Asymptotic behaviour of higher eigenfunctions of the p-Laplacian as p goes to 1

Inaugural-Dissertation
zur
Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Universität zu Köln

vorgelegt von
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Köln
2009

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Tag der mündlichen Prüfung: 23.11.2009

Abstract

Subject of this thesis is the asymptotic behaviour of the higher eigenvalues of the p-Laplacian operator as p goes to 1. The limit setting depends only on the geometry of the domain. In the particular case of a planar disc, it is possible to show that the second eigenfunctions are nonradial if p is close enough to 1. Moreover, it is shown that second eigenfunctions of $-\Delta_p$ can be obtained as limit of least energy nodal solutions of a p-superlinear problem.

Zusammenfassung

Gegenstand dieser Dissertation ist das asymptotische Verhalten höherer Eigenwerte des p-Laplace Operators für p gegen 1. Der Limes hängt nur von der Geometrie des Gebietes ab. Im besonderen Fall einer Kreisscheibe, gelingt der Nachweis, dass die zweiten Eigenfunktionen nicht radialsymmetrisch sind, falls p nah genug an 1 liegt. Außerdem wird gezeigt, dass zweite Eigenfunktionen von $-\Delta_p$ als Grenzwert von vorzeichenwechselnden Funktionen mit kleinster Energie eines p-superlinearen Problems erhalten werden können.

"Considerate la vostra semenza:
fatti non foste a viver come bruti,
ma per seguir virtute e canoscenza"

(Dante Alighieri, "Divina Commedia")

Acknowledgements

First of all I would like to thank my advisor, Prof. Bernd Kawohl, for suggesting the research topic of the present thesis, and for his constant advice and encouragement. The realization of this work would have been impossible without an adequate financial support; in this regard, I have to thank the Deutsche Forschungsgemeinschaft and the University of Cologne.

A special thank goes also to all my colleagues in Cologne for the stimulating discussions and for the nice moments spent together in these years.

For the support received I am grateful to my family and to all the people that have been by my side along all this time.

Contents

Introduction

Eigenvalue problems have been for many years an important part of the mathematical landscape. One of the most known and investigated is surely the eigenvalue problem for the Laplacian operator:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and

$$\Delta : u \mapsto \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

is the *Laplacian operator*. A real number λ is called *eigenvalue* if the equation (1) admits a solution $u \not\equiv 0$, which will be called *eigenfunction*. One could be tempted to look for solutions directly in the function space $\mathcal{C}^2(\overline{\Omega})$ or $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, but this approach does not work. Instead, a common procedure is the following:

- introduce the Sobolev space $W_0^{1,2}(\Omega)$ as the subset of $L^2(\Omega)$ consisting of those function which admit weak partial derivatives in $L^2(\Omega)$;
- define a *weak solution* of (1) as a function $u \in W_0^{1,2}(\Omega)$ such that

  $$\int_\Omega \nabla u \nabla v = \lambda \int_\Omega uv$$

  for every $v \in W_0^{1,2}(\Omega)$;
- find weak solutions of (1), usually by means of variational methods;
- investigate the regularity properties of weak solutions. If one can prove that they belong to some Sobolev space whose order is high enough, then the solutions are *classical*, that is, they belong to $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$.

The eigenvalues of the Laplacian are given by a sequence

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$$

such that $\lambda_k \to +\infty$ as $k \to \infty$. Moreover, the eigenfunctions are analytic and thus they are in particular classical solutions.

In some kind of applications - such as fluid dynamics, nonlinear elasticity and glaciology - the following problem is of relevant interest:

$$\begin{cases} -\Delta_p u &= \lambda |u|^{p-2} u \quad \text{in } \Omega \\ u &= 0 \qquad\qquad \text{on } \partial\Omega \end{cases} \tag{2}$$

where $1 < p < +\infty$ and

$$\Delta_p : u \mapsto \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

is the *p-Laplacian operator*. Remark that $\Delta_2 = \Delta$. Problem (2) is structurally different from (1), since the equation is nonlinear: if u and v solve the equation, then $u + v$ needs not be a solution anymore. However, the problem is $(p-1)$-homogenous, which implies that if u is a solution, then also tu ($t \in \mathbb{R}$) solves the equation. Since (2) shares some of the properties of linear problems, it makes sense to introduce the concept of eigenfunction also in this case - although the idea must be necessarily interpreted in a generalized sense. One defines an eigenfunction as a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$ of (2), that is a function such that, for a fixed $\lambda \in \mathbb{R}$ (which will be again called *eigenvalue*),

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v = \lambda \int_\Omega |u|^{p-2} uv$$

for every $v \in W_0^{1,p}(\Omega)$.
Most of the methods which one can use in the linear case do not find immediate application to this problem. Nevertheless, it is possible to show the existence of a sequence of eigenvalues

$$0 < \lambda_1(p;\Omega) < \lambda_2(p;\Omega) \leq \lambda_3(p;\Omega) \leq \ldots$$

such that $\lambda_k(p;\Omega) \to +\infty$ as $k \to \infty$. The eigenfunctions of the p-Laplacian share many properties with those of the ordinary Laplacian: for instance, the first eigenfunction has constant sign and is unique up to multiplication by a nonzero constant. Eigenfunctions corresponding to higher eigenvalues must be sign-changing, and in particular the second eigenfunction has exactly two nodal domains. Moreover, the first eigenvalue is isolated, which means that there does not exist any eigenvalue between $\lambda_1(p;\Omega)$ and $\lambda_2(p;\Omega)$.

The investigation of the higher eigenvalues of the p-Laplacian is however far from being complete. One of the most interesting and difficult questions is to understand if other eigenvalues exist, apart from the above mentioned sequence, if $p \neq 2$. Other properties of the eigenfunctions - for instance whether they satisfy the so-called "unique continuation property" - are still an open problem. For a better understanding of all these issues it seems sensible to look at the behaviour of eigenvalues and eigenfunctions in the limit cases $p \to 1$ and $p \to +\infty$. In the latter case Juutinen and Lindqvist could prove that

$$\lim_{p\to+\infty} \lambda_1(p;\Omega)^{\frac{1}{p}} = \Lambda_1(\Omega)$$

and

$$\lim_{p\to+\infty} \lambda_2(p;\Omega)^{\frac{1}{p}} = \Lambda_2(\Omega)$$

where

$$\Lambda_k(\Omega)^{-1} := \sup\{r \,|\, \text{there exist } k \text{ disjoint balls of radius } r \text{ contained in } \Omega\}.$$

Moreover, the first (resp. the second) eigenfunctions converge uniformly to a viscosity solution of

$$\begin{cases} F_\Lambda(u, \nabla u, D^2 u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where

$$F_\Lambda(s,\xi,X) = \begin{cases} \min\{|\xi| - \Lambda s, -X\xi\cdot\xi\} & \text{if } s>0 \\ -X\xi\cdot\xi & \text{if } s=0 \\ \max\{-\Lambda s - |\xi|, -X\xi\cdot\xi\} & \text{if } s<0 \end{cases}$$

for $\Lambda = \Lambda_1(\Omega)$ (resp. $\Lambda = \Lambda_2(\Omega)$).

In the present thesis I focus on the case $p \to 1$. The aim is to extend the results found by Kawohl and Fridman, who showed that

$$\lim_{p\to 1} \lambda_1(p;\Omega) = h_1(\Omega)$$

where

$$h_1(\Omega) := \inf_{E\subset\overline{\Omega}} \frac{Per(E;\mathbb{R}^n)}{V(E)}$$

is the so-called *Cheeger constant*. Here $V(E)$ is the n-dimensional Lebesgue measure of E, while $Per(E;\mathbb{R}^n)$ is the perimeter of E measured with respect to $\mathbb{R}^n$, defined in the sense of De Giorgi. I am able to show that a similar result holds also for the second eigenvalue; namely, it will be shown that

$$\lim_{p\to 1} \lambda_2(p;\Omega) = h_2(\Omega)$$

where $h_2(\Omega)$ is defined as

$$h_2(\Omega) := \inf\left\{\lambda\in\mathbb{R}^+ \,\middle|\, \exists\, E_1, E_2 \subset \Omega\ ,\, E_1\cap E_2 = \emptyset\ ,\, \max_{i=1,2} \frac{Per(E_i)}{V(E_i)} \le \lambda\right\}.$$

The geometrical properties of the sets for which $h_2(\Omega)$ is attained are investigated, and in particular I can compute the value of the constant when Ω is a planar disc. As a consequence, it is possible to deduce the nonradiality of the second eigenfunctions if p is sufficiently close to 1.

In the last chapter I show that it is possible to obtain second eigenfunctions of the p-Laplacian as a limit of the following p-superlinear problem:

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

where $1 < p < q < p^*$. Equation (3) admits a sign-changing solution of least energy, whose limit as $q \to p$ is a second eigenfunction of $-\Delta_p$.

Chapter 1

Multiple Cheeger sets

In this chapter we will introduce a geometrical problem which generalizes the well-known Cheeger problem.

1.1 Some results on the Cheeger problem

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Lipschitz boundary. Let us define the *Cheeger constant* of Ω as

$$h_1(\Omega) := \inf_{E \subset \overline{\Omega}} \frac{Per(E;\mathbb{R}^n)}{V(E)}$$

where $Per(E;\mathbb{R}^n)$ is the distributional perimeter of E measured in $\mathbb{R}^n$ (see Definition B.4), and $V(E)$ is the volume of E, that is its n-dimensional Lebesgue measure. A set for which the infimum is attained is called a *Cheeger set* for Ω. For the sake of simplicity, in the following we will set $Per(E) := Per(E;\mathbb{R}^n)$.

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open domain with boundary of class Lipschitz. Then there exists at least one Cheeger set for Ω.*

Proof. A proof is given in Appendix B (Proposition B.12). □

Proposition 1.2. *Let $E \subset \mathbb{R}^n$ a set of finite perimeter. Then there exists a sequence of sets of finite perimeter $\{E_k\}_{k=1}^{+\infty}$ such that:*

1. *∂E_k is smooth for every k;*
2. *$E_k \subset\subset E$ for every k;*
3. *$\chi_{E_k} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^n)$ as $k \to +\infty$;*
4. *$Per(E_k) \to Per(E)$ as $k \to +\infty$.*

Proof. The proof can be found in [45]. □

Proposition 1.3. *The following equalities hold:*

$$\inf_{E\subset\overline{\Omega}} \frac{Per(E)}{V(E)} = \inf_{E\subset\subset\Omega} \frac{Per(E)}{V(E)} = \inf_{\substack{E\subset\subset\Omega \\ \partial E \text{ smooth}}} \frac{Per(E)}{V(E)}.$$

Proof. It is clear that

$$\inf_{E\subset\overline{\Omega}} \frac{Per(E)}{V(E)} \leq \inf_{E\subset\subset\Omega} \frac{Per(E)}{V(E)} \leq \inf_{\substack{E\subset\subset\Omega \\ \partial E \text{ smooth}}} \frac{Per(E)}{V(E)}.$$

Let F be a Cheeger set for Ω; applying Proposition 1.2 we obtain

$$\inf_{\substack{E\subset\subset\Omega \\ \partial E \text{ smooth}}} \frac{Per(E)}{V(E)} \leq \frac{Per(F)}{V(F)} = \inf_{E\subset\overline{\Omega}} \frac{Per(E)}{V(E)}$$

so that the claim is proved. □

In the following we will mention some geometric properties of Cheeger sets.

Proposition 1.4. *Let E be a Cheeger set for Ω; then $\partial E \cap \partial\Omega \neq \emptyset$.*

Proof. Let us suppose that this is not the case. Then E is compactly contained in Ω, which means that there exists a number $\lambda > 1$ such that the set $\lambda E = \{\lambda x \,|\, x \in E\}$ is contained in Ω. But then

$$\frac{Per(\lambda E)}{V(\lambda E)} = \frac{1}{\lambda}\frac{Per(E)}{V(E)} < \frac{Per(E)}{V(E)}$$

which contradicts the fact that E is a Cheeger set. □

Proposition 1.5. *Let E be a Cheeger set for Ω; then:*

1. *$\partial E \cap \Omega$ is analytical, up to a singular set of Hausdorff dimension $n-8$.*
2. *The mean curvature in every regular point of $\partial E \cap \Omega$ is equal to $h_1(\Omega)$.*
3. *Let $x \in \partial E \cap \partial\Omega$ be a regular point for $\partial\Omega$; then x is a regular point for ∂E.*

Proof. The proof can be found in [30]. As a consequence, if ∂E meets $\partial\Omega$ in a regular point of the latter, this must happen tangentially. □

Proposition 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex domain. Then there exists a unique Cheeger set E for Ω. Moreover, E is convex.*

Proof. A proof of the existence of a convex Cheeger set can be found in [38, Remark 10]. Uniqueness has been proved in [39] for the case $n = 2$, and in [1] for general n and Ω of class $\mathcal{C}^{1,1}$. □

Remark 1.7. The hypothesis of convexity can not be dropped: there are examples of star-shaped domains which admit infinitely many Cheeger sets (see [48]). However, it was proved that "almost all" bounded domains admit a unique Cheeger set (see [17]).

Remark 1.8. If $n = 2$ and Ω is convex, then the Cheeger set is the union of balls of suitable radius contained in Ω. This property holds no longer true in higher dimensions (see [39]).

We will often make use of the following property.

Proposition 1.9. *Let $\Omega \subset \mathbb{R}^n$ be bounded, and let $B \subset \mathbb{R}^n$ be a ball such that $|B| = |\Omega|$. Then*

$$h_1(B) \leq h_1(\Omega).$$

Proof. The proof is a consequence of the well-known isoperimetric property of the ball (see for instance [21]). A quantitative version of this theorem is stated in [26]. □

1.2 A continuity result for the Cheeger constant

In the following theorem we prove that $h_1(\Omega)$ is continuous with respect to the L^1 convergence of sets, if we restrict ourselves to the class of convex subsets of $\mathbb{R}^n$.

Theorem 1.10. *Let $\Omega, \Omega_k \subset \mathbb{R}^n$ be bounded convex sets such that $\Omega_k \to \Omega$ in the L^1-topology as $k \to \infty$. Suppose that there exist two bounded set $D, F \subset \mathbb{R}^n$ such that $D \subset \Omega \subset F$ and $D \subset \Omega_k \subset F$ for every k. Then, after possibly passing to a subsequence,*

$$h_1(\Omega_k) \to h_1(\Omega).$$

Proof. In order to prove the claim we will make use of the notion of Γ-convergence (see Appendix C). Let Σ_k and Σ be the families of convex subsets of Ω_k and Ω respectively. Let us define the functionals

$$\Phi_k(C) := \frac{Per(C)}{V(C)} \quad \text{for } C \in \Sigma_k$$

and

$$\Phi(C) := \frac{Per(C)}{V(C)} \quad \text{for } C \in \Sigma.$$

Notice that the elements of Σ_k and Σ are convex subsets of F, so that we can actually define Φ_k and Φ on the family of convex subsets of F, endowed with the metric inherited by the L^1-convergence. Moreover, observe that

$$h_1(\Omega_k) = \inf_{C\in\Sigma_k} \frac{Per(C)}{V(C)}$$

since every convex domain admits a convex Cheeger set (see Proposition 1.6). We are now ready to prove the Γ-convergence of the functionals Φ_k to Φ.
liminf inequality. Let $C \in \Sigma$ and $C_k \in \Sigma_k$ such that $C_k \to C$ in the L^1-topology. Of course we have $V(C_k) \to V(C)$, while from the lower semicontinuity of the perimeter (Proposition B.5) we obtain $Per(C) \leq \liminf_{k\to\infty} Per(C_k)$. In conclusion we get (see also Proposition A.3)

$$\Phi(C) \leq \liminf_{k\to\infty} \Phi_k(C_k).$$

limsup inequality. Let $C \in \Sigma$, and let us define $C_k := C \cap \Omega_k$. The sets C_k are convex sets contained in Ω_k, and are such that $C_k \to C$ in the L^1-topology. From [15, Lemma 4.4] one has $Per(C_k) \to Per(C)$, so that

$$\Phi(C) = \lim_{k\to\infty} \Phi_k(C_k).$$

Equicoercivity. Let $\widetilde{C}_k$ be a convex Cheeger set for Ω_k. From $D \subset \Omega_k \subset F$ we obtain

$$\frac{Per(\widetilde{C}_k)}{V(\widetilde{C}_k)} \leq h_1(D) \Rightarrow Per(\widetilde{C}_k) \leq h_1(D) \cdot V(\widetilde{C}_k) \leq h_1(D) \cdot V(F).$$

So the characteristic functions of the sets $\widetilde{C}_k$ are uniformly bounded in $BV(F)$ and hence they are contained in a compact set of $L^1(F)$.
From the properties of the Γ-convergence we obtain that, after possibly passing to a subsequence,

$$h_1(\Omega_k) \to h_1(\Omega)$$

and there exists a sequence of Cheeger sets $\widetilde{C}_k$ for Ω_k converging in the L^1-topology to a Cheeger set $\widetilde{C}$ for Ω. $\square$

1.3 Multiple Cheeger sets

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Lipschitz boundary. We define, for $k \in \mathbb{N}$,

$$h_k(\Omega) := \inf \left\{ \lambda \in \mathbb{R}^+ \,\middle|\, \exists\, E_1, ..., E_k \subset \Omega \text{ , } E_i \cap E_j = \emptyset \text{ for } i \neq j, \right.$$
$$\left. \max_{i=1,\dots,k} \frac{Per(E_i)}{V(E_i)} \leq \lambda \right\}$$

with the convention that

$$\frac{Per(E)}{V(E)} = +\infty$$

whenever $V(E) = 0$. We will call $h_k(\Omega)$ the *k-th Cheeger constant* for Ω. Notice that, for $k = 1$, we recover the definition of the Cheeger constant $h_1(\Omega)$. By Proposition 1.2 it is possible to take the infimum on sets compactly contained in Ω, or even on sets compactly contained in Ω with smooth boundary.

Theorem 1.11. *For every k, there exist k pairwise disjoint subsets $E_1, ..., E_k$ contained in Ω such that*

$$\max_{i=1,\dots,k} \frac{Per(E_i)}{V(E_i)} \leq h_k(\Omega).$$

Proof. Let us consider minimizing sequences of pairwise disjoint sets $E_{1,n}, ..., E_{k,n}$ for $n = 1, 2, ...$, corresponding to the value μ_n, where

$$\mu_n = \max_{i=1,\dots,k} \frac{Per(E_{i,n})}{V(E_{i,n})}.$$

Set $\chi_{i,n} = \chi_{E_{i,n}}$ for $i = 1, ..., k$. Fix R as the radius of k equal disjoint balls of fixed arbitrary volume $V_0 > 0$ contained in Ω. We are going to show that we can consider $V(E_{i,n}) \geq V_0$ for every i, n. Indeed, if we had $V(E_{\widehat{i},\widehat{n}}) < V_0$ for some values of $\widehat{i}$ and $\widehat{n}$, then by Proposition 1.9 we would surely have

$$\frac{Per(E_{\widehat{i},\widehat{n}})}{V(E_{\widehat{i},\widehat{n}})} \geq h_1(B_r)$$

where B_r is a ball with the same volume as $V(E_{\widehat{i},\widehat{n}})$ and so of radius $r < R$. As a consequence, $\mu_{\widehat{n}} > h_1(B_R)$, which means that we can actually discard the k-tuple of sets $E_{1,\widehat{n}}, ..., E_{k,\widehat{n}}$. Because of the compact embedding of $BV(\Omega)$ in $L^1(\Omega)$ (see Theorem B.9), there exist $E_1, ..., E_k$ such that, up to a subsequence, $\chi_{i,n} \to \chi_{E_i}$ almost everywhere on Ω. Moreover, $V(E_i) \geq V_0 > 0$. Denote with

N the negligible set of non-convergence. From the lower semicontinuity of the total variation (Theorem B.5), it follows that

$$\frac{Per(E_i)}{V(E_i)} \leq h_k(\Omega)$$

for every $i = 1, ..., k$. We are going to show that the E_i are pairwise disjoint: suppose $i \neq j$, then $x \in E_i \setminus N \Rightarrow \chi_{E_i}(x) = 1$, which implies $\chi_{i,n}(x) = 1$ definitely; this means $\chi_{j,n}(x) = 0$ definitely, hence $\chi_{E_j}(x) = 0$, that is $x \notin E_j \setminus N$. If $x \in N$, we can assign arbitrary values to the characteristic functions (this does not affect the total variation). Hence we obtain the claim.

□

Definition 1.12. Any k-tuple of sets $E_1, ..., E_k$ as in Theorem 1.11 will be called a k-tuple of *multiple Cheeger sets*. If $k = 2$, we will also speak of *coupled Cheeger sets*.

Remark 1.13. The proof of the theorem shows that we can always consider a minimizing sequence of k-tuples of sets for $h_k(\Omega)$, where the volumes of the sets are uniformly bounded from below.

Remark 1.14. Proceeding as in Proposition 1.4, one can show that at least one of the minimizing sets must touch the boundary.

Let us define

$$\Lambda_k(\Omega) := \inf \left\{ \frac{1}{r} \mid \exists\ k \text{ disjoint balls } B_1, ..., B_k \subset \Omega \text{ of radius } r \right\}.$$

According to [37], $\Lambda_1(\Omega)$ and $\Lambda_2(\Omega)$ are the first two eigenvalues of the ∞-Laplacian, defined as

$$\Delta_\infty u := \langle D^2 u \cdot \nabla u,\, \nabla u \rangle.$$

We are then able to state the following

Proposition 1.15.

$$h_k(\Omega) \leq n\Lambda_k(\Omega).$$

Proof. Fix $\varepsilon > 0$ and consider k disjoint balls B_1, ..., B_k of radius $(\Lambda_k(\Omega)+\varepsilon)^{-1}$. Then

$$h_k(\Omega) \leq \frac{Per(B_1)}{V(B_1)} = \frac{n\omega_n r^{n-1}}{\omega_n r^n} = n(\Lambda_k(\Omega) + \varepsilon).$$

The claim follows letting ε tend to 0. □

In the following we will give a different characterization of the higher Cheeger constants.

Remark 1.16. The constant $h_k(\Omega)$ can also be defined as

$$h_k(\Omega) := \inf \left\{ \lambda \in \mathbb{R}^+ \,\middle|\, \exists\, E_1, ..., E_k \subset\subset \Omega \;, E_i \cap E_j = \emptyset \text{ for } i \neq j, \right.$$
$$\left. \frac{Per(E_i)}{V(E_i)} = \lambda \text{ for every } i = 1, ..., k \right\}.$$

This is a consequence of the following observation. If $\frac{Per(E)}{V(E)} < \lambda$, it is possible to find a subset $F \subset E$ with $\frac{Per(F)}{V(F)} = \lambda$ by the following procedure: fix a point $x_0 \in E$ and set $R := \sup\{r > 0 \,|\, B_r(x_0) \subset E\}$. Set $E_r := E \setminus B_r(x_0)$; the function

$$[0, R] \mapsto \frac{Per(E_r)}{V(E_r)} = \frac{Per(E) - 2\pi r}{V(E) - \pi r^2}$$

is then continuous with respect to r and monotonously increasing. One can repeat the procedure with E_R instead of E as often as wished, so that the perimeter of the set obtained increases, while its volume tends to zero. This yields the claim.

