Comparison and sign preserving properties of bilaplace boundary value problems in domains with corners

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Abstract

This work is focused on the study of the Kirchhoff-Love model for thin, transversally loaded plates with corner singularities on the boundary. The former consists in finding a function $u : \overline{\Omega} \to \mathbb{R}$, where the bounded set $\Omega \subset \mathbb{R}^2$ represents the shape of the plate and $u(x)$ its vertical deflection at the point $x \in \Omega$. This makes sense since we are in the framework of linear elasticity, that is, the model assumes that no horizontal deformation takes place. The function u is found as the minimizer of the Kirchhoff energy functional in different subsets of the Sobolev space $W^{2,2}(\Omega)$, incorporating the boundary conditions. One can distinguish the following cases: *(i) clamped:* $u = |\nabla u| = 0$ on $\partial\Omega$, *(ii) hinged:* $u = 0$ on $\partial\Omega$ and *(iii) supported:* $u \geq 0$ on $\partial\Omega$. A hinged plate will additionally satisfy a set of natural boundary conditions, whereas a solution in the supported case will exist only if we assume that the load f pushes the plate down effectively; in that case a set of natural boundary conditions will be again fulfilled. It is however common within the mathematical and engineering literature to confuse the hinged and supported plates. This originates from the expectation that when pressed down, a supported plate, like a supported beam, will have a zero deflection on the boundary. Here we prove the contrary: If the domain has a corner, then a hinged plate cannot be in general a minimizer of the energy functional if we allow variations with positive boundary values. Moreover, we illustrate that a hinged plate with $C^{2,1}$ boundary satisfies a comparison principle: If $f \geq 0$ then $u > 0$ in Ω. In the last chapter we consider the problem of decoupling a clamped plate into a system of second order equations. This approach is very important for numerical procedures, since one can then use standard piecewise linear elements. We show that such a decomposition yields the correct solutions only if the domain has convex corners; when a concave corner is present then the system has no solution.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Untersuchung des Kirchhoff-Love-Modells für dünne, tranversal beladene Platten mit Eck-Singularitäten auf dem Rand. Es besteht darin, eine Funktion $u : \overline{\Omega} \to \mathbb{R}$ zu finden, so dass die beschränkte Menge $\Omega \subset \mathbb{R}^2$ die Form einer Platte und $u(x)$ deren vertikale Auslenkung in einem Punkt $x \in \Omega$ repräsentiert. Dies ist sinnvoll, da sich das Modell im Rahmen linearer Elastizität befindet, das heißt es wird angenommen, dass keine horizontale Deformation stattfindet. Die Funktion u findet sich als Minimierer des Kirchhoff-Energie-Funktionals in verschiedenen Teilmengen des Sobolev-Raumes $W^{2,2}(\Omega)$ wieder, die die Randbedingungen erfüllen. Man kann die folgenden Fälle unterscheiden: *(i) eine eingespannte Platte:* $u = |\nabla u| = 0$ auf $\partial\Omega$, *(ii) eine gelenkig gelagerte Platte:* $u = 0$ auf $\partial\Omega$ und *(iii) eine lose aufliegende Platte:* $u \geq 0$ auf $\partial\Omega$. Eine gelenkig gelagerte Platte wird zusätzlich eine Menge natürlicher Randbedingungen erfüllen, während eine Lösung im Fall einer lose aufliegenden Platte nur unter der Annahme existiert, dass die Belastung f die Platte tatsächlich herunterdrückt. Allerdings ist es in der mathematischen und Ingenieur-Literatur weit verbreitet, die Begriffe der gelenkig gelagerten und der lose aufliegenden Platte zu verwechseln. Dies kommt von der Erwartung, dass sich eine lose aufliegende Platte beim Herunterdrücken wie ein lose aufliegender Balken verhält, nämlich ohne positive Auslenkung auf dem Rand. Hier beweisen wir das Gegenteil: Hat das Gebiet eine Ecke, kann eine gelenkig gelagerte Platte im Allgemeinen kein Minimierer des Energie-Funktionals sein, wenn wir Variationen mit positiven Randwerten zulassen. Darüber hinaus zeigen wir, dass eine gelenkig gelagerte Platte mit $C^{2,1}$-Rand das folgende Vergleichsprinzip erfüllt: Falls $f \geq 0$, dann ist $u > 0$ in Ω. Im letzten Kapitel betrachten wir das Problem der Entkopplung einer eingespannten Platte in ein System von Differentialgleichungen zweiter Ordnung. Dieser Ansatz ist für numerische Abläufe sehr bedeutend, da dann stückweise lineare Elemente verwendet werden können. Wir zeigen, dass eine solche Zerlegung nur dann die richtige Lösung liefert, wenn das Gebiet konvexe Ecken hat; existiert eine konkave Ecke, so hat das System keine Lösung.

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Köln, August 2010 Athanasios Stylianou

Contents

Chapter 1

Introduction

The theory of second order elliptic boundary value problems has been significantly well developed over the past years. Many powerful tools have been constructed and used in order to prove qualitative and quantitative properties of solutions to such problems. The maximum principle (i.e. the assertion that a solution of an elliptic equation in a bounded domain will attain its maximum value on the boundary) is maybe one of the most basic tools. It can be used to obtain estimates, prove symmetry of solutions and even define certain types of them (the use of viscosity solutions for example is based on the presence of a comparison principle).

On the other hand, when one considers higher order elliptic equations, such basic tools do not exist, even for simple cases. A model operator in this case is the bilaplace operator. It arises naturally in the theory of plate bending and Stokes flow. It thus becomes reasonable to seek specific configurations where such maximum principle-type results hold true.

The main goal of this work is to address issues of sign-preserving and regularity properties of boundary value problems involving the bilaplace operator, arising from linear elasticity, in planar domains with corners. In the rest of this introduction we are going to describe the various models under consideration and the corresponding problems to be addressed.

Notational conventions. Points in $\mathbb{R}^n$ will be denoted by $x = (x_1, .., x_n)$ whereas in the planar case we also write $(x, y) \in \mathbb{R}^2$. Moreover, ds will generally denote the $(n-1)$-dimensional boundary Lebesgue measure. Differentiation will be denoted with subscripts (e.g. u_x, u_y, u_{xy}, $u_{x_1 x_2}$) or using the standard multi-index notation: for $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = \alpha_1 + ... + \alpha_n$ we define

$$\partial_\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

To avoid confusion, directional derivatives will be denoted exlusively by ∂_γ where γ is the corresponding direction. Moreover, n and τ will denote the exterior unit normal and tangent vector (counter-clockwise oriented in the planar case) respectively and the L^p and $W^{m,p}$ norms will be written $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$ whenever there is no confusion on the domain of integration.

1.1 The mathematical setting

The Kirchhoff-Love model for thin elastic plates can be considered as the Euler-Lagrange equation that arises in the following minimization problem:

> *Find* $u_0 : \Omega \to \mathbb{R}$ *in an appropriate family of functions* $\mathcal{V}$*, with*
>
> $$J_\sigma(u_0) = \min_{u \in \mathcal{V}} J_\sigma(u),$$
>
> *where*
>
> $$J_\sigma(u) := \int_\Omega \left(\tfrac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) - fu\right) dxdy.$$

Here $\Omega \subset \mathbb{R}^2$ represents the shape of the plate and $u_0(x)$ the vertical deflection at $x \in \Omega$ under the load density $f(x)$. Note that we assume that the plate exhibits no horizontal deformation and has zero thickness. The parameter σ denotes the Poisson ratio, which is defined by $\sigma := \frac{\lambda}{2(\lambda+\mu)}$. The constants λ and μ are named after G. Lamé and depend on the material. Usually $\lambda \geq 0$ and $\mu > 0$ and hence $0 \leq \sigma < \frac{1}{2}$. The minimal value of the functional corresponds to the elastic energy of the deformed plate. Introducing the boundary conditions through an appropriate set of functions $\mathcal{V}$, one models the different cases. For references concerning the above discussion and the derivation of the model see the book [17] or the survey [55].

The Poisson ratio gives a measure of the tendency of materials to expand or contract in the other directions when they are forced to expand or contract in one direction respectively. Most materials tend to expand when forcefully contracted in one direction and thus possess a positive Poisson ratio; cork has almost zero and metals are close to 0.3, whereas for some exotic foam polymers $\sigma < 0$, i.e. they can contract in all directions when they are forced to do so only in one (see [55] and references therein).

Assume that $f \in L^2(\Omega)$, the minimizer is smooth enough, and that the set $\mathcal{V}$ is open. Then one can integrate by parts the weak Euler-Lagrange equation $J'_\sigma(u;\varphi) = 0$, that is

$$\int_\Omega \Big(\Delta u \Delta\varphi + (1-\sigma)\left(2u_{xy}\varphi_{xy} - u_{xx}\varphi_{yy} - u_{yy}\varphi_{xx}\right) - f\varphi\Big)\, dxdy = 0$$

for all $\varphi \in \mathcal{V}$, to obtain, for a sufficiently smooth domain, the differential equation and the corresponding natural boundary conditions. After the aforementioned integration by parts, the equation $J'_\sigma(u;\varphi) = 0$ becomes

$$\begin{aligned} 0 \;=\; & \int_{\partial\Omega} \Big(\sigma\Delta u + (1-\sigma)\,\partial_{nn}u\Big)\,\varphi_n\,ds - \int_{\partial\Omega} \left((1-\sigma)\,\partial_{\tau\tau n}u + \partial_n\Delta u\right)\varphi\,ds \\ + \; & \int_\Omega \left(\Delta^2 u - f\right)\varphi\,dx \quad \text{for all } \varphi \in \mathcal{V} \end{aligned} \tag{1.1}$$

and one can distinguish the following cases:

- The clamped plate problem, the pure Dirichlet case, with $\mathcal{V} = W_0^{2,2}(\Omega)$ is as follows:
$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

- For the hinged plate we have $\mathcal{V} = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and the boundary value problem becomes:
$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \sigma\Delta u + (1-\sigma)\partial_{nn}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

- For the supported plate we have $\mathcal{V} = \{u \in W^{2,2}(\Omega);\ \min(u,0) \in W_0^{1,2}(\Omega)\}$. In this case we are minimizing in a closed subset of $W^{2,2}(\Omega)$ and the solution will satisfy a variational inequality. Applying local arguments (for more details see Chapter 2, Example 2.4.7) one sees that a (smooth enough) solution u should satisfy:
$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \\ \sigma\Delta u + (1-\sigma)\partial_{nn}u = 0 & \text{on } \partial\Omega, \\ \begin{array}{c} u(x) = 0 \text{ or} \\ \partial_n\left(\Delta u(x)\right) + (1-\sigma)\left(\partial_{n\tau\tau}u\right)(x) = 0 \end{array} & \text{for } x \in \partial\Omega. \end{cases} \tag{1.4}$$

- If a smooth connected subset $\Gamma \subset \partial\Omega$ is such that no physical constraints are imposed on the plate (i.e. one has a "free edge"), the natural boundary conditions that appear are
$$\sigma\Delta u + (1-\sigma)\partial_{nn}u = \partial_n\Delta u + (1-\sigma)\partial_{n\tau\tau}u = 0 \text{ on } \Gamma. \tag{1.5}$$

The last boundary condition in (1.3) and (1.4) can be rewritten as

$$\sigma\Delta u + (1-\sigma)\partial_{nn}u = \Delta u - (1-\sigma)\partial_{\tau\tau}u - (1-\sigma)\kappa\partial_n u \text{ on } \partial\Omega$$

and $\partial_{\tau\tau}u|_{\partial\Omega} = 0$, when $u|_{\partial\Omega} = 0$. The function κ is the signed curvature of the boundary (positive on strictly convex boundary parts).

1.2 Supported versus hinged boundary conditions

In the mathematical and engineering literature the (simply) supported and hinged boundary conditions are often confused. Let us differentiate once more between these two conditions:

- *hinged:* the deflection of the plate is zero on the boundary;
- *supported:* the deflection of the plate cannot become negative on the boundary.

The reason for the above confusion lies mostly in the fact that in the case of the bending of a slender beam the two conditions above are trivially identical. It is however natural that in the case of a plate one should seek an answer to the following question:

> *Does a plate which is supported at its boundary by walls of constant height and is pushed downwards, touch its supporting structure everywhere?*

It is widely accepted in the engineering literature that a rectangular supported plate will lift at the corners when pushed downwards. A rule of thumb is described by Figure 1.1. One approximates a thin plate by a configuration of 9 rigid tiles, elastically connected to each other, and supposes that the force is distributed over 12 points at the boundary. Pushed downwards by a uniformly distributed weight of size 1, the forces working on these 12 points act as depicted in Figure 1.1. That is, upward forces appear at the corners which, if the roof is not fixed to its supporting walls, will tend to move the plate upwards. See also [8, pp. 178-182].

In Chapter 3 we will show that a similar result comes out from the continuous formulation. Within the framework of the Kirchhoff-Love model, a negatively loaded, simply supported plate will exhibit bending moments, concentrated at the corners, which will force the plate to lift there.

1.3 A preferred strategy for finite element approximations

A common numerical practice to approximate solutions for higher order boundary value problems is to rewrite these equations as systems of at most second order equations (see for example [4]). The reason for this is that the weak formulation of second order equations allows the solution to be approximated by piecewise affine functions. Hence, one is able to approximate the solution by taking a sufficiently fine mesh with corresponding piecewise affine finite elements in $C^{0,1}$. Proceeding directly with the fourth order problem would need finite elements in $C^{1,1}$. Although the theory is well known for higher order finite elements, there are several reasons why piecewise affine elements are preferred. To name two: *(i)* generating appropriate

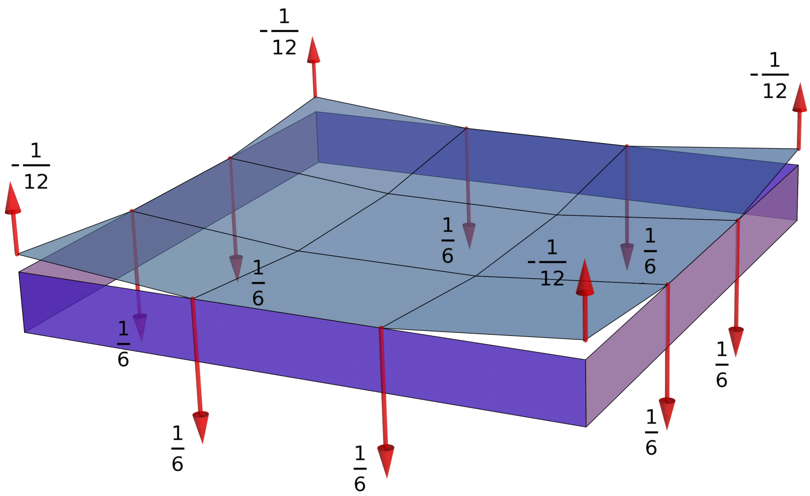

Figure 1.1: A discretized square roof with homogeneous weight distribution lifts at the corners.

higher order finite elements is much more elaborate and is not well represented in the available numerical packages; *(ii)* for elements in $C^{1,1}$ one needs more elements and also there is much larger overlap in the supports. This leads to bigger and less sparse matrices.

The aim of this work is to compare such solutions coming from decoupling a fourth order equation from linear elasticity, in planar domains with corner type singularities. As a model problem we will consider the biharmonic (or plate) equation

$$\Delta^2 u = f \text{ in } \Omega \tag{1.6}$$

with two different types of boundary conditions:

- Navier boundary conditions:

$$u = \Delta u = 0 \text{ on } \partial\Omega. \tag{1.7}$$

  On straight boundary parts and away from corners these also represent the boundary conditions satisfied by a hinged plate.

- Dirichlet boundary conditions:

$$u = \partial_n u = 0 \text{ on } \partial\Omega. \tag{1.8}$$

  These occur when the plate is clamped at the boundary.

Such so called "mixed formulations" have been extensively studied from the numerical point of view and error estimates have been proven when the solution of the boundary value problem is smooth enough. A starting point in the case of bilaplace problems was the work by Ciarlet and Raviart in [12]. Many works have since followed; see [19, 44, 45, 52]. C. Davini in [16] uses a dual mesh to produce an unconstrained method out of a mixed one, whereas in [15] a way to treat distributions over mesh-sides using only piecewise affine elements was introduced.

The analysis of the fourth order system usually starts with proving existence in $W^{2,2}(\Omega)$ and continues by using regularity results to find that the solution is smoother. For a corresponding second order system one proves the existence of a pair $(u, w) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$. In case of smooth domains one may use those regularity estimates to show that both solutions are more regular and hence coincide. In domains with corners however the different settings might each have a unique solution, but generically these solutions are different. Polygonal domains Ω are natural for the plate equation. Precisely for those domains the decomposition of a fourth order equation to a second order system, namely $-\Delta u = w$ and $-\Delta w = f$, changes the natural mathematical setting since standard regularity results for smooth domains do not usually hold in the presence of corners.

Concerning the Navier case, which is sometimes also referred to as hinged boundary conditions, we may refer to results of Nazarov et al. in [48] or to [57]. The Dirichlet problem has the additional difficulty, that, due to the boundary conditions, the resulting system of second order equations cannot be solved directly: One obtains boundary conditions only for u and none for v. This difficulty is overcome in [45] where the author studies a different kind of decomposition. However, one notices that the Shapiro-Lopatinksiĭ complementary condition is satisfied for the system and thus, one expects that the problem is well defined.

We will assume $f \in L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$ a bounded domain but would like to emphasize that even for $f \in C_0^\infty(\Omega)$ the same phenomenon can be observed. Indeed, the regularity problems do not come from the non-smoothness of f but are consequences of the nonsmooth boundary.

Chapter 2

Tools

In this chapter we are going to develop some of the theory that we are going to use later. Some tools are standard and thus briefly stated. Proofs are given when the results are less known or new.

2.1 Elliptic boundary value problems in domains with corner singularities

We will start first with a definition concerning the smoothness of the boundary of domains in $\mathbb{R}^n$. We will consider domains which have a smooth boundary with the exception of a finite number of conical points (or corners in the case of planar domains).

Definition 2.1.1 *We say that an open, connected and bounded domain $\Omega \subset \mathbb{R}^n$ is piecewise smooth (or C^k) with conical boundary singularities (or corners in the planar case) if the following hold:*

(i) There exists a set $\mathcal{S} \subset \partial\Omega$ of finite cardinality, such that $\partial\Omega \setminus \mathcal{S}$ is smooth (or C^k).

(ii) If $x_0 \in \mathcal{S}$, we assume that there exists $\varepsilon > 0$, depending on x_0, such that

$$B_\varepsilon(x_0) \cap \Omega = B_\varepsilon(x_0) \cap \mathcal{K}_{x_0}.$$

Here $\mathcal{K}_{x_0}$ is the following cone

$$\mathcal{K}_{x_0} := \left\{ x \in \mathbb{R}^n;\ 0 < |x - x_0| \text{ and } \frac{x - x_0}{|x - x_0|} \in \Omega' \right\}, \tag{2.1}$$

where $\Omega' \subset \mathbb{S}^{n-1}$ (the $(n-1)$-dimensional sphere) is open, with $\partial\Omega' \in C^\infty$. The point x_0 is called a conical point (or corner in the planar case) of $\partial\Omega$.

We generally consider problems of the type

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $\mathcal{L}$ is a differential operator of order $2l$ and $\mathcal{B} = \{\mathcal{B}_1, ..., \mathcal{B}_l\}$ a vector of boundary differential operators, each of order $m_j < 2l$:

$$\mathcal{L}u = \sum_{|\alpha|\leq 2l} a_\alpha(x)\, \partial_\alpha u \text{ and} \tag{2.3}$$

$$\mathcal{B}_j u = \sum_{|\alpha|\leq m_j} b_{j,\alpha}(x)\, \partial_\alpha u \text{ for } j = 1, ..., m, \tag{2.4}$$

where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ and $\partial_\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. For the coefficients we assume that $a_\alpha(x), b_{j,\alpha}(x) \in C^\infty(\overline{\Omega})$ for all $i = 1, ..., N$ and $j = 1, ..., l$. Moreover, for every $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ we define $\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. Important properties of the above operators are described through the following definitions.

Definition 2.1.2 *$\mathcal{L}$ is said to be properly elliptic at $x \in \overline{\Omega}$ if for all $\xi \in \mathbb{R}^n \setminus \{0\}$ holds that*

$$\sigma_{\mathcal{L}}(\xi) := \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha = \sum_{|\alpha|=2m} a_\alpha(x)\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \neq 0.$$

The form $\sigma_{\mathcal{L}}(\xi)$ is called the (principal) symbol of $\mathcal{L}$.

Definition 2.1.3 *A system $\mathcal{B} = (\mathcal{B}_1, .., \mathcal{B}_l)$ of boundary differential operators is called normal at $x \in \partial\Omega$ if one has for the orders m_j of the operators $\mathcal{B}_j$ that*

$$0 \leq m_1 < m_2 < \cdots < m_{l-1} < m_l < 2l$$

and for all $j = 1, ..., l$ that

$$\sum_{|\alpha|=m_j} b_{j,\alpha}(x) n(x)^\alpha \neq 0,$$

where $n(x)$ denotes the exterior normal vector at the point $x \in \partial\Omega$.

The complementary condition. It is also known as the *Shapiro - Lopatinskiĭ condition* and was established independently by Ya. B. Shapiro and Z. Ya. Lopatinskiĭ in 1953 (see [36, Note 4.5.1]). It concerns the transformation of problem (2.2) into a system of integral equations on $\partial\Omega$ and the normal solvability of the original boundary value problem. In order to state the condition in its algebraic form we need to make a preliminary discussion: Let $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, linearly independent, and $x \in \overline{\Omega}$. If (2.2) is elliptic, we can factorize the following polynomial for $z \in \mathbb{R}$

$$\sum_{|\alpha|=2m} a_\alpha(x)(\xi + z\eta)^\alpha = \alpha^+_{x,\xi,\eta}(z)\alpha^-_{x,\xi,\eta}(z),$$

such that the zeros $\{z_i^+\}_{i=1}^l$ of $\alpha^+_{x,\xi,\eta}$ and the zeros $\{z_i^-\}_{i=1}^l$ of $\alpha^-_{x,\xi,\eta}$ satisfy $\operatorname{Re} z_i^+ > 0$ and $\operatorname{Re} z_i^- < 0$ respectively. If $\tau(x)$ and $n(x)$ denote a tangent and the normal vector at $x \in \partial\Omega$, we define the polynomials

$$p_j(z) = p_j\left(x, \tau(x), n(x)\right) := \sum_{|\alpha|=m_j} b_{j,\alpha}(x)\left(\tau(x) + zn(x)\right)^\alpha.$$

Now we can state the condition.

Definition 2.1.4 *Problem* (2.2) *is said to satisfy the complementary condition at $x \in \partial\Omega$, if for each tangent direction $\tau(x)$ and the exterior normal vector $n(x)$, the polynomials $p_j(z)$ are linearly independent modulo $\alpha^+_{x,\tau(x),n(x)}(z)$.*

Concerning the different forms of the complementary condition, one can see [36, Definition 2.2.2, p. 40] for the case that Ω is a half-space, or [31, Definition 20.1.1, p. 233] for the case Ω being smooth.

The following definitions summarize the essential properties of problem (2.2). Let $\mathcal{S}$ denote the set of conical points of $\partial\Omega$.

Definition 2.1.5 *The boundary value problem* (2.2) *is called regular at $\overline{\Omega} \setminus \mathcal{S}$ if the following conditions hold:*

(i) $\mathcal{L}$ is properly elliptic for all $x \in \overline{\Omega} \setminus \mathcal{S}$,

(ii) $\mathcal{B}$ is a normal system of boundary operators for all $x \in \partial\Omega \setminus \mathcal{S}$ and

(iii) problem (2.2) *satisfies the complementary condition for all $x \in \partial\Omega \setminus \mathcal{S}$.*

The next definition is in line with the assumptions considered in [35].

Definition 2.1.6 *The boundary value problem* (2.2) *is called admissible if it is regular at $\overline{\Omega} \setminus \mathcal{S}$ and for all $x_0 \in \mathcal{S}$, the problem*

$$\begin{cases} \displaystyle\sum_{|\alpha|=2l} a_\alpha(x_0)\,\partial_\alpha u = \hat{f} & \text{in } \mathcal{K}_{x_0}, \\ \displaystyle\sum_{|\alpha|=m_j} b_{j,\alpha}(x_0)\,\partial_\alpha u = 0 & \text{on } \partial\mathcal{K}_{x_0}, \text{ for } j = 1, ..., m, \end{cases}$$

where

$$\hat{f}(x) := \begin{cases} f(x) & \text{in } \Omega \cap \mathcal{K}_{x_0}, \\ 0 & \text{in } \mathcal{K}_{x_0} \setminus \Omega, \end{cases}$$

satisfies the complementary condition for all $x \in \partial\mathcal{K}_{x_0} \setminus \{x_0\}$, where K_{x_0} is as in (2.1).

Next we are going to define weighted Sobolev spaces in which problems of the type (2.2) are to be solved. First we need the following: Let $\mathcal{S}$ be again the set of all conical points of $\partial\Omega$ and define

$$C_0^\infty(\overline{\Omega}\setminus\mathcal{S}) := \left\{u \in C(\overline{\Omega});\ \exists v \in C^\infty(\mathbb{R}^n):\ v_{|\overline{\Omega}} = u \text{ and } \operatorname{supp} u \subset \overline{\Omega}\setminus\mathcal{S}\right\}.$$

Definition 2.1.7 (Weighted Sobolev spaces on cones) *Let (r,θ) be a spherical coordinate system centered at $x_0 \in \mathbb{R}^n$, $m \in \mathbb{N}^n$ and $\mathcal{K}_{x_0}$ as in (2.1). Then define*

$$\|u\|_{W_\alpha^k(\mathcal{K}_{x_0})} := \left(\sum_{|m|\le k}\int_{\mathcal{K}_{x_0}} r^{\alpha-2(k-|m|)}|\partial_m u|^2\,dx\right)^{\frac{1}{2}} \tag{2.5}$$

and the following weighted space on $\mathcal{K}_{x_0}$:

$$W_\alpha^k(\mathcal{K}_{x_0}) := \overline{C_0^\infty(\overline{\mathcal{K}}_{x_0}\setminus\mathcal{S})}^{\|\cdot\|_{W_\alpha^k(\mathcal{K}_{x_0})}}. \tag{2.6}$$

Definition 2.1.8 (Weighted Sobolev spaces on domains) *Let $\mathcal{S}$ denote the set of conical points of $\partial\Omega$. For every $x \in \mathcal{S}$ we let $\varepsilon_x > 0$ be as in (ii) of Definition 2.1.1 and $\zeta_x \in C^\infty(\overline{\Omega})$ a cut-off function with*

$$\zeta_x = \begin{cases} 1 & \text{in } \overline{\Omega}\cap B_{\varepsilon_x/2}(x),\\ 0 & \text{in } \overline{\Omega}\setminus B_{\varepsilon_x}(x).\end{cases}$$

Moreover, set $\zeta = 1 - \sum_{x\in\mathcal{S}}\zeta_x$ and

$$\widehat{\zeta_x u} = \begin{cases} \zeta_x u & \text{in } \overline{\Omega}\cap\overline{\mathcal{K}}_x,\\ 0 & \text{in } \overline{\mathcal{K}}_x\setminus\left(\overline{\Omega}\cap\overline{\mathcal{K}}_x\right).\end{cases}$$

Then we define

$$W_\alpha^k(\Omega) := \overline{C_0^\infty(\overline{\Omega}\setminus\mathcal{S})}^{\|\cdot\|_{W_\alpha^k(\Omega)}}, \tag{2.7}$$

where

$$\|u\|_{W_\alpha^k(\Omega)} := \|\zeta u\|_{W^{k,2}(\Omega)} + \sum_{x\in\mathcal{S}}\|\widehat{\zeta_x u}\|_{W_\alpha^k(\mathcal{K}_x)}. \tag{2.8}$$

Definition 2.1.9 *We define $W_\alpha^{k-1/2}(\partial\Omega)$ to be the space of traces on $\partial\Omega$ of functions in $W_\alpha^k(\Omega)$ with the norm*

$$\|u\|_{W_\alpha^{k-1/2}(\partial\Omega)} := \inf_{v\in W_\alpha^k(\Omega)}\left\{\|v\|_{W_\alpha^k(\Omega)};\ v_{|\partial\Omega} = u\right\}.$$

The latter are Banach spaces with the endowed norms. Following [23, 35] one has the following embedding properties:

Proposition 2.1.10 *The following hold true:*

(i) If $\alpha - \alpha_1 < 2(k-k_1) < 0$, then $W_\alpha^k(\Omega) \hookrightarrow W_{\alpha_1}^{k_1}(\Omega)$ compactly.

(ii) $W_\alpha^k(\Omega) \hookrightarrow W^{k,2}(\Omega)$ if and only if $\alpha \le 0$.

(iii) $W^{k,2}(\Omega) \hookrightarrow W_\alpha^k(\Omega)$ if and only if $\alpha \ge 2k$.

2.1.1 Solvability in an infinite cone

We will sketch the argumentation in [35] in order to find a solution to the following problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \mathcal{K}_0, \\ \mathcal{B}u = 0 & \text{on } \partial\mathcal{K}_0, \end{cases} \tag{2.9}$$

where $\mathcal{K}_0$ is an infinite cone with its vertex at the origin (see (2.1)). We assume that (2.9) is admissible in $\mathcal{K}_0$. Let (r, θ) be a system of spherical coordinates in $\mathbb{R}^n$. Applying a change of variables with $x \to (r, \theta)$ and $r = e^{-t}$ in (2.9), we arrive at the equivalent problem

$$\begin{cases} \mathcal{L}_1 u = f_1 & \text{in } D, \\ \mathcal{B}_1 u = 0 & \text{on } \partial D, \end{cases} \tag{2.10}$$

where D is an infinite cylinder, that is, we have pushed the singularity at the origin towards $+\infty$. Note that since (2.9) admissible, that is elliptic away from the vertex of the cone, and the transformation is a diffeomorphism, problem (2.10) is elliptic in D. Now, one applies the following partial Fourier transform

$$\tilde{u}(\theta) := \mathcal{F}_t\left(u(t, \theta)\right) := \int_{-\infty}^{+\infty} e^{-i\lambda t} u(t, \theta)\, dt$$

to arrive at the problem

$$\begin{cases} \mathcal{L}_\lambda \tilde{u} = \tilde{f}_1 & \text{in } D', \\ \mathcal{B}_\lambda \tilde{u} = 0 & \text{on } \partial D', \end{cases} \tag{2.11}$$

where D' is the profile of the cylinder D. Note that $\partial D' \in C^\infty$ and then for each $\lambda \in \mathbb{C}$ problem (2.11) is regular in $\overline{D'}$.

