Hyperbolic Hubbard-Stratonovich transformations and bosonisation of granular fermionic systems

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Zusammenfassung

Die vorliegende Arbeit besteht aus zwei Teilen. Der erste Teil beschäftigt sich mit hyperbolischen Hubbard-Stratonovich-Transformationen. Solche Transformationen werden z.B. im Bereich der ungeordneten Elektronensysteme benötigt, um nichtlineare Sigma-Modelle herzuleiten, die das Niederenergieverhalten dieser Systeme beschreiben. Der mathematische Status hyperbolischer Hubbard-Stratonovich-Transformationen vom Pruisken-Schäfer-Typ war lange ungeklärt. Kürzlich wurden zwei Spezialfälle, nämlich die pseudounitärer und pseudoorthogonaler Symmetrie, bewiesen [10, 11, 12]. In dieser Arbeit wird nun der Fall einer allgemeinen (im wesentlichen halbeinfachen) Symmetriegruppe bewiesen. Der Beweis ist anschaulich und zeigt explizit den Zusammenhang mit Standard-Gauß-Integralen.

Im zweiten Teil wird eine eine neuartige Methode entwickelt, um wechselwirkende granular fermionische Systeme zu bosonisieren. Die Methode ist nicht mit der bekannten Bosonisierung $(1 + 1)$-dimensionaler Systeme verwandt, sondern eher im Bereich der kohärenten Zustände anzusiedeln. Ein Zugang ist, die Grassmann-Pfadintegraldarstellung einer großkanonischen Zustandssumme durch mehrfache Anwendung der Colour-Flavour-Transformation in eine Form zu bringen, welche die Eliminierung der Grassmannvariablen erlaubt. Das Resulat ist ein Pfadintegral in generalisierten kohärenten Zuständen mit speziellen Randbedingungen.

Abstract

The present work consists of two parts. The first part deals with hyperbolic Hubbard-Stratonovich transformations. Such transformations are used to derive non-linear sigma models that describe the low energy behaviour of disordered electron systems. For a long time the mathematical status of hyperbolic Hubbard-Stratonovich transformations of Pruisken-Schäfer type remained unclear. Only recently the two special cases of pseudounitary and pseudoorthogonal symmetry were proven [10, 11, 12]. In this thesis we prove the transformation for a general (essentially semisimple) symmetry group. The proof is descriptive and shows explicitly the connection to the standard Gaussian integrals.

In the second part we develop a novel method to bosonise granular fermionic systems. The method is related to the method of coherent states. In particular it is not based on the well known bosonisation of $(1 + 1)$-dimensional systems. One approach is to use the colour-flavour transformation to transform the Grassmann path integral representation of a grand canonical partition function in a way that allows to eliminate the Grassmann variables. The result is a path integral in generalised coherent states with special boundary conditions.

Contents

Introduction

To obtain an adequate description of a physical system, and to compute quantities of interest, it is often necessary to replace the microscopic degrees of freedom of the system by physically more relevant 'collective' degrees of freedom. Two prominent methods to introduce collective variables in the field of many particle physics are Hubbard-Stratonovich transformations and bosonisation. In this work we discuss special variants of both methods. The first part of this work clarifies the mathematical status of a class of hyperbolic Hubbard-Stratonovich transformations, whereas in the second part a new kind of bosonisation is developed. The focus of this work is rather on methodology than on applications.

Let us start with a more detailed introduction to the first part of the thesis. First, we explain where hyperbolic Hubbard-Stratonovich transformations are commonly used. A natural area of application of hyperbolic Hubbard-Stratonovich transformations are disordered electron systems [1] and their description in the form of non-compact non-linear sigma models. The corresponding formalism was pioneered by Wegner [3], Schäfer & Wegner [4], and Pruisken & Schäfer [5]. Efetov [2] developed the more rigorous supersymmetry method, which avoids the use of the replica trick, to derive (supersymmetric) non-linear sigma models. The supersymmetry method has a wide range of applications [15]. Examples are the description of single electron motion in a disordered or chaotic mesoscopic system [16], chaotic scattering [6], and Anderson localisation [17]. Traditional derivations of non-linear sigma models in the supersymmetry formalism rely crucially on hyperbolic Hubbard-Stratonovich transformations. To describe what hyperbolic Hubbard-Stratonovich transformations are we briefly review the case of (mathematically trivial) ordinary Hubbard-Stratonovich transformations. These transformations are frequently used throughout condensed matter field theory. From a mathematical point of view such a Hubbard-Stratonovich transformation consists of applying a Gaussian integral formula backwards, i.e., introducing additional integrations. Such a scheme converts a quartic interaction term in the original variables into a quadratic term coupled linearly to the newly introduced integration variables. The word 'hyperbolic' indicates a non-compact symmetry group of the original system. In such a situation the standard Gaussian integral formula cannot be applied due to issues of convergence.[1] A solution to this problem was given by Schäfer and Wegner [4]. They found a contour of integration for which the Gaussian integral formula holds and convergence is guaranteed. Nevertheless the majority of the physics community uses a different contour suggested by Pruisken and Schäfer [5] which, in contrast to the Schäfer-Wegner solution, preserves the full symmetry of the original system. However, until recently there existed no proof of the validity of the Pruisken-Schäfer transformation. The main difficulty is that the Pruisken-Schäfer domain has a boundary. This prevents an easy proof similar to the standard Gaussian integral and to the Schäfer-Wegner domain. Recently, several cases of the Pruisken-Schäfer transformation have been made rigorous by Fyodorov, Wei and Zirnbauer. Fyodorov [10] gave a proof for pseudounitary symmetry by using methods of semiclassical exactness. After that Fyodorov and Wei [11] proved a variant of the Pruisken-Schäfer transformation for the case of $\mathrm{O}(1,1)$ and $\mathrm{O}(2,1)$ symmetry by direct calculation, and proposed a result for the full $\mathrm{O}(p,q)$ case. They conjectured that the Gaussian integral decomposes into differerent parts that have to be weighted with certain alternating sign factors to obtain the right result. This conjecture indicates that the Pruisken-Schäfer transformation for the pseudoorthogonal case is not correct in its original form. Finally Fyodorov, Wei and Zirnbauer [12] proved the conjecture by reducing the calculation to the $\mathrm{O}(1,1)$ case and showing explicitly that all relevant boundary contributions vanish.

The motivation for our work is twofold. First, we want to obtain a better understanding of the somewhat mysterious alternating sign factors that appear in the $\mathrm{O}(p,q)$ case, and second, we want to generalise the transformation to more symmetry classes. The basic idea we follow is that in some sense, the Pruisken-Schäfer domain should be a deformation of the standard Gaussian domain. The problem of the boundary of the Pruisken-Schäfer contour is overcome by extending it, such that the integral remains unchanged and the boundary is moved to infinity. This leads to a proof of a variant of the Pruisken-Schäfer transformation for a general symmetry group. The proof shows that it is possible to deform the Pruisken-Schäfer integration contour into the standard Gaussian contour without changing the value of the integral. Actually the same can be done with the Schäfer-Wegner contour.

The structure of chapter one is as follows: First we give a more detailed motivation and a description of the convergence problems one encounters when applying the Gaussian integral in case of a non-compact symmetry. Next we discuss a two dimensional example that gives a road map for the general proof. Then we state our result and give its proof. Finally we show how to obtain the pseudounitary and pseudoorthogonal cases as special cases

[1] A detailed discussion of this issue is given at the beginning of chapter one.

of the general result.

The second part of this work explores a new method of bosonisation of granular fermionic systems. The terminology 'granular fermionic' indicates the structure of a fermionic vector model. In the following we list some examples: The well known Gross-Neveu models [22] and all fermionic models having an orbital degeneracy are in this class. An exactly solvable toy model is the Lipkin-Meshkov-Glick model [21]. A more complicated example is the many orbital generalisation of the Hubbard-model. A class of models which is currently intensively studied in mesoscopic physics are arrays of quantum dots or granular metals [24]. Each quantum dot is described by the universal Hamiltonian, which has a large orbital degeneracy [23]. Note that granularity, or equivalently large orbital degeneracy, implies the existence of a natural large N limit. Such large N limits are classical limits. For Gross-Neveu models this was investigated by Berezin [33] and for a much larger class of models by Yaffe [34]. In our work we will restrict ourselves to discrete (lattice) models that have either orthogonal, unitary or unitary symplectic symmetry. This contains all relevant possibilities for the universal Hamiltonian [23]. The term 'bosonisation' does not refer to the well known (non) Abelian bosonisation [19], which is limited to $(1+1)$ dimensional models, but rather to the natural geometric approach through generalised coherent state path integrals [35, 36].[2] It is interesting to note that these path integrals lead to a generalised Holstein-Primakoff transformation [18].

The restriction to granular fermionic systems with a classical Lie group as symmetry group gives access to powerful results from the theory of Howe dual pairs [27, 28]. One important tool that relies on the theory of Howe dual pairs is the colour-flavour transformation [29, 30]. Within our method we put the available structure to use in the calculation of the grand canonical partition function of a granular fermionic system. The result we obtain is a path integral representation of the grand canonical partition function of the granular fermionic system in terms of bosonic, i.e. commuting variables. The representation is essentially a path integral in generalised coherent states with certain boundary conditions. However, we cannot apply generalised coherent states directly in this context, since this would yield a path integral only for a subspace of Fock space.

The structure of the second part is as follows: We consecutively discuss two different derivations of the bosonic path integral representation of the grand canonical partition function. Furthermore we calculate the contribution of fluctuations in the semiclassical limit in terms of classical quantities.

[2] There have also been attempts to use coherent state path integrals for loop groups [20] to bosonise $(1+1)$ dimensional models.

Chapter 1

Hyperbolic Hubbard-Stratonovich transformations

1.1 Motivation

Non-compact non-linear sigma models are important and extensively used tools in the study of disordered electron systems. As mentioned in the introduction, the corresponding formalism was pioneered by Wegner [3], Schäfer & Wegner [4], and Pruisken & Schäfer [5]. Shortly afterwards Efetov [2] improved the formalism. He developed the supersymmetry method to derive non-linear sigma models. Many applications of the supersymmetry method can be found in the textbook by Efetov [15].

There are different ways to derive non-linear sigma models from microscopic models, for an introduction see [8]. The traditional approach is to use a Hubbard-Stratonovich transformation, i.e., a transformation of the form

$$c_0 \, e^{-\operatorname{Tr} A^2} = \int_{\mathrm{D}} e^{-\operatorname{Tr} Q^2 - 2i \operatorname{Tr} QA} \, |dQ| \, , \tag{1.1}$$

where $c_0 \in \mathbb{R}$. We leave the domain of integration D unspecified for now. $|dQ|$ denotes Lebesgue measure of a normed vector space.

Let us discuss the case of pseudoorthogonal symmetry $\mathrm{O}(p, q)$ as an example. Then A is given by $A_{ij} = \sum_{a=1}^{N} \Phi_{a,i} \Phi_{a,j} s_{jj}$ with $s = \mathrm{Diag}(\mathbb{1}_p, -\mathbb{1}_q)$ and $\Phi_{a,j} \in \mathbb{R}$. The $\Phi_{a,j}$ represent the microscopic degrees of freedom. Using equation (1.1) and integrating out ϕ gives a description in terms of the effective degrees of freedom Q. Thus the task is to find a domain of integration D for which identity (1.1) holds and the term $\exp(-2i \operatorname{Tr} QA)$ stays bounded. The latter allows to perform the Φ integrals after applying identity (1.1). Note that the real matrices A fulfil the symmetry relation $A = sA^t s$. A naive choice of the domain of integration D to keep the term

$\exp(-2i \operatorname{Tr} QA)$ bounded would be the domain of all real matrices satisfying $Q = sQ^t s$. This choice of D is not valid since then the quadratic form $\operatorname{Tr} Q^2 = \operatorname{Tr} QsQ^t s$ is of indefinite sign.

Schäfer and Wegner [4] found a domain D=SW, and gave a proof that it solves the problem. Nonetheless another domain D=PS was proposed in later work by Pruisken and Schäfer [5]. Until recently the mathematical status of identity (1.1) for D=PS was unclear. The main difficulty in proving identity (1.1) for D=PS is that the PS domain has a boundary. This prevents an easy proof by completing the square and shifting the contour as is possible for the standard Gauss integral and for the SW domain. Nevertheless the PS domain was used in most applications worked out by the mesoscopic physics community. The reason might be that it inherits the full symmetry of the domain of A matrices.

Recently Fyodorov, Wei and Zirnbauer [10, 11, 12] proved special cases of the Pruisken-Schäfer transformation.

In the following we state the result for the $\mathrm{O}(p, q)$ case that was obtained in [12]. Choose D as the subspace of matrices $Q = sQ^t s$ that can be diagonalised by an element of $\mathrm{O}(p, q)$. The domain D can be seen as the union of $\binom{p+q}{p}$ subdomains D_σ. The domains D_σ are labeled as follows. Up to a set of measure zero $Q \in D$ has p 'space-like' eigenvectors $\{v_i\}_{i \le p}$ with $v_i^t s v_i > 0$ and q 'time-like' eigenvectors $\{v_i\}_{q < i \le p}$ with $v_i^t s v_i < 0$. Again up to a set of measure zero in D, the eigenvalues of Q can be arranged in decreasing order. We translate this ordered sequence into a binary sequence by writing the symbol '$\bullet$' for space-like and '$\circ$' for time-like eigenvalues.[1] Furthermore, let

$$|dQ| = \prod_{i \le j} dQ_{ij} \tag{1.2}$$

denote flat integration measure on all domains D_σ, and let $\operatorname{sgn}(\sigma)$ be the parity of the number of transpositions $\bullet \leftrightarrow \circ$ needed to reduce the binary sequence σ to the extremal form $\sigma_0 = \bullet \cdots \bullet \circ \cdots \circ$. Then the following theorem [12] holds:

Theorem 1.1. *There exists some choice of cutoff function $Q \mapsto \chi_\epsilon(Q)$ (converging pointwise to unity as $\epsilon \to 0$), and a unique choice of sign function $\sigma \to \operatorname{sgn}(\sigma) \in \{\pm 1\}$ and a constant $C_{p,q}$ such that*

$$C_{p,q} \lim_{\epsilon \to 0} \sum_\sigma \operatorname{sgn}(\sigma) \int_{\mathrm{D}_\sigma} e^{-\operatorname{Tr} Q^2 - 2i \operatorname{Tr} AQ} \chi_\epsilon(Q) |dQ| = e^{-\operatorname{Tr} A^2} \tag{1.3}$$

holds true for all matrices $A = sA^t s$ with the positivity property $As > 0$.

The alternating sign factor was already conjectured in [11]. Note that in the large N limit only one D_σ contributes to the results. Therefore earlier

[1] The properties of the domains D_σ are discussed in more detail in [12].

works using (1.3) without the alternating sign lead to correct results in the large N limit.

In this chapter we show a variant of theorem 1.1 for a general symmetry group. The proof shows that it is possible to deform the Pruisken-Schäfer (PS) integration contour into the standard Gaussian contour without changing the value of the integral. Actually the same can be done with the Schäfer-Wegner (SW) contour. The problem of the boundary of PS is overcome by extending the PS domain in a way that leaves the integral unchanged and moves the boundary to infinity. The proof also demonstrates the origin of the alternating sign factors in equation 1.3

For convenience of the reader, the main ideas of the proof are first illustrated in a simple two dimensional example. Then we state the general results and give their proof. Finally some applications are discussed. In particular, it is discussed how the cases of pseudounitary and orthogonal symmetry fit into the general setting.

1.2 Two dimensional example

As a first step towards a general theorem of hyperbolic Hubbard-Stratonovich transformations, we discuss a two dimensional example. In this simplified setting the general result and main ideas of its proof can be nicely illustrated.

First we have to fix the setting. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be a basis of $\mathbb{C}^2$ and $dq_i(\mathbf{e}_j) = \delta_{ij}$. D denotes the parametrisation of a two dimensional surface in $\mathbb{C}^2$ and $dq_1 \wedge dq_2$ is a holomorphic two form on $\mathbb{C}^2$. Now consider the simple Gaussian type integral identity

$$\int_D e^{-q_1^2+q_2^2-2ia_1q_1+2ia_2q_2} dq_1 \wedge dq_2 = i\pi e^{-a_1^2+a_2^2} \ . \tag{1.4}$$

At this stage a_1 and a_2 may be arbitrary complex numbers. In the following we discuss different parametrisations of domains of integration D for which the identity holds. Eventually this requires imposing additional restrictions on a_1 and a_2. Since we are integrating differential forms, the domains of integration must have an (inner) orientation. Note that we discern between italic D and non italic D. The former denotes the parametrisation of the domain D.

Euclidean domain of integration

Obviously identity (1.4) holds for the standard Euclidean domain of integration

$$\begin{aligned} Euclid : \mathbb{R}^2 &\to \mathbb{C}^2 \\ (r, s) &\mapsto r\,\mathbf{e}_1 + is\,\mathbf{e}_2 \ . \end{aligned}$$

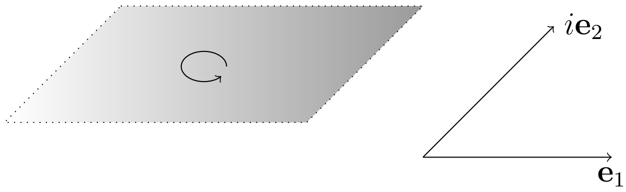

Figure 1.1: Standard Euclidean domain of integration.

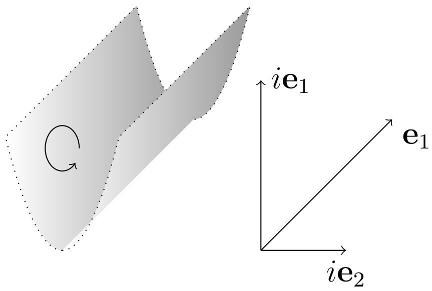

Figure 1.2: Schäfer-Wegner domain of integration.

Here the orientation comes from choosing an orientation on the domain of definition $\mathbb{R}^2$ and declaring *Euclid* to be orientation preserving. Note that the orientation of the two other domains of integration, which we discuss below, will be choosen in the same way. The orientation of *Euclid* is indicated by a sense of circulation in figure 1.1.

Schäfer-Wegner domain of integration

Demanding that $a_i \in \mathbb{R}$ and that $a_1 > a_2 \geq 0$, it can be checked by direct calculation that identity (1.4) also holds for the Schäfer-Wegner family of domains of integration, which is given by

$$\begin{aligned} SW &: \mathbb{R}^2 \to \mathbb{C}^2 \\ &(r, s) \mapsto r\,\mathbf{e}_1 - ib\cosh(s)\,\mathbf{e}_1 - ib\sinh(s)\,\mathbf{e}_2\,, \end{aligned}$$

where $b > 0$.

Pruisken-Schäfer domain of integration

The Pruisken-Schäfer domain is given by

$$\begin{aligned} PS &: \mathbb{R}^2 \to \mathbb{R}^2 \\ &(r, s) \mapsto r\cosh(s)\,\mathbf{e}_1 + r\sinh(s)\,\mathbf{e}_2\,. \end{aligned} \tag{1.5}$$

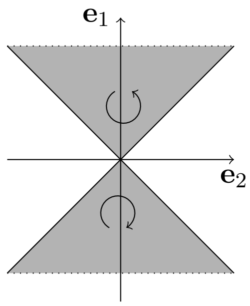

Figure 1.3: Pruisken-Schäfer domain of integration. The orientation is induced by the parametrisation (1.5).

Identity (1.4) holds for $D = PS$ only in a regularised form. For $|a_1| > |a_2|$ and $\chi_\epsilon(\vec{q}) = \exp(-\epsilon q_2^2)$ it can be checked by direct calculation that

$$\lim_{\epsilon \to 0} \int_D e^{-q_1^2 + q_2^2 - 2ia_1q_1 + 2ia_2q_2} \chi_\epsilon(q) dq_1 \wedge dq_2 = i\pi e^{-a_1^2 + a_2^2} \tag{1.6}$$

holds.

Note that by introducing a Minkowski scalar product $B(\vec{a}, \vec{q}) = a_1q_1 - a_2q_2$ with $\vec{a} = (a_1, a_2)^t$ and $\vec{q} = (q_1, q_2)^t$, (1.6) can be rewritten to

$$\lim_{\epsilon \to 0} \int_D e^{-B(\vec{q},\vec{q}) - 2iB(\vec{a},\vec{q})} \chi_\epsilon(\vec{q}) dq = i\pi e^{-B(\vec{a},\vec{a})} \ . \tag{1.7}$$

Here we have used $dq := dq_1 \wedge dq_2$. B has an $O(1,1)$ symmetry group, acting on $\mathbb{R}^2$. The Pruisken-Schäfer domain of integration is invariant under the action of the group. In this case also the domain of $\vec{a}$ for which (1.6) holds has this invariance. Both domains (for $\vec{a}$ and $\vec{q}$) are given by forward and backward lightcones as depicted in figure 1.3, or, put differently, by all timelike vectors ($B(\vec{a}, \vec{a}) > 0$ and $B(\vec{q}, \vec{q}) > 0$).

Moreover, in order for (1.6) to hold it is important that the upper and lower cone (see figure 1.3) have opposite orientations. If one wants to integrate Lebesgue measure $|dq|$ rather than a differential form, (1.6) has to be modified accordingly to a difference of integrals. Defining

$$g(\vec{q}, \vec{a}) := e^{-B(\vec{q},\vec{q}) - 2iB(\vec{a},\vec{q})} \ ,$$

identity (1.6) can be reformulated as

$$\lim_{\epsilon \to 0} \int_{D_+} g(\vec{q}, \vec{a}) \chi_\epsilon(q) |dq| - \lim_{\epsilon \to 0} \int_{D_-} g(\vec{q}, \vec{a}) \chi_\epsilon(q) |dq| = i\pi e^{-a_1^2 + a_2^2} \ ,$$

where D_+ stands for the upper and D_- for the lower cone in figure 1.3.

It was already mentioned in the introduction that for more general cases it is very difficult to prove identity (1.6) for the Pruisken-Schäfer domain by direct calculation. The idea of the proof for the general case is now discussed in the two dimensional case.

Main idea of the proof

The main idea is to deform the PS domain into the Euclidean domain without changing the value of the integral. As an easy example for such a deformation scheme we can use the SW domain of integration. Consider the deformation (or homotopy) given by

$$\begin{aligned} DSW &: [0,1] \times \mathbb{R}^2 \to \mathbb{C}^2 \\ &(t,r,s) \mapsto r\mathbf{e}_1 + ib(1-t)\cosh(s)\mathbf{e}_1 + ib\sinh(s)\mathbf{e}_2 \ , \end{aligned}$$

which is just a smooth projection onto the plane spanned by $\mathbf{e}_1$ and $i\mathbf{e}_2$. Note that $DSW(t=1) = Euclid$ and that the parametrisation given by DSW is a nice integration chain. Hence, $\partial DSW = Euclid - SW$ and we can apply Stokes theorem:

$$0 = \int_{DSW} \underbrace{d\left(g(\vec{q},\vec{a})dq_1 \wedge dq_2\right)}_{=0} = \int_{\partial DSW} g(\vec{q},\vec{a})dq_1 \wedge dq_2 \ .$$

Thus we have

$$\int_{SW} g(\vec{q},\vec{a})dq_1 \wedge dq_2 = \int_{Euclid} g(\vec{q},\vec{a})dq_1 \wedge dq_2 = i\pi e^{-B(\vec{a},\vec{a})} \ . \qquad (1.8)$$

The PS domain has a boundary, which seems to prevent an analogous proof of (1.6).

Proof

The following proof of (1.6) for $a_1 > a_2 \geq 0$ mimics the proof of the higher dimensional relatives of (1.6). Therefore the proof should be seen as a road map for the more complicated proof in the next section.

A suitable parametrisation: A different parametrisation of the PS domain is obviously given by

$$\begin{aligned} PS &: [-1,1] \times \mathbb{R} \to \mathbb{R}^2 \\ &(h,x) \mapsto x(\mathbf{e}_1 + h\mathbf{e}_2) \ . \end{aligned}$$

The boundary operator ∂ gives a nonzero result on $[-1,1]$. The other contributions vanish since the integrals to be considered are exponentially convergent for $\epsilon > 0$. We therfore have $\partial PS = PS(1) - PS(-1)$.

