\\ &= \int_{s-\theta}^{\infty} \varphi_0''(x)dx\\ &= 2(s-\theta)^3\varphi_0(s-\theta).\end{aligned} \tag{5.10}$$
Putting together (5.7), (5.8), and (5.10), we finally get for the inner integral in (5.5)
$$\left|\int_{-\infty}^{s-\theta} \big(9x^4-\tau^2\big)\varphi_0(x)dx\right| \leq \left(\frac{\tau^2}{2(s-\theta)^3}+2(s-\theta)^3\right)\varphi_0(s-\theta) \tag{5.11}$$

for all $s > K + \theta$.
For analysing the integral with respect to the measure p in (5.5), we observe that $\varphi_0(s-\theta)dp(s) = cds$ with some $c > 0$. Consider for $y > 0$ the following decomposition:

$$\begin{aligned}
&\int_0^y \left| \int_{-\infty}^{s-\theta} (9x^4 - \tau^2)\varphi_0(x)dx \right| dp(s) \\
&\quad = \int_0^y 1_{[0,K+\theta]}(s) \left| \int_{-\infty}^{s-\theta} (9x^4 - \tau^2)\varphi_0(x)dx \right| dp(s) \qquad (5.12) \\
&\quad + \int_0^y 1_{(K+\theta,\infty)}(s) \left| \int_{-\infty}^{s-\theta} (9x^4 - \tau^2)\varphi_0(x)dx \right| dp(s).
\end{aligned}$$

The first integral on the right hand side of (5.12) is bounded by some constant for all $y > 0$. According to (5.11), the second integral is bounded by

$$c \int_{(K+\theta,y]} \left(\frac{\tau^2}{2(s-\theta)^3} + 2(s-\theta)^3 \right) ds. \qquad (5.13)$$

If $y \leq 0$, we can proceed in a similar way: there exists a number $c' > 0$ such that

$$\begin{aligned}
&\int_y^0 \left| \int_{-\infty}^{s-\theta} (9x^4 - \tau^2)\varphi_0(x)dx \right| dp(s) \\
&\quad \leq c' + \int_{[y,\theta-K)} \left| \int_{-\infty}^{s-\theta} \left(9x^4 - \tau^2\right) \varphi_0(x)dx \right| dp(s).
\end{aligned} \qquad (5.14)$$

Similar to (5.11), from (5.6) and (5.9) follows that

$$\begin{aligned}
&\int_y^0 \left| \int_{-\infty}^{s-\theta} \left(9x^4 - \tau^2\right) \varphi_0(x)dx \right| dp(s) \\
&\quad \leq c' - \int_{[y,\theta-K)} 2(s-\theta)^3\varphi_0(s-\theta)dp(s).
\end{aligned} \qquad (5.15)$$

Note that the right hand side of the last equation represents some polynomial. Therefore, (5.12), (5.13), and (5.15) imply

$$\left| \int_y^0 \int_{-\infty}^s \big(9(z-\theta)^4 - \tau^2\big)\varphi_\theta(z)\, dz\, dp(s) \right| \le h(y) \quad \forall\, y \in \mathbb{R}$$

for some function $h \in \mathcal{P}$.
For completing the proof, observe that the product $f \cdot h$ is integrable with respect to $\mu(\theta)$ because the density φ_θ has exponential decrease.

□

It remains to prove that $(q - E_{\mu(\theta)}q) \in \mathcal{M}(b(\theta,\cdot),\sigma)$ where q is a function chosen for constructing the EMM according to Subsection 2.2.1.
We follow Kutoyants (2004), Section 2.4.2, and take the function $q(z) = z$, $z \in \mathbb{R}$.

Lemma 5.4. *Let* $\sigma = 1$ *and* $0 < \alpha \le \beta$. *Then the function*

$$f(z) = q(z) - E_{\mu(\theta)}q = z - \theta, \quad z \in \mathbb{R},$$

satisfies

$$\int_{\mathbb{R}} \left| f(y) \int_y^0 \int_{-\infty}^s f(z)\varphi_\theta(z)\, dz\, dp(s) \right| \varphi_\theta(y)dy < \infty$$

for any $\theta \in [\alpha,\beta]$.