Definition 2.1.11 *An eigenvalue of* (2.11) *is a complex number λ, such that the problem*

$$\begin{cases} \mathcal{L}_\lambda \tilde{u} = 0 & \textit{in } D', \\ \mathcal{B}_\lambda \tilde{u} = 0 & \textit{on } \partial D' \end{cases}$$

has a nonzero solution u_λ. The dimension of the space of these nonzero solutions is called the (algebraic) multiplicity of the eigenvalue λ. The nonzero, linearly independent solutions $u_{\lambda,s}$ for $s = 1, 2, ...,$ are called the eigenfunctions of (2.11).

Now one can prove the following (see [35, Theorem 1.1])

Theorem 2.1.12 *Let $k \geq 0$, $\alpha \in \mathbb{R}$ and $f \in W^k_\alpha(\mathcal{K}_0)$. Then, if there are no eigenvalues of* (2.11) *lying on the line*

$$\operatorname{Im} \lambda = \frac{2k + 4l - \alpha - n}{2},$$

problem (2.9) *has a unique solution $u \in W^{k+2l}_\alpha(\mathcal{K}_0)$ and there exists a positive constant C, such that*

$$\|u\|_{W^{k+2l}_\alpha(\mathcal{K}_0)} \leq C \|f\|_{W^k_\alpha(\mathcal{K}_0)} .$$

If extra regularity is present one can state the following

Theorem 2.1.13 *Let $k_1 \geq k$, $\alpha, \alpha_1 \in \mathbb{R}$ and $f \in W^{k_1}_{\alpha_1}(\mathcal{K}_0)$. Moreover, define the numbers*

$$h := \frac{2k + 4l - \alpha - n}{2} \quad \text{and} \quad h_1 := \frac{2k_1 + 4l - \alpha_1 - n}{2}.$$

If $h < h_1$, if there exist no eigenvalues of (2.11) on the line $\operatorname{Im} \lambda = h_1$ and if the unique solution u of (2.9) satisfies $u \in W^{k+2l}_{\alpha}(\mathcal{K}_0)$, then

$$u = \sum_{h < \operatorname{Im} \lambda < h_1} \sum_{s=0}^{M_\lambda} c_{\lambda,s} r^{-i\lambda} (\ln r)^s u_{\lambda,s} + w,$$

where

$$\|w\|_{W^{k_1+2l}_{\alpha_1}(\mathcal{K}_0)} \leq C \left(\|f\|_{W^{k_1}_{\alpha_1}(\mathcal{K}_0)} + \|u\|_{W^{k+2l}_{\alpha}(\mathcal{K}_0)} \right)$$

and w satisfies

$$\begin{cases} \mathcal{L}w = f & \text{in } \mathcal{K}_0, \\ \mathcal{B}w = 0 & \text{on } \partial\mathcal{K}_0. \end{cases}$$

The numbers M_λ are the multiplicities of the eigenvalues λ and for the eigenfunctions one obtains that $u_{\lambda,s} \in C^\infty(\overline{D'})$.

2.1.2 Solvability in bounded domains.

Now we turn to the case when Ω is such as in Definition 2.1.1. Here we cannot prove existence of solutions in a direct way, but we can guarantee that if a solution exists and posseses some extra smoothness, then it has a certain expansion near a corner. This can, however, be useful in proving nonsmoothness of solutions as we will illustrate in an example. In the following theorem we set

$$(r_{x_0}, \theta_{x_0}) := \left(|x - x_0| \, , \frac{x - x_0}{|x - x_0|} \right),$$

a spherical coordinate system centered at $x_0 \in \mathcal{S}$.

Theorem 2.1.14 *Let $h, h_1, k, k_1, \alpha, \alpha_1$ as in Theorem 2.1.13 and assume that $k \leq k_1$, $h \leq h_1$ and $f \in W^{k_1}_{\alpha_1}(\Omega)$. Moreover, assume that there are no eigenvalues of (2.11) lying on the line $\operatorname{Im} \lambda = h_1$ and that there exists a function $u \in W^{k+2l}_{\alpha}(\Omega)$ which solves (2.2). Then, for every $x_0 \in \mathcal{S}$ there exist $\varepsilon, C, C' > 0$, $c_{\lambda,s} \in \mathbb{R}$, $q \in \mathbb{Z}$, functions of the form $P_{\lambda,s,q}(t) = \sum_{i=0}^{h_1 - \operatorname{Im} \lambda} a_i(\theta_{x_0}) t^i$, where a_i are smooth functions of θ_{x_0} and $w \in W^{k_1+2l}_{\alpha_1}(\Omega)$, such that for all $x \in B_\varepsilon(x_0) \cap \Omega$ the expansion*

$$u(x) = \sum_{h < \operatorname{Im} \lambda < h_1} \sum_{s=0}^{M_\lambda} c_{\lambda,s} r_{x_0}^{-i\lambda} (\ln r_{x_0})^s P_{\lambda,s,q} \left(r_{x_0} (\ln r_{x_0})^q \right) + w(x)$$

and the following estimates hold true:

$$\|w\|_{W^{k_1+2l}_{\alpha_1}(\Omega)} \le C\left(\|f\|_{W^{k_1}_{\alpha_1}(\Omega)} + \|u\|_{W^{k+2l}_{\alpha}(\Omega)}\right),$$

$$\|\mathcal{L}w - f\|_{W^{k_1}_{\alpha_1-2}(\Omega)} \le C'\left(\|f\|_{W^{k_1}_{\alpha_1}(\Omega)} + \|u\|_{W^{k+2l}_{\alpha}(\Omega)}\right) \text{ and}$$

$$\left((r_{x_0},\theta_{x_0}) \mapsto \mathcal{L}\left\{r_{x_0}^{-i\lambda}(\ln r_{x_0})^s P_{\lambda,s,q}\left(r_{x_0}(\ln r_{x_0})^q\right)\right\}\right) \in W^{k_1}_{\alpha_1}(\Omega),$$

where M_λ is the multiplicity of the eigenvalue λ.

Proof. See [35, Theorem 3.3]. ■

Example 2.1.15 *Let Ω be an open bounded subset of $\mathbb{R}^2$ such that $(0,0) \in \partial\Omega$ and $\partial\Omega$ is smooth (locally the graph of a C^∞ function) except at $(0,0)$. For $\omega \in (0,2\pi)$ define*

$$\mathcal{K} := \{(x,y) \in \mathbb{R}^2;\ (x,y) = (r\cos\theta, r\sin\theta), r > 0, \theta \in (0,\omega)\}.$$

Concerning the form of $\partial\Omega$ at the origin, we assume that there exists $\varepsilon > 0$ such that $\Omega \cap B_\varepsilon = \mathcal{K} \cap B_\varepsilon$, where $B_\varepsilon := \{(x,y) \in \mathbb{R}^2;\ \sqrt{x^2+y^2} < \varepsilon\}$. Consider the Poisson problem for the Laplace operator:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Let $f \in L^2(\Omega)$. We claim that if there exists a unique solution $u \in W^2_2(\Omega)$ of (2.12) then it has the following expansion inside $B_{\varepsilon/2} \cap \Omega$:

$$u(r,\theta) = \sum_{0<\frac{j\pi}{\omega}<1} C_j r^{\frac{j\pi}{\omega}} \sin\frac{j\pi}{\omega}\theta + w(r,\theta),\ j \in \mathbb{Z}\setminus\{0\}$$

in polar coordinates (r,θ), where $w \in W^{2,2}(B_{\varepsilon/2}\cap\Omega)$ and C_j are some real constants.

To prove the claim, consider the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \mathcal{K}, \\ u = 0 & \text{on } \partial\mathcal{K}. \end{cases} \tag{2.13}$$

Writing the Laplacian in polar coordinates (r,θ) and setting $\tau = \ln\dfrac{1}{r}$ and $v = e^\tau u(r,\theta)$, we obtain

$$\begin{aligned} \Delta u(r,\theta) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u(r,\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial\theta^2} \\ &= e^\tau\left(\frac{\partial^2 v}{\partial\theta^2} + \frac{\partial^2 v}{\partial\tau^2}\right) \end{aligned}$$

and (2.13) becomes equivalent to

$$\begin{cases} \dfrac{\partial^2 v}{\partial \theta^2} + \dfrac{\partial^2 v}{\partial \tau^2} = 0 & \text{for } (\tau, \theta) \in \mathbb{R} \times (0, \omega), \\ v(\tau, 0) = v(\tau, \omega) = 0 & \text{for } \tau \in \mathbb{R}. \end{cases} \tag{2.14}$$

Next we apply the partial Fourier transform

$$\tilde{v}(\lambda, \theta) = \mathcal{F}_\tau [v(\tau, \theta)] := \int_{-\infty}^{\infty} e^{-i\lambda\tau} v(\tau, \theta)\, d\tau$$

to obtain the following boundary value problem

$$\begin{cases} V''(\theta) - \lambda^2 V(\theta) = 0 & \text{for } \theta \in (0, \omega), \\ V(0) = V(\omega) = 0, \end{cases} \tag{2.15}$$

where $V(\theta) = V_\lambda(\theta) := \tilde{v}(\lambda, \theta)$. *Problem (2.15) has nonzero eigenvalues only when* $Re(\lambda) = 0$. *In this case they are given by*

$$\lambda_j = \pm \frac{ij\pi}{\omega},\ j \in \mathbb{Z} \setminus \{0\} \tag{2.16}$$

with corresponding eigenfunctions

$$\Phi_j = \sin(-i\lambda_j \theta),\ j \in \mathbb{Z} \setminus \{0\}. \tag{2.17}$$

Now let $\chi \in C^\infty(\overline{\Omega})$ *be a cut-off function, such that*

$$\chi(x, y) = \begin{cases} 1, & \text{when } (x, y) \in B_{\varepsilon/2} \cap \Omega, \\ 0, & \text{when } (x, y) \in \Omega \setminus B_\varepsilon \end{cases}$$

and calculate

$$\begin{aligned} -\Delta(\chi u) &= -u\Delta\chi - \chi\Delta u - 2\nabla u \cdot \nabla\chi \\ &= -u\Delta\chi + \chi f - 2\nabla u \cdot \nabla\chi \\ &=: f_1. \end{aligned}$$

Moreover, using the elementary inequality $(a + b + c)^2 \leq 4(a^2 + b^2 + c^2)$ *and the fact that*

$$|\partial_m \chi| \leq c \frac{1}{\varepsilon^{|m|}}$$

(see [30, p. 25]), we obtain

$$\begin{aligned} \int_{B_\varepsilon \cap \Omega} |f_1|^2\, dxdy \quad &\leq \quad 4 \int_{B_\varepsilon \cap \Omega} |u\Delta\chi|^2\, dxdy + 4 \int_{B_\varepsilon \cap \Omega} |\chi f|^2\, dxdy \\ &\quad +16 \int_{B_\varepsilon \cap \Omega} |\nabla u \cdot \nabla\chi|^2\, dxdy \end{aligned}$$

$$\begin{aligned}
&= 4\int_{(B_\varepsilon\cap\Omega)\setminus B_{\varepsilon/2}} |u\Delta\chi|^2\,dxdy + 4\int_{B_\varepsilon\cap\Omega} |\chi f|^2\,dxdy \\
&\quad +16\int_{(B_\varepsilon\cap\Omega)\setminus B_{\varepsilon/2}} |\nabla u\cdot\nabla\chi|^2\,dxdy \\
&\le 4\sup_{x\in(B_\varepsilon\cap\Omega)\setminus B_{\varepsilon/2}} |\Delta\chi|^2 \int_{(B_\varepsilon\cap\Omega)\setminus B_{\varepsilon/2}} |u|^2\,dxdy \\
&\quad +4\int_{B_\varepsilon\cap\Omega} |f|^2\,dxdy \\
&\quad +16\sup_{x\in(B_\varepsilon\cap\Omega)\setminus B_{\varepsilon/2}} |\nabla\chi|^2 \int_{(B_\varepsilon\cap\Omega)\setminus B_{\varepsilon/2}} |\nabla u|^2\,dxdy \\
&\le \frac{c_1}{\varepsilon^4}\,\|u\|^2_{L^2\left((B_\varepsilon\cap\Omega)\setminus B_{\varepsilon/2}\right)} + 4\,\|f\|^2_{L^2(B_\varepsilon\cap\Omega)} \\
&\quad +\frac{c_2}{\varepsilon^2}\,\|\nabla u\|^2_{L^2\left((B_\varepsilon\cap\Omega)\setminus B_{\varepsilon/2}\right)}.
\end{aligned} \tag{2.18}$$

Since

$$\|\chi u\|_{W_2^2(\Omega)} = \|\chi u\|_{W_2^2(\Omega\cap B_\varepsilon)} \le c_3\,\|u\|_{W_2^2(\Omega\cap B_\varepsilon)},$$

that is, the second order derivatives of $u_1 := \chi u$ *are well defined in a weak sense over* $\partial B_\varepsilon \cap \Omega$*, we can extend the function* u_1 *to a function* $\widehat{u}_1 \in W_2^2(\mathcal{K})$ *in the following way:*

$$\widehat{u}_1(x,y) := \begin{cases} u_1(x,y), & \text{when } (x,y)\in B_\varepsilon\cap\Omega, \\ 0, & \text{when } (x,y)\in\mathcal{K}\setminus B_\varepsilon. \end{cases}$$

Then, $\widehat{u}_1$ *satisfies*

$$\begin{cases} -\Delta\widehat{u}_1 = \widehat{f}_1 & \text{in } \mathcal{K}, \\ \widehat{u}_1 = 0 & \text{on } \partial\mathcal{K}, \end{cases} \tag{2.19}$$

where

$$\widehat{f}_1(x,y) := \begin{cases} f_1(x,y), & \text{when } (x,y)\in B_\varepsilon\cap\Omega, \\ 0, & \text{when } (x,y)\in\mathcal{K}\setminus B_\varepsilon. \end{cases}$$

Using estimate (2.18), one sees that $\widehat{f}_1 \in W_0^0(\mathcal{K}) = L^2(\mathcal{K})$*. Now apply [35, Theorem 1.2] to problem (2.19) with* $\alpha = 2$*,* $\alpha_1 = 0$*,* $k_1 = k = 0$*,* $l = 1$ *and* $n = 2$ *to obtain that there exist nonzero constants* C_j*, such that*

$$\widehat{u}_1 = \sum_{0<\frac{j\pi}{\omega}<1} C_j r^{\frac{j\pi}{\omega}} \sin\frac{j\pi}{\omega}\theta + w,\ j\in\mathbb{Z}\setminus\{0\}, \tag{2.20}$$

where $w \in W_0^2(\mathcal{K})$*, when* $\omega \neq j\pi$*. Moreover, one has that* $W_0^2(\mathcal{K}\cap B_{\varepsilon/2}) \subset W^{2,2}(\mathcal{K}\cap B_{\varepsilon/2})$ *and since* $\widehat{u}_1|_{B_{\varepsilon/2}\cap\mathcal{K}} \equiv u$*, the proof is complete.*

Remark 2.1.16 (An example of nonsmoothness) *Let $\omega > \pi$. Then one term appears in the asymptotic sum, that is the term*

$$U_1 := r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta.$$

We calculate

$$\begin{aligned}\int_0^{\varepsilon/2} (r^{\frac{\pi}{\omega}-2})^2 r\, dr &= \int_0^{\varepsilon/2} r^{\frac{2\pi}{\omega}-3}\, dr \\ &= \frac{1}{\frac{2\pi}{\omega}-2} \int_0^{\varepsilon/2} \left(r^{\frac{2\pi}{\omega}-2}\right)' dr \\ &= \frac{\omega}{2(\pi-\omega)} \left[r^{2(\pi-\omega)}\right]_0^{\varepsilon/2} = \infty,\end{aligned}$$

since $\pi - \omega < 0$. That means that the second order derivatives of U_1 blow up in $L^2(\Omega \cap B_{\varepsilon/2})$ and thus U_1 does not belong in $W^{2,2}(\Omega \cap B_{\varepsilon/2})$.

Remark 2.1.17 *Assuming more regularity for f, one can obtain more than one eigenfunctions in the asymptotic sum. That is, if $f \in W^{k_1}_{\alpha_1}(\Omega)$, then the sum ranges over all eigenvalues λ_j of (2.15), such that*

$$0 < Im(\lambda_j) < \frac{2k_1 - \alpha_1 + 2}{2}.$$

Note, however, that the regularity of the solution depends strongly on the existence of the eigenfunctions in the asymptotic sum. If at least one (that is U_1) is present, then the regularity is governed by the regularity of U_1. If one does not wish to have any eigenfunctions in the sum, then restrictions appear due to [36, Theorem 6.6.1]. Hence, the opening angle ω will always yield the maximum possible regularity of the solution.

2.2 A Sobolev approximation theorem

In this section we are going to deal with the question of approximating a Sobolev function with zero boundary traces with a sequence of smoother ones, sharing the same trace. This is not clear by standard theorems (see for example [1, Theorem 3.22]), since the construction of the approximating sequence there does not preserve the boundary behaviour of the approximated function. To be more precise, the sequence $\{u_n\}_{n\in\mathbb{N}}$ of restrictions of functions in $C^\infty(\mathbb{R}^n)$, which is constructed as in [1, Theorem 3.22] to approximate a function $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, does not have zero traces.

In the case of planar domains with corner boundary singularities, one can consider the approach of [28, Section 1.6.2]: There exists a complete characterization of traces of Sobolev functions in terms of compatibility conditions at corners. Thus, one is able to prove the density of the corresponding trace spaces and one arrives to the following

Theorem 2.2.1 *Let $k \geq 3$ and $\Omega \subset \mathbb{R}^2$ be bounded and C^k diffeomorphic to a polygon. Then*

$$\overline{W^{k,2}(\Omega) \cap W_0^{1,2}(\Omega)}^{\|\cdot\|_{2,2}} = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Proof. It is a direct consequence of [28, Theorem 1.6.2]. ∎

Here we give an alternative approach: we use an extension operator in order to prove that smooth up to the boundary functions with zero boundary traces are dense in the space $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. The extension operator to be used is a rather simple one: the reflection. Its properties, however, make it suitable for constructing an extension of a function $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ in a bigger set $\Omega' \supset \Omega$, such that its mollification will yield the desired approximation sequence.

This method has its advantages and drawbacks. It can be used when $\Omega \subset \mathbb{R}^n$ is C^k diffeomorphic to a convex polytope, but it is inapplicable when a concave boundary singularity is present.

Define the following hyperoctant-type sets for $j = 0, ..., n$ by

$$S = S_0 := \{x = (x_1, ..., x_n) \in \mathbb{R}^n;\ x_1, ..., x_n > 0\}$$

(the positive cone in $\mathbb{R}^n$) and

$$S_j := \left(\overline{S_{j-1} \cup \left\{ (x_1, ..., x_{j-1}, -x_j, x_{j+1}, ..., x_n) \in \mathbb{R}^n;\ (x_1, ..., x_n) \in S_{j-1} \right\}}\right)^\circ$$

that is, S_j is the set that consists of S_{j-1} and its reflection over the hyperplane $\{x_j = 0\}$. Note that $S_n = \mathbb{R}^n$. Moreover, define the corresponding reflection operators

$$(E_j u)(x) := \begin{cases} u(x), & \text{for } x \in S_{j-1}, \\ -u(x_1, ..., x_{j-1}, -x_j, x_{j+1}, ..., x_n), & \text{for } x \in S_j \setminus S_{j-1} \end{cases} \tag{2.21}$$

for any function $u : \overline{S}_{j-1} \to \mathbb{R}$. A straightforward calculation yields that

$$E_j : C^1(\overline{S}_{j-1}) \cap C_0(\overline{S}_{j-1}) \to C^1(\overline{S}_j) \cap C_0(\overline{S}_j)$$

is continuous and, moreover, one has the following

Lemma 2.2.2 *The operators*

$$E_j : W^{2,2}(S_{j-1}) \cap W_0^{1,2}(S_{j-1}) \longrightarrow W^{2,2}(S_j) \cap W_0^{1,2}(S_j)$$

are bounded and linear.

Proof. Without loss of generality we will prove the claim for the operator

$$(E_1 u)(x) := \begin{cases} u(x), & \text{when } x \in S, \\ -u(-x_1, x_2, ..., x_n), & \text{when } x \in S_1 \setminus S. \end{cases} \tag{2.22}$$

The proof for the rest of the E_j's is done similarly. It suffices to show that if $u \in W^{2,2}(S) \cap W_0^{1,2}(S)$, then all weak derivatives of $E_1 u$ up to order 2 belong in $L^2(S_1)$, which will be straightforward after integration by parts and use of the boundary behaviour of the functions in the domain. Continuity is then straightforward. Define $u_1 := E_1 u$ and

$$v_j(x) := \begin{cases} \partial_{x_j} u(x), & \text{when } x \in S, \\ \partial_{x_j} \left(-u(-x_1, x_2, ..., x_n)\right), & \text{when } x \in S_1 \setminus S \end{cases}$$

for $j = 1, ..., n$, to obtain that $v_j \in L^2(\Omega)$. Let $j \neq 1$, $\varphi \in C_0^\infty(S_1)$ and calculate

$$\begin{aligned}
\int_{S_1} u_1 \partial_{x_j} \varphi \, dx = & \int_{(\mathbb{R}^+)^{n-2}} \left\{ \int_{-\infty}^0 \left(\int_0^\infty -u(-x_1, x_2, ..., x_n) \partial_{x_j} \varphi \, dx_j \right) dx_1 \right\} d\bar{x} \\
& + \int_{(\mathbb{R}^+)^{n-2}} \left\{ \int_0^\infty \left(\int_0^\infty u \, \partial_{x_j} \varphi \, dx_j \right) dx_1 \right\} d\bar{x} \\
= & \int_{(\mathbb{R}^+)^{n-2}} \left(\int_{-\infty}^0 [-u(-x_1, x_2, ..., x_n)\varphi]_{x_j=0}^\infty \, dx_1 \right) d\bar{x} \\
& - \int_{(\mathbb{R}^+)^{n-1}} \left(\int_{-\infty}^0 \partial_{x_j} \left(-u(-x_1, x_2, ..., x_n)\right) \varphi \, dx_1 \right) dx_2 ... dx_n \\
& + \int_{(\mathbb{R}^+)^{n-1}} [u \, \varphi]_{x_j=0}^\infty \, dx_1 d\bar{x} \\
& - \int_{(\mathbb{R}^+)^{n-1}} \left(\int_0^\infty \partial_{x_j} u \, \varphi \, dx_1 \right) dx_2 ... dx_n \\
= & \int_{(\mathbb{R}^+)^{n-2}} \left(\int_{-\infty}^0 \left(u(-x_1, x_2, ..., x_n)\varphi\right)_{|x_j=0} \, dx_1 \right) d\bar{x} \\
& - \int_{(\mathbb{R}^+)^{n-1}} (u \, \varphi)_{|x_j=0} \, dx_1 d\bar{x} - \int_{S_1} v_j \varphi \, dx,
\end{aligned} \tag{2.23}$$

where $d\bar{x} := dx_2...dx_{j-1}dx_{j+1}...dx_n$. One has that

$$u(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) = 0$$

for all $j = 1, .., n$ almost everywhere and thus

$$\begin{aligned}\int_{(\mathbb{R}^+)^{n-1}} (u\,\varphi)_{|x_j=0}\, dx_1 d\bar{x} &= \int_{(\mathbb{R}^+)^{n-2}} \left(\int_{-\infty}^{0} \big(u(-x_1, x_2, ..., x_n)\varphi\big)_{|x_j=0}\, dx_1 \right) d\bar{x} \\ &= 0\end{aligned}$$

for all $j = 2, .., n$. On the other hand, when $j = 1$ the calculation is similar, that is

$$\begin{aligned}\int_{\mathbb{R}^n} u_1 \partial_{x_1}\varphi\, dx &= \int_{(\mathbb{R}^+)^{n-1}} \left(\int_{-\infty}^{0} -u(-x_1, x_2, ..., x_n)\partial_{x_1}\varphi\, dx_1 \right) dx_2...dx_n \\ &\quad + \int_{(\mathbb{R}^+)^{n-1}} \left(\int_{0}^{\infty} u\,\partial_{x_1}\varphi\, dx_1 \right) dx_2...dx_n \\ &= \int_{(\mathbb{R}^+)^{n-1}} [-u(-x_1, x_2, ..., x_n)\varphi]_{x_1=-\infty}^{0}\, dx_2...dx_n \\ &\quad - \int_{(\mathbb{R}^+)^{n-1}} \left(\int_{-\infty}^{0} \partial_{x_1}\left(-u(-x_1, x_2, ..., x_n)\right)\varphi\, dx_1 \right) dx_2...dx_n \\ &\quad + \int_{(\mathbb{R}^+)^{n-1}} [u\,\varphi]_{x_1=0}^{\infty}\, dx_2...dx_n \\ &\quad - \int_{(\mathbb{R}^+)^{n-1}} \left(\int_{0}^{\infty} \partial_{x_1} u\,\varphi\, dx_1 \right) dx_2...dx_n \\ &= \int_{(\mathbb{R}^+)^{n-1}} \big(-u(-x_1, x_2, ..., x_n)\varphi\big)_{|x_1=0}\, dx_2...dx_n \\ &\quad - \int_{(\mathbb{R}^+)^{n-1}} (u\,\varphi)_{|x_1=0}\, dx_2...dx_n - \int_{S_1} v_1\varphi\, dx. \qquad (2.24)\end{aligned}$$

Since $u_{|x_1=0} = 0$ almost everywhere, we have $v_j(x) = \partial_{x_j}u_1(x)$ in the sense of distributions, that is $\partial_{x_j}u \in L^2(S_1)$ for all $j = 1, ..., n$. In the same spirit, let $i, j = 1, ..., n$ and define

$$v_{ij}(x) := \begin{cases} \partial_{x_i x_j} u(x), & \text{when } x \in S, \\ \partial_{x_i x_j}\big(-u(-x_1, x_2, ..., x_n)\big), & \text{when } x \in S_1 \setminus S. \end{cases} \qquad (2.25)$$

We claim that $\partial_{x_i x_j} u_1 = v_{ij}$. To that end we consider the following cases:

(1) $i \neq j$ and $j \neq 1$: Similarly to (2.23) we obtain

$$\begin{aligned}\int_{S_1} \partial_{x_i}u_1\, \partial_{x_j}\varphi\, dx &= \int_{(\mathbb{R}^+)^{n-2}} \left(\int_{-\infty}^{0} \big(\partial_{x_i}\left(-u(-x_1, x_2, ..., x_n)\right)\varphi\big)_{|x_j=0}\, dx_1 \right) d\bar{x} \\ &\quad - \int_{(\mathbb{R}^+)^{n-1}} (\partial_{x_i}u\,\varphi)_{|x_j=0}\, dx_1 d\bar{x} - \int_{S_1} v_{ij}\varphi\, dx. \qquad (2.26)\end{aligned}$$

Since $u(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) = 0$ almost everywhere,

$$\partial_{x_i} u(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) = 0 \quad \text{for all} \quad i = 1, ..., n \tag{2.27}$$

almost everywhere and the claim follows.

<u>*(2)* $i = j \neq 1$</u> : Here the integration by parts yields

$$\begin{aligned}\int_{S_1} \partial_{x_j} u_1 \, \partial_{x_j} \varphi \, dx = & \int_{(\mathbb{R}^+)^{n-2}} \left(\int_{-\infty}^{0} \left((\partial_{x_j} u)(-x_1, x_2, ..., x_n) \varphi \right)_{|x_j=0} dx_1 \right) d\bar{x} \\ & - \int_{(\mathbb{R}^+)^{n-1}} \left(\partial_{x_j} u \, \varphi \right)_{|x_j=0} dx_1 d\bar{x} - \int_{S_1} v_{ij} \varphi \, dx \end{aligned} \tag{2.28}$$

and the claim follows since $\phi \in C_0^\infty(S_1)$ imlies that $\phi_{|x_j=0} = 0$.

<u>*(3)* $i = j = 1$</u> : The calculation goes similar to (2.24), that is

$$\begin{aligned}\int_{S_1} \partial_{x_1} u_1 \, \partial_{x_1} \varphi \, dx = & \int_{(\mathbb{R}^+)^{n-1}} \Big(\partial_{x_1} \big(-u(-x_1, x_2, ..., x_n) \big) \varphi \Big)_{|x_1=0} dx_2 ... dx_n \\ & - \int_{(\mathbb{R}^+)^{n-1}} \left(\partial_{x_1} u \, \varphi \right)_{|x_1=0} dx_2 ... dx_n - \int_{S_1} v_{ij} \varphi \, dx \\ = & - \int_{S_1} v_{11} \varphi \, dx, \end{aligned} \tag{2.29}$$

since the boundary terms cancel each other.