Extending the PS domain: Before deforming the PS domain, the boundary problem has to be dealt with first. The idea is to attach halfplanes to the boundary lines spanned by $e_1 \pm e_2$ as illustrated in figure 1.4. Here it is again crucial that the upper and lower cone have opposite orientations to allow the attachment of the halfplanes in a consistent way. A parametrisation of the halfplanes is given by

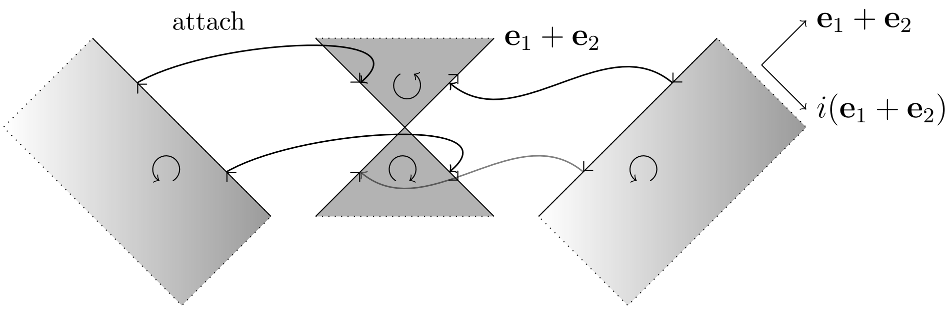

Figure 1.4: Attaching halfplanes that do not contribute to the integral.

$$\begin{aligned} hp_\pm : \mathbb{R}_+ \times \mathbb{R} &\to \mathbb{C}^2 \\ (h,x) &\mapsto \pm x\mathbf{e}_\pm - ih\mathbf{e}_\pm \ , \end{aligned}$$

where $\mathbf{e}_\pm = \mathbf{e}_1 \pm \mathbf{e}_2$. A good motivation for this special choice is that

$$dq_1 \wedge dq_2(\mathbf{e}_\pm, i\mathbf{e}_\pm) = idq_1 \wedge dq_2(\mathbf{e}_\pm, \mathbf{e}_\pm) = 0$$

and hence the halfplanes do not contribute to the integral. This is not yet a completely rigorous argument since existence and convergence of the integral still needs to be discussed. The extension of PS is defined as $ePS := PS + hp_+ + hp_-$, which is to be understood as a sum of integration chains.

Equivalence of PS and Euclid: The deformation is given by

$$\begin{aligned} DPS : [0,1] \times [-1,1] \times \mathbb{R} &\to \mathbb{R}^2 \\ (t,h,x) &\mapsto x(\mathbf{e}_1 + (1-t)h\mathbf{e}_2) \ , \\ Dhp_\pm : [0,1] \times \mathbb{R}_+ \times \mathbb{R} &\to \mathbb{C}^2 \\ (t,h,x) &\mapsto \pm x[\mathbf{e}_1 \pm (1-t)\mathbf{e}_2] - ih[(1-t)\mathbf{e}_1 \pm \mathbf{e}_2] \ , \end{aligned}$$

which defines then $DePS = DPS + Dhp_+ + Dhp_-$. For $t = 1$ the deformed PS surface degenerates into a line and both halfplanes $hp_\pm$ are projected into the plane spanned by $\mathbf{e}_1$ and $i\mathbf{e}_2$ as shown in figure 1.5. Note that $Dhp_\pm(t=1) : (h,x) \mapsto \pm x\mathbf{e}_1 \mp ih\mathbf{e}_2$. Thus we have $DePS(t=1) = Euclid$.

Now, we want to apply Stokes' theorem. We use $\partial DePS^\epsilon = Euclid - DePS(\epsilon)$, where $DePS^\epsilon := DePS|_{t\in[\epsilon,1]}$. The idea is to deform Euclid, as far as convergence of the integral allows, into PS:

$$\begin{aligned} \int_{DePS(\epsilon)} g(\vec{q},\vec{a})dq &= -\int_{DePS^\epsilon} g(\vec{q},\vec{a})d(dq) + \int_{Euclid} g(\vec{q},\vec{a})dq \\ &= \int_{Euclid} g(\vec{q},\vec{a})dq \ . \end{aligned} \tag{1.9}$$

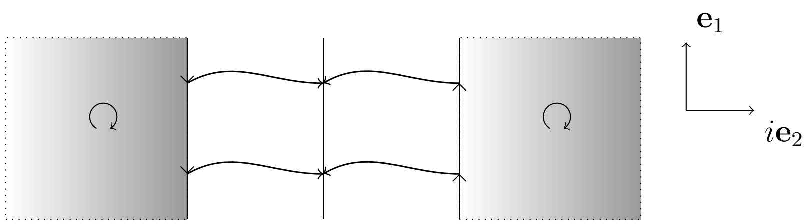

Figure 1.5: Deformation of PS to Euclid. The two halfplanes are the (deformed) attached surfaces and the cones are deformed into the vertical line.

Next we have to discuss the limit $\epsilon \to 0$ carefully.

First consider the PS part. Here, we may apply Fubini's theorem and perform the x integration first, since for $\epsilon > 0$ the integral is exponentially convergent. We define

$$I_{PS}(\epsilon, h) := \sqrt{\frac{\pi}{1 - h^2(1-\epsilon)^2}}\; e^{-\frac{(a_1 - a_2 h(1-\epsilon))^2}{1-h^2(1-\epsilon)^2}} \; .$$

Then we obtain

$$\lim_{\epsilon \to 0} \int_{DPS(\epsilon)} g(\vec{q}, \vec{a}) dq = \lim_{\epsilon \to 0} \int_{-1}^{1} dh \; I_{PS}(\epsilon, h) \; .$$

The limit $\lim_{\epsilon \to 0} I_{PS}(\epsilon, h)$ is uniform, since $(a_1 - a_2 h(1-\epsilon))^2 > 0$ for $1 \geq \epsilon \geq 0$ and $a_1 > a_2 \geq 0$. Thus the h integral and $\lim_{\epsilon \to 0}$ commute.

In particular, this is also true if we replace $I_{PS}(\epsilon, h)$ by

$$\tilde{I}_{PS}(\epsilon, h) := \sqrt{\frac{\pi}{1 - h^2(1-\epsilon)^2}} e^{-\frac{(a_1 - a_2 h)^2}{1-h^2(1-\epsilon)^2}} \; .$$

Then we have the following series of equalities:

$$\begin{aligned} \lim_{\epsilon \to 0} \int_{DPS(\epsilon)} g(\vec{q}, \vec{a}) dq &= \int_{-1}^{1} dh \; \lim_{\epsilon \to 0} I_{PS}(\epsilon, h) \\ &= \lim_{\epsilon \to 0} \int_{-1}^{1} dh \; \tilde{I}_{PS}(\epsilon, h) \\ &= \lim_{\epsilon' \to 0} \int_{PS} g(\vec{q}, \vec{a}) \chi_{\epsilon'}(q) dq \; , \end{aligned}$$

where we shift the ϵ dependence of the domain of integration to the integrand by introducing a regulating function $\chi_{\epsilon'}(q) = \exp(-\epsilon' q_2^2)$. To be more precise, we identify $\epsilon' = 2\epsilon - \epsilon^2$.

It remains to show that the contribution from the $hp_\pm$ parts vanish. This is done using similar arguments as above. Consider

$$\lim_{\epsilon \to 0} \int_{Dhp_\pm(\epsilon)} g(\vec{q}, \vec{a}) dq = \lim_{\epsilon \to 0} \int_{0}^{\infty} dh \; I_{hp_\pm}(\epsilon, h) \; ,$$

where we define

$$I_{hp_\pm}(\epsilon,h) := \sqrt{\frac{\pi}{1-(1-\epsilon)^2}}\; e^{-\frac{(\pm a_1-(1-\epsilon)a_2)^2}{1-(1-\epsilon)^2}}\; e^{-2h((1-\epsilon)a_1\mp a_2)} e^{-h^2(2\epsilon-\epsilon^2)} \; .$$

The last two factors ensure exponential convergence in h. Hence the integral exists and

$$\lim_{\epsilon\to 0} I_{hp_\pm}(\epsilon,h) = 0$$

holds uniformly in h. Thus we can conclude that

$$\lim_{\epsilon\to 0}\int_{Dhp_\pm(\epsilon)} g(\vec{q},\vec{a})dq = 0 \; .$$

The proof is finished, since we now have

$$\lim_{\epsilon\to 0}\int_{DePS(\epsilon)} g(\vec{q},\vec{a})dq = \lim_{\epsilon\to 0}\int_{PS} g(\vec{q},\vec{a})\chi_\epsilon(q)dq \; . \tag{1.10}$$

1.3 General setting and theorem

In this section we present our results in a rather general form. First, we describe the setting and then we state our theorem. Let us note in advance that appendix A contains a systematic discussion of the structures that are used to formulate the theorem and its proof.

All constructions take place in $\mathfrak{gl}(n,\mathbb{C})$, the space of all complex $n\times n$ matrices. The following results also apply to the case where $\mathfrak{gl}(n,\mathbb{C})$ is replaced by a complex Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$. Let $s\in\mathfrak{gl}(n,\mathbb{C})$ be hermitian with the property $s^2=\mathbb{1}$. s leads to two involutions[2] $\theta(X) = sXs^{-1}$ and $\gamma(X) = -sX^\dagger s^{-1}$ on $\mathfrak{gl}(n,\mathbb{C})$. In addition we assume that we are given some involutions τ_i on $\mathfrak{gl}(n,\mathbb{C})$, which commute with each other and with θ and γ. Then we can define a subspace $\mathcal{Q}$ of $\mathfrak{gl}(n,\mathbb{C})$ as

$$\mathcal{Q} = \{Q\in\mathfrak{gl}(n,\mathbb{C}) | Q = -\gamma(Q) \text{ and } \forall i \;:\; Q = \sigma_i\tau_i(Q)\} \; , \tag{1.11}$$

where $\sigma_i\in\{\pm 1\}$ and the τ_i have to be such that $s\in\mathcal{Q}$. The Lie algebra of the relevant symmetry group of $\mathcal{Q}$ is given by[3]

$$\mathfrak{g} = \{X\in\mathfrak{gl}(n,\mathbb{C}) | X = \gamma(X) \text{ and } \forall i \;:\; X = \tau_i(X)\} \; . \tag{1.12}$$

The decomposition of $\mathcal{Q}$ into the plus and minus one eigenspaces of θ gives a decomposition into the hermitian and antihermitian parts denoted by $\mathcal{Q}_+$ and $\mathcal{Q}_-$. Similarly θ gives the decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$, where $\mathfrak{k}$ is the plus

[2]In this context involution means an involutive automorphism of Lie algebra.
[3]See also section A.1.2 in appendix A.

one eigenspace and $\mathfrak{p}$ the minus one eigenspace. The commutation relations of these spaces are[4]

$$\begin{aligned} &[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}\ , && [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}\ , && [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}\ , && [\mathcal{Q}_+,\mathcal{Q}_-] \subset \mathfrak{p}\ , \\ &[\mathcal{Q}_\pm,\mathcal{Q}_\pm] \subset \mathfrak{k}\ , && [\mathfrak{k},\mathcal{Q}_\pm] \subset \mathcal{Q}_\pm\ , && [\mathfrak{p},\mathcal{Q}_\pm] \subset \mathcal{Q}_\mp\ . \end{aligned} \tag{1.13}$$

Then the parametrisation of the Pruisken-Schäfer domain is given by

$$\begin{aligned} PS : \mathfrak{p} \oplus \mathcal{Q}_+ &\to \mathcal{Q} \\ (Y,X) &\mapsto e^Y X e^{-Y}\ . \end{aligned} \tag{1.14}$$

The parametrisation of the Euclidean domain is given by

$$\begin{aligned} Euclid : \mathcal{Q}_- \oplus \mathcal{Q}_+ &\to \mathcal{Q}^{\mathbb{C}} \\ (Y,X) &\mapsto X + iY\ , \end{aligned}$$

where $\mathcal{Q}^{\mathbb{C}} = \mathcal{Q} \oplus i\mathcal{Q}$. The parametrisation of the Schäfer-Wegner domain is given by

$$SW :\ \mathfrak{p} \oplus \mathcal{Q}_+ \to \mathcal{Q}^{\mathbb{C}} \tag{1.15}$$

$$(Y,X) \mapsto X - ibe^Y s e^{-Y}\ , \tag{1.16}$$

where b is a positive real number. The orientation for PS, $Euclid$ and SW is provided by choosing an orientation of the domain of definition. This induces an orientation on the corresponding domain of integration.

Theorem 1.2. *If in the setting above $\mathfrak{g}$ is the direct sum[5] of a semisimple and an Abelian Lie algebra and $A \in \mathcal{Q}$ with $As > 0$, then*

$$\lim_{\epsilon \to 0} \int_{PS} e^{-\operatorname{Tr}(Q^2) - 2i\operatorname{Tr}(QA)} \chi_\epsilon(Q) dQ = c\, e^{-\operatorname{Tr}(A^2)} \tag{1.17}$$

holds. Here, $\chi_\epsilon(Q) = \exp(\frac{\epsilon}{4}\operatorname{Tr}(Q - \theta Q)^2)$ is a regulating function and dQ denotes a constant volume form on $\mathcal{Q}$. $c \in \mathbb{C} \setminus \{0\}$ is a constant that does not depend on A.

The result will be proved by showing that the PS domain can first be extended and then deformed into a standard Euclidean integration domain without changing the value of the integral. Therefore we view DQ as holomorphic $\dim\mathcal{Q}$ form on $\mathcal{Q}^{\mathbb{C}}$. In addition it will be shown that the SW domain can also be deformed into this Euclidean integral. Hence the PS and SW domains are contour deformations of the same simple Euclidean Gaussian domain.

The following corollary is the analogue of corollary 1 in [12]:

---

[4] See section A.1.3 in appendix A.

[5] This means in particular that the semisimple and the Abelian part commute with each other.

Corollary 1.1. *Let $\mathfrak{k}\oplus\mathcal{Q}_+$ be the direct sum of a semisimple and an Abelian Lie algebra and let $\mathfrak{h}$ be a maximal Abelian subalgebra of $\mathcal{Q}_+$. Furthermore let $\mathfrak{g}$ be semisimple and define $G = \exp(\mathfrak{p})\exp(\mathfrak{k})$*[6]*. Then*

$$\lim_{\epsilon\to 0}\int_{\mathfrak{h}}\left(\int_G e^{-2i\,\mathrm{Tr}(g\lambda g^{-1}A)}\chi_\epsilon(g\lambda g^{-1})|dg|\right)\, e^{-\,\mathrm{Tr}\,\lambda^2}J'(\lambda)|d\lambda| = \tilde{c}\; e^{-\,\mathrm{Tr}(A^2)}\ ,$$

holds, where

$$J'(\lambda) = \prod_{\alpha\in\Sigma_+(\mathfrak{p}\oplus\mathcal{Q}_-,\mathfrak{h})}\alpha(\lambda)^{d_\alpha}\prod_{\alpha\in\Sigma_+(\mathfrak{k}\oplus\mathcal{Q}_+,\mathfrak{h})}|\alpha(\lambda)^{d_\alpha}|\ .$$

$\Sigma_+(V,\mathfrak{h})$ denotes the sets of positive weights with respect to the adjoint action of $\mathfrak{h}$ with weight spaces in V.[7] *d_α are the dimensions of the weight spaces. $|dg|$ denotes Haar measure on G and $|d\lambda|$ denotes Lebesgue measure on the vector space $\mathfrak{h}$. $\tilde{c}\in\mathbb{C}\setminus\{0\}$ is a constant that does not depend on A.*

The following corollary is the analogue of theorem 1 in [12]:

Corollary 1.2. *If the parametrisation PS is nearly everywhere injective and regular, then*

$$\lim_{\epsilon\to 0}\int_{\mathrm{Im}PS} e^{-\,\mathrm{Tr}(Q^2)-2i\,\mathrm{Tr}(QA)}\;\chi_\epsilon(Q)\;\;\mathrm{sgn}(J'(\lambda))\;|dQ| = \tilde{c}' e^{-\,\mathrm{Tr}(A^2)}$$

holds. $\mathrm{Im}PS$ *denotes the image of PS and the mapping from* $\mathrm{Im}PS$ *to $\mathfrak{h}$ sending Q to λ is well defined up to a set of measure zero. $|dQ|$ denotes Lebesgue measure on $\mathcal{Q}$. $\tilde{c}'\in\mathbb{C}\setminus\{0\}$ is a constant that does not depend on A.*

1.4 Proof of the theorem

For simplicity we restrict ourselves to the case where $\mathfrak{g}$ is semisimple. The extension to the more general case is straightforward. The proof is divided into three parts. The first part, 1.4.1, contains the derivation of a new parametrisation of the PS domain, which makes its boundary *visible* in the domain of definition. Decomposing this parametrisation suitably as a sum of integration cells one obtains an adequate description of the boundary of the PS domain. The second part of the proof, 1.4.2, deals with the extension of the PS domain to a domain ePS without boundary. First we identify good directions into which the PS domain can be extended. Then an extension of PS that does not change the value of the integral is given. Finally in section 1.4.3 we construct a deformation $DePS$ of the extended PS domain to the Euclidean domain. The deformation satisfies $\partial DePS = Euclid{-}ePS$.

[6] G is the unique analytic subgroup of $GL(n,\mathbb{C})$ with Lie algebra $\mathfrak{g}$.

[7] See also section A.1.4 in appendix A.

Essentially we want to make rigorous the following schematic application of Stokes:

$$\begin{aligned}\int_{PS} g(Q,A)dQ &= \int_{ePS} g(Q,A)dQ \\ &= -\int_{DePS} \underbrace{d(g(Q,A)dQ)}_{=0} + \int_{Euclid} g(Q,A)dQ\ ,\end{aligned}$$

where we define $g(Q,A) := e^{-\operatorname{Tr}(Q^2)-2i\operatorname{Tr}(QA)}$. Note that the first term in the second line is identically zero since $g(Q,A)$ is holomorphic in Q. At this point a **warning** is in order: In this form the upper expressions do not make sense. In order for the integrals over PS, ePS and $DePS$ to exist we have to include some regularisation. This delicate issue is discussed in detail in the last part of section 1.4.3.

1.4.1 A suitable parametrisation of the PS domain

First, we perform a series of reparametrisations to derive a more convenient parametrisation which allows to use the geometric intuition from the two dimensional example for the PS domain and its boundary. Finally we decompose the parametrisation into different parts, allowing the application of Stokes' theorem. In the following we use standard results from Lie theory. A good reference is [14]. In addition appendix A gives a detailed description of the constructions we use.

Reparametrisation I: Decomposition of $\mathfrak{p}$

The goal of the next three reparametrisations is to evaluate $PS(Y,X) = \operatorname{Ad}(e^Y)X$ in more detail. Key to this is choosing a maximal Abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$, whose adjoint action on $\mathcal{Q} = \mathcal{Q}_+ \oplus \mathcal{Q}_-$ can be diagonalised simultaneously.[8] To begin with, we decompose the parameter space $\mathfrak{p}$ to see the algebra $\mathfrak{a}$. Therefore we define the compact group $K := \exp(\mathfrak{k})$ and the centraliser $Z_K(\mathfrak{a})$ of $\mathfrak{a}$ in K. Furthermore $\mathfrak{a}_+^o \subset \mathfrak{a}$ denotes the interior of a fixed Weyl chamber.[9] Consider the mapping

$$\begin{aligned}R_I : \mathfrak{a}_+^o \times K/Z_K(\mathfrak{a}) &\to \mathfrak{p} \\ (H,[k]) &\mapsto kHk^{-1}\ .\end{aligned}$$

R_I is obviously well defined. In appendix A.2.1 it is shown that R_I is injective and regular. Hence R_I is a diffeomorphism onto. Note, that $\mathfrak{p} \setminus \operatorname{Im}(R_I)$ is a set of measure zero since $\mathfrak{p} = \cup_{k\in K} k\mathfrak{a}k^{-1}$ [10] and $\operatorname{Im}(R_I) = \cup_{k\in K} k(\mathfrak{a}\setminus\cup_\alpha \ker\alpha)k^{-1}$ where α are the restricted roots with respect to $\mathfrak{a}$.[11]

[8] See section A.1.4 in appendix A.

[9] See [14] for a definition and properties of Weyl chambers.

[10] This standard result, which can be found in [14], is used without further explanation.

[11] See section A.1.4 in appendix A.

To be precise we want to use the parametrisation

$$\begin{aligned} PS \circ R_I : \mathfrak{a}_+ \times K/Z_K(\mathfrak{a}) \times \mathcal{Q}_+ &\to \mathcal{Q} \\ (H, [k], X) &\mapsto e^{kHk^{-1}} X e^{-kHk^{-1}} . \end{aligned}$$

The orientation of PS is given by an orientation of $\mathfrak{p} \oplus \mathcal{Q}_+$. Declaring R_I to be orientation preserving induces an orientation on $\mathfrak{a}_+^o \times K/Z_K(\mathfrak{a}) \times \mathcal{Q}_+$. To keep the notation simple we call each new parametrisation again PS.

Reparametrisation II: Twisting $K/Z_K(\mathfrak{a})$ and $\mathcal{Q}_+$

In this section we prepare further evaluation of the $\mathfrak{a}$ action in the next subsection. Consider the reparametrisation

$$\begin{aligned} R_{II} : K \times_{Z_K(\mathfrak{a})} \mathcal{Q}_+ &\to K/Z_K(\mathfrak{a}) \times \mathcal{Q}_+ \\ [kz^{-1}, zXz^{-1}] &\mapsto ([k], kXk^{-1}) . \end{aligned}$$

Note that $z \in Z_K(\mathfrak{a})$ in the expression $[kz^{-1}, zXz^{-1}]$ indicates group actions of $Z_K(\mathfrak{a})$ on K and on $\mathcal{Q}_+$. These group actions are used to define the bundle $K \times_{Z_K(\mathfrak{a})} \mathcal{Q}_+$. The inverse of R_{II} is given by

$$\begin{aligned} R_{II}^{-1} : K/Z_K(\mathfrak{a}) \times \mathcal{Q}_+ &\to K \times_{Z_K(\mathfrak{a})} \mathcal{Q}_+ \\ ([k], X) &\mapsto [k, k^{-1}Xk] . \end{aligned}$$

R_{II} is obviously a diffeomorphism and therefore can be used as a reparametrisation to obtain

$$\begin{aligned} PS \circ R_{II} : \mathfrak{a}_+^o \times K \times_{Z_K(\mathfrak{a})} \mathcal{Q}_+ &\to \mathcal{Q} \\ (H, [kz, z^{-1}Xz]) &\mapsto e^{kHk^{-1}} kXk^{-1} e^{-kHk^{-1}} \\ &= ke^H X e^{-H} k^{-1} \end{aligned}$$

as new parametrisation.

Reparametrisation III: Decomposition of $\mathcal{Q}_+$

The weight decomposition of $\mathcal{Q}$ with respect to the adjoint action of $\mathfrak{a}$ is given by

$$\mathcal{Q} = \mathcal{Q}_0 \oplus \bigoplus_{\alpha \in \Sigma_+(\mathcal{Q}, \mathfrak{a})} (\mathcal{Q}_\alpha \oplus \mathcal{Q}_{-\alpha}) .$$

Here $\Sigma_+(\mathcal{Q}, \mathfrak{a})$ denotes the set of posive weights. Defining

$$\mathcal{Q}_{+,\alpha} := \mathrm{Fix}_\theta(\mathcal{Q}_\alpha \oplus \mathcal{Q}_{-\alpha}) \ \text{ and } \ \mathcal{Q}_{+,0} = \mathrm{Fix}_\theta(\mathcal{Q}_0) ,$$

we obtain the decomposition

$$\mathcal{Q}_+ = \mathcal{Q}_{+,0} \oplus \bigoplus_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha} \ .$$

For more details and properties of this decomposition see appendix A.1.4.