Proposition 1.17. *Let $\mathcal{P}_k$ be the set of all partitions of Ω with k subsets $E_1, ..., E_k$. Then*

$$h_k(\Omega) = \inf_{\mathcal{P}_k} \max_{i=1,...,k} h_1(E_i).$$

Proof. Set $\widehat{h}_k(\Omega) := \inf_{\mathcal{P}_k} \max_{i=1,...,k} h_1(E_i)$. Let us suppose $\widehat{h}_k(\Omega) < h_k(\Omega)$; then there exists a partition $E_1, ..., E_k$ of Ω such that

$$\max_{i=1,...,k} h_1(E_i) < h_k(\Omega)$$

which is a contradiction. Thus $\widehat{h}_k(\Omega) \geq h_k(\Omega)$. On the other hand, if $C_1, ..., C_k$ are the sets realizing $h_k(\Omega)$, then we can find a partition $E_1, ..., E_k$ of Ω with the property that $C_i \subset E_i$ for every $i = 1, ..., k$. Hence, for every i,

$$h_1(E_i) \leq \frac{Per(C_i)}{V(C_i)} \leq h_k(\Omega)$$

and consequently

$$\max_{i=1,...,k} h_1(E_i) \leq h_k(\Omega)$$

that is

$$\widehat{h}_k(\Omega) \leq h_k(\Omega)$$

which finally yields

$$\widehat{h}_k(\Omega) = h_k(\Omega).$$

□

Remark 1.18. The proof of the proposition also states that there exists a partition realizing $\widehat{h}_k(\Omega)$; that is, we can also write

$$h_k(\Omega) = \min_{\mathcal{P}_k} \max_{i=1,\dots,k} h_1(E_i)$$

or also

$$h_k(\Omega) = \min_{\mathcal{P}_k} \max_{i=1,\dots,k} \min_{C\subset \overline{E}_i} \frac{Per(C)}{V(C)}$$

or also

$$h_k(\Omega) = \min_{\mathcal{P}_k} \max_{i=1,\dots,k} \min_{u\in BV(E_i)} \frac{\|Du\|(\mathbb{R}^n)}{\|u\|_1}.$$

Remark 1.19. The sets realizing $h_k(\Omega)$ can be supposed to be connected. Indeed, if E is disconnected, i.e. $E = E_1 \cup E_2$, with $\overline{E}_1 \cap \overline{E}_2 = \emptyset$, we have

$$\frac{Per(E)}{V(E)} = \frac{Per(E_1) + Per(E_2)}{V(E_1) + V(E_2)} \geq \min\left\{\frac{Per(E_1)}{V(E_1)}, \frac{Per(E_2)}{V(E_2)}\right\}.$$

This follows from Proposition A.4. So one connected component of E has a lower or equal ratio perimeter/area. If $E_1 \cap E_2 = \emptyset$, but $\overline{E}_1 \cap \overline{E}_2 \neq \emptyset$, we modify E on a set of measure zero (this does not affect the total variation) to get a connected set E' defined as

$$E' = E_1 \cup E_2 \cup (\partial E_1 \cap \partial E_2).$$

Theorem 1.20. *There exist multiple Cheeger sets such that the part of their boundary contained in Ω is a piecewise smooth hypersurface of piecewise constant mean curvature.*

Proof. We will give the proof for the case $k = 2$: let E_1 and E_2 be two coupled Cheeger sets, which exist according to Theorem 1.11. Since E_1 minimizes perimeter (measured in $\mathbb{R}^n$) in $\Omega \setminus \overline{E_2}$ with a volume constraint, it will have interior regularity according to [30]. More precisely, $\partial E_1 \cap (\Omega \setminus \overline{E_2})$ is an analytic hypersurface up to a singular set with Hausdorff dimension $n - 8$, whose regular points have constant mean curvature. The same can be stated for E_2. Then we have to consider the possibly nonempty contact surface: also in this case [30, Theorem 2] can be applied to state that the contact surface (if it exists) enjoys the same regularity as the interior boundary of the two sets and has constant mean curvature. □

Definition 1.21. Let E_1 and E_2 be a pair of coupled Cheeger sets. The *free boundary* of E_1 is defined as $\partial E_1 \cap (\Omega \setminus \overline{E_2})$ (analogously for E_2). The *contact surface* between E_1 and E_2 is $\partial E_1 \cap \partial E_2 \cap \Omega$.

Theorem 1.22. *It is possible to find two coupled Cheeger sets such that the following holds. Suppose that $\partial E_1 \cap \partial E_2 \neq \emptyset$. Let us denote by c_1 the mean curvature of the free boundary of E_1, by c_2 the mean curvature of the free boundary of E_2, and by c_3 the mean curvature of the contact surface, measured from E_1. Then the relation*

$$c_1 - c_2 - 2c_3 = 0 \tag{1.1}$$

holds.

Proof. We follow [14, pp. 10-11]. Take $x_1 \in (\partial E_1 \setminus \partial E_2) \cap \Omega$, $x_2 \in (\partial E_2 \setminus \partial E_1) \cap \Omega$ and $x_3 \in \partial E_1 \cap \partial E_2 \cap \Omega$. Suppose that the boundaries of E_1 and E_2 can be locally described by the graph of a function u defined in an open subset $\omega = \omega_1 \cup \omega_2 \cup \omega_3$ of $\mathbb{R}^{n-1}$, where ω_1, ω_2 and ω_3 are disjoint open neighborhoods of x_1, x_2 and x_3 respectively. For $i = 1, 2, 3$, let v_i be a function defined in ω_i with compact support and such that the following conditions are satisfied:

$$\int_{\omega_1} v_1 + \int_{\omega_3} v_3 = 0, \tag{1.2}$$

$$\int_{\omega_2} v_2 - \int_{\omega_3} v_3 = 0. \tag{1.3}$$

Since E_1 and E_2 are coupled Cheeger sets, we can suppose that u is such that

$$\int_{\omega_1 \cup \omega_3} \sqrt{1 + |\nabla u|^2} \leq \int_{\omega_1 \cup \omega_3} \sqrt{1 + |\nabla u + \varepsilon \nabla (v_1 + v_3)|^2}$$

and

$$\int_{\omega_2 \cup \omega_3} \sqrt{1 + |\nabla u|^2} \leq \int_{\omega_2 \cup \omega_3} \sqrt{1 + |\nabla u + \varepsilon \nabla (v_2 + v_3)|^2}$$

for small $\varepsilon > 0$. It follows that

$$0 \leq \int_{\omega_1} \frac{\nabla u \nabla v_1}{\sqrt{1 + |\nabla u|^2}} + \int_{\omega_2} \frac{\nabla u \nabla v_2}{\sqrt{1 + |\nabla u|^2}} + 2 \int_{\omega_3} \frac{\nabla u \nabla v_3}{\sqrt{1 + |\nabla u|^2}} =$$

$$= - \int_{\omega_1} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) v_1 - \int_{\omega_2} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) v_2 -$$

$$-2 \int_{\omega_3} \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) v_3 = -c_1 \int_{\omega_1} v_1 - c_2 \int_{\omega_2} v_2 - 2c_3 \int_{\omega_3} v_3.$$

Since also the functions $-v_1$, $-v_2$ and $-v_3$ are admissible, it follows that

$$c_1 \int_{\omega_1} v_1 + c_2 \int_{\omega_2} v_2 + 2c_3 \int_{\omega_3} v_3 = 0$$

for arbitrary v_1, v_2, v_3 satisfying the conditions (1.2) and (1.3); hence we obtain

$$c_1 - c_2 - 2c_3 = 0.$$

$\square$

Remark 1.23. The condition on the mean curvatures is similar to the one given in [35] for the double bubble problem: find two regions in $\mathbb{R}^n$ which enclose two given amounts of volume, such that they minimize the sum of the surface measures. However, in that problem the quantity to minimize is slightly different, so also the condition on the mean curvatures differs and reads $c_1 - c_2 - c_3 = 0$.

Proposition 1.24. *Let $\Omega \subset \mathbb{R}^2$ be a convex planar domain; then it is possible to find two coupled Cheeger sets E_1, E_2 such that they satisfy condition (1.1) in Theorem 1.22 and such that, if $\partial E_1 \cap \partial E_2 \neq \emptyset$, then their boundaries meet tangentially.*

Proof. We can suppose that c_1, $c_2 \geq 0$; otherwise, since Ω is convex, it would be possible to modify the sets suitably in order to decrease their perimeter and increase their volume. As a consequence, at least one of the two sets (say E_1) is convex. Let us suppose that ∂E_1 and ∂E_2 meet each other in a non-smooth way. Then one could consider the Cheeger set C_1 of E_1, which is convex and has a $\mathcal{C}^1$ boundary, and then find a perimeter-minimizing set C_2 in $\Omega \setminus \overline{C}_1$ under the volume constraint $|C_2| = |E_2|$. The boundaries ∂C_1 and ∂C_2 will meet tangentially as proved in [30]. Then one can apply again Theorem 1.22 to get the condition on the curvatures. $\square$

Proposition 1.25. *Let $\Omega \subset \mathbb{R}^n$ admit a unique Cheeger set. Then*

$$h_1(\Omega) < h_2(\Omega).$$

Proof. Let us suppose that $h_1(\Omega) = h_2(\Omega)$; then there exist two disjoint subsets C_1, $C_2 \subset \Omega$ such that

$$\max\left\{\frac{Per(C_1)}{V(C_1)}, \frac{Per(C_2)}{V(C_2)}\right\} = h_1(\Omega)$$

which means, by definition of $h_1(\Omega)$,

$$\frac{Per(C_1)}{V(C_1)} = \frac{Per(C_2)}{V(C_2)} = h_1(\Omega).$$

This is a contradiction to the uniqueness of the Cheeger set for Ω. $\square$

Remark 1.26. It is worth noting that there exist nonconvex domains for which $h_1(\Omega) = h_2(\Omega)$; think for example of a "barbell domain" made of two identical rectangles connected by a thin pipe. To be more precise, consider the planar set

$$\Omega = \{(0,1) \times (0,2)\} \cup \{[1,2] \times (0,\varepsilon)\} \cup \{(2,3) \times (0,2)\}$$

where $\varepsilon > 0$ is small enough.

Proposition 1.27. *Let us denote by ω_n the volume of the unit ball in $\mathbb{R}^n$. Then*

$$h_k(\Omega) \geq n \left(\frac{k\omega_n}{|\Omega|} \right)^{\frac{1}{n}}.$$

Proof. Let $E_1, ..., E_k$ be a family of multiple Cheeger sets, so that

$$\max_{i=1,\ldots,k} h_1(E_i) \leq h_k(\Omega).$$

The volume of each E_i can not be smaller than the volume of a ball with Cheeger constant $h_k(\Omega)$, which is exactly $\omega_n \left(\frac{n}{h_k(\Omega)} \right)^n$. In fact, let $\widetilde{B}$ be a ball such that $|E_i| = |\widetilde{B}|$, and B a ball such that $h_1(B) = h_k(\Omega)$; if $|\widetilde{B}| < |B|$ we would have, applying Proposition 1.9,

$$h_k(\Omega) = h_1(B) < h_1(\widetilde{B}) \leq h_1(E_i) \leq h_k(\Omega)$$

which is a contradiction. So we obtain

$$k\omega_n \left(\frac{n}{h_k(\Omega)} \right)^n \leq |\Omega| \Rightarrow h_k(\Omega) \geq n \left(\frac{k\omega_n}{|\Omega|} \right)^{\frac{1}{n}}.$$

□

Corollary 1.28.

$$h_k(\Omega) \to +\infty$$

as $k \to +\infty$.

Remark 1.29. The lower bound in Proposition 1.27 for $k = 1$ follows directly from Proposition 1.9, and is obviously optimal if Ω is a ball. For the higher Cheeger constants, it can be easily seen that the estimate is optimal for the union of k balls with equal radii. If we try to minimize $h_k(\Omega)$ among connected sets, it turns out that the infimum is the same (consider a family of k balls of equal radii connected by thin strips whose width goes to 0). An interesting question would be to minimize $h_k(\Omega)$ among plane convex sets of given area. If we focus on $h_2(\Omega)$, it seems that a stadium (the convex hull of two tangent balls with both radii equal to R) is very near to be a minimizer; namely, it is possible to show that

$$\frac{1.874}{R} \leq h_2(\Omega) \leq \frac{1.912}{R}.$$

The lower bound follows directly from Proposition 1.27. To obtain the upper bound, one can note that the common tangent divides Ω into two equal convex halves, whose Cheeger set E is given by the union of balls of constant radius $x \leq R$. E satisfies then the conditions

$$Per(E) = 4R + \pi R - 4x + \pi x$$

$$V(E) = \frac{1}{2}\pi R^2 + 2R^2 - 2x^2 + \frac{1}{2}\pi x^2$$

and since it must be

$$\frac{Per(E)}{V(E)} = \frac{1}{x}$$

we get $x = 0.523R$. This yields the estimate from above. However, it should be mentioned that the stadium does not minimize the second eigenvalue of the Laplacian among convex planar domains, as proved in [32].

1.4 Coupled Cheeger sets for a planar disc

In this section we will determine the coupled Cheeger sets of a disc $\Omega \subset \mathbb{R}^2$ with radius r. As a first step we will compute the Cheeger set E for a half-disc Ω' of same radius. According to the results in Section 1.1, the Cheeger set must have the geometry shown in the picture.

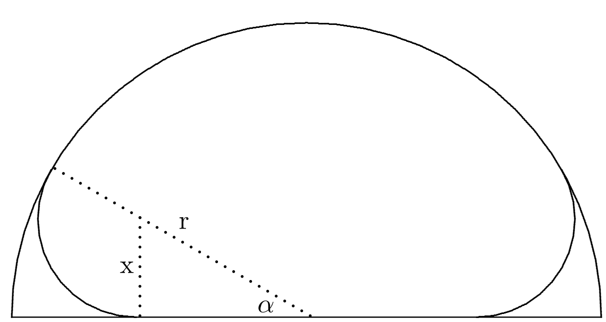

Figure 1.1: The candidate Cheeger set for a half-disc.

We will denote by α the inner angle and by x the radius of the inner arc. Thus we have the relation

$$(r-x)\sin\alpha = x$$

which gives the existence condition $0 \leq x \leq \frac{r}{2}$. Then

$$Per(E) = 2(r-x)\cos\alpha + 2x\left(\frac{\pi}{2}+\alpha\right) + r(\pi - 2\alpha),$$

$$V(E) = x(r-x)\cos\alpha + x^2\left(\frac{\pi}{2}+\alpha\right) + \frac{r^2}{2}(\pi - 2\alpha).$$

Remember that $\alpha = \arcsin\left(\frac{x}{r-x}\right)$ and $\cos\alpha = \sqrt{1-\left(\frac{x}{r-x}\right)^2}$, since we consider $0 \leq \alpha \leq \frac{\pi}{2}$. Numerical resolution of the equation

$$\frac{Per(E)}{V(E)} = \frac{1}{x} \quad (= \text{possible } h_1(\Omega'))$$

gives, for $r = 1$,

$$x = 0.317028...$$

which means

$$h_1(\Omega') = 3.15429...$$

This is the best configuration with convex subsets to compute $h_2(\Omega)$; indeed, a convex partition of a convex set can be obtained only cutting the set with hyperplanes (otherwise we would have a point of non-zero curvature which gives convexity from one side but concavity from the other one). The Cheeger sets of each of the two partitioning subsets are then convex. Conversely, two convex subsets can be separated by a hyperplane thanks to the Hahn-Banach Theorem. The Cheeger constant of a circular segment strictly contained in a half-disc is then strictly higher, due to uniqueness reasons. So the above configuration is the best among convex subsets of the disc.
Observe that the two coupled Cheeger sets E_1 and E_2 must have a contact surface. If it was not the case, we can suppose without loss of generality that

$$\frac{Per(E_1)}{V(E_1)} \leq \frac{Per(E_2)}{V(E_2)}$$

and that E_1 is a Cheeger set for $\Omega \setminus E_2$. Notice that E_2 is automatically a Cheeger set for $\Omega \setminus E_1$. Due to the properties of Cheeger sets, the free boundaries of E_1 and E_2 must be circular arcs which meet $\partial\Omega$ tangentially. The only possibility is that E_1 and E_2 are discs, and the best configuration is

given by to equal discs with radius $\frac{r}{2}$, which is clearly not optimal for Ω.
We are going to prove that the contact surface can not be closed; if it was the case, then one of the two coupled Cheeger sets, which we denote by E_1, would be a disc of radius $r' < r$, as in Figure 1.2. The other set E_2 will be then contained in $\Omega \setminus E_1$. Suppose that E_2 has a free boundary consisting of arcs with constant curvature $c_2 \geq 0$. An easy computation shows that the case $c_2 = 0$ is never optimal; so we can suppose that the arcs have constant curvature $c_2 > 0$. Due to the fact that ∂E_1 is the contact surface, these arcs can not start on $\partial\Omega$ and end on ∂E_1; the only possibility is that the free boundary "encloses" E_1 as the dashed line in Figure 1.2. But in this case, the choice $E_2 = \Omega \setminus E_1$ would give a lower ratio perimeter/area. So the optimal choice is the pair consisting of E_1 and its complement. By modifying r' suitably, one can easily convince himself that the optimal configuration is achieved when the ratios perimeter/area of E_1 and E_2 are equal. This implies

$$\frac{Per(E_1)}{V(E_1)} = \frac{Per(E_2)}{V(E_2)} \Rightarrow \frac{2}{r'} = \frac{2}{r - r'} \Rightarrow r' = \frac{r}{2}$$

which yields, for $r = 1$,

$$h_1(E_1) = h_1(E_2) = 4.$$

This gives a worse configuration than the one found before. As a consequence, the contact surface can not be a closed line.

We will now use the regularity results about the coupled Cheeger sets; in particular, by Remark 1.24 we can suppose that the boundary of each of the two sets meets the boundary of the other set tangentially. Suppose that the separating surface is an arc PQ with constant curvature c_3. From the point P two arcs of curvature c_1 and c_2 respectively will depart, in such a way that the centres of curvature lie on the chord AB orthogonal to PQ and such that $P \in AB$. Notice that we can suppose, without loss of generality, that c_1, $c_2 \geq 0$.

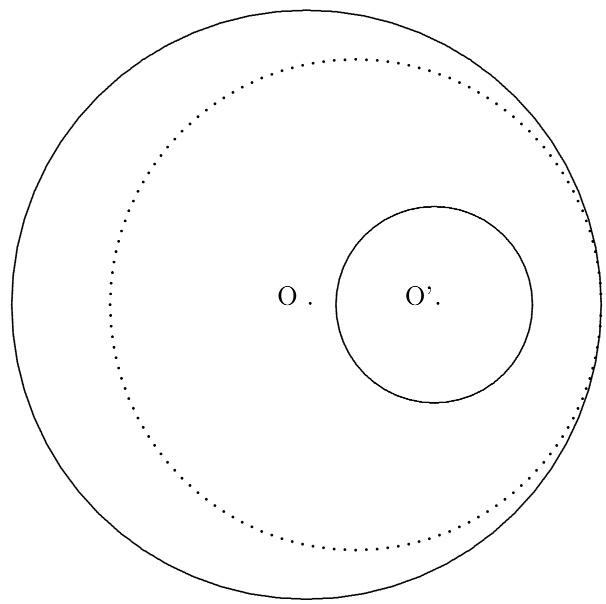

Figure 1.2: The contact surface can not be closed.

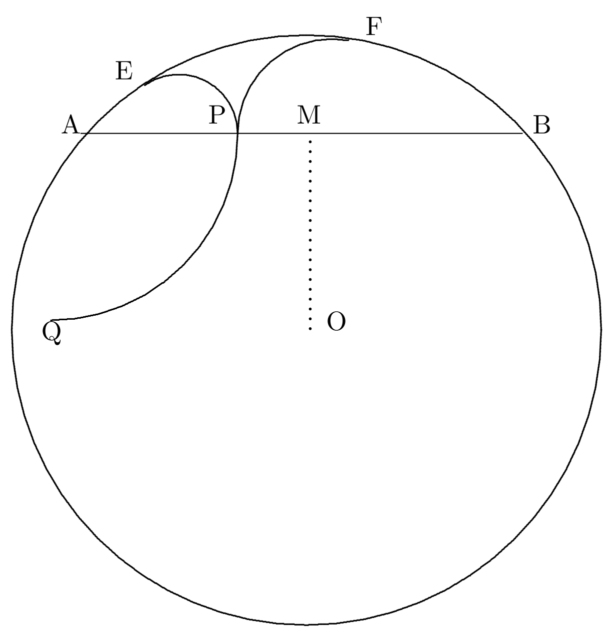

Figure 1.3: The case $c_3 \geq 0$.

Let E_1 be the "candidate" Cheeger set containing the segment AP and such that the curvature of its free boundary is c_1; let E_2 be the set containing the segment PB and with curvature of the free boundary equal to c_2. Without loss of generality, we can suppose that $\overline{AP} \leq \overline{PB}$. Let M be the middle point of the segment AB. If $P \neq M$, it is impossible that $c_3 \geq 0$ (as in Figure 1.3); indeed, since E_1 would be a subset of a circular segment strictly contained in a half-disc, this would contradict the fact that the configuration of the Cheeger sets of the two half-discs is better. So it must be $c_3 < 0$.

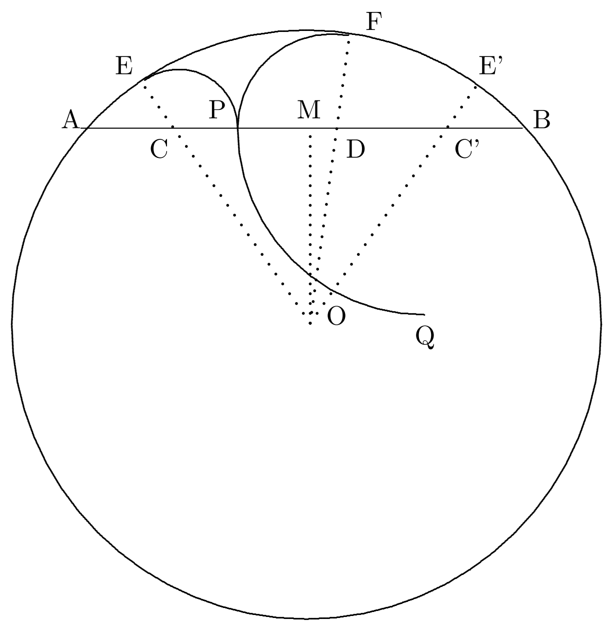

Figure 1.4: The case $c_3 < 0$.