<u>*(4)* $i \neq j$ *and* $j = 1$</u> : As in the previous case

$$\begin{aligned}\int_{S_1} \partial_{x_i} u_1 \, \partial_{x_1} \varphi \, dx \quad = & \quad \int_{(\mathbb{R}^+)^{n-1}} \big(-(\partial_{x_i} u)(-x_1, x_2, ..., x_n) \varphi \big)_{|x_1=0} dx_2 ... dx_n \\ & - \int_{(\mathbb{R}^+)^{n-1}} \left(\partial_{x_i} u \, \varphi \right)_{|x_1=0} dx_2 ... dx_n - \int_{S_1} v_{ij} \varphi \, dx \\ = & \quad - \int_{S_1} v_{11} \varphi \, dx, \end{aligned} \tag{2.30}$$

since $u(0, x_2, ..., x_n) = 0$ almost everywhere implies that

$$(\partial_{x_i} u)(0, x_2, ..., x_n) = 0 \quad \text{for all} \quad i = 2, ..., n.$$

Thus the claim is proved for all cases. ■

Corollary 2.2.3 *For* $j = 0, .., n-1$, *the operator*

$$\tilde{E}_j := E_n \circ ... \circ E_{j+1} : W^{2,2}(S_j) \cap W_0^{1,2}(S_j) \to W^{2,2}(\mathbb{R}^n) \cap W_0^{1,2}(\mathbb{R}^n)$$

is continuous.

Next we move on to construct the approximation sequence. Define

$$C_0(\overline{\Omega}) := \left\{ u \in C(\overline{\Omega}) \;\; \text{with} \;\; u_{|\partial\Omega} = 0 \right\}.$$

Theorem 2.2.4 *Let $\Omega \subset \mathbb{R}^n$ be bounded and open such that for every $x \in \partial\Omega$ there exist a $j \in \{0, ..., n-1\}$, $\varepsilon > 0$ and a C^k diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ with $k \geq 2$, such that the following hold:*

(i) $\Phi(x) = 0$,

(ii) $\Phi\left(B_\varepsilon(x) \cap \Omega\right) \subset S_j$ *and*

(iii) $\Phi\left(B_\varepsilon(x) \cap \partial\Omega\right) \subset \partial S_j$,

where the sets S_j were defined in the begining of this subsection. Then

$$\overline{C^k(\overline{\Omega}) \cap C_0(\overline{\Omega})}^{\|\cdot\|_{2,2}} = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

Remark 2.2.5 *The assumptions imposed on Ω imply that $\partial\Omega \in C^k$ with the exception of a finite number of points where it is C^k diffeomorphic to one of the S_j's. Indeed, since $\partial\Omega$ is compact, one needs only a finite number of diffeomorphisms to draw the boundary. Moreover, $\partial\Omega$ is Lipschitz and this allows us, using Stein's extension Theorem (see [1]), to identify a function $u \in C^k(\overline{\Omega})$ with its smooth extension in $\mathbb{R}^n$, i.e.*

$$C^k(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega});\; \exists v \in C^k(\mathbb{R}^n), \;\; \textit{such that } u = v_{|\overline{\Omega}} \right\}.$$

Remark 2.2.6 *In case of planar domains, the prerequisites of the previous Theorem imply that the only singularities that can occur on $\partial\Omega$ are convex corners.*

Proof of Theorem 2.2.4. Let $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and fix $x \in \partial\Omega$ and its corresponding Φ, S_j and $\varepsilon > 0$ according to the assumptions of the Theorem. Since Φ is a diffeomorphism, there exists $\varepsilon' > 0$, such that $\Phi^{-1}\left(B_{\varepsilon'}(0) \cap S_j\right) \subset B_\varepsilon(x) \cap \Omega$. For $\rho > 0$ we define the standard mollifier in $\mathbb{R}^n$, that is $\eta_\rho(x) = \rho^{-n}\eta(x/\rho)$, where

$$\eta(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}}, & \text{when } |x| < 1, \\ 0, & \text{when } |x| \geq 1. \end{cases}$$

Define the coordinate transformation operator

$$\mathbf{\Phi} : W^{2,2}\left(\Phi^{-1}\left(B_{\varepsilon'}(0) \cap S_j\right)\right) \longrightarrow W^{2,2}\left(B_{\varepsilon'}(0) \cap S_j\right)$$

with $\mathbf{\Phi}u = \left(\mathbf{\Phi}u\right)(y) = u\left(\Phi^{-1}(y)\right)$. Following [1, Theorem 3.41], one has that $\mathbf{\Phi}$ is well defined and bounded with a bounded inverse $\mathbf{\Phi}^{-1}u = \left(\mathbf{\Phi}^{-1}u\right)(x) = u\left(\Phi(x)\right)$.

Now apply the $\tilde{E}_j$ extension operator defined in Corollary 2.2.3 to obtain a function $\tilde{E}_j\,\mathbf{\Phi}u \in W^{2,2}\left(B_{\varepsilon'}(0)\right)$, symmetric with respect to the hyperplanes

$$\left\{x \in \mathbb{R}^n;\ x_{n-i} = 0\right\} \quad \text{for} \quad i = 0, .., n-j.$$

This implies that for ρ small enough

$$\eta_\rho * (\tilde{E}_j\,\mathbf{\Phi}u)_{|B_{\varepsilon'}(0)\cap\partial S_j} = 0, \tag{2.31}$$

since η_ρ is radially symmetric. Now we define

$$u_\rho := \mathbf{\Phi}^{-1}\left(\eta_\rho * (\tilde{E}_j\,\mathbf{\Phi}u)\right) \tag{2.32}$$

and claim that there exists $\varepsilon'' > 0$, such that:

(i) $u_\rho \in C^k\left(\overline{B_{\varepsilon''}(x) \cap \Omega}\right)$,

(ii) $u_{\rho|B_{\varepsilon''}(x)\cap\partial\Omega} = 0$ and

(iii) $\|u_\rho - u\|_{W^{2,2}(B_{\varepsilon''}(x)\cap\Omega)} \to 0$ for $\rho \to 0$.

Assuming that the claim holds and since $\partial\Omega$ is compact, one obtains for $i = 1, .., N$, $x_i \in \partial\Omega$ and the corresponding $\varepsilon''_i > 0$, that $\partial\Omega \subset \bigcup_{i=1}^N B_{\varepsilon''_i}(x_i)$ and that there exists $\Omega' \subset\subset \Omega$ such that $\Omega \subset \Omega' \cup \left(\bigcup_{i=1}^N B_{\varepsilon''_i}(x_i)\right)$. Now let $\{\zeta_i\}_{i=1}^N$ be a smooth partition of unity subordinate to the covering $\left\{\Omega', \left\{B_{\varepsilon''_i}(x_i)\right\}_{i=1}^N\right\}$ and define $v_\rho = \sum_{i=1}^N \zeta_i u_{\rho,i}$, where

$$u_{\rho,i} := \mathbf{\Phi}_i^{-1}\left(\eta_\rho * (\tilde{E}_j^i\,\mathbf{\Phi_i}u)\right)$$

and $\mathbf{\Phi}_i$, $\tilde{E}_j^i$ are the coordinate transformation and extension operators corresponding to x_i. Then, by standard argumentation, *(i)* and *(iii)* yield that $v_\rho \in C^k(\overline{\Omega})$ and $\|v_\rho - u\|_{W^{2,2}(\Omega)} \to 0$ as $\rho \to 0$, whereas *(ii)* implies that $v_\rho \in C_0(\overline{\Omega})$ which yields the result.

Proof of the claim: (i) Since $\tilde{E}_j\,\mathbf{\Phi}u \in W^{2,2}\left(B_{\varepsilon'}(0)\right)$, we obtain that $\eta_\rho * (\tilde{E}_j\,\mathbf{\Phi}u) \in C^k\left(\overline{B_{\varepsilon'}(0)}\right)$. Moreover, Φ is a C^k diffeomorphism which implies that there exists $\varepsilon'' > 0$ with $\varepsilon'' < \varepsilon'$, such that $\Phi\left(B_{\varepsilon''}(x)\right) \subset B_{\varepsilon'}(0)$ and more specifically $\Phi\left(B_{\varepsilon''}(x) \cap \Omega\right) \subset B_{\varepsilon'}(0) \cap S_j$. Thus the function $u_\rho : \overline{B_{\varepsilon''}(x) \cap \Omega} \to \mathbb{R}$ is well defined and the C^k continuity follows immediately since it is a composition of C^k functions.

(ii) It follows from (2.31) and the fact that $\Phi\left(B_{\varepsilon''}(x) \cap \partial\Omega\right) \subset B_{\varepsilon'}(0) \cap \partial S_j$.

(iii) We have that

$$\eta_\rho * (\tilde{E}_j\,\mathbf{\Phi}u) \to \tilde{E}_j\,\mathbf{\Phi}u \ \text{ strongly in } \ W^{2,2}\left(\Phi\left(B_{\varepsilon''}(x) \cap \Omega\right)\right).$$

Since both $\mathbf{\Phi}, \mathbf{\Phi}^{-1}$ are linear and bounded, they are continuous which implies that

$$\mathbf{\Phi}^{-1}\left(\eta_\rho * (E_j\, \mathbf{\Phi} u)\right) \to \mathbf{\Phi}^{-1}\big(E_j\, \mathbf{\Phi} u\big)$$

strongly in $W^{2,2}\big(B_{\varepsilon''}(x) \cap \Omega\big)$. However, one has that $E_j\, \mathbf{\Phi} u_{|B_{\varepsilon''}(x)\cap\Omega} = \mathbf{\Phi} u_{|B_{\varepsilon''}(x)\cap\Omega}$ and the claim follows. ■

2.3 Norm Inequalities

Since weak solutions for fourth order equations lie in $W^{2,2}$, one has a number of different norms to work with. The equivalence of these is generally depending on the smoothness of the boundary. In this section we prove some of these equivalences.

2.3.1 A higher order Poincaré type inequality

It is well known that for a bounded domain Ω it holds that $\|u\|_{L^2(\Omega)} \le c_\Omega \,\||\nabla u|\|_{L^2(\Omega)}$ for $u \in W_0^{1,2}(\Omega)$, but it is less known that $\||\nabla u|\|_{L^2(\Omega)} \le c_\Omega \left\||\nabla^2 u|\right\|_{L^2(\Omega)}$ for $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. We will, thus, include a proof. Consider $x = (x_1, ..., x_n) \in \Omega$ and write dx and ds for the n-D Lebesgue and $(n-1)$-D surface measure.

Define the following semi-norm

$$|u|_{2,2} := \left\||\nabla^2 u|\right\|_2 ,$$

where $\nabla^2 u$ denotes the Hessian matrix of u.

Lemma 2.3.1 *Let $\Omega \subset \mathbb{R}^n$ be a simply connected, open and bounded domain. Then there exists a constant $c > 0$, such that*

$$\int_\Omega |\nabla u(x)|^2 \, dx \le c \ |u|_{2,2}^2 \quad \textit{for all } u \in W^{2,2}(\Omega) \cap C^2(\Omega) \cap C_0(\overline{\Omega}).$$

Proof. First we consider the 1D case and assume that $u \in W^{2,2}(0,l) \cap C^2(0,l) \cap C_0[0,l]$. Since $u(0) = u(l) = 0$, the mean value Theorem shows that there exists $x_0 \in (0,l)$, such that $u'(x_0) = 0$. Using twice Hölder's inequality, one has that

$$\begin{aligned}
\int_{x_0}^{l} \left|u'(x)\right|^2 dx &= \int_{x_0}^{l} \left|\int_{x_0}^{x} u''(t)\,dt\right|^2 dx \ \le\ \int_{x_0}^{l} \left(\int_{x_0}^{x} |u''(t)|\,dt\right)^2 dx \\
&\le \int_{x_0}^{l} \left[\left(\int_{x_0}^{x} \left|u''(t)\right|^2 dt\right)\left(\int_{x_0}^{x} dt\right)\right] dx \\
&\le \left(\int_{x_0}^{l} \left|u''(t)\right|^2 dt\right) \int_{x_0}^{l} (x - x_0)\,dx \\
&= \frac{1}{2}\,(l - x_0)^2 \int_{x_0}^{l} \left|u''(t)\right|^2 dt.
\end{aligned}$$

Since $\int_0^l \left|u'(x)\right|^2 dx = \int_0^{x_0} \left|u'(x)\right|^2 dx + \int_{x_0}^{l} \left|u'(x)\right|^2 dx$, one obtains the desired result.

In the $n-$dimensional case we will proceed using Fubini's Theorem and the above result for one variable at a time, that is

$$\int_\Omega |\nabla u|^2 \, dx \ = \ \int_{x_1}\int_{x_2} ... \int_{x_n} |\nabla u(x_1, x_2, ..., x_n)|^2 \, dx_n ... dx_2 dx_1$$

$$
\begin{aligned}
&= \int_{x_1}\int_{x_2}...\int_{x_n}\left(|u_{x_1}|^2+|u_{x_2}|^2+...+|u_{x_n}|^2\right)dx_n...dx_2dx_1 \\
&\leq c\int_{x_1}\int_{x_2}...\int_{x_n}\left(|u_{x_1x_1}|^2+...+|u_{x_nx_n}|^2\right)dx_n...dx_2dx_1 \\
&\leq c\,|u|_{2,2}^2 .
\end{aligned}
$$

Thus the assertion is proved. ■

Remark 2.3.2 *A related result will hold for functions with nonzero boundary conditions. Let Ω be a simply connected, bounded domain of $\mathbb{R}^n$. Then there exists $c_1 > 0$, such that*

$$\int_\Omega |\nabla u|^2\,dx \leq c_1\left(|u|_{2,2}^2+\int_{\partial\Omega}|\nabla u|^2\,ds\right)$$

for all $u \in W^{2,2}(\Omega)\cap C^2(\Omega)\cap C^1(\overline{\Omega})$.

Based on a remark on a theorem of Meyers and Serrin (see [1, Theorem 3.17]), stated in [25], one can prove the following

Proposition 2.3.3 *Let $\Omega \subset \mathbb{R}^n$ be bounded, with Lipschitz boundary and define $C_0^k(\overline{\Omega})$ as the space of k-differentiable functions whose derivatives up to order k are zero on $\partial\Omega$. Then, for $m \geq k+1$ it holds that*

$$\overline{W^{m,p}(\Omega)\cap C^\infty(\Omega)\cap C_0^k(\overline{\Omega})}^{\|\cdot\|_{m,p}} = W^{m,p}(\Omega)\cap W_0^{k+1,p}(\Omega).$$

Proof. For consistency reasons we will first outline the proof given by Meyers and Serrin. Let $k \in \mathbb{N}$ and define

$$\Omega_k := \left\{x\in\Omega;\ |x|<k \text{ and } d(x,\partial\Omega)>\frac{1}{k}\right\},$$

$$\Omega_{-1}=\Omega_0:=\emptyset.$$

Setting $U_k := \Omega_{k+1}\cap\left(\overline{\Omega_{k-1}}\right)^c$ one sees that $\{U_k\}_{k\in\mathbb{N}}$ is an open covering of Ω, so there exists a partition of unity Ψ subordinate to $\{U_k\}_{k\in\mathbb{N}}$. Moreover, let ψ_k be the sum of the finitely many functions $\psi\in\Psi$ with support in U_k and η_ε the standard mollifier. Fixing

$$0<\varepsilon<\frac{1}{(k+1)(k+2)},$$

one can see that $\eta_\varepsilon*(\psi_k u)$ has support in $V_k := \Omega_{k+2}\cap(\Omega_{k-2})^c \subset\subset \Omega$. Choose ε_k small enough, such that

$$\|\eta_{\varepsilon_k}*(\psi_k u)-\psi_k u\|_{W^{m,p}(\Omega)} = \|\eta_{\varepsilon_k}*(\psi_k u)-\psi_k u\|_{W^{m,p}(V_k)} < \frac{\varepsilon}{2^k} \tag{2.33}$$

and set

$$\phi_\varepsilon := \sum_{k=1}^{\infty} \eta_{\varepsilon_k} * (\psi_k u).$$

Then $\phi_\varepsilon \in C^\infty(\Omega)$ and, since ε is independent of V_k, one obtains that $\|\phi_\varepsilon - u\|_{m,p,\Omega} < \varepsilon$.

Following [25, Remark 1.18, p. 16] we consider $\delta > 0$, $\rho > 0$, $x_0 \in \partial\Omega$ and $k_0 = \left\lceil \frac{1}{\rho} \right\rceil - 2$. Then one has that

$$\phi_\delta(x) - u(x) = \sum_{k=k_0}^{\infty} \Big(\eta_{\delta_k} * \big(\psi_k(x) u(x)\big) - \psi_k(x) u(x) \Big)$$

for all $x \in B_\rho(x_0) \cap \Omega$. Now, estimate (2.33) yields

$$\begin{aligned} \|u - \phi_\delta\|_{W^{m,p}\left(B_\rho(x_0)\cap\Omega\right)} &\leq \sum_{k=k_0}^{\infty} \|J_{\delta_k} * (\psi_k u) - \psi_k u\|_{W^{m,p}\left(B_\rho(x_0)\cap\Omega\right)} \\ &\leq \delta \sum_{k=k_0}^{\infty} \frac{1}{2^k} \\ &\leq 2^{-\frac{1}{\rho}} 8\delta. \end{aligned}$$

Assuming that the norm on the left is not identically zero, there exist constants $c(\delta), \rho(\delta) > 0$, such that for all positive $\rho < \rho(\delta)$ one has

$$c(\delta)\rho^2 \leq \|u - \phi_\delta\|_{W^{m,p}\left(B_\rho(x_0)\cap\Omega\right)} \leq 8\delta\, 2^{-\frac{1}{\rho}},$$

which cannot be true. Thus for ρ small enough

$$\|u - \phi_\delta\|_{W^{m,p}\left(B_\rho(x_0)\cap\Omega\right)} = 0,$$

which together with Lebesgue's differentation Theorem implies that the traces of the $m-1$ order derivatives of the approximating sequence φ_δ agree in an L^p-sense with the ones of u (which are well defined since $\partial\Omega$ is Lipschitz) and the claim is proved. ■

Combining Lemma 2.3.1 and Proposition 2.3.3, we obtain the desired result.

Corollary 2.3.4 *Let $\Omega \subset \mathbb{R}^n$ be bounded with a Lipschitz boundary. Then $|\cdot|_{2,2}$ and $\|\cdot\|_{2,2}$ are equivalent norms on $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.*

2.3.2 The "second fundamental inequality"

We would like to consider the cases when the Laplacian of a function is a norm. The matter is not simple and is tightly related to uniqueness and regularity of solutions for Poisson's equation. The "smooth" version of the result needs no convexity assumption on the domain, no matter what the dimension is. For simplicity, we give here only the planar case:

Lemma 2.3.5 *Let $\Omega \subset \mathbb{R}^2$ be bounded and piecewise smooth with the exception of a finite number of corners. Then there exists a positive constant C such that*

$$\|u\|_{2,2} \leq C\,\|\Delta u\|_2 \ \textit{ for all } u \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega). \tag{2.34}$$

Proof. First assume that $u \in W^{3,2}(\Omega)$ and integrate by parts

$$\begin{aligned}\int_\Omega u_{xx}u_{yy}\,dxdy &= \int_{\partial\Omega}(u_{xx}u_yn_2 - u_{xy}u_yn_1)ds + \int_\Omega u_{xy}^2\,dxdy \\ &= \int_{\partial\Omega}(u_xu_{yy}n_1 - u_xu_{xy}n_2)ds + \int_\Omega u_{xy}^2\,dxdy.\end{aligned}$$

Since $u = 0$ on $\partial\Omega$, one obtains that $u_x|_{\partial\Omega} = n_1\partial_n u$ and $u_y|_{\partial\Omega} = n_2\partial_n u$ and we calculate

$$\begin{aligned}\int_\Omega (\Delta u)^2\,dxdy &= \int_\Omega \left(u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2\right)\,dxdy \\ &= \int_{\partial\Omega}(u_{xx}u_yn_2 + u_{yy}u_xn_1 - u_{xy}u_xn_2 - u_{xy}u_yn_1)\,ds \\ &\quad + \left\||\nabla^2 u|\right\|_2^2 \\ &= \int_{\partial\Omega}\partial_n u\ \left(u_{xx}n_2^2 + u_{yy}n_1^2 - 2u_{xy}n_1n_2\right)ds + \left\||\nabla^2 u|\right\|_2^2 \\ &= \int_{\partial\Omega}\partial_n u\ \left(\Delta u - u_{xx}n_1^2 - u_{yy}n_2^2 - 2u_{xy}n_1n_2\right)ds \\ &\quad + \left\||\nabla^2 u|\right\|_2^2 \\ &= \int_{\partial\Omega}\kappa(s)\ (\partial_n u)^2\ ds + \left\||\nabla^2 u|\right\|_2^2,\end{aligned}$$

where we have taken into consideration the fact that $u_{nn} = n_1^2u_{xx} + n_2^2u_{yy} + 2n_1n_2u_{x,y}$ and $\Delta u = \partial_{nn}u + \partial_{\tau\tau}u + \kappa(s)\partial_n u$ (see [54]), that is we have obtained

$$\int_\Omega (\Delta u)^2\,dxdy = \int_{\partial\Omega}\kappa(s)\ (\partial_n u)^2\ ds + \left\||\nabla^2 u|\right\|_2^2. \tag{2.35}$$

Define $\underline{\kappa} := \operatorname{ess\,inf}_{s\in\partial\Omega}\kappa(s)$. Note that because of the assumption on the smoothness of the domain $\underline{\kappa} > -\infty$. If the latter is convex, that is $\underline{\kappa} \geq 0$, then one obtains immediately that

$$\|\Delta u\|_2^2 \geq \left\||\nabla^2 u|\right\|_2^2.$$

Assume now that $\underline{\kappa} < 0$. We claim that for all $\varepsilon > 0$ and $u \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$ the following inequality holds

$$\int_{\partial\Omega} (\partial_n u)^2 \, ds \leq C \left(\varepsilon^{\frac{4}{3}} \left\| |\nabla^2 u| \right\|_2^2 + \frac{1}{\varepsilon^4} \| |\nabla u| \|_2^2 \right)$$

for a positive constant C independent of u. Then (2.35) will yield

$$\int_\Omega (\Delta u)^2 \, dxdy \geq (1 - C\varepsilon^{\frac{4}{3}} |\underline{\kappa}|) \left\| |\nabla^2 u| \right\|_2^2 - C|\underline{\kappa}| \frac{1}{\varepsilon^4} \| |\nabla u| \|_2^2$$

or written otherwise

$$(1 - C\varepsilon^{\frac{4}{3}} |\underline{\kappa}|) \left\| |\nabla^2 u| \right\|_2^2 \leq \int_\Omega (\Delta u)^2 \, dxdy + C|\underline{\kappa}| \frac{1}{\varepsilon^4} \| |\nabla u| \|_2^2 . \tag{2.36}$$

However, one can integrate by parts and use Poincaré's inequality to obtain

$$\begin{aligned} \int_\Omega |\nabla u|^2 \, dxdy &= \int_\Omega \nabla u \cdot \nabla u \, dxdy = -\int_\Omega u \Delta u \, dxdy \\ &\leq \|u\|_2 \|\Delta u\|_2 \leq c_\Omega \| |\nabla u| \|_2 \|\Delta u\|_2 \end{aligned}$$

for a positive constant c_Ω, that is

$$\| |\nabla u| \|_2 \leq c_\Omega \|\Delta u\|_2 . \tag{2.37}$$

Thus, for $\varepsilon^{\frac{4}{3}} < \frac{1}{C|\underline{\kappa}|}$, use (2.36) and (2.37) to obtain the desired result.

Proof of the claim. Since $u \in W^{3,2}(\Omega)$ one has that $|\nabla u| \in W^{1,2}(\Omega)$ and applying [1, Theorem 5.10] to $|\nabla u|$ with $k = m = 1$, $q = 4$ $\left(\text{and thus } p = \frac{4}{3}\right)$ yields that there exists a positive constant C, such that

$$\| |\nabla u| \|_{L^2(\partial\Omega)}^2 \leq C \| |\nabla u| \|_{L^4(\Omega)} \| |\nabla u| \|_{W^{1,\frac{4}{3}}(\Omega)} .$$

Now apply [1, Theorem 5.8] with $p = 2$, $m = 1$ and $q = 4$ to obtain

$$\| |\nabla u| \|_{L^4(\Omega)} \leq C_1 \| |\nabla u| \|_{L^2(\Omega)}^{\frac{1}{2}} \| |\nabla u| \|_{W^{1,2}(\Omega)}^{\frac{1}{2}}$$

and thus, using Corollary 2.3.4 and the embedding $W^{1,2}(\Omega) \hookrightarrow W^{1,\frac{4}{3}}(\Omega)$ we get

$$\| |\nabla u| \|_{L^2(\partial\Omega)}^2 \leq C_2 \left\| |\nabla^2 u| \right\|_{L^2(\Omega)}^{\frac{3}{2}} \| |\nabla u| \|_{L^2(\Omega)}^{\frac{1}{2}} .$$

The claim is finally proven by application of the elementary inequality

$$ab \leq \frac{\varepsilon^p a^p}{p} + \frac{b^q}{\varepsilon^q q}, \tag{2.38}$$

where a, b, ε are any positive numbers and $\frac{1}{p} + \frac{1}{q} = 1$, with $p = \frac{4}{3}$ and $q = 4$. To prove (2.38), one writes $x = p \ln(a\varepsilon)$, $y = q \ln \frac{b}{\varepsilon}$, $t = \frac{1}{p}$ and computes

$$ab = (a\varepsilon)(b\varepsilon^{-1}) = e^{tx+(1-t)y} \leq te^x + (1-t)e^y = \frac{\varepsilon^p a^p}{p} + \frac{b^q}{\varepsilon^q q}.$$

■

To complete the picture, we need a density argument. This is not straightforward. In the case of planar domains, there exists a complete characterization of traces of functions in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, found in the proof of Theorem 2.2.1. For higher dimensions one can use the results of section 2.2, specifically Theorem 2.2.4.

Corollary 2.3.6 *Let $\Omega \in \mathbb{R}^2$ be C^3 diffeomorphic to a polygon. Then there exists a positive constant $C = C(\Omega)$, such that*

$$\|u\|_{2,2} \leq C(\Omega) \|\Delta u\|_2 \quad \textit{for all} \quad u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega). \tag{2.39}$$

Proof. It is a direct consequence of (2.34) and Theorem 2.2.1. ■

Remark 2.3.7 *When Ω has only convex corners, one arrives to the same result by using regularity properties of solutions for the Dirichlet Laplace problem. The presence, however, of a concave corner may yield some confusion. In that case, one can construct a polygonal domain Ω_0 and a function $f_0 \in L^2(\Omega_0)$, such that there exists a unique $u_0 \in W_0^{1,2}(\Omega_0)$ which satisfies*

$$\begin{cases} -\Delta u_0 = f_0, & \textit{in } \Omega_0 \\ u_0 = 0, & \textit{on } \partial\Omega_0, \end{cases}$$

in a weak sense and we have that $u_0 \notin W^{2,2}(\Omega_0)$. Thus, u_0 does not satisfy (2.39). *However, estimate* (2.39) *holds for any function $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ on any polygon Ω, no matter if it is concave or convex. We will return to this matter in section 4.3.2.*

2.4 Variational inequalities

Here we summarize the basic definitions and results concerning variational inequalities (see [34]). For the following, unless otherwise stated, X denotes a reflexive Banach space, X^* its dual and $\langle \cdot, \cdot \rangle$ the corresponding duality brackets. Moreover, K will be a closed and convex subset of X.

Definition 2.4.1 *A mapping $A : K \to X^*$ is called monotone, whenever*

$$\langle Au - Av, u - v \rangle \geq 0 \ \ \textit{for all} \ \ u, v \in K.$$

Furthermore, a monotone mapping is called strictly monotone, when

$$\langle Au - Av, u - v \rangle = 0 \ \ \textit{implies} \ \ u = v.$$

Definition 2.4.2 *A mapping $A : K \to X^*$ is called continuous on finite dimensional subspaces, when for any finite dimensional subspace $M \subset X$ the mapping $A : K \cap M \to X^*$ is weakly continuous.*

Definition 2.4.3 *A mapping $A : K \to X^*$ is called coercive, if there exists $v \in K$ such that for any sequence $\{u_n\}_{n \in \mathbb{N}} \subset K$ with $\lim\limits_{n \to \infty} \|u_n\|_X = \infty$ one has that*

$$\lim_{n \to \infty} \frac{\langle Au_n - Av, u_n - v \rangle}{\|u_n - v\|_X} = +\infty.$$

Next we state two important results. See [34, Chapter III].

Lemma 2.4.4 (G. J. Minty) *Let $A : K \to X^*$ be monotone and continuous on finite dimensional subspaces. Then an element $u \in K$ satisfies*

$$\langle Au, v - u \rangle \geq 0 \ \ \textit{for all } v \in K$$

if and only if it satisfies

$$\langle Av, v - u \rangle \geq 0 \ \ \textit{for all } v \in K.$$

Theorem 2.4.5 *Let K be a nonempty closed and convex subset of X and let $A : K \to X^*$ be monotone, coercive and continuous on finite dimensional subspaces. Then there exists $u \in K$, such that*

$$\langle Au, v - u \rangle \geq 0 \ \ \textit{for all } v \in K.$$

If moreover A is strictly monotone, then u is also unique.

The next lemma illustrates the connection between variational inequalities and minimization problems.

Lemma 2.4.6 *Let X be a Banach space, $F \in C^1(X;\mathbb{R})$ a convex functional, i.e.*

$$F\left(u + t\left(v-u\right)\right) \leq F\left(u\right) + t\left(F\left(v\right) - F\left(u\right)\right) \text{ for } u, v \in X \text{ and } t \in [0,1],$$

and let $K \subset X$ be closed and convex. For $u \in K$ the following statements are equivalent.

(i) $F(u) = \min_{v \in K} F(v)$,

(ii) $F'(u; v-u) \geq 0$ *for all* $v \in K$,

where $F'(u;h)$ denotes the Gâteaux derivative of F at u in the direction of h.