Since $Z_K(\mathfrak{a})$ is a subset of K, and, by definition, commutes with the $\mathrm{ad}(\mathfrak{a})$ action on $\mathcal{Q}$, the decomposition is compatible with the bundle structure of $K \times_{Z_K(\mathfrak{a})} \mathcal{Q}_+$. Again, the reparametrisation

$$\begin{aligned} R_{III} : \mathfrak{a}_+ \times K \times_{Z_K(\mathfrak{a})} \Big(\mathcal{Q}_{+,0} \oplus \bigoplus_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha}\Big) &\to \ \mathfrak{a}_+ \times K \times_{Z_K(\mathfrak{a})} \mathcal{Q}_+ \\ (H, [kz, z^{-1}Mz, z^{-1}X_\alpha z]) &\mapsto \ \Big(H, \Big[kz, z^{-1}\Big(M + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} X_\alpha\Big)z\Big]\Big) \ , \end{aligned}$$

is an orientation preserving diffeomorphism.

We define a mapping $\phi : \mathcal{Q}_+ \to \mathcal{Q}_-$ implicitly through

$$[H, X_\alpha] = \alpha(H)\phi(X_\alpha) \ \text{ and } \ [H, \phi(X_\alpha)] = \alpha(H)X_\alpha \tag{1.18}$$

for all $H \in \mathfrak{a}$. See also appendix A.1.3. Using (1.18) a short calculation gives

$$e^{\mathrm{ad}(H)}X_\alpha = \cosh(\alpha(H))X_\alpha + \sinh(\alpha(H))\phi(X_\alpha) \ .$$

Moreover, the parametrisation can be rewritten to

$$\begin{aligned} &PS \circ R_{III} : \mathfrak{a}_+ \times K \times_{Z_K(\mathfrak{a})} \Big(\mathcal{Q}_{+,0} \oplus \bigoplus_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha}\Big) \to \mathcal{Q} \\ &(H, [k, M, X_\alpha]) \\ &\qquad \mapsto \mathrm{Ad}(k)\Big[M + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \big[\cosh(\alpha(H))X_\alpha + \sinh(\alpha(H))\phi(X_\alpha)\big]\Big] \ . \end{aligned}$$

In the following $PS \circ R_{III}$ is denoted simply by PS.

Reparametrisation IV: Transfer boundary to $\mathfrak{a}_+$

As a motivation for the next reparametrisation of $\mathcal{Q}_{+,\alpha}$ one might imagine the coordinate line belonging to $\lambda H \in \mathfrak{a}_+$ as a hyperbola. We want to have a simple Euclidean picture of the situation. Thus we change our parametrisation in a way that these coordinate lines are straight lines, see figure 1.6. Most importantly such a reparametrisation simplifies the view on the boundary of the PS domain, as is discussed in the next subsection.

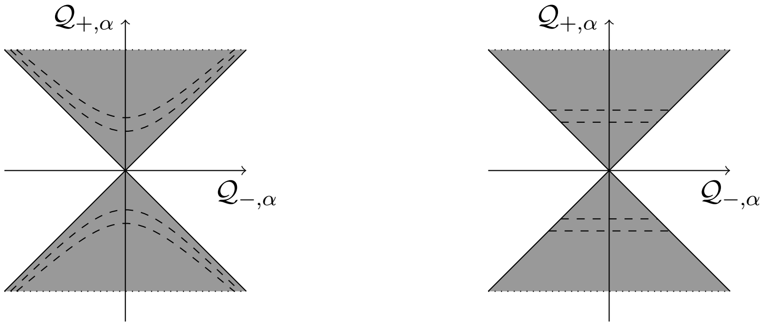

Figure 1.6: Motivation for the third reparametrisation step. The dashed lines are coordinate lines of $\lambda H \in \mathfrak{a}_+$.

Thus, the fourth reparametrisation we use is given by

$$\begin{aligned} R_{IV}:\ \mathfrak{a}_+ \times K \times_{Z_K(\mathfrak{a})} \Big(\mathcal{Q}_{+,0} \oplus \bigoplus_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha}\Big) \\ \to \mathfrak{a}_+ \times K \times_{Z_K(\mathfrak{a})} \Big(\mathcal{Q}_{+,0} \oplus \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha}\Big) \\ (H,[k,M,X_\alpha]) \mapsto \Big(H, \Big[k, M, \frac{1}{\cosh(\alpha(H))} X_\alpha\Big]\Big)\,, \end{aligned}$$

which is also an orientation preserving diffeomorphism. We obtain

$$\begin{aligned} PS \circ R_{IV} : \mathfrak{a}_+ \times K \times_{Z_K(\mathfrak{a})} \Big(\mathcal{Q}_{+,0} \oplus \bigoplus_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha}\Big) \to \mathcal{Q} \\ (H,[k,M,X_\alpha]) \mapsto \mathrm{Ad}(k)\Big[M + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \big[X_\alpha + \tanh(\alpha(H))\phi(X_\alpha)\big]\Big]\,, \end{aligned} \tag{1.19}$$

which we again call PS in the following.

Boundary of the PS domain

In this subsection we explain why parametrisation (1.19) is useful to get an intuition for the geometry and especially the boundary of the PS domain. This discussion is not meant to be rigorous but motivates the next steps of the proof.

The geometry can be described using $\mathrm{Tr}(XY^\dagger)$ as an $\mathrm{Ad}(K)$-invariant scalar product on $\mathcal{Q}$. Thus for the moment we forget about the $\mathrm{Ad}(K)$

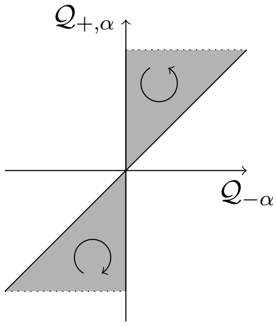

Figure 1.7: *Interior* of the PS domain. The rest is generated by the K action.

action and restrict ourselves to the inner part

$$iPS : \mathfrak{a}_+ \times \Big(\mathcal{Q}_{+,0} \oplus \bigoplus_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha}\Big) \to \mathcal{Q}$$

$$(H, M, X_\alpha) \mapsto M + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \big[X_\alpha + \tanh(\alpha(H))\phi(X_\alpha)\big] \ .$$

Note that the different $\mathcal{Q}_{+,\alpha}$ and also $\mathcal{Q}_{-,\alpha}$ are all orthogonal to each other. Hence we consider only one $\alpha \in \Sigma_+(\mathcal{Q}, \mathfrak{a})$ at a time:

$$X_\alpha + \tanh(\alpha(H))\phi(X_\alpha) \ ,$$

for which figure 1.7 is a good two dimensional picture. Except for the additional presence of the Weyl group, figure 1.7 is in agreement with figure 1.3 of the two dimensional example.

It is clear that the boundary is reached when some $\alpha(H)$ goes to $\pm\infty$ and hence tanh goes to ± 1. To put it differently, the boundary of the domain of integration can be reached through a limit in the parameter space $\mathfrak{a}_+$. Note that acting with K on the boundary, identified in the *inner* part of the PS parametrisation, the full boundary is generated.

Since we want to get rid of the boundary by attaching halfplanes to the PS domain, we want to give a parametrisation which reaches all boundary points. This implies performing the limit explicitly and thus making the boundary visible in the domain of definition.

Decomposition of the parametrisation PS

The problem we face is to perform the limit in $\mathfrak{a}_+$ in a well defined way. Therefore we have to discuss the weights $\alpha \in \Sigma_+(\mathcal{Q}, \mathfrak{a})$ in more detail.

First note that $\alpha(H)$ might change sign on $\mathfrak{a}_+$ since $\mathfrak{a}_+$ is defined with respect to the restricted roots $\beta \in \Sigma(\mathfrak{g}, \mathfrak{a})$. Within this subsection α will always denote a weight in $\Sigma_+(\mathcal{Q}, \mathfrak{a})$.

The main idea is to decompose the PS parametrisation by decomposing $\mathfrak{a}_+$ into different cones on which $\mathrm{sgn}(\alpha(H))$ stays constant or $\alpha(H)$ goes to zero for all $\alpha \in \Sigma_+(\mathcal{Q}, \mathfrak{a})$.

The closures of the connected components of $\mathfrak{a}_+ \setminus \cup_\alpha \ker(\alpha)$ are pointed polyhedral cones, whose edges lie in the intersections of hyperplanes defined by the kernels of the weights in $\Sigma_+(\mathcal{Q}, \mathfrak{a})$. Let us consider one of these pointed cones. It can also be defined as an intersection of halfspaces or as non-negative linear combination of some generators $H_i \in \mathfrak{a}$. The generators are the edges of the cone. In general the number of generators might be greater than $n := \dim \mathfrak{a}$. But each pointed cone can be triangulated (without introducing new vertices) into simplicial cones, i.e., cones where the number of generators equals $\dim \mathfrak{a}$. See for example [13]. Now we fix triangulations for each original pointed polyhedral cone. Thus we obtain a decomposition of $\mathfrak{a}_+$ into simplicial cones, which we denote by $\mathfrak{a}_{+,c}$ and

$$\mathfrak{a}_+ = \bigcup_{c \in C} \mathfrak{a}_{+,c} \ , \tag{1.20}$$

where C is an index set for the different cones. Denote the generators/edges of $\mathfrak{a}_{+,c}$ by $H_{i,c}$. Then we can represent $H \in \mathfrak{a}_{+,c}$ uniquely as

$$H = \sum_{i=1}^{n} h^i H_{i,c} \tag{1.21}$$

with coefficients $h^i \in \mathbb{R}^+$. The important thing is that the sign of all α on a given simplicial cone stays constant. However it is still allowed that α vanishes at the boundary of the simplicial cone. Thus we decompose our parametrisation as follows:

$$PS = \sum_{c \in C} PS|_{\mathfrak{a}_{+,c} \times K \times_{Z_K(\mathfrak{a})} (\mathcal{Q}_{+,0} \oplus \bigoplus_{\alpha \in \Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha})} \ .$$

In the following we hide the index c of $H_{i,c}$ and use the notation $H = \sum_i h^i H_i$ for $H \in \mathfrak{a}_+$ which implies the choice of H_i as described above. By construction we then have $h^i \geq 0$.

Reparametrisation V: Making the boundary visible

To make the boundary visible in the domain of definition we define

$$\mathfrak{a}_{B,c} = \left\{ \sum_{i=1}^{n \equiv \dim \mathfrak{a}} h^i H_i \in \mathfrak{a}_{+,c} | \forall i \ : \ 0 \leq h^i \leq 1 \right\} \ .$$

and use the reparametrisation

$$\begin{aligned} R_{V,c} : \ \mathfrak{a}^o_{B,c} &\to \mathfrak{a}_{+,c} \\ \sum_{i=1}^{n} h^i H_i &\mapsto \sum_{i=1}^{n} \frac{h^i}{1-h^i} H_i \ , \end{aligned}$$

which is an (orientation preserving) diffeomorphism onto for each simplicial cone. The reparametrisation is visualised in figure 1.8. Note that there is an obvious diffeomorphism between $[0,1]^n$ and $\mathfrak{a}_{B,c}$. Defining for each cone $c \in C$

$$\begin{aligned} PS_c : [0,1]^n \times K \times_{Z_K(\mathfrak{a})} \Big(\mathcal{Q}_{+,0} \oplus \bigoplus_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha}\Big) &\to \mathcal{Q} \\ (h^i, [k, M, X_\alpha]) &\mapsto \\ \mathrm{Ad}(k)\Big[M + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \Big(X_\alpha + \tanh\Big(\sum_i \frac{h^i}{1-h^i}\alpha(H_i)\Big)\phi(X_\alpha)\Big)\Big] \,, \end{aligned} \tag{1.22}$$

we have $PS = \sum_{c\in C} PS_c$ as an equation of integration chains. It is important to keep in mind that $[0,1]^n$ is essentially $\mathfrak{a}_{B,c}$, which can be seen as a truncated cone in $\mathfrak{a}_+$.

In the following we want to give the notion 'boundary of the PS domain' a precise meaning. For integration cells, i.e., differentiable mappings defined on a cube, the boundary operator ∂ is defined as usual. ∂ can also be applied to integration chains, i.e. formal linear combinations of cells. In principle we would have to decompose each PS_c into cells to apply ∂. In the following we argue that we can treat each PS_c effectively as cell with the boundary operator ∂ acting just on the $[0,1]^n$ part of the domain of definition.

First we have to show that PS_c can be extended to a neighbourhood of $[0,1]$ on which it is still differentiable. This property is included in the definition of an integration cell. It is needed to define the orientation of the boundary. Let us first concentrate on the tanh term in the PS_c parametrisation. Since

$$\lim_{h^j\to 1} \partial_{h^j} \tanh\Big[\sum_{i=1}^n \frac{h^i}{1-h^i}\alpha(H_i)\Big] = 0\,, \tag{1.23}$$

generalises to all higher (and mixed) partial derivatives, we extend PS_c for $h^i > 1$ by setting it constant in that direction. To be more precise we define for $h^i > 0$

$$PS_c(\dots, h^i, \dots) := PS_c(\dots, 1, \dots)\ .$$

For $h^i < 0$ just analytically continue the parametrisation PS_c. Using (1.23) it can be easily shown that the extension of PS_c to a neighbourhood of $[0,1]^n$ is differentiable.

Now we present an argument that we can treat each PS_c effectively as cell and that it is enough to let the boundary operator ∂ act on the $[0,1]^n$ part of the domain of definition. Since K is a closed compact manifold it suffices to discuss boundary contributions arising from a decomposition of $\mathcal{Q}_{+,0} \oplus \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha}$ into cells. Inspecting our parametrisation we see

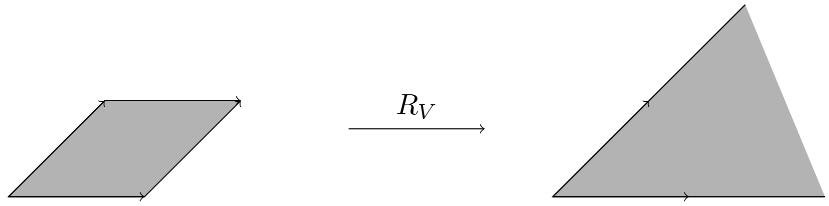

Figure 1.8: $su(2,2)$ example for R_V mapping $\mathfrak{a}_B$ on the left hand side to $\mathfrak{a}_+$ on the right side. In this example there is only one simplicial cone, i.e. $|C| = 1$.

that going to infinity in the domain of definition implies going to infinity in the domain of integration. In section 1.4.3 we show that the integrand converges exponentially on this domain and hence all possible boundary contributions vanish.

Only the part $[0,1]^n$, or equivalently, the part $\mathfrak{a}_{B,c}$ of the domain of definition, leads to a nontrivial boundary. The boundary parts coinciding with boundaries of $\mathfrak{a}_+$ are of codimension at least two, and thus do not contribute. For a detailed argument concerning this point see appendix A.2.3.

1.4.2 Extending the PS domain

In this section we construct an extension of the PS domain that has no relevant boundary. This means we have to attach additional domains to the boundary of the PS domain. The idea is to attach a halfline to each boundary point. The direction of this halfline should be a convergent one, and it should also guarantee that the attached domain does not contribute to the integral when integrated against $f(Q)\, dQ$. First we determine such a direction, and then give a parametrisation of the attached domains. In the following it is often convenient to view $B(X,Y) := \mathrm{Tr}(XY)$ as a bilinear form on $\mathcal{Q}^{\mathbb{C}}$.

Good directions

We want to extend the PS domain from the boundary into an imaginary null directions E_i. The terminology 'null direction' conveys two things. First E_i is a null direction in the sense that $B(E_i, E_i) = 0$. And second the extension in this direction does not contribute to the integral since the volume form vanishes for these directions. E_i shall also be a convergent direction. It is reasonable to expect that the term $\exp(-2iB(Q,A))$ guarantees convergence in this situation if

$$\Re\,[iB(\mathrm{Ad}(k)E_i, A)] = \Re[i\,\mathrm{Tr}(s^{-1}\,\mathrm{Ad}(k)E_i A s)] > 0\;. \tag{1.24}$$

In order for (1.24) to hold it suffices that $is\mathrm{Ad}(k)E_i \geq 0$ and $E_i \neq 0$, since $As > 0$ by assumption. For our definition of E_i below it is important that for $Y \in \mathfrak{p}$

$$s^{-1}e^{Y}se^{-Y} = e^{-2Y} > 0$$

holds. The natural choice for E_i is

$$E_i := -2i \lim_{t\to\infty} \frac{\mathrm{Ad}(e^{tH_i})s}{\max_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} e^{|\alpha(tH_i)|}} \ . \tag{1.25}$$

Note that $Z_K(\mathfrak{a})$ acts trivially on E_i. Let us also mention again that we have hidden the dependence of H_i on $c \in C$. Thus E_i also depends on $c \in C$.

It is instructive to give a more explicit form of E_i. Since $s \in \mathcal{Q}_+$ we have the decomposition

$$s = M_s + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} X_{s,\alpha} \ , \tag{1.26}$$

where $M_s \in \mathcal{Q}_{+,0}$ and $X_{s,\alpha} \in \mathcal{Q}_{+,\alpha}$. A short calculation gives

$$\mathrm{Ad}(e^{H})s = M_s + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \cosh(\alpha(H))X_{s,\alpha} + \sinh(\alpha(H))\phi(X_{s,\alpha}) \ .$$

This shows that the limit in (1.25) exists. In addition $E_i \neq 0$ because $\mathfrak{a} \to \mathcal{Q}_-, H \mapsto [H, s]$ is injective. The properties of this mapping are discussed in detail in appendix A.1.3. Note also that E_i depends on the chosen simplicial cone c, and that $B(E_i, E_i) = 0$. Most importantly inequality (1.24) holds (even without taking the real part).

$is^{-1}E_i$ can be regarded as an orthogonal projection on the weight space (in the vector space on to which $\mathfrak{g}$ acts) with the largest eigenvalue with respect to the e^{H_i} action.

Parametrisation of the extension

In this section we suggest an extension of the PS domain and then check that it has all the desired properties. The extended PS domain (ePS) is parametrised by

$$\begin{aligned} ePS : \mathfrak{a}_+ \times K \times_{Z_K(\mathfrak{a})} \Big(\mathcal{Q}_{+,0} \oplus \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha}\Big) &\to \mathcal{Q}^{\mathbb{C}} \\ (H, [k, M, X_\alpha]) &\mapsto \\ \mathrm{Ad}(k)\Big[M + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \Big[X_\alpha &+ \tanh\Big(\sum_j \xi(h^j)\alpha(H_j)\Big)\phi(X_\alpha)\Big] \\ &+ \sum_{j=1}^{n} \Theta(h^j - 1)(h^j - 1)E_j\Big] \ , \end{aligned}$$

where we use

$$\xi(h) := \begin{cases} \frac{h}{1-h} & h < 1 \\ \infty & h \geq 1 \end{cases}$$

Θ is the step function. The attached surface starts as soon as some $h^i > 1$.

The limit contained in the expression $\tanh(\sum_j \xi(h^j)\alpha(H_j))$ is well defined because $\operatorname{sgn}(\alpha(H_j))$ is the same for all i with $\alpha(H_i) \neq 0$. Note that the mapping ePS is well defined since $Z_K(\mathfrak{a})$ acts trivially on the E_i.

In the following we decompose the parametrisation ePS into different pieces, which can be treated as integration cells. Remember that we already have the corresponding decomposition $PS = \sum_{c \in C} PS_c$. For ePS the situation is slightly more involved as we have to account explicitly for the limit being taken, i.e., which h^i are larger than one. The different possibilities are characterised by subsets $L \subset \{1, 2, \ldots, n = \dim \mathfrak{a}\}$. The extended parametrisation decomposes into several pieces $ePS_{L,c}$. The situation is visualised in figure 1.9. To be able to write down $ePS_{L,c}$ explicitly we define

$$\Sigma_{L,\neq} := \{\alpha \in \Sigma_+(\mathcal{Q}, \mathfrak{a}) | \exists i \in L \,:\, \alpha(H_i) \neq 0\}$$

and

$$\Sigma_{L,=} := \{\alpha \in \Sigma_+(\mathcal{Q}, \mathfrak{a}) | \forall i \in L \,:\, \alpha(H_i) = 0\} \,.$$

Then we have for each $L \subset \{1, 2, \ldots, n = \dim\mathfrak{a}\}$ and $c \in C$

$$\begin{aligned}
&ePS_{L,c} : [0,1]^{n-|L|} \times [1,\infty)^{|L|} \times K \times_{Z_K(\mathfrak{a})} \Big(\mathcal{Q}_{+,0} \oplus \sum_{\alpha \in \Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha}\Big) \to \mathcal{Q}^{\mathbb{C}} \\
&\quad (h^i, [k, M, X_\alpha]) \mapsto \\
&\quad \mathrm{Ad}(k)\Big[M + \sum_{\alpha \in \Sigma_{L,\neq}} (X_\alpha + \operatorname{sgn}(\alpha(H_i))\phi(X_\alpha)) \\
&\qquad + \sum_{\alpha \in \Sigma_{L,=}} \Big[X_\alpha + \tanh\Big(\sum_{j=1}^n \xi(h^j)\alpha(H_j)\Big)\phi(X_\alpha)\Big] \\
&\qquad + \sum_{j \in L} (h^j - 1)E_j\Big] \,.
\end{aligned}$$

Next we present an argument that $ePS_{L,c}$ can be treated as an integration cell and that the boundary operator ∂ only acts on $[0,1]^{n-|L|} \times [1,\infty)^{|L|}$. First we discuss the extension of $ePS_{L,c}$ to a neighbourhood of $[0,1]^{n-|L|} \times [1,\infty)^{|L|}$. For $i \notin L$ and $h^i > 0$ we define

$$ePS_{L,c}(\ldots, h^i, \ldots) := ePS_{L,c}(\ldots, 1, \ldots) \,,$$

and for $h^i < 0$ or $i \in L$ and $h^i < 1$ we analytically continue the parametrisation $ePS_{L,c}$. The boundary operator ∂ applied to $ePS_{L,c}$ is evaluated using

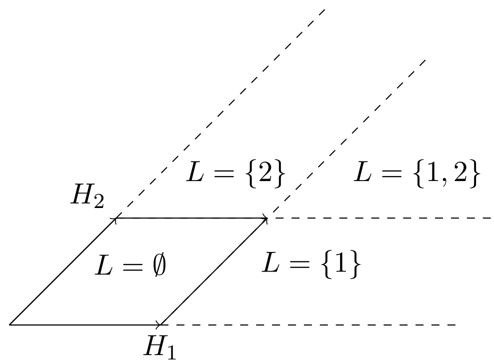

Figure 1.9: This figure shows $\mathfrak{a}_+$ for $\mathfrak{g} = \mathfrak{su}(2,2)$. $\mathfrak{a}_+$ is decomposed into the domains of definition for the different mappings $ePS_{L,c}$ in $\mathfrak{a}_+$ with $L \subset \{1, 2\}$.

exactly the same reasoning as for the PS_c mappings. Thus ∂ acts only on $[0, 1]^{n-|L|} \times [1, \infty)^{|L|}$. To see that

$$\partial ePS = \sum_{c \in C} \sum_{L \subset \{1,2,\ldots,n\}} \partial ePS_{L,c} = 0$$

holds, note that the different integration cells $ePS_{L,c}$ fit together by definition, i.e., the induced orientation on the boundaries between two neighbouring cells is just opposite. In section A.2.3 in appendix A it is shown that the contributions from $\partial \mathfrak{a}_+$ are of codimension at least two. Thus a contribution can only come from h^i going to infinity, but the integral is convergent and hence these terms do not contribute either.

To get some intuition for the situation it is useful to note that the halflines which are glued to boundary points of the PS domain point into a direction within $\oplus_{\alpha \in \Sigma(\mathcal{Q}, \mathfrak{a})} i\mathcal{Q}_\alpha$, and hence cannot coincide with tangent vectors to PS which live in $\mathcal{Q}$.

Extensions are nullsurfaces

Now we want to motivate that the extension does not contribute to the integral. The following argument is not rigorous as it does not refer to existence and convergence of the integrals involved.