Let C and D the centers of curvature of the free boundaries of E_1 and E_2 respectively, and E, F as in Figure 1.4 such that $\overline{CP} = \overline{EC}$ and $\overline{PD} = \overline{DF}$. Since $c_3 < 0$, from Theorem 1.22 it must be $c_1 < c_2$, that is $\overline{PC} > \overline{PD}$. This is impossible for geometrical reasons: indeed, take a point C' on AB such that $\overline{AC} = \overline{C'B}$; it follows $\overline{PC'} > \overline{PC} > \overline{PD}$, which means that the point D must lie between P and C'. If E' is the intersection of the circle with the line OC', it is clear that $\overline{DF} > \overline{C'E'}$. This is a contradiction because we would have $\overline{C'E'} = \overline{CE} > \overline{DF} > \overline{C'E'}$.
It follows that necessarily $P = M$. For symmetry reasons, this implies $c_1 = c_2$ and hence, again from Theorem 1.22, $c_3 = 0$. So we recover the optimal configuration consisting of the Cheeger sets of the two half-discs.

Chapter 2

Eigenvalues under Dirichlet boundary condition

2.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ a bounded open domain with Lipschitz boundary. We are interested in the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u &= \lambda |u|^{p-2}u \quad \text{in } \Omega \\ u &= 0 \qquad\qquad \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\lambda \in \mathbb{R}$ and $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the *p-Laplacian* operator. A real number λ is said to be an *eigenvalue* if there exists a function $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ (called *eigenfunction*) satisfying (2.1) in the weak sense, which means

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \int_\Omega |u|^{p-2}uv \qquad \forall\ v \in W_0^{1,p}(\Omega).$$

For $p = 2$ we recover the well-known eigenvalue problem for the Laplacian:

$$\begin{cases} -\Delta u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

From standard results of linear functional analysis it is known that all eigenvalues of the Laplacian are given by a sequence $\{\lambda_k(2;\Omega)\}_{k=1}^{+\infty}$ such that

$$\lambda_1(2;\Omega) < \lambda_2(2;\Omega) \le ... \le \lambda_k(2;\Omega) \to +\infty.$$

In order to obtain the first eigenvalue one can use the direct method of Calculus of Variations by minimizing the so-called *Rayleigh quotient*, which means

$$\lambda_1(2;\Omega) := \inf_{v \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega |v|^2}.$$

In the case $p \neq 2$, the operator Δ_p is no longer linear, so that it is impossible to use techniques from linear functional analysis. In fact, if u and v are two eigenfunctions associated to the same eigenvalue, then $u+v$ need not be necessarily an eigenfunction. However, for every $c \neq 0$, the function cu will still be an eigenfunction. Somehow surprisingly, many of the results valid in the linear case extend also to the p-Laplacian. Indeed, the first eigenvalue can be obtained in an analogous way to the case $p = 2$:

$$\lambda_1(p;\Omega) := \inf_{v \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p}.$$

We have the following

Proposition 2.1. *Let u be an eigenfunction of the p-Laplacian associated to $\lambda \in \mathbb{R}$. Then $u \in \mathcal{C}^{1,\alpha}_{loc}(\Omega)$ where $\alpha \in (0,1)$ depends only on p and n.*

Proof. The claim follows from the estimate

$$\|u\|_\infty \leq 4^n \cdot \lambda^{\frac{n}{p}} \cdot \|u\|_1,$$

whose proof can be found in [44], and from the regularity results in [22]. □

Proposition 2.2. *There exists, up to a nonzero multiplicative constant, one and only one eigenfunction $e_{1,p}$ associated to $\lambda_1(p;\Omega)$. Moreover, $e_{1,p}$ is of only one sign and therefore it can be considered to be strictly positive in Ω.*

Proof. Suppose that $e_{1,p}$ is a first eigenfunction of the p-Laplacian; observe that, by Proposition 2.1, $e_{1,p}$ is in particular continuous. Since

$$\frac{\int_\Omega |\nabla |e_{1,p}||^p}{\int_\Omega |e_{1,p}|^p} = \frac{\int_\Omega |\nabla e_{1,p}|^p}{\int_\Omega |e_{1,p}|^p}$$

then also the function $v := |e_{1,p}|$ will be a first eigenfunction. From Harnack's inequality it follows that $v > 0$ in Ω, which in turns implies that $e_{1,p}$, due to its continuity, is one-signed. Having this in mind, one can prove the simplicity of $e_{1,p}$ following for instance the proof in [8]. □

$\lambda_1(p;\Omega)$ is not the only eigenvalue of the p-Laplacian. Indeed, it is possible to build a sequence of eigenvalues

$$\lambda_1(p;\Omega) < \lambda_2(p;\Omega) \leq \lambda_3(p;\Omega) \leq ... \leq \lambda_k(p;\Omega) \to +\infty$$

using the following minimax principle, as shown for instance in [27] and explained in Appendix D.

Definition 2.3. Let X be a Banach space, $A \subset X$ a closed, symmetric subset. The *Krasnoselskii genus* $\gamma(A)$ is defined as

$$\gamma(A) := \min\{m \in \mathbb{N} \,|\, \exists\ \varphi : A \to \mathbb{R}^m \setminus \{0\},\ \varphi \text{ is continuous and odd}\}.$$

Definition 2.4. We denote by Γ_k the set

$$\Gamma_k := \left\{ A \subset W_0^{1,p}(\Omega)\backslash\{0\} \,\middle|\, A \cap \{\|u\|_p = 1\} \text{ is compact}, A \text{ symmetric}, \gamma(A) \geq k \right\}.$$

It is possible to prove that, for every $k \in \mathbb{N}$, the following numbers are eigenvalues:

$$\lambda_k(p;\Omega) := \inf_{A \in \Gamma_k} \max_{u \in A} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}.$$

In the literature they are sometimes called *variational eigenvalues*. It can be easily seen that the two definitions of $\lambda_1(p;\Omega)$ given so far coincide. It is still an open question, whether other eigenvalues can exist. We mention that the existence of nonvariational eigenvalues was proved in [9] for the problem

$$\begin{cases} -\Delta_p u &= \lambda q |u|^{p-2} u \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 \qquad\qquad\quad \text{on } \partial\Omega \end{cases}$$

for some $q \in \mathcal{C}^1(\overline{\Omega})$, $q > 0$, $1 < p \neq 2$, in the case where Ω has a particular shape (for instance a planar annulus).

We recall some results about higher eigenfunctions.

Proposition 2.5. *Eigenfunctions associated to higher eigenvalues of the p-Laplacian must be sign-changing.*

Proof. A proof can be found in [43, Lemma 3.1]. □

Proposition 2.6. *There does not exist any eigenvalue between $\lambda_1(p;\Omega)$ and $\lambda_2(p;\Omega)$, which means that $\lambda_1(p;\Omega)$ is isolated.*

Proof. A proof can be found in [5]. □

The second eigenvalue has also the following mountain-pass characterization, which turns out to be very useful in the numerical investigation of the problem (see [33]).

Proposition 2.7. *Let $e_{1,p}$ be a first eigenfunction of the p-Laplacian. Then*

$$\lambda_2(p;\Omega) = \inf_{\gamma \in A} \sup_{u \in \gamma[0,1]} \int_\Omega |\nabla u|^p$$

where

$$A := \{\gamma \in \mathcal{C}([0,1]; W_0^{1,p}(\Omega)) \,|\, \|\gamma(t)\|_p = 1,\ \gamma(0) = e_{1,p},\ \gamma(1) = -e_{1,p}\}.$$

Proof. The proof can be found in [20, Corollary 3.2]. □

A *nodal domain* of a function $u : \Omega \to \mathbb{R}$ is a connected component of the set $\{x \in \Omega \,|\, u(x) \neq 0\}$. It is not known whether the zero set of an eigenfunction of the p-Laplacian has Hausdorff dimension $n-1$, or if it can be even an open subset. The following result generalizes Courant's nodal domain Theorem for the eigenfunctions of the Laplacian.

Proposition 2.8. *Let u be an eigenfunction associated to $\lambda_k(p;\Omega)$. Then u has at most $2k-2$ nodal domains.*

Proof. The proof can be found in [23, Theorem 3.3]. □

As an easy consequence of the previous proposition it follows that any second eigenfunction has exactly two nodal domains.

2.2 A convergence result for higher eigenvalues

First of all we prove an approximation result for functions of bounded variation. We will denote by $BV(\Omega)$ the space of functions of bounded variation on a set $\Omega \subset \mathbb{R}^n$. If $u \in BV(\Omega)$, the symbol $\|Du\|(\Omega)$ will stand for the *total variation* of u measured in Ω (as defined in B.1), while $\|Du\|(\mathbb{R}^n)$ will be the total variation of u measured in $\mathbb{R}^n$. It holds

$$\|Du\|(\mathbb{R}^n) = \|Du\|(\Omega) + \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1}.$$

Theorem 2.9. *Assume $\Omega \in \mathbb{R}^n$ is bounded and $\partial\Omega$ is C^1. Let $u \in BV(\Omega)$. Then there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset C^\infty(\overline{\Omega})$, converging strictly to u.*

Proof. By a known approximation result (see [25, Chapter 5, Theorem 2]) there exists a sequence $\{v_k\}_{k=1}^{\infty} \subset C^\infty(\Omega) \cap BV(\Omega)$ converging strictly to u. Every v_k belongs in particular to $W^{1,1}(\Omega)$, and so by [24, Section 5.3, Theorem 3] there exists a sequence $w_{k,m}$ in $C^\infty(\overline{\Omega})$ converging to v_k in $W^{1,1}(\Omega)$ as $m \to \infty$; in particular, $\|Dw_{k,m}\|(\Omega) \to \|Dv_k\|(\Omega)$. By a diagonal procedure we obtain the claim. □

Remark 2.10. Since the trace operator is continuous from $BV(\Omega)$ (endowed with the topology of the strict convergence) to $L^1(\partial\Omega;\mathcal{H}^{n-1})$ (see [6, Theorem B.11]), the functions u_k of the previous theorem are such that

$$\|Du_k\|(\mathbb{R}^n) \to \|Du\|(\mathbb{R}^n).$$

Theorem 2.11. *Assume $\Omega \in \mathbb{R}^n$ is bounded and $\partial\Omega$ is C^2. Let $u \in BV(\Omega)$. Then there exists a sequence $\{v_k\}_{k=1}^{\infty} \subset W_0^{1,\infty}(\Omega)$, such that*

$$v_k \to u \text{ in } L^1(\Omega)$$

and

$$\|Dv_k\|(\mathbb{R}^n) \to \|Du\|(\mathbb{R}^n)$$

as $k \to \infty$.

Proof. Set

$$d_\varepsilon(x) := \begin{cases} \varepsilon^{-1} dist(x, \partial\Omega) & \text{if } dist(x,\partial\Omega) < \varepsilon \\ 1 & \text{if } dist(x,\partial\Omega) \geq \varepsilon. \end{cases}$$

Let u_k be the approximating sequence in $C^\infty(\overline{\Omega})$ given by Theorem 2.9; the claim will follow if we prove that every u_k can be approximated by a sequence in $W_0^{1,\infty}(\Omega)$ converging strictly in $BV(\mathbb{R}^n)$. To this end, fix w as such a u_k and set $v_\varepsilon := w d_\varepsilon$. Clearly, $v_\varepsilon \in W_0^{1,\infty}(\Omega) \cap C(\overline{\Omega})$. Moreover,

$$v_\varepsilon \to w \text{ in } L^1(\Omega)$$

as $\varepsilon \to 0$, so that

$$\|Dw\|(\mathbb{R}^n) \leq \liminf_{\varepsilon \to 0} \|Dv_\varepsilon\|(\mathbb{R}^n).$$

Then

$$\begin{aligned} \|Dv_\varepsilon\|(\mathbb{R}^n) &= \int_\Omega |\nabla v_\varepsilon| = \int_\Omega |\nabla(w d_\varepsilon)| = \int_\Omega |w \nabla d_\varepsilon + d_\varepsilon \nabla w| \\ &\leq \int_\Omega |w \nabla d_\varepsilon| + \int_\Omega |d_\varepsilon \nabla w|. \end{aligned}$$

Denote by Ω_ε the set

$$\Omega_\varepsilon := \{x \in \Omega \,|\, dist(x, \partial\Omega) < \varepsilon\}.$$

Then we have

$$\|Dv_\varepsilon\|(\mathbb{R}^n) \leq \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} |w| + \|\nabla w\|_\infty \cdot |\Omega \setminus \Omega_\varepsilon| + \int_{\Omega_\varepsilon} |\nabla w|.$$

For $\varepsilon \to 0$ it follows (see also Lemma 3.1)

$$\limsup_{\varepsilon \to 0} \|Dv_\varepsilon\|(\mathbb{R}^n) \leq \int_{\partial\Omega} |w| \, d\mathcal{H}^{n-1} + \int_\Omega |\nabla w| = \|Dw\|(\mathbb{R}^n).$$

Hence

$$\lim_{\varepsilon \to 0} \|Dv_\varepsilon\|(\mathbb{R}^n) = \|Dw\|(\mathbb{R}^n).$$

□

Theorem 2.12. *Let Ω have a boundary of class $\mathcal{C}^2$. Define the following functional on $L^1(\Omega)$*

$$F_p(u) := \begin{cases} \|\nabla u\|_p^p & \text{for } u \in W_0^{1,p}(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus W_0^{1,p}(\Omega). \end{cases}$$

Then the functionals F_p Γ-converge in $L^1(\Omega)$, as $p \to 1$, to the functional

$$F_1(u) := \begin{cases} \|Du\|(\mathbb{R}^n) & \text{for } u \in BV(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus BV(\Omega). \end{cases}$$

Proof. liminf inequality. Let $u_p \to u$ in $L^1(\Omega)$. If only a finite number of the u_p's are in $W_0^{1,p}(\Omega)$, then $\liminf\limits_{p\to 1} F_p(u_p) = +\infty$ and there is nothing to prove. If $u_{p_j} \in W_0^{1,p_j}(\Omega)$ for a sequence, then $u \in BV(\Omega)$. From the lower semicontinuity of the total variation it follows

$$\begin{aligned} \|Du\|(\mathbb{R}^n) &\le \liminf_{j\to\infty} \|Du_j\|(\mathbb{R}^n) = \liminf_{j\to\infty} \int_\Omega |\nabla u_j| \\ &\le \liminf_{j\to\infty} \left(\int_\Omega |\nabla u_j|^{p_j}\right)^{\frac{1}{p_j}} |\Omega|^{\frac{1}{p_j'}} \\ &\le \liminf_{j\to\infty} \left[\left(\int_\Omega |\nabla u_j|^{p_j}\right) + |\Omega| \cdot \frac{p_j^{-\frac{p_j'}{p_j}}}{p_j'}\right] \\ &\le \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^{p_j} + \limsup_{j\to\infty} \left(|\Omega| \cdot \frac{p_j^{-\frac{p_j'}{p_j}}}{p_j'}\right) \\ &= \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^{p_j}. \end{aligned}$$

limsup inequality. First of all, if $u = 0$ the proof is trivial. Let us suppose in the following $u \neq 0$. If $u \notin BV(\Omega)$, there is nothing to prove. If $u \in BV(\Omega)$, by Theorem 2.11 we can find a sequence of functions u_k in $W_0^{1,\infty}(\Omega)$ such that

$u_k \to u$ in $L^1(\Omega)$ and $\|Du_k\|(\mathbb{R}^n) \to \|Du\|(\mathbb{R}^n)$. It follows that

$$\begin{aligned}\|Du\|(\mathbb{R}^n) &= \lim_{k\to+\infty} \|Du_k\|(\mathbb{R}^n) = \lim_{k\to+\infty} \int_\Omega |\nabla u_k| \\ &= \lim_{k\to+\infty} \|\nabla u_k\|_\infty \int_\Omega \frac{|\nabla u_k|}{\|\nabla u_k\|_\infty} \\ &\geq \limsup_{k\to+\infty} \|\nabla u_k\|_\infty^{1-p_k} \int_\Omega |\nabla u_k|^{p_k} \\ &\geq \left(\liminf_{k\to+\infty} \|\nabla u_k\|_\infty^{1-p_k}\right)\left(\limsup_{k\to\infty} \int_\Omega |\nabla u_k|^{p_k}\right).\end{aligned}$$

If $\liminf\limits_{k\to\infty} \|\nabla u_k\|_\infty = c > 0$, we obtain

$$\|Du\|(\mathbb{R}^n) \geq \limsup_{k\to\infty} \int_\Omega |\nabla u_k|^{p_k}$$

which is the claim. If $\liminf\limits_{k\to\infty} \|\nabla u_k\|_\infty = 0$, we would have

$$\|Du\|(\mathbb{R}^n) \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k| \leq \liminf_{k\to\infty} \|\nabla u_k\|_\infty \cdot |\Omega| = 0$$

and thus $u = 0$, case which we ruled out.

□

Corollary 2.13. *Let Ω have a boundary of class $\mathcal{C}^2$. Define*

$$F_p(u) := \begin{cases} \|\nabla u\|_p & \text{for } u \in W_0^{1,p}(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus W_0^{1,p}(\Omega). \end{cases}$$

Then the functionals F_p Γ-converge in $L^1(\Omega)$, as $p \to 1$, to the functional

$$F_1(u) := \begin{cases} \|Du\|(\mathbb{R}^n) & \text{for } u \in BV(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus BV(\Omega). \end{cases}$$

Proof. The liminf inequality can also follow from the fact that $\liminf a_n b_n \leq (\liminf a_n)(\limsup b_n)$. Otherwise one can argue that $\lim a_n = \lim a_n^{p_n}$ as $p_n \to 1$. □

Now we consider a slightly different family of functionals, where the space $W_0^{1,p}(\Omega)$ is replaced by $W^{1,p}(\Omega)$. We will show that the a very similar result holds, where the quantity $\|Du\|(\mathbb{R}^n)$ is substituted by $\|Du\|(\Omega)$.

Proposition 2.14. *Let Ω have a boundary of class $\mathcal{C}^2$. Define*

$$F_p(u) := \begin{cases} \|\nabla u\|_p^p & \text{for } u \in W^{1,p}(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus W^{1,p}(\Omega). \end{cases}$$

Then the functionals F_p Γ-converge in $L^1(\Omega)$, as $p \to 1$, to the functional

$$F_1(u) := \begin{cases} \|Du\|(\Omega) & \text{for } u \in BV(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus BV(\Omega). \end{cases}$$

Proof. liminf inequality. The same proof as in Theorem 2.12, with $\|Du\|(\Omega)$ instead of $\|Du\|(\mathbb{R}^n)$.
limsup inequality. First of all, if $u = const$ the proof is trivial. Let us suppose in the following $u \neq const$. If $u \notin BV(\Omega)$, there is nothing to prove. If $u \in BV(\Omega)$, by Theorem 2.9 we can find a sequence of functions u_k in $W^{1,\infty}(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$ and $\|Du_k\|(\Omega) \to \|Du\|(\Omega)$. It follows that

$$\begin{aligned}
\|Du\|(\Omega) &= \lim_{k\to+\infty} \|Du_k\|(\Omega) = \lim_{k\to+\infty} \int_\Omega |\nabla u_k| \\
&= \lim_{k\to+\infty} \|\nabla u_k\|_\infty \int_\Omega \frac{|\nabla u_k|}{\|\nabla u_k\|_\infty} \\
&\geq \limsup_{k\to+\infty} \|\nabla u_k\|_\infty^{1-p_k} \int_\Omega |\nabla u_k|^{p_k} \\
&\geq \left(\liminf_{k\to+\infty} \|\nabla u_k\|_\infty^{1-p_k} \right) \left(\limsup_{k\to\infty} \int_\Omega |\nabla u_k|^{p_k} \right).
\end{aligned}$$

If $\liminf_{k\to\infty} \|\nabla u_k\|_\infty = c > 0$, we obtain

$$\|Du\|(\Omega) \geq \limsup_{k\to\infty} \int_\Omega |\nabla u_k|^{p_k}$$

which is the claim. If $\liminf_{k\to\infty} \|\nabla u_k\|_\infty = 0$, we would have

$$\|Du\|(\Omega) \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k| \leq \liminf \|\nabla u_k\|_\infty \cdot |\Omega| = 0$$

and thus $u = const$, case which we ruled out. □

Corollary 2.15. *Let Ω have a boundary of class $\mathcal{C}^2$. Define*

$$F_p(u) := \begin{cases} \|\nabla u\|_p & \text{for } u \in W^{1,p}(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus W^{1,p}(\Omega). \end{cases}$$

Then the functionals F_p Γ-converge in $L^1(\Omega)$, as $p \to 1$, to the functional

$$F_1(u) := \begin{cases} \|Du\|(\Omega) & \text{for } u \in BV(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus BV(\Omega). \end{cases}$$

In the following we will prove the main result of this section. Our aim is to modify slightly the results of [19], in order to adapt them to our setting. We will consider the family of functionals

$$F_p(u) := \begin{cases} \|\nabla u\|_p & \text{for } u \in W_0^{1,p}(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus W_0^{1,p}(\Omega) \end{cases}$$

with $p > 1$. Let us denote by F_1 the functional defined as

$$F_1(u) := \begin{cases} \|Du\|(\mathbb{R}^n) & \text{for } u \in BV(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus BV(\Omega). \end{cases}$$

We also define, for $p > 1$,

$$\Sigma_p^k := \left\{ A \subset W_0^{1,p}(\Omega) \,\middle|\, A \subset \{\|v\|_p = 1\}, \text{ symmetric, compact, } \gamma(A) \geq k \right\}$$

and

$$\Sigma_1^k := \left\{ A \subset BV(\Omega) \,\middle|\, A \subset \{\|v\|_1 = 1\}, \text{ symmetric, compact, } \gamma(A) \geq k \right\}.$$

Moreover it will be

$$\mathcal{K}_s := \{A \subset L^1(\Omega) \,|\, A \text{ symmetric, compact in } L^1(\Omega)\}.$$

It turns out that $\Sigma_p^k \subset \mathcal{K}_s$, and the genus of a set in Σ_p^k is the same as the genus as an element of $\mathcal{K}_s$ (see [19, Lemma 3.2]). We define, for $p \geq 1$, the following functional on $\mathcal{K}_s$:

$$J_p^k(G) := \begin{cases} \sup_{v \in G} F_p(v) & \text{if } G \in \Sigma_p^k \\ +\infty & \text{otherwise.} \end{cases}$$

Again from [19] one has, for $p > 1$,

$$\lambda_k(p; \Omega) = \inf_{G \in \mathcal{K}_s} J_p^k(G).$$

Then we define

$$\lambda_k(1; \Omega) := \inf_{G \in \mathcal{K}_s} J_1^k(G).$$

It is still not known whether the $\lambda_k(1; \Omega)$ can be considered as higher eigenvalues of the 1-Laplacian, defined formally as

$$\Delta_1 u := \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

It is not clear neither what the eigenvalue equation should look like; however, it was proved in [46] that there exists a sequence of eigenvalues obtained using abstract results of nonsmooth analysis which make use of the concept of

Ljusternik-Schnirelman category.