Proof. $(i) \Rightarrow (ii)$. Assume that u minimizes F in K and let $v \in K$. Since K is convex it holds that $u + t(v-u) \in K$ for all $t \in [0,1]$. This implies that the C^1 function

$$g(t) := F\left(u + t(v-u)\right) \text{ with } t \in [0,1]$$

attains its minimum at $t = 0$, i.e. $g'(0) \geq 0$ or

$$F'(u; v-u) \geq 0.$$

$(ii) \Rightarrow (i)$. Since F is convex, one has that

$$F'(u; v-u) = \lim_{t \downarrow 0} \frac{F\left(u + t(v-u)\right) - F(u)}{t} \leq F(v) - F(u) \text{ for all } v \in K,$$

i.e. $F(u) \leq F(v)$ for all $v \in K$. ■

Example 2.4.7 *Maybe the most classical example of a variational inequality is the so called obstacle problem. Let Ω be a smooth bounded domain in $\mathbb{R}^2$, $h \in C^2(\overline{\Omega})$ with $h_{|\partial\Omega} \leq 0$ and define*

$$K := \{v \in W_0^{1,2}(\Omega);\ v \geq h \text{ almost everywhere in } \Omega\}.$$

Then one has that K is a closed and convex subset of $W_0^{1,2}(\Omega)$. The function h is called the obstacle. For $f \in L^2(\Omega)$, the problem consists of minimizing the functional

$$\mathcal{I}(u) = \int_\Omega \left(\tfrac{1}{2}|\nabla u|^2 - fu\right) dxdy \tag{2.40}$$

among all functions $u \in K$. Applying the previous argumentation to the operator $\mathcal{I}' : K \to W^{-1,2}(\Omega)$, one obtains that there exists a unique $u \in K$, such that $\mathcal{I}'(u; v-u) \geq 0$ for all $v \in K$ or

$$\int_\Omega \nabla u \cdot \nabla(u-v)\, dxdy \leq \int_\Omega f(u-v)\, dxdy \text{ for all } v \in K.$$

If $f \in L^\infty(\Omega)$ *and* $h_{|\partial\Omega} < 0$ *, we can apply [34, Theorem 3.6] to obtain that* $u \in W^{2,p}(\Omega)$ *for all* $1 \leq p < \infty$ *and then, since the function* $u - h$ *is continuous, the set*

$$I := \{(x,y) \in \Omega;\ u(x,y) = h(x,y)\}$$

is closed, as its relative complement

$$\Omega \setminus I := \{(x,y) \in \Omega;\ u(x,y) > h(x,y)\}$$

is open. The set I *is called the coincidence set of the solution* u *and its boundary* ∂I *is called the free boundary of the problem. Let* $v \in C_0^\infty(\Omega \setminus I)$*. Then there exists* $C > 0$ *such that for* $t \in \mathbb{R}$ *with* $|t| \leq C$ *the function* $w := u + tv$ *belongs in* K*. Since* $\mathcal{I}'(u; w-u) \geq 0$*, we obtain*

$$t \int_\Omega (\nabla u \cdot \nabla v - fv)\, dxdy \geq 0.$$

But then, since $t \in [-C, C]$*, it follows that*

$$-\Delta u = f \quad \textit{almost everywhere in } \Omega \setminus I.$$

On the other hand, if $t > 0$*, then for* $v \in C_0^\infty(\Omega)$*, the function* $w := u + tv$ *belongs in* K*. Thus, integrating by parts as before, yields that*

$$-\Delta u \geq f \quad \textit{almost everywhere in } \Omega.$$

Chapter 3

Comparing hinged and supported Kirchhoff plates

We will start this chapter with two results concerning the existence and uniqueness of solutions to the problems under consideration.

3.1 Existence Results

Let Ω be a planar and bounded Lipschitz domain and $f \in L^2(\Omega)$. We seek minimizers of the functional

$$J_\sigma(u) = \int_\Omega \left(\tfrac{1}{2}(\Delta u)^2 + (1-\sigma)(u_{xy}^2 - u_{xx}u_{yy}) - fu \right) dxdy \tag{3.1}$$

in $\mathcal{V} \subset W^{2,2}(\Omega)$ for the following cases:

- hinged: $\mathcal{V} = H_0(\Omega) := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$,
- supported: $\mathcal{V} = H_+(\Omega) := \left\{ u \in W^{2,2}(\Omega);\ \min(u,0) \in W_0^{1,2}(\Omega) \right\}$.

Here $\sigma \in (-1,1)$ is the Poisson ratio as described in the introduction. We will also use the following semi-norm

$$|u|_{2,2} = \left\| |\nabla^2 u| \right\|_2 ,$$

introduced in Chapter 2. Note that as shown in Corollary 2.3.4, $|\cdot|_{2,2}$ is a norm in $H_0(\Omega)$ which is equivalent to $\|\cdot\|_{2,2}$.

Remark 3.1.1 *One should notice that $H_+(\Omega)$ is closed in $W^{2,2}(\Omega)$. Indeed, let $\{u_n\}_{n\in\mathbb{N}} \subset H_+(\Omega)$ with $u_n \to u \in W^{2,2}(\Omega)$ in the $W^{2,2}$ norm topology. Then, $u_n \to u$ in $W^{1,2}(\Omega)$ which implies that $W_0^{1,2}(\Omega) \ni u_n^- \to u^-$. Since $W_0^{1,2}(\Omega)$ is closed in $W^{1,2}(\Omega)$, it follows that $u^- \in W_0^{1,2}(\Omega)$.*

3.1.1 The hinged case

When the plate is hinged, existence is a straightforward and standard result, but we include it here for the sake of completeness (see the classical work of Friedrichs [21] for the case $\sigma = 0$).

Theorem 3.1.2 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Let J_σ be as in (3.1) with $-1 < \sigma < 1$ and $f \in L^2(\Omega)$. Then J_σ possesses a unique minimizer in $H_0(\Omega)$.*

Proof. Defining the bilinear form

$$\alpha_\sigma(u,v) := \int_\Omega \big(\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})\big)\, dxdy, \tag{3.2}$$

we can write

$$J_\sigma(u) = \frac{1}{2}\alpha_\sigma(u,u) - \int_\Omega fu\; dxdy \quad \text{and} \quad J'_\sigma(u;v) = \alpha_\sigma(u,v) - \int_\Omega fv\; dxdy.$$

One has the following estimate

$$\begin{aligned}
\alpha_\sigma(u,u) &= \int_\Omega \big((\Delta u)^2 + 2(1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})\big)\, dxdy \\
&= 2\int_\Omega \big(\tfrac{1}{2}u_{xx}^2 + \tfrac{1}{2}u_{yy}^2 + (1-\sigma)u_{xy}^2 + \sigma\, u_{xx}u_{yy}\big)\, dxdy \\
&\geq 2(1-|\sigma|)\int_\Omega \big(\tfrac{1}{2}\left(u_{xx}^2 + u_{yy}^2\right) + u_{xy}^2\big)\, dxdy \\
&= (1-|\sigma|)\,|u|_{2,2}^2
\end{aligned} \tag{3.3}$$

and coercivity is implied by Corollary 2.3.4. Thus, J_σ is convex, continuous and coercive in $H_0(\Omega)$ and the proof follows from the direct method of calculus of variations. ∎

3.1.2 The supported case and the variational inequality

Proving the existence of minimizers in $H_+(\Omega)$ is not so straightforward. Two problems appear that make it more appropriate to consider an alternative approach: the nature of the boundary conditions and the fact that the corresponding bilinear form is not obviously coercive. Thus, we will prove existence by studying the corresponding variational inequality and its regularization.

The connection between variational inequalities and minimization problems is well known (see [34]) and illustrated by Lemma 2.4.6 from the previous section.

Theorem 3.1.3 (Existence for the supported case) *Let $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary and $-1 < \sigma < 1$. Moreover, assume that $0 \not\equiv f \in L^2(\Omega)$, such that*

$$\int_\Omega f\zeta \, dxdy < 0 \quad \text{for all} \quad 0 \not\equiv \zeta \in H_+(\Omega) \text{ with } \alpha_\sigma(\zeta,\zeta) = 0. \tag{3.4}$$

Then there exists a minimizer $u_\sigma \in H_+(\Omega)$ of J_σ.

Remark 3.1.4 *(i) The functions ζ which satisfy (3.4) are nothing more than the affine functions with nonnegative boundary values. They represent the rigid motions of an unloaded plate. Translating assumption (3.4) above, we demand that the force density is such that among all rigid motions $u = 0$ is minimal in J_σ. For the condition (3.4) to be satisfied, it is not necessary that f is nonpositive everywhere.*

(ii) The condition also implies that there exists $x_0 \in \partial\Omega$ such that $u_\sigma(x_0) = 0$: Assume that the plate does not touch $\partial\Omega$ and thus

$$h = \min_{x \in \partial\Omega} u_\sigma(x) > 0.$$

Then $u_\sigma - h \in H_+(\Omega)$ and

$$J_\sigma(u_\sigma - h) = J_\sigma(u_\sigma) + \int_\Omega fh \, dxdy < J_\sigma(u_\sigma),$$

which is a contradiction, since u_σ is supposed to be a minimizer. In fact, the same argument shows that $h = 0$ is the only affine function such that $u_\sigma - h \in H_+(\Omega)$.

(iii) A failure to fulfill (3.4) will result in the existence of multiple minimizers or even the nonexistence of such. To see this, assume that u_σ is a minimizer and that ζ_0 is a nontrivial affine function in $H_+(\Omega)$ for which $\int_\Omega f\zeta_0 \, dxdy \geq 0$. Then

$$J_\sigma(u_\sigma + \zeta_0) = J_\sigma(u_\sigma) - \int_\Omega f\zeta_0 \, dxdy \leq J_\sigma(u_\sigma).$$

Hence u_σ, if it exists, is not unique. If $\int_\Omega f\zeta_0 \, dxdy > 0$, then no minimizer exists, since $u_\sigma + t\,\zeta_0 \in H_+(\Omega)$ for all $t \geq 0$ and

$$\lim_{t \to \infty} J_\sigma(u_\sigma + t\zeta_0) = -\infty.$$

Proof of Theorem 3.1.3. Following Lemma 2.4.6 a minimizer is a function $u_\sigma \in H_+(\Omega)$ such that

$$J'_\sigma\,(u_\sigma; v - u_\sigma) \geq 0 \text{ for all } v \in H_+(\Omega).$$

Since the functional J_σ is not coercive on $W^{2,2}(\Omega)$ or $H_+(\Omega)$ we are going to consider an elliptic regularization of α_σ.

Define the inner product $(\!(\cdot,\cdot)\!)$ on $W^{2,2}(\Omega)$ by

$$(\!(u,v)\!) := \int_\Omega \left(uv + \nabla u \cdot \nabla v + \nabla^2 u \cdot \nabla^2 v\right) dxdy$$

and consider for $\varepsilon > 0$:

$$\alpha_{\sigma,\varepsilon}(u,v) := \alpha_\sigma(u,v) + \varepsilon\,(\!(u,v)\!) \text{ for } u,v \in W^{2,2}(\Omega). \tag{3.5}$$

Let $J_{\sigma,\varepsilon}$ be the corresponding regularized functional, i.e.

$$J_{\sigma,\varepsilon}(u) := \frac{1}{2}\alpha_{\sigma,\varepsilon}(u,u) - \int_\Omega fu\;dxdy$$

and let $J'_{\sigma,\varepsilon} : H_+(\Omega) \to \left(W^{2,2}(\Omega)\right)^*$ denote the Gâteaux derivative of $J_{\sigma,\varepsilon}$ given by

$$J'_{\sigma,\varepsilon}(u;v) = \alpha_{\sigma,\varepsilon}(u,v) - \int_\Omega fv\;dxdy.$$

Since $u \mapsto \sqrt{\alpha_{\sigma,\varepsilon}(u,u)}$ is a norm on $W^{2,2}(\Omega)$, the mapping $u \mapsto J'_{\sigma,\varepsilon}(u;\cdot)$ is continuous and strictly monotone, that is

$$J'_{\sigma,\varepsilon}(u;u-v) - J'_{\sigma,\varepsilon}(v;u-v) = \alpha_{\sigma,\varepsilon}(u-v,u-v) \geq 0 \quad \text{for} \quad u,v \in H_+(\Omega)$$

with a strict inequality for $u \neq v$, and coercive, i.e.

$$\lim_{\substack{u\in H_+(\Omega)\\ \|u\|_{2,2}\to\infty}} \frac{J'_{\sigma,\varepsilon}(u;u)}{\|u\|_{2,2}} = +\infty$$

(take $\varphi \equiv 0$ in Definition 2.4.3). We also have that $H_+(\Omega)$ is closed (see Remark 3.1.1) and convex in $W^{2,2}(\Omega)$. Then Theorem 2.4.5 implies the existence of $u_\varepsilon \in H_+(\Omega)$ satisfying

$$J'_{\sigma,\varepsilon}(u_\varepsilon; v-u_\varepsilon) \geq 0 \quad \text{for all} \quad v \in H_+(\Omega). \tag{3.6}$$

By the strict monotonicity u_ε is unique. Rephrased (3.6) means that

$$\alpha_{\sigma,\varepsilon}(u_\varepsilon, u_\varepsilon - v) \leq \int_\Omega f(u_\varepsilon - v)\;dxdy \quad \text{for all} \quad v \in H_+(\Omega) \tag{3.7}$$

or, using Minty's Lemma (see 2.4.4),

$$\begin{aligned} \alpha_{\sigma,\varepsilon}(v, u_\varepsilon - v) &= \alpha_{\sigma,\varepsilon}(u_\varepsilon, u_\varepsilon - v) - \alpha_{\sigma,\varepsilon}(u_\varepsilon - v, u_\varepsilon - v) \\ &\leq \int_\Omega f(u_\varepsilon - v)\;dxdy \quad \text{for all} \quad v \in H_+(\Omega). \end{aligned} \tag{3.8}$$

So we have a unique minimizer u_ε of $J_\sigma(u) + \frac{1}{2}\varepsilon\,(\!(u,u)\!)$ in $H_+(\Omega)$.

What happens if we let $\varepsilon \downarrow 0$? If $\|u_\varepsilon\|_{2,2}$ is uniformly bounded, then, since bounded sets in $W^{2,2}(\Omega)$ are weakly precompact, there exists a weakly convergent subsequence $u_{\varepsilon_n} \rightharpoonup u_\sigma$ and the weak lower semicontinuity of $J_\sigma(u)$ implies

$$\begin{aligned} J_\sigma(u_\sigma) &\leq \liminf_{n\to\infty} J_\sigma(u_{\varepsilon_n}) = \liminf_{n\to\infty}\left(J_{\sigma,\varepsilon_n}(u_{\varepsilon_n}) - \tfrac{1}{2}\varepsilon_n\,(\!(u_{\varepsilon_n}, u_{\varepsilon_n})\!)\right) \\ &= \liminf_{n\to\infty} J_{\sigma,\varepsilon_n}(u_{\varepsilon_n}) \leq \liminf_{n\to\infty} J_{\sigma,\varepsilon_n}(v) = J_\sigma(v) \end{aligned}$$

for any $v \in H_+(\Omega)$. So, J_σ has a minimizer u_σ. See also [34, Theorem 2.1, p. 88].

Now suppose that $\|u_\varepsilon\|_{2,2}$ is not uniformly bounded, that is, there exists a subsequence $\{u_{\varepsilon_n}\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ and $\|u_{\varepsilon_n}\|_{2,2} \to \infty$ for $n \to \infty$. Setting $w_n = \|u_{\varepsilon_n}\|_{2,2}^{-1} u_{\varepsilon_n} \in H_+(\Omega)$, there exists a subsequence, again denoted by w_n, that weakly converges in $W^{2,2}(\Omega)$, say $w_n \rightharpoonup w$. Since $H_+(\Omega)$ is closed and convex it is also weakly closed by Mazur's Lemma (see [40, Theorem 6, p. 103]). Hence $w \in H_+(\Omega)$. Now, use (3.7) and estimate (3.3) with $v = 0$, $\varepsilon = \varepsilon_n$ and divide by $\|u_{\varepsilon_n}\|_{2,2}^2$ to get

$$\begin{aligned}
0 \le (1-|\sigma|)\,|w_n|_{2,2}^2 &\le \alpha_\sigma(w_n, w_n) = \frac{a_{\sigma,\varepsilon}\,(u_{\varepsilon_n}, u_{\varepsilon_n}) - \varepsilon_n \,\|u_{\varepsilon_n}\|_{2,2}^2}{\|u_{\varepsilon_n}\|_{2,2}^2} \\
&\le \frac{1}{\|u_{\varepsilon_n}\|_{2,2}} \int_\Omega f w_n \, dxdy - \varepsilon_n \le \frac{\|f\|_2\, \|w_{\varepsilon_n}\|_2}{\|u_{\varepsilon_n}\|_{2,2}} - \varepsilon_n \\
&\le \frac{\|f\|_2\, \|w_{\varepsilon_n}\|_{2,2}}{\|u_{\varepsilon_n}\|_{2,2}} - \varepsilon_n = \frac{\|f\|_2}{\|u_{\varepsilon_n}\|_{2,2}} - \varepsilon_n. \qquad (3.9)
\end{aligned}$$

Thus, it follows that $|w_n|_{2,2} \to 0$ for $n \to \infty$. Moreover, the functional $|\cdot|_{2,2} : W^{2,2}(\Omega) \to \mathbb{R}$ is weakly lower semicontinuous. Indeed

$$\begin{aligned}
&\int_\Omega |\partial_\alpha w_n|^2 \, dxdy - \int_\Omega |\partial_\alpha w|^2 \, dxdy \\
&= \int_\Omega (\partial_\alpha w_n - \partial_\alpha w)\,(\partial_\alpha w_n + \partial_\alpha w)\, dxdy \\
&= \int_\Omega (\partial_\alpha w_n - \partial_\alpha w)\,(\partial_\alpha w_n - \partial_\alpha w + 2\partial_\alpha w)\, dxdy \\
&= \int_\Omega (\partial_\alpha w_n - \partial_\alpha w)^2 \, dxdy + 2\int_\Omega (\partial_\alpha w_n - \partial_\alpha w)\, \partial_\alpha w \, dxdy \\
&\ge 2\int_\Omega (\partial_\alpha w_n - \partial_\alpha w)\, \partial_\alpha w \, dxdy
\end{aligned}$$

for any multi-index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| = 2$. Since

$$\int_\Omega (\partial_\alpha w_n - \partial_\alpha w)\, \partial_\alpha w \, dxdy \to 0$$

as $n \to \infty$, we obtain the claim and thus $|w|_{2,2} = 0$. Hence w is affine.

Dividing (3.8) by $\|u_{\varepsilon_n}\|_{2,2}$, we find

$$\alpha_{\sigma,\varepsilon_n}\left(v, w_n - \|u_{\varepsilon_n}\|_{2,2}^{-1} v\right) \le \int_\Omega f \,\left(w_n - \|u_{\varepsilon_n}\|_{2,2}^{-1} v\right)\, dxdy \qquad (3.10)$$

for all $v \in H_+(\Omega)$. Since $w_n \rightharpoonup w$ in $W^{2,2}(\Omega)$ and $\|w_n\|_{2,2} = 1$, we get that

$$\alpha_{\sigma,\varepsilon_n}(v, w_n) \to a_\sigma(v, w) \quad \text{for} \quad n \to \infty.$$

Moreover, as $\|u_{\varepsilon_n}\|_{2,2}^{-1} \to 0$ and $\int_\Omega f w_n \, dxdy \to \int_\Omega f w \, dxdy$ for $n \to \infty$, one finds from (3.10) that

$$\alpha_\sigma(v, w) \le \int_\Omega f \; w \; dxdy \quad \text{for all} \quad v \in H_+(\Omega). \tag{3.11}$$

We have that w is affine and thus

$$0 = \alpha_\sigma(v, w) \le \int_\Omega f \; w \; dxdy \le 0 \text{ for all } v \in H_+(\Omega)$$

with a strict inequality and hence a contradiction unless $w \equiv 0$. So $w \equiv 0$.

By (3.9) we have $|w_n|_{2,2} \to 0$ for $n \to \infty$. The compact embedding of $W^{2,2}(\Omega)$ into $W^{1,2}(\Omega)$ implies that $w_n \to w$ strongly in $W^{1,2}(\Omega)$ and it follows that $\|w_n\|_{1,2} \to 0$ for $n \to \infty$. Since $\|\cdot\|_{1,2} + |\cdot|_{2,2}$ and $\|\cdot\|_{2,2}$ are equivalent norms, one finds that $\|w_n\|_{2,2} \to 0$ for $n \to \infty$ which contradicts $\|w_n\|_{2,2} = 1$. ∎

Next we show the uniqueness of the minimizer.

Proposition 3.1.5 *Having the same assumptions as in Theorem 3.1.3, the minimizer u_σ of J_σ is unique in $H_+(\Omega)$.*

Proof. Let $u, v \in H_+(\Omega)$. Then one has that

$$\begin{aligned}
J'_\sigma(u; u-v) - J'_\sigma(v; u-v) = \; & \int_\Omega \Big((u-v)_{xx}^2 + 2\sigma(u-v)_{xx}(u-v)_{yy} \\
& + (u-v)_{yy}^2 + 2(1-\sigma)(u-v)_{xy}^2 \Big) \, dxdy \\
\ge \; & (1-|\sigma|) \int_\Omega \big((u-v)_{xx}^2 + (u-v)_{yy}^2 \\
& + 2(u-v)_{xy}^2 \big) \, dxdy \\
= \; & (1-|\sigma|)|u-v|_{2,2} \ge 0
\end{aligned}$$

with equality if and only if $u - v$ is affine. Now let $v_\sigma \in H_+(\Omega)$ with $v_\sigma \not\equiv u_\sigma$, such that

$$J'_\sigma(v_\sigma; v - v_\sigma) \ge 0 \quad \text{for all} \quad v \in H_+(\Omega). \tag{3.12}$$

Assume that $u_\sigma - v_\sigma$ is not affine. Then, using (3.12) we obtain

$$J'_\sigma(v_\sigma; u_\sigma - v_\sigma) < J'_\sigma(u_\sigma; u_\sigma - v_\sigma) = -J'_\sigma(u_\sigma; v_\sigma - u_\sigma) \le 0$$

which is contradicting (3.12) since $J'_\sigma(u_\sigma; v - u_\sigma) \geq 0$ for all $v \in H_+(\Omega)$. This means that $w := u_\sigma - v_\sigma$ is affine and one has

$$J_\sigma(u_\sigma) = J_\sigma(v_\sigma + w) = J_\sigma(v_\sigma) - \int_\Omega fw \, dxdy > J_\sigma(u_\sigma)$$

if and only if $w \not\equiv 0$. ■

3.2 The sign preserving property of smooth hinged plates

In this section we are going to show a comparison type result for a hinged plate in the case Ω is convex with a sufficiently smooth boundary.

3.2.1 Preliminaries

The following lemma is analogous to Theorems 2.2.1 and 2.2.4. Since we assume some more smoothness on $\partial\Omega$, one can deploy a simpler argument using elliptic regularity to get the desired result.

Lemma 3.2.1 *Let $\Omega \subset \mathbb{R}^n$ be bounded with $\partial\Omega \in C^{2,1}$. Then the space $W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$ is densely embedded in $H_0(\Omega)$.*

Proof. Let $u \in H_0(\Omega)$. Then $f := -\Delta u \in L^2(\Omega)$ and u is the unique weak solution of the equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $C_0^\infty(\Omega)$ such that $f_k \to f$ in $L^2(\Omega)$ as $k \to +\infty$. In particular, one has $f_k \in W^{1,2}(\Omega)$. Consider now the solutions u_k to the family of problems

$$\begin{cases} -\Delta u_k = f_k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

From [24, Theorem 9.19] one has that $u_k \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and applying [24, Lemma 9.17] to the function $u_k - u$, one obtains that $u_k \to u$ in H_0, which completes the proof. ∎

Using the above lemma we can derive the boundary conditions for the hinged plate in the case that Ω has a $C^{2,1}$ smooth boundary.

Lemma 3.2.2 *Set*

$$\mathcal{K}(u) := \int_\Omega \det(\nabla^2 u)\, dxdy, \quad \textit{for } u \in W^{2,2}(\Omega).$$

Then, for $\Omega \subset \mathbb{R}^2$ bounded with $\partial\Omega \in C^{2,1}$, the following hold:

(i) For all $u \in H_0(\Omega)$ and $\varphi \in W^{3,2}(\Omega)$:

$$\mathcal{K}'(u;\varphi) = \int_{\partial\Omega} \left(\kappa(s)\, \partial_n\varphi\, \partial_n u + \partial_{\tau\tau}\varphi\, \partial_n u\right)\, ds.$$

(ii) For all $u \in H_0(\Omega)$:

$$\mathcal{K}(u) = \frac{1}{2}\int_{\partial\Omega} \kappa(s)(\partial_n u)^2\, ds.$$

Proof. Let n_i, τ_i, $i = 1, 2$, denote the ith coordinate of the normal and tangent vector, respectively.

(i) First we suppose $u \in H_0(\Omega)$ and $\varphi \in C^\infty(\overline{\Omega})$. Integrating by parts, one obtains

$$\begin{aligned}\int_\Omega \varphi_{xy} u_{xy}\, dxdy &= \int_{\partial\Omega} (\varphi_{xy} u_x n_2 - \varphi_{xyy} u n_1)\, ds + \int_\Omega \varphi_{xxyy} u\, dxdy \\ &= \int_{\partial\Omega} (\varphi_{xy} u_y n_1 - \varphi_{xxy} u n_2)\, ds + \int_\Omega \varphi_{xxyy} u\, dxdy\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \varphi_{xx} u_{yy}\, dxdy &= \int_{\partial\Omega} (\varphi_{xx} u_y n_2 - \varphi_{xxy} u n_2)\, ds + \int_\Omega \varphi_{xxyy} u\, dxdy, \\ \int_\Omega \varphi_{yy} u_{xx}\, dxdy &= \int_{\partial\Omega} (\varphi_{yy} u_x n_1 - \varphi_{xyy} u n_1)\, ds + \int_\Omega \varphi_{xxyy} u\, dxdy.\end{aligned}$$

Since $u \in H_0(\Omega)$, we get that $u_x|_{\partial\Omega} = n_1 \partial_n u$ and $u_y|_{\partial\Omega} = n_2 \partial_n u$. Hence

$$\begin{aligned}\mathcal{K}'(u;\varphi) &= \int_\Omega (\varphi_{xx} u_{yy} + \varphi_{yy} u_{xx} - 2\varphi_{xy} u_{xy})\, dxdy \\ &= \int_{\partial\Omega} (\varphi_{xx} u_y n_2 + \varphi_{yy} u_x n_1 - \varphi_{xy} u_x n_2 - \varphi_{xy} u_y n_1)\, ds \\ &= -\int_{\partial\Omega} (\varphi_{xy} u_x n_2 + \varphi_{xy} u_y n_1 + \varphi_{yy} u_y n_2 + \varphi_{xx} u_x n_1)\, ds \\ &\quad + \int_{\partial\Omega} \Delta\varphi\, \partial_n u\, ds \\ &= \int_{\partial\Omega} \Delta\varphi\, \partial_n u\, ds - \int_{\partial\Omega} \partial_{nn}\varphi\, \partial_n u\, ds \\ &= \int_{\partial\Omega} (\kappa(s)\, \partial_n\varphi\, \partial_n u + \partial_{\tau\tau}\varphi\, \partial_n u)\, ds,\end{aligned}$$

where we have taken into consideration the fact that $\partial_{nn}\varphi = n_1^2 \varphi_{xx} + n_2^2 \varphi_{yy} + 2 n_1 n_2 \varphi_{xy}$ and $\Delta\varphi = \partial_{nn}\varphi + \partial_{\tau\tau}\varphi + \kappa \partial_n \varphi$ (see, for example, [54]).

We now consider the case $u \in W^{2,2}(\Omega)$ and $\varphi \in W^{3,2}(\Omega)$. To that end, take a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ in $C^\infty(\overline{\Omega})$ converging to φ in $W^{3,2}(\Omega)$ (such a sequence exists when the domain is smooth enough and satisfies, for example, a segment condition; see [24, section 7.6] and references therein). Then, since $\partial_{n\tau}\varphi_k \to \partial_{n\tau}\varphi$ and $\partial_{\tau\tau}\varphi_k \to \partial_{\tau\tau}\varphi$ in $L^2(\partial\Omega)$, passing to the limit in the above relation as $k \to +\infty$ completes the proof.

(ii) The claim follows from Lemma 3.2.1 by taking $u \in H_0(\Omega)$, a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ in $W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$ which approximates u (so that $\partial_\tau u = 0$ and $\partial_{\tau\tau}\varphi_k = 0$ on $\partial\Omega$) and observing that

$$\mathcal{K}(u) = \frac{1}{2}\mathcal{K}'(u;u) \quad \text{for all } v \in H_0(\Omega).$$

∎

Remark 3.2.3 *In this case the functional J_σ has the boundary form*

$$J_\sigma(u) = \int_\Omega \left(\tfrac{1}{2}(\Delta u)^2 - fu\right)\, dxdy - \frac{(1-\sigma)}{2}\int_{\partial\Omega} \kappa(s)(\partial_n u)^2\, ds$$

and a hinged plate will satisfy the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa \partial_n u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

Proposition 3.2.4 *For every $u \in H_0(\Omega)$ one has that*

$$\int_\Omega (\Delta u)^2\, dxdy \geq 2\int_{\partial\Omega} \kappa(s)(\partial_n u)^2\, ds.$$

Proof. From Lemma 3.2.2 and (3.3) we get that

$$\int_\Omega (\Delta u)^2\, dxdy - (1-\sigma)\int_{\partial\Omega} \kappa(s)(\partial_n u)^2\, ds \geq (1-|\sigma|)|u|_{2,2}^2 \geq 0.$$

Setting $\sigma = -1$ one obtains the claim. ■

3.2.2 Positivity preserving property

In view of Proposition 3.2.4 the following can be proved for the hinged plate problem.

Theorem 3.2.5 *Assume that $\Omega \subset \mathbb{R}^2$ is bounded and convex with $\partial\Omega \in C^{2,1}$. Let $-1 < \sigma < 1$ and $f \in L^2(\Omega)$. Then the minimizer u_σ of J_σ is the unique weak solution in $H_0(\Omega)$ of (3.13). If, moreover, $f \geq 0$ and $f \not\equiv 0$, then there exists a positive constant c_f such that*

$$u_\sigma(x) > c_f d(x, \partial\Omega),$$

where $d(\cdot, \partial\Omega)$ is the distance to the boundary.