The holomorphic volume form gives zero if two linearly dependent (over $\mathbb{C}$) vectors are inserted. Thus it is enough to show that we can find two tangent vectors of the extension which are linearly dependent over $\mathbb{C}$. Let $h^j > 1$, then one tangent vector is $\mathrm{Ad}(k)E_j(H)$. The latter can be expanded as follows:

$$E_j = -i \sum_{\alpha \in \Sigma_{\{j\},\neq}} e_j^\alpha (X_{s,\alpha} + \mathrm{sgn}(\alpha(H_j))\phi(X_{s,\alpha})) \ , \qquad (1.27)$$

where $e_j^\alpha \in \{0,1\}$. Within the boundary parametrisation the following terms are contained:

$$\mathrm{Ad}(k) \sum_{\alpha \in \Sigma_{\{j\},\neq}} (X_\alpha + \mathrm{sgn}(\alpha(H_j))\, \phi(X_\alpha)) \ .$$

It is clear that by differentiation in $\sum_{\alpha \in \Sigma_+(\mathcal{Q}_+,\mathfrak{a})} \mathcal{Q}_{+,\alpha}$ in the direction of $\sum_{\alpha \in \Sigma_{\{i\},\neq}} e_i^\alpha X_{s,\alpha}$ we obtain a tangent vector parallel (over $\mathbb{C}$) to $\mathrm{Ad}(k)E_i$. This implies that the volume form vanishes, and that the extension of PS does not contribute to the integral. This argument does not incorporate the function which is integrated, but only on the integration chain and the volume form.

1.4.3 Equivalence of PS and $Euclid$

Finally we want to show that the integral over PS equals the integral over $Euclid$. First we give a deformation of ePS into $Euclid$. Then we apply Stokes' theorem. To get the desired equation we first show the existence of the appearing integrals with regularisation $\epsilon > 0$. A careful discussion of the limit ϵ going to zero yields the theorem.

Deformation of ePS into $Euclid$

The idea is to deform ePS into the subspace $\mathcal{Q}_+ \oplus i[\mathfrak{p}, s]$ of $\mathcal{Q}^\mathbb{C}$ where B is positive definite. Note that in appendix A.1.3 it is shown that $[\mathfrak{p}, s] = \mathcal{Q}_-$. Along the deformation we have to show that the integral remains convergent, so that no boundary terms at infinity are generated.

The deformation is given by

$$\begin{aligned} DePS : [0,1] \times \mathfrak{a}_+ \times K \times_{Z_K(\mathfrak{a})} \Big(\mathcal{Q}_{+,0} \oplus \sum_{\alpha \in \Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha} \Big) &\to \mathcal{Q}^\mathbb{C} \ , \\ (t, H, [k, M, X_\alpha]) \mapsto & \\ \underbrace{\mathrm{Ad}(k) \Big[M + \sum_{\alpha \in \Sigma_+(\mathcal{Q},\mathfrak{a})} \Big(X_\alpha + (1-t) \tanh \Big(\sum_{j=1}^n \xi(h^j) \alpha(H_j) \Big) \phi(X_\alpha) \Big) \Big]}_{\Xi} & \\ + \underbrace{\mathrm{Ad}(k) \Big[\sum_{j=1}^n \Theta(h^j - 1)(h^j - 1)(E_j - t\theta(E_j) + t\Delta X_{s,j}) \Big]}_{\Upsilon} \ , & \qquad (1.28) \end{aligned}$$

where we use

$$\Delta X_{s,i} := -2 \sum_{\alpha \in \Sigma_+(\mathcal{Q},\mathfrak{a})} \alpha(H_i) i\phi(X_{s,\alpha}) - (E_i - \theta(E_i)) \ .$$

Note also that $DePS$ is well defined, since the involution θ commutes with the action of the centraliser $Z_K(\mathfrak{a})$. To proceed we decompose $DePS$ similarly as ePS:

$$\begin{aligned} DePS_{L,c} : [0,1]^{1+n-|L|} \times [1,\infty)^{|L|} \times K \times_{Z_K(\mathfrak{a})} \Big(\mathcal{Q}_{+,0} \oplus \sum_{\alpha \in \Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha} \Big) \to \mathcal{Q}^{\mathbb{C}} \\ (t, h^i, [k, M, X_\alpha]) \mapsto \mathrm{Ad}(k) \Big[M \\ + \sum_{\alpha \in \Sigma_{L,\neq}} (X_\alpha + (1-t) \operatorname{sgn}(\alpha(H_i)) \phi(X_\alpha)) \\ + \sum_{\alpha \in \Sigma_{L,=}} \Big[X_\alpha + (1-t) \tanh \Big(\sum_j \xi(h^j) \alpha(H_j) \Big) \phi(X_\alpha) \Big] \\ + \sum_{j \in L} (h^j - 1)(E_j - t\theta(E_j) + t\Delta X_{s,j}) \Big] \ . \end{aligned}$$

Using similar reasoning as for $ePS_{L,c}$ each parametrisation $DePS_{L,c}$ can be seen as integration cell with ∂ acting only on the $[0,1]^{1+n-|L|} \times [1,\infty)^{|L|}$ part. In particular we have

$$\partial DePS = \sum_{c \in C} \sum_{L \subset \{1,\ldots,n\}} \partial DePS_{L,c} = DePS(t=1) - DePS(t=0) \ ,$$

where we used in the last equality that the contributions from boundaries of $\mathfrak{a}_+$ vanish.[12]

Inspecting our previous arguments it is clear that for $t \in [0,1)$, $DePS_{L,c}(t)$ are integration chains with the same properties as $ePS_{L,c}$. For $t=1$ and $L \neq \{1,\ldots,n\}$ the parametrisation $DePS(1)$ degenerates, i.e., for $i \notin L$ we have $\partial_{h^i} DePS(1) = 0$. Hence we have the following equation for integration chains:

$$DePS(1) = \sum_{c \in C} DePS_{\{1,\ldots,n\},c}(1) \ .$$

In the following we establish the connection between $DePS(1)$ and $Euclid$. Therefore it is useful to introduce the mapping

$$\begin{aligned} \psi : \mathbb{R}_+^n \to [1,\infty)^n \\ (\ldots, h^i, \ldots) \mapsto (\ldots, h^i + 1, \ldots) \ . \end{aligned}$$

[12] See section A.2.3 in appendix A.

The precise statement we show is:

$$Euclid = \sum_{c\in C} DePS_{\{1,\dots,n\},c}(1) \circ (\psi, id) \circ R_{III}^{-1} \circ R_{II}^{-1} \circ R_{I}^{-1} \ ,$$

where id denotes the identity on $K \times_{Z_K(\mathfrak{a})} (\mathcal{Q}_{+,0} \oplus \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+,\alpha})$. Since $[\mathfrak{p}, s] = \mathcal{Q}_-$, the following calculation suffices:

$$\begin{aligned} DePS(1,\psi(H),[k,M,X_\alpha]) &= \mathrm{Ad}(k)\Big(M + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} X_\alpha - i \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a}),i} 2h^i\alpha(H_i)\phi(X_{s,\alpha})\Big) \\ &= \mathrm{Ad}(k)\Big(M + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} X_\alpha - 2i \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} [H, X_{s,\alpha}]\Big) \\ &= \mathrm{Ad}(k)\Big(M + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} X_\alpha\Big) - 2i\,\mathrm{Ad}(k)[H,s] \\ &= \mathrm{Ad}(k)\Big(M + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} X_\alpha\Big) - 2i[\mathrm{Ad}(k)H,s] \\ &= X - 2i[Y,s] \ . \end{aligned}$$

In the third equality we use equation (1.26) and in the last line the reparametrisations R_{III}, R_{II} and R_I are understood to be undone.

Application of Stokes' theorem

In our application of Stokes' theorem we essentially want to use $\partial DePS = Euclid - ePS$. To that end, note that the boundary operator ∂ acts only on the $[0,1]^{1+n-|L|} \times [1,\infty)^{|L|}$ part of the domain of definition of $DePS_{L,c}$. The boundary parts coinciding with boundaries of $\mathfrak{a}_+$ are again of codimension two and do not contribute (see also appendix A.2.3). The only nonvanishing contribution comes from $t = 0$ (ePS) and $t = 1$ ($Euclid$). Since we want to postpone issues of convergence to the next two subsections, we use $\partial DePS^\epsilon = Euclid - DePS(\epsilon)$, where $DePS^\epsilon := DePS|_{t\in[\epsilon,1]}$. The idea is to deform Euclid into PS as far as convergence of the integral allows, i.e. we apply Stokes in the following way:

$$\begin{aligned} \int_{DePS(\epsilon)} g(Q,A)dQ &= -\int_{DePS^\epsilon} \underbrace{d(g(Q,A)dQ)}_{=0} + \int_{Euclid} g(Q,A)dQ \\ &= \int_{Euclid} g(Q,A)dQ \ . \end{aligned}$$

Existence of the integral for $\epsilon > 0$

In this subsection we show that the integral

$$\int_{DePS(\epsilon)} g(Q,A)dQ$$

exists for $1 \geq \epsilon > 0$. The limit $\epsilon \to 0$ requires more care and is discussed in the next subsection. Thus we have to evaluate the terms in the exponent of $g(Q, A)$ in more detail. To this end we note some useful relations that are derived in section A.1.5 in appendix A. For $X_\alpha \in \mathcal{Q}_{+,\alpha}$, $X, X' \in \bigoplus_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{+\alpha}$ hold:

$$B(X_\alpha, X_\beta) = -B(\phi(X_\alpha), \phi(X_\beta)) = \delta_{\alpha,\beta} B(X_\alpha, X_\alpha) \;, \tag{1.29}$$

$$B(X, X') = -B(\phi(X), \phi(X')) \;, \tag{1.30}$$

$$B(X_\alpha \pm \phi(X_\alpha), X_\beta \pm \phi(X_\beta)) = 0 \;, \tag{1.31}$$

$$B(X_\alpha + \phi(X_\alpha), X_\beta - \phi(X_\beta)) = \delta_{\alpha,\beta} 2B(X_\alpha, X_\alpha) \;. \tag{1.32}$$

For $\epsilon > 0$ it suffices to discuss the $B(Q, Q)$ term. Referring to (1.28), $B(\Xi, \Xi)$ can be rewritten to

$$\begin{aligned}
&-B(\Xi, \Xi) \\
&\quad = \mathrm{Tr}(M^2) + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathrm{Tr}\left[\left(X_\alpha + (1-\epsilon)\tanh\Big[\sum_j \xi(h^j)\alpha(H_j)\Big]\phi(X_\alpha)\right)^2\right] \\
&\quad \underset{(1.29)}{=} B(M, M) + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} B(X_\alpha, X_\alpha)\left(1 - (1-\epsilon)^2 \tanh^2\Big[\sum_j \xi(h^j)\alpha(H_j)\Big]\right) .
\end{aligned}$$

For $h^i \in [0, 1]$ this term guarantees convergence of the integral. The cross term $B(\Xi, \Upsilon)$ is imaginary and therefore leads only to a phase factor.

Before we turn to $B(\Upsilon, \Upsilon)$ let us make two observations. First note that $B(E_i, E_j) = 0$, which can be seen from

$$\begin{aligned}
&\sum_{\alpha\in\Sigma_{\{j\},\{i\},\neq}} e_i^\alpha e_j^\alpha B(X_{s,\alpha} + \mathrm{sgn}(\alpha(H_i))\phi(X_{s,\alpha}), X_{s,\alpha} + \mathrm{sgn}(\alpha(H_j))\phi(X_{s,\alpha})) \\
&\qquad \underset{(1.31)}{=} 0 \;,
\end{aligned} \tag{1.33}$$

where we define $\Sigma_{\{j\},\{i\},\neq} := \Sigma_{\{i\},\neq} \cap \Sigma_{\{j\},\neq}$. For the equality (1.33) we use $\mathrm{sgn}(\alpha(H_i)) = \mathrm{sgn}(\alpha(H_j))$. Second we have that

$$E_j - \epsilon\theta(E_j) + \epsilon\Delta X_{s,j} = (1-\epsilon)E_j - \epsilon \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \alpha(H_j) i\phi(X_{s,\alpha})$$

holds. Since $B(\cdot, \cdot)$ is negative definite on on $\mathcal{Q}_-$, the following computation is enough to show $B(\Upsilon, \Upsilon) > 0$ for $\epsilon \in (0, 1]$ and thus convergence in the h^i directions for all $X_\alpha \in \mathcal{Q}_{+,\alpha}$:

$$\begin{aligned}
&B\Big(-i\sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \alpha(H_j)\phi(X_{s,\alpha}), E_i\Big) \\
&\qquad \underset{(1.27)}{=} -\sum_{\alpha,\beta\in\Sigma_+(\mathcal{Q},\mathfrak{a})} e_i^\beta \,\mathrm{sgn}(\beta(H_i))\alpha(H_j) B(\phi(X_{s,\alpha}), \phi(X_{s,\beta})) \\
&\qquad = -\sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} e_i^\alpha |\alpha(H_j)| B(\phi(X_{s,\alpha}), \phi(X_{s,\alpha})) > 0 \;.
\end{aligned}$$

Thus we conclude that the integral over $DePS(\epsilon)$ exists for $\epsilon > 0$.

Existence of $\lim_{\epsilon\to 0}$

We first show for $|L| > 0$ that

$$\lim_{\epsilon\to 0}\int_{DePS_{L,c}(\epsilon)} g(Q,A)dQ = 0\ . \tag{1.34}$$

The reason for this result is a very general one. The convergence of the integral at the boundary of PS is brought about by an oscillatory term along the boundary. In our case, integrating along the boundary lines yields essentially a regularised delta distribution. Our parametrisation is well suited to show this mechanism explicitly. Using $DePS(\epsilon)$, we pull back dQ to $\mathfrak{a}_+ \times K \times_{Z_K(\mathfrak{a})} \mathcal{Q}_+$. Choosing an appropriate set of charts and a decomposition of unity it is enough for our argument to consider integrals over finitely many subsets $\mathfrak{a}_+ \times U_i \times \mathcal{Q}_+$, with $U_i \subset K/Z_K(\mathfrak{a})$. Here, it is important that $K/Z_K(\mathfrak{a})$ is compact. Then the integration is decomposed into an outer part over $\mathfrak{a}_+ \times K/Z_K(\mathfrak{a})$, and an inner part over $\mathcal{Q}_+$. This is possible since the integrals are exponentially convergent ($\epsilon > 0$), and Fubini's theorem can be applied. The $\mathcal{Q}_+$ integrations are essentially Gaussian times a polynomial coming from the Jacobian. We show that it is possible to perform the limit $\epsilon \to 0$ after doing the inner Gaussian integrations. In the following Einstein's summation convention is in place. Schematically, the Gaussian integrations over $\mathcal{Q}_+$ for each simplicial cone $c \in C$ are of the form

$$I_{L,c}(\epsilon, H, [k]) := e^{-h^i\tilde{g}_i}\int\limits_{\mathbb{R}^{\dim\mathcal{Q}_+}}\prod_l dx^l\ e^{-(x^l)^2 f_l - 2ix^l g_l}\ P\ ,$$

where $\tilde{g}_i(\epsilon,[k])$, $f_l(\epsilon, h^i, [k])$ and $g_l(\epsilon, h^i, [k])$ are functions of $[k] \in K/Z_K(\mathfrak{a})$, ϵ and $H = h^iH_i \in \mathfrak{a}_+$ to be specified later on. P is a polynomial in ϵ, x^l, h^i,$[k]$ and $\partial^r \tanh(\xi(h^i)\alpha(H_i))$, where ∂^r represents arbitrary partial derivatives with respect to h_i. Note that $i = 1,\dots,\dim\mathfrak{a}$ and $l = 1,\dots,\dim\mathcal{Q}_+$. Now it is possible to introduce source terms, and to perform the integral

$$\begin{aligned}
I_{L,c}(\epsilon, H, [k]) &= e^{-h^i\tilde{g}_i}\ P'(\partial_{j_l}|_{j_l=0},\dots)\int dx^l e^{-f_l(x^l)^2-2ix^l(g_l+j_l)} \\
&= e^{-h^i\tilde{g}_i}\ P''(\partial_{j_l}|_{j_l=0},\dots)\prod_l\sqrt{\frac{\pi}{f_l}}e^{-\frac{(g_l+j_l)^2}{f_l}} \\
&= e^{-h^i\tilde{g}_i}\ P'''(\frac{1}{f_l},\dots)\prod_l\sqrt{\frac{\pi}{f_l}}e^{-\frac{g_l^2}{f_l}}\ ,
\end{aligned}\tag{1.35}$$

where primes just indicate that these are different polynomials, and the dots represent a dependence on ϵ, h^i,$[k]$ and $\partial^r \tanh(\xi(h^i)\alpha(H_i))$. We will show that for $\epsilon = 0$ we have $f_1 = 0$ and $g_1 \neq 0$. Furthermore we show that $f_l \geq 0$ and $g_l \in \mathbb{R}$ for all l. Then the exponential dominates the polynomial, and (1.35) is zero for $\epsilon = 0$. In addition we show that $\tilde{g}_i > 0$, and hence the remaining integrals over $\mathfrak{a}_+$ are convergent.

Thus the issue of convergence is reduced to a discussion of the functions $\tilde{g}_i$, f_l and g_l. $\tilde{g}_i$ is read off from

$$2iB(\Upsilon, A) = (h^i - 1 + \epsilon)\tilde{g}_i \ ,$$

which gives

$$\tilde{g}_i = 2i\theta(h^i - 1)\, B(\mathrm{Ad}(k)(E_i - \epsilon\theta(E_i) + \epsilon\Delta X_{s,i}), A) \ .$$

Remembering inequality (1.24) we conclude that $\tilde{g}_i > 0$ for small enough ϵ.

In the following we restrict ourselves to the $\mathcal{Q}_{+,\alpha}$ integrations, since the integrations over $\mathcal{Q}_{+,0}$ are trivially convergent. The functions f_l and g_l depend on the choice of basis of $\mathcal{Q}_+$. For a good choice, inequality (1.24) is important. Let $\Pi_{\mathcal{Q}_+}$ denote the orthogonal projection onto $\mathcal{Q}_+$, then we choose $j \in L$ and define

$$X_1 := i\Pi_{\mathcal{Q}_+}(E_j) = \sum_{\alpha \in \Sigma_{\{j\},\neq}} e_j^\alpha X_{s,\alpha} \qquad (1.36)$$

as the first basis vector, and extend this to an orthonormal basis of $\oplus_{\alpha(H_{j,c})\neq 0}\mathcal{Q}_{+,\alpha}$, which fixes the first m basis vectors. Extend this to an orthonormal basis of $\mathcal{Q}_+$ that respects the root decomposition for the root spaces with $\alpha(H_j) = 0$. For all X_l and $X_{l'}$ we have the equality

$$B(X_l + \phi(X_l), X_{l'} + \phi(X_{l'})) = 0 \ .$$

This is important as it makes $B(\Xi, \Xi)$ proportional to $\delta_{l,l'}$, i.e. that the Gaussian integrals are diagonal. We read off

$$f_l = \sum_{\alpha \in \Sigma_{\{l\},\neq}} e_l^\alpha |B(X_{s,\alpha}, X_{s,\alpha})| \big(1 - (1-\epsilon)^2 \tanh^2(\xi(h^i)\alpha(H_i))\big) \ .$$

In particular we have for $l = 1$:

$$f_1 = \epsilon(2 - \epsilon)B(X_1, X_1) \ .$$

Similarly it is easy to check that the g''_l defined by

$$-2B(\Xi, \Upsilon) = -2ix^l g''_l$$

vanish in the limit ($\epsilon = 0$ and $h^l \geq 1$) or are identically zero. Therefore we neglect g''_l in the following. Now, we turn to $2iB(Q, A)$. $B(\Xi, A)$ contains x^l only linearly and thus we define

$$-2iB(\Xi, A) = -2ix^l {g'}_l \ ,$$

which gives

$$g'{}_l = \sum_{\alpha\in\Sigma_{\{l\},\neq}} e_l^\alpha B(X_{s,\alpha} + (1-\epsilon)\tanh(\xi(h^i)\alpha(H_i))\phi(X_{s,\alpha}), k^{-1}Ak) \ ,$$

and $g_l = g'_l + g''_l$. This leads to

$$\lim_{\epsilon\to 0} g_1 = B(-iE_j, \mathrm{Ad}(k^{-1})A) < 0 \ .$$

Putting everything together we obtain that

$$\lim_{\epsilon\to 0} I_{L,c}(\epsilon, H, [k]) = 0 \ .$$

In particular $\int_{\mathfrak{a}_{+,L,c}} I_{L,c}(\epsilon, H, [k])$ is a bounded function, and the limit $\lim_{\epsilon\to 0} I_{L,c}(\epsilon, H, [k])$ is uniform. Thus the limit commutes with the outer integrals. Hence the discussion of $I_{L,c}(\epsilon, H, [k])$ directly yields the existence of the limit.

Reaching PS

The information about $I_{L,c}(\epsilon, H, [k])$ helps us finish the proof with the following two equations:

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{DePS(\epsilon)} g(Q,A)dQ &= \lim_{\epsilon\to 0}\int_{\sum_{c\in C} DePS_{\emptyset,c}(\epsilon)} g(Q,A)dQ \\ &= \lim_{\epsilon\to 0}\int_{PS} g(Q,A)\chi_\epsilon(Q)dQ \ .\end{aligned}$$

The first equality sign holds since the contributions of all attached surfaces $|L| > 0$ vanish. For the last equality sign it is important to note that for $\sum_{c\in C} DePS_{\emptyset,c}(\epsilon)$ only the ϵ contained in the $B(\Xi,\Xi)$ term is important. All other ϵ can be set to zero even before executing the Gaussian integrations. This procedure affects the terms $B(\Xi,\Upsilon)$ and $B(Q,A)$. The last equality holds by identifying $\chi_{\epsilon'} = \exp(+\frac{\epsilon'}{4}\operatorname{Tr}(Q-\theta Q)^2)$ and $\epsilon' = 2\epsilon - \epsilon^2$.

Remark 1.1. *To obtain the theorem when $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ is the direct sum of a semisimple and an Abelian Lie algebra, let $\mathfrak{a}'\oplus\mathfrak{a}$ denote a maximal Abelian subalgebra of $\mathfrak{p}$ and replace $\mathfrak{a}_+$ by $\mathfrak{a}'\times\mathfrak{a}_+$ and H by $H'+H$ everywhere in the proof. In addition let $\mathfrak{k}'$ denote the semisimple part of $\mathfrak{k}$ and replace $\mathfrak{k}$ by $\mathfrak{k}'$ everywhere in the proof.*

Remark 1.2. *It is possible to choose different regularisation functions χ_ϵ. Nevertheless, the choice made here seems natural, as it has the highest invariance possible.*

Remark 1.3. *The convergence properties can be seen quite clearly in the discussion of $I_c(\epsilon, H, [k])$. The convergence is not uniform in A. To have uniform convergence, we need $As > \delta$ for fixed $\delta > 0$.*

Remark 1.4. *In applications with $As \geq 0$, one has to substitute A by $A+\delta s$. For fixed $\delta > 0$ this gives uniform convergence in A.*

Remark 1.5. *In the proof, we do not have to require that the extensions of PS are nullsurfaces. However, this idea is needed as a guiding principle for finding the extension.*

1.4.4 Equivalence of SW and $Euclid$

Let us note that the SW domain and the validity of the corresponding hyperbolic Hubbard-Stratonovich transformation is discussed in detail in [7]. Nevertheless we give a different proof by deforming SW into $Euclid$. Using some of the constructions of the proof for the PS transformation this deformation can be stated very explicitely.

First we briefly discuss convergence of the Gaussian integral over

$$\begin{aligned} SW:\ &\mathfrak{p} \oplus \mathcal{Q}_+ \to \mathcal{Q}^{\mathbb{C}} \\ &(Y, X) \mapsto X - ibe^Y se^{-Y}\ . \end{aligned}$$

For $X \in \mathcal{Q}_+$ and $Y \in \mathfrak{p}$ we have that $B(X, X) > 0$ and

$$B(ibe^Y se^{-Y}, ibe^Y se^{-Y}) = -b^2 B(s, s)$$

is constant. Furthermore $B(X, ibe^Y se^{-Y})$ is purely imaginary and

$$-iB(-ibe^Y se^{-Y}, A) = -b\,\mathrm{Tr}(e^{-2Y} As) < 0$$

yields convergence in the $\mathfrak{p}$ directions.