In the following we will denote by $d_{\mathcal{H}}$ the Hausdorff distance between two compact sets E and F, defined as

$$d_{\mathcal{H}}(E,F) := \sup_{x\in F} dist(x,E) + \sup_{y\in E} dist(y,F).$$

It turns out that $(\mathcal{K}_s, d_{\mathcal{H}})$ is a metric space.

Theorem 2.16.

$$\lim_{p\to 1} \lambda_k(p;\Omega) = \lambda_k(1;\Omega).$$

Proof. We will follow the scheme of [19, Theorem 3.3]. We divide the proof in three steps.

Step 1. We prove that the family of functionals $\{J_p^k\}_{1<p<p_0}$ is equicoercive in $\mathcal{K}_s$ for a $p_0 > 1$. Let $p < p_0$ and $G_p \in \mathcal{K}_s$ be such that $J_p^k(G_p) \le C$. By definition of J_p^k we obtain the estimate

$$\|u\|_{W_0^{1,1}} \le C|\Omega|^{\frac{p-1}{p}} \le K$$

for every $u \in G_p$. By [19, Proposition 2.5] the sublevels $\{J_p^k \le C\}$ are contained in a common compact subset of $(\mathcal{K}_s, d_H)$ for $p < p_0$, so that the family $\{J_p^k\}_{1<p<p_0}$ is equicoercive.

Step 2. We show the Γ-liminf estimate. Take $G \in \mathcal{K}_s$ and $\{G_p\}_{p>1}$ such that $G_p \to G$ in the Hausdorff topology. We want to prove that

$$J_1^k(G) \le \liminf_{p\to 1} J_p^k(G_p).$$

Without loss of generality, we may assume that there exists a constant $C > 0$ such that $J_p^k(G_p) \le C$ for every $p > 1$. Let us first show that $\gamma(G) \ge k$. By [19, Proposition 2.4] there exists an open symmetric neighbourhood N of G in $L^1(\Omega)$ such that $\gamma(\overline{N}) = \gamma(G)$. We then infer from [19, Lemma 2.8] that $G_p \subset N \subset \overline{N}$ for p near enough to 1. By the second property in [19, Remark 2.3], for such a p we get

$$k \le \gamma(G_p) \le \gamma(\overline{N}) = \gamma(G).$$

Let now $u \in G$, by the sequential characterisation of the Hausdorff convergence of compact sets, there exists a (generalised) sequence $u_p \in G_p$ converging to u in $L^1(\Omega)$. By the Γ-liminf inequality for the functionals F_p we have

$$F_1(u) \le \liminf_{p\to 1} F_p(u_p) \le \liminf_{p\to 1} \left(\sup_{G_p} F_p \right) = \liminf_{p\to 1} J_p^k(G_p).$$

Taking the supremum on all $u \in G$ we obtain the claim.
Step 3. It only remains to prove that

$$\limsup_{p\to 1}\left(\inf_{G\in\mathcal{K}_s} J_p^k(G)\right) \leq \inf_{G\in\mathcal{K}_s} J_1^k(G).$$

Without loss of generality we can assume that $\inf\limits_{G\in\mathcal{K}_s} J_1^k(G) < +\infty$. Fix $\delta > 0$, and let $G_0 \in \mathcal{K}_s$ be such that

$$\inf_{G\in\mathcal{K}_s} J_1^k(G) \geq J_1^k(G_0) - \delta.$$

Since G_0 is compact in $BV(\Omega)$, by the compact embedding theorem G_0 is also compact in $L^1(\Omega)$; so there exists a finite family $\{u^i\}_{i=1,...,m}$ in G_0 such that

$$G_0 \subset \bigcup_{i=1}^m B_{L^1(\Omega)}\left(u^i, \frac{\delta}{5}\right).$$

From the Γ-limsup inequality for F_p there exists, for every $i = 1, ..., m$, a family $\{u_p^i\}_p$ in $L^1(\Omega)$ such that

$$u_p^i \to u^i \qquad \text{in } L^1(\Omega)$$

and

$$F_p(u_p^i) \to F_1(u^i)$$

as $p \to 1$. Taking p_0 as in step 1, for any $p \in (1, p_0)$ we define C_p to be the convex closure of the finite symmetric set $\{\pm u_p^i \,|\, i = 1, ..., m\}$. We may assume that $F_p(u_p^i) < +\infty$ for any i and any $p \in (1, p_0)$, so that the finite dimensional set C_p is a compact convex subset both of $W_0^{1,p}(\Omega)$ and $L^1(\Omega)$. We denote by Q_p the unique projection onto C_p for the L^1-norm (with respect to which C_p is compact) satisfying the property

$$\|Q_p(v)\|_{\frac{2N}{2N-1}} = \min\left\{\|w\|_{\frac{2N}{2N-1}} : \|w - v\|_1 = \min\{\|v - w'\|_1 : w' \in C_p\}\right\}.$$

Moreover we notice that for any $v \in G_0$ there exists $i = 1, ..., m$ such that $\|v - u^i\|_1 \leq \frac{\delta}{5}$. Therefore

$$\begin{aligned}\|Q_p(v)\|_1 &\geq \|u_p^i\|_1 - \|Q_p(u^i) - u_p^i\|_1 - \|Q_p(v) - Q_p(u^i)\|_1 \\ &\geq \|u_p^i\|_1 - \|u^i - u_p^i\|_1 - \frac{\delta}{5}.\end{aligned}$$

Since $u_p^i \to u^i$ in $L^1(\Omega)$, for p close enough to 1 we have

$$Q_p(G_0) \subset C_p \setminus B_{L^1(\Omega)}\left(0, 1 - \frac{\delta}{2}\right).$$

Moreover, the element $Q_p(G_0)$ of $\mathcal{K}_s$ satisfies $\gamma(Q_p(G_0)) \geq k$. Then consider the functional $\varphi_p : Q_p(G_0) \to W_0^{1,p}(\Omega)$ defined as $\varphi_p(v) := \frac{v}{\|v\|_p}$ and set, for every $p \in (0, p_0)$,

$$G_p := \varphi_p(Q_p(G_0)).$$

Since φ_p is continuous on $Q_p(G_0)$, G_p belongs to Σ_p^k (notice that it is finite-dimensional). Moreover one has, for every $v \in Q_p(G_0)$,

$$1 - \frac{\delta}{2} \leq \|v\|_1 \leq \|v\|_p |\Omega|^{1-\frac{1}{p}}.$$

As a consequence we get

$$\begin{aligned}
J_p^k(G_p) &= \sup \left\{ F_p\left(\frac{v}{\|v\|_p}\right) \mid v \in Q_p(G_0) \right\} \\
&\leq \frac{|\Omega|^{1-\frac{1}{p}}}{1-\frac{\delta}{2}} \sup\{F_p(v) \mid v \in Q_p(G_0)\} \\
&\leq \frac{2|\Omega|^{1-\frac{1}{p}}}{2-\delta} \sup\{F_p(v) \mid v \in C_p\} \\
&= \frac{2|\Omega|^{1-\frac{1}{p}}}{2-\delta} \max_{i=1,\dots,m} \{F_p(u_p^i)\}.
\end{aligned}$$

Thus

$$\limsup_{p \to 1} \left(\inf_{G \in \mathcal{K}_s} J_p^k(G) \right) \leq \limsup_{p \to 1} J_p^k(G_p) \leq \frac{2}{2-\delta} \left(\inf_{G \in \mathcal{K}_s} J_1^k(G) + \delta \right).$$

The claim follows letting δ go to 0.

□

2.3 Continuity of $\lambda_k(p;\Omega)$ with respect to p

In [34] it was proved that the first two eigenvalues of the p-Laplacian are continuous with respect to p. To show that also the higher eigenvalues are continuous functions of p, a possibility could be to prove that eigenfunctions corresponding to different eigenvalues are linearly independent, which is still an open question. However, the result can be obtained as an application of the results in [19].

Theorem 2.17. *Let Ω have a boundary of class Lipschitz. Let $p, q > 1$. Define*

$$F_q(u) := \begin{cases} \|\nabla u\|_q & \text{for } u \in W_0^{1,q}(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus W_0^{1,q}(\Omega) \end{cases}$$

Then the functionals F_q Γ-converge in $L^1(\Omega)$, as $q \to p^+$, to the functional F_p.

Proof. liminf inequality. Let $u_q \to u$ in $L^1(\Omega)$ for $q \to p^+$; if $\liminf_{q\to p^+} F_q(u_q) = +\infty$ there is nothing to prove. If $\liminf_{q\to p^+} F_q(u_q) = c < +\infty$ then the u_q's are uniformly bounded in $W_0^{1,p}(\Omega)$ by Hölder's inequality; hence there exists a sequence u_{q_k} such that $q_k \to p^+$ as $k \to \infty$, $\lim_{k\to\infty} F_{q_k}(u_{q_k}) = c$ and $u_{q_k} \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. From the weak lower semicontinuity of the norm it follows

$$\begin{aligned}\int_\Omega |\nabla u|^p &\le \liminf_{k\to\infty} \int_\Omega |\nabla u_{q_k}|^p \\ &\le \liminf_{k\to\infty} \left(\int_\Omega |\nabla u_{q_k}|^{q_k}\right)^{\frac{p}{q_k}} |\Omega|^{\frac{q_k-p}{q_k}} \\ &\le \liminf_{k\to\infty} \left(\int_\Omega |\nabla u_{q_k}|^{q_k}\right)^{\frac{p}{q_k}} \cdot \limsup_{k\to\infty} |\Omega|^{\frac{q_k-p}{q_k}} \\ &= \liminf_{k\to\infty} \left(\int_\Omega |\nabla u_{q_k}|^{q_k}\right)^{\frac{p}{q_k}}\end{aligned}$$

so that

$$F_p(u) \le \liminf_{k\to\infty} F_{q_k}(u_{q_k}) = \liminf_{q\to p^+} F_q(u_q).$$

limsup inequality. If $u \notin W_0^{1,p}(\Omega)$, there is nothing to prove. Let us suppose $u \in W_0^{1,p}(\Omega)$; if $u = 0$, simply take $u_k = 0$. If $u \neq 0$, we can find a sequence of functions u_k in $C_c(\Omega)$ (and hence in $W_0^{1,\infty}(\Omega)$) such that $u_k \to u$ in the $W^{1,p}$-norm. Set $q_k \to p^+$. It follows that

$$\begin{aligned}\left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}} &= \lim_{k\to+\infty} \left(\int_\Omega |\nabla u_k|^p\right)^{\frac{1}{p}} = \lim_{k\to+\infty} \|\nabla u_k\|_\infty \left(\int_\Omega \frac{|\nabla u_k|^p}{\|\nabla u_k\|_\infty^p}\right)^{\frac{1}{p}} \\ &\ge \limsup_{k\to+\infty} \|\nabla u_k\|_\infty \left(\int_\Omega \frac{|\nabla u_k|^{q_k}}{\|\nabla u_k\|_\infty^{q_k}}\right)^{\frac{1}{p}} \\ &\ge \limsup_{k\to+\infty} (\|\nabla u_k\|_\infty)^{\frac{p-q_k}{p}} \left(\int_\Omega |\nabla u_k|^{q_k}\right)^{\frac{1}{p}} \\ &\ge \liminf_{k\to+\infty} (\|\nabla u_k\|_\infty)^{\frac{p-q_k}{p}} \cdot \limsup_{k\to\infty} \left(\int_\Omega |\nabla u_k|^{q_k}\right)^{\frac{1}{p}}.\end{aligned}$$

If $\liminf_{k\to\infty} \|\nabla u_k\|_\infty = c > 0$, we obtain

$$\left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}} \ge \limsup_{k\to\infty} \left(\int_\Omega |\nabla u_k|^{q_k}\right)^{\frac{1}{p}} = \limsup_{k\to\infty} \left(\int_\Omega |\nabla u_k|^{q_k}\right)^{\frac{1}{q_k}}$$

which is the claim. If $\liminf_{k\to\infty} \|\nabla u_k\|_\infty = 0$, we would have, by the liminf inequality,

$$\left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}} \le \liminf_{k\to\infty}\left(\int_\Omega |\nabla u_k|^{q_k}\right)^{\frac{1}{q_k}} \le \liminf \|\nabla u_k\|_\infty \cdot |\Omega|^{\frac{1}{q_k}} = 0$$

and thus $u = 0$, case which we ruled out.

□

Theorem 2.18. *For a Lipschitz domain Ω the eigenvalues $\lambda_k(p;\Omega)$ are continuous from the right with respect to p, that is*

$$\lim_{q\to p^+} \lambda_k(q;\Omega) = \lambda_k(p;\Omega)$$

Proof. The theorem is a consequence of the results in [19, Theorem 3.3]. □

Theorem 2.19. *Let Ω have a boundary of class Lipschitz. Let $p, q > 1$. Define*

$$F_q(u) := \begin{cases} \|\nabla u\|_q & \text{for } u \in W_0^{1,q}(\Omega) \\ +\infty & \text{for } u \in L^1(\Omega) \setminus W_0^{1,q}(\Omega). \end{cases}$$

Then the functionals F_q Γ-converge in $L^1(\Omega)$, as $q \to p^-$, to the functional F_p.

Proof. liminf inequality. Let $u_q \to u$ in $L^1(\Omega)$ for $q \to p^-$ and fix $\varepsilon > 0$; if $\liminf_{q\to p^-} F_q(u_q) = +\infty$ there is nothing to prove. If $\liminf F_q(u_q) = c < +\infty$ then the u_q's are uniformly bounded in $W_0^{1,p-\varepsilon}(\Omega)$ by Hölder's inequality; hence there exists a sequence u_{q_k} such that $q_k \to p^-$ as $k \to \infty$, $\lim_{k\to\infty} F_{q_k}(u_{q_k}) = c$ and $u_{q_k} \rightharpoonup u$ weakly in $W_0^{1,p-\varepsilon}(\Omega)$. From the weak lower semicontinuity of the norm it follows

$$\begin{aligned}\int_\Omega |\nabla u|^{p-\varepsilon} &\le \liminf_{k\to\infty}\int_\Omega |\nabla u_{q_k}|^{p-\varepsilon} \\ &\le \liminf_{k\to\infty}\left(\int_\Omega |\nabla u_{q_k}|^{q_k}\right)^{\frac{p-\varepsilon}{q_k}} |\Omega|^{\frac{q_k-p+\varepsilon}{q_k}} \\ &\le \liminf_{k\to\infty}\left(\int_\Omega |\nabla u_{q_k}|^{q_k}\right)^{\frac{p-\varepsilon}{q_k}} \cdot \limsup_{k\to\infty} |\Omega|^{\frac{q_k-p+\varepsilon}{q_k}} \\ &= |\Omega|^{\frac{\varepsilon}{p}} \liminf_{k\to\infty}\left(\int_\Omega |\nabla u_{q_k}|^{q_k}\right)^{\frac{p-\varepsilon}{q_k}}\end{aligned}$$

so that

$$F_{p-\varepsilon}(u) \le |\Omega|^{\frac{\varepsilon}{p}} \liminf_{k\to\infty} F_{q_k}(u_{q_k}) = |\Omega|^{\frac{\varepsilon}{p}} \liminf_{q\to p^-} F_q(u_q).$$

Notice that the value $\liminf_{q\to p^-} F_q(u_q)$ does not depend on the choice of the particular subsequence, and so does not depend on ε. Letting ε tend to 0, we obtain

$$F_p(u) \leq \liminf_{k\to\infty} F_q(u_q).$$

limsup inequality. Set $q_k \to p^-$. If $u \notin W_0^{1,p}(\Omega)$, there is nothing to prove. If $u \in W_0^{1,p}(\Omega)$, then it belongs in particular to $W^{1,q_k}(\Omega)$ for every k and so we can simply consider the constant sequence $u_k := u$ for every k; then of course

$$F_p(u) = \lim_{k\to\infty} F_{q_k}(u_k). \qquad \square$$

Theorem 2.20. *For a Lipschitz domain Ω the eigenvalues $\lambda_k(p;\Omega)$ are continuous from the left with respect to p, that is*

$$\lim_{q\to p^-} \lambda_k(q;\Omega) = \lambda_k(p;\Omega).$$

Proof. The theorem is a consequence of the results in [19, Theorem 3.3]. $\square$

Theorem 2.21. *For a Lipschitz domain Ω the eigenvalues $\lambda_k(p;\Omega)$ are continuous functions with respect to p.*

Proof. The theorem is a consequence of Theorems 2.18 and 2.20. $\square$

2.4 The second eigenfunction

2.4.1 The second eigenvalue as $p \to 1$

Lemma 2.22. *Let $E \subset \mathbb{R}^n$ be a set with Lipschitz boundary, and let E^ε be, for $\varepsilon > 0$, the ε-strip around E defined as*

$$E^\varepsilon := \{x \in \mathbb{R}^n \setminus \overline{E} \,|\, dist(x, \partial E) < \varepsilon\}.$$

Then

$$V(E^\varepsilon) = \varepsilon Per(E) + o(\varepsilon)$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$.

Proof. The proof can be found in [4]. $\square$

Theorem 2.23.

$$\limsup_{p\to 1} \lambda_2(p;\Omega) \leq h_2(\Omega).$$

Proof. Let C_1, $C_2 \subset\subset \Omega$ be two subsets such that $C_1 \cap C_2 = \emptyset$, and

$$\frac{Per(C_1)}{V(C_1)}, \frac{Per(C_2)}{V(C_2)} \leq h_2(\Omega) + \frac{1}{2k}.$$

It is possible to find E_1, E_2 with the property that, for $i = 1, 2$, $E_i \subset\subset C_i$, ∂E_i is smooth, and

$$\frac{Per(E_i)}{V(E_i)} \leq h_2(\Omega) + \frac{1}{k}.$$

Let $\varepsilon > 0$, and let v_i $(i = 1, 2)$ be two functions such that: $v_i = 1$ on E_i, $v_i = 0$ outside a ε-neighbourhood of E_i, and $|\nabla v_i| = \varepsilon^{-1}$ on the ε-strip E_i^ε outside E_i. ε should be chosen in a way that $(E_1 \cup E_1^\varepsilon) \cap (E_2 \cup E_2^\varepsilon) = \emptyset$. Set

$$A_0 := \left\{ \alpha v_1 + \beta v_2 \,\middle|\, |\alpha|^p + |\beta|^p = 1 \right\}.$$

Then $A_0 \in \Gamma_2$ (see also [34, Lemma 2.1]). Thus we have

$$\begin{aligned}
\lambda_2(p; \Omega) &\leq \sup_{u \in A_0} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \leq \sup_{|\alpha|^p + |\beta|^p = 1} \frac{\varepsilon^{-p}|\alpha|^p V(E_1^\varepsilon) + \varepsilon^{-p}|\beta|^p V(E_2^\varepsilon)}{|\alpha|^p V(E_1) + |\beta|^p V(E_2)} \\
&= \sup_{|\alpha|^p + |\beta|^p = 1} \frac{\varepsilon^{1-p}|\alpha|^p Per(E_1) + \varepsilon^{1-p}|\beta|^p Per(E_2) + \varepsilon^{-p} o(\varepsilon)}{|\alpha|^p V(E_1) + |\beta|^p V(E_2)} \\
&\leq \varepsilon^{1-p} \left(h_2(\Omega) + \frac{1}{k} \right) + \frac{\varepsilon^{-p} o(\varepsilon)}{\min\{V(E_1), V(E_2)\}}
\end{aligned}$$

as we have

$$V(E_i^\varepsilon) = \varepsilon Per(E_i) + o(\varepsilon)$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$ (see Lemma 2.22). Note that the last inequality is true because of Proposition A.4. If we send $p \to 1$, we obtain

$$\limsup_{p \to 1} \lambda_2(p; \Omega) \leq h_2(\Omega) + \frac{1}{k} + \frac{\varepsilon^{-1} o(\varepsilon)}{\min\{V(E_1), V(E_2)\}}$$

and if $\varepsilon \to 0$

$$\limsup_{p \to 1} \lambda_2(p; \Omega) \leq h_2(\Omega) + \frac{1}{k}.$$

The claim follows if we send $k \to \infty$. The fact that E_1 and E_2 depend from k does not constitute a problem, since in any case we can estimate $V(E_i)$ uniformly from below, as a consequence of Proposition 1.9. □

Remark 2.24. The theorem can be easily generalised to the k-th variational eigenvalue obtaining

$$\limsup_{p \to 1} \lambda_k(p; \Omega) \leq h_k(\Omega).$$

Theorem 2.25. *The following Cheeger-type inequality holds:*

$$\lambda_2(p;\Omega) \geq \left(\frac{h_2(\Omega)}{p}\right)^p.$$

Proof. Let $e_{2,p}$ be a second eigenfunction of the p-Laplacian. From [37] we know that $e_{2,p}$ has exactly two nodal domains $N_{1,p}$, $N_{2,p}$. $e_{2,p}$ is also a first eigenfunction on each of the two nodal domains; from Cheeger's inequality it follows, for $i = 1, 2$,

$$\lambda_2(p;\Omega) = \lambda_1(p; N_{i,p}) \geq \left(\frac{h_1(N_{i,p})}{p}\right)^p.$$

But as $N_{1,p} \cap N_{2,p} = \emptyset$, we have

$$\max\{h_1(N_{1,p}), h_1(N_{2,p})\} \geq h_2(\Omega)$$

due to the definition of $h_2(\Omega)$. So we obtain the claim. □

Remark 2.26. It is worth noting that, if λ is an eigenvalue such that there exists an associated eigenfunction with k nodal domains, then

$$\lambda \geq \left(\frac{h_k(\Omega)}{p}\right)^p.$$

Theorem 2.27.

$$\lim_{p\to 1} \lambda_2(p;\Omega) = h_2(\Omega).$$

Proof. The claim follows easily from Theorems 2.23 and 2.25. □

2.4.2 Nodal domains as $p \to 1$

In the following we prove a result about the asymptotic behaviour of the nodal domains of second eigenfunctions as $p \to 1$ and draw some consequences about the shape of the nodal line if Ω is a planar disc or a square.