Proof. Let the centred interval $[-K,K]$, $K > 0$, be large enough such that for $|z-\theta| > K$

$$4(z-\theta)^6 - 6(z-\theta)^2 > 0 \quad \text{and} \quad |z-\theta| \le 4(z-\theta)^6 - 6(z-\theta)^2 \qquad (5.16)$$

hold. Similar to the proof of Lemma 5.3, consider the decomposition

$$\begin{aligned} &\int_y^0 \int_{-\infty}^s (z-\theta)\, \varphi_\theta(z)\, dz\, dp(s) \\ &\quad = \int_y^0 \Bigg(1_{[-K,K+\theta]}(s) \int_{-\infty}^s (z-\theta)\, \varphi_\theta(z) dz \qquad (5.17) \\ &\qquad + 1_{[-K,K+\theta]^c}(s) \int_{-\infty}^s (z-\theta)\varphi_\theta(z)dz \Bigg) dp(s). \end{aligned}$$

We have to distinguish between the signs of y. If $y \leq 0$, there exists a constant $c_1 \geq 0$ such that by (5.16) and the assumption that $\theta > 0$ one obtains

$$\int_y^0 \left| \int_{-\infty}^s (z-\theta)\,\varphi_\theta(z)dz \right| dp(s)$$
$$\leq c_1 + \int_{[y,-K)} \int_{-\infty}^s \left(4(z-\theta)^6 - 6(z-\theta)^2\right) \varphi_\theta(z)dz\,dp(s).$$

It follows from (5.9) that

$$\int_y^0 \left| \int_{-\infty}^s (z-\theta)\,\varphi_\theta(z)dz \right| dp(s) \leq c_1 + \int_{[y,-K)} \int_{-\infty}^s \varphi_\theta''(z)dz\,dp(s)$$
$$= c_1 + \int_{[y,-K)} \varphi_\theta'(s)dp(s).$$

A second application of (5.9) yields for $y \leq 0$

$$\int_y^0 \left| \int_{-\infty}^s (z-\theta)\,\varphi_\theta(z)dz \right| dp(s) \leq c_1 - \int_{[y,-K)} 2c(s-\theta)^3 ds \tag{5.18}$$

where we have used that $\varphi_\theta(s)dp(s) = cds$ for some $c > 0$.

Now let $y > 0$. Similar to equation (5.7), consider for $s > K + \theta$

$$\left| \int_{-\infty}^s (z-\theta)\varphi_\theta(z)dz \right| = \left(\int_{-\infty}^s z\varphi_\theta(z)dz - \theta F_\theta(s) \right)^+$$
$$+ \left(\int_{-\infty}^s z\varphi_\theta(z)dz - \theta F_\theta(s) \right)^- .$$

If

$$\int_{-\infty}^s z\varphi_\theta(z)dz - \theta F_\theta(s) \geq 0,$$

we have by Lemma 5.2

$$\int_{-\infty}^{s} z\varphi_\theta(z)dz - \theta F_\theta(s) \le \theta\big(1 - F_0(s-\theta)\big) \\ < \theta\frac{\varphi_0(s-\theta)}{2(s-\theta)^3}. \tag{5.19}$$

In case of

$$\theta F_\theta(s) - \int_{-\infty}^{s} z\varphi_\theta(z)dz > 0,$$

we obtain

$$\theta F_\theta(s) - \int_{-\infty}^{s} z\varphi_\theta(z)dz \le \theta - \int_{-\infty}^{s} z\varphi_\theta(z)dz \\ = \int_{s}^{\infty} z\varphi_\theta(z)dz.$$

Note that $z > s$ implies $z - \theta > K$. Apply (5.16), (5.9), and Lemma 5.2 in order to get

$$\int_{s}^{\infty} z\varphi_\theta(z)dz \le \int_{s}^{\infty} \big[\varphi''_\theta(z) + \theta\varphi_\theta(z)\big]dz \\ = -\varphi'_\theta(s) + \theta\big(1 - F_0(s-\theta)\big) \\ < \varphi_0(s-\theta)\left(2(s-\theta)^3 + \frac{\theta}{2(s-\theta)^3}\right). \tag{5.20}$$

Combining (5.17), (5.19), and (5.20), we have for $y > 0$

$$\int_{0}^{y}\left|\int_{-\infty}^{s}(z-\theta)\varphi_\theta(z)dz\right| dp(s) \le c_2 + c\int_{(K+\theta,y]}\left(\frac{\theta}{(s-\theta)^3} + 2(s-\theta)^3\right)ds \tag{5.21}$$

where c_2 is a positive constant. From (5.18) and (5.21) results that

$$y \longmapsto f(y)\int_{y}^{0}\int_{-\infty}^{s} f(z)\varphi_\theta(z)\,dz\,dp(s), \quad y \in \mathbb{R},$$

is dominated by a function of at most polynomial growth. Hence, f satisfies the integrability property stated in the lemma.