To see the properties of SW more explicitely we use the reparametrisation R_I and the decomposition of s to obtain:

$$\begin{aligned} SW \circ R_I:\ &\mathfrak{a}_+ \times K/Z_K(\mathfrak{a}) \times Q_+ \to \mathcal{Q}^{\mathbb{C}} \\ &(H, [k], X) \mapsto X - ib\,\mathrm{Ad}(k)\Big[M_s + \\ &\sum_{\alpha \in \Sigma_+(\mathcal{Q},\mathfrak{a})} [\cosh(\alpha(H)) X_{s,\alpha} + \sinh(\alpha(H))\,\phi(X_{s,\alpha})]\Big]\ . \end{aligned} \tag{1.37}$$

Since the image of the boundary of $\mathfrak{a}_+$ is again of codimension at least two, the parametrisation (1.37) clearly shows that $\partial SW = 0$.

A suitable deformation is given by

$$\begin{aligned} DSW:\ &[0,1] \times \mathfrak{a}_+ \times K/Z_K(\mathfrak{a}) \times \mathcal{Q}_+ \to \mathcal{Q}^{\mathbb{C}} \\ &(t, H, [k], X) \mapsto X - ib\,\mathrm{Ad}(k)\Big[(1-t)M_s + \\ &\sum_{\alpha \in \Sigma_+(\mathcal{Q},\mathfrak{a})} [(\cosh((1-t)\alpha(H)) - 1) X_{s,\alpha} + \frac{\sinh((1-t)\alpha(H))}{1-t}\,\phi(X_{s,\alpha})]\Big] \end{aligned}$$

Note that $DSW(t=0) = SW$ and $DSW(1, H, [k], X) = X + i[kHk^{-1}, s]$. Since $[\mathfrak{p}, s] = \mathcal{Q}_-$ we obtain $DSW(1) = Euclid$.

To complete the argument we show that the integral over DSW is convergent. Therefore note that

$$B(Q,Q) = \underbrace{B(X,X)}_{>0} + \sum_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} (2t-t^2)\frac{\sinh^2((1-t)\alpha(H))}{(1-t)^2}\underbrace{B(X_{s,\alpha}, X_{s,\alpha})}_{>0} + \dots ,$$

where the dots represent unimportant terms. These are terms which are purely imaginary, terms which are linear in sinh and all terms containing M_s. Thus we have convergence for $t > 0$. For $t = 0$ convergence is generated by the $B(Q, A)$ term, as discussed above for the SW parametrisation.

1.4.5 Different representations of the integral

In this section we discuss different possibilities to represent the integral over the PS domain. In particular we want to derive corollary 1.1 and 1.2.

Assume that $\mathfrak{k} \oplus \mathcal{Q}_+$ is the direct sum of an Abelian and a semisimple Lie algebra. Then choose a maximal Abelian subalgebra $\mathfrak{h}$ of $\mathcal{Q}_+$ and let $\mathfrak{h} = \mathfrak{h}' \oplus \tilde{\mathfrak{a}}$ be the corresponding decomposition into the Abelian and semisimple part. Then we have the following reparametrisation:

$$\begin{aligned}\tilde{R}:\ \mathfrak{p}\times K/Z_K(\tilde{\mathfrak{a}})\times\tilde{a}^o_+\times\mathfrak{h}' &\to \mathfrak{p}\oplus\mathcal{Q}_+\\ (Y,[k],\tilde{H},J') &\mapsto Y + k(\tilde{H}+H')k^{-1}\ .\end{aligned}$$

If $\mathfrak{g}$ is semisimple we can use Cartan decomposition to define a semisimple Lie group as $G := e^{\mathfrak{p}}K$. This yields the reparametrisation

$$\begin{aligned}R:\ G/Z_K(\tilde{\mathfrak{a}})\times\tilde{a}^o_+\times\mathfrak{h}' &\to \mathfrak{p}\oplus\mathcal{Q}_+\\ (e^Y[k],\tilde{H},H') &\mapsto Y + k(\tilde{H}+H')k^{-1}\ .\end{aligned}$$

The parametrisation of the PS domain which is most frequently used in the literature, is

$$\begin{aligned}PS\circ R:\ G/Z_K(\tilde{\mathfrak{a}})\times\tilde{a}^o_+\times\mathfrak{h}' &\to \mathcal{Q}\\ ([g],\tilde{H},H') &\mapsto g(\tilde{H}+H')g^{-1}\ .\end{aligned}$$

In the following we present a derivation of corollary 1.1. This is done in a detailed way that clearly exhibits the origin and form of J' in 1.1. In section A.2.2 we show that the pullback of dQ by $PS\circ R$ is given by

$$(PS\circ R)^*dQ = \Delta(\tilde{H}+H')\ d\mu([g])\wedge dH\ ,$$

where $d\mu([g])$ is a left invariant volume form on $G/Z_K(\tilde{\mathfrak{a}})$ and Δ is given by

$$\Delta(\tilde{H}+H') = \prod_{\alpha\in\Sigma_+(\mathfrak{p}\oplus\mathcal{Q}_-,\mathfrak{h})} \alpha(\tilde{H}+H')^{d_\alpha} \cdot \prod_{\alpha\in\Sigma_+(\mathfrak{k}\oplus\mathcal{Q}_+,\mathfrak{h})} \alpha(\tilde{H})^{d_\alpha} \,. \qquad (1.38)$$

This needs further explanation: d_α denotes the dimension of the weight space corresponding to α. The weights $\alpha(\tilde{H}+H')$ are real since $[\tilde{H}+H',\cdot]$ is hermitian with respect to $\mathrm{Tr}(XY^\dagger)$. Furthermore $\alpha(H')=0$ for $\alpha\in\Sigma_+(\mathfrak{k}\oplus\mathcal{Q}_+,\mathfrak{h})$.

It is important that Δ differs from J' in corollary 1.1 only by taking the modulus of the weights in $\Sigma_+(\mathfrak{k}\oplus\mathcal{Q}_+,\mathfrak{h})$. But the roots $\alpha\in\Sigma_+(\mathfrak{k}\oplus\mathcal{Q}_+,\mathfrak{h})$ are positive when evaluated on $\tilde{\mathfrak{a}}_+^o$. Therefore we have the following equality:

$$\int_{PS\circ R} f(Q)\, dQ = \int_{id} f(g(\tilde{H}+H')g^{-1}) \;\cdot\; J'(\tilde{H}+H')\, d\mu([g])\wedge dH \;,$$

where id denotes the identity on $G/Z_K(\tilde{\mathfrak{a}})\times\tilde{a}_+^o\times\mathfrak{h}'$. Now it is possible to replace the volume form $d\mu([g])$ by the left invariant measure $|d\mu([g])|$ and dH by Lebesgue measure $|dH|$ on $\mathfrak{h}$:

$$\int\limits_{PS\circ R} f(Q)\, dQ = \int\limits_{G/Z_K(\tilde{\mathfrak{a}})\times\tilde{a}_+^o\times\mathfrak{h}'} f(g(\tilde{H}+H')g^{-1}) \;\cdot\; J'(\tilde{H}+H')\, |d\mu([g])||dH| \;.$$

Replacing $G/Z_K(\tilde{\mathfrak{a}})$ by G introduces only a constant factor $c'\in\mathbb{R}\setminus\{0\}$: $c'\in\mathbb{R}\setminus\{0\}$:

$$\int\limits_{PS\circ R} f(Q)\, dQ = c'\int\limits_{G\times\tilde{a}_+^o\times\mathfrak{h}'} f(g(\tilde{H}+H')g^{-1}) \;\cdot\; J'(\tilde{H}+H')\, |d\mu(g)||dH| \;,$$

where $|d\mu(g)|$ denotes the invariant measure on G. Since now $|d\mu(g)|$ is in particular right invariant we can use that the action of the Weyl group on $\tilde{\mathfrak{a}}_+$ generates $\tilde{\mathfrak{a}}$. To exploit this property we need that J' is also invariant under the action of the Weyl group. Recall that J' is given by

$$J'(\lambda) = \prod_{\alpha\in\Sigma_+(\mathfrak{p}\oplus\mathcal{Q}_-,\mathfrak{h})} \alpha(\lambda)^{d_\alpha} \prod_{\alpha\in\Sigma_+(\mathfrak{k}\oplus\mathcal{Q}_+,\mathfrak{h})} |\alpha(\lambda)^{d_\alpha}| \,.$$

The second factor is trivially invariant, whereas for the first factor an additional argument is needed. Therefore note that we used s to define a notion of positivity for the weights $\alpha\in\Sigma_+(\mathfrak{p}\oplus\mathcal{Q}_-,\mathfrak{h})$. Since s is $\mathrm{Ad}(K)$ invariant we conclude that the action of the Weyl group only permutes the weights in $\Sigma_+(\mathfrak{p}\oplus\mathcal{Q}_-,\mathfrak{h})$ and hence it is invariant. Thus we have

$$\int\limits_{PS\circ R} f(Q)\, dQ = c''\int\limits_{G\times\mathfrak{h}} f(g(\tilde{H}+H')g^{-1}) \;\cdot\; J'(\tilde{H}+H')\, |d\mu(g)||dH| \;,$$

where $c'' \in \mathbb{R}\{0\}$ is a constant. Setting $f = g \cdot \chi_\epsilon$ we obtain corollary 1.1.

If PS is nearly everywhere injective and regular, then so is $PS \circ R$. This allows application of the change of variable theorem, which yields corollary 1.2.

1.5 Examples

1.5.1 $U(p, q)$ symmetry

This case has been proven by Fyodorov [10] using different methods. The general theorem (1.2) can by applied by choosing $\mathfrak{gl}(p+q, \mathbb{C})$ as complex Lie algebra and defining $s = \mathrm{Diag}(\mathbb{1}_q, -\mathbb{1}_p)$. No additional involutions τ_i are needed. In this setting, the maximal Abelian subalgebra $\mathfrak{h} \subset \mathcal{Q}_+$ is given by the real diagonal matrices. In addition we have

$$\mathfrak{k} \oplus \mathcal{Q}_+ = \{x \in \mathfrak{gl}(n, \mathbb{C}) | X = sXs\} \text{ and } \mathfrak{p} \oplus \mathcal{Q}_- = \{x \in \mathfrak{gl}(n, \mathbb{C}) | X = -sXs\}$$

In the following $\lambda := \mathrm{Diag}(\lambda_1, \ldots, \lambda_{p+q}) \in \mathfrak{h}$ denotes a real diagonal matrix. The weights are given by $f_i - f_j$ where $i \neq j$ and $f_i(\lambda) = \lambda_i$. The corresponding weight spaces are given by the matrix E_{ij}, which is nonzero only in the i, jth entry. Thus every weight has a two dimensional weight space, i.e., complex one dimensional. Thus we have

$$J'(\lambda) = \prod_{i<j} |\lambda_i - \lambda_j|^2 .$$

1.5.2 $O(p, q)$ symmetry

This case has been proven by Fyodorov, Wei and Zirnbauer [12]. In addition to the involutions of the pseudounitary setting we have an involution $\tau_1(X) = -sX^t s$ and $\sigma_1 = -1$. The additional involution just requires all matrices to be real. Therefore the weight spaces are now one dimensional, and give rise to non trivial signs:

$$\begin{aligned}
J'(\lambda) &= \prod_{\alpha \in \Sigma_+(\mathfrak{p} \oplus \mathcal{Q}_-, \mathfrak{h})} \alpha(\lambda) \prod_{\alpha \in \Sigma_+(\mathfrak{k} \oplus \mathcal{Q}_+, \mathfrak{h})} |\alpha(\lambda)| \\
&= \prod_{i \le p < j \le p+q} (\lambda_{ii} - \lambda_{jj}) \prod_{i<j\le p, p<i<j\le p+q} |\lambda_i - \lambda_j| \\
&= \prod_{i<j} |\lambda_i - \lambda_j| \prod_{i=1}^{p} \prod_{j=p+1}^{p+q} \mathrm{sgn}(\lambda_i - \lambda_j) ,
\end{aligned}$$

which is precisely corollary one in [12].

1.5.3 Two dimensional case

The two dimensional example considered at the beginning of the chapter can also be put into the general framework. Consider the $O(1,1)$ case, and require all matrices to be traceless. This means exchanging $\mathfrak{gl}(2,\mathbb{C})$ by $\mathfrak{sl}(2,\mathbb{C})$. Then all relevant spaces are:

$$\mathcal{Q}_+ = \mathbb{R}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \mathcal{Q}_- = \mathbb{R}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$
$$\mathfrak{p} = \mathbb{R}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \mathfrak{k} = \{0\}\ .$$

Hence there are only two roots $\pm(f_1 - f_2) \in \Sigma(\mathfrak{p} \oplus \mathcal{Q}_-, \mathfrak{h})$, and J' contains only one factor. But this factor is vital since it yields a nontrivial sign. Identifying

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \text{ and } \mathbf{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

we obtain that $B(\mathbf{e}_i, \mathbf{e}_j) = \mathrm{Tr}(\mathbf{e}_i\mathbf{e}_j) = (-1)^{i+1}\delta_{ij}$.

1.5.4 Simple system of interacting bosons

In the following we sketch a simple application of a hyperbolic Hubbard-Stratonovich transformation in the context of interacting bosons. Let $a^\dagger$ and a denote bosonic creation and annihilation operators and define charge and pair annihilation operators:

$$Q := \sum_{i=1}^{N} a_i^\dagger a_i \text{ and } P := \sum_{i=1}^{N} a_i a_i\ .$$

Then we consider the simple interaction Hamiltonian given by

$$\hat{H}_{int} = eQ^2 - bP^\dagger P\ ,$$

which has to satisfy the stability condition $e > b$. Using boson coherent states, and neglecting normal ordering terms for the moment, the Hamiltonian is given by

$$H_{int}(\bar{z}, z) = e\left(\sum_{i=1}^{N} \bar{z}_i z_i\right) - b\left(\sum_{i=1}^{N} \bar{z}_i\bar{z}_i\right)\left(\sum_{i=1}^{N} z_i z_i\right)\ .$$

Introducing the matrix

$$A(\bar{z}, z) := \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{e}\sum_{i=1}^{N}\bar{z}_i z_i & \sqrt{b}\sum_{i=1}^{N}\bar{z}_i\bar{z}_i\\ \sqrt{b}\sum_{i=1}^{N} z_i z_i & -\sqrt{e}\sum_{i=1}^{N}\bar{z}_i z_i\end{pmatrix}$$

the interaction Hamiltonian is equal to:

$$H_{int}(\bar{z}, z) = \mathrm{Tr}\, A(\bar{z}, z)^2 \, .$$

The stability condition translates to $As > 0$, with $s = \mathrm{Diag}(1, -1)$. Furthermore, A satisfies the symmetry relations $A = sA^\dagger s$ and $A = -\Omega^t A^t \Omega$, with

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \, .$$

Let us now state how this fits into the general setting of theorem 1.2: Choose $\mathfrak{sl}(2, \mathbb{C})$ as complex Lie algebra and the additional involution $\tau_1(X) = -\Omega^t X^t \Omega$, with $\sigma_1 = 1$. The relevant spaces are

$$\mathfrak{k} = i\mathbb{R} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathfrak{p} = \mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\mathbb{R} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$\mathcal{Q}_+ = \mathbb{R} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{Q}_- = \mathbb{R} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + i\mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now it is possible to apply theorem 1.2 to decouple the interaction term in the boson coherent state path integral representation of the partition function of the system. Note however that we have to make the usual continuum approximation that $\bar{z}(t)$ is complex conjugate to $z(t)$.[13] After the application of theorem 1.2 the z and $\bar{z}$ integrations can be performed. We do not proceed any further in this direction, since the intention was only to illustrate a possible application to many body systems with bosonic degrees of freedom.

[13]This is an approximation since $z(t)$ and $\bar{z}(t)$ stem from neighbouring time steps in the discrete time path integral.

Chapter 2

Bosonisation of granular fermionic systems

In this chapter we derive a bosonic path integral representation of the grand canonical partition function of a granular fermionic system.[1] In the following we give an overview of the organisation of this chapter.

In section 2.2 we give a derivation of the bosonic path integral representation of the grand canonical partition function of a granular fermionic system starting from a Grassmann coherent state path integral representation of the grand canonical partition function. Then colour-flavour transformation is applied iteratively for each time step, which leads to a factorisation of the Grassmann integrals. Integrating out the Grassmann variables leads to the bosonic path integral representation. In section 2.3 we give another derivation based on a suitably enlarged Fock space. In this larger Fock space generalised coherent states can be used to obtain the result. Finally in section 2.4 we calculate the contribution of fluctuations in the semiclassical limit. Essentially this amounts to calculate the fluctuation determinant for generalised coherent state path integrals. To the best of our knowledge there exists no derivation for the general case. We do the calculation in discrete time using a method invented by Forman [32].

2.1 Granular bosonisation via colour-flavour transformation

In the first subsection we fix the setting and discuss three different symmetry classes. Furthermore we state a colour-flavour transformation for each symmetry class. This transformation is used in the next section to decouple the time evolution within the Grassmann coherent state path integral,

[1] For a detailed description of the context and the motivation for this chapter see the second part of the introduction.

i.e. to convert the 'normalisation factors' $\exp(-\sum_k \bar{\psi}_k\psi_k)$ into expressions containing only pairs $\bar{\psi}_k\bar{\psi}_k$,$\psi_k\psi_k$ and $\bar{\psi}_k\psi_{k-1}$, that are invariants of the corresponding group.[2] In the third subsection we integrate out the Grassmann variables. This leads to an effective action which allows a formal continuum limit.

2.1.1 Setting and symmetry classes

Our starting point is the discrete time Grassmann functional integral representation of the grand canonical partition function [35]:

$$\mathrm{Tr}(\exp(-\beta\hat{H})) = \lim_{M\to\infty}\int \mathcal{D}_M(\bar{\psi},\psi)\exp(-S_M[\bar{\psi},\psi]) \ , \tag{2.1}$$

where $\hat{H}$ denotes the Hamiltonian of the granular system. Note that we absorb the chemical potential into the Hamiltonian to simplify notation. The discrete time action S_M is given by

$$-S_M[\bar{\psi},\psi] = -\sum_{k=0}^{M-1}\bar{\psi}_k\psi_k + \sum_{k=1}^{M}\bar{\psi}_k\psi_{k-1} - \frac{\beta}{M}\sum_{k=1}^{M} H\left(\bar{\psi}_k,\psi_{k-1}\right) \ , \tag{2.2}$$

with antiperiodic boundary conditions $\bar{\psi}_M = -\bar{\psi}_0$. Let us first hide the time step index. The remaining index structure is given by

$$\begin{aligned}\bar{\psi} &= (\bar{\psi}_1^1, ..., \bar{\psi}_\alpha^i, ..., \bar{\psi}_{N_e}^{N'})\\ \psi &= (\psi_1^1, ..., \psi_\alpha^i, ..., \psi_{N_e}^{N'})^t \ .\end{aligned}$$

The upper indices are called internal and the lower external. To distinguish between external and time indices the former are denoted by greek and the latter by latin letters. If the internal or external indices are not explicitly shown, they are summed over. Internal indices will not be denoted by t so there is no confusion with transposition.

The Hamiltonian is required to have a unitary, orthogonal or unitary symplectic symmetry. This has the consequence that the Hamiltonian is a polynomial in the generators of the Howe dual group (see appendix B.2.4 and theorem B.1). The generators are exactly the bilinear invariants of the symmetry group $K \in \{\mathrm{U}(N'), \mathrm{O}(N'), \mathrm{USp}(N')\}$. See table 2.1 for a list of fermionic Howe dual pairs (K, G). Let $k \in K$, then the group action is given by

$$\begin{aligned}\bar{\psi}_j &\mapsto \bar{\psi}_j k^{-1} \equiv \bar{\psi}_j(\mathbb{1}_{N_e}\otimes k^{-1}) = \sum_{i=1}^{N'}\bar{\psi}_{j,\alpha}^i k_{im}^{-1}\\ \psi_l &\mapsto k\psi_l \equiv (\mathbb{1}_{N_e}\otimes k)\psi_l = \sum_{i=1}^{N'} k_{mi}\psi_{l,\alpha}^i \ ,\end{aligned} \tag{2.3}$$

[2] Here the index structure has been suppressed deliberately. It will be presented in detail in the next section.

SO(2N)	K	G	H
$N = (p+q)N_i$	$\mathrm{U}(N_i)$	$\mathrm{U}(p+q)$	$\mathrm{U}(p) \times \mathrm{U}(q)$
$N = N_e N_i$	$\mathrm{O}(N_i)$	$\mathrm{SO}(2N_e)$	$\mathrm{U}(N_e)$
$N = 2N_e N_i$	$\mathrm{USp}(2N_i)$	$\mathrm{USp}(2N_e)$	$\mathrm{U}(N_e)$

Table 2.1: (K, G) denotes a Fermionic Howe dual pair.

with time indices j and l. In the following we discuss the situation for each symmetry group. In the case of **unitary** symmetry we set $N' = N_i$. The quadratic invariants of $\mathrm{U}(N_i)$ are given by

$$\bar{\psi}_\alpha \psi_\beta = \sum_{i=1}^{N_i} \bar{\psi}_\alpha^i \psi_\beta^i \ .$$

This corresponds to the special case of unitary Howe dual pairs where $p = N_e$ and $q = 0$ (see table 2.1). We restrict ourselves to this case since we are not aware of a physical system that requires an arbitrary p and q. However the following constructions readily generalise to the case of arbitrary p and q. For **orthogonal** symmetry we set $N' = N_i$ and the invariants are given by

$$\bar{\psi}_\alpha \psi_\beta = \sum_{i=1}^{N_i} \bar{\psi}_\alpha^i \psi_\beta^i \ , \ \psi_\alpha^t \psi_\beta = \sum_{i=1}^{N_i} \psi_\alpha^i \psi_\beta^i \text{ and } \bar{\psi}_\alpha^t \bar{\psi}_\beta = \sum_{i=1}^{N_i} \bar{\psi}_\alpha^i \bar{\psi}_\beta^i \ .$$

Moreover, for **unitary symplectic** symmetry we set $N' = 2N_i$. The upper index i is seen as a composite index (i, s), with $s = \pm 1$. The invariants are given by

$$\bar{\psi}_\alpha \psi_\beta = \sum_{i=1}^{N_i} \sum_{s=\pm 1} \bar{\psi}_\alpha^{i,s} \psi_\beta^{i,s} \ , \ \psi_\alpha^t J' \psi_\beta = \sum_{i=1}^{N_i} \sum_{s=\pm 1} \psi_\alpha^{i,s} s \psi_\beta^{i,-s}$$
$$\text{and } \ \bar{\psi}_\alpha^t J' \bar{\psi}_\beta = \sum_{i=1}^{N_i} \sum_{s=\pm 1} \bar{\psi}_\alpha^{i,s} s \bar{\psi}_\beta^{i,-s} \ ,$$

and J' is the symplectic unit acting on the upper indices. Note that for $k \in \mathrm{USp}(2N_i)$ we have $k^t J' = J' k^{-1}$.