Theorem 2.28. *Let $N_{1,p}$, $N_{2,p}$ the nodal domains of the second eigenfunction of the p-Laplacian. Then*

$$\lim_{p\to 1} \max\{h_1(N_{1,p}), h_1(N_{2,p})\} \to h_2(\Omega).$$

Proof. By definition of $h_2(\Omega)$ we have

$$h_2(\Omega) \leq \max\{h_1(N_{1,p}), h_1(N_{2,p})\}.$$

It remains to prove that for every $\varepsilon > 0$, there exists $p_0 > 1$ such that for every $1 < p < p_0$,

$$\max\{h_1(N_{1,p}), h_1(N_{2,p})\} \leq h_2(\Omega) + \varepsilon.$$

Suppose that this is not the case; then there exists $\varepsilon > 0$ such that, without loss of generality, $h_1(N_{1,p_k}) > h_2(\Omega) + \varepsilon$ for a subsequence $p_k \to 1$. From Cheeger's inequality

$$\lambda_2(p_k; \Omega) \geq \left(\frac{h_1(N_{1,p_k})}{p_k}\right)^{p_k} > \left(\frac{h_2(\Omega) + \varepsilon}{p_k}\right)^{p_k} > h_2(\Omega) + \frac{\varepsilon}{2}$$

for k large enough. But this contradicts the fact that $\lim\limits_{p\to 1} \lambda_2(p; \Omega) = h_2(\Omega)$. Hence the claim follows. $\square$

Corollary 2.29. *For $p \to 1$, the volume of each of the nodal sets is uniformly bounded from below by $\omega_n \left(\frac{n}{2h_2(\Omega)}\right)^n$.*

Proof. From the preceding theorem there exists $p_0 > 1$ such that, for every $1 < p < p_0$,

$$\max\{h_1(N_{1,p}), h_1(N_{2,p})\} \leq 2h_2(\Omega).$$

Arguing as in Proposition 1.27, the volume of the nodal sets can not be smaller than the volume of a ball with Cheeger constant $2h_2(\Omega)$, which is exactly $\omega_n \left(\frac{n}{2h_2(\Omega)}\right)^n$. Thus, for $i = 1, 2$,

$$|N_{i,p}| \geq |B| = \omega_n \left(\frac{n}{2h_2(\Omega)}\right)^n$$

as claimed. $\square$

2.4.3 The second eigenfunction as $p \to 1$

We are now going to investigate the asymptotic behaviour of the second eigenfunction as $p \to 1$. First, we state some technical lemmas.

Lemma 2.30. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set with Lipschitz boundary, $p_j \to 1$ as $j \to \infty$ ($p_j \geq 1$), $u_j \in W_0^{1,p_j}(\Omega)$ for every j, $u_j \to u$ in $L^1(\Omega)$ as $j \to \infty$. Then*

$$\|Du\|(\mathbb{R}^n) \leq \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^{p_j}.$$

Proof. Since $\partial\Omega$ is Lipschitz, the functions u_j are in particular in $BV(\mathbb{R}^n)$. Let us denote by p'_j the exponent conjugate to p_j; by Theorem B.5, Hölder's inequality and [24, page 622, letter d] we have

$$\begin{aligned}
\|Du\|(\mathbb{R}^n) &\leq \liminf_{j\to\infty} \|Du_j\|(\mathbb{R}^n) = \liminf_{j\to\infty} \int_\Omega |\nabla u_j| \\
&\leq \liminf_{j\to\infty} \left(\int_\Omega |\nabla u_j|^{p_j}\right)^{\frac{1}{p_j}} |\Omega|^{\frac{1}{p'_j}} \\
&\leq \liminf_{j\to\infty} \left[\left(\int_\Omega |\nabla u_j|^{p_j}\right) + |\Omega| \cdot \frac{p_j^{-\frac{p'_j}{p_j}}}{p'_j}\right] \\
&\leq \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^{p_j} + \limsup_{j\to\infty} |\Omega| \cdot \frac{p_j^{-\frac{p'_j}{p_j}}}{p'_j} \\
&= \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^{p_j}.
\end{aligned}$$

□

Lemma 2.31. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set, $p_j \to 1$ as $j \to \infty$ ($p_j \geq 1$), $0 < \|u_j\|_{L^\infty(\Omega)} \leq c$ for every j ($c > 0$), $u \in L^1(\Omega)$, and $u_j \to u$ in $L^1(\Omega)$ as $j \to \infty$. Then*

$$\lim_{j\to\infty} \int_\Omega |u_j|^{p_j} = \int_\Omega |u|.$$

Proof. Let us denote by p'_j the exponent conjugate to p_j. By Hölder's inequality and [24, page 622, letter d], we have

$$\begin{aligned}
\int_\Omega |u| = \lim_{j\to\infty} \int_\Omega |u_j| &\leq \liminf_{j\to\infty} \left(\int_\Omega |u_j|^{p_j}\right)^{\frac{1}{p_j}} |\Omega|^{\frac{1}{p'_j}} \\
&\leq \liminf_{j\to\infty} \left[\left(\int_\Omega |u_j|^{p_j}\right) + |\Omega| \cdot \frac{p_j^{-\frac{p'_j}{p_j}}}{p'_j}\right] \\
&= \liminf_{j\to\infty} \int_\Omega |u_j|^{p_j} + \limsup_{j\to\infty} |\Omega| \cdot \frac{p_j^{-\frac{p'_j}{p_j}}}{p'_j} \\
&= \liminf_{j\to\infty} \int_\Omega |u_j|^{p_j}. \qquad (2.3)
\end{aligned}$$

On the other hand from $0 < \|u_j\|_{L^\infty(\Omega)} \leq c$ and $p_j \geq 1$ we have

$$\int_\Omega \frac{|u_j|}{\|u_j\|_\infty} \geq \int_\Omega \left(\frac{|u_j|}{\|u_j\|_\infty}\right)^{p_j}$$

so that

$$\int_\Omega |u| = \lim_{j\to\infty} \int_\Omega |u_j| \geq \limsup_{j\to\infty} \|u_j\|_\infty^{1-p_j} \cdot \int_\Omega |u_j|^{p_j} \geq \limsup_{j\to\infty} \int_\Omega |u_j|^{p_j}.$$

The last equation and (2.3) end the proof.

□

Lemma 2.32. *Let $e_{2,p}$ be a second eigenfunction of the p-Laplacian. Then*

$$\|e_{2,p}\|_\infty \leq 4^n \cdot \lambda_2(p;\Omega)^{\frac{n}{p}} \cdot \|e_{2,p}\|_1.$$

Proof. The proof can be found in [44]. □

Theorem 2.33. *Let $e_{2,p}$ be second eigenfunctions of the p-Laplacian such that $\|e_{2,p}\|_p = 1$. Then (after possibly passing to a subsequence) $e_{2,p}$ converge, as $p \to 1$, in $L^1(\Omega)$ and hence pointwise a.e. to a function $u \in BV(\Omega)$ such that $\|u\|_1 = 1$ and $\|Du\|(\mathbb{R}^n) \leq h_2(\Omega)$. Moreover, u can not be strictly positive or strictly negative.*

Proof. From Lemma 2.32 and Hölder's inequality, $e_{2,p}$ are uniformly bounded in $L^\infty(\Omega)$. Moreover, we have

$$\|De_{2,p}\|(\mathbb{R}^n) = \int_\Omega |\nabla e_{2,p}| \leq \left(\int_\Omega |\nabla e_{2,p}|^p\right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} = \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p'}}$$

where p' is the exponent conjugate to p. Since $\lambda_2(p;\Omega) \to h_2(\Omega)$, the functions are uniformly bounded in $BV(\Omega)$; hence there exists a subsequence converging in $L^1(\Omega)$ to a function $u \in BV(\Omega)$. From Proposition B.5 we have

$$\begin{aligned}\|Du\|(\mathbb{R}^n) &\leq \liminf_{p\to 1} \|De_{2,p}\|(\mathbb{R}^n) \leq \liminf_{p\to 1} \left(\int_\Omega |\nabla e_{2,p}|^p\right)^{\frac{1}{p}} |\Omega|^{\frac{1}{p'}} \\ &= \liminf_{p\to 1} \lambda_2(p;\Omega)^{\frac{1}{p}} \cdot |\Omega|^{\frac{1}{p'}} = h_2(\Omega).\end{aligned}$$

Finally, Lemma 2.31 yields $\|u\|_1 = 1$.
The fact that u can not be strictly positive or strictly negative is a consequence of Corollary 2.29. Note that it is possible that $\|e_{2,p}^-\|_1 \to 0$ as $p \to 1$ although $|\{e_{2,p} < 0\}|$ is uniformly bounded away from zero. □

2.5 The cases of the disc and of the square

In this section we will apply the previously found results to the particular case where the domain $\Omega \subset \mathbb{R}^2$ is a disc or a square. In particular, we are able to state that, if p is sufficiently close to 1, then every second eigenfunction in a planar disc must be nonradial. Let us recall that the existence of radial eigenfunctions was shown in [52]; in this case, one has to solve the ordinary differential equation

$$\begin{cases} -(r^{n-1}|u'|^{p-2}u')' & = & \lambda r^{n-1}|u|^{p-2}u \quad \text{in (0,R)} \\ u'(0) & = & 0 \\ u(R) & = & 0 \end{cases}$$

Let us mention that no result about the symmetry properties of the second eigenfunction of the p-Laplacian seems to be known so far (except for the case $p = 2$).

Proposition 2.34. *Let $\Omega \subset \mathbb{R}^2$ be a disc of radius $R > 0$. Then*

$$\lim_{p\to 1} \lambda_2(p;\Omega) = \frac{3.15429}{R}.$$

Proof. The claim follows from Theorem 2.23 and the results in section 1.4. □

Theorem 2.35. *For p close to 1, the second eigenfunction of the p-Laplacian in a disc $\Omega \subset \mathbb{R}^2$ can not have a circular centered nodal domain. In particular, it can not be radial.*

Proof. Fix $R = 1$. From Proposition 2.34 there exists $p_0 > 1$ such that

$$\lambda_2(p;\Omega) \le 3.5$$

for $1 < p < p_0$. Let us suppose that there exists a second eigenfunction of the p-Laplacian whose nodal domains are a ball B_r of radius r $(0 < r < 1)$, compactly contained in Ω, and $A := \Omega \setminus B_r$. If we restrict ourselves to the case $p < 1.1$, Cheeger's inequality allows us to state that

$$\lambda_2(p;\Omega) \ge \left(\frac{h_1(B_r)}{p}\right)^p = \left(\frac{2}{rp}\right)^p \ge \left(\frac{1.818}{r}\right)^p \ge \frac{1.818}{r}$$

and

$$\lambda_2(p;\Omega) \ge \left(\frac{h_1(A)}{p}\right)^p = \left(\frac{2}{(1-r)p}\right)^p \ge \left(\frac{1.818}{1-r}\right)^p \ge \frac{1.818}{1-r}.$$

Then we have the following compatibility conditions:

$$\frac{1.818}{r} \le 3.5 \Rightarrow r \ge 0.519$$

and
$$\frac{1.818}{1-r} \leq 3.5 \Rightarrow 1-r \geq 0.519 \Rightarrow r \leq 0.481$$
which are incompatible. Hence we obtain the claim.

□

Theorem 2.36. *For p close to 1, the second eigenfunction of the p-Laplacian in a square Ω can not have the diagonal as nodal line.*

Proof. The proof is similar as in the preceding theorem. In fact, one notices that, if the $\Omega = [-1,1]^2$, the Cheeger constant of each rectangle obtained cutting Ω along a cartesian axis is 2.842, while the Cheeger constant of the triangle obtained cutting along the diagonal is 2.970. □

2.6 The one-dimensional case

In the one-dimensional case (with $\Omega = (a,b)$) the eigenvalue problem for the p-Laplacian reads

$$\begin{cases} -(|u'|^{p-2}u')' &= \lambda |u|^{p-2}u \quad \text{in } (a,b) \\ u(a) &= 0 \\ u(b) &= 0 \end{cases}$$

It is known (see [44]) that the first eigenvalue is explicitly given by the expression
$$\lambda_1(p;(a,b)) = (p-1)\left(\frac{2\pi}{p(b-a)\sin\frac{\pi}{p}}\right)^p$$
and that
$$\lambda_k(p;(a,b)) = k^p \lambda_1(p;(a,b)).$$
The sequence $\{\lambda_k(p;(a,b))\}_{k=1}^{+\infty}$ exhausts the spectrum (see [10]). Moreover, every eigenvalue is simple, and the eigenfunction $e_{k,p}$ associated to $\lambda_k(p;(a,b))$ has exactly $k-1$ zeros in (a,b), which means that it has exactly k nodal domains. Arguing as in the previous sections, one can obtain the following "abstract" result:

Theorem 2.37. *Let $\Omega = (a,b)$. Then*
$$\lim_{p\to 1} \lambda_k(p;\Omega) = h_k(\Omega).$$

The result can be actually obtained by direct calculation, once one observes that
$$h_k((a,b)) = \frac{2k}{b-a}.$$

Indeed, the optimal configuration for $h_k((a,b))$ is given by k disjoint intervals $I_1, ..., I_k$ of equal length, so that $Per(I_i) = 2$ and $V(I_i) = \frac{b-a}{k}$ for every $i = 1, ..., k$.

2.7 Other results

Theorem 2.38. *Let $\Omega \subset \mathbb{R}^n$ be such that $h_k(\Omega) < h_{k+j}(\Omega)$ for a $j \in \mathbb{N}$. Then there exists $p_0 > 1$ such that every eigenfunction relative to the eigenvalue $\lambda_k(p;\Omega)$ with $p < p_0$ has at most $k+j-1$ nodal domains.*

Proof. Assume that there exists a sequence of values $p_m \searrow 1$ for which the eigenfunctions e_{k,p_m} corresponding to $\lambda_k(p_m;\Omega)$ have at least $k+j$ nodal domains. Then, according to Remark 2.26 and Theorem 2.23, we would have

$$\liminf_{p_m \to 1} \lambda_k(p_m;\Omega) \geq h_{k+j}(\Omega) > h_k(\Omega) \geq \limsup_{p_m \to 1} \lambda_k(p_m;\Omega)$$

which is a contradiction. So we obtain the claim.

□

Proposition 2.39. *Let Ω be of class $C^{2,\alpha}$. Let λ be an eigenvalue of the p-Laplacian, and e_p an associated eigenfunction such that $\|e_p\|_p = 1$. Let M be the maximum of $\left|\frac{\partial e_p}{\partial \nu}\right|$ on $\partial\Omega$. Then*

$$M \geq \frac{h_1(\Omega)}{[(p-1)\cdot c(\Omega)]^{\frac{1}{p}}\, p^{(1-\frac{1}{p})}}.$$

Proof. M is well defined because, under these hypotheses, $e_p \in C^{1,\beta}(\overline{\Omega})$ for a $\beta \in (0,1)$ (see [42]) and so $\frac{\partial e_p}{\partial \nu}$ is continuous ($\partial\Omega$ is compact). Then, by the generalised Rellich identity (see [40]), we obtain

$$\frac{2p}{p-1}\left(\frac{h_1(\Omega)}{p}\right)^p \leq \frac{2p\lambda}{p-1} \leq M^p \int_{\partial\Omega} \frac{\partial(r^2)}{\partial \nu}\, d\mathcal{H}^{n-1} =: M^p\, c(\Omega). \qquad \square$$

Corollary 2.40. *Let Ω be of class $C^{2,\alpha}$. Let λ be an eigenvalue of the p-Laplacian, and e_p an associated eigenfunction. Then*

$$\max_{x \in \partial\Omega} \left|\frac{\partial e_p(x)}{\partial \nu}\right| \to +\infty$$

as $p \to 1$.

Chapter 3

Extensions

The aim of this chapter is to show that the results previously found essentially hold also when other differential operators are involved. Complete proofs of the statements will not always be given; instead, it will be pointed out which modifications are necessary in order to obtain the results.

3.1 The weighted problem

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Lipschitz boundary. We consider the problem

$$\begin{cases} -\mathrm{div}(g|\nabla u|^{p-2}\nabla u) &= \lambda f|u|^{p-2}u \quad \text{in } \Omega \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega \end{cases}$$

where $f \in \mathcal{C}(\overline{\Omega})$, $g \in \mathcal{C}^1(\overline{\Omega})$ are such that

$$0 < f_0 \leq f \leq \|f\|_\infty,$$

$$0 < g_0 \leq g \leq \|g\|_\infty.$$

We define the *weighted volume*

$$V^f(E) := \int_E f(x)\,dx$$

and the *weighted perimeter* (measured in $\mathbb{R}^n$)

$$Per^g(E) := \|D\chi_E(x)\|_g(\mathbb{R}^n)$$

(see also [16]) where

$$\|Du\|_g(\mathbb{R}^n) := \sup\left\{\int_{\mathbb{R}^n} u(x)\,\mathrm{div}\,(g(x)\varphi(x))\,\middle|\,\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n),\ |\varphi| \leq 1\right\}.$$

The *weighted Cheeger constants* are defined as

$$h_k^{g,f}(\Omega) := \inf\left\{\lambda \in \mathbb{R} \,\middle|\, \exists E_1, ..., E_k \subset\subset \Omega, E_i \cap E_j = \emptyset \,\forall\, i \neq j, \right.$$
$$\left. \partial E_i \text{ smooth } \forall\ i = 1, ...k,\ \max_{i=1,...,k} \frac{Per^g(E_i)}{V^f(E_i)} \leq \lambda \right\}.$$

Similarly to the case $f \equiv 1$, $g \equiv 1$, the following values are eigenvalues, as proved in [41]:

$$\lambda_k^{g,f}(p;\Omega) := \inf_{A \in \Gamma_k^w} \max_{u \in A} \frac{\int_\Omega g|\nabla u|^p}{\int_\Omega f|u|^p}$$

where

$$\Gamma_k^w := \left\{ A \subset W_0^{1,p}(\Omega) \setminus \{0\} \ \mid\ A \cap \left\{ \int_\Omega f|u|^p = 1 \right\} \text{ compact}, \right.$$
$$\left. A \text{ symmetric},\ \gamma(A) \geq k \right\}.$$

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $D \subset\subset \Omega$ be a subset with boundary of class $\mathcal{C}^2$, and set*

$$D^\varepsilon : \{x \in \Omega \setminus D \,|\, dist(x, \partial D) \leq \varepsilon\}.$$

Let $g : \Omega \to \mathbb{R}$ be a continuous function. Then

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{D^\varepsilon} g(x)\, dx = \int_{\partial D} g(x)\, d\mathcal{H}^{n-1}(x).$$

Proof. Fix $\widetilde{\varepsilon} > 0$. Since g is a uniformly continuous function, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \widetilde{\varepsilon}$ for every $x, y \in \Omega$. Let $0 < \varepsilon \leq \delta$. For every $y \in D^\varepsilon$, let us denote by x_y the projection of y on ∂D; such a projection is unique provided ε is small enough. Then

$$\frac{1}{\varepsilon} \left| \int_{D^\varepsilon} g(y)\, dy - \int_{D^\varepsilon} g(x_y)\, dy \right| \leq \frac{\widetilde{\varepsilon}\,|D^\varepsilon|}{\varepsilon}.$$

From [4, Proposition 19] we can deduce that

$$\int_{D^\varepsilon} g(x_y)\, dy = (\varepsilon + o(\varepsilon)) \int_{\partial D} g(x)\, d\mathcal{H}^{n-1}.$$

The claim follows easily letting ε tend to 0 if we recall that

$$\lim_{\varepsilon \to 0} \frac{|D^\varepsilon|}{\varepsilon} = Per(D).$$

$\square$

Lemma 3.2. *Let $g \in \mathcal{C}^0(\Omega)$, $D \subset\subset \Omega$ such that ∂D is of class $\mathcal{C}^2$. Denote with $\overline{g}(E)$ the mean value of g on E, and let $D^\varepsilon \subset\subset \Omega$ be ε-strips around ∂D. Then*

$$\overline{g}(D^\varepsilon) \to \overline{g}(\partial D)$$

as $\varepsilon \to 0$.

Proof. One has

$$\begin{aligned}\overline{g}(D^\varepsilon) &= \frac{1}{|D^\varepsilon|}\int_{D^\varepsilon} g(x)\,dx = \frac{1}{\varepsilon Per(D) + o(\varepsilon)}\int_{D^\varepsilon} g(x)\,dx \\ &= \frac{1}{Per(D) + \varepsilon^{-1}o(\varepsilon)} \cdot \frac{1}{\varepsilon}\int_{D^\varepsilon} g(x)\,dx.\end{aligned}$$

From Lemma 3.1 we obtain

$$\lim_{\varepsilon \to 0} \overline{g}(D^\varepsilon) = \frac{1}{Per(D)}\int_D g(x)\,d\mathcal{H}^{n-1}(x) = \overline{g}(\partial D).$$

□

Now we try to extend the approximation result proved in [45] to this setting. We recall that we can define the weighted total variation as

$$\|Du\|_g(\Omega) := \sup\left\{ \int_\Omega u(x)\,\mathrm{div}\,(g(x)\varphi(x)) \,\middle|\, \varphi \in \mathcal{C}_c^\infty(\Omega),\ |\varphi| \le 1 \right\}$$

(see [16]). The weighted total variation is L^1-lower semicontinuous and a coarea formula is available. One has to prove the passage in [45] from (2.28) to (2.29), but this can be done using the following lemma.

Lemma 3.3. *Let $\tau \ge 0$ be a test function defined on $\mathbb{R}^n$ such that $\tau(x) = \tau(|x|)$, $\tau(x) = 0$ if $|x| \ge 1$, $\int \tau(x)\,dx = 1$. Set $\tau_h(x) := h^n\tau(hx)$ and $\psi_h(x) := \tau_h \star \chi_\Omega$. Then*

$$\|D\psi_h\|_g(\mathbb{R}^n) \to \|D\chi_\Omega\|_g(\mathbb{R}^n)$$

as $h \to \infty$.

Proof. Since $\tau_h \to \chi_\Omega$ in $L^1_{loc}(\mathbb{R}^n)$, we have

$$\|D\chi_\Omega\|_g(\mathbb{R}^n) \le \liminf_{h\to\infty} \|D\psi_h\|_g(\mathbb{R}^n).$$

Moreover we have

$$\begin{aligned}\int_{\mathbb{R}^n} \psi_h\,\mathrm{div}\,(g\varphi) &= \int_{\mathbb{R}^n} (\tau_h \star \chi_\Omega)\,\mathrm{div}\,(g\varphi) = \int_{\mathbb{R}^n} \chi_\Omega\,\mathrm{div}\,(\tau_h \star (g\varphi)) \\ &= \int_{\mathbb{R}^n} \chi_\Omega\,\mathrm{div}\,(g(\tau_h \star \varphi)) \le \|D\chi_\Omega\|_\Phi(\mathbb{R}^n)\end{aligned}$$

since

$$|\tau_h \star \varphi(x)| \leq \int_{\mathbb{R}^n} \tau_h(x-y)|\varphi(y)|\,dy \leq \int_{\mathbb{R}^n} \tau_h(x-y)\,dy = 1.$$

So

$$\limsup_{h\to\infty} \|D\psi_h\|_g(\mathbb{R}^n) \leq \|D\chi_\Omega\|_g(\mathbb{R}^n)$$

from which the claim follows. □

In order to prove the desired result, we need to verify relation (3.21) in [45]; this can be proved as in [47], Appendix, knowing that if E is a set of finite perimeter, there exists a sequence of smooth functions ψ_n approximating χ_E in the strong topology, such that $0 \leq \psi_n \leq 1$. See also the approximation result in [36]. We are then able to state the following proposition.