□

Chapter 6

Alternative approach

6.1 Strong approximation of the estimator process

Now we consider the statistical model

$$(C[0,\infty),\, \mathcal{B},\, P_\theta,\, \theta \in \Theta)$$

from Section 2.2 where $\mathcal{B}$ represents the Borel σ-algebra, Θ a compact interval, and P_θ the distribution of the unique solution $X(\theta)$ to the Ito stochastic equation

$$dX_s = b(\theta, X_s)ds + \sigma(X_s)dW_s, \quad X_0 \sim \mu(\theta), \quad s \geq 0. \tag{6.1}$$

Let $[0,t],\, t \geq 0,$ be the observation period and $\mu(\theta)$ be the stationary distribution with density f_θ. Recall from Subsection 2.2.2 that the one-step MLE for the parameter θ was defined to be

$$\hat{\theta}_{0,t} = \bar{\theta}_{0,t} + \frac{\psi_{0,t}(\bar{\theta}_{0,t})}{tI(\bar{\theta}_{0,t})},$$

where $\bar{\theta}_{0,t}$ denotes a consistent starting estimator, $I(\theta)$ denotes the Fisher information given by

$$I(\theta) = E_{\mu(\theta)}\left|\frac{\dot{b}(\theta,\cdot)}{\sigma}\right|^2,$$

and where $\theta \mapsto \psi_{0,t}(\theta)$ represents the function

$$\psi_{0,t}(\theta) = \partial_\theta \log \frac{dP_\theta|_{\mathcal{B}_{0,t}}}{dP_{\theta_*}|_{\mathcal{B}_{0,t}}}(X) - \frac{\partial_\theta f_\theta}{f_\theta}(X_0).$$

We follow some ideas of Gerencsér (1991a) in order to approximate the estimator process $(t(\hat{\theta}_{0,t} - \theta_0)\colon t \geq 0)$ P_{θ_0}-a.s. by a Wiener process.

Denote by θ_0 the true parameter value. Assume that there exists a positive, increasing function $t \mapsto \varphi_t$, $t \geq 0$, with $\lim_{t\to\infty} \varphi_t = \infty$ and some $t_0 \geq 0$ such that

$$\sup_{t \geq t_0} \varphi_t \left| \bar{\theta}_{0,t} - \theta_0 \right| < \infty \quad P\text{-a.s.} \tag{6.2}$$

Proposition 6.1. *Let Assumption $\mathcal{A}_0(\Theta)$ and $I > 0$ be satisfied. Suppose that $b(\theta,\cdot)$, $\partial_\theta \partial_x b(\theta,\cdot)$, $\partial_\theta^2 \partial_x b(\theta,\cdot) \in \mathcal{P}$ uniformly in θ and that $(|\partial_\theta b(\theta_0,\cdot)/\sigma|^2 - I(\theta_0)) \in \mathcal{M}(b(\theta_0,\cdot),\sigma)$ for $\theta_0 \in \Theta$.*
If (6.2) holds, then we have

$$t\big(\hat{\theta}_{0,t} - \theta_0\big) - \frac{\psi_{0,t}(\theta_0)}{I(\theta_0)} = \mathcal{O}\left(\frac{\sqrt{t \log_2 t}}{\varphi_t} \right) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty \tag{6.3}$$

where

$$\psi_{0,t}(\theta_0) = \int_0^t \frac{\dot{b}(\theta_0, X_s)}{\sigma(X_s)} dW(s) \quad \forall\, t \geq 0.$$

Proof. The definition of the one-step MLE gives

$$t\big(\hat{\theta}_{0,t} - \theta_0\big) = t(\bar{\theta}_{0,t} - \theta_0) + \frac{\psi_{0,t}(\bar{\theta}_{0,t})}{I(\bar{\theta}_{0,t})}.$$