Proof. From the assumptions on the domain one obtains that $0 \leq \kappa \in C^{0,1}(\partial\Omega)$ with $\kappa \not\equiv 0$. We define

$$\delta_{1,\kappa} := \inf_{u \in H_0(\Omega)} \frac{\displaystyle\int_\Omega (\Delta u)^2\, dxdy}{\displaystyle\int_{\partial\Omega} \kappa(s)(\partial_n u)^2\, ds}$$

with the convention that

$$\frac{\displaystyle\int_\Omega (\Delta u)^2\, dxdy}{\displaystyle\int_{\partial\Omega} \kappa(s)(\partial_n u)^2\, ds} = +\infty \quad \text{whenever} \quad \int_{\partial\Omega} \kappa(s)(\partial_n u)^2\, ds = 0.$$

Then, in view of [22, Theorem 4.1], it suffices to prove that

$$(1-\sigma)\kappa \leq \delta_{1,\kappa}\kappa \quad \text{and} \quad (1-\sigma)\kappa \not\equiv \delta_{1,\kappa}\kappa.$$

Indeed, from Proposition 3.2.4 one has that $\delta_{1,\kappa} \geq 2$, and since $\kappa \geq 0$ and $\kappa \not\equiv 0$ the assertion is proved. ■

Remark 3.2.6 *The difficulties that arise from the absence of a general maximum principle for fourth order problems can, in some special cases, be directly overcome: If $\sigma = 1$ and the domain is smooth enough, then one decouples the problem into the following elliptic system*

$$\begin{cases} -\Delta u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

A recursive application of the classical maximum principle for second order elliptic operators yields the claim. However, the presence of boundary singularities can make things complicated. An example where the system solution on a domain with a concave corner does not coincide with the one of the fourth order problem will be given in Chapter 4.

Remark 3.2.7 *As a consequence of the results in [51] and [20], it is possible to show a lower bound for $\delta_{1,\kappa}$ similar to the one stated in Proposition 3.2.4. In fact, Fichera's principle of duality (see [20]) states that*

$$\inf_{u \in H_0(\Omega)\setminus W_0^{2,2}(\Omega)} \frac{\displaystyle\int_\Omega (\Delta u)^2 \, dxdy}{\displaystyle\int_{\partial\Omega} (\partial_n u)^2 \, ds} = \inf_{u \in \mathcal{H}\setminus\{0\}} \frac{\displaystyle\int_{\partial\Omega} u \, ds}{\displaystyle\int_\Omega u \, dxdy},$$

where Ω is a domain with C^2 boundary and $\mathcal{H}$ is defined as the closure of the set $\{u \in C^2(\overline{\Omega});\ \Delta u = 0 \text{ in } \Omega\}$ with respect to the norm $\|\cdot\|_{L^2(\partial\Omega)}$. From [51] we have that

$$\inf_{u \in \mathcal{H}\setminus\{0\}} \frac{\displaystyle\int_{\partial\Omega} u \, ds}{\displaystyle\int_\Omega u \, dxdy} \geq 2 \cdot \min_{\partial\Omega} \kappa$$

and hence

$$\delta_{1,\kappa} \geq 2 \cdot \frac{\min_{\partial\Omega} \kappa}{\max_{\partial\Omega} \kappa}.$$

The result of Proposition 3.2.4 and equivalently the bound $\delta_{1,\kappa} \geq 2$ is optimal: One can show that if Ω is a disc, then $\delta_{1,\kappa} = 2$ (see [7]).

3.3 Comparison of hinged and supported plates

Next we turn our attention to the problem of a simply supported plate. In this section we are going to prove that the latter does not in general satisfy a sign preserving property, that is, we give an answer to the question from the introduction:

> *A plate which is supported at its boundary by walls of constant height and zero thickness and is pushed downwards will in general* ***not*** *touch its supporting structure everywhere.*

We focus our attention to plates with corners and show the result in this case.

3.3.1 A rectangular plate

We first consider the case of angles of 90 degrees. This is a special case but it cannot be included in the more general argumentation of the following subsections and has to be considered separately. It is also the most common shape one could imagine for a plate.

The regularity of hinged rectangular plates.

It is essential to note that the nonsmoothness of the boundary affects negatively the regularity of the solution to any elliptic problem. In our case and as it will be shown in a later section, the presence of angles has generically the following effect: The smaller (measured from the inside) the angle the smoother the solution. The case of special corners, however, exists: For $f \in L^2$ and an opening angle $\omega > \frac{\pi}{2}$, the solution of the hinged plate problem is $W^{s,2}$ with $s < 3$ and for $\frac{\pi}{3} < \omega < \frac{\pi}{2}$ one gets smoothness of order $3 < s < 4$ (see [26]). We will, however, prove that for $\omega = \frac{\pi}{2}$ the solution lies in $W^{4,2}$.

An extension and a density lemma. When one considers functions on $\overline{\Omega}$ that are 0 on $\partial\Omega$, corners in the boundary $\partial\Omega$ may imply loss of regularity or demand extra conditions for the behaviour of the function near such corners. Usually regularity near the boundary is obtained by defining an extension operator on functions on $\overline{\Omega}$ to those that live on a neighbourhood of Ω. For domains with corners, such an extension operator was constructed in Theorem 2.2.4. The straight angles of a rectangle, however, allow the following straightforward extension:

Lemma 3.3.1 *Set*

$$\mathcal{R} = (0, a) \times (0, b) \tag{3.15}$$

with $a, b > 0$. For $u : \overline{\mathcal{R}} \to \mathbb{R}$ let us define

$$Eu\,(x,y) := \begin{cases} u\,(x,y) & \textit{for } (x,y) \in \mathcal{R}, \\ -u\,(-x,y) & \textit{for } (-x,y) \in \mathcal{R}, \\ -u\,(x,-y) & \textit{for } (x,-y) \in \mathcal{R}, \\ u\,(-x,-y) & \textit{for } (-x,-y) \in \mathcal{R}, \\ 0 & \textit{elsewhere.} \end{cases}$$

Hence Eu defines a function from $[-a,a] \times [-b,b]$ to $\mathbb{R}$. Set $\mathcal{R}_0 = (-a,a) \times (-b,b)$. Then the following hold:

(i) Let $\gamma \in [0,1]$. Then the operator $E : C^{1,\gamma}\left(\overline{\mathcal{R}}\right) \cap C_0\left(\overline{\mathcal{R}}\right) \to C^{1,\gamma}\left(\overline{\mathcal{R}}_0\right) \cap C_0\left(\overline{\mathcal{R}}_0\right)$ is continuous.

(ii) The operator $E : W^{2,2}\left(\mathcal{R}\right) \cap W_0^{1,2}\left(\mathcal{R}\right) \to W^{2,2}\left(\mathcal{R}_0\right) \cap W_0^{1,2}\left(\mathcal{R}_0\right)$ is continuous.

(iii) Let $G_{\mathcal{R}_0}$ denote the solution operator for the Dirichlet Laplace problem. Then

$$EG_{\mathcal{R}} = G_{\mathcal{R}_0}E. \tag{3.16}$$

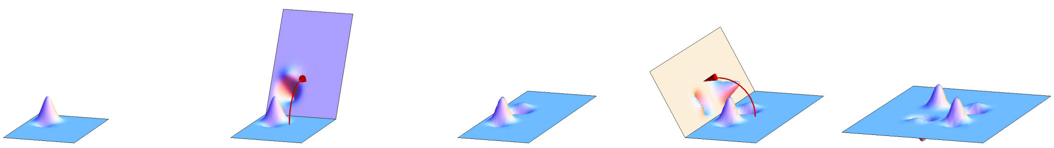

Figure 3.1: The construction of this extension can be viewed as unfolding a bulging doubly folded piece of paper.

Proof. We will prove that the range of E is well defined and contained in the appropriate spaces. The continuity is then immediate. Let us consider

$$E_1u\,(x,y) = \begin{cases} u\,(x,y) & \text{for } (x,y) \in \mathcal{R}, \\ -u\,(-x,y) & \text{for } (-x,y) \in \mathcal{R}, \\ 0 & \text{elsewhere,} \end{cases}$$

which defines a first antisymmetric reflection to $[-a,a] \times [0,b]$. With a similarly defined E_2 in the y-direction, one finds $E = E_2 \circ E_1$. It is thus enough to give the proof for E_1.

For the first item it is sufficient to notice that due to $u\,(0,y) = 0$ the function E_1u and its first derivatives are continuous over $\{0\} \times [0,b]$.

A short proof of the second and third item uses elliptic regularity. Set

$$f\,(x,y) := \begin{cases} -\left(\Delta u\right)(x,y) & \text{for } (x,y) \in \mathcal{R}, \\ \left(\Delta u\right)(-x,y) & \text{for } (-x,y) \in \mathcal{R}, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f \in L^2(\Omega)$. Let $\mathcal{R}_1 = (-a,a) \times (0,b)$ and consider the following

$$\begin{cases} -\Delta \tilde{u} = f & \text{in } \mathcal{R}_1, \\ \tilde{u} = 0 & \text{on } \partial\mathcal{R}_1. \end{cases} \tag{3.17}$$

Problem (3.17) has a unique weak solution $\tilde{u} \in W_0^{1,2}(\mathcal{R}_1)$ and since $\mathcal{R}_1$ is convex, one even finds $\tilde{u} \in W^{2,2}(\mathcal{R}_1)$ (see [33]). Now define

$$\hat{u}(x,y) := -\tilde{u}(-x,y).$$

Then $\hat{u} \in W^{2,2}(\mathcal{R}_1)$ and satisfies (3.17). Since strong solutions of (3.17) are unique (see [24]), we find $\tilde{u} \equiv \hat{u}$ and thus

$$\tilde{u}(0,y) = \hat{u}(0,y) = -\tilde{u}(0,y), \text{ i.e. } \tilde{u}(0,y) = 0.$$

Thus $\tilde{u} - u \in W^{2,2}(\mathcal{R})$ and since $-\Delta(\tilde{u} - u) = 0$ in $\mathcal{R}$ and $\tilde{u} = u = 0$ on $\partial\mathcal{R}$, we find by uniqueness that $\tilde{u} \equiv u$ in $\overline{\mathcal{R}}$, that is $\tilde{u} \equiv E_1 u \in W^{2,2}(\mathcal{R}_1)$. ■

We recall the following definitions:

$$\begin{aligned} C^k(\overline{\mathcal{R}}) &:= \Big\{u \in C^k(\mathcal{R});\ \partial_\alpha u \text{ bounded, uniformly continuous in } \mathcal{R} \\ &\qquad \text{for all } \alpha \in \mathbb{N} \times \mathbb{N} \text{ with } |\alpha| \le k\Big\}, \\ C^\infty(\overline{\mathcal{R}}) &:= \bigcap_{k=0}^{\infty} C^k(\overline{\mathcal{R}}), \\ C_0(\overline{\mathcal{R}}) &:= \left\{u \in C(\overline{\mathcal{R}});\ u = 0 \text{ on } \partial\mathcal{R}\right\}. \end{aligned}$$

Since $\mathcal{R}$ is bounded and has a Lipschitz boundary, there exists a total extension operator for $\mathcal{R}$ (see [1, Theorem 5.24, p. 154]) and thus $C^\infty(\overline{\mathcal{R}})$ coincides with the space of functions in $C^\infty(\mathbb{R}^2)$ restricted to $\overline{\mathcal{R}}$.

Corollary 3.3.2 $\overline{C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}})}^{\|\cdot\|_{2,2}} = H_0(\mathcal{R})$.

Proof. Since $C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}}) \subset H_0(\mathcal{R})$ and $H_0(\mathcal{R})$ is closed it is immediate that

$$\overline{C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}})}^{\|\cdot\|_{2,2}} \subset H_0(\mathcal{R}).$$

Let us write $\mathcal{R}_{00} := (-2a,2a) \times (-2b,2b)$. For the other inclusion we may use Lemma 3.3.1 twice to define an extension operator

$$\tilde{E} : \mathbb{R}^{\mathcal{R}} \to \mathbb{R}^{\mathcal{R}_{00}}, \tag{3.18}$$

that is, from functions on $\mathcal{R}$ to functions on $\mathcal{R}_{00}$. First we extend a function defined in $\overline{\mathcal{R}}$ in an odd way, as for E, from $[0,a] \times [0,b]$ to $[0,2a] \times [0,2b]$ and next again in an odd way, which could also be called a periodic extension, from $[0,2a] \times [0,2b]$

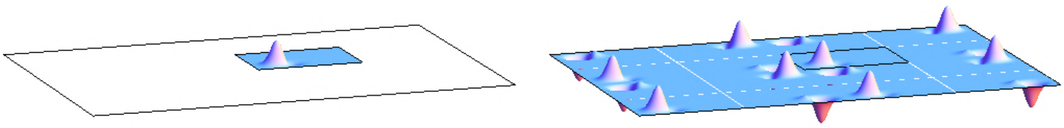

Figure 3.2: $\tilde{E}$ extends a function on $\mathcal{R}$ to a function on $\mathcal{R}_{00}$ by "unfolding" respectively to east, north, west and south.

to $[-2a, 2a] \times [-2b, 2b]$ (see Figure 3.2). By Lemma 3.3.1, $\tilde{E}$ is continuous as an operator from $W^{2,2}(\mathcal{R}) \cap W_0^{1,2}(\mathcal{R})$ to $W^{2,2}(\mathcal{R}_{00}) \cap W_0^{1,2}(\mathcal{R}_{00})$.
Next we define a function $\chi \in C^\infty(\mathcal{R}_{00})$ with $0 \le \chi \le 1$ which satisfies

$$\chi = \begin{cases} 1 & \text{in } \left(-\frac{4}{3}a, \frac{4}{3}a\right) \times \left(-\frac{4}{3}b, \frac{4}{3}b\right), \\ 0 & \text{in } \mathcal{R}_{00} \setminus \left(-\frac{5}{3}a, \frac{5}{3}a\right) \times \left(-\frac{5}{3}b, \frac{5}{3}b\right). \end{cases}$$

If $u \in H_0(\mathcal{R})$, then $\tilde{E}u \in H_0(\mathcal{R}_{00})$ and $\chi\tilde{E}u \in W_0^{2,2}(\mathcal{R}_{00})$. Using the standard mollifier with $z = (x, y) \in \mathbb{R}^2$, that is

$$\varphi_1(z) := \begin{cases} ce^{-\frac{1}{1-|z|^2}} & \text{for } |z| < 1, \\ 0 & \text{for } |z| \ge 1 \end{cases}$$

with $c^{-1} = \int_{\mathbb{R}^2} e^{-\frac{1}{1-|z|^2}} \, dxdy$ and $\varphi_\varepsilon(z) = \varepsilon^{-2}\varphi_1(z/\varepsilon)$, we find for the convolution

$$\varphi_\varepsilon * \chi\tilde{E}u \in C_0^\infty(\mathcal{R}_{00}) \quad \text{for} \quad \varepsilon < \operatorname{dist}\left(\operatorname{supp}\chi\tilde{E}u, \partial\mathcal{R}_{00}\right) = \min\left(\frac{a}{3}, \frac{b}{3}\right)$$

and

$$\left\|\varphi_\varepsilon * \chi\tilde{E}u - \tilde{E}u\right\|_{W^{2,2}(\mathcal{R}_{00})} \to 0 \quad \text{for} \quad \varepsilon \downarrow 0.$$

It follows that

$$\left\|\left(\varphi_\varepsilon * \chi\tilde{E}u\right)_{|\mathcal{R}} - u\right\|_{W^{2,2}(\mathcal{R})} \to 0 \quad \text{for} \quad \varepsilon \downarrow 0.$$

By the symmetry of $\tilde{E}$ and φ_ε, and the fact that $\chi = 1$ near $\partial\mathcal{R}$, it follows that $\varphi_\varepsilon * \chi\tilde{E}u = 0$ on $\partial\mathcal{R}$ for ε small enough. Hence $\left(\varphi_\varepsilon * \chi\tilde{E}u\right)_{|\overline{\mathcal{R}}} \in C_0\left(\overline{\mathcal{R}}\right)$ for those small ε. ■

Regularity of hinged plates. The argument is proven indirectly, by showing that a hinged rectangular plate solves the Navier bilaplace problem. To that end we take the following steps.

Dirichlet Laplace. In the proof of Lemma 3.3.1 we have used properties of the solution of the Dirichlet-Laplace problem. Indeed, if $f \in L^2(\mathcal{R})$, then the solution u of

$$\begin{cases} -\Delta u = f & \text{in } \mathcal{R}, \\ u = 0 & \text{on } \partial\mathcal{R} \end{cases} \tag{3.19}$$

satisfies $u \in W^{2,2}(\mathcal{R}) \cap W_0^{1,2}(\mathcal{R})$. If $f \in C^\gamma(\mathcal{R})$ with $\gamma \in (0,1)$, then $\tilde{E}f \in L^\infty(\mathcal{R}_{00})$ and in general $\tilde{E}f \notin C^\gamma(\mathcal{R}_{00})$. Thus we may conclude using interior regularity on $\mathcal{R}_{00}$ that $u \in W^{2,p}(\mathcal{R})$ for any $p \in (1,\infty)$ and through a Sobolev embedding that $u \in C^{1,\theta}\left(\overline{\mathcal{R}}\right)$ for any $\theta \in (0,1)$. This is the optimal regularity if we refrain from putting additional restrictions on the function f.

Iterated Dirichlet Laplace and Navier bilaplace. Concerning the Navier boundary conditions for the bilaplace operator, i.e. the problem

$$\begin{cases} \Delta^2 u = f & \text{in } \mathcal{R}, \\ u = \Delta u = 0 & \text{on } \partial\mathcal{R}, \end{cases} \tag{3.20}$$

an iterated use of the regularity for (3.19) yields a function $u \in W^{2,2}(\mathcal{R})$ with $\Delta u \in W^{2,2}(\mathcal{R})$ satisfying

$$\begin{cases} -\Delta u = w & \text{in } \mathcal{R}, \\ u = 0 & \text{on } \partial\mathcal{R} \end{cases} \text{ and } \begin{cases} -\Delta w = f & \text{in } \mathcal{R}, \\ w = 0 & \text{on } \partial\mathcal{R}. \end{cases} \tag{3.21}$$

But this does not give a priori the optimal result: For any bounded domain Ω and $f \in L^2(\Omega)$ one obtains a unique weak solution $\hat{u} \in H_0(\Omega)$ of (3.20) by minimizing the functional

$$J_1(u) := \int_\Omega \left(\tfrac{1}{2}(\Delta u)^2 - fu\right) dxdy.$$

On smooth boundary parts one finds that this solution $\hat{u}$ satisfies $\Delta\hat{u} = 0$. However, this function is not necessarily the same as the system solution: $\Delta u \in W^{2,2}(\Omega)$ does not in general imply $u \in W^{4,2}(\Omega)$ (see [48]).

On the other hand, if $f = 0$ on $\partial\mathcal{R}$ and $f \in C^\gamma(\mathcal{R})$, then $\tilde{E}f \in C^\gamma(\mathcal{R}_{00})$. This implies that when we consider (3.20) as an iterated Dirichlet Laplacian, a better regularity result is available for the second step.

Lemma 3.3.3 *If $f \in L^2(\mathcal{R})$, then the weak solution $(u,w) \in W_0^{1,2}(\mathcal{R}) \times W_0^{1,2}(\mathcal{R})$ of (3.21) satisfies $u \in W^{4,2}(\mathcal{R})$ and thus $w = -\Delta u$.*

Proof. Assuming that $\tilde{E}$ and $\mathcal{R}_{00}$ are as in (3.18), we have that $\tilde{E}f \in L^2(\mathcal{R}_{00})$. Solving

$$\begin{cases} \Delta^2 \tilde{u} = \tilde{E}f & \text{in } \mathcal{R}_{00}, \\ \Delta\tilde{u} = \tilde{u} = 0 & \text{on } \partial\mathcal{R}_{00}, \end{cases} \tag{3.22}$$

one finds for the weak solution, by standard regularity theory (see [2]), that $\tilde{u} \in W^{4,2}_{loc}(\Omega)$ for any domain Ω with $\overline{\Omega} \subset \mathcal{R}_{00}$. This implies that $\tilde{u}_{|\mathcal{R}} \in W^{4,2}(\mathcal{R})$. Since $\tilde{E}f$ is antisymmetric and the weak solution of (3.22) is unique, applying an argument similar to that in the proof of Lemma 3.3.1(2) one finds that $\tilde{u}$ satisfies

$$\tilde{u}(x,y) = -\tilde{u}(-x,y) = -\tilde{u}(x,-y) = \tilde{u}(-x,-y)$$

for $x \in \mathcal{R}_0 = (0,2a) \times (0,2b)$. Thus

$$\Delta\tilde{u}(x,y) = -\Delta\tilde{u}(-x,y) = -\Delta\tilde{u}(x,-y) = \Delta\tilde{u}(-x,-y),$$

which implies that $\tilde{u} = \Delta\tilde{u} = 0$ on $\partial\mathcal{R}$. Thus we have found that $\left(\tilde{u}_{|\mathcal{R}}, -\Delta\tilde{u}_{|\mathcal{R}}\right)$ is a solution to (3.20) and we may conclude that $u \equiv \tilde{u}_{|\mathcal{R}} \in W^{4,2}(\mathcal{R})$. ■

A hinged rectangular plate. Let us use the following notation:

$$\mathcal{K}(u) := \int_{\mathcal{R}} \det(\nabla^2 u)\, dxdy, \text{ for } u \in W^{2,2}(\mathcal{R}),$$

where $\nabla^2 u$ is the Hessian matrix of u and $\det(\nabla^2 u) = u_{xx}u_{yy} - u_{xy}^2$. Defining

$$J_1(u) := \int_{\mathcal{R}} \left(\tfrac{1}{2}(\Delta u)^2 - fu\right) dxdy,$$

we obtain the following decomposition of the energy functional:

$$J_\sigma(u) = J_1(u) - (1-\sigma)\mathcal{K}(u). \tag{3.23}$$

As we have seen for the case of smooth domains (see Lemma 3.2.2), $\mathcal{K}(u)$ turns out to be a boundary term and its behaviour is going to yield the corresponding natural boundary conditions for a Kirchhoff plate. Here we use a more straightforward integration by parts and Corollary 3.3.2.

Lemma 3.3.4 *Let $u \in C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}})$. Then for all $v \in C^\infty(\overline{\mathcal{R}})$*

$$\mathcal{K}'(u;v) = -\int_{\partial\mathcal{R}} \partial_{\tau n} u\, \partial_\tau v\, ds = 2\left[u_{xy}v\right]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)} + \int_{\partial\mathcal{R}} \partial_{\tau\tau n} u\, v\, ds,$$

where $[\Psi]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)} := \Psi(a,0) + \Psi(0,b) - \Psi(0,0) - \Psi(a,b)$.

Proof. Standard use of Fubini and integrating by parts gives

$$\begin{aligned}
\int_{\mathcal{R}} u_{xy}v_{xy}\, dxdy &= \int_0^a \left[u_{xy}(x,y)v_x(x,y)\right]_{y=0}^b dx \\
&\quad - \int_0^a \int_0^b u_{xyy}(x,y)v_x(x,y)\, dy\, dx \\
&= \int_0^b \left[u_{xy}(x,y)v_y(x,y)\right]_{x=0}^a dy \\
&\quad - \int_0^b \int_0^a u_{xxy}(x,y)v_y(x,y)\, dx\, dy.
\end{aligned}$$

Since $\mathcal{R}$ has boundary parts parallel to the axes,

$$u(x,0) = u(x,b) = u(0,y) = u(a,y) = 0$$

implies that

$$u_{xx}(x,0) = u_{xx}(x,b) = u_{yy}(0,y) = u_{yy}(a,y) = 0.$$

Thus one obtains

$$\begin{aligned}\int_{\mathcal{R}} u_{xx}v_{yy}\;dxdy &= \int_0^a \left[u_{xx}(x,y)v_y(x,y)\right]_{y=0}^b dx \\ &\quad -\int_0^a\int_0^b u_{xxy}(x,y)v_y(x,y)\;dy\,dx \\ &= -\int_0^a\int_0^b u_{xxy}(x,y)v_y(x,y)\;dy\,dx\end{aligned}$$

and similarly

$$\begin{aligned}\int_{\mathcal{R}} u_{yy}v_{xx}\;dxdy &= \int_0^b \left[u_{yy}(x,y)v_x(x,y)\right]_{x=0}^a dy \\ &\quad -\int_0^b\int_0^a u_{xyy}(x,y)v_x(x,y)\;dx\,dy \\ &= -\int_0^b\int_0^a u_{xyy}(x,y)v_x(x,y)\;dx\,dy.\end{aligned}$$

Hence, a direct calculation yields that

$$\begin{aligned}\mathcal{K}'(u;v) &= \int_{\mathcal{R}} \left(u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}\right)\,dxdy \\ &= -\int_0^a \left[u_{xy}(x,y)v_x(x,y)\right]_{y=0}^b dx \\ &\quad -\int_0^b \left[u_{xy}(x,y)v_y(x,y)\right]_{x=0}^a dy \\ &= -\int_{\partial\mathcal{R}} \partial_{\tau n}u\;\partial_\tau v\;ds.\end{aligned}$$

Moreover, by

$$\begin{aligned}-\int_0^a u_{xy}(x,b)v_x(x,b)dx &= -\left[u_{xy}v\right]_{(0,b)}^{(a,b)} + \int_0^a u_{xxy}(x,b)v(x,b)dx, \\ \int_0^a u_{xy}(x,0)v_x(x,0)dx &= \left[u_{xy}v\right]_{(0,0)}^{(a,0)} - \int_0^a u_{xxy}(x,b)v(x,b)dx, \\ -\int_0^b u_{xy}(a,y)v_y(a,y)dy &= -\left[u_{xy}v\right]_{(a,0)}^{(a,b)} + \int_0^b u_{xyy}(a,y)v(a,y)dy, \\ \int_0^b u_{xy}(0,y)v_y(0,y)dy &= \left[u_{xy}v\right]_{(0,0)}^{(0,b)} - \int_0^b u_{xyy}(0,y)v(0,y)dy,\end{aligned}$$

we find that

$$\mathcal{K}'(u;v) = 2\left[u_{xy}v\right]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)} + \int_{\partial\mathcal{R}} \partial_{\tau\tau n} u \; v \; ds$$

and the Lemma is proved. ∎

Corollary 3.3.5 *It holds that $\mathcal{K}(u) = 0$ for all $u \in H_0(\mathcal{R})$.*

Proof. Let $u \in H_0(\mathcal{R})$. Using Corollary 3.3.2 we can find a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}})$, such that $u_k \to u$ in $H_0(\overline{\mathcal{R}})$ for $k \to \infty$. For $u_k \in C^\infty(\overline{\mathcal{R}}) \cap C_0(\overline{\mathcal{R}})$ one finds that

$$\mathcal{K}(u_k) = \frac{1}{2}\mathcal{K}'\left(u_k; u_k\right)$$

and by Lemma 3.3.4

$$\mathcal{K}'\left(u_k; u_k\right) = -\int_{\partial\mathcal{R}} \partial_{\tau n} u_k \; \partial_\tau u_k \; ds = 0.$$

Since $(u,v) \mapsto \mathcal{K}'(u;v)$ is continuous on $H_0(\mathcal{R}) \times H_0(\mathcal{R})$, one has that

$$\mathcal{K}(u) = \frac{1}{2}\mathcal{K}'\left(u;u\right) = \frac{1}{2}\lim_{k\to\infty} \mathcal{K}'\left(u_k; u_k\right) = 0.$$

∎

Remark 3.3.6 *Thus, in the case of a rectangular plate with fixed boundary, the total energy functional becomes*

$$J_\sigma(u) = J_1(u) = \int_{\mathcal{R}} \left(\tfrac{1}{2}(\Delta u)^2 - fu\right) dxdy.$$

Corollary 3.3.7 *If $\tilde{u}$ is a minimizer of J_σ in $H_0(\mathcal{R})$ with $f \in L^2(\mathcal{R})$, then $u \in W^{4,2}(\mathcal{R})$.*

Proof. It is a direct result of Remark 3.3.6 and Lemma 3.3.3. ∎

The comparison argument.

Finally we can state the main result of this section. The right angles of our rectangle will allow us to deploy an argument based on Serrin's corner point Lemma ([53]).

Theorem 3.3.8 *Let $f \in L^2(\mathcal{R})$ with $0 \not\equiv f \leq 0$. Then, the minimizer $\tilde{u}$ of J_1 in $H_0(\mathcal{R})$ cannot be a minimizer of J_σ in $H_+(\mathcal{R})$.*

Proof. We proceed by contradiction and assume that $\tilde{u} \in H_0(\mathcal{R})$ minimizes J_σ also in $H_+(\mathcal{R})$. By Corollary 3.3.7 we find that $\tilde{u} \in W^{4,2}(\mathcal{R})$. Consequently, Sobolev's embedding Theorem implies that $\tilde{u} \in C^{2,\theta}(\overline{\mathcal{R}})$ for $0 < \theta < 1$ and that the traces of 3rd order derivatives of $\tilde{u}$ are well defined in $L^2(\partial\mathcal{R})$.