Proposition 3.4. *Let $F \subset \Omega$ be a set of finite perimeter. Then there exists a sequence of smooth sets $\{F_h\}$ such that: $F_h \subset\subset F$, $\chi_{F_h} \to \chi_F$, and $Per^g(F_h) \to Per^g(F)$ as $h \to \infty$.*

Corollary 3.5. *We have*

$$h_k^{g,f}(\Omega) = \inf\left\{\lambda \in \mathbb{R} \,\middle|\, \exists E_1, ..., E_k \subset \overline{\Omega}\,, E_i \cap E_j = \emptyset\,\forall\, 1 \leq i,j \leq k,\right.$$
$$\left.\frac{Per^g(E_i)}{V^f(E_i)} \leq \lambda\,\forall\; i = 1, ...k\right\}.$$

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Then*

$$\limsup_{p\to 1} \lambda_k^{g,f}(p;\Omega) \leq h_k^{g,f}(\Omega).$$

Proof. We give the proof for $k = 2$. It is possible to find E_1, E_2 with the property that, for $i = 1, 2$, $E_i \subset\subset \Omega$, ∂E_i is smooth, and

$$\frac{Per(E_i)}{V(E_i)} \leq h_2^{g,f}(\Omega) + \frac{1}{k}.$$

Let $\varepsilon > 0$, and let v_i $(i = 1, 2)$ be two functions such that: $v_i = 1$ on E_i, $v_i = 0$ outside a ε-neighbourhood of E_i, and $|\nabla v_i| = \varepsilon^{-1}$ on the ε-strip E_i^ε outside E_i. ε should be chosen in a way that $(E_1 \cup E_1^\varepsilon) \cap (E_2 \cup E_2^\varepsilon) = \emptyset$. Set

$$A_0 := \left\{\alpha v_1 + \beta v_2 \,\middle|\, |\alpha|^p + |\beta|^p = 1\right\}.$$

Then $A_0 \in \Gamma_2^w$ (see also [34, Lemma 2.1]). Thus we have

$$\begin{aligned}
\lambda_2^{g,f}(p;\Omega) &\le \sup_{u\in A_0} \frac{\int_\Omega g|\nabla u|^p}{\int_\Omega f|u|^p} \le \sup_{|\alpha|^p+|\beta|^p=1} \frac{\varepsilon^{-p}|\alpha|^p V^g(E_1^\varepsilon) + \varepsilon^{-p}|\beta|^p V^g(E_2^\varepsilon)}{|\alpha|^p V^f(E_1) + |\beta|^p V^f(E_2)} \\
&\le \sup_{|\alpha|^p+|\beta|^p=1} \frac{\varepsilon^{-p}[|\alpha|^p \overline{g}(E_1^\varepsilon)V(E_1^\varepsilon) + |\beta|^p \overline{g}(E_2^\varepsilon)V(E_2^\varepsilon)]}{|\alpha|^p V^f(E_1) + |\beta|^p V^f(E_2)} \\
&= \sup_{|\alpha|^p+|\beta|^p=1} \frac{\varepsilon^{1-p}[|\alpha|^p \overline{g}(E_1^\varepsilon)Per(E_1) + |\beta|^p \overline{g}(E_2^\varepsilon)Per(E_2) + \varepsilon^{-1}o(\varepsilon)]}{|\alpha|^p V^f(E_1) + |\beta|^p V^f(E_2)} \\
&\le \varepsilon^{1-p}\left[\max_{i=1,2} \frac{\overline{g}(E_i^\varepsilon)Per(E_i)}{V^f(E_i)} + \frac{\varepsilon^{-1}o(\varepsilon)}{\min\{V^f(E_1), V^f(E_2)\}}\right]
\end{aligned}$$

as we have from Lemma 2.22

$$V(E_i^\varepsilon) = \varepsilon Per(E_i) + o(\varepsilon)$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. Note that the last inequality is true because of Proposition A.4. If we send $p \to 1$, we obtain

$$\limsup_{p\to 1} \lambda_2^{g,f}(p;\Omega) \le \max_{i=1,2} \frac{\overline{g}(E_i^\varepsilon)Per(E_i)}{V^f(E_i)} + \frac{\varepsilon^{-1}o(\varepsilon)}{\min\{V^f(E_1), V^f(E_2)\}}$$

and if $\varepsilon \to 0$

$$\limsup_{p\to 1} \lambda_2(p;\Omega) \le \max_{i=1,2} \frac{\overline{g}(\partial E_i)Per(E_i)}{V^f(E_i)} = \max_{i=1,2} \frac{Per^g(E_i)}{V^f(E_i)} \le h_2^{g,f}(\Omega) + \frac{1}{k}.$$

The claim follows if we send $k \to \infty$. The fact that E_1 and E_2 depend from k does not constitute a problem; in fact we can estimate $V^f(E_i)$ from below: if it were $V^f(E_i) \to 0$ we would have

$$\frac{Per^g(E_i)}{V^f(E_i)} \ge \frac{g_0}{\|f\|_\infty}\frac{Per(E_i)}{V(E_i)} \ge \frac{g_0}{\|f\|_\infty}\frac{Per(B_i)}{V(B_i)} = \frac{g_0}{\|f\|_\infty}\frac{n}{R_i} \to +\infty$$

where B_i is a ball with radius R_i such that $V(B_i) = V(E_i)$. □

Theorem 3.7.

$$\lambda_1^{g,f}(p;\Omega) \ge \left(\frac{f_0}{\|g\|_\infty}\right)^{\frac{p}{q}} \cdot \left(\frac{h_1^{g,f}(\Omega)}{p}\right)^p.$$

Proof. By means of the weighted Cavalieri principle and the weighted coarea formula, which are available also in this case (see [16]), we generalize [38, Theorem 3] and obtain at the end the following inequality:

$$\begin{aligned} h_1^{g,f}(\Omega) &\le \frac{\int_\Omega g|\nabla w|}{\int_\Omega f|w|} \le p\frac{\int_\Omega g|v|^{p-1}|\nabla v|}{\int_\Omega f|v|^p} \le p\frac{(\int_\Omega g|v|^p)^{\frac{1}{q}}(\int_\Omega g|\nabla v|^p)^{\frac{1}{p}}}{(\int_\Omega f|v|^p)^{\frac{1}{q}}(\int_\Omega f|v|^p)^{\frac{1}{p}}} \\ &\le p\frac{\|g\|_\infty^{\frac{1}{q}}(\int_\Omega |v|^p)^{\frac{1}{q}}(\int_\Omega g|\nabla v|^p)^{\frac{1}{p}}}{f_0^{\frac{1}{q}}(\int_\Omega |v|^p)^{\frac{1}{q}}(\int_\Omega f|v|^p)^{\frac{1}{p}}} \le p\left(\frac{\|g\|_\infty}{f_0}\right)^{\frac{1}{q}}\frac{(\int_\Omega g|\nabla v|^p)^{\frac{1}{p}}}{(\int_\Omega f|v|^p)^{\frac{1}{p}}}. \end{aligned}$$

We recall that Sard's Theorem assures us that almost all of the level sets of a smooth function have a smooth boundary. □

Corollary 3.8.

$$\lim_{p\to 1} \lambda_1^{g,f}(p;\Omega) = h_1^{g,f}(\Omega).$$

Remark 3.9. If $g \le f$ we can also proceed as follows:

$$\begin{aligned} h_1^{g,f}(\Omega) &\le \frac{\int_\Omega g|\nabla w|}{\int_\Omega f|w|} \le p\frac{\int_\Omega g|v|^{p-1}|\nabla v|}{\int_\Omega f|v|^p} \le p\frac{(\int_\Omega g|v|^p)^{\frac{1}{q}}(\int_\Omega g|\nabla v|^p)^{\frac{1}{p}}}{\int_\Omega f|v|^p} \\ &\le p\frac{(\int_\Omega f|v|^p)^{\frac{1}{q}}(\int_\Omega g|\nabla v|^p)^{\frac{1}{p}}}{\int_\Omega f|v|^p} \le p\frac{(\int_\Omega g|\nabla v|^p)^{\frac{1}{p}}}{(\int_\Omega f|v|^p)^{\frac{1}{p}}} \end{aligned}$$

to obtain

$$\lambda_1^{g,f}(p;\Omega) \ge \left(\frac{h_1^{g,f}(\Omega)}{p}\right)^p$$

which is a better estimate in the case $f_0 < \|g\|_\infty$, which can occur even if $g \le f$.

Remark 3.10. The proof of the existence of a function $u \in BV(\Omega)$ minimizing the ratio

$$\frac{\|Dv\|_g(\mathbb{R}^n)}{\int_\Omega f|v|}$$

among all functions $v \in BV(\Omega)$, as well as the existence of a weighted Cheeger set for Ω (with the minimum of the ratio above equal to $h_1^{g,f}(\Omega)$) can be found in [16].

Remark 3.11. The first eigenfunction(s) of the weighted p-Laplacian can be considered to be strictly positive; indeed, Harnack's inequality is available according to [51] or [49]; the results of those articles are in fact still valid if one sets, using their notation,

$$A(x,u,\nabla u) = -\frac{1}{g_0}\operatorname{div}(g(x)|\nabla u|^{p-2}\nabla u)$$

and

$$B(x, u, \nabla u) = \frac{1}{g_0}(f(x)|u|^{p-2}u)$$

since $g(x) \geq g_0 > 0$.

Remark 3.12. Since f, g are positive and bounded from below and from above, [43, Lemma 3.1] could be used to state that the first eigenvalue of the weighted p-Laplacian is simple. From the same computations it follows (see the remark following that Lemma) that the higher eigenfunctions have to change their sign.

Theorem 3.13. *The following inequality holds:*

$$\lambda_2^{g,f}(p;\Omega) \geq \left(\frac{f_0}{\|g\|_\infty}\right)^{\frac{p}{q}} \cdot \left(\frac{h_2^{g,f}(\Omega)}{p}\right)^p.$$

Proof. There exists a second eigenfunction $e_2^{g,f}$ which admits at least two nodal domains N_1 and N_2. One can then proceed as in the case $f, g \equiv 1$. □

Corollary 3.14.

$$\lim_{p\to 1} \lambda_2^{g,f}(p;\Omega) = h_2^{g,f}(\Omega).$$

3.2 The pseudo-p-Laplacian

It is worth mentioning that the results of the preceding chapter hold also for the eigenvalue problem for the *pseudo-p-Laplace* operator, defined as

$$\widetilde{\Delta}_p u := \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right).$$

For $n = 1$ we have $\widetilde{\Delta}_p u = \Delta_p u$, while $\widetilde{\Delta}_2 u = \Delta u$ for every n. The pseudo-p-Laplacian admits a sequence of eigenvalues

$$\widetilde{\lambda}_1(p;\Omega) < \widetilde{\lambda}_2(p;\Omega) \leq ... \leq \widetilde{\lambda}_k(p;\Omega) \to +\infty$$

which can be obtained by means of a minimax principle, similarly as for the p-Laplacian. We define

$$\begin{aligned}\|Du\|_1(\mathbb{R}^n) &:= \sup\left\{\sum_{i=1}^n \int_\Omega u\,\frac{\partial \varphi_i}{\partial x_i}\,\middle|\, \varphi_i \in C_c^\infty(\Omega;\mathbb{R}),\ |\varphi_i| \leq 1\right\}\\ &= \sup\left\{\int_\Omega u\,\mathrm{div}\varphi\,\middle|\,\varphi \in C_c^\infty(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \leq 1\right\}.\end{aligned}$$

The difference with the ordinary total variation is that there we required the condition $|\varphi| \leq 1$, i.e. $\|\varphi\|_2 \leq 1$. By the equivalence of the norms in $\mathbb{R}^n$ we have

$$\|Du\|(\mathbb{R}^n) < \infty \Leftrightarrow \|Du\|_1(\mathbb{R}^n) < \infty.$$

For a set $E \subset \Omega$ we define

$$Per_1(E) = Per_1(E;\mathbb{R}^n) := \sup\left\{\int_{\mathbb{R}^n} \chi_E \,\mathrm{div}\varphi \,\middle|\, \varphi \in C_c^\infty(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \leq 1\right\}$$

and the Cheeger constant

$$\widetilde{h}_1(\Omega) := \inf_{E\subset\overline{\Omega}} \frac{Per_1(E;\mathbb{R}^n)}{V(E)}.$$

The L^1-lower semicontinuity of $\|Du\|_1$ and the existence of a coarea formula (see [2]) yield the existence of a minimizer for the Rayleigh quotient, as well as the existence of a Cheeger set for every $\Omega \subset \mathbb{R}^n$. In [7, Theorem 3.7] it was proved that

$$\lim_{p\to 1} \widetilde{\lambda}_1(p;\Omega) = \widetilde{h}_1(\Omega).$$

It is easily seen that the results of Chapter 2 can be extended also to this setting. In the following we will only extend a useful approximation result for Cheeger sets, which was proved in [45] in the standard case.

Lemma 3.15. *Let $\tau \geq 0$ be a test function defined on $\mathbb{R}^n$ such that $\tau(x) = \tau(|x|)$, $\tau(x) = 0$ if $|x| \geq 1$, $\int \tau(x)\,dx = 1$. Set $\tau_h(x) := h^n\tau(hx)$ and $\psi_h(x) := \tau_h \star \chi_\Omega$. Then*

$$\|D\psi_h\|_1(\mathbb{R}^n) \to \|D\chi_\Omega\|_1(\mathbb{R}^n)$$

as $h \to \infty$.

Proof. Since $\tau_h \to \chi_\Omega$ in $L^1_{loc}(\mathbb{R}^n)$, we have

$$\|D\chi_\Omega\|_1(\mathbb{R}^n) \leq \liminf_{h\to\infty} \|D\psi_h\|_1(\mathbb{R}^n).$$

Moreover we have

$$\int_{\mathbb{R}^n} \psi_h \,\mathrm{div}\,\varphi = \int_{\mathbb{R}^n} (\tau_h \star \chi_\Omega)\,\mathrm{div}\,\varphi = \int_{\mathbb{R}^n} \chi_\Omega \,\mathrm{div}\,(\tau_h \star \varphi) \leq \|D\chi_\Omega\|_1(\mathbb{R}^n)$$

since

$$\|\tau_h \star \varphi(x)\|_\infty \leq \int_{\mathbb{R}^n} \tau_h(x-y)\|\varphi(y)\|_\infty\,dy \leq \int_{\mathbb{R}^n} \tau_h(x-y)\,dy = 1.$$

So

$$\limsup_{h\to\infty} \|D\psi_h\|_1(\mathbb{R}^n) \leq \|D\chi_\Omega\|_1(\mathbb{R}^n)$$

from which the claim follows. $\square$

Proposition 3.16. *Let $F \subset \Omega$ be a set of finite perimeter. Then there exists a sequence of smooth sets $\{F_h\}$ such that: $F_h \subset\subset F$, $\chi_{F_h} \to \chi_F$, and $Per_1(F_h) \to Per_1(F)$ as $h \to \infty$.*

Proof. The results in [45] about the approximation of Caccioppoli sets from the inside can be extended also in this case, since the article makes use only of the lower semicontinuity and of the coarea formula, which are available also in this case. One modification is needed in (3.21): in this case it is possible to use the results of [47, Appendix] by modifying the classical proof using Lemma 3.15. The same lemma is useful in order to prove the step between (2.28) and (2.29). $\square$

Chapter 4

Lane-Emden problem and Dirichlet eigenfunctions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We consider the Lane-Emden equation for the p-Laplacian, that is

$$\begin{cases} -\Delta_p u &= \lambda |u|^{q-2}u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Here is $\lambda > 0$, $1 < p < q < p^*$ (with $p^* = \frac{np}{n-p}$ if $p < n$, and $p^* = +\infty$ otherwise). We are interested in the existence and the asymptotic behaviour, as $q \to p$, of the positive and sign-changing solutions with minimal energy. It will be proved that, for suitable values of λ, such solutions converge to eigenfunctions of the p-Laplacian.

The results of this chapter were obtained in collaboration with Christopher Grumiau and have appeared in [31]. However, the proof of Proposition 4.9 is given here in a simplified version obtained together with Fernando Charro.

We denote by $\|.\|$ the norm in $W_0^{1,p}(\Omega)$ defined as

$$\|u\| = \left(\int_\Omega |\nabla u|^p \right)^{\frac{1}{p}}.$$

In order to simplify the notation, we will set $\lambda_1 := \lambda_1(p;\Omega)$ and $\lambda_2 := \lambda_2(p;\Omega)$.

4.1 Existence of solutions

Let us fix $1 < p < +\infty$ and $p < q < p^*$. We will prove the existence of at least two non-trivial solutions to the Lane-Emden problem (4.1). In particular we prove the existence of a ground state solution (non-trivial solution with minimum energy) and a least energy nodal solution (sign-changing solution with minimum energy). We recall that the existence question in the case $p = 2$ and $1 < q < 2^*$ was already studied in 1973 by Ambrosetti and Rabinowitz in [3], where it was shown that the problem admits a positive ground state solution. The existence of a sign-changing solution with minimal energy was proved in [18] by Castro, Cossio and Neuberger in 1997.

We introduce the energy functional

$$\varphi_q(u) := \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q$$

defined on $W_0^{1,p}(\Omega)$. A function u is a solution of (4.1) if and only if it is a critical point of φ_q. Remark that φ_q is a $\mathcal{C}^2$ functional for $p \geq 2$ and $\mathcal{C}^1$ functional for $1 < p < 2$.

Let us define the first variation of φ_q at u in direction v

$$\mathrm{d}\varphi_q(u)(v) := \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v - \lambda\int_\Omega |u|^{q-2}uv$$

and the *Nehari manifold*

$$\mathcal{N}_q := \{u \in W_0^{1,p}(\Omega) \setminus \{0\} \,|\, \mathrm{d}\varphi_q(u)(u) = 0\}.$$

Clearly, all the non-trivial solutions belong to $\mathcal{N}_q$. We will also make use of the *positive Nehari manifold*

$$\mathcal{N}_q^+ := \{u \in \mathcal{N}_q \,|\, u \geq 0\},$$

of the *negative Nehari manifold*

$$\mathcal{N}_q^- := \{u \in \mathcal{N}_q \,|\, u \leq 0\}$$

and of the *nodal Nehari set*

$$\mathcal{M}_q := \{u \in \mathcal{N}_q \,|\, u^+ \in \mathcal{N}_q^+,\, u^- \in \mathcal{N}_q^-\},$$

where we defined the positive part $u^+ := \max(0, u)$ and the negative part $u^- := \min(0, u)$.

Notice that by definition the functions belonging to $\mathcal{M}_q$ are sign-changing. Moreover, all sign-changing solutions of the problem belong to $\mathcal{M}_q$. The following results prove that ground state solutions are characterized by functions minimizing the energy functional in $\mathcal{N}_q$ and least energy nodal solutions are characterized by functions minimizing the energy functional in $\mathcal{M}_q$.

Proposition 4.1. *For every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exists one and only one $t_q^* > 0$ such that $t_q^* u \in \mathcal{N}_q$. Moreover,*

$$\varphi_q(t_q^* u) = \max_{t>0} \varphi_q(tu).$$

Proof. For $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we have

$$tu \in \mathcal{N}_q \Leftrightarrow \int_\Omega |\nabla(tu)|^p - \lambda \int_\Omega |tu|^q = 0 \Leftrightarrow t^p \int_\Omega |\nabla u|^p - \lambda t^q \int_\Omega |u|^q = 0.$$

The last equation admits

$$t_q^* := \left(\frac{\int_\Omega |\nabla u|^p}{\lambda \int_\Omega |u|^q} \right)^{\frac{1}{q-p}} \tag{4.2}$$

as unique positive solution. For $t \geq 0$ we define

$$\psi(t) := \varphi_q(tu) = \frac{1}{p} \int_\Omega |\nabla(tu)|^p - \frac{\lambda}{q} \int_\Omega |tu|^q = \frac{t^p}{p} \int_\Omega |\nabla u|^p - \frac{\lambda t^q}{q} \int_\Omega |u|^q.$$

We have

$$\psi'(t) = t^{p-1} \int_\Omega |\nabla u|^p - \lambda t^{q-1} \int_\Omega |u|^q,$$

so that the only positive critical point is $t = t_q^*$. Since $\psi(0) = 0$ and $\psi(t) \to -\infty$ as $t \to +\infty$, t_q^* must be a maximum point, which means

$$\varphi_q(t_q^* u) = \max_{t>0} \varphi_q(tu).$$

□

By the previous result and since the support of u^+ and u^- are disjoint, we obtain

Corollary 4.2. *For every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, the numbers $t_q^+, t_q^- > 0$ such that $t_q^+ u^+ + t_q^- u^- \in \mathcal{M}_q$ are uniquely defined.*

Proposition 4.3. *The Nehari manifold $\mathcal{N}_q$ is closed in $W_0^{1,p}(\Omega)$.*

Proof. Since φ_q is of class $\mathcal{C}^1$, it is clear that $\mathcal{N}_q \cup \{0\}$ is closed. So we must prove that 0 is not an accumulation point for $\mathcal{N}_q$; this follows from the fact that the $W_0^{1,p}$-norm of every function $u \in \mathcal{N}_q$ is uniformly bounded from below. Indeed, from Sobolev's embedding Theorem we have

$$\|\nabla v\|_p \geq C \|v\|_q \qquad \forall\ v \in W_0^{1,p}(\Omega).$$

For $v \in W_0^{1,p}(\Omega) \setminus \{0\}$ the unique positive multiplicative function $t_q^* v \in \mathcal{N}_q$ (with t_q^* as in (4.2)) satisfies

$$\|\nabla(t_q^* v)\|_p \geq C\|t_q^* v\|_q = C\left(\frac{\|\nabla v\|_p^p}{\lambda\|v\|_q^q}\right)^{\frac{1}{q-p}}\|v\|_q = C\lambda^{-\frac{1}{q-p}}\left(\frac{\|\nabla v\|_p}{\|v\|_q}\right)^{\frac{p}{q-p}}$$
$$\geq C^{\frac{q}{q-p}}\lambda^{-\frac{1}{q-p}}.$$

□

The following result proves that we can compute the minimum of the energy on the positive and negative Nehari manifold, and on the nodal Nehari set. The idea for it is the same as the one used by Castro, Cossio and Neuberger in [18].