We need some preparations to be able to state the colour-flavour transformations, which we want to use in the next section. First we define

$$\bar{\Psi} := \begin{pmatrix} \bar{\psi}_j & \psi_l^t \end{pmatrix}$$
$$\Psi := \begin{pmatrix} \psi_j \\ \bar{\psi}_l^t \end{pmatrix}$$

for fixed time indices j and l. The colour-flavour transformations for the three different symmetry classes are derived in appendix B.2.4. In the following we just state without derivation all objects that are needed to define the colour-flavour transformation. In our application of the colour-flavour transformation we consider two neighbouring time steps. For this reason we have take twice as many degrees of freedom into account in the application of the colour-flavour transformation. Therefore we use the colour-flavour transformation corresponding to the groups summarised in table 2.2 instead of the ones in table 2.1. In addition we need a complex vector space W

SO($2N$)	K	G	H
$N = 2N_eN_i$	U(N_i)	U($2N_e$)	U(N_e) $\times$ U(N_e)
$N = 2N_eN_i$	O(N_i)	SO($4N_e$)	U($2N_e$)
$N = 4N_eN_i$	USp($2N_i$)	USp($4N_e$)	U($2N_e$)

Table 2.2: List of Howe dual pairs needed for the colour-flavour transformations we use.

and the measure $d\mu(Z_e^\dagger, Z_e)$ defined in table 2.3. W gives a parametrisation

K	W	$f(Z_e^\dagger, Z_e)$
U(N_i)	$\{Z_e \in W_{2N_e} \vert sZ_es = -Z_e\}$	$\det^{-N_e}(1 + Z_e^\dagger Z_e)$
O(N_i)	W_{2N_e}	$\det^{-2N_e}(1 + Z_e^\dagger Z_e)$
USp($2N_i$)	$\{Z_e \in W_{2N_e} \vert sZ_es = -Z_e, JZ_eJ^t = Z_e\}$	$\det^{-N_e}(1 + Z_e^\dagger Z_e)$

Table 2.3: $W_n := \{Z_e \in \mathrm{End}(\mathbb{C}^n) | Z_e^t = -Z_e\}$, $s = \mathbb{1}_{N_e} \otimes \sigma_z$ and $J = i\mathbb{1}_{N_e} \otimes \sigma_y$. W parametrises the coset space G/H. Let dw denote Lebesgue measure on W then we define $d\mu(Z_e^\dagger, Z_e) := f(Z_e^\dagger, Z_e)dw$.

of the coset space G/H, and $d\mu(Z_e^\dagger, Z_e)$ denotes the left invariant measure. Now, we can state the colour-flavour transformation:

$$\int_K dk\ \exp\left[\bar{\Psi}\begin{pmatrix} k & 0 \\ 0 & k^{-1t}\end{pmatrix}\Psi\right] = \int_W d\mu(Z_e^\dagger, Z_e)\ \det{}^{-N_i/2}(1 + Z_e^\dagger Z_e)\ \exp\left[\frac{1}{2}\bar{\Psi}Z\bar{\Psi}^t + \frac{1}{2}\Psi^t Z^\dagger \Psi\right], \tag{2.4}$$

where $Z := Z_e \otimes \mathbb{1}_{N_i}$ and $Z_e \in W$. We need for the derivation of granular bosonisation that Z commutes with k.

2.1.2 Application of the colour-flavour transformation

In this section we promote the global symmetry (2.3) of the Hamiltonian to a local one, which leads to a factorisation of the Grassmann integrals. We proceed iteratively, i.e., for one time step after the other. First we introduce an average over the symmetry group, and then we apply colour-flavour transformation.

First step: We perform transformation (2.3) for $j = 1$ and $l = 0$ in the discrete time path integral representation and average over K. The corresponding integration variable is called k_1. Only the first two summands in the first term in (2.2) are affected by this transformation. The two summands are given by

$$-\bar{\psi}_1 k_1^{-1} \psi_1 - \bar{\psi}_0 k_1 \psi_0 = \begin{pmatrix} -\bar{\psi}_0 & \psi_1^t \end{pmatrix} \begin{pmatrix} k_1 & 0 \\ 0 & k_1^{-1\,t} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \bar{\psi}_1^t \end{pmatrix} .$$

The right hand side above is exactly the left hand side of the colour-flavour transformation (2.4). Applying (2.4) yields

$$\frac{1}{2} \begin{pmatrix} -\bar{\psi}_0 & \psi_1^t \end{pmatrix} Z_1 \begin{pmatrix} -\bar{\psi}_0^t \\ \psi_1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \psi_0^t & \bar{\psi}_1 \end{pmatrix} Z_1^\dagger \begin{pmatrix} \psi_0 \\ \bar{\psi}_1^t \end{pmatrix} . \tag{2.5}$$

Next we use the well known identity

$$f(\bar{\xi}) = \int \mathrm{d}\bar{\eta} \mathrm{d}\eta \, e^{-\bar{\eta}\eta + \bar{\xi}\eta} f(\bar{\eta}) \tag{2.6}$$

to change the term $\bar{\psi}_1 \psi_0$ to $-\bar{\eta}_1 \eta_1 + \bar{\psi}_1 \eta_1 + \bar{\eta}_1 \psi_0$. Adding this to (2.5) we obtain

$$\frac{1}{2} \begin{pmatrix} -\bar{\psi}_0 & \psi_1^t \end{pmatrix} Z_1 \begin{pmatrix} -\bar{\psi}_0^t \\ \psi_1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \bar{\eta}_1 & \eta_1^t & \psi_0^t & \bar{\psi}_1 \end{pmatrix} \begin{pmatrix} -J & \sigma_3 \\ -\sigma_3 & Z_1^\dagger \end{pmatrix} \begin{pmatrix} \bar{\eta}_1^t \\ \eta_1 \\ \psi_0 \\ \bar{\psi}_1^t \end{pmatrix} . \tag{2.7}$$

The transformation $\bar{\eta}_1 \mapsto -\bar{\eta}_1$ brings the second term to a more convenient form. In the last step we will make a similar transformation that compensates signs factors which might arise from the first transformation. The first term in (2.7) is dealt with in the next step.

Second step: Now we do the transformation (2.3) for $j = 2$ and $l = 1$ and average over K with integration variable k_2. This transformation changes the first term in (2.7) and the term $-\bar{\psi}_2 \psi_2$ in (2.2) into

$$\begin{aligned} &\frac{1}{2} \begin{pmatrix} -\bar{\psi}_0 & (k_2 \psi_1)^t \end{pmatrix} Z_1 \begin{pmatrix} -\bar{\psi}_0^t \\ k_2 \psi_1 \end{pmatrix} - \bar{\psi}_2 k_2^{-1} \psi_2 \\ &= \frac{1}{2} \begin{pmatrix} -\bar{\psi}_0 k_2 & \psi_1^t \end{pmatrix} Z_1 \begin{pmatrix} -k_2^t \bar{\psi}_0^t \\ \psi_1 \end{pmatrix} - \bar{\psi}_2 k_2^{-1} \psi_2 . \end{aligned}$$

Using (2.6) to decouple $-\bar{\psi}_0 k_2$ gives

$$\begin{aligned} &-\bar{\eta}_2 \eta_2 - \bar{\psi}_0 k_2 \eta_2 + \frac{1}{2} \begin{pmatrix} \bar{\eta}_2 & \psi_1^t \end{pmatrix} Z_1 \begin{pmatrix} \bar{\eta}_2^t \\ \psi_1 \end{pmatrix} - \bar{\psi}_2 k_2^{-1} \psi_2 \\ &= -\bar{\eta}_2 \eta_2 + \begin{pmatrix} -\bar{\psi}_0 & \psi_2^t \end{pmatrix} \begin{pmatrix} k_2 & 0 \\ 0 & k_2^{-1\,t} \end{pmatrix} \begin{pmatrix} \eta_2 \\ \bar{\psi}_2^t \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \bar{\eta}_2 & \psi_1^t \end{pmatrix} Z_1 \begin{pmatrix} \bar{\eta}_2^t \\ \psi_1 \end{pmatrix} \end{aligned}$$

Now colour-flavour transformation can be applied:

$$\frac{1}{2}\begin{pmatrix}\eta_2^t & \bar\psi_2\end{pmatrix} Z_2^\dagger \begin{pmatrix}\eta_2 \\ \bar\psi_2^t\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\bar\eta_2 & \psi_1^t\end{pmatrix} Z_1 \begin{pmatrix}\bar\eta_2^t \\ \psi_1\end{pmatrix} - \bar\eta_2\eta_2 \\ + \frac{1}{2}\begin{pmatrix}-\bar\psi_0 & \psi_2^t\end{pmatrix} Z_2 \begin{pmatrix}-\bar\psi_0^t \\ \psi_2\end{pmatrix} .$$

The term in the second line will be treated in step three. Combining the other terms with $\bar\psi_2\psi_1$ we obtain

$$\frac{1}{2}\begin{pmatrix}\bar\eta_2 & \psi_1^t & \eta_2^t & \bar\psi_2\end{pmatrix}\begin{pmatrix}Z_1 & -\mathbb{1} \\ \mathbb{1} & Z_2^\dagger\end{pmatrix}\begin{pmatrix}\bar\eta_2^t \\ \psi_1 \\ \eta_2 \\ \bar\psi_2^t\end{pmatrix} .$$

Third step: Now we have nearly the same starting point as in step two, and we can iterate the procedure until step $M-1$.

Last (Mth) step: Our starting point is

$$\frac{1}{2}\begin{pmatrix}-\bar\psi_0 & \psi_{M-1}^t\end{pmatrix} Z_{M-1}\begin{pmatrix}-\bar\psi_0^t \\ \psi_{M-1}\end{pmatrix} + \bar\psi_M\psi_{M-1} \\ = \frac{1}{2}\begin{pmatrix}\bar\psi_M & \psi_{M-1}^t\end{pmatrix} Z_{M-1}\begin{pmatrix}\bar\psi_M \\ \psi_{M-1}\end{pmatrix} + \bar\psi_M^t\psi_{M-1} ,$$

where we have used that $\bar\psi_M = -\bar\psi_0$. These are the antiperiodic boundary conditions that arise in the Grassmann coherent state representation of the trace. Application of identity (2.6) yields

$$\frac{1}{2}\begin{pmatrix}\bar\psi_M & \psi_{M-1}^t\end{pmatrix} Z_{M-1}\begin{pmatrix}\bar\psi_M^t \\ \psi_{M-1}\end{pmatrix} - \bar\eta_M\eta_M + \bar\psi_M\eta_M + \bar\eta_M\psi_{M-1} ,$$

which can be written as

$$\frac{1}{2}\begin{pmatrix}\bar\psi_M & \psi_{M-1}^t & \eta_M^t & \bar\eta_M\end{pmatrix}\begin{pmatrix}Z_{M-1} & \sigma_3 \\ -\sigma_3 & J\end{pmatrix}\begin{pmatrix}\bar\psi_M^t \\ \psi_{M-1}^t \\ \eta_M \\ \bar\eta_M^t\end{pmatrix} .$$

As in the first step, we can make an additional transformation $\eta_M \mapsto -\eta_M$. Thus we have obtained a path integral representation which allows a factorisation of the Grassmann integrals. We have introduced new Grassmann variables η and $\bar\eta$ and new bosonic variables Z and $Z^\dagger$. Note that the path integral representation is local in discrete time. However at this stage it is not clear how to perform the (formal) continuum limit $M \to \infty$. Obtaining such a continuum limit is the aim of the next section. To simplify the notation we set $Z_0 = J$ and $Z_M^\dagger = J^\dagger = -J$.

2.1.3 Effective Hamiltonian and the continuum limit

In the last section the time evolution of the Grassmann variables was decoupled at the expense of introducing new fields. In this section we want to integrate out the Grassmann variables to obtain an effective Hamiltonian in the new fields. Therefore we define the mean value of a function $F(\xi)$ of Grassmann variables ξ as

$$\langle F\rangle_X = \frac{\int \mathrm{d}\xi e^{\frac{1}{2}\xi^t X\xi} F(\xi)}{\int \mathrm{d}\xi e^{\frac{1}{2}\xi^t X\xi}} \ .$$

In this situation we have the following Wick theorem for Grassmann variables:

$$\langle \exp(\chi\xi)\rangle_X = \exp\left(-\frac{1}{2}\chi^t X^{-1}\chi\right) \ .$$

Let us first define

$$X_k := \begin{pmatrix} Z_{k-1} & -\mathbb{1} \\ \mathbb{1} & Z_k^\dagger \end{pmatrix}$$

as an important building block. To this end note the basic Gaussian integral

$$\int \mathrm{d}\xi \; e^{\frac{1}{2}\xi^t X_k \xi} = \det{}^{1/2}(\mathbb{1} + Z_k^\dagger Z_{k-1}) \ ,$$

and the useful identity

$$X_k^{-1} = \begin{pmatrix} Z_{k-1} & -\mathbb{1} \\ \mathbb{1} & Z_k^\dagger \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbb{1} + Z_k^\dagger Z_{k-1})^{-1} Z_k^\dagger & (\mathbb{1} + Z_k^\dagger Z_{k-1})^{-1} \\ -(\mathbb{1} + Z_{k-1} Z_k^\dagger)^{-1} & Z_{k-1}(\mathbb{1} + Z_k^\dagger Z_{k-1})^{-1} \end{pmatrix} \ .$$

Now, integrating out the Grassmann variables yields

$$\int \mathcal{D}_M \mu(Z^\dagger, Z) \frac{\det^{1/2}(\mathbb{1} + Z_1^\dagger J) \det^{1/2}(\mathbb{1} + J^\dagger Z_{M-1})}{\det^{1/2}(\mathbb{1} + Z_1^\dagger Z_1)}$$
$$\prod_{k=2}^{M-1} \frac{\det^{1/2}(\mathbb{1} + Z_k^\dagger Z_{k-1})}{\det^{1/2}(\mathbb{1} + Z_k^\dagger Z_k)} \left(1 - \frac{\beta}{M}\langle H\rangle_{X_k}\right) \ ,$$

where we defined

$$\mathcal{D}_M \mu(Z^\dagger, Z) := \prod_{k=1}^{M-1} \mathrm{d}\mu(Z_k^\dagger, Z_k) \ .$$

Recalling $Z_M^\dagger = J^\dagger$ and $Z_0 = J$, we obtain

$$\mathrm{Tr}\exp(-\beta \hat{H}) = \lim_{M\to\infty} \int \mathcal{D}_M \mu(Z^\dagger, Z) \exp(-S_M[Z^\dagger, Z])$$

with

$$
\begin{aligned}
-S_M[Z^\dagger, Z] = \frac{1}{2}\operatorname{tr}\ln(1 + Z_M^\dagger Z_{M-1}) + \frac{1}{2}\sum_{k=1}^{M-1} \operatorname{tr}\ln\frac{1 + Z_k^\dagger Z_{k-1}}{\mathbb{1} + Z_k^\dagger Z_k} \\
- \frac{\beta}{M}\sum_k \langle H\rangle_{X_k} \; .
\end{aligned}
$$

This expression allows for a formal continuum limit. Defining

$$
H(Z^\dagger, Z) := \langle H\rangle_{\left(\begin{smallmatrix} Z & -1 \\ 1 & Z^\dagger \end{smallmatrix}\right)}
$$

the continuum limit of $S_M[Z^\dagger, Z]$ can be identified as

$$
\begin{aligned}
-S[Z^\dagger, Z] =& \frac{N_i}{2}\operatorname{tr}(\ln(\mathbb{1} + Z^\dagger(\beta)Z(\beta)) \\
&- \frac{N_i}{2}\int_0^\beta \mathrm{d}\tau \operatorname{tr}(Z^\dagger \partial_\tau Z(\mathbb{1} + Z^\dagger Z)) - \int_0^\beta \mathrm{d}\tau H(Z^\dagger, Z) \; . \qquad (2.8)
\end{aligned}
$$

The boundary conditions are $Z(0) = J$ and $Z^\dagger = J^\dagger$. Variation of (2.8) leads to the following saddle point equations

$$
\begin{aligned}
\partial_\tau Z^\dagger &= \frac{2}{N_i}(\mathbb{1} + Z^\dagger Z)(\partial_{Z^t} H)(\mathbb{1} + ZZ^\dagger) \\
\partial_\tau Z &= -\frac{2}{N_i}(1 + ZZ^\dagger)(\partial_{\bar{Z}} H)(\mathbb{1} + Z^\dagger Z) \; .
\end{aligned}
$$

Let us summarise what we have achieved so far: We have obtained a path integral representation of the grand canonical partition function in terms of purely bosonic variables. We use the word ‘granular bosonisation’ for this way of representing a grand canonical partition function of a granular fermionic system. The representation closely resembles generalised coherent state path integrals [35] with rather strange boundary conditions and a classical limit that is controlled by N_i.

In the next section the connection to generalised coherent state path integral is made precise. This connection sheds some light on the origin and physical meaning of the boundary conditions. Furthermore we make contact with the large body of literature concerning coherent states and semiclassical limits.

2.2 Fock space approach to granular bosonisation

In the previous section we obtained a path integral description in bosonic variables of the grand canonical partition function of a granular fermionic system. In particular, the representation suggests that the large N_i limit is a classical limit, and that the classical phase space is G/H. However, it is not so clear how to interpret a classical state Z and the boundary conditions in terms of the original system. The aim of this section is to propose an answer to these questions. In addition, this leads to a different derivation of the path integral representation.

Let us first give a rough motivation for the course of action we take: The path integral representation of the last section is very similar to well known generalised coherent state path integrals. Such coherent states are used in the derivation of colour-flavour transformation to represent the projector P in appendix B.2.4. These coherent states come with a Fock space containing twice as many fermions as the original system. This Fock space can be seen as the tensor product of two copies of the original Fock space of our system. We will take the view that one is the state space of our system and the other is an ancilla system. The details are discussed in subsection 2.2.1. Granular bosonisation is derived and discussed in subsection 2.2.2.

2.2.1 Coherent states and a semiclassical limit

The Fock space of system and ancilla system is generated by fermionic creation and annihilation operators ${c^i_\alpha}^\dagger$ and c^i_α, with $\alpha = 1, \ldots, 2N_e$ and $i = 1, \ldots, N'$. Let $|0\rangle$ denote the vacuum state. The state space of the system $\mathcal{H}_S$ is generated by creation operators with $\alpha \leq N_e$, and the state space $\mathcal{H}_A$ of the ancilla system is generated by creation operators with $\alpha > N_e$. The coherent states are defined as

$$|Z\rangle := \exp\left(\sum_{\alpha,\beta,i} {c^i_\alpha}^\dagger Z_{\alpha\beta} {c^i_\beta}^\dagger\right)|0\rangle\ .$$

Z is an element of the complex vector space W defined in table 2.3 for each symmetry class. In addition we know that

$$P = \int_W \frac{\mathrm{d}\mu(Z^\dagger, Z)}{\langle Z|Z\rangle}|Z\rangle\langle Z|$$

projects onto the space of states I_0 which are invariant with respect to the symmetry group K. In the following we will work only with the subspace I_0. It is well known that in this setting the limit of large N_i is a classical limit [33, 34]. This implies that expectation values factorise, and the Schrödinger equation becomes Hamiltons equation of motion. The phase space is G/H, which comes with a natural symplectic structure. The heuristic way to see this is to use P as a representation of unity on I_0 to obtain a path integral representation. Let us stress again that this representation of P is only possible since I_0 carries an irreducible representation of G. This follows from the fact that we consider Howe dual pairs (K, G). Next we define the reduced density matrix of pure states $|Z\rangle$:

$$\rho(Z^\dagger, Z) = \mathrm{Tr}_{\mathcal{H}_A}\left(\frac{|Z\rangle\langle Z|}{\langle Z|Z\rangle}\right) .$$

The expectation value of an observable A_s of the system in state $\rho(Z^\dagger, Z)$ is given by

$$\mathrm{Tr}_{\mathcal{H}_S}(A_s \rho(Z^\dagger, Z)) = \frac{\mathrm{Tr}(A_s \otimes \mathbb{1}_{\mathcal{H}_A}|Z\rangle\langle Z|)}{\langle Z|Z\rangle} = \frac{\langle Z|A_s \otimes \mathbb{1}_{\mathcal{H}_A}|Z\rangle}{\langle Z|Z\rangle} .$$

2.2.2 Derivation of granular bosonisation

The key to the derivation of granular bosonisation is hidden in the boundary conditions of the path integral representation in the last section. The boundary conditions are $Z(0) = J$ and $Z^\dagger(\beta) = J^\dagger$. This motivates a further investigation of $|J\rangle$. An easy calculation shows that

$$\rho(J^\dagger, J) = \frac{\mathbb{1}_{\mathcal{H}_S}}{\mathrm{dim}\mathcal{H}_S} .$$

Thus, in the state $|J\rangle$ system and ancilla system are entangled in such a way that the reduced density gives uniform weight to all states. Such a state is called maximally entangled state. Now, the grand canonical partition function is given by

$$\mathrm{Tr}_S(e^{-\beta\hat{H}}) = \mathrm{dim}\mathcal{H}_S\, \mathrm{Tr}_S(e^{-\beta\hat{H}}\rho(J^\dagger, J)) = \langle J|e^{-\beta\hat{H}} \otimes \mathbb{1}_{\mathcal{H}_A}|J\rangle .$$

Since $\hat{H}$ has K as a symmetry group its action leaves I_0 invariant. Therefore we can use P as a resolution of unity to derive a path integral representation.

To interpret the states Z in the path integral let us first summarise the results of the above. The combined system has a well defined classical limit and its phase space is given by the coset space G/H. For each point Z in phase space there exists a coherent state $|Z\rangle$. The state $|Z\rangle$ can be interpreted in terms of the original system as a reduced density matrix $\rho(Z^\dagger, Z)$. The boundary condition $Z(0) = J$ corresponds to a state of the joint system whose reduced density matrix is the state of maximal entropy for the system.

2.3 Contributions of fluctuations

In this section we calculate the contributions of fluctuations to the semiclassical limit of the partition function. In essence this implies the calculation of a fluctuation determinant. Although a rather general result by Kochetov [40] can be applied to our case, we make some effort to rederive it from the discrete time definition of a generalised coherent state path integral. The reasons for this approach are as follows: First, to the best of our knowledge no derivation of the result is available. Only the special case of the spin path integral is treated in [40, 41]. And even this derivation had to be improved in [42]. Second, there are subtleties involved in the semiclassical limit of generalised coherent state path integrals which are not well known. Third, the method we use seems to be relatively unknown and deserves a detailed exposition.

First we explain some of the subtleties to be dealt with and point to some references. After that we state the precise setting and the result. Finally we use Forman's method to derive the contributions of fluctuations.

2.3.1 Subtleties concerning the evaluation of coherent state path integrals

Calculating the contribution of fluctuations is in principle straightforward: After determining the saddlepoint, i.e., the solution of the classical equations of motion, we calculate the second variation of the action, perform the corresponding gaussian integral, and finally compute the determinant which is produced.

In the case of generalised coherent state path integrals there are some obstacles to this scheme. First of all the quantum problem imposes boundary conditions for $Z(0)$ and $Z^\dagger(T)$. But a classical solution is already fully determined by only one of these conditions. Two solutions have been proposed to the 'problem of overspecification'. The first, suggested by Faddeev [38], is to take Z and $Z^\dagger$ as independent variables. The second solution, proposed by Klauder [37], consists of adding a small second order derivative term. We follow the first suggestion. This means that even in real time, the solutions $Z(t)$ and $Z^\dagger(t)$ of the classical equations of motions are in general not complex conjugate to each other. These solutions can be reached via a contour deformation into a larger (complexified) space, which is obtained by making Z and $Z^\dagger$ independent. This leads to a problem when calculating the second variation, since the variation should take place in some surface within the enlarged space. When this issue is ignored the result turns out to be correct only up to a phase factor.

Let us first give a short review of the relevant results for the spin path integral. Note that for the spin path integral Z is just a complex number. Solari [39] calculated the fluctuation determinant for the spin path integral using the discrete time action. He discovered an additional *phase factor* which had been previously overlooked. A few years later this factor was rediscovered by Kochetov [41]. He calculated the fluctuations for the spin path integral from the continuous action. In [40] Kochetov suggests a result for arbitrary generalised spin coherent states. His derivation for the spin path integral was extended and improved by Stone et al. [42].

For the path integral of a canonical degree of freedom the situation is similar. We refer to an exhaustive discussion by Baranger et al. [43], where the fluctuation determinant is calculated in discrete time.

There are only very few discussions of the contribution of fluctuations in the case of more general coherent state path integrals. As noted above there is the statement of a result without proof by Kochetov [40]. In addition there is work by Ribeiro et al. [44]. They derive the fluctuation determinant for a spin coupled to a canonical degree of freedom in discrete time. Their method is rather cumbersome and not tailored for higher dimensional generalisation. In fact their result is a special case of the result by Kochetov.

The situation can be summarised as follows: for a spin and a canonical degree of freedom the contributions of fluctuations is well understood. Nevertheless for the case of arbitrary generalised coherent states a clear derivation confirming the result of Kochetov [40] is still lacking. In the following we provide such a derivation using context and notation of the discussion of granular bosonisation.

2.3.2 Setting and result

The generalised coherent states we used in the Fock space approach to granular bosonisation are points in a Kähler manifolds. See for example the review [26] for an overview of the topic. For our purposes it is not necessary to elaborate on these structures. However the objects we introduce and the way we perform the calculations are motivated by the available structure.