Letting $v \in C^\infty(\overline{\mathcal{R}})$ and integrating by parts the corresponding variational inequality, we find that

$$\begin{aligned} J'_\sigma(\tilde{u}; v - \tilde{u}) &= J'_\sigma(\tilde{u}; v) - J'_\sigma(\tilde{u}; \tilde{u}) \\ &= J'_\sigma(\tilde{u}; v) \\ &= J'_1(\tilde{u}; v) - (1-\sigma)\mathcal{K}'(\tilde{u}; v) \\ &= \int_{\mathcal{R}} (\Delta\tilde{u}\Delta v - fv)\, dxdy - (1-\sigma)\mathcal{K}'(\tilde{u}; v). \end{aligned} \tag{3.24}$$

Moreover, since $\tilde{u}$ also minimizes J_σ in $H_0(\mathcal{R})$, Remark 3.3.6 yields that $\Delta\tilde{u} = 0$ on $\partial\mathcal{R}$. Hence we have

$$\begin{aligned} \int_{\mathcal{R}} \Delta\tilde{u}\Delta v\, dxdy &= \int_0^a [\Delta\tilde{u}\ v_y - \Delta\tilde{u}_y\ v]_{y=0}^b\, dx + \int_0^b [\Delta\tilde{u}\ v_x - \Delta\tilde{u}_x\ v]_{x=0}^a\, dy \\ &\quad + \int_{\mathcal{R}} \Delta^2\tilde{u}\ v\ dxdy \\ &= \int_{\mathcal{R}} \Delta^2\tilde{u}\ v\ dxdy - \int_0^a [\Delta\tilde{u}_y\ v]_{y=0}^b\, dx - \int_0^b [\Delta\tilde{u}_x\ v]_{x=0}^a\, dy \\ &= \int_{\mathcal{R}} \Delta^2\tilde{u}\ v\ dxdy - \int_{\partial\mathcal{R}} \partial_n(\Delta\tilde{u})\, v\, ds. \end{aligned}$$

Using Lemma 3.3.4, Corollary 3.3.2 and the density of smooth functions into $L^2(\partial\mathcal{R})$ we find

$$\begin{aligned} J'_\sigma(\tilde{u}; v - \tilde{u}) \quad = \quad & \int_{\mathcal{R}} \left(\Delta^2\tilde{u} - f\right) v\, dxdy - \int_{\partial\mathcal{R}} \partial_n\left(\Delta\tilde{u} + (1-\sigma)\,\partial_{\tau\tau}\tilde{u}\right) v\, ds \\ & -2\,(1-\sigma)\,[\tilde{u}_{xy}v]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)}. \end{aligned}$$

We have assumed that $\tilde{u}$ is a minimizer in $H_+(\mathcal{R})$ and, thus, one has $J'_\sigma(\tilde{u}; v - \tilde{u}) \geq 0$ for all $v \geq 0$ on $\partial\mathcal{R}$. Hence $\Delta^2\tilde{u} = f$ and

$$\int_{\partial\mathcal{R}} \partial_n\left(\Delta\tilde{u} + (1-\sigma)\,\partial_{\tau\tau}\tilde{u}\right) v\, ds + 2\,(1-\sigma)\,[\tilde{u}_{xy}v]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)} \leq 0. \tag{3.25}$$

Since $\tilde{u} = 0$ on $\partial\mathcal{R}$ and $f \leq 0$ is nontrivial, then $\tilde{u}_{xy}$ will have a sign at these corners. We claim that $\tilde{u}_{xy}(0,0) < 0$. As $\tilde{u} \in W^{4,2}(\mathcal{R})$, the function $\tilde{u}$ solves

$$\begin{cases} -\Delta\tilde{u} = w & \text{in } \mathcal{R}, \\ \tilde{u} = 0 & \text{on } \partial\mathcal{R}, \end{cases}$$

where w solves

$$\begin{cases} -\Delta w = f & \text{in } \mathcal{R}, \\ w = 0 & \text{on } \partial\mathcal{R}. \end{cases}$$

By the maximum principle it follows for $f \le 0$ and nontrivial, that $w < 0$ in $\mathcal{R}$. An application of the maximum principle to $\tilde{u}$ implies $\tilde{u} < 0$ and by Hopf's boundary point Lemma even that $\partial_n \tilde{u} > 0$ away from corners. At corners, the fact that $\tilde{u} = 0$ on $\partial\mathcal{R}$ and the C^2 smoothness imply $|\nabla \tilde{u}| = 0$. Hence we may use Serrin's corner point Lemma (see [53]) which implies that $(\partial_{\gamma\gamma}\tilde{u})(0,0) < 0$ for all directions γ, entering $\mathcal{R}$ non-tangentially. Taking $\gamma = \left(\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$ we have

$$(\partial_{\gamma\gamma}\tilde{u})(0,0) = \tfrac{1}{2}\tilde{u}_{xx}(0,0) + \tilde{u}_{xy}(0,0) + \tfrac{1}{2}\tilde{u}_{yy}(0,0) = \tilde{u}_{xy}(0,0).$$

Thus we find $\tilde{u}_{xy}(0,0) < 0$. Let $\varepsilon > 0$ and consider the test function

$$v_\varepsilon(x,y) = e^{-\left(x^2+y^2\right)/\varepsilon}.$$

One obtains

$$\int_{\partial\mathcal{R}} \partial_n \left(\Delta\tilde{u} + (1-\sigma)\,\partial_{\tau\tau}\tilde{u}\right) v_\varepsilon \, ds = \mathcal{O}(\varepsilon) \ \text{ for } \varepsilon \downarrow 0$$

and

$$2(1-\sigma)\left[\tilde{u}_{xy}v_\varepsilon\right]_{(a,b)}^{(a,0)\,\&\,(0,b)} = \mathcal{O}\left(e^{-\min(a,b)^2/\varepsilon}\right) \le \mathcal{O}(\varepsilon) \ \text{ for } \varepsilon \downarrow 0.$$

However,

$$2(1-\sigma)\left[\tilde{u}_{xy}v_\varepsilon\right]_{(0,0)} = -2(1-\sigma)\,\tilde{u}_{xy}(0,0) > 0$$

and we find

$$\begin{aligned} \int_{\partial\mathcal{R}} \partial_n \left(\Delta\tilde{u} + (1-\sigma)\,\partial_{\tau\tau}\tilde{u}\right) v_\varepsilon \, ds + 2(1-\sigma)\left[\tilde{u}_{xy}v_\varepsilon\right]_{(0,0)\,\&\,(a,b)}^{(a,0)\,\&\,(0,b)} \\ = \mathcal{O}(\varepsilon) - 2(1-\sigma)\,\tilde{u}_{xy}(0,0) > 0 \end{aligned}$$

for ε sufficiently small, which is a contradiction to (3.25). ■

3.3.2 A plate with corners of arbitrary opening angle

Here we consider the general case where the corners of the plate have an arbitrary opening angle $\omega \in (0, 2\pi)$, measured from the inside. Note that one does not expect that the solution will have the regularity that a rectangular plate exhibits, unless some orthogonality conditions are fulfilled. Following the theory developed by Kondrat'ev and Williams [35, 56], the solutions will have an expansion near the corner consisting of a regular part and a singular one. The coefficients of these "singular eigenfunctions" depend on the domain and f; they are going to be zero only when f is orthogonal to a set of "adjoint eigenfunctions". For a full development of the theory see [26, 28, 36, 37, 42, 43, 47].

The boundary value problem

We start with two useful observations that are going to enable us to compare minimizers and solutions for the hinged plate boundary value problem in weighted spaces. The goal is to show that a minimizer of J_σ in $H_0(\Omega)$ is a solution to a boundary value problem away from the cornerpoints in a weighted Sobolev space and vice versa.

Lemma 3.3.9 *Let u_σ be a minimizer of J_σ in $H_+(\Omega)$. Then*

$$J'(u_\sigma; u_\sigma) = 0 \quad \text{and } J'(u_\sigma; v) \geq 0 \text{ for all } v \in H_+(\Omega). \tag{3.26}$$

Proof. Since u_σ is a minimizer, it satisfies the variational inequality

$$J'(u_\sigma; v - u_\sigma) \geq 0 \text{ for all } v \in H_+(\Omega).$$

Taking $v = 2u_\sigma \in H_+(\Omega)$ and $v \equiv 0 \in H_+(\Omega)$ completes the proof. ■ In the next lemma we consider for $\omega \in (0, 2\pi)$ the unit circular sector of radius 1:

$$\Omega_\omega := \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2;\ 0 < r < 1 \text{ and } 0 < \theta < \omega \right\}.$$

Lemma 3.3.10 *Let $\partial_{\mathbf{i}}$, for $\mathbf{i} = \mathbf{1}, ..., \mathbf{4}$ a multi-index with $i = |\mathbf{i}|$, denote any partial derivative of order i. Then the bilinear forms*

(i) $b_1(u,v) := \int_{\partial\Omega_\omega} \partial_{\mathbf{2}} u \, \partial_{\mathbf{1}} v \, ds,$

(ii) $b_2(u,v) := \int_{\Omega_\omega} \partial_{\mathbf{2}} u \, \partial_{\mathbf{2}} v \, dxdy$ *and*

(iii) $b_3(u,v) := \int_{\Omega_\omega} \partial_{\mathbf{4}} u \, v \, dxdy$

are continuous in $W_4^4(\Omega_\omega) \times H_0(\Omega_\omega)$.

Proof. *(i)* Since $u \in W_4^4(\Omega_\omega)$, one gets that $\partial_{\mathbf{2}} u \in W_4^2(\Omega_\omega)$, that is $\partial_{\mathbf{2}} u \in W_4^{3/2}(\partial\Omega_\omega) \subset W_1^0(\partial\Omega_\omega)$ (see [36, Lemma 6.1.2]). On the other hand, we have that

$$H_0(\Omega_\omega) \subset W_0^2(\Omega_\omega) \cap W_{-2}^1(\Omega_\omega)$$

(see [48, Lemma 3.4]) and thus $\partial_{\mathbf{1}} v \in W_0^{1/2}(\partial\Omega) \subset W_{-1}^0(\partial\Omega)$. Then, using the Cauchy-Schwarz inequality, one gets

$$\begin{aligned} |b_1(u,v)| &= \int_{\partial\Omega_\omega} (r^{\frac{1}{2}} \partial_{\mathbf{2}} u) \, (r^{-\frac{1}{2}} \partial_{\mathbf{1}} v) \, ds \\ &\leq \left(\int_{\partial\Omega_\omega} r \, (\partial_{\mathbf{2}} u)^2 \, ds \right)^{\frac{1}{2}} \left(\int_{\partial\Omega_\omega} r^{-1} (\partial_{\mathbf{1}} v)^2 \, ds \right)^{\frac{1}{2}} \\ &\leq c \, \|\partial_{\mathbf{2}} u\|_{W_4^2(\Omega_\omega)} \, \|v\|_{W_0^2(\Omega_\omega)} . \end{aligned}$$

(ii) It is immediate since $\partial_{\mathbf{2}} u \in W_4^2(\Omega_\omega) \subset W_0^0(\Omega_\omega) = L^2(\Omega_\omega)$.

(iii) One has the following estimate

$$\begin{aligned} |b_3(u,v)| &= \int_{\Omega_\omega} (r^2 \partial_{\mathbf{4}} u)(r^{-2} v)\, dxdy \\ &\le \left(\int_{\Omega_\omega} r^4 |\partial_{\mathbf{4}} u|^2\, dxdy \right)^{\frac{1}{2}} \left(\int_{\Omega_\omega} r^{-4} |v|^2\, dxdy \right)^{\frac{1}{2}} \\ &= A \cdot B. \end{aligned} \tag{3.27}$$

Moreover, since $u \in W_4^4(\Omega_\omega)$, we get that

$$\begin{aligned} \|u\|^2_{W_4^4(\Omega_\omega)} &= \sum_{|m|\le 4} r^{4-2(4-|m|)} |\partial_m u|^2\, dxdy \\ &\ge \int_{\Omega_\omega} r^4 |\partial_{\mathbf{4}} u|^2\, dxdy = A^2 \end{aligned} \tag{3.28}$$

and since $v \in H_0(\Omega) \subset W_0^2(\Omega_\omega)$ (see [48, Lemma 3.4]) it holds that

$$\begin{aligned} \|v\|^2_{2,2} &\ge c\, \|v\|^2_{W_0^2(\Omega_\omega)} = \sum_{|m|\le 2} r^{-2(2-|m|)} |\partial_m v|^2\, dxdy \\ &\ge \int_{\Omega_\omega} r^{-4} |v|^2\, dxdy = B^2. \end{aligned} \tag{3.29}$$

Combining (3.27), (3.28) and (3.29) completes the proof. ■

The above Lemma enables us to integrate by parts functions which belong to a weighted space.

Corollary 3.3.11 *Let $\Omega \subset \mathbb{R}^2$ be bounded, piecewise smooth with corner boundary singularities, $u \in W_4^4(\Omega)$ and $v \in H_0(\Omega)$. Then the following Green's identity holds:*

$$\int_\Omega \Delta u \Delta v\, dxdy = \int_\Omega \Delta^2 u\, v\, dxdy + \int_{\partial\Omega} \Delta u\, \partial_n v\, ds. \tag{3.30}$$

Proof. Let $\mathcal{S}$ be the set of cornerpoints of $\partial\Omega$ and $\{u_k\}_{k\in\mathbb{N}} \subset C_0^\infty\big(\overline{\Omega} \setminus \mathcal{S}\big)$, such that $\lim_{k\to\infty} \|u_k - u\|_{W_4^4(\Omega)} = 0$. Then, (3.30) holds true for $u \equiv u_k$ and Lemma 3.3.10 allows us to take the limit as $k \to \infty$ to complete the proof. ■

As in previous sections, we define

$$\mathcal{K}(u) := \int_\Omega \det(\nabla^2 u)\, dxdy$$

for $u \in W^{2,2}(\Omega)$, where $\nabla^2 u$ denotes the Hessian matrix of u.

Corollary 3.3.12 *Let $\Omega \subset \mathbb{R}^2$ be bounded and piecewise smooth with corner boundary singularities and let $\mathcal{S}$ be the set containing the corners of $\partial\Omega$. Then the following hold true:*

(i) For all $u \in H_0(\Omega)$ and $v \in W^{3,2}(\Omega)$ it holds that

$$\mathcal{K}'(u;v) = \int_{\partial\Omega} \left(\kappa(s)\, \partial_n u\, \partial_n v + \partial_n u\, \partial_{\tau\tau} v\right)\, ds. \tag{3.31}$$

(ii) For all $u \in H_0(\Omega)$ we have

$$\mathcal{K}(u) = \frac{1}{2} \int_{\partial\Omega} \kappa(s)(\partial_n u)^2\, ds. \tag{3.32}$$

(iii) $\mathcal{K}'(u;v)$ is a continuous bilinear form in $W_4^4(\Omega) \times H_0(\Omega)$.

Proof. Using the same argumentation as in the proof of Lemma 3.2.2, we directly get that (3.31) holds. Applying Theorem 2.2.1 we are able to pass to the limit in (3.31) and complete the proof of *(i)* and *(ii)*.

Concerning *(iii)*, one gets

$$\mathcal{K}'(u;v) = \int_{\partial\Omega} \left(\kappa(s)\, \partial_n u\, \partial_n v + \partial_n v\, \partial_{\tau\tau} u\right)\, ds$$

for $u \in C_0^\infty\left(\overline{\Omega} \setminus \mathcal{S}\right)$ and $v \in H_0(\Omega)$. The result follows then by density with the help of Lemma 3.3.10. ■

Now we are able to give the relationship between the minimization and the boundary value problem for a hinged plate.

Corollary 3.3.13 *Let $f \in L^2(\Omega)$ and $-1 < \sigma < 1$.*

(i) A hinged plate, i.e. the unique minimizer of J_σ in $H_0(\Omega)$, lies in $W^{4,2}(\Omega_1)$ for any open Ω_1 with $\overline{\Omega}_1 \subset \overline{\Omega} \setminus \mathcal{S}$, and satisfies

$$\begin{cases} \Delta^2 u = f & a.e. \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \Delta u - (1-\sigma)\kappa\partial_n u = 0 & \text{on } \partial\Omega \setminus \mathcal{S}, \end{cases} \tag{3.33}$$

where $\mathcal{S}$ is the set of corners of $\partial\Omega$.

(ii) If $u \in W_4^4(\Omega)$ satisfies (3.33) then it is a minimizer of J_σ in $H_0(\Omega)$.

Proof. Let $\varepsilon > 0$ and define

$$B(\mathcal{S}) := \bigcup_{x \in \mathcal{S}} B_\varepsilon(x).$$

Since Problem (3.33) is regular on $\partial\Omega \setminus \mathcal{S}$ and the boundary of the domain is smooth away from the corners, one can show, using standard regularity techiniques, that the minimizer $u \in H_0(\Omega)$ of J_σ lies in $W^{4,2}\big(\Omega \setminus \overline{B(\mathcal{S})}\big)$. Then one has that $J'(u;\varphi) = 0$ for all $\varphi \in C^\infty(\overline{\Omega}) \cap C_0(\overline{\Omega})$ compactly supported with $\operatorname{supp}\varphi \subset \overline{\Omega} \setminus \overline{B(\mathcal{S})}$ and an integration by parts is allowed:

$$\begin{aligned} 0 &= \int_{\partial\Omega\setminus\overline{B(\mathcal{S})}} (\Delta u \Delta\varphi - f\varphi)\, dxdy - (1-\sigma)\mathcal{K}'(u;\varphi) \\ &= \int_{\Omega\setminus\overline{B(\mathcal{S})}} \left(\Delta^2 u - f\right)\varphi\, dx + \int_{\partial\Omega\setminus\overline{B(\mathcal{S})}} \Big(\Delta u - (1-\sigma)\,\kappa\partial_n u\Big)\,\partial_n\varphi\, ds \\ &\quad - \int_{\partial\Omega\setminus\overline{B(\mathcal{S})}} \big((1-\sigma)\,\partial_{\tau\tau n} u + \partial_n\Delta u\big)\,\varphi\, ds. \end{aligned} \tag{3.34}$$

Thus, one obtains the differential equation in $\Omega \setminus \overline{B(\mathcal{S})}$ and the natural boundary condition on $\partial\Omega \setminus \overline{B(\mathcal{S})}$. Letting $\varepsilon \to 0$ we get that

$$\Delta^2 u = f \text{ in } \Omega \quad \text{and} \quad \Delta u - (1-\sigma)\kappa\partial_n u = 0 \text{ on } \partial\Omega \setminus \mathcal{S}.$$

On the other hand, (3.31) and Green's identity (3.30) imply that if for the solution u to (3.33) holds that $u \in W_4^4(\Omega)$, then u will satisfy the weak Euler-Lagrange equation $J_\sigma(u;v) = 0$ for all $v \in H_0(\Omega)$. ■

Kondrat'ev's expansion near a corner

Let $\omega \in (0, 2\pi)$ and define

$$\mathcal{K}_\omega := \left\{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2;\ r > 0 \text{ and } 0 < \theta < \omega\right\}, \tag{3.35}$$

an infinite circular sector of $\mathbb{R}^2$, centered at the origin with an opening angle ω. Consider the following problem

$$\begin{cases} \Delta^2 u = f & \text{in } \mathcal{K}_\omega, \\ u = \Delta u = 0 & \text{on } \partial\mathcal{K}_\omega. \end{cases} \tag{3.36}$$

According to [35], for solving (3.36) one needs to find the nonzero solutions for the following two-point boundary value problem

$$\begin{cases} v''''(\theta) + 2(2 - 2\lambda + \lambda^2)v''(\theta) + \lambda^2(\lambda-2)^2 v(\theta) = 0 \quad \text{in } (0,\omega), \\ v(0) = v''(0) = 0, \\ v(\omega) = v''(\omega) = 0. \end{cases} \tag{3.37}$$

We assume that Ω is smooth with the exception of N corners with interior opening angles $\omega_i \in (0, 2\pi)$ for $i = 1, .., N$. Using the results of the previous section we are able to prove a regularity assertion for a hinged plate.

Proposition 3.3.14 *Let $f \in L^2(\Omega)$ and assume for all $i = 1, .., N$ that $\omega_i \in (0, 2\pi)$. Then the weak solution u for (3.33), i.e. the minimizer of J_σ in $H_0(\Omega)$, belongs in $W_4^4(\Omega)$.*

Proof. We first show the existence of a solution for (3.33) in $W_4^4(\Omega)$, when f belongs in a larger space than $L^2(\Omega)$. Assume, for the time being, that $f \in W_4^0$ and define the operator

$$L : W_4^4(\Omega) \to W_4^0(\Omega) \quad \text{with} \quad Lu := \Delta^2 u \quad \text{and}$$
$$\mathcal{D}(L) := \{u \in W_4^4(\Omega);\ u = 0 \text{ on } \partial\Omega,\ \Delta u - (1-\sigma)\kappa\partial_n u = 0 \text{ on } \partial\Omega \setminus \mathcal{S}\}.$$

Note that functions in $W_4^4(\Omega)$ are C^2 up to the boundary, away from the corners and thus, the operator L is well defined. Now we apply [36, Theorem 6.3.3] to find that L is Fredholm when $\lambda \neq 1$, where λ is any eigenvalue of problem (3.37). We can directly solve (3.37) by assuming exponential type solutions, to obtain for each j a pair of eigenvalues λ_j, μ_j corresponding to the same type of eigenfunctions Φ_j:

$$\lambda_j = \frac{j\pi}{\omega} \text{ and } \mu_j = \frac{j\pi}{\omega} + 2 \text{ with } \Phi_j = \sin\left(\frac{j\pi}{\omega}\theta\right).$$

Thus, L is Fredholm when $\omega_i \neq 0, \pi, 2\pi$ and its range coincides with the set of all functions $f \in W_4^0(\Omega)$ such that

$$\int_\Omega fv\ dxdy = 0 \quad \text{for all } v \in \ker L^\dagger,$$

where the operator $L^\dagger : W_4^4(\Omega) \to W_4^0(\Omega)$ is defined similarly to L for the formally adjoint problem to (3.33): Let $u, v \in C_0^\infty(\overline{\Omega} \setminus \mathcal{S})$ (for the definition see Chapter 2) and calculate

$$\begin{aligned}
&\int_\Omega \left(\Delta^2 u\right) v\ dxdy + \int_{\partial\Omega} \left(\Delta u - (1-\sigma)\kappa\partial_n u\right) \partial_n v\ ds \\
&= \int_\Omega \Delta u \Delta v\ dxdy - (1-\sigma) \int_{\partial\Omega} \kappa\partial_n u\ \partial_n v\ ds \\
&= \int_\Omega \left(\Delta^2 v\right) u\ dxdy + \int_{\partial\Omega} \left(\Delta v - (1-\sigma)\kappa\partial_n v\right) \partial_n u\ ds.
\end{aligned}$$

Thus, in view of [36, Section 6.2.3], we obtain $L = L^\dagger$, that is, the problem (3.33) is formally self-adjoint. To complete the proof of this step we need to show that $\ker L = \{0\}$. Let $u \in W_4^4(\Omega)$, such that

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\kappa\partial_n u = 0 & \text{on } \partial\Omega \setminus \mathcal{S}. \end{cases} \tag{3.38}$$

Since $W_4^4(\Omega) \subset W_0^2(\Omega) \subset W^{2,2}(\Omega)$, Corollary 3.3.13 implies that u is the unique minimizer of J_σ in $H_0(\Omega)$ with $f \equiv 0$, that is $u \equiv 0$. ∎

Corollary 3.3.15 *Assume that $u \in W_4^4(\Omega)$ solves (3.33) for $f \in C_0^\infty(\overline{\Omega} \setminus \mathcal{S})$. Then, for each corner of opening angle $\omega_i \in (0, 2\pi) \setminus \{\frac{3\pi}{2}\}$, there exists $k = k_{\omega_i}$ arbitrarily large, such that in a neighbourhood of this corner u has the following expansion:*

$$u = \sum_{0<\frac{j\pi}{\omega}<k+3} c_j r^{\frac{j\pi}{\omega}} \sin\left(\frac{j\pi}{\omega}\theta\right) + \sum_{0<\frac{j\pi}{\omega}+2<k+3} c_j' r^{\frac{j\pi}{\omega}+2} \sin\left(\frac{j\pi}{\omega}\theta\right) + w \tag{3.39}$$

with $w \in W_0^{k+4}(\Omega)$. Moreover, if u_l denotes the lowest order term in the above expansion, then for

- $\omega \in (0, \pi) : u_l = c_1 r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right)$,
- $\omega \in \left(\pi, \frac{3\pi}{2}\right) : u_l = -c_{-1}' r^{2-\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right)$,
- $\omega \in \left(\frac{3\pi}{2}, 2\pi\right) : u_l = c_2 r^{\frac{2\pi}{\omega}} \sin\left(\frac{2\pi}{\omega}\theta\right)$.

Note that in the last case the lowest order term is sign-changing.

Proof. Since $f \in C_0^\infty(\overline{\Omega} \setminus \mathcal{S})$, we have that $f \in W_0^k(\Omega)$ for all $k \in \mathbb{N}$. A solution of the hinged plate problem satisfies $u \in W_4^4(\Omega) \subset W_6^4(\Omega)$ and thus one can apply [35, Theorem 3.3] with $k_1 = k = \alpha_1 = 0$, and $\alpha = 6$ to obtain that the solution will have the expansion

$$u = \sum_{0<\frac{j\pi}{\omega}<k+3} c_j r^{\frac{j\pi}{\omega}} \sin\left(\frac{j\pi}{\omega}\theta\right) + \sum_{0<\frac{j\pi}{\omega}+2<k+3} c_j' r^{\frac{j\pi}{\omega}+2} \sin\left(\frac{j\pi}{\omega}\theta\right) + w \tag{3.40}$$

whenever $\frac{j\pi}{\omega} \neq k+3$ with $w \in W_0^{k+4}(\Omega)$. Since $\omega \neq \pi, 2\pi$, we can always choose k as large as needed, such that

$$j = \frac{\omega}{\pi}(k+3) \tag{3.41}$$

is not a positive integer: If $\frac{\omega}{\pi} \in \mathbb{Q}$ and $\frac{\omega}{\pi}(k+3) \in \mathbb{N}$, then $\frac{\omega}{\pi}(k+1+3) \notin \mathbb{N}$. Moreover, there will exist at least one term in the above sum when

$$\omega > \frac{\pi}{k+3}. \tag{3.42}$$

Summing up, for a given opening angle ω we choose k such that (3.42) holds and (3.41) gives that j is not a positive integer. ■

The coefficients in the expansion. Before we move on with the comparison of the hinged and supported plate, it is important to have a certain estimate on the coefficients of the lowest order terms in the expansion (3.39). We would wish to have a general answer to the sign of the coefficients of the lowest order terms. This depends highly on Ω and f and thus a general answer is not to be expected. However, when the boundary of the domain has only convex corners, then one can give the following estimate based on the maximum principle.

Lemma 3.3.16 *Assume that Ω is a convex polygon and let $f \in C_0^\infty(\overline{\Omega} \setminus \mathcal{S})$ with $f \leq 0$ and $f \not\equiv 0$. Then $c_1 < 0$.*

Proof. If all corners of the boundary are convex, one obtains that u has the expansion

$$u = c_1 r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right) + c_1' r^{\frac{\pi}{\omega}+2} \sin\left(\frac{\pi}{\omega}\theta\right) + \text{higher order terms}$$

in an ε-neighbourhood of a corner, where the first term is harmonic and the higher order terms smoother than the first two. Then we have

$$\int_0^\varepsilon \left(r^{\frac{\pi}{\omega}-2}\right)^2 r\, dr < \infty,$$

which implies that $\Delta u \in W^{2,2}(\Omega)$. Thus, a hinged plate will satisfy

$$\begin{cases} -\Delta u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{3.43}$$

and an iterated application of the maximum principle yields that $u < 0$ in Ω. Now u as a solution of the Dirichlet Laplacian with right hand side v has the expansion

$$u = \sum_{0<j<(k+2)\frac{\omega}{\pi}} c_j r^{\frac{\pi}{\omega}} \sin\left(\frac{j\pi}{\omega}\theta\right) + w$$

with $w \in W_0^{k+2}(\Omega)$ (see [36, Section 6.6.1]). Note that $W_0^{k+2}(\Omega) \subset W_{-2k}^2(\Omega)$ and one can apply [48, Lemma 6.7] with $\gamma = -k$ to find that there exists a positive constant C, such that

$$|w| \leq C r^{1+k}$$

sufficiently close to the corner. Moreover, the functions $r^{\frac{\pi}{\omega}} \sin\left(\frac{j\pi}{\omega}\theta\right)$ are sign changing for $j > 1$. Since k can be taken arbitrarily large, we get that $u = o\Big(r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right)\Big)$ and thus $c_1 < 0$. ∎

A comparison argument using Kondrat'ev's "singular eigenfunctions"

A criterion for checking whether a hinged plate is also a solution to the supported problem is given by the following

Lemma 3.3.17 *Let $f \in C_0^\infty(\overline{\Omega} \setminus \mathcal{S})$ and assume that the minimizer $u \in H_0(\Omega)$ of J_σ is also a minimizer in $H_+(\Omega)$. Then*

$$\partial_n \Delta u + (1-\sigma)\partial_{n\tau\tau} u \leq 0 \text{ on } \partial\Omega \setminus \mathcal{S}, \tag{3.44}$$

where $\mathcal{S}$ is the set containing the corners of $\partial\Omega$.

Proof. Similarly as in the proof of Corollary 3.3.13 one can show that for $f \in C_0^\infty(\overline{\Omega}\setminus\mathcal{S}) \subset W^{1,2}(\Omega)$ we get $u \in W^{5,2}(\Omega\setminus\overline{B(\mathcal{S})})$ and thus all third order derivatives of u are continuous on the boundary. Testing the variational inequality $J'(u;v) \geq 0$ with functions v nonnegative on the boundary and supported away from the corners proves the Lemma. ■

Now we move on to compare the hinged and supported plates. For simplicity we assume that $\partial\Omega$ contains only one corner at the origin, of opening angle ω.

Theorem 3.3.18 *Let $\omega \in (0, 2\pi) \setminus \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ and $f \in C_0^\infty(\overline{\Omega} \setminus \{0\})$. Moreover, we make the following assumptions on the coefficients of the lowest order terms in* (3.39)*:*

(i) for $\omega \in (0, \pi) \setminus \left\{\frac{\pi}{2}\right\}$, we assume $c_1 < 0$ and

(ii) for $\omega \in \left(\pi, \frac{3\pi}{2}\right)$, we assume $c'_{-1} > 0$.