Proposition 4.4. *The infima*

$$\inf_{u\in\mathcal{N}_q^+}\varphi_q(u), \quad \inf_{u\in\mathcal{N}_q^-}\varphi_q(u), \quad \inf_{u\in\mathcal{M}_q}\varphi_q(u)$$

are attained.

Proof. We will give a proof only for $\mathcal{M}_q$, since the arguments are the same for $\mathcal{N}_q^+$ and $\mathcal{N}_q^-$. Let us define $c := \inf_{\mathcal{M}_q}\varphi_q$ and consider $\{u_n\}_{n=1}^{\infty} \subset \mathcal{M}_q$ such that $\varphi_q(u_n) \to c$. Since

$$\varphi_q(v) = \left(\frac{1}{p} - \frac{1}{q}\right)\|v\|$$

for any $v \in \mathcal{N}_q$, we obtain that $\{u_n\}_{n=1}^{\infty}$ is bounded in $W_0^{1,p}(\Omega)$. So, up to a subsequence, there exist u, v and w such that $u_n \rightharpoonup u$, $u_n^+ \rightharpoonup v$ and $u_n^- \rightharpoonup w$ in $W_0^{1,p}(\Omega)$. By Sobolev's embedding Theorem and as the functions $u \mapsto u^+$ and $u \mapsto u^-$ are continuous, we obtain that $u^+ = v$ and $u^- = w$.

By Proposition 4.3, the Nehari manifold $\mathcal{N}_q$ is closed in $W_0^{1,p}(\Omega)$. We obtain that

$$\lambda\int_\Omega |u^+|^q = \lambda\lim_{n\to+\infty}\int_\Omega |u_n^+|^q = \lim_{n\to+\infty}\left\|u_n^+\right\|^p > 0.$$

So u is a sign-changing function.

It remains to verify that $u \in \mathcal{M}_q$ and $u_n \to u$ in $W_0^{1,p}(\Omega)$. In fact, it suffices to prove that $u_n^+ \to u^+$ and $u_n^- \to u^-$ in $W_0^{1,p}(\Omega)$. Suppose by contradiction that this is not the case; without loss of generality, we can assume that u_n^+ does not converge to u^+. Then

$$\left\|u^+\right\|^p < \liminf_{n\to+\infty}\left\|u_n^+\right\|^p$$

(see [13, Proposition 3.30]), which implies that $\mathrm{d}\varphi_q(u^+)(u^+) < 0$. So u^+ does not belong to the Nehari manifold. By Proposition 4.1, there exist $0 < \alpha < 1$ and $0 < \beta \leq 1$ such that $\alpha u^+ + \beta u^-$ belongs to $\mathcal{M}_q$. In fact, we have

$$\varphi_q(\alpha u^+ + \beta u^-) < \liminf_{n\to+\infty}(\varphi_q(\alpha u_n^+) + \varphi_q(\beta u_n^-)) \leq \liminf_{n\to+\infty} \varphi_q(u_n) = c,$$

which is a contradiction. So the minimum of the energy on $\mathcal{M}_q$ is attained in u.

□

The following results show that the functions found in Proposition 4.4 are solutions of the problem (4.1). Remark that, as the positive part and the negative part of a solution belong to the Nehari manifold and as the energy of the positive or negative part is strictly less than the energy of the solution, we obtain that the functions which minimize energy on the positive Nehari manifold or negative Nehari manifold are ground state solutions of the problem (4.1). We will make use of the following lemma, also known as Miranda's theorem.

Lemma 4.5. *Let $B \subset \mathbb{R}^n$ be a closed ball, let $f : B \to \mathbb{R}^n$ be a continuous function. If f points inside B on ∂B, then f possesses a zero in B.*

Proof. A proof of this theorem can be found for instance in [24, Section 9.1]. □

Proposition 4.6. *If $u_q \in \mathcal{M}_q$ (resp. $\mathcal{N}_q^+$ or $\mathcal{N}_q^-$) is such that $\varphi_q(u_q) = \inf_{u\in\mathcal{M}_q} \varphi_q(u)$ (resp. $\inf_{u\in\mathcal{N}_q^+} \varphi_q(u)$ or $\inf_{u\in\mathcal{N}_q^-} \varphi_q(u)$), then u_q is a critical point for φ_q.*

Proof. We give the proof for $\mathcal{M}_q$. The arguments are essentially the same for the two other cases: we only need to think that a minimum on $\mathcal{N}_q^+$ or $\mathcal{N}_q^-$ is a minimum on $\mathcal{N}_q$. So, for the two other cases, we do not need that the deformation used in the next part of the proof stays in the positive Nehari or negative Nehari manifold.

Fix $c := \min_{\mathcal{M}_q} \varphi_q$. Let us suppose that u_q is not a critical point for φ_q. Since φ_q is of class $\mathcal{C}^1$, it is possible to find a ball B with $u_q \in B$ and such that, for $\varepsilon > 0$,

$$c - \varepsilon \leq \varphi_q(u) \leq c + \varepsilon \quad \forall\, u \in B$$

and

$$\|\,\mathrm{d}\varphi_q(u)\|_{(W_0^{1,p})'} \geq \frac{\varepsilon}{2} \quad \forall\, u \in B.$$

Let us consider the quarter of a hyperplane π defined as

$$\pi := \{\alpha u_q^+ + \beta u_q^- \mid \alpha, \beta > 0\}.$$

Notice that, from Proposition 4.1, u_q is the unique global maximum of φ_q on π. By the deformation Lemma (see [28, Proposition 5.1.25]) there exists a deformation Γ such that

1. $\varphi_q(\Gamma(t, u)) < c$ for $u \in B \cap \pi$ and $t \in [0, 1]$,
2. $\Gamma(t, u) = u$ for $u \in \partial B \cap \pi$ and $t \in [0, 1]$, and
3. $\|\Gamma(t, u) - u\| \leq 8t$ for $u \in B \cap \pi$ and $t \in [0, 1]$.

Because of the compactness of $B \cap \pi$, it is possible to find $t^* > 0$ such that $\Gamma(t^*, u)$ is a sign-changing function for every $u \in B \cap \pi$. Now we consider the application $\psi : \pi \to \mathbb{R} \times \mathbb{R}$ defined as

$$\psi : v \mapsto (\, \mathrm{d}\varphi_q(\Gamma(t^*, v)^+)(\Gamma(t^*, v)^+),\, \mathrm{d}\varphi_q(\Gamma(t^*, v)^-)(\Gamma(t^*, v)^-)).$$

Since $\Gamma(t^*, v) = v$ on ∂B, we obtain that the vector field points inwards on ∂B. Using Lemma 4.5 we obtain that there exists $w \in B \cap \pi$ such that $\Gamma(t^*, w) \in \mathcal{M}_q$. This is a contradiction because $\varphi_q(\Gamma(t^*, w)) < c$. □

4.2 Convergence results

In this Section we study the asymptotic behaviour of ground state solutions u_q (resp. least energy nodal solutions) of the Problem (4.1) when q goes to p. We prove that there exist suitable positive constants C_1 and C_2 such that

$$C_1 \left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{q-p}} \leq \|u_q\| \leq C_2 \left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{q-p}}$$

if u_q is a ground state solution, and

$$C_1 \left(\frac{\lambda_2}{\lambda}\right)^{\frac{1}{q-p}} \leq \|u_q\| \leq C_2 \left(\frac{\lambda_2}{\lambda}\right)^{\frac{1}{q-p}}$$

if u_q is a least energy nodal solution. We are able to state the following result.

Theorem 4.7. *As $q \to p$, the ground state solutions of Problem (4.1):*
(i) diverge to infinity, up to a subsequence, if $\lambda < \lambda_1$;
(ii) converge to a first eigenfunction of the p-Laplacian, up to a subsequence, if $\lambda = \lambda_1$;
(iii) converge to zero, up to a subsequence, if $\lambda > \lambda_1$.

Theorem 4.8. *As $q \to p$, the least energy nodal solutions of Problem (4.1):*
(i) diverge to infinity, up to a subsequence, if $\lambda < \lambda_2$;
(ii) converge to a second eigenfunction of the p-Laplacian, up to a subsequence, if $\lambda = \lambda_2$;
(iii) converge to zero, up to a subsequence, if $\lambda > \lambda_2$.

We mention that the case $\lambda < \lambda_1$ in Theorem 4.7 was already investigated in [34].
Let us first remark that statements (i) and (iii) of Theorems 4.7 and 4.8 can be derived from (ii) as follows. If v_q is a ground state solution of (4.1) for $\lambda = \lambda_1$, then the function

$$u_q := \left(\frac{\lambda_1}{\mu}\right)^{\frac{1}{q-p}} v_q$$

will be a ground state solution for $\lambda = \mu$. So for $\lambda < \lambda_1$, the function

$$u_q = \left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{q-p}} v_q$$

goes to infinity as $q \to p$, while for $\lambda > \lambda_1$ it goes to zero. The proof of Theorem 4.8 (i) and (iii) is virtually identical. It remains to consider the case $\lambda = \lambda_1$ for ground state solutions, and $\lambda = \lambda_2$ for least energy nodal solutions. Remark that the energy functional of problem (4.1) is given by

$$\varphi_q(u) := \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{q}\int_\Omega |u|^q$$

where $\lambda = \lambda_1$ (resp. λ_2). We denote by $\mathcal{N}_{\lambda,q}$ the associated Nehari manifold and $\mathcal{M}_{\lambda,q}$ the associated nodal Nehari set. The family $\{u_{q,1}\}_{q>p}$ will denote a family of ground state solutions for the problem (4.1) with $\lambda = \lambda_1$, while $\{u_{q,2}\}_{q>p}$ will be a family of least energy nodal solutions for the same problem with $\lambda = \lambda_2$. We prove that, up to a subsequence, $\{u_{q,1}\}_{q>p}$ (resp. $\{u_{q,2}\}_{q>p}$) converge in $L^p(\Omega)$ to a first (resp. second) eigenfunction of $-\Delta_p$.

Let us fix a first eigenfunction e_1 and a second eigenfunction e_2 of $-\Delta_p$.

Proposition 4.9. *The families $\{u_{q,1}\}_{q>p}$ and $\{u_{q,2}\}_{q>p}$ are uniformly bounded in $W_0^{1,p}(\Omega)$.*

Proof. We give the proof only for the family $\{u_{q,2}\}_{q>p}$. The arguments are easier in the other case. As $u_{q,2}$ belongs to the Nehari manifold, $\mathrm{d}\varphi_q(u_{q,2})(u_{q,2}) =$

0, which means $\|\nabla u_{q,2}\|_p^p = \lambda_2 \|u_{q,2}\|_q^q$. On one hand we have

$$\begin{aligned}\left(\frac{1}{p}-\frac{1}{q}\right)\|\nabla u_{q,2}\|_p^p &= \varphi_q(u_{q,2}) \\ &= \inf_{u\in\mathcal{M}_q}\varphi_q(u) \\ &\le \varphi_q(t_q^+e_2^+ + t_q^-e_2^-) \\ &= \varphi_q(t_q^+e_2^+) + \varphi_q(t_q^-e_2^-).\end{aligned}$$

On the other hand we have

$$\varphi_q(t_q^+e_2^+) = \frac{1}{p}(t_q^+)^p\|\nabla e_2^+\|_p^p - \frac{\lambda_2}{q}(t_q^+)^q\|e_2^+\|_q^q = \left(\frac{1}{p}-\frac{1}{q}\right)(t_q^+)^p\|\nabla e_2^+\|_p^p$$

and analogously for $\varphi_q(t_q^-e_2^-)$. So we obtain

$$\|\nabla u_{q,2}\|_p^p \le (t_q^+)^p\|\nabla e_2^+\|_p^p + (t_q^-)^p\|\nabla e_2^-\|_p^p$$

where

$$t_q^+ = \left(\frac{\|\nabla e_2^+\|_p^p}{\lambda_2\|e_2^+\|_q^q}\right)^{\frac{1}{q-p}} = \left(\frac{\|e_2^+\|_p^p}{\|e_2^+\|_q^q}\right)^{\frac{1}{q-p}}$$

and similarly for t_q^-. By Hölder's inequality one has

$$t_q^+ \le \left(\frac{|\Omega|^{\frac{q}{p}-1}\|e_2^+\|_p^p}{\|e_2^+\|_p^q}\right)^{\frac{1}{q-p}} = \frac{|\Omega|^{\frac{1}{p}}}{\|e_2^+\|_p}.$$

Substituting we obtain

$$\|\nabla u_{q,2}\|_p^p \le 2\cdot\lambda_2\cdot|\Omega|$$

so that

$$\|\nabla u_{q,2}\|_p \le (2\cdot\lambda_2\cdot|\Omega|)^{\frac{1}{p}}.$$

□

The two following results prove that the sequence of ground state solutions (resp. least energy nodal solutions) of problem (4.1) stays away from the zero function.

Theorem 4.10. *Let $\{u_{q,1}\}_{q>p}$ be a family of ground state solutions of the Lane-Emden problem (4.1) for $\lambda=\lambda_1$. Then*

$$\liminf_{q\to p}\|\nabla u_{q,1}\|_p > 0.$$

Proof. Fix $r > 0$ such that $p < r < p^*$, and set $s := \frac{r(q-p)}{q(r-p)}$. By interpolation of Hölder's inequality (see Proposition A.5) we obtain

$$\|u_{q,1}\|_q^p \leq \|u_{q,1}\|_p^{p-ps} \|u_{q,1}\|_r^{ps}.$$

By definition of λ_1 we have

$$\lambda_1 \|u_{q,1}\|_p^p \leq \|\nabla u_{q,1}\|_p^p.$$

On the other hand, since $\{u_{q,1}\}_{q>p}$ belongs to the Nehari manifold $\mathcal{N}_q$, we have

$$\|\nabla u_{q,1}\|_p^p = \lambda_1 \|u_{q,1}\|_q^q$$

and, since $r < p^*$, by Sobolev's embedding Theorem, we know that there exists a constant C such that

$$\|u_{q,1}\|_r^p \leq C \|\nabla u_{q,1}\|_p^p.$$

So it follows that

$$\|\nabla u_{q,1}\|_p \geq \lambda_1^{\frac{-p+q-sq}{pq-p^2}} C^{-\frac{sq}{pq-p^2}}$$

which means, recalling the definition of s,

$$\|\nabla u_{q,1}\|_p \geq \lambda_1^{\frac{1}{p-r}} C^{\frac{r}{p(p-r)}}.$$

Since this estimate does not depend on q, we obtain the claim.

$\square$

Theorem 4.11. *Let $\{u_{q,2}\}_{q>p}$ be a family of least energy nodal solutions of the Lane-Emden problem (4.1) for $\lambda = \lambda_2$. Then*

$$\liminf_{q \to p} \|\nabla u_{q,2}\|_p > 0.$$

Proof. Since $u_{q,2}$ is sign-changing we can write $u_{q,2} = u_{q,2}^+ + u_{q,2}^-$, with $u_{q,2}^+, u_{q,2}^- \neq 0$. Define

$$A := \left\{ v \in W_0^{1,p}(\Omega) \setminus \{0\} \;\middle|\; v = \alpha u_{q,2}^+ + \beta u_{q,2}^-,\ (\alpha, \beta) \neq (0,0) \right\}.$$

It can be proved that $A \in \Gamma_2$, where Γ_2 is as in Definition 2.4. By definition of λ_2 we have

$$\lambda_2 \leq \max_{(\alpha,\beta)\neq(0,0)} \frac{|\alpha|^p \|\nabla u_{q,2}^+\|_p^p + |\beta|^p \|\nabla u_{q,2}^-\|_p^p}{|\alpha|^p \|u_{q,2}^+\|_p^p + |\beta|^p \|u_{q,2}^-\|_p^p} \leq \max \left\{ \frac{\|\nabla u_{q,2}^+\|_p^p}{\|u_{q,2}^+\|_p^p}, \frac{\|\nabla u_{q,2}^-\|_p^p}{\|u_{q,2}^-\|_p^p} \right\}.$$

The last inequality follows from Proposition A.4. Let us assume, without loss of generality, that the maximum is attained for $u_{q,2}^+$. Then we have

$$\lambda_2 \|u_{q,2}^+\|_p^p \leq \|\nabla u_{q,2}^+\|_p^p.$$

Fix $r > 0$ such that $p < r < p^*$ and set $s := \frac{r(q-p)}{q(r-p)}$. By interpolation of Hölder's inequality (see Proposition A.5) we obtain

$$\|u_{q,2}^+\|_q^p \leq \|u_{q,2}^+\|_p^{p-ps} \|u_{q,2}^+\|_r^{ps}.$$

On the other hand, since $\{u_{q,2}^+\}_{q>p}$ belongs to the Nehari manifold $\mathcal{N}_q$, we have

$$\|\nabla u_{q,2}^+\|_p^p = \lambda_2 \|u_{q,2}^+\|_q^q$$

and since $r < p^*$ by Sobolev's embedding Theorem we get

$$\|u_{q,2}^+\|_r^p \leq C \|\nabla u_{q,2}^+\|_p^p.$$

So it follows that

$$\|\nabla u_{q,2}^+\|_p \geq \lambda_2^{\frac{-p+q-sq}{pq-p^2}} C^{-\frac{sq}{pq-p^2}}$$

and if we recall the definition of s

$$\|\nabla u_{q,2}^+\|_p \geq \lambda_2^{\frac{1}{p-r}} C^{\frac{r}{p(p-r)}}.$$

From the relation

$$\|\nabla u_{q,2}\|_p \geq \|\nabla u_{q,2}^+\|_p$$

and since the estimate does not depend on q we obtain the claim. □

Theorem 4.12. *Let $\{u_{q,1}\}_{q>p}$ be a family of ground state solutions of the Lane-Emden problem (4.1) for $\lambda = \lambda_1$ (resp. $\{u_{q,2}\}_{q>p}$ be a family of least energy nodal solutions for $\lambda = \lambda_2$). Then, up to a subsequence, $u_{q,1} \to u_*$ (resp. $u_{q,2} \to u_*$) in $L^p(\Omega)$ as $q \to p$, where the function u_* is a first (resp. second) eigenfunction of the p-Laplacian.*

Proof. We give the proof for the family of least energy nodal solutions. The idea is the same for the family of ground state solutions. Let $v \in W_0^{1,p}(\Omega)$. Because of the uniform boundedness of the family $\{u_{q,2}\}_{q>p}$ in $W_0^{1,p}(\Omega)$, there exists $u_* \in W_0^{1,p}(\Omega)$ such that $u_{q,2} \rightharpoonup u_*$ in $W_0^{1,p}(\Omega)$ and $u_{q,2} \to u_*$ in $L^p(\Omega)$ for $q \to p$ (up to a subsequence). By Lebesgue's dominated convergence Theorem we also have

$$|u_{q,2}|^{q-2} u_{q,2} \to |u_*|^{p-2} u_* \ \text{ in } L^p(\Omega).$$

So

$$\begin{aligned}
\int_\Omega |\nabla u_*|^{p-2} \nabla u_* \nabla v &= \lim_{q \to p} \int_\Omega |\nabla u_{q,2}|^{p-2} \nabla u_{q,2} \nabla v \\
&= \lim_{q \to p} \lambda_2 \int_\Omega |u_{q,2}|^{q-2} u_{q,2} v \\
&= \lambda_2 \int_\Omega |u_*|^{p-2} u_* v.
\end{aligned}$$

By Theorem 4.11 $u_* \neq 0$. Hence, u_* is a second eigenfunction of $-\Delta_p$. □

Appendix A

Inequalities

Proposition A.1. *Let $\{a_n\}$, $\{b_n\}$ be sequences. Then*

$$\limsup_{n\to\infty}(a_n b_n) \geq \left(\liminf_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right).$$

Proof. Set $a := \liminf_{n\to\infty} a_n$, $b := \limsup_{n\to\infty} b_n$. Let $\{b_{n_k}\}$ be a subsequence such that $b_{n_k} \to b$. Then

$$a \leq \liminf_{k\to\infty} a_{n_k}.$$

Let $\{a_{n'_k}\}$ a subsequence of a_{n_k} such that $a_{n'_k} \to a' := \liminf_{k\to\infty} a_{n_k}$. Then $b_{n'_k} \to b$ and

$$ab \leq a'b = \left(\lim_{k\to\infty} a_{n'_k}\right)\left(\lim_{k\to\infty} b_{n'_k}\right) = \lim_{k\to\infty}\left(a_{n'_k} b_{n'_k}\right) \leq \limsup_{n\to\infty} a_n b_n.$$

$\square$

Proposition A.2. *Let $\{a_n\}$, $\{b_n\}$ be sequences. Then*

$$\liminf_{n\to\infty}(a_n b_n) \leq \left(\liminf_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right).$$

Proof. Set $a := \liminf_{n\to\infty} a_n$, $b := \limsup_{n\to\infty} b_n$. Let $\{a_{n_k}\}$ be a subsequence such that $a_{n_k} \to a$. Then

$$b \geq \limsup_{k\to\infty} b_{n_k}.$$

Let $\{b_{n'_k}\}$ a subsequence of b_{n_k} such that $b_{n'_k} \to b' := \limsup_{k\to\infty} b_{n_k}$. Then $a_{n'_k} \to a$ and

$$ab \geq ab' = \left(\lim_{k\to\infty} a_{n'_k}\right)\left(\lim_{k\to\infty} b_{n'_k}\right) = \lim_{k\to\infty}\left(a_{n'_k} b_{n'_k}\right) \geq \liminf_{n\to\infty} a_n b_n.$$

$\square$

Proposition A.3. *Let* $\{a_n\}$, $\{b_n\}$ *be sequences such that* $a_n, b_n \geq 0$. *Then*

$$\frac{\liminf_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \liminf_{n\to\infty} \frac{a_n}{b_n}.$$

Proof. Set $a := \liminf_{n\to\infty} a_n$, $b := \lim_{n\to\infty} b_n$. Let $\{a_{n_k}\}$ be a subsequence such that $a_{n_k} \to a$. Then

$$\liminf_{n\to\infty} \frac{a_n}{b_n} \leq \liminf_{k\to\infty} \frac{a_{n_k}}{b_{n_k}} = \frac{a}{b}.$$

On the other side, set $c := \liminf_{n\to\infty} \frac{a_n}{b_n}$, and let $\left\{\frac{a_{n_k}}{b_{n_k}}\right\}$ be a subsequence such that $\frac{a_{n_k}}{b_{n_k}} \to c$; then $a_{n_k} \to bc$. So

$$\frac{\liminf_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \leq \frac{\liminf_{k\to\infty} a_{n_k}}{b} = \frac{bc}{b} = c.$$

$\square$

Proposition A.4. *Let* $a, b, c, d > 0$. *Then*

$$\min\left\{\frac{a}{c}, \frac{b}{d}\right\} \leq \frac{a+b}{c+d} \leq \max\left\{\frac{a}{c}, \frac{b}{d}\right\}.$$

Proof. The claim follows from the fact that

$$\frac{a+b}{c+d} \leq (\geq)\frac{a}{c} \Leftrightarrow ac + bc \leq (\geq) ac + ad \Leftrightarrow \frac{b}{d} \leq (\geq)\frac{a}{c}.$$

$\square$

Proposition A.5. *Let* $\Omega \subset \mathbb{R}^n$, $1 \leq p \leq q \leq +\infty$, *and let* $u \in L^p(\Omega) \cap L^q(\Omega)$. *Then* $u \in L^r(\Omega)$ *for every* $r \in [p, q]$ *and*

$$\|u\|_r \leq \|u\|_p^\alpha \, \|u\|_q^{1-\alpha}$$

where $0 \leq \alpha \leq 1$ *and*

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

Proof. The proof can be found in [13, Chapter 4]. $\square$

Appendix B

Functions of bounded variation

The functions of bounded variation build a generalisation of Sobolev functions. Most of the following results can be found in [29] or in [25].