To make the notation easier we define

$$F(\bar{Z}, Z) := \det(1 + Z^\dagger Z)\ . \tag{2.9}$$

Note that $\ln F$ is called the Kähler potential. Furthermore we introduce composite indices $\alpha = (a_1, a_2)$ such that $Z_\alpha := Z_{a_1 a_2}$. In the following we use Einstein summation convention, summation over a composite index implies summation over its two subindices. Nevertheless, k still denotes the time steps, whereas the greek indices correspond to the composite indices. The metric is given by

$$g_{\alpha\bar{\beta}} := \partial_{Z_\alpha} \partial_{\bar{Z}_\beta} \ln F\ .$$

Here we have to distinguish between indices belonging to Z or to $\bar{Z}$. Indices belonging to $\bar{Z}$ are written with a bar whenever confusion might otherwise arise. Then, an easy calculation confirms that the volume element is given by the determinant of the metric as it should. The inverse of the metric $g^{\alpha\bar{\gamma}}$ is defined by $g_{\alpha\bar{\beta}} g^{\alpha\bar{\gamma}} = \delta_{\bar{\beta}}^{\bar{\gamma}}$. Using (2.9) the continuous action (2.8) can be rewritten as

$$\begin{aligned}-S =& \frac{N_i}{4} \left[\ln F(\bar{Z}(\beta), Z(\beta)) + \ln F(\bar{Z}(0), Z(0))\right] \\ &+ \frac{N_i}{4} \int_0^\beta d\tau \left(\partial_{\bar{Z}_\beta} \ln F\, \partial_\tau \bar{Z}_\beta - \partial_{Z_\alpha} \ln F\, \partial_\tau Z_\alpha\right) - \int_0^\beta d\tau H\ .\end{aligned}$$

In the following we state what the contributions of fluctuations look like. We derive the result in the next subsection using the discrete time representation.

Theorem 2.1. *The fluctutation determinant with respect to* (2.8) *is given by*

$$c \cdot \left[\frac{\det(\partial_{Z(0)}\partial_{\bar{Z}(\beta)} 2S/N_i)}{g^{1/2}(\beta) g^{1/2}(0)}\right]^{1/2} e^{\frac{1}{2N_i}\int_0^\beta d\tau\, [\partial_{Z_\beta}\left(g^{\beta\bar{\gamma}}\partial_{\bar{Z}_\gamma} H\right) + \partial_{\bar{Z}_\beta}(g^{\gamma\bar{\beta}}\partial_{Z_\gamma} H)]} ,$$

where $c \in \mathbb{C}$ is a proportionality constant that can be fixed by setting $\beta = 0$.

Remark 2.1. *The unusual exponential factor is called Solari-Kochetov phase in the literature (although, in general, it is not a phase factor).*

2.3.3 Derivation using Forman's method

To be sure that the result does not depend on a specific regularisation, we calculate the fluctuation determinant using the discretisation given by the problem. Our result confirms a suggestion by Kochetov [40]. We use Forman's method to calculate the determinant. Using the notation introduced in the last subsection, the action is given by

$$-S = \frac{N_i}{2}\sum_{k=1}^{M-1} \ln \frac{F(\bar{Z}_k, Z_{k-1})}{F(\bar{Z}_k, Z_k)} + \frac{N_i}{2}\ln F(\bar{Z}_M, Z_{M-1}) - \frac{\beta}{M}\sum_{k=1}^{M} H(\bar{Z}_k, Z_{k-1}) .$$

In the following we use the notation $F_{k,l} = F(\bar{Z}_k, Z_l)$ and $H_k := H(\bar{Z}_k, Z_{k-1})$. Hence, the second derivatives evaluate to

$$\begin{aligned}
A_{k,\alpha\beta} &= -\partial_{Z_{k,\alpha}}\partial_{Z_{k,\beta}} 2S/N_i = \partial_{Z_{k,\alpha}}\partial_{Z_{k,\beta}}\left(\ln\frac{F_{k+1,k}}{F_{k,k}} - \frac{2\beta}{N_i M}H_{k+1}\right)\\
B_{k,\bar{\alpha}\bar{\beta}} &= -\partial_{\bar{Z}_{k,\alpha}}\partial_{\bar{Z}_{k,\beta}} 2S/N_i = \partial_{\bar{Z}_{k,\alpha}}\partial_{\bar{Z}_{k,\beta}}\left(\ln\frac{F_{k,k-1}}{F_{k,k}} - \frac{2\beta}{N_i M}H_{k}\right)\\
C_{k,\alpha\bar{\beta}} &= -\partial_{Z_{k,\alpha}}\partial_{\bar{Z}_{k,\beta}} 2S/N_i = \partial_{Z_{k,\alpha}}\partial_{\bar{Z}_{k,\beta}}(-1)\ln F_{k,k} = -g_{\alpha\bar{\beta}}\\
D_{k,\alpha\bar{\beta}} &= -\partial_{Z_{k-1,\alpha}}\partial_{\bar{Z}_{k,\beta}} 2S/N_i = \partial_{Z_{k-1,\alpha}}\partial_{\bar{Z}_{k,\beta}}\left(\ln F_{k,k-1} - \frac{2\beta}{N_i M}H_{k}\right) .
\end{aligned}$$

In particular, the Hessian of $2S/N_i$ is given by

$$Q := \begin{pmatrix}
A_{M-1} & -g_{M-1} & & & & & \\
-g_{M-1} & B_{M-1} & D_{M-1} & & & & \\
& D_{M-1} & A_{M-2} & -g_{M-2} & & & \\
& & & \ddots & & & \\
& & & & D_2 & A_1 & -g_1 \\
& & & & & -g_1 & B_1
\end{pmatrix} .$$

Integrating the fluctuations gives a contribution proportional to $\mathrm{Det}^{-1/2}\, Q$. Next we combine the volume factor coming from the measure with $\mathrm{Det}^{-1/2}\, Q$. Therefore we define

$$g_{k,\alpha\bar{\beta}} := \partial_{Z_{k,\alpha}} \partial_{\bar{Z}_{k,\beta}} \ln F(Z_k, \bar{Z}_k)\ .$$

For the kth time step the volume factor is given by $\det(g_{k,\alpha\bar{\beta}})$. Defining the blockdiagonal matrix

$$R := \mathrm{BlockDiag}(-g_{M-1}, g_{M-1}, \ldots, -g_1, g_1)\ , \qquad (2.10)$$

the contribution from the measure can be written as $\mathrm{Det}^{1/2}\, R$. This leads to the definition of $L_A := R^{-1}Q$. Setting $A'_k := g_k^{-1} A_k$ and similarly for B, C and D we have

$$L_A = \begin{pmatrix} -A'_{M-1} & 1 & & & & & \\ -1 & B'_{M-1} & D'_{M-1} & & & & \\ & -D'_{M-1} & -A'_{M-2} & 1 & & & \\ & & & \ddots & & & \\ & & & -D'_2 & -A'_1 & 1 \\ & & & & -1 & B'_1 \end{pmatrix} .$$

In addition, we define the matrix

$$L := \begin{pmatrix} -D'_M & -A'_{M-1} & 1 & & & & & \\ & -1 & B'_{M-1} & D'_{M-1} & & & & \\ & & -D'_{M-1} & -A'_{M-2} & 1 & & & \\ & & & & \ddots & & & \\ & & & & -D'_2 & -A'_1 & 1 & \\ & & & & & -1 & B'_1 & D'_0 \end{pmatrix} .$$

Next, we view L and L_A as matrix representations of finite difference operators on the vectorspace V of functions that live on the links of the time lattice. The set of links is indexed by $N := \{1, 3, \ldots, M\}$, and the vector space of functions is defined as $V := \{f : N \to \mathbb{C}^{2N_e^2}\}$. Furthermore, let $e_\alpha, e_{\bar{\alpha}}$ denote an orthonormal basis of $\mathbb{C}^{N_e^2} \times \mathbb{C}^{N_e^2}$ with respect to a scalar product $\langle \cdot, \cdot \rangle$. We define a basis $f_{Z_\alpha,i}, f_{\bar{Z}_\alpha,i}$ of V which satisfies $f_{Z_\alpha,i}(j) = e_\alpha \delta_{i,j}$ and $f_{\bar{Z}_\alpha,i}(j) = e_{\bar{\alpha}} \delta_{i,j}$. Demanding that this basis is orthonormal defines a scalar product on V. Now, we view L in the obvious way as the matrix representation of a linear operator on V with respect to the given basis. In addition we can define the subspace $A = \{f \in V | \langle e_{\bar{\alpha}}, f(M) \rangle = 0 = \langle e_\alpha, f(1) \rangle\}$. Let $\pi_A : V \to A$ be the orthogonal projection onto A. Then, it can be easily checked that $L_A = \pi_A L \pi_A$. Thus we see that the matrix L_A, of which we want to calculate the determinant, corresponds to a finite difference operator on V with certain boundary conditions imposed.

Forman's theorem

Forman's theorem [32] can be formulated as follows:

Theorem 2.2. *Let V be a finite dimensional Euclidean or Hermitian vector space and let $L : V \to V$ denote a linear mapping with kernel $K \subset V$. Moreover, let $C \subset V$ be a subspace of V with the property $V = C \oplus K$. Define a restriction of L via $L_A := \pi_A L \pi_A : A \to A$ where $A \subset V$, and π_A is the orthogonal projection onto A. Let $\rho_K^C : V \to K$, $v \mapsto \rho_K^C(v)$ denote the projection operator onto K with respect to the decomposition $V = C \oplus K$. Furthermore $A^\perp$ denotes the orthogonal complement of A within V. Finally, $vol(V) = vol(A^\perp) \wedge vol(A) = vol(C) \wedge vol(C^\perp)$ denote the volume forms that are used to calculate the determinants. In this situation one has:*

$$\begin{aligned} \mathrm{Det}\, L_A = \det(&\pi_{A^\perp}\rho_K^C : C^\perp \to A^\perp) \\ &\times \mathrm{Det}(L : C \to \mathrm{Image}(L))\,\mathrm{Det}(\pi_A : \mathrm{Image}(L) \to A)\ . \end{aligned} \tag{2.11}$$

The theorem is useful if $\mathrm{Det}\, L : C \to \mathrm{Image}(L)$ and the kernel of L are easy to compute.

Application of Forman's theorem

Now we proceed by using Forman's method to calculate $\mathrm{Det}(L_A)$. For this we have to define the remaining objects. In particular, C is given by $C = \{f \in V | f(M) = 0\}$. To define the volume forms we introduce the notation $f_{Z,k} := f_{Z_1,k} \wedge \cdots \wedge f_{Z_N,k}$, and similar for $f_{\bar{Z},k}$. Then the volume forms are

$$\begin{aligned} vol(A) &= f_{\bar{Z},1} \wedge \cdots \wedge f_{Z,M}\ , \\ vol(A^\perp) &= f_{Z,1} \wedge f_{\bar{Z},M}\ , \\ vol(C) &= f_{Z,1} \wedge \cdots \wedge f_{Z,M-1}\ , \\ vol(C^\perp) &= f_{Z,M} \wedge f_{\bar{Z},M}\ , \\ vol(V) &= vol(A^\perp) \wedge vol(A) = vol(C) \wedge vol(C^\perp)\ . \end{aligned}$$

Note that we have $\mathrm{Image}(L) = A$ and hence $\mathrm{Det}(\pi_A : \mathrm{Image}(L) \to A) = 1$. Our choice of C renders the matrix representation of $L : C \to \mathrm{Im}(L)$ lower triangular. The matrix is given by

$$\begin{pmatrix} 1 & & & & & & \\ B'_{M-1} & D'_{M-1} & & & & & \\ -D'_{M-1} & -A'_{M-2} & 1 & & & & \\ & & \ddots & & & & \\ & & -D'_2 & -A'_1 & 1 & & \\ & & & -1 & B'_1 & D'_0 & \end{pmatrix},$$

and $\mathrm{Det}(L : C \to \mathrm{Im}(L))$ is simply the product of the determinants of the blockmatrices on the diagonal:

$$\begin{aligned}\lim_{M\to\infty}\prod_{k=0}^{M-1}\det(D'_k) &= \lim_{M\to\infty}\prod_{k=0}^{M-1}\det\left(\delta^{\alpha}_{\beta}-\frac{2\beta}{N_iM}\partial_{Z_\beta}\left(g^{\alpha\bar{\gamma}}\partial_{\bar{Z}_\gamma}H\right)\right)\\ &= \lim_{M\to\infty}\prod_{k=0}^{M-1}\left(1-\frac{\beta}{M}\frac{2}{N_i}\partial_{Z_\beta}\left(g^{\beta\bar{\gamma}}\partial_{\bar{Z}_\gamma}H\right)\right)\\ &= \exp\left(-\frac{2}{N_i}\int_0^\beta d\tau\;\partial_{Z_\beta}\left(g^{\beta\bar{\gamma}}\partial_{\bar{Z}_\gamma}H\right)\right)\;.\end{aligned}$$

This is already part of the Solari-Kochetov phase, the other part comes from the determinants of the projection operators.

It remains to calculate $\det(\pi_{A^\perp}\rho^C_K : C^\perp \to A^\perp)$. For that we assume that $V = C \oplus K$ holds. To compute the determinant, which is defined with respect to the given volume forms, we have to calculate the images of the basis vectors of $C^\perp$ under $\pi_{A^\perp}\circ\rho^C_K$. For $f \in C^\perp$ and $\tilde{f} := \rho^C_K(f) \in K$ it follows that

$$\tilde{f}(M) = f(M)\;. \tag{2.12}$$

Thus ρ^C_K can be seen as the identification of an element $\tilde{f}$ in the kernel of L with its initial condition $f(M)$.

The image of $f_{Z_\alpha,M}$ is given by

$$\begin{aligned}k^{Z_\alpha} &:= k^{Z_\alpha}_{Z_\beta}(1)f_{Z_\beta,1} + k^{Z_\alpha}_{\bar{Z}_\beta}(M)f_{\bar{Z}_\beta,M}\\ &:= \pi_{A^\perp}\circ\rho^C_K(f_{Z_\alpha,M})\\ &= k^{Z_\alpha}_{Z_\beta}(1)f_{Z_\beta,1}\;,\end{aligned}$$

where we have used that we have by definition of ρ^C_K that

$$\begin{aligned}k^{Z_\alpha}(M) = f_{Z_\alpha,M}(M) = e_\alpha&\\ \Rightarrow k^{Z_\alpha}_{\bar{Z}_\beta}(M) = 0&\;.\end{aligned}$$

And similarly the image of $f_{\bar{Z}_\alpha,M}$ is given by

$$\begin{aligned}k^{\bar{Z}_\alpha} &:= k^{\bar{Z}_\alpha}_{Z_\beta}(1)f_{Z_\beta,1} + k^{\bar{Z}_\alpha}_{\bar{Z}_\beta}(M)f_{\bar{Z}_\beta,M}\\ &:= \pi_{A^\perp}\circ\rho^C_K(f_{\bar{Z}_\alpha,M})\\ &= k^{\bar{Z}_\alpha}_{Z_\beta}(1)f_{Z_\beta,1} + f_{\bar{Z}_\alpha,M}\;.\end{aligned}$$

Now the determinant can be read off from

$$\begin{aligned}\det(\pi_{A^\perp}\circ\rho_K^C : C^\perp \to A^\perp)vol(A^\perp)\\ &= (\pi_{A^\perp}\circ\rho_K^C : C^\perp \to A^\perp)^* vol(C^\perp)\\ &= \prod_\alpha k^{Z_\alpha}_{Z_\beta}(1) f_{Z_\beta,1} \wedge \prod_\alpha (k^{\bar Z_\alpha}_{Z_\beta}(1) f_{Z_\beta,1} + f_{\bar Z_\alpha,M})\\ &= \det(k^{Z_\alpha}_{Z_\beta}(1)) vol(A^\perp)\ .\end{aligned}$$

We now put everything together to obtain

$$\lim_{M\to\infty} \operatorname{Det} L_A = \det(\tilde k^{Z_\alpha}_{Z_\beta}(0)) \exp\left(-\frac{2}{N_i}\int_0^\beta d\tau\, \partial_{Z_\beta}\left(g^{\beta\bar\gamma}\partial_{\bar Z_\gamma} H\right)\right)\ , \qquad (2.13)$$

where $\tilde k^{Z_\alpha}(0) = \lim_{M\to\infty} k^{Z_\alpha}(1)$. The strategy of the next three subsections will be to connect $\tilde k^{Z_\alpha}_{Z_\beta}(0)$ to the continuous action. To that end we first calculate the continuum limit of L.

Continuum limit of L

To obtain the continuum limit of L we have to calculate the matrix entries up to first order in β/M. We begin with a rather detailed calculation for A_k:

$$-(g_k^{-1}A_k)_{\bar\alpha\beta} = g_k^{\bar\alpha\gamma} A_{k,\gamma\beta} \approx \frac{2\beta}{N_i M}\partial_{Z_\beta}\left(g^{\bar\alpha\gamma}\partial_{Z_\gamma} H\right)\ ,$$

where we used that

$$\ln\frac{F_{k+1,k}}{F_{k,k}} \approx \partial_{\bar Z_{k,\delta}} \ln F_{k,k}\ \Delta \bar Z_{k+1,\delta}$$

and

$$g^{\bar\alpha\gamma}\partial_{Z_\gamma}\partial_{Z_\beta}\partial_{\bar Z_\delta}\ln F\ \partial_\tau \bar Z_\delta = -\frac{2}{N_i}\left(\partial_{Z_\beta} g^{\bar\alpha\gamma}\right)\ \partial_{Z_\gamma} H\ .$$

Similar calculations yield:

$$\begin{aligned}(g_k^{-1}B_k)_{\alpha\bar\beta} &\approx -\frac{2\beta}{N_i M}\,\partial_{\bar Z_\beta}\left(g^{\alpha\bar\gamma}\partial_{\bar Z_\gamma} H\right)\ ,\\ (g_k^{-1}D_k)_{\alpha\beta} &\approx \delta^\alpha_\beta - \frac{2\beta}{N_i M}\partial_{Z_\beta}\left(g^{\alpha\bar\gamma}\partial_{\bar Z_\gamma} H\right)\ ,\\ (g_{k-1}^{-1}D_k)_{\bar\alpha\bar\beta} &\approx \delta^{\bar\alpha}_{\bar\beta} - \frac{2\beta}{N_i M}\partial_{\bar Z_\beta}\left(g^{\bar\alpha\gamma}\partial_{Z_\gamma} H\right)\ .\end{aligned}$$

Up to a factor $-\beta/M$ this leads to the first order differential operator

$$\tilde L := \begin{pmatrix} \frac{2}{N_i}\partial_{\bar Z_\gamma}(g^{\alpha\bar\beta}\partial_{\bar Z_\beta} H) & \delta^\gamma_\alpha\partial_\tau + \frac{2}{N_i}\partial_{Z_\gamma}(g^{\alpha\bar\beta}\partial_{\bar Z_\beta} H)\\ \delta^{\bar\gamma}_{\bar\beta}\partial_\tau - \frac{2}{N_i}\partial_{\bar Z_\gamma}(g^{\alpha\bar\beta}\partial_{Z_\alpha} H) & -\frac{2}{N_i}\partial_{Z_\gamma}(g^{\alpha\bar\beta}\partial_{Z_\alpha} H)\end{pmatrix}\ .$$

To make the connection with the continuous action (2.8) we consider its first variation, including variation of the boundary values. This leads to

$$\begin{aligned}\delta S =& \frac{N_i}{2}\left[\partial_{\bar{Z}_\beta}\ln F(\beta)\delta\bar{Z}_\beta + \partial_{Z_\alpha}\ln F(0)\delta Z_\alpha(0)\right]\\ &+\frac{N_i}{2}\int_0^\beta d\tau\delta Z_\alpha\left(\partial_{Z_\alpha}\partial_{\bar{Z}_\beta}\ln F\;\partial_\tau\bar{Z}_\beta - \partial_{Z_\alpha}H\right)\\ &-\frac{N_i}{2}\int_0^\beta d\tau\delta\bar{Z}_\beta\left(\partial_{\bar{Z}_\beta}\partial_{Z_\alpha}\ln F\;\partial_\tau Z_\alpha + \partial_{\bar{Z}_\beta}H\right)\;. \end{aligned} \tag{2.14}$$

The equations of motion are

$$\begin{aligned}\partial_\tau\bar{Z}_\gamma &= \frac{2}{N_i}g^{\alpha\bar{\gamma}}\partial_{Z_\alpha}H\\ \partial_\tau Z_\gamma &= -\frac{2}{N_i}g^{\gamma\bar{\beta}}\partial_{\bar{Z}_\beta}H\;.\end{aligned}$$

In particular, the linearised equations of motion are given by

$$\tilde{L}\begin{pmatrix}\delta\bar{Z}_\gamma\\ \delta Z_\gamma\end{pmatrix} = 0\;.$$

Next we view $\tilde{k}^{Z_\alpha}_{Z_\beta}$ as an element of the kernel of $\tilde{L}$. This allows us to relate $\tilde{k}^{Z_\alpha}_{Z_\beta}(0)$ to solutions of the classical equations of motion and in particular to the action of these solutions.

Relating $\tilde{k}^{Z_\alpha}_{Z_\beta}(0)$ to classical solutions

The kernel of $\tilde{L}$ is spanned by $\partial_{Z_\alpha(0)}(Z_\gamma(t)e_\gamma+\bar{Z}_\gamma(t)e_{\bar{\gamma}})$ and $\partial_{\bar{Z}_\alpha(\beta)}(Z_\gamma(t)e_\gamma+\bar{Z}_\gamma(t)e_{\bar{\gamma}})$, where $Z(t)$ and $\bar{Z}(t)$ are solutions of the classical equations of motion, which are seen as functions of their initial values $Z(0)$ and $\bar{Z}(\beta)$.

We are interested in the special element that fulfils $\tilde{k}^{Z_\alpha}(\beta) = e_\alpha$. It can be written as (no summation convention)

$$\tilde{k}^{Z_\alpha}(t) = \sum_\gamma(\partial_{Z_\alpha(0)}Z_\gamma(\beta))^{-1}\partial_{Z_\alpha(0)}(Z_\gamma(t)e_\gamma + \bar{Z}_\gamma(t)e_{\bar{\gamma}})\;.$$

In the last step we express $\det(\tilde{k}^{Z_\alpha}_{Z_\beta}(0))$ by second derivatives of the action. Therefore consider

$$\begin{aligned}\partial_{Z_\alpha(0)}\partial_{\bar{Z}_\beta(\beta)}2S/N_i &= \partial_{Z_\alpha(0)}\left(\partial_{\bar{Z}_\beta}\ln F\big|_{Z=Z(\beta),\bar{Z}=\bar{Z}(\beta)}\right)\\ &= \left(\partial_{Z_\gamma}\partial_{\bar{Z}_\beta}\ln F\right)\big|_{Z=Z(\beta),\bar{Z}=\bar{Z}(\beta)}\;\partial_{Z_\alpha(0)}Z_\gamma(\beta)\\ &= g_{\gamma\bar{\beta}}(\beta)\,\partial_{Z_\alpha(0)}Z_\gamma(\beta)\\ \Rightarrow \partial_{Z_\alpha(0)}Z_\delta(\beta) &= g^{\delta\bar{\beta}}(\beta)\partial_{Z_\alpha(0)}\partial_{\bar{Z}_\beta(\beta)}2S/N_i\;,\end{aligned}$$

where we used (2.14). This leads to

$$\tilde{k}^{Z_\alpha}_{Z_\delta}(0) = (\partial_{Z_\alpha(0)} Z_\delta(\beta))^{-1} = \left(g^{\delta\bar{\beta}}(\beta) \partial_{Z_\alpha(0)} \partial_{\bar{Z}_\beta(\beta)} 2S/N_i \right)^{-1} .$$

Hence we have

$$\lim_{M\to\infty} \det L_A = \det \left(g^{\delta\bar{\beta}}(\beta) \partial_{Z_\alpha(0)} \partial_{\bar{Z}_\beta(\beta)} 2S/N_i \right)^{-1} \times \exp\left(-\frac{2}{N_i} \int_0^\beta d\tau \; \partial_{Z_\beta} \left(g^{\beta\bar{\gamma}} \partial_{\bar{Z}_\gamma} H \right) \right) .$$

After some additional manipulations we obtain the symmetrised version of the inverse square root of the determinant

$$\lim_{M\to\infty} \det^{-1/2}(L_A) = \left[\frac{\det(\partial_{Z(0)} \partial_{\bar{Z}(\beta)} 2S/N_i)}{g^{1/2}(\beta) g^{1/2}(0)} \right]^{1/2} \times \exp\left(\frac{1}{2N_i} \int_0^\beta d\tau \; [\partial_{Z_\beta}(g^{\beta\bar{\gamma}} \partial_{\bar{Z}_\gamma} H) + \partial_{\bar{Z}_\beta}(g^{\gamma\bar{\beta}} \partial_{Z_\gamma} H)] \right) , \qquad (2.15)$$

which is precisely the statement (2.1).