Then the minimizer $u \in H_0(\Omega)$ of J_σ cannot be a minimizer in $H_+(\Omega)$.

Remark 3.3.19 *Note that f should also satisfy assumption* (3.4) *on the existence of a minimizer of J_σ in $H_+(\Omega)$. Otherwise the result of the Theorem is trivial: There would exist no minimizer in $H_+(\Omega)$.*

Proof of Theorem 3.3.18. We will show that the "supported" boundary condition

$$N(u) := \partial_n \Delta u + (1-\sigma)\partial_{n\tau\tau} u \leq 0 \text{ on } \partial\Omega \setminus \{0\} \tag{3.45}$$

cannot be satisfied sufficiently close to the origin by a hinged plate. That is, if we assume that a minimizer of J_σ in $H_0(\Omega)$ is also a minimizer in $H_+(\Omega)$, then we are lead to a contradiction. To that end, if u_l denotes the lowest order term in all cases of Corollary 3.3.15, we will show that $N(u_l)$ is also the leading term of $N(u)$. Thus, calculating $N(u_l)$, we will see that it does not satisfy the supported condition near the origin.

A hinged plate, i.e. the minimizer of J_σ in $H_0(\Omega)$ has the following expansion in a neighbourhood of the origin:

$$u = u_l + \text{higher order terms} + w,$$

where $w \in W_0^{k+4}(\Omega)$ with k arbitrarily large. Thus one has for its third order derivatives that $\partial_{\mathbf{3}} w \in W_0^{k+1}(\Omega) \subset W^2_{-2(k-1)}$ and therefore

$$|(\partial_{\mathbf{3}} w)(r,\theta)| \leq c\,\|w\|_{W_0^5(\Omega)}\, r^k \tag{3.46}$$

(see [48, Lemma 6.7]). Hence, there exists a sufficiently large k such that $N(u) \sim N(u_l)$. For $\theta = \omega$ we get $\partial_n = \partial_\theta$ and $\partial_{\tau\tau} = \partial_{rr}$. Consider the following cases:

(1) $\omega \in (0,\pi)$: We get

$$\begin{aligned} N\left(c_1 r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right)\right) &= (1-\sigma)\ \frac{c_1\pi^2\left(\frac{\pi}{\omega}-1\right)}{\omega^2} r^{\frac{\pi}{\omega}-2}\cos\left(\frac{\pi\theta}{\omega}\right)\bigg|_{\theta=\omega} \\ &= (1-\sigma)\frac{c_1\pi^2\left(1-\frac{\pi}{\omega}\right)}{\omega^2} r^{\frac{\pi}{\omega}-2} > 0. \end{aligned}$$

(2) $\omega \in \left(\pi, \frac{3\pi}{2}\right)$: We calculate

$$\begin{aligned} N\left(c'_{-1} r^{2-\frac{\pi}{\omega}} \sin\left(-\frac{\pi}{\omega}\theta\right)\right) &= -\frac{c'_{-1}\pi(\omega-\pi)\big(\sigma(2\omega-\pi)+\pi-6\omega\big)}{\omega^3} r^{-\frac{\pi}{\omega}} \\ &\longrightarrow +\infty, \end{aligned}$$

since for $\sigma < 1 \le \frac{6\omega-\pi}{2\omega-\pi}$ we find that $\sigma(2\omega-\pi)+\pi-6\omega < 0$.

(3) $\omega \in \left(\frac{3\pi}{2}, 2\pi\right)$: Similar to the previous cases we calculate for $\theta = \omega$ that

$$\begin{aligned} N\left(c_2 r^{\frac{2\pi}{\omega}} \sin\left(\frac{2\pi}{\omega}\theta\right)\right)\bigg|_{\theta=\omega} &= (1-\sigma)\ \frac{4c_2\pi^2\left(\frac{2\pi}{\omega}-1\right)}{\omega^2} r^{\frac{2\pi}{\omega}-2}\cos\left(\frac{2\pi\theta}{\omega}\right)\bigg|_{\theta=\omega} \\ &= (1-\sigma)\frac{4c_2\pi^2\left(\frac{2\pi}{\omega}-1\right)}{\omega^2} r^{\frac{2\pi}{\omega}-2}, \end{aligned}$$

whereas for $\theta = 0$ we have that $\partial_n = -\partial_\theta$ and thus

$$\begin{aligned} N\left(c_2 r^{\frac{2\pi}{\omega}} \sin\left(\frac{2\pi}{\omega}\theta\right)\right)\bigg|_{\theta=0} &= (1-\sigma)\ \frac{4c_2\pi^2\left(1-\frac{2\pi}{\omega}\right)}{\omega^2} r^{\frac{2\pi}{\omega}-2}\cos\left(\frac{2\pi\theta}{\omega}\right)\bigg|_{\theta=0} \\ &= -(1-\sigma)\frac{4c_2\pi^2\left(1-\frac{2\pi}{\omega}\right)}{\omega^2} r^{\frac{2\pi}{\omega}-2} \\ &= -\ N\left(c_2 r^{\frac{2\pi}{\omega}} \sin\left(\frac{2\pi}{\omega}\theta\right)\right)\bigg|_{\theta=\omega}. \end{aligned}$$

In this case we need no assumption on the coefficient. Since for $\frac{3\pi}{2} < \omega < 2\pi$ one has that $\frac{2\pi}{\omega} - 2 < 0$ and hence $N(u_l) \to +\infty$ as $r \to 0$ either on the one side of the corner ($\theta = \omega$) or on the other ($\theta = 0$), depending on the sign of c_2. ■

Chapter 4

Decoupling fourth order equations into second order systems

In this part we are going to illustrate how the nonsmoothness of the domain can create problems when one tries to split a fourth order equation into a second order system. An outline of this chapter is as follows: In the first section we recall existence and uniqueness for the Navier and Dirichlet bilaplace problems in smooth domains, aiming to illustrate the equivalence between the original boundary value problem and its corresponding system splitting. In the second section we illustrate the use of piecewise linear finite elements for approximating the system solutions in both cases. Next we consider issues of existence of the Navier bilaplace problem and prove an existence and nonexistence result for the Dirichlet system approach in domains with respectively convex and concave corners. The last section is addressing issues of convergence of the numerical scheme for the Dirichlet bilaplace system.

Here again, Ω will denote an open, bounded subset of $\mathbb{R}^n$ unless noted otherwise.

4.1 Recalling existence of solutions on smooth domains

In this section we explain several ways on how to obtain a solution for the fourth order equation by going through second order systems.

4.1.1 Existence for the Navier case

With $-\Delta u = w$ problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

changes into

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = w & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Definition 4.1.1 *A function u is called a weak solution of (4.1) if*

(i) $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ *and*

(ii) $\int_\Omega (\Delta u \Delta \varphi - f\varphi)\, dx = 0$ *for all* $\varphi \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

Remark 4.1.2 *The weak Euler-Lagrange equation for the minimization problem*

$$\begin{aligned} &\min\left\{J_e(u)\,; u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\right\} \text{ with} \\ &J_e(u) := \int_\Omega \left(\tfrac{1}{2}(\Delta u)^2 - fu\right) dx \end{aligned} \tag{4.3}$$

coincides with (ii) of the definition above.

Definition 4.1.3 *The pair (u, w) is called a weak solution of (4.2) if*

(i) $u, w \in W_0^{1,2}(\Omega)$,

(ii) $\int_\Omega (\nabla u \cdot \nabla \varphi - w\varphi)\, dx = 0$ *and* $\int_\Omega (\nabla w \cdot \nabla \psi - f\psi)\, dx = 0$ *for all* $\varphi, \psi \in W_0^{1,2}(\Omega)$.

Remark 4.1.4 *Considering successively the minimization problems*

$$\min\left\{J_1(w)\,; w \in W_0^{1,2}(\Omega)\right\} \text{ with } J_1(w) := \int_\Omega \left(\tfrac{1}{2}|\nabla w|^2 - fw\right) dx$$

and

$$\min\left\{J_2(u)\,; u \in W_0^{1,2}(\Omega)\right\} \text{ with } J_2(u) := \int_\Omega \left(\tfrac{1}{2}|\nabla u|^2 - wu\right) dx,$$

one finds the second part as the corresponding weak Euler-Lagrange equations. Alternatively one may look for a stationary point of

$$H(u, w) := \int_\Omega \left(\nabla u \cdot \nabla w - f\ u - \tfrac{1}{2}w^2\right) dx \quad \text{for } u, w \in W_0^{1,2}(\Omega) \tag{4.4}$$

and will also find the equations in the definition.

The existence for (4.1) as well as for (4.2) follows from Riesz' representation Theorem. The form $\langle\cdot,\cdot\rangle$ defined by

$$\langle\varphi, \psi\rangle := \int_\Omega \nabla\varphi \cdot \nabla\psi\ dx$$

is an inner product on $W_0^{1,2}(\Omega)$ when Ω is bounded. For $\langle\langle\cdot,\cdot\rangle\rangle$ defined by

$$\langle\langle\varphi, \psi\rangle\rangle := \int_\Omega \Delta\varphi\ \Delta\psi\ dx \tag{4.5}$$

to be an inner product, the domain needs to satisfy some additional regularity: If $\partial\Omega \in C^2$, then one obtains that the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.6}$$

has a unique solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ for $f \in L^2(\Omega)$ satisfying $\|u\|_{2,2} \leq C \|\Delta u\|_2$ with a positive constant C depending only on the domain (see [18]). In the case of less smooth domains one still obtains existence but needs more intricate arguments. We will return to this problem in a later section.

Whenever we do have existence, then for domains which are sufficiently smooth, C^4 suffices (see again [18] for second order differential operators), standard regularity arguments apply and an integration by parts, that is

$$\int_\Omega (\Delta u \Delta\varphi - f\varphi)\, dx = \int_{\partial\Omega} \Delta u\, \partial_n \varphi\, ds + \int_\Omega \left(\Delta^2 u - f\right) \varphi\, dx,$$

shows that the weak solution for (4.1) satisfies the differential equation and also the second boundary condition $\Delta u = 0$ on $\partial\Omega$. Hence the solutions u of both definitions coincide for the smooth case.

4.1.2 Existence for the Dirichlet case

Here we consider

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

A standard way to obtain a solution to (4.7) is by minimizing

$$J_e(u) = \int_\Omega \left(\tfrac{1}{2} (\Delta u)^2 - f\ u \right) dx \tag{4.8}$$

over $u \in W_0^{2,2}(\Omega)$. Such a minimizer exists and is unique since the functional is coercive and strictly convex (hence weakly lower semicontinuous) over the Hilbert space $W_0^{2,2}(\Omega)$. Note that, in contrary to the Navier case, proving coercivity does not need any regularity of the boundary. The unique minimizer satisfies the weak Euler-Lagrange equation

$$0 = \partial J_e(u; \varphi) = \int_\Omega (\Delta u \Delta \varphi - f\varphi)\, dx \quad \text{for all} \quad \varphi \in W_0^{2,2}(\Omega). \tag{4.9}$$

Definition 4.1.5 *The function u is called a weak solution of (4.7) if*

(i) $u \in W_0^{2,2}(\Omega)$ and

(ii) u satisfies (4.9).

With $-\Delta u = w$, problem (4.7) changes into

$$\left\{ \begin{array}{ll} -\Delta w = f & \text{in } \Omega, \\ \text{——} & \text{on } \partial\Omega \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} -\Delta u = w & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{4.10}$$

For this system there is no obvious way of solving (see [45] for another approach). Nevertheless, there is an appropriate weak formulation. Following Remark 4.1.4, consider the functional

$$H(u,w) = \int_\Omega \left(\nabla u \cdot \nabla w - f\ u - \tfrac{1}{2} w^2 \right) dx \quad \text{for} \quad (u,w) \in \mathcal{H}, \tag{4.11}$$

where

$$\mathcal{H} := W_0^{1,2}(\Omega) \times W^{1,2}(\Omega). \tag{4.12}$$

A stationary point $(u,w) \in \mathcal{H}$ is a solution to the Euler-Lagrange equation

$$\begin{aligned} \partial H(u,w;\varphi,\psi) &= \int_\Omega (\nabla u \cdot \nabla \psi + \nabla \varphi \cdot \nabla w - f\ \varphi - w\ \psi)\, dx \\ &= 0 \quad \text{for all} \quad (\varphi,\psi) \in \mathcal{H}, \end{aligned} \tag{4.13}$$

which is equivalent to

$$\int_\Omega (\nabla w \cdot \nabla \varphi - f\ \varphi)\, dx = 0 \quad \text{for all} \quad \varphi \in W_0^{1,2}(\Omega) \tag{4.14}$$

$$\int_\Omega (\nabla u \cdot \nabla \psi - w\ \psi)\, dx = 0 \quad \text{for all} \quad \psi \in W^{1,2}(\Omega). \tag{4.15}$$

Definition 4.1.6 *The pair (u,w) is called a weak solution of (4.10) if*

(i) $(u,w) \in \mathcal{H} = W_0^{1,2}(\Omega) \times W^{1,2}(\Omega)$ *and*

(ii) (u,w) *satisfies (4.14)-(4.15).*

Let $u \in W_0^{1,2}(\Omega)$ be a nontrivial function. Then

$$H\left(t^2 u, tu\right) = \int_\Omega \left(t^3 |\nabla u|^2 - t^2 \left(f\ u + \tfrac{1}{2} u^2 \right) \right) dx \left\{ \begin{array}{ll} \to +\infty & \text{for } t \to +\infty, \\ \to -\infty & \text{for } t \to -\infty, \end{array} \right.$$

shows the unboundedness of H. So, generically, one expects that when a stationary point exists, it would be a saddle point.

We will not be able to show the existence of a stationary point of (4.11) directly. Instead, we will show that in the case of convex domains, there exists a one to one correspondence between stationary points of (4.11) and solutions of (4.7). On the other hand, the presence of concave corners will enable us to find an example where no stationary points exist. To that end we have the following

Proposition 4.1.7 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $f \in L^2(\Omega)$.*

(i) If $(u, w) \in \mathcal{H}$ is a stationary point of (4.11) and $u \in W^{2,2}(\Omega)$, then u satisfies the weak Euler-Lagrange equation (4.9).

(ii) If $u \in W_0^{2,2}(\Omega)$ satisfies (4.9) and $\Delta u \in W^{1,2}(\Omega)$, then $(u, -\Delta u)$ is a stationary point of (4.11).

Proof. *(i)* Let $(u, w) \in \mathcal{H}$ be a stationary point of (4.11), i.e. both (4.14) and (4.15) are satisfied. If $u \in W^{2,2}(\Omega)$, then for all $\psi \in W^{1,2}(\Omega)$ it holds that

$$\int_\Omega (-\Delta u - w)\,\psi\, dx - \int_{\partial\Omega} \partial_n u\, \psi\, ds = \int_\Omega (\nabla u \cdot \nabla\psi - w\ \psi)\, dx = 0,$$

which implies $-\Delta u = w$ in Ω and $\partial_n u = 0$ on $\partial\Omega$ in a weak sense. Hence $u \in W_0^{2,2}(\Omega)$. Moreover, by density,

$$\int_\Omega (-\Delta u - w)\,\psi dx = 0 \text{ for all } \psi \in L^2(\Omega).$$

Taking $\varphi \in W_0^{2,2}(\Omega)$ and using $\psi = -\Delta\varphi$, we find with (4.15) that

$$\begin{aligned} 0 &= \int_\Omega (-\Delta u - w)(-\Delta\varphi)\, dx = \int_\Omega (\Delta u\ \Delta\varphi - \nabla w \cdot \nabla\varphi)\, dx \\ &= \int_\Omega (\Delta u\ \Delta\varphi - f\ \varphi)\, dx. \end{aligned}$$

(ii) If (4.9) holds true, $u \in W_0^{2,2}(\Omega)$ and $\Delta u \in W^{1,2}(\Omega)$, then

$$\begin{aligned} 0 &= \int_\Omega (\Delta u \Delta\varphi - f\varphi)\, dx \\ &= \int_\Omega (-\nabla\Delta u \cdot \nabla\varphi - f\varphi)\, dx \quad \text{for all} \quad \varphi \in W_0^{2,2}(\Omega). \end{aligned}$$

Since $W_0^{2,2}(\Omega)$ is dense in $W_0^{1,2}(\Omega)$, one has

$$0 = \int_\Omega (-\nabla\Delta u \cdot \nabla\varphi - f\varphi)\, dx \quad \text{for all} \quad \varphi \in W_0^{1,2}(\Omega)$$

and it follows that

$$\begin{aligned} \partial H(u, -\Delta u; \varphi, \psi) &= \int_\Omega (\nabla u \cdot \nabla\psi - \nabla\varphi \cdot \nabla\Delta u - f\ \varphi + \Delta u\ \psi)\, dx \\ &= \int_\Omega (\nabla u \cdot \nabla\psi + \Delta u\ \psi)\, dx \\ &= \int_{\partial\Omega} \partial_n u\ \psi\ ds = 0 \end{aligned}$$

holds for all $(\varphi, \psi) \in \mathcal{H}$. ∎

4.2 An approximate solution by piecewise linear finite elements

4.2.1 Numerics for the Navier system

The procedure in this case is rather straightforward. One replaces $W_0^{1,2}(\Omega)$ by a finite dimensional space V_N consisting of piecewise linear functions on a triangulation of Ω. Then the corresponding finite elements will be used to find unique solutions w_N and u_N of the discretized version of *(ii)* of Definition 4.1.3. Refining in a uniform way the triangularization ($N \to \infty$), these functions (u_N, w_N) converge in $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ to a solution (u, w) of (4.2). In other words, these numerical approximations converge to the weak system solution.

4.2.2 Numerics for the Dirichlet system

The finite element approximation of the setting explained in Section 4.1.2 can be treated similarly. Approximating the curvilinear domain from the inside with polygonal domains needs only a mildly regular boundary. Different kinds of methods and their convergence have been studied; see [19, 44, 52] for the case of smooth domains. The authors of [16] adapted a method of Raviart and Ciarlet (see [12]) by using a dual mesh to avoid complications.

Proposition 4.2.1 *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded polygonal domain and let $T_{N,K}$ be a triangulation of Ω with N internal nodes und K boundary nodes. Suppose the family $\{e_i\}_{i=1}^{N+K}$ denotes the corresponding piecewise linear elementary functions (Lipschitz on Ω, affine in each triangle of $T_{N,K}$ and supported in the triangles adjacent to the i-th node). Set*

$$\langle e_i, e_j \rangle = \int_\Omega \nabla e_i \ \nabla e_j \ dx \quad \text{and} \quad (e_i, e_j) = \int_\Omega e_i \ e_j \ dx. \tag{4.16}$$

Then for a given $f \in L^2(\Omega)$ there exists a unique solution $\{u_i\}_{i=1}^N, \{w_i\}_{i=1}^{N+K}$ of

$$\sum_{i=1}^{N+K} \langle e_i, e_j \rangle w_i = \sum_{i=1}^{N} (f, e_j) \ \text{for } j = 1, \ldots, N, \tag{4.17}$$

$$\sum_{i=1}^{N} \langle e_i, e_j \rangle u_i = \sum_{i=1}^{N+K} (e_i, e_j)\, w_i \ \text{for } j = 1, \ldots, N+K. \tag{4.18}$$

Remark 4.2.2 *Note that the system (4.17)-(4.18) is just the discrete version of (4.13). Indeed, letting $u = \sum_{i=1}^N u_i e_i$, $w = \sum_{i=1}^{N+K} w_i e_i$ and allowing the test functions $\varphi = \sum_{j=1}^N \varphi_j e_j$ and $\psi = \sum_{j=1}^{N+K} \psi_j e_j$, equations (4.17)-(4.18) are just (4.14)-(4.15) for given $f \in L^2(\Omega)$.*

Proof of Proposition 4.2.1. In order to see that (4.17)-(4.18) has a solution, let us start with $\{e_i\}_{i=1}^N$ being a basis in the finite dimensional space that serves as an approximation of $W_0^{1,2}(\Omega)$. Then we add additional elements $\{e_i\}_{i=N+1}^{N+K}$, corresponding to the boundary nodes, to form a basis in the finite approximation space for $W^{1,2}(\Omega)$. By a first Gram-Schmidt process we normalize these elements with respect to the inner product $\langle \cdot, \cdot \rangle$. So the modified $\{e_i\}_{i=1}^N$ is still a basis in the finite dimensional space that serves as an approximation of $W_0^{1,2}(\Omega)$. The set $\{e_{N+1}, \dots, e_{N+K}\}$ is no longer localised near the boundary but that will not produce a problem. So we may assume

$$\langle e_i, e_j \rangle = \delta_{ij}.$$

We apply the Gram-Schmidt process a second time but now with respect to $(\cdot, \cdot)$ for $\{e_{N+1}, \dots, e_{N+K}\}$. Indeed, we define the elements e^*_{N+k} with $k = 1, 2, ..., K$ by

$$e^*_{N+k} = e_{N+k} - \sum_{i=1}^{k-1} \frac{(e_{N+k}, e^*_{N+i})}{(e_{N+k}, e_{N+k})} e_{N+k}.$$

Then we find

$$\left(e^*_i, e^*_j\right) = 0 \text{ for } i, j \in \{N+1, \dots, N+K\} \text{ with } i \neq j.$$

Moreover, for all $k = 1, \dots, K$ and $j = 1, \dots, N$ it still holds that

$$\langle e^*_{N+k}, e_j \rangle = \langle e_{N+k}, e_j \rangle - \sum_{i=1}^{k-1} \frac{(e_{N+k}, e^*_{N+i})}{(e_{N+k}, e_{N+k})} \langle e_{N+k}, e_j \rangle = 0.$$

After these prelimiminaries we can solve the equations. Since $\{e_i\}_{i=1}^N$ is a basis, for the given $f \in L^2(\Omega)$, there exists a unique set $\{w_i\}_{i=1}^N$ solving the system

$$\sum_{i=1}^N \langle e_i, e_j \rangle w_i = \sum_{i=1}^N (f, e_j) \quad \text{for} \quad j = 1, \dots, N.$$

Let us write $\bar{w} = \sum_{i=1}^N w_i e_i$. The solutions of (4.17) are precisely given by

$$w = \bar{w} + \sum_{i=N+1}^{N+K} c_i e^*_j,$$

where $c_i \in \mathbb{R}$, $i \in \{N+1, ..., N+k\}$. Thus, from (4.18) for u, that is $\{u_i\}_{i=1}^N$, and the yet unknown $\{c_i\}_{i=N+1}^{N+K}$ one finds the $N+K$ equations

$$\sum_{i=1}^N \langle e_i, e_j \rangle u_i = \sum_{i=1}^N \left(e_i, e_j^{(*)}\right) w_i + \sum_{i=N+1}^{N+K} \left(e^*_i, e_j^{(*)}\right) c_i \text{ for } j = 1, \dots, N+K,$$

where the superscripts $^{(*)}$ are related to the N internal elements and K "boundary" elements. Separating the first N from the remaining K one finds

$$\sum_{i=1}^{N} \langle e_i, e_j \rangle u_i = \sum_{i=1}^{N} (e_i, e_j)\, w_i + \sum_{i=N+1}^{N+K} (e_i^*, e_j)\, c_i \text{ for } j = 1, \ldots, N$$

and

$$0 = \sum_{i=1}^{N} \left(e_i, e_j^*\right) w_i + \sum_{i=N+1}^{N+K} \left(e_i^*, e_j^*\right) c_i \text{ for } j = N+1, \ldots, N+K.$$

Due to the orthonormality of the elements with respect to the appropriate inner products, the c_i in the last equations are uniquely determined and one may compute these first and then plug them in the first set of equations in order to yield a unique set of u_i. ∎

4.3 Existence and decoupling in the presence of corners

4.3.1 The second order Dirichlet Laplace problem

Let $f \in L^2(\Omega)$. Using Riesz' representation Theorem or variational methods, it is well known (see for example [24]) that the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.19}$$

has a unique weak solution in $W_0^{1,2}(\Omega)$ for any bounded domain Ω. One can even prove (see [26, 33]) that in the case of convex domains (or of those that can be mapped to a convex domain via a C^2 diffeomorphism) the solution lies in $W^{2,2}(\Omega)$. However, whenever one seeks less regular solutions, it is folklore that this uniqueness is no longer true in general. Let us illustrate this with an example. For $\Omega = \Omega_\omega$ with

$$\Omega_\omega := \left\{ (r\cos\theta, r\sin\theta)\, ; 0 < r < 1 \text{ and } |\theta| < \tfrac{1}{2}\omega \right\}, \tag{4.20}$$

an open pacman domain with an opening angle of $\omega > \pi$, solutions of (4.19) in $W_0^{1,p}(\Omega_\omega)$ with

$$p < \frac{2\omega}{\omega + \pi}$$

are no longer unique. Indeed:

Lemma 4.3.1 *For $\omega > \pi$, the function*

$$u_\omega(x_1, x_2) = \left(r^{-\pi/\omega} - r^{\pi/\omega} \right) \sin\left(\frac{\pi}{\omega}\theta \right) \tag{4.21}$$

with

$$x_1 = r\cos\theta \text{ and } x_2 = r\sin\theta$$

lies in $W_0^{1,p}(\Omega_\omega)$, whenever $p \in \left(1, \frac{2\omega}{\omega+\pi}\right)$, and satisfies $\Delta u = 0$ in Ω.

Remark 4.3.2 *Later on, we will see that the function in (4.21) plays a special role in the comparison of solutions to (4.1) and (4.2).*

Proof. It is immediate that $u_\omega = 0$ on $\partial\Omega_\omega \backslash \{0\}$ and $\Delta u_\omega = 0$ in Ω_ω. To see that $u_\omega \in W^{1,p}(\Omega_\omega)$ one checks that $\omega > \pi$ and $p < \frac{2\omega}{\omega+\pi}$ imply

$$\int_0^1 \left(r^{-\pi/\omega - 1} \right)^p r\, dr < \infty \quad \text{and} \quad \int_0^1 \left(r^{\pi/\omega - 1} \right)^p r\, dr < \infty.$$

Since $u_\omega \in W^{1,p}(\Omega_\omega) \cap C(\overline{\Omega}_\omega \backslash \{0\})$ and $u_\omega = 0$ on $\partial\Omega_\omega \backslash \{0\}$, it follows that $u_\omega \in W_0^{1,p}(\Omega_\omega)$. ∎

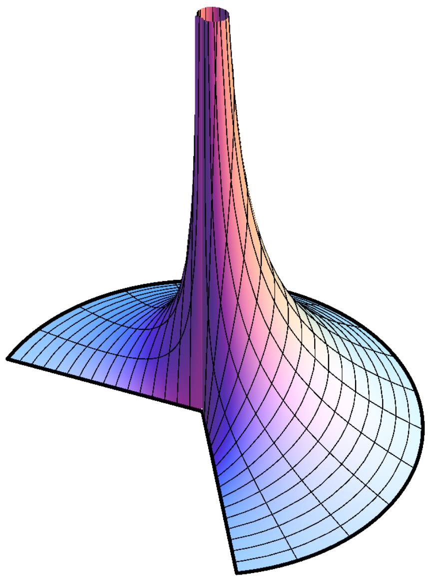

Figure 4.1: For $\omega = \frac{4}{3}\pi$ a graph of the harmonic function u_ω from (4.21) that satisfies zero Dirichlet boundary conditions in the following sense: $u_\omega \in W_0^{1,p}(\Omega_\omega)$ for all $p \in [1, \frac{8}{7})$.

4.3.2 The Navier case for the biharmonic

The inner product

The question of existence of a weak solution for (4.1) on nonsmooth domains is relevant to studying the regularity of the Dirichlet problem for the Laplacian. In case of C^2 smooth domains one can prove straightforwardly by a local "straightening" of the boundary that the solution of (4.19) lies in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and satisfies the estimate $\|u\|_{2,2} \le C \|\Delta u\|_2$. When the boundary of the domain has singularities, one can distinguish the following cases:

- In case the boundary of the domain has corner-like singularities (for planar domains Ω), one can use directly Grisvard's density argument (see Theorem 2.2.1 and Corollary 2.3.6). When the domain $\Omega \subset \mathbb{R}^n$ can be mapped onto a convex domain via a C^2 diffeomorphism (the corners are "convex"), one can also apply an approximation argument: In this case the domain Ω can be approximated from the inside by a sequence of smooth domains. Then the solution for the problem in these domains is smooth enough and converges to a solution of the original problem in $W_0^{1,2}(\Omega)$ and the zero extension of the second order derivatives converges in $L^2(\mathbb{R}^n)$. It follows that the solution to the original problem lies in $W^{2,2}(\Omega)$ (see [26, 33]).

- When the boundary of the domain has exterior cusps, one can use a transformation argument and results by Dore and Venni on the closedness of the sum of two closed operators to yield the result (see [27] and references therein).

- The presence of a reentrant corner yields negative results: For the domain $\Omega = \Omega_\omega$ and $f \in L^2(\Omega_\omega)$, one has that there exists a ball B centered at the origin, such that the weak solution of (4.19) has the expansion

$$u = cr^{\pi/\omega} \sin\left(\frac{\pi}{\omega}\theta\right) + U,$$

where c is a nonzero constant and $U \in W^{2,2}(\Omega)$ (see [26, 35, 36, 37, 48]). When $\omega > \pi$, then one obtains that the weak solution u will not in general be in $W^{2,2}(\Omega_\omega)$. For such a function u, there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{2,2}(\Omega_\omega)$ such that, for $k \to \infty$, $u_k \to u$ in $L^2(\Omega_\omega)$ and $\Delta u_k \to \Delta u$ in $L^2(\Omega_\omega)$ (see [26, 28]), but $\|u_k\|_{2,2} \to \infty$. Thus the L^2 norm of the Laplacian is not a norm in $W^{2,2}(\Omega_\omega)$.