The mean value theorem

$$\frac{\psi_{0,t}(\bar{\theta}_{0,t})}{I(\bar{\theta}_{0,t})} = \frac{\psi_{0,t}(\theta_0)}{I(\theta_0)} + \partial_\theta \Big(\frac{\psi_{0,t}}{I} \Big)(\xi_t)(\bar{\theta}_{0,t} - \theta_0),$$

where $|\xi_t - \theta_0| \leq |\bar{\theta}_{0,t} - \theta_0|$, implies

$$t\big(\hat{\theta}_{0,t} - \theta_0\big) = \frac{\psi_{0,t}(\theta_0)}{I(\theta_0)} + \left[t + \partial_\theta \Big(\frac{\psi_{0,t}}{I} \Big)(\xi_t) \right] (\bar{\theta}_{0,t} - \theta_0). \tag{6.4}$$

Since the functions

$$(\theta, t) \longmapsto \psi_{0,t}(\theta), \quad (\theta, t) \longmapsto \dot{\psi}_{0,t}(\theta), \quad (\theta, t) \in \Theta \times [0, \infty),$$

are P-a.s. continuous, we obtain by the continuity of I and $\partial_\theta I$

$$\lim_{t\to\infty} \left[\partial_\theta \Big(\frac{\psi_{0,t}}{I} \Big)(\xi_t) - \frac{\dot{\psi}_{0,t}(\theta_0) I(\theta_0) - \dot{I}(\theta_0)\psi_{0,t}(\theta_0)}{I^2(\theta_0)} \right] = 0 \quad P\text{-a.s.} \tag{6.5}$$

It remains to investigate the process

$$t + \frac{\dot{\psi}_{0,t}(\theta_0)}{I(\theta_0)} - \frac{\dot{I}(\theta_0)}{I^2(\theta_0)} \psi_{0,t}(\theta_0), \quad t \geq 0, \tag{6.6}$$

where, according to Remark 2.20, we have

$$\dot{\psi}_{0,t}(\theta_0) = \int_0^t \frac{\ddot{b}(\theta_0, X_s)}{\sigma(X_s)} dW_s - \int_0^t \left| \frac{\dot{b}(\theta_0, X_s)}{\sigma(X_s)} \right|^2 ds$$

We can apply the LIL of Lemma 2.1 on both stochastic integrals in (6.6) as well as the LIL of Remark 2.2 on the process

$$tI(\theta_0) - \int_0^t \left| \frac{\dot{b}(\theta_0, X_s)}{\sigma(X_s)} \right|^2 ds, \quad t \geq 0.$$

It follows that

$$t + \frac{\dot{\psi}_{0,t}(\theta_0)}{I(\theta_0)} - \frac{\dot{I}(\theta_0)}{I^2(\theta_0)} \psi_{0,t}(\theta_0) = \mathcal{O}(\sqrt{t \log_2 t}) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty. \tag{6.7}$$

Combining (6.2), (6.4), (6.5), and (6.7), we obtain (6.3).

□

Now, since under the assumptions of Proposition 6.1 the strong invariance principle of Theorem 2.3 is also valid, we obtain a strong approximation of the estimator process by a Wiener process:

Theorem 6.2. *Let Assumption $\mathcal{A}_0(\Theta)$ and $I > 0$ be satisfied. Suppose that $b(\theta, \cdot)$, $\partial_\theta \partial_x b(\theta, \cdot)$, $\partial_\theta^2 \partial_x b(\theta, \cdot) \in \mathcal{P}$ uniformly in θ and that $(|\partial_\theta b(\theta_0, \cdot)/\sigma|^2 - I(\theta_0)) \in \mathcal{M}(b(\theta_0, \cdot), \sigma)$ for $\theta_0 \in \Theta$.*
If (6.2) holds, then there exists a Wiener process B such that

$$t\big(\hat{\theta}_{0,t} - \theta_0\big) - \frac{B_t}{\sqrt{I(\theta_0)}} = \mathcal{O}\left(\max \left\{ \frac{(t \log_2 t)^{1/2}}{\varphi_t}, (t\log_2 t)^{1/4} (\log t)^{1/2} \right\} \right)$$

P-a.s. as $t \to \infty$.