Definition B.1. Let $\Omega \subset \mathbb{R}^n$ an open set. The *total variation* $\|Du\|(\Omega)$ of a function $u \in L^1(\Omega)$ is defined as

$$\|Du\|(\Omega) := \sup \left\{ \int_\Omega u \operatorname{div} \varphi \;\middle|\; \varphi \in C_c^\infty(\Omega; \mathbb{R}^n) \ , |\varphi| \leq 1 \right\}.$$

A function $u \in L^1(\Omega)$ is said to have *bounded total variation* if $\|Du\|(\Omega) < \infty$. This quantity can also be indicated by $\int_\Omega |Du|$ or $\|u\|_{TV}$.

We denote by $BV(\Omega)$ the space of function of bounded variation (also called *BV functions*). For every $u \in BV(\Omega)$, we define

$$\|u\|_{BV(\Omega)} := \|u\|_1 + \|Du\|(\Omega) \tag{B.1}$$

Remark B.2. It is easy to see that $\|\cdot\|_{BV(\Omega)}$ is a norm on the space $BV(\Omega)$. Moreover, it can be shown that $\|Du\|$ is a Radon measure on Ω, defining

$$\|Du\|(U) := \sup \left\{ \int_U u \operatorname{div} \varphi \;\middle|\; \varphi \in C_c^\infty(U; \mathbb{R}^n) \ , |\varphi| \leq 1 \right\}$$

where U is a subset of Ω.

Remark B.3. It can be shown that $W^{1,1}(\Omega) \subset BV(\Omega)$, but $W^{1,1}(\Omega) \neq BV(\Omega)$. As a counterexample take for instance $\Omega = (0,1) \subset \mathbb{R}$, $E = (0, \frac{1}{2})$, and $u = \chi_E$. Then clearly $u \in L^1(\Omega)$ and if we take $\varphi \in C_c^\infty([0,1]; \mathbb{R})$ with $|\varphi| \leq 1$ we obtain

$$\int_0^1 \chi_E \, \varphi' \ = \ \int_0^{\frac{1}{2}} \varphi' \ = \ \varphi\left(\frac{1}{2}\right) - \varphi(0) = \varphi\left(\frac{1}{2}\right).$$

Taking the supremum on every admissible φ we have $\|Du\|(\Omega) = 1$; hence $u \in BV(\Omega)$. However, $u \notin W^{1,1}(\Omega)$.

We recall the following

Definition B.4. A Lebesgue-measurable subset $E \subset \mathbb{R}^n$ has *finite perimeter* in Ω if

$$\chi_E \in BV(\Omega).$$

The quantity $\|D\chi_E\|(\Omega)$ is called the *perimeter* of E in Ω and can also be denoted by $\|\partial E\|(\Omega)$.

For a set E with sufficiently smooth boundary (for instance of class Lipschitz) we have

$$\|\partial E\|(\Omega) = \mathcal{H}^{n-1}(\partial E)$$

where the symbol $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure.

Theorem B.5. *(Semicontinuity of the total variation) Let $\Omega \in \mathbb{R}^n$ an open set, and $\{u_k\}_{k=1}^{\infty}$ a sequence of functions in $BV(\Omega)$, converging in $L^1_{loc}(\Omega)$ to a function u. Then $u \in BV(\Omega)$, and*

$$\|Du\|(\Omega) \ \leq \ \liminf_{k\to\infty} \|Du_k\|(\Omega).$$

Definition B.6. Let u, $u_k \in BV(\Omega)$ $(k = 1, ...)$. We say that the sequence $\{u_k\}_{k=1}^{\infty}$ *converges strictly* to u if, as $k \to \infty$:

1. $u_k \to u \quad \text{in } L^1(\Omega)$, and
2. $\|Du_k\|(\Omega) \to \|Du\|(\Omega)$.

Remark B.7. It should be noted that BV-norm convergence implies strict convergence, but the converse is in general not true.

Theorem B.8. *Let $u \in BV(\Omega)$. Then there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset C^{\infty}(\Omega) \cap BV(\Omega)$, converging strictly to u.*

Theorem B.9. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with $\partial\Omega$ of class Lipschitz. Assume $\{u_k\}_{k=1}^{\infty}$ is a sequence in $BV(\Omega)$ satisfying*

$$\sup_k \|u_k\|_{BV(\Omega)} \leq M$$

for some $M > 0$ Then there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and a function $u \in BV(\Omega)$ such that

$$u_{k_j} \to u \ \ in \ L^1(\Omega)$$

as $j \to \infty$.

Theorem B.10. *(Coarea formula) Let $u \in BV(\Omega)$, and define*

$$E_t := \{x \in \Omega \,|\, u(x) > t\}.$$

Then:

- *E_t has finite perimeter for almost every $t \in \mathbb{R}$.*
- *$\|Du\|(\Omega) = \int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega)\, dt$.*
- *Conversely, if $u \in L^1(\Omega)$, and*

$$\int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega)\, dt < \infty$$

then $u \in BV(\Omega)$.

We consider now two minimization problems. Let us define

$$\lambda_1(\Omega) := \inf_{u \in BV(\Omega)} \frac{\|Du\|(\mathbb{R}^n)}{\|u\|_1} \tag{B.2}$$

and

$$h(\Omega) := \inf_{E \subset \overline{\Omega}} \frac{Per(E)}{V(E)} \tag{B.3}$$

as in Chapter 1. The connections between the two problems will be shown in the next two theorems (see also [38])

Proposition B.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open domain with Lipschitz boundary. Then there exists a function $u \in BV(\Omega)$ such that*

$$\frac{\|Du\|(\mathbb{R}^n)}{\|u\|_1} = \lambda_1(\Omega).$$

Proof. Clearly, $\lambda_1(\Omega) \geq 0$. Let $\{u_k\}_{k=1}^{\infty}$ be a minimizing sequence for (B.2). Without loss of generality, we can suppose $\|u_k\|_1 = 1$. For every k big enough we have

$$\|Du_k\|(\mathbb{R}^n) \leq \lambda_1(\Omega) + 1.$$

It follows

$$\|u_k\|_{BV(\Omega)} \leq \lambda_1(\Omega) + 2.$$

According to Theorem B.9, there exists a subsequence (still denoted by $\{u_k\}$), such that there exists $u \in BV(\Omega)$ with

$$u_k \to u \quad \text{in} \;\; L^1(\Omega).$$

Using Theorem B.5 we get

$$\lambda_1(\Omega) \le \|Du\|(\mathbb{R}^n) \le \liminf_{k\to\infty} \|Du_k\|(\mathbb{R}^n) = \lim_{k\to\infty} \|Du_k\|(\mathbb{R}^n) = \lambda_1(\Omega)$$
$$\Rightarrow \|Du\|(\mathbb{R}^n) = \lambda_1(\Omega).$$

As $\|u\|_1 = 1$, we obtain the claim.

□

Proposition B.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open domain with Lipschitz boundary. Then there exists a set $E \subset \overline{\Omega}$ such that*

$$\frac{Per(E)}{V(E)} = h(\Omega).$$

Moreover, $\lambda_1(\Omega) = h(\Omega)$.

Proof. We begin to observe that $\lambda_1(\Omega) \le h(\Omega)$: this is true, as $h(\Omega)$ can be considered as the same infimum in (B.2), taken only on all the characteristic functions of sets of finite perimeter in Ω (and $\|D\chi_E\|(\mathbb{R}^n) = Per(E)$ according to Definition B.4). Let $u \in BV(\Omega)$ be as in Theorem B.11; using the coarea formula B.10 and Cavalieri's principle we get

$$\lambda_1(\Omega) = \frac{\|Du\|(\mathbb{R}^n)}{\|u\|_1} = \frac{\int_{-\infty}^{+\infty} \|\partial E_t\|(\mathbb{R}^n)\,dt}{\int_{-\infty}^{+\infty} V(E_t)\,dt} = \frac{\int_{-\infty}^{+\infty} Per(E_t)\,dt}{\int_{-\infty}^{+\infty} V(E_t)\,dt}$$

hence

$$\int_{-\infty}^{+\infty} [Per(E_t) - \lambda_1(\Omega)V(E_t)]\,dt = 0.$$

As $\lambda_1(\Omega) \le h(\Omega)$, we have

$$Per(E_t) - \lambda_1(\Omega)V(E_t) \ge 0$$

for every $t \in \mathbb{R}$ Hence, for almost every $t \in \mathbb{R}$, it must be

$$Per(E_t) - \lambda_1(\Omega)V(E_t) = 0$$

that is

$$\frac{Per(E_t)}{V(E_t)} = \lambda_1(\Omega).$$

It follows

$$\lambda_1(\Omega) \ge h(\Omega) \Rightarrow \lambda_1(\Omega) = h(\Omega)$$

as well as the existence of a minimizing set for (B.3). □

Appendix C

Γ-convergence

We will give in the following the basic definitions and results of Γ-convergence of functionals; our reference text for this purpose will be [12].

Definition C.1. Let X be a metric space. We say that a sequence of functions $f_j : X \to [-\infty, +\infty]$ Γ-*converges* to $f_\infty : X \to [-\infty, +\infty]$ if for every $x \in X$ we have:
(i) (liminf inequality) for every sequence $\{x_j\}_{j=1}^{\infty}$ converging to x we have

$$f_\infty(x) \le \liminf_{j\to+\infty} f_j(x_j).$$

(ii) (limsup inequality) there exists a sequence $\{x_j\}_{j=1}^{\infty}$ converging to x such that

$$f_\infty(x) \ge \limsup_{j\to+\infty} f_j(x_j).$$

The function f_∞ is called the Γ-*limit* of $\{f_j\}$, and we write

$$f_\infty = \Gamma - \lim_{j\to+\infty} f_j.$$

The sequence $\{x_j\}_{j=1}^{\infty}$ is called a *recovery sequence* for x.

Remark C.2. Consider the case of a constant sequence $f_j = f$ for every $j \in \mathbb{N}$; if these sequence has a Γ-limit, but f is not lower semicontinuous, it can not be true that $f = \Gamma - \lim_{j\to+\infty} f_j$; in fact, by (i) in Definition C.1 we would have, for every $x \in X$ and for every $x_j \to x$,

$$f(x) \le \liminf_{j\to+\infty} f(x_j)$$

which contradicts the fact that f is not lower semicontinuous. So even for a constant sequence the Γ-limit may differ from the pointwise limit.

Remark C.3. If we have a continuous family of functions $f_p : X \to [-\infty, +\infty]$, $p \in \mathbb{R}$, we can still define Γ-convergence by asking that, for every sequence of indices $\{p_j\}_{j=1}^{\infty}$, $p_j \to \infty$, Definition C.1 holds for the sequence of functions f_{p_j}.

Definition C.4. A function $f : X \to [-\infty, +\infty]$ is *coercive* if for all $t \in \mathbb{R}$ the set $\{f \le t\}$ is precompact. A function $f : X \to [-\infty, +\infty]$ is *mildly coercive* if there exists a non-empty compact set K such that $\inf_X f = \inf_K f$. A sequence of functions $f_j : X \to [-\infty, +\infty]$ ($j \in \mathbb{N}$) is *equi-mildly coercive* if there exists a non-empty compact set K such that $\inf_X f_j = \inf_K f_j$ for all j.

We are now ready to state one of the main results about Γ-convergence.

Lemma C.5. *Let f_j,$f_\infty : X \to [-\infty, +\infty]$ be functions. Then we have:*
(i) if Definition C.1 (i) is satisfied for all $x \in X$, and $K \subset X$ is a compact set, then we have

$$\inf_K f_\infty \le \liminf_{j\to+\infty} \inf_K f_j.$$

(ii) if Definition C.1 (ii) is satisfied for all $x \in X$, and $U \subset X$ is an open set, then we have

$$\inf_U f_\infty \ge \limsup_{j\to+\infty} \inf_U f_j.$$

Proof. (i) Let $\{\widetilde{x}_j\}_{j=1}^{\infty}$ be a sequence such that $\liminf_{j\to+\infty} \inf_K f_j = \liminf_{j\to+\infty} f_j(\widetilde{x}_j)$. For the compactness of K we can extract a subsequence $\{\widetilde{x}_{j_k}\}_{k=1}^{\infty}$ such that $\widetilde{x}_{j_k} \to \overline{x}$ and

$$\lim_k f_{j_k}(\widetilde{x}_{j_k}) = \liminf_{j\to+\infty} \inf_K f_j.$$

If we set

$$x_j = \begin{cases} \widetilde{x}_{j_k} & \text{if } j = j_k \\ \overline{x} & \text{if } j \ne j_k \text{ for every } k \end{cases}$$

then

$$\begin{aligned} \inf_K f_\infty \le f_\infty(\overline{x}) &\le \liminf_{j\to+\infty} f_j(x_j) \le \liminf_k f_{j_k}(x_{j_k}) \\ &= \lim_k f_{j_k}(\widetilde{x}_{j_k}) = \liminf_{j\to+\infty} \inf_K f_j \end{aligned} \tag{C.1}$$

as required.
(ii) Let $\delta > 0$ be fixed, and let $x \in U$ be such that $f_\infty(x) \le \inf_U f_\infty + \delta$. Then, if $\{x_j\}_{j=1}^{\infty}$ is a recovery sequence for x, we have

$$\inf_U f_\infty + \delta \ge f_\infty(x) \ge \limsup_{j\to+\infty} f_j(x_j) \ge \limsup_{j\to+\infty} \inf_U f_j. \tag{C.2}$$

The claim follows from the arbitrariness of δ. □

Theorem C.6. *Let X be a metric space, let $\{f_j\}_{j=1}^{\infty}$ a sequence of equi-mildly coercive functions on X, and let $f_\infty = \Gamma - \lim_{j\to+\infty} f_j$. Then $\min_X f_\infty$ exists, and*

$$\min_X f_\infty = \lim_{j\to+\infty} \inf_X f_j.$$

Moreover, if $\{x_j\}_{j=1}^{\infty}$ is a precompact subsequence such that $\lim_{j\to+\infty} f_j(x_j) = \lim_{j\to+\infty} \inf_X f_j$, then every limit of a subsequence of $\{x_j\}_{j=1}^{\infty}$ is a minimum point for f_∞.

Proof. The proof follows from Lemma C.5. Let $\overline{x}$ be as in the proof of Lemma C.5 (i); then by (C.1) and (C.2) with $U = X$, and by the equi-mild coerciveness condition we get

$$\begin{aligned} \inf_X f_\infty \le \inf_K f_\infty \le f_\infty(\overline{x}) &\le \liminf_{j\to+\infty} \inf_K f_j \\ &= \liminf_{j\to+\infty} \inf_X f_j \le \limsup_{j\to+\infty} \inf_X f_j \le \inf_X f_\infty. \end{aligned} \tag{C.3}$$

As the first and the last terms coincide, we obtain the claim. $\square$

Appendix D

Nonlinear eigenvalues

Let X be a Banach space, $A \subset X$ a closed, symmetric subset. The *Krasnoselskii genus* $\gamma(A)$ is defined as

$$\gamma(A) := \min\{m \in \mathbb{N} \,|\, \exists\, \varphi : A \to \mathbb{R}^m \setminus \{0\},\, \varphi \text{ is continuous and odd}\}.$$

Let us denote by Γ_k the set

$$\Gamma_k := \left\{ A \subset W_0^{1,p}(\Omega)\setminus\{0\} \,\middle|\, A \cap \{\|u\|_p = 1\} \text{ is compact}, A \text{ symmetric}, \gamma(A) \geq k \right\}$$

and by $\widetilde{\Gamma}_k$ the set

$$\widetilde{\Gamma}_k := \left\{ A \subset W_0^{1,p}(\Omega) \cap \{\|u\|_p = 1\} \,\middle|\, A \text{ is compact and symmetric}, \gamma(A) \geq k \right\}.$$

We define for every $k \in \mathbb{N}$ the numbers

$$\lambda_k(p;\Omega) := \inf_{A \in \Gamma_k} \max_{u \in A} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}$$

and

$$\widetilde{\lambda}_k(p;\Omega) := \inf_{A \in \widetilde{\Gamma}_k} \max_{u \in A} \int_\Omega |\nabla u|^p.$$

Proposition D.1. *For every $k \in \mathbb{N}$,*

$$\lambda_k(p;\Omega) = \widetilde{\lambda}_k(p;\Omega).$$

Proof. It is clear that $\widetilde{\Gamma}_k \subset \Gamma_k$, so that $\lambda_k(p;\Omega) \leq \widetilde{\lambda}_k(p;\Omega)$. Let $A \in \Gamma_k$. Define $\varphi : W_0^{1,p}(\Omega) \setminus \{0\} \to W_0^{1,p}(\Omega) \setminus \{0\}$ as

$$\varphi(u) = \frac{u}{\|u\|_p}.$$

Then $\widetilde{A} := \varphi(A)$ belongs to $\widetilde{\Gamma}_k$, and

$$\max_{u\in A}\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} = \max_{u\in\widetilde{A}}\int_\Omega |\nabla u|^p.$$

It follows $\widetilde{\lambda}_k(p;\Omega) \le \lambda_k(p;\Omega)$ and hence the claim. □

We now define

$$\widehat{\Gamma}_k := \{A \subset W_0^{1,p}(\Omega)\setminus\{0\} \mid A \text{ is compact }, A \text{ symmetric }, \gamma(A)\ge k\}$$

and

$$\widehat{\lambda}_k(p;\Omega) := \inf_{A\in\widehat{\Gamma}_k}\max_{u\in A}\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}.$$

Proposition D.2. *For every $k \in \mathbb{N}$,*

$$\lambda_k(p;\Omega) = \widetilde{\lambda}_k(p;\Omega) = \widehat{\lambda}_k(p;\Omega).$$

Proof. The claim follows from the Proposition D.1 and from the fact that $\widetilde{\Gamma}_k \subset \widehat{\Gamma}_k \subset \Gamma_k$ which implies $\lambda_k(p;\Omega) \le \widehat{\lambda}_k(p;\Omega) \le \widetilde{\lambda}_k(p;\Omega)$. □

Now our aim will be to find critical points of the functional

$$F(u) := \int_\Omega |\nabla u|^p$$

subject to the constraint $G(u) = 1$, where

$$G(u) := \int_\Omega |u|^p.$$

By Lagrange's multiplier rule it is clear that constrained critical points of F are weak eigenfunctions of $-\Delta_p$.

Definition D.3. Let X be a Banach space, $M \subset X$ a $\mathcal{C}^1$ manifold, and let $c \in \mathbb{R}$. A functional $F \in \mathcal{C}^1(M;\mathbb{R})$ satisfies the Palais-Smale condition at level c if every sequence $\{u_n\}$ in M such that

- $F(u_n) \to c$, and
- $\|dF(u_n)\| \to 0$,

where dF is the differential of F, admits a converging subsequence (see [50] for more details).

Remark D.4. The functional F defined above satisfies the Palais-Smale condition for every $c \in \mathbb{R}$ (see for instance [41]).

The following proposition is a version of the deformation lemma useful in the setting we are considering, and can be found in [11, Theorem 2.5].

Proposition D.5. *Let X be a Banach space, $G \in \mathcal{C}^1(X;\mathbb{R})$. Let $M := G^{-1}(1)$ be a $\mathcal{C}^1$ manifold, F a $\mathcal{C}^1$ functional defined on a neighbourhood of M which satisfies the Palais-Smale condition, and let c be a noncritical value of F. Then there exists $\hat{\varepsilon} > 0$ such that for every $\varepsilon < \hat{\varepsilon}$, there exists a homeomorphism $h : M \to M$ such that:*

- *$h(u) = u$ if $F(u) \notin [c - \hat{\varepsilon}, c + \hat{\varepsilon}]$;*
- *$F(h(u)) \leq F(u)$ for every $u \in M$;*
- *$F(h(u)) \leq c - \varepsilon$ if $F(u) \leq c + \varepsilon$;*
- *if $M = -M$ and $F(u) = F(-u)$ for every $u \in M$, then $h(-u) = -h(u)$ for every $u \in M$.*

Theorem D.6. *For every $k \in \mathbb{N}$, the numbers $\lambda_k(p;\Omega)$ are eigenvalues of the p-Laplacian.*

Proof. Fix $\varepsilon > 0$, and set

$$F(u) := \int_\Omega |\nabla u|^p,$$

$\lambda_k := \lambda_k(p;\Omega)$, $X := W_0^{1,p}(\Omega) \setminus \{0\}$ and

$$M := \left\{ u \in X \;\middle|\; \int_\Omega |u|^p = 1 \right\}.$$

By definition, there exists a set $C_k \in \widetilde{\Gamma}_k$ such that $F(u) \leq \lambda_k + \varepsilon$ for every $u \in C_k$. Suppose that λ_k is not a critical value of F; then by Proposition D.5 there exists a homeomorphism $h : M \to M$ such that

- $h(-u) = -h(u)$ for every $u \in M$;
- $F(h(u)) \leq \lambda_k - \varepsilon$ if $F(u) \leq \lambda_k + \varepsilon$.

In particular, the set $\widetilde{C}_k := h(C_k)$ belongs to $\widetilde{\Gamma}_k$ and is such that $F(u) \leq \lambda_k - \varepsilon$ for every $u \in \widetilde{C}_k$, a contradiction. Hence λ_k is a critical point for F.

□

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In Zusammenarbeit mit C. Grumiau: "On the asymptotics of solutions of the Lane-Emden problem for the p-Laplacian". Erschienen in "Archiv der Mathematik", 91 (2008), pp. 354-365.