A counterexample where solutions differ

In this section we recall a result from Nazarov and Sweers in [48] and tailor it for the domain Ω_ω defined in (4.20). Let $G : L^2(\Omega_\omega) \longrightarrow W_0^{1,2}(\Omega_\omega)$ be the solution operator of (4.19), that is

$$\int_{\Omega_\omega} (\nabla (Gf) \cdot \nabla\varphi - f\ \varphi)\, dx = 0 \quad \text{for all} \quad \varphi \in W_0^{1,2}(\Omega_\omega).$$

Proposition 4.3.3 *Let Ω_ω be as above with $\omega > \pi$ and let $f \in L^2(\Omega_\omega)$. Then the following hold:*

(i) System (4.2) possesses a unique solution in $W_0^{1,2}(\Omega_\omega) \times W_0^{1,2}(\Omega_\omega)$, namely

$$(u, v) = (G\,(Gf)\,, Gf)\,.$$

(ii) Equation (4.1) possesses a unique solution in $W^{2,2}(\Omega_\omega) \cap W_0^{1,2}(\Omega_\omega)$, namely

$$u = G\,(Gf) - \frac{\displaystyle\int_{\Omega_\omega} u_\omega\, Gf\ dx}{\displaystyle\int_{\Omega_\omega} u_\omega^2\ dx}\, Gu_\omega,$$

where u_ω is defined by (4.21).

Remark 4.3.4 *To distinguish between the two solutions we call them respectively u_{system} and $u_{equation}$. They have the following properties:*

(i) $u_{system} = u_{equation}$ *if and only if*

$$\int_{\Omega_\omega} u_\omega \, Gf \; dx = 0.$$

(ii) *Since* $u_\omega \notin W_0^{1,2}(\Omega_\omega)$ *and* $Gf \in W_0^{1,2}(\Omega_\omega)$, *it follows* $\Delta u_{equation} \notin W_0^{1,2}(\Omega_\omega)$ *unless*

$$\int_{\Omega_\omega} u_\omega \, Gf \; dx = 0,$$

contrary to $\Delta u_{system} = -v \in W_0^{1,2}(\Omega_\omega)$.

(iii) *Since* $\omega > \pi$, *one finds* $u_{system} \notin W^{2,2}(\Omega_\omega)$ *unless*

$$\int_{\Omega_\omega} u_\omega \, Gf \; dx = 0.$$

(iv) *If* $f \geq 0$ *and* $f \not\equiv 0$ *(abbr.* $f \gneqq 0$*), then the maximum principle implies* $u_{system} > 0$. *For* $u_{equation}$ *there is not such a sign preserving result (see [48]). For* $f \gneqq 0$ *one may conclude* $u_{system} \geq u_{equation}$.

4.3.3 The Dirichlet case for the biharmonic

Here we deal with the existence of weak solution for (4.10) in domains with corners and see what can go wrong. We first state a useful observation.

Lemma 4.3.5 *Let* $w \in L^2(\Omega)$, *where* $\Omega \subset \mathbb{R}^2$ *is open and bounded with* $\partial\Omega$ *being smooth with the exception of a finite number of corners. If there exists* $u \in W_0^{1,2}(\Omega)$ *satisfying*

$$\int_\Omega (\nabla u \cdot \nabla \psi - w\psi) \, dx = 0 \;\; \text{for all} \;\; \psi \in W^{1,2}(\Omega), \tag{4.22}$$

then $u \in W_0^{2,2}(\Omega)$.

Proof. If such a function u exists, it coincides with the weak solution of the problem

$$\begin{cases} -\Delta u = w & \text{in } \Omega, \\ \quad\; u = 0 & \text{on } \partial\Omega. \end{cases}$$

When Ω is convex one finds that $u \in W^{2,2}(\Omega)$ (see [33]) and an integration by parts implies that $\partial_n u = 0$ on $\partial\Omega$. Thus we only need to treat the case when $\partial\Omega$ has a corner of opening angle $\omega > \pi$. In that case, following [36, Examples 6.6.1 and 6.6.2], there exists a ball of radius $\varepsilon > 0$ centered at this corner, which is denoted by B_ε and constants $c_0, c_1, \tilde{c}_1$, such that any function $u \in W_0^{1,2}(\Omega)$ satisfying the assumptions of the Lemma will have the expansion

$$u = c_0 + c_1 \, r^{\pi/\omega} \cos\left(\frac{\pi}{\omega}\theta\right) + u_r \;\; \text{in} \;\; B_\varepsilon \cap \Omega,$$

as a weak solution to the Neumann problem (according to [36, Example 6.6.2]), as well as

$$u = \tilde{c}_1\, r^{\pi/\omega} \sin\left(\frac{\pi}{\omega}\theta\right) + \tilde{u}_r \quad \text{in} \quad B_\varepsilon \cap \Omega,$$

as a weak solution to the Dirichlet problem (according to [36, Example 6.6.1]). Here, (r, θ) is a polar coordinate system centered at the corner point and $u_r, \tilde{u}_r \in W_0^2(\Omega)$. Thus, since $W_0^2(\Omega) \subset W^{2,2}(\Omega)$ (see Proposition 2.1.10), we get

$$c_0 + r^{\pi/\omega}\left(c_1 \cos\left(\frac{\pi}{\omega}\theta\right) - \tilde{c}_1 \sin\left(\frac{\pi}{\omega}\theta\right)\right) = \tilde{u}_r - u_r \in W^{2,2}(\Omega)$$

and since

$$\int_0^\varepsilon \left(r^{\pi/\omega - 2}\right)^2 r\, dr = \infty$$

implies that

$$r^{\pi/\omega}\left(c_1 \cos\left(\frac{\pi}{\omega}\theta\right) - \tilde{c}_1 \sin\left(\frac{\pi}{\omega}\theta\right)\right) \in W^{1,2}(\Omega) \setminus W^{2,2}(\Omega),$$

one obtains that $c_1 = \tilde{c}_1 = 0$. On the other hand, applying [48, Lemma 6.7] we get that there exists a constant $C > 0$, such that

$$|c_0| = |u_r - \tilde{u}_r| \le Cr.$$

Thus $c_0 = 0$ and the Lemma is proved. ■

Existence under the assumption of extra regularity

We shall begin with a brief description of the geometry and the behaviour of critical points of (4.11). The following lemma is complementary to Proposition 4.1.7.

Lemma 4.3.6 *The functional (4.11) is affine in $u \in W_0^{1,2}(\Omega)$ and concave in $w \in W^{1,2}(\Omega)$. Moreover, the following hold:*

(i) Fix $u \in W_0^{1,2}(\Omega)$. If $u \in W_0^{2,2}(\Omega)$ and $\Delta u \in W^{1,2}(\Omega)$, then the supremum of $w \mapsto H(u, w)$ is attained for $w = -\Delta u$.

(ii) Fix $w \in W^{1,2}(\Omega)$. If $\int_\Omega (\nabla w \cdot \nabla \psi - f\, \psi) = 0$ for all $\psi \in W^{1,2}(\Omega)$, then $u \mapsto H(u, w)$ is constant on $W_0^{1,2}(\Omega)$.

Proof. One has for $t \in (0, 1)$ that

$$H(tu_1 + (1-t)u_2, w) = tH(u_1, w) + (1-t)H(u_2, w).$$

Since

$$\begin{aligned} H(u, tw_1 + (1-t)w_2) &= tH(u, w_1) + (1-t)H(u, w_2) \\ &\quad -\frac{1}{2}t(1-t)\int_\Omega (w_1 - w_2)^2\, dx \\ &\le tH(u, w_1) + (1-t)H(u, w_2), \end{aligned}$$

the functional $w \mapsto H(u,w)$ is concave for every $u \in W_0^{1,2}(\Omega)$.

(i) Moreover, whenever $u \in W_0^{2,2}(\Omega)$ and $\Delta u \in W^{1,2}(\Omega)$ holds, one finds for any direction $0 \not\equiv \psi \in W^{1,2}(\Omega)$, that

$$\begin{aligned} H(u, -\Delta u + t\psi) &= \int_\Omega \Big(-\nabla u \cdot \nabla \Delta u + t \nabla u \cdot \nabla \psi - fu - \tfrac{1}{2}(-\Delta u + t\psi)^2\Big) dx \\ &= \int_\Omega \Big(-\nabla u \cdot \nabla \Delta u - fu - \tfrac{1}{2}(\Delta u)^2 - \tfrac{1}{2} t^2 \psi^2\Big) dx \\ &\leq H(u, -\Delta u) \end{aligned}$$

with equality if and only if $t = 0$. Thus, the supremum is attained for $w = -\Delta u$.

(ii) If $w \in W^{1,2}(\Omega)$ and $-\Delta w = f$ in the weak sense, then

$$H(u,w) = \int_\Omega \left(-\nabla u \cdot \nabla w - f\ u - \frac{1}{2} w^2\right) dx = -\frac{1}{2} \int_\Omega w^2 dx,$$

which is independent of u. ∎

Regularity in the neighbourhood of a cornerpoint

As we have seen, existence is implied when solutions have some extra regularity. We will assume for simplicity that the boundary of the domain $\Omega \subset \mathbb{R}^2$ has only one corner, namely at 0, which is locally the apex of a cone. That is, in polar coordinates (r,θ), there exists $\varepsilon > 0$, such that

$$\Omega \cap B_\varepsilon(0) = \left\{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2;\ 0 < r < \varepsilon \text{ and } \theta \in (0,\omega)\right\}, \tag{4.23}$$

where $\omega \in (0, 2\pi)$. In order to state the main regularity theorem, we need the following eigenvalue problem:

$$\begin{cases} \Delta^2 \left(r^{\lambda+1}\Phi(\theta)\right) = 0 & \text{in } \mathbb{R}^+ \times A, \\ \Phi(\theta) = \frac{\partial}{\partial \nu}\Phi(\theta) = 0 & \text{on } \partial A. \end{cases} \tag{4.24}$$

What follows is a special version of Kondrat'ev's regularity Theorem (see [35]) for general elliptic problems, found in [26].

Theorem 4.3.7 (V. A. Kondrat'ev) *Let $\Omega \subset \mathbb{R}^2$ be a domain satisfying (4.23), where the opening angle of the conical point is not equal to π or 2π and let $u \in W_0^{2,2}(\Omega)$ be the unique weak solution of (4.7) with $f \in L^2(\Omega)$. Let λ_j be the eigenvalues of (4.24) and assume that $Re(\lambda_j) \neq 2$. Then, there exists a $w \in W^{4,2}(\Omega)$ and positive constants c_{jl}, such that u has the representation*

$$u = w + \chi(r) \sum_{0 < Re(\lambda_j) < 2} \sum_{l=0}^{\kappa_j - 1} c_{jl} r^{\lambda_j + 1} (\ln r)^l\ u_{jl}(\lambda_j, \theta), \tag{4.25}$$

where κ_j is the algebraic multiplicity of the j-th eigenvalue. Moreover, the functions u_{jl} are the corresponding eigenfunctions and they are infinitely differentiable. The function $\chi(r)$ is a cutoff function equal to one near the cornerpoint and is supported sufficiently close to it.

Remark 4.3.8 *Following [26], one can show that a solution to the eigenvalue problem (4.24) exists, precisely when there exists λ that solves the transcendental equation*

$$\sin(\omega\lambda)^2 = \lambda^2 \sin(\omega)^2, \tag{4.26}$$

where $\omega \in I := (0,\pi)\cup(\pi,2\pi)$ is the opening angle of the conical point. Moreover, for $(\omega, Re(\lambda)) \in I \times (0,2)$, the above equation has only simple and double roots. Double roots with geometric multiplicity 1 occur when

$$\omega \sin(\omega\lambda)\cos(\omega\lambda) = \lambda \sin(\omega)^2, \tag{4.27}$$

that is, where the strictly complex (dashed) curves join the real valued λ (see Figure 4.2). Combining (4.26) and (4.27) it follows that

$$\lambda = \pm\sqrt{\frac{1}{\sin^2\omega} - \frac{1}{\omega^2}}.$$

Existence for domains with convex corners

As illustrated in Proposition 4.1.7, a minimizer of (4.8) will be a stationary point if one assumes some extra regularity. We will show that in the case of a bounded piecewise smooth domain with finitely many convex corners this is indeed the case.

Lemma 4.3.9 *Assume $0 < \alpha < 1$ and let $\Omega \subset \mathbb{R}^2$ be a piecewise C^4 domain with a finite number of corners, each of which has opening angle, measured from the inside, less than π. Then, the minimizer u of the clamped plate functional (4.8) satisfies $\Delta u \in W^{1,2}(\Omega)$.*

Proof. Following Theorem 4.3.7, one sees that the regularity of the solution is governed by the exponents in the asymptotic sums of the expression (4.25). We will assume for simplicity that the domain is like (4.23) and the angle of the conical point is such that no double roots of the corresponding eigenvalue problem exist. Then, in this case, the Laplacian of the solution will have the form

$$\Delta u = \Delta w + \sum_{0<Re(\lambda_j)<2} c_j r^{\lambda_j - 1}\left((\lambda_j+1)^2 u_j(\lambda_j,\theta) + \partial_{\theta\theta} u_j(\lambda_j,\theta)\right) \tag{4.28}$$

with $w \in W^{4,2}(\Omega)$ in the neighbourhood of the origin. From the values for λ shown in Figure (4.2), one finds that for $R > 0$:

$$\omega < \pi \Longrightarrow \lambda > 1 \Longrightarrow \int_0^R \left(r^{(\lambda-1)-1}\right)^2 r\, dr < \infty$$

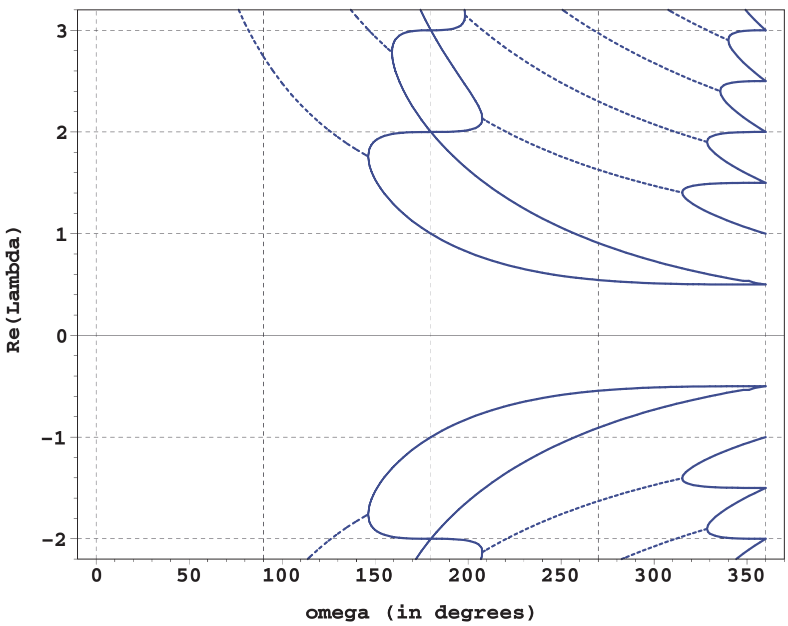

Figure 4.2: The relation between opening angle ω and Re(λ) in the singular solutions for the clamped problem. The dashed lines correspond with (double) strictly complex λ. The figure is taken from [23].

and thus $\Delta u \in W^{1,2}(\Omega)$. ■

Corollary 4.3.10 *Assume that the domain Ω is as above. Then (4.11) possesses a critical point in $\mathcal{H}$.*

Proof. It is enough to apply Lemma 4.3.9 with Proposition 4.1.7 to obtain that the unique minimizer of (4.8) is a critical point of (4.11). ■

Troubles with reentrant corners

We will now proceed with the comparison of the two solutions. In the case of the existence of concave corners the system approach will, in general, not agree with the minimization problem. For simplicity we let $\omega \in (0, 2\pi)$ and assume $\Omega = \Omega_\omega$ to be a pacman domain as in (4.20). Moreover, we define the subdomains (see Figure 4.3)

$$\Omega_{\omega,1} := \left\{x \in \Omega;\ |x| < \tfrac{1}{3}\right\},\ \Omega_{\omega,2} := \left\{x \in \Omega;\ \tfrac{1}{3} < |x| < \tfrac{2}{3}\right\} \quad \text{and}$$
$$\Omega_{\omega,3} := \left\{x \in \Omega;\ \tfrac{2}{3} < |x| < 1\right\}.$$

Theorem 4.3.11 *Let $\Omega = \Omega_\omega \subset \mathbb{R}^2$ be the pacman domain* (4.20) *with opening angle $\omega > \pi$. Then, there exists a right hand side $f \in L^2(\Omega)$, such that the unique minimizer of (4.8) is not a stationary point of (4.11) in $\mathcal{H}$.*

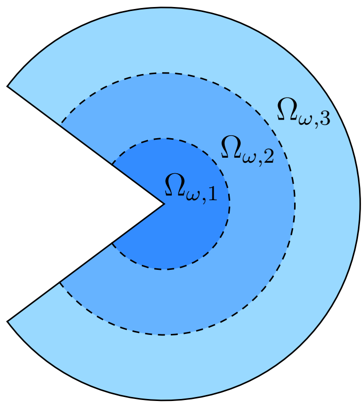

Figure 4.3: The sliced pacman

Proof. Let $\chi \in C^\infty[0,\epsilon]$ be a function such that $\chi \equiv 1$ in $\Omega_{\omega,1}$, $\chi \not\equiv 0$ in $\Omega_{\omega,2}$ and $\chi \equiv 0$ in $\Omega_{\omega,3}$ (see Figure 4.3). The relation between θ and λ is found in Figure 4.2. Since

$$\int_0^1 \left(r^{\lambda+1-k}\right)^2 rdr < \infty \text{ if and only if } k < \lambda + 2$$

and similarly if $\ln r$ is included, one finds that for $k \in \{0,1,2,3,...\}$, that

$$\psi_j(r,\theta) := r^{\lambda_j+1} u_j(\lambda_j,\theta) \in W^{k,2}(\Omega) \text{ if and only if } k < \lambda_j + 2.$$

Let ψ_1 be the first positive singular eigenfunction. We set

$$u^* = \chi\psi_1.$$

One finds $u^* = 0$ on $\partial\Omega$ and $\partial_n u^* = 0$ on $\partial\Omega\backslash\{0\}$ and since $\psi_1 \in W^{2,2}(\Omega)$ and $\psi_1 \notin W^{3,2}(\Omega)$ for $\omega > \pi$ (see Figure 4.2) we find

$$u^* \in W_0^{2,2}(\Omega)\backslash W^{3,2}(\Omega).$$

Since $u^* = \psi_1$ on Ω_1 and ψ_1 is a biharmonic function, it follows that $f^* := \Delta^2 u_1 \in C^\infty(\overline{\Omega})$. Taking $f = f^*$ we have u^* as the unique minimizer of (4.8).

Let us assume that $(u^*, -\Delta u^*)$ is a stationary point in the sense of (4.13). Then

$$u^* \in W_0^{1,2}(\Omega) \quad \text{and} \quad \Delta u^* \in W^{1,2}(\Omega). \tag{4.29}$$

Applying Theorem 4.3.7, one gets that the solution $u^* \in W_0^{1,2}(\Omega)$ possesses the asymptotic representation

$$u^* = \tilde{u}_1 + c_{-2}\psi_{-2} + c_{-1}\psi_{-1} + c_1\psi_1 + c_2\psi_2 + \text{higher order terms},$$

where $\tilde{u}_1 \in W^{4,2}(\Omega)$ and

$$\psi_{-2}, \psi_{-1} \in W^{1,2}(\Omega) \backslash W^{2,2}(\Omega) \text{ and } \psi_1, \psi_2 \in W^{2,2}(\Omega) \backslash W^{3,2}(\Omega).$$

The higher order terms lie in $W^{3,2}(\Omega)$. Moreover, since the functions ψ_i are not harmonic, indeed

$$\Delta \psi_i = r^{\lambda_i - 1} \left((\lambda_i + 1)^2 u_j (\lambda_i, \theta) + \partial_{\theta\theta} u_j (\lambda_i, \theta) \right) \neq 0,$$

one obtains

$$\Delta \psi_{-2}, \Delta \psi_{-1} \notin L^2(\Omega) \text{ and } \Delta \psi_1, \Delta \psi_2 \notin W^{1,2}(\Omega)$$

and that they do not cancel each others' singularity near 0. Hence, if $\Delta u_1 \in W^{1,2}(\Omega)$, then by the second condition in (4.29), one is forced to set $c_{-2} = c_{-1} = c_1 = c_2 = 0$ to find $u_1 \in W^{3,2}(\Omega) \cap W_0^{2,2}(\Omega)$, a contradiction. ■

In fact we can also prove the nonexistence of a critical point in this case.

Corollary 4.3.12 *Let $\Omega = \Omega_\omega \subset \mathbb{R}^2$ be the pacman domain of Theorem 4.3.11. Then for f as in Theorem 4.3.11, the functional (4.11) possesses no critical point.*

Proof. Applying Lemma 4.3.5, we obtain that if there exists a critical point $(u, w) \in \mathcal{H}$ of (4.11), then it must hold that $u \in W_0^{2,2}(\Omega)$. But then, applying Proposition 4.1.7, we have that u is a minimizer of (4.8). However, according to Theorem 4.3.11 this cannot be true. ■

Remark 4.3.13 *The numerical computations to*

$$\sin(\omega\lambda)^2 - \lambda^2 \sin(\omega)^2 = 0 \quad \textit{for} \quad \omega \in (0, 2\pi],$$

that produce Figure 4.2 are performed with the Maple 9.5 package in the following way: for every $\omega_n = \frac{60}{180}\pi + \frac{1}{60}\pi n$, $n = 0, ..., 100$ we compute the entries of the set $\{\lambda_j\}_{j=1}^N$. Here, N is determined by the conditions $\mathrm{Re}\,(\lambda_N) \leq 3.200$ and $\mathrm{Re}\,(\lambda_{N+1}) > 3.200$. The points (ω, λ) where λ_j transits from the complex plane to the real one, or vice-versa, are solutions to the system $P(\omega, \lambda) = 0$, $\frac{\partial P}{\partial \lambda}(\omega, \lambda) = 0$. For $\omega = \pi$ and $\lambda \in \{1, 2, 3\}$ one obtains the standard polynomials y^2, xy^2, y^3, xy^3 and $y^2 \left(3x^2 - y^2\right)$.

Remark 4.3.14 *The asymptotic analysis of the original problem and the system approach produces the same boundary value problems and imposes naturally the same boundary conditions on the angular eigenfunctions. This produces exactly the same singular eigenfunctions ψ_j.*

Remark 4.3.15 *Following the argumentation of the previous theorem, one sees that when $\omega < \pi$, the stationary point is indeed in $W^{2,2}(\Omega)$. Thus, Proposition 4.1.7 implies that it is unique.*

4.4 Convergence of the corresponding finite element approximation scheme

The equation with Navier boundary conditions can be decoupled with the boundary conditions splitting to form an iterated Dirichlet Laplace problem which can be treated in a standard way. Here we consider the clamped plate problem where we have to take the more tedious approach which is described in Section 4.2.2. Proving convergence of the numerical scheme is closely dependent on an approximation property of the finite element spaces.

Let Ω be an open and bounded polygonal domain of $\mathbb{R}^2$. Let $\mathcal{T}$ be the family of all uniform triangulations of Ω, that is, each triangle involved has angles uniformly bounded away from 0. Let $T := T_{N,K} \in \mathcal{T}$ be a triangulation of Ω with N internal nodes, K boundary nodes and $h := h(T)$ the maximum of the diameters of the triangles. Moreover, $\{e_i\}_{i=1}^{N+K}$ will denote the basis elements corresponding to the triangulation T: Lipschitz in Ω, affine in each triangle and supported in the triangles adjacent to the ith node.

By refining the mesh in a uniform way (see e.g. [9]), we can construct a family of triangulations $\{T_{N_n,K_n}\}_{n=0}^{\infty} \subset \mathcal{T}$ with $\lim_{n\to\infty} h(T_{N_n,K_n}) = 0$. On each of these triangulations $T_n := T_{N_n,K_n}$ we define the finite element spaces

$$\mathring{V}_n := \operatorname{span}\left(\{e_{n,i}\}_{i=1}^{N_n}\right) \quad \text{and} \quad V_n := \operatorname{span}\left(\{e_{n,i}\}_{i=1}^{N_n+K_n}\right).$$

In that context one can prove the following

Lemma 4.4.1 *Let Ω be an open and bounded polygonal domain of $\mathbb{R}^2$ and let T_n, $\mathring{V}_n$ and V_n be as above. Then for every $v \in W^{4,2}(\Omega) \cap W_0^{1,2}(\Omega)$ (resp. $W^{4,2}(\Omega)$) there exists a family $\{v_n\}_{n=0}^{\infty}$ with $v_n \in \mathring{V}_n$ (resp. $v_n \in V_n$) such that $\|v_n - v\|_{1,2} \to 0$ as $n \to \infty$.*

Proof. See [5, 9]. ∎

Using Proposition 4.2.1, one can prove for each $n \in \mathbb{N}$, the existence of a unique pair $(u_n, w_n) \in \mathring{V}_n \times V_n$, satisfying

$$\int_\Omega (\nabla u_n \cdot \nabla \psi - w_n \psi)\, dx = 0 \quad \text{for all} \quad \psi \in V_n, \tag{4.30}$$

$$\int_\Omega (\nabla \varphi \cdot \nabla w_n - f\, \varphi)\, dx = 0 \quad \text{for all} \quad \varphi \in \mathring{V}_n. \tag{4.31}$$

The main result of this section is the following

Proposition 4.4.2 *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded polygonal domain. Then if $f \in L^2(\Omega)$, there exists a pair $(u, w) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$, such that $u_n \to u$ strongly in $W_0^{1,2}(\Omega)$, $w_n \rightharpoonup w$ weakly in $L^2(\Omega)$ as $n \to \infty$ and $u \in W^{2,2}(\Omega)$, where $(u_n, w_n) \in \mathring{V}_n \times V_n$ solves (4.30)-(4.31).*

Remark 4.4.3 *The above result comes somehow unexpected. Although the functional (4.11) possesses in general no critical points when the domain has concave corners, the corresponding finite element analysis of its Euler-Lagrange equations will yield a convergent sequence of solutions.*

Proof of Theorem 4.4.2. A unique solution $(u_n, w_n) \in \mathring{V}_n \times V_n$ can be obtained similarly as in the proof of Proposition 4.2.1. To prove convergence we need some estimates. Take $\varphi \equiv u_n$ in (4.31) and $\psi \equiv w_n$ in (4.30). Then one obtains that

$$\int_\Omega f u_n \, dx = \int_\Omega w_n^2 \, dx. \tag{4.32}$$

Now, taking $\psi \equiv u_n$ in (4.30) and using (4.32) yields

$$\begin{aligned} \int_\Omega |\nabla u_n|^2 \, dx &= \int_\Omega u_n w_n \, dx \leq \|u_n\|_2 \, \|w_n\|_2 \\ &= \|u_n\|_2 \left(\int_\Omega f u_n \, dx \right)^{\frac{1}{2}} \leq \|u_n\|_2^{\frac{3}{2}} \, \|f\|_2^{\frac{1}{2}} . \end{aligned}$$

Apply now Poincaré's inequality to obtain

$$\|\nabla u_n\|_2^2 \leq C(\Omega)^{\frac{3}{2}} \, \|\nabla u_n\|_2^{\frac{3}{2}} \, \|f\|_2^{\frac{1}{2}} \Longrightarrow \|\nabla u_n\|_2 \leq C(\Omega)^3 \, \|f\|_2 \tag{4.33}$$

and, consequently,

$$\|u_n\|_2 \leq C(\Omega)^4 \, \|f\|_2 \, . \tag{4.34}$$

Moreover, from (4.32) we have

$$\|w_n\|_2 = \left(\int_\Omega f u_n \, dx \right)^{\frac{1}{2}} \leq C(\Omega)^2 \, \|f\|_2 \, . \tag{4.35}$$

Through (4.33), (4.34) and (4.35) we conclude that there exists $(u, w) \in W_0^{1,2}(\Omega) \times L^2(\Omega)$ and subsequences $\{u_{n_k}\}_{k\in\mathbb{N}}$, $\{w_{n_k}\}_{k\in\mathbb{N}}$ such that $u_{n_k} \rightharpoonup u$ in $W_0^{1,2}(\Omega)$ and $w_{n_k} \rightharpoonup w$ in $L^2(\Omega)$ for $k \to \infty$. Now let $\psi \in W^{4,2}(\Omega)$. Using Lemma 4.4.1, there exists a sequence of functions $\psi_{n_k} \in V_{n_k}$ for $k \in \mathbb{N}$, such that $\|\psi_{n_k} - \psi\|_{1,2} \to 0$ as $k \to \infty$. Taking the limit we obtain that u and w satisfy

$$\int_\Omega (\nabla u \cdot \nabla \psi - w\, \psi) \, dx = 0 \text{ for all } \psi \in W^{4,2}(\Omega)$$

and by the density of $W^{4,2}(\Omega)$ into $W^{1,2}(\Omega)$ (see [1, Theorem 3.17]) we get that

$$\int_\Omega (\nabla u \cdot \nabla \psi - w\, \psi) \, dx = 0 \text{ for all } \psi \in W^{1,2}(\Omega). \tag{4.36}$$

Thus, from Lemma 4.3.5 one obtains that $u \in W^{2,2}(\Omega)$.

Finally we show that $u_{n_k} \to u$ strongly in $W_0^{1,2}(\Omega)$. Due to the compactness of the embedding of $W_0^{1,2}(\Omega)$ into $L^2(\Omega)$ one finds that $u_{n_k} \to u$ strongly $L^2(\Omega)$ and taking $\psi \equiv u_{n_k}$ in (4.30) and $\psi \equiv u$ in (4.36) we have

$$\int_\Omega |\nabla u_{n_k}|^2 \, dx = \int_\Omega u_{n_k} w_{n_k} \, dx \to \int_\Omega uw \, dx = \int_\Omega |\nabla u|^2 \, dx \text{ as } k \to \infty,$$

that is, $\|u_{n_k}\|_{1,2} \to \|u\|_{1,2}$, which, since $W^{1,2}(\Omega)$ is locally convex, proves the claim (see [10, Proposition III.30]). ■

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