Corollary 6.3. *Suppose that the starting estimator $\bar{\theta}_{0,t}$, $t \geq 0$, is chosen to be the estimator of the method of moments (see Subsection 2.2.1) where the determining function q belongs to $\mathcal{D}(\Theta)$ and satisfies $(q - a(\theta_0)) \in \mathcal{M}(b(\theta_0, \cdot), \sigma)$ for the true parameter value $\theta_0 \in \Theta$.*
Then, under the assumptions of Theorem 6.2, we have

$$t\big(\hat{\theta}_{0,t} - \theta_0\big) - \frac{B_t}{\sqrt{I(\theta_0)}} = \mathcal{O}\left((t\log_2 t)^{1/4} (\log t)^{1/2} \right) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty. \tag{6.8}$$

Proof. According to Proposition 2.16, (6.2) holds with $\varphi_t = \sqrt{t}/\sqrt{\log_2 t}$ for $t \geq e^e$.

□

6.2 Asymptotics under the hypothesis

By means of the strong approximation in (6.8) we can give an alternative and very easy proof for Corollary 3.8. Suppose that X follows the model given in Section 1.1 (see equations (1.1), (1.2), (1.3)) and that the EMM is chosen to be the starting estimator in the one-step procedure. Recall that $\hat{\theta}_{m,t},\ m,t \geq 0$, represents the one-step estimator based on the observation during the time interval $[m, m+t]$.

Theorem 6.4. *Let* $0 \leq \gamma < 1/4,\ I > 0,\ q \in \mathcal{D}(\Theta)$ *as well as* $(|\partial_\theta b(\theta_0, \cdot)/\sigma|^2 - I(\theta_0)),\ (q - a(\theta_0)) \in \mathcal{M}(b(\theta_0, \cdot), \sigma)$ *for* $\theta_0 \in \Theta$. *In addition, suppose that* $b(\theta, \cdot),\ \partial_\theta \partial_x b(\theta, \cdot),\ \partial_\theta^2 \partial_x b(\theta, \cdot),\ \partial_\theta^3 \partial_x b(\theta, \cdot) \in \mathcal{P}$ *uniformly in* θ.
Then, under H_0 *and Assumption* $\mathcal{A}_0(\Theta)$, *we have*

$$\sup_{t>0} \frac{\sqrt{I(\hat{\theta}_{0,m})} \cdot t}{g_m(t)} \big|\hat{\theta}_{m,t} - \hat{\theta}_{0,m}\big| \xrightarrow{D} \sup_{0<t\leq 1} \frac{|W_t|}{t^\gamma} \quad \textit{as} \quad m \to \infty,$$

where W *represents an arbitrary Wiener process.*

Proof. Since the Fisher information I is continuous and the one-step estimator is consistent, by Slutsky's lemma it is sufficient to prove that

$$\sup_{t>0} \frac{\sqrt{I(\theta_0)} \cdot t}{g_m(t)} \big|\hat{\theta}_{m,t} - \hat{\theta}_{0,m}\big| \xrightarrow{D} \sup_{0<t\leq 1} \frac{|W_t|}{t^\gamma} \quad \text{as} \quad m \to \infty.$$

First, we approximate the processes

$$\big(at(\hat{\theta}_{m,t} - \theta_0) \colon m, t \geq 0\big), \quad \big(am(\hat{\theta}_{0,m} - \theta_0) \colon m \geq 0\big),$$

by

$$\big(\psi_{m,t}(\theta_0) \colon m, t \geq 0\big) \quad \text{and} \quad \big(\psi_{0,m}(\theta_0) \colon m \geq 0\big),$$

where $a = \sqrt{I(\theta_0)}$, as follows:

$$\sup_{t>0} \frac{1}{\log_2 t} \left| at(\hat{\theta}_{m,t} - \theta_0) - \frac{\psi_{m,t}(\theta_0)}{a} \right| = \mathcal{O}_P(1) \quad \text{as} \quad m \to \infty \tag{6.9}$$

and P-a.s.

$$\sup_{t>0} \frac{t}{m g_m(t)} \left| am(\hat{\theta}_{0,m} - \theta_0) - \frac{\psi_{0,m}(\theta_0)}{a} \right| = o(1) \quad \text{as} \quad m \to \infty.$$

Equation (6.9) is obtained by means of the Markov property of $X_{m+(\cdot)}$ using the same argumentation as in the proof of Theorem 2.6.
In the second step, we apply Lemma 3.2. In conclusion, for any $m > 0$ there exist independent Wiener processes $B^{(m)}$ and $(B_s \colon 0 \leq s \leq m)$ such that

$$\sup_{t>0} \frac{1}{g_m(t)} \left| at\big(\hat{\theta}_{m,t} - \hat{\theta}_{0,m}\big) - \Big(B_t^{(m)} - \frac{t}{m} B_m\Big) \right| = o_P(1) \quad \text{as} \quad m \to \infty.$$

Then equation (3.14) completes the proof.

□

6.3 Perspectives

6.3.1 Moving sum type procedure

A moving window procedure is determined by a chosen window size $h = h(m)$ and by the detection process

$$S_t^m \approx \big(\hat{\theta}_{m+t-h,h} - \hat{\theta}_{0,m}\big), \quad t, m \geq 0. \tag{6.10}$$

I.e., at each time t of the online observation we compare the estimator based on $(X_{m+t-s}\colon 0 \leq s \leq h)$ to the estimator based on $(X_s\colon 0 \leq s \leq m)$. If the estimator in (6.10) is constructed by means of partial sums, typically the expression "moving sum procedure" is used.
So far, it was common in the analysis of moving sum procedures to approximate the estimator process $(h(\hat{\theta}_{m+t-h,h} - \theta_0)\colon t \geq 0)$ by the process of increments $(W_{m+t} - W_{m+t-h}\colon t \geq 0)$ of a Wiener process W as $t \to \infty$. Unfortunately, it is an open problem how this approximation can be carried out for estimators which do not have the structure of partial sums.

However, we suggest a moving sum type statistic where we replace $\hat{\theta}_{m+t-h,h}$ by the weighted increment

$$\frac{(m+t)\hat{\theta}_{0,m+t} - (m+t-h)\hat{\theta}_{0,m+t-h}}{h}, \quad t, m \geq 0.$$

Subtracting $(m+t-h)\hat{\theta}_{0,m+t-h}/h$, we hope that the contribution of the observation period $[m, m+t-h]$ is compensated. We set

$$S_t^m = \frac{\sqrt{h}}{g(t/h)}\left(\frac{(m+t)\hat{\theta}_{0,m+t} - (m+t-h)\hat{\theta}_{0,m+t-h}}{h} - \hat{\theta}_{0,m}\right), \quad t, m \geq 0,$$

where g and the weighting $h^{1/2}/g$ are borrowed from Aue et al. (2009). The function g should belong to a suitable class of functions.

It is highly probable that one can derive from the strong invariance principle of Corollary 6.3 the limit distribution of $\sup_{t>0} |S_t^m|$ under H_0. One should follow Horváth et al. (2008).

6.3.2 Strong approximation under the alternative

In order to study the cumulative window procedure (see Section 1.1) as well as the proposed moving sum type procedure under the alternative, it is necessary to prove a strong approximation result for the estimator process under H_1. However, this problem does not seem to be difficult. The strong rate of the

EMM under H_1 proven in Proposition 2.17 should be applied in order to obtain, under suitable conditions, for any change-point $t^* > 0$

$$t\big(\hat{\theta}_{0,t} - \theta_1\big) - \frac{B_t}{\sqrt{I(\theta_1)}} = \mathcal{O}\left((t\log_2 t)^{1/4}(\log t)^{1/2}\right) \quad P\text{-a.s.} \quad \text{as} \quad t \to \infty$$

where θ_1 is the true parameter value after the change and B is some Wiener process.

6.3.3 Multidimensional parameter set

Under the approach of Chapters 3 and 4 it is possible to extend the change-point problem presented in Section 1.1 to a multidimensional parameter set $\Theta \in \mathbb{R}^d$, $d \geq 2$, if the existence of a starting estimator with the properties of Propositions 2.16 and 2.17 is assumed. Choosing the estimator of the method of moments to be the starting estimator, the strong rate can be proven similarly, but it might be more difficult to show the measurability. Θ should be chosen to be compact and convex.

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Stefan Mihalache