Conclusion

In the first chapter we proved a general version of the Pruisken-Schäfer hyperbolic Hubbard-Stratonovich transformation. Previous results concerning pseudounitary and pseudoorthogonal symmetry are obtained as special cases. The method of the proof also shows why the transformation holds: The Pruisken-Schäfer domain can be seen as a deformation of the standard Gaussian domain of integration. Thus the transformation is valid, since the integrand is holomorphic and the domain of integration can be deformed without changing the value of the integral. Deformation of the domain of integration requires integration of chains against forms. In this setting, the alternating sign factors appearing in the pseudoorthogonal case have a natural explanation. More precisely, we find that the alternating sign factors are induced by the sign of the product of all weights in $\Sigma_+(\mathfrak{p} \oplus \mathcal{Q}_-, \mathfrak{h})$.

Apart from the Pruisken-Schäfer transformation we also considered the well-established Schäfer-Wegner transformation. We showed that the latter is also a deformation of the standard Gaussian domain.

To sum up, the first chapter yields a unified view on the different Hubbard-Stratonovich identities, and our result allows to rigorously apply the Pruisken-Schäfer transformation. Using these transformation implies in particular the possibility of obtaining results beyond the large N limit.

In the second chapter we developed a path integral representation of the grand canonical partition function for an interacting granular fermionic system in terms of bosonic variables. In particular, we have discussed two different derivations of this representation. The first derivation uses the colour-flavour transformation and the time discrete Grassmann path integral, whereas the second derivation illuminates the structure of the underlying Hilbert spaces. Furthermore we have computed contributions of quadratic fluctuations to the large N limit in terms of the corresponding classical system. The result, concerning the contributions of fluctuations, applies to a wide class of generalised coherent state path integrals. An interesting prospect for the future is to establish applications of the novel bosonisation method developed in chapter two.

Appendix A

Techniques needed in chapter one

A.1 Basic constructions and useful relations

A.1.1 Setting

To make the appendix self contained we repeat the setting of theorem 1.2. For any hermitian $s \in \mathfrak{gl}(n, \mathbb{C})$, with $s^2 = \mathbb{1}$, we have two involutions $\theta(X) = sXs^{-1}$ and $\gamma(X) = -sX^\dagger s^{-1}$ on $\mathfrak{gl}(n, \mathbb{C})$. Let τ_i be additional involutions on $\mathfrak{gl}(n, \mathbb{C})$, which have to commute with each other and with θ and γ. This leads to the definition

$$\mathcal{Q} := \{Q \in \mathfrak{gl}(n, \mathbb{C}) | Q = -\gamma(Q) \text{ and } \forall i \, : \, Q = \sigma_i \tau_i(Q)\} \; ,$$

where $\sigma_i \in \{\pm 1\}$, and the τ_i have to be such that $s \in \mathcal{Q}$.

A.1.2 Symmetries of $\mathcal{Q}$

Define a Lie algebra

$$\mathfrak{g} := \{X \in \mathfrak{gl}(n, \mathbb{C}) | X = \gamma(X) \text{ and } \forall i \, : \, X = \tau_i(X)\} \; ,$$

and let τ be either τ_i or γ. For $X \in \mathfrak{g}$ and $X' \in \mathcal{Q}$, the basic calculation

$$\tau([X, X']) = [\tau(X), \tau(X')] = [X, \tau(X')]$$

shows that $\mathfrak{g}$ is the Lie algebra of the symmetry group of $\mathcal{Q}$.

A.1.3 Commutation relations

The decomposition of $\mathcal{Q}$ into the plus and minus one eigenspaces of θ gives a decomposition into the hermitian and antihermitian parts denoted by $\mathcal{Q}_+$ and $\mathcal{Q}_-$. Similarly, θ gives the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the plus one eigenspace and $\mathfrak{p}$ the minus one eigenspace. This leads to the following commmutation relations:

$$\begin{array}{llll} [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \;, & [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \;, & [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k} \;, & [\mathcal{Q}_+,\mathcal{Q}_-] \subset \mathfrak{p} \;, \\ [\mathcal{Q}_\pm,\mathcal{Q}_\pm] \subset \mathfrak{k} \;, & [\mathfrak{k},\mathcal{Q}_\pm] \subset \mathcal{Q}_\pm \;, & [\mathfrak{p},\mathcal{Q}_\pm] \subset \mathcal{Q}_\mp \;. & \end{array} \tag{A.1}$$

Defining

$$\begin{aligned} f : \mathfrak{p} \oplus \mathcal{Q}_- &\to \mathfrak{p} \oplus \mathcal{Q}_- \\ X &\mapsto \frac{1}{2}[X,s] \;, \end{aligned}$$

it is easy to check that the inverse is given by

$$\begin{aligned} f^{-1} : \mathfrak{p} \oplus \mathcal{Q}_- &\to \mathfrak{p} \oplus \mathcal{Q}_- \\ X &\mapsto \frac{1}{2}[X,s^{-1}] \;. \end{aligned}$$

In particular we have

$$f(\mathfrak{p}) = \mathcal{Q}_- \;\text{ and }\; f(\mathcal{Q}_-) = \mathfrak{p} \;. \tag{A.2}$$

Note that we always have $s \in \mathcal{Q}_+$.

A.1.4 Decompositions

For $H, X, Y \in \mathfrak{gl}(n,\mathbb{C})$ with $H = H^\dagger$ we have

$$\mathrm{Tr}([H,X]Y^\dagger) = \mathrm{Tr}(X[Y^\dagger,H]) = \mathrm{Tr}(X([H,Y])^\dagger) \;.$$

This shows that $[H,\cdot]$ is hermitian with respect to $\mathrm{Tr}(XY^\dagger)$, which is a hermitian scalar product on the space of complex matrices. Hence $[H,\cdot]$ has real eigenvalues, and the corresponding eigenspaces are orthogonal. In the following we choose sets of commuting hermitian matrices, and diagonalise the commutator or adjoint action simultaneously.

Decomposition of $\mathfrak{g}$

The first set we consider is a maximal Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. We consider the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$. This leads to the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha \;.$$

In this setting the eigenvalues α are called roots. The set of roots is denoted by $\Sigma(\mathfrak{g},\mathfrak{a})$. For $Z \in \mathfrak{g}_\alpha$ and $H \in \mathfrak{a}$ we have

$$[H,\theta(Z)] = -\theta([H,Z]) = -\alpha(H)\theta(Z) \;,$$

and hence $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$. This shows that

$$\mathfrak{g}_{\pm,\alpha} := \mathrm{Fix}_{\pm\theta}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \tag{A.3}$$

is well defined. Choosing $H_o \in \mathfrak{a}$ with $\alpha(H_o) \neq 0$ for all α, we have a notion of positivity for the roots α. Let $\Sigma_+(\mathfrak{g}, \mathfrak{a})$ denote the set of positive roots. Then we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma_+(\mathfrak{g},\mathfrak{a})} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \ . \tag{A.4}$$

Note that $\mathfrak{g}_0 = Z_K(\mathfrak{a}) \oplus \mathfrak{a}$. Furthermore we can also define a mapping ϕ as

$$\begin{aligned} \phi : \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} &\to \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \\ Z + Z' &\mapsto \frac{1}{\alpha(H_o)}[H_o, Z + Z'] = Z - Z' \ , \end{aligned} \tag{A.5}$$

which has the properties

$$\phi \circ \phi = id \ \text{ and } \ \phi(\mathfrak{g}_{\pm,\alpha}) = \mathfrak{g}_{\mp,\alpha} \ . \tag{A.6}$$

The mapping ϕ obviously extends to a mapping on $\bigoplus_{\alpha\in\Sigma_+(\mathfrak{g},\mathfrak{a})} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Furthermore for $X \in \mathfrak{g}_{\pm,\alpha}$ and $H \in \mathfrak{a}$ we have

$$[H, X] = \alpha(H)\phi(X) \text{ and } [H, \phi(X)] = \alpha(H)X \ . \tag{A.7}$$

In addition we define orthogonal projections

$$\pi_{\pm,\alpha} : \mathfrak{g} \to \mathfrak{g}_{\pm,\alpha} \ . \tag{A.8}$$

Next we discuss additional decompositions. For each decomposition we define the appropriate mappings ϕ and $\pi_{\pm,\alpha}$. Although they are also called ϕ and $\pi_{\pm,\alpha}$, it is always clear from the context which mapping is meant.

Decomposition of $\mathcal{Q}$

Next, we consider the adjoint action of $\mathfrak{a}$ on $\mathcal{Q}$, which gives us the decomposition

$$\mathcal{Q} = \mathcal{Q}_0 \oplus \bigoplus_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} (\mathcal{Q}_\alpha \oplus \mathcal{Q}_{-\alpha}) \ . \tag{A.9}$$

Using the same reasoning as above we obtain that

$$\mathcal{Q}_{\pm,\alpha} := \mathrm{Fix}_{\pm\theta}(\mathcal{Q}_\alpha \oplus \mathcal{Q}_{-\alpha}) \subset \mathcal{Q}_\pm \tag{A.10}$$

is well defined. The same is true for $\mathcal{Q}_{\pm,0} := \mathrm{Fix}_{\pm\theta}(\mathcal{Q}_0) \subset \mathcal{Q}_\pm$. Choosing $H_o \in \mathfrak{a}$ with $\alpha(H_o) \neq 0$ for all $\alpha \in \Sigma(\mathcal{Q}, \mathfrak{a})$, we again have a notion of positivity. Hence we have the decompositions

$$\mathcal{Q}_\pm = \mathcal{Q}_{\pm,0} \oplus \bigoplus_{\alpha\in\Sigma_+(\mathcal{Q},\mathfrak{a})} \mathcal{Q}_{\pm,\alpha} \; . \tag{A.11}$$

Now we can define

$$\begin{aligned} \phi : \mathcal{Q}_\alpha \oplus \mathcal{Q}_{-\alpha} &\to \mathcal{Q}_\alpha \oplus \mathcal{Q}_{-\alpha} \\ Z + Z' &\mapsto \frac{1}{\alpha(H_o)}[H_o, Z + Z'] = Z - Z' \; . \end{aligned} \tag{A.12}$$

Note that ϕ satisfies the corresponding relations to (A.6) and (A.7). In addition we define orthogonal projections π_α:

$$\pi_{\pm,\alpha} : \mathcal{Q} \to \mathcal{Q}_{\pm,\alpha} \; . \tag{A.13}$$

Decomposition of $\mathfrak{k} \oplus \mathcal{Q}_+$

Secondly, we choose a maximal Abelian subalgebra $\mathfrak{h}$ of $\mathcal{Q}_+$ such that $s \in \mathfrak{h}$ and consider its adjoint action on

$$\tilde{\mathfrak{g}} := \mathfrak{k} \oplus \mathcal{Q}_+ \; . \tag{A.14}$$

Similar to the above we obtain the decomposition

$$\tilde{\mathfrak{g}} = Z_{\mathfrak{k}}(\mathfrak{h}) \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Sigma_+(\tilde{\mathfrak{g}},\mathfrak{h})} (\tilde{\mathfrak{g}}_\alpha \oplus \tilde{\mathfrak{g}}_{-\alpha}) \; . \tag{A.15}$$

In addition we define the mapping ϕ as

$$\begin{aligned} \phi : \tilde{\mathfrak{g}}_\alpha \oplus \tilde{\mathfrak{g}}_{-\alpha} &\to \tilde{\mathfrak{g}}_\alpha \oplus \tilde{\mathfrak{g}}_{-\alpha} \\ Z + Z' &\mapsto \frac{1}{\alpha(H_o)}[H_o, Z + Z'] = Z - Z' \; . \end{aligned} \tag{A.16}$$

Note that H_0 is of course choosen such that $\alpha(H_0) \neq 0$ for all $\alpha \in \Sigma(\tilde{\mathfrak{g}}, \mathfrak{h})$ and ϕ satisfies the analogous relations to (A.6) and (A.7). Defining

$$\tilde{\mathfrak{g}}_{\pm,\alpha} := \mathrm{Fix}_{\pm\gamma}(\tilde{\mathfrak{g}}_\alpha \oplus \tilde{\mathfrak{g}}_{-\alpha}) \tag{A.17}$$

the orthogonal projection operators are denoted by

$$\pi_{\pm,\alpha} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_{\pm,\alpha} \; . \tag{A.18}$$

Decomposition of $\mathfrak{p} \oplus \mathcal{Q}_-$

Next we consider the adjoint action of $\mathfrak{h}$ on

$$\tilde{\mathcal{Q}} := \mathfrak{p} \oplus \mathcal{Q}_- \ . \tag{A.19}$$

In the following we list the analogous definitions and objects: The decomposition of $\tilde{\mathcal{Q}}$ is given by

$$\tilde{\mathcal{Q}} = \bigoplus_{\alpha \in \Sigma_+(\tilde{\mathcal{Q}},\mathfrak{h})} (\tilde{\mathcal{Q}}_\alpha \oplus \tilde{\mathcal{Q}}_{-\alpha}) \ . \tag{A.20}$$

Note that, using $s \in \mathfrak{h}$ and (A.2), it can be easily checked that $\tilde{\mathcal{Q}}_0$ is trivial. Since we have $\alpha(s) \neq 0$ for all α it is useful to use s to obtain a notion of positivity for the weights. This special choice implies in particular that the property $\alpha > 0$ is invariant under the action of the Weyl group, since s is $\mathrm{Ad}(K)$ invariant. Furthermore we define

$$\tilde{\mathcal{Q}}_{\pm,\alpha} := \mathrm{Fix}_{\pm\gamma}(\tilde{\mathcal{Q}}_\alpha \oplus \tilde{\mathcal{Q}}_{-\alpha}) \ , \tag{A.21}$$

and

$$\begin{aligned} \phi : \tilde{\mathcal{Q}}_\alpha \oplus \tilde{\mathcal{Q}}_{-\alpha} &\to \tilde{\mathcal{Q}}_\alpha \oplus \tilde{\mathcal{Q}}_{-\alpha} \\ Z + Z' &\mapsto \frac{1}{\alpha(s)}[s, Z + Z'] = Z - Z' \ , \end{aligned} \tag{A.22}$$

which satisfies the corresponding relations to (A.6) and (A.7). In addition we define

$$\tilde{\mathcal{Q}}_{\pm,\alpha} := \mathrm{Fix}_{\pm\gamma}(\tilde{\mathcal{Q}}_\alpha \oplus \tilde{\mathcal{Q}}_{-\alpha}) \tag{A.23}$$

and the orthogonal projections

$$\pi_{\pm,\alpha} : \tilde{\mathcal{Q}} \to \tilde{\mathcal{Q}}_{\pm,\alpha} \ . \tag{A.24}$$

A.1.5 Orthogonality relations

We discuss orthogonality with respect to $\mathrm{Tr}(XY^\dagger)$. In particular we are interested in the corresponding relations for $B(X,Y) = \mathrm{Tr}(XY)$. A typical argument for $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$ goes as follows:

$$B(X,Y) = \mathrm{Tr}(XY^\dagger) = \mathrm{Tr}(sXssY^\dagger s) = -\mathrm{Tr}(XY^\dagger) = 0 \ . \tag{A.25}$$

The same holds for $X \in \mathcal{Q}_+ \oplus Y \in \mathcal{Q}_-$. This implies that the decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathcal{Q} = \mathcal{Q}_+ \oplus \mathcal{Q}_-$ are orthogonal. The following relations for $X_\alpha \in \mathcal{Q}_{+,\alpha}$, $X, X' \in \bigoplus_{\alpha>0} \mathcal{Q}_{+\alpha}$ hold:

a) $B(X_\alpha, X_\beta) = -B(\phi(X_\alpha), \phi(X_\beta)) = \delta_{\alpha,\beta} B(X_\alpha, X_\alpha)$

b) $B(X, X') = -B(\phi(X), \phi(X'))$

c) $B(X_\alpha \pm \phi(X_\alpha), X_\beta \pm \phi(X_\beta)) = 0$

d) $B(X_\alpha + \phi(X_\alpha), X_\beta - \phi(X_\beta)) = \delta_{\alpha,\beta} 2B(X_\alpha, X_\alpha)$.

Indeed, note that $X_\alpha \pm \phi(X_\alpha) \in \mathcal{Q}_{\pm\alpha}$, and that for $X, Y \in \mathcal{Q}$ $\mathrm{Tr}(XY^\dagger) = \mathrm{Tr}(X\theta(Y))$ holds. This immediately gives c). Using (A.25), gives a) and hence d). b) is an immediate consequence of a).

A.2 Three additional arguments

A.2.1 Reparametrisation R_I

Here we want to show that R_I is injective and regular. Note that $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is the Weyl group, which acts simply transitive on the Weyl chambers [14]. Let $H \in \mathfrak{a}_+^o$ and $k \in K$, then we have

$$\begin{aligned} kHk^{-1} &= k'H'k'^{-1} \\ \Leftrightarrow\ k'^{-1}kH(k'^{-1}k)^{-1} &= H' \ . \end{aligned}$$

This shows that $\tilde{k} := k'^{-1}k \in N_K(\mathfrak{a})$ and thus $\tilde{k} = wz$ with w in the Weyl group and $z \in Z_K(\mathfrak{a})$. It follows that $wHw^{-1} = H'$, which implies that $w = \mathbb{1}$, $H = H'$, and hence $[\tilde{k}] = [\mathbb{1}]$. This proves injectivity. The differential is given by

$$\begin{aligned} d(kHk^{-1}) &= \mathrm{Ad}(k)\left([k^{-1}dk, H] + dH\right) \\ &= \mathrm{Ad}(k)\left(\sum_{\alpha\in\Sigma_+(\mathfrak{g},\mathfrak{a})} \alpha(H)\ \pi_{-,\alpha} \circ \phi(k^{-1}dk) + dH\right) \ . \end{aligned}$$

All $\alpha(H) \neq 0$, and hence the differential has full rank.

A.2.2 Pullback of dQ

In this subsection we use

$$\begin{aligned} PS \circ R :\ G/Z_K(\tilde{\mathfrak{a}}) \times \tilde{a}_+^o \times \mathfrak{h}' &\to \mathcal{Q} \\ ([g], \tilde{H}, H') &\mapsto g(\tilde{H} + H')g^{-1} \ . \end{aligned}$$

to pullback the constant volume form dQ on $\mathcal{Q}$. In the following we use the notation and the constructions of section A.1.4. Consider the adjoint action of $\mathfrak{h}$ on

$$\mathfrak{g} \oplus \mathcal{Q} = \tilde{\mathfrak{g}} \oplus \tilde{\mathcal{Q}} \ ,$$

which leads to the decomposition

$$\mathcal{Q} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Sigma_+(\tilde{\mathfrak{g}},\mathfrak{h})} \tilde{\mathfrak{g}}_{-,\alpha} \oplus \bigoplus_{\alpha\in\Sigma_+(\tilde{\mathcal{Q}},\mathfrak{h})} \tilde{\mathcal{Q}}_{-,\alpha} \; . \tag{A.26}$$

The corresponding set of weights is given by

$$\Sigma_+ := \Sigma_+(\tilde{\mathfrak{g}},\mathfrak{h}) \cup \Sigma_+(\tilde{\mathcal{Q}},\mathfrak{h}) \; .$$

For $\alpha \in \Sigma_+$ we define a basis $\{e_{\alpha,i}\}$ of $\tilde{\mathfrak{g}}_{-,\alpha}$ and $\tilde{\mathcal{Q}}_{-,\alpha}$. Note that $i \in \{1,\ldots,d_\alpha\}$, where d_α denotes the dimension of the corresponding weight space of α. Furthermore let dH denote a constant volume form on $\mathfrak{h}$. Now we can compute $(PS\circ R)^*dQ$:

$$\begin{aligned}
&(PS\circ R)^*dQ\\
&\quad = dQ[\mathrm{Ad}(g)([g^{-1}dg, \tilde{H}+H'] + d\tilde{H} + dH')]\\
&\quad = dQ[\sum_{\alpha\in\Sigma_+} \alpha(\tilde{H}+H')\phi\circ\pi_{+,\alpha}(g^{-1}dg) + d\tilde{H} + dH']\\
&\quad = \prod_{\alpha\in\Sigma_+} \alpha(\tilde{H}+H')^{d_\alpha} \bigwedge_{\alpha\in\Sigma}\bigwedge_{i=1}^{d_\alpha} \left[de_{\alpha,i}\circ\phi\circ\pi_{+,\alpha}(g^{-1}dg)\right] \wedge dH\\
&\quad =: \Delta(\tilde{H}+H')\, d\mu([g]) \wedge dH \; .
\end{aligned} \tag{A.27}$$

The first equality in (A.27) is obtained by elementary calculation. For the second equality we have used equations (A.20), (A.15) and (A.26) and the fact that $\det(\exp([X,\cdot])) = 1$ for $X \in \mathfrak{g}$. The third equality just uses the basis $\{e_{\alpha,i}\}$ we defined above. Note that comparison of the second and the last line in (A.27) shows that $d\mu([g])$ is a left invariant volume form on $G/Z_K(\tilde{\mathfrak{a}})$.

Note that $\alpha(\tilde{H}+H') \in \mathbb{R}$, since $[\tilde{H}+H',\cdot]$ is hermitian with respect to $\mathrm{Tr}(XY^\dagger)$. For $\alpha \in \Sigma_{\mathfrak{k}\oplus\mathcal{Q}_+}$ we have $\alpha(H') = 0$. Therefore we obtain:

$$\Delta(\tilde{H}+H') = \prod_{\alpha\in\Sigma_{\mathfrak{p}\oplus\mathcal{Q}_-},\alpha>0} \alpha(\tilde{H}+H')^{d_\alpha} \cdot \prod_{\alpha\in\Sigma_{\mathfrak{k}\oplus\mathcal{Q}_+},\alpha>0} \alpha(\tilde{H})^{d_\alpha} \; .$$

A.2.3 Contributions from $\partial\mathfrak{a}_+$

Here we give the detailed argument showing the contributions from $\partial\mathfrak{a}_+$ are irrelevant for PS, ePS and $DePS$, since they are at least of codimension two. In doing so we recall the argument for the well definedness of these parametrisations.

Let $H_i \in \mathfrak{a}_{+,c} \cap \partial\mathfrak{a}_+$ be an edge of $\mathfrak{a}_+$. We show that the restriction of PS_c to the domain of definition with $h^i = 0$ yields a domain of at least codimension two. It is well known that the dimension of the isotropy group of $\mathfrak{a}$ changes at the boundary of $\mathfrak{a}_+$. In particular $\dim Z_K(\{\sum_{j\neq i} h^j H_j\}) > \dim Z_K(\mathfrak{a})$. This can be seen from the fact that to each face of the boundary of the Weyl chamber there is associated a restricted root α with root space $\mathfrak{g}_\alpha$ and $\alpha(\sum_{j\neq i} h^j H_j) = 0$. The group generated by $\mathrm{Fix}_\theta(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ leaves the face invariant, and $\mathrm{Fix}_\theta(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \not\subset Z_{\mathfrak{k}}(\mathfrak{a})$. If we restrict $PS \circ R_I$ to such a face of $\mathfrak{a}_+$, we can replace $K/Z_K(\mathfrak{a})$ by the lower dimensional space $K/Z_{K,i}$, with $Z_{K,i} := Z_K(\{\sum_{j\neq i} h^j H_j\})$ without changing the image of the parametrisation. In fact the eigenspace decomposition of $\mathcal{Q}$ with respect to $\mathfrak{a}$ restricts to an eigenspace decomposition with respect to the smaller algebra $\{\sum_{j\neq i} h^j H_j\}$. Hence all parametrisations restrict to parametrisations where the quotient group is $Z_{K,i}$. This completes the argument for PS_c.

For ePS we have to argue that the analogous restriction is well defined. For that it is enough to note that for $X \in Lie(Z_{k,i})$ we have $[X, E_j] = 0$ if $j \neq i$. The analogous restriction of $DePS$ is well defined for the same reason. Thus we see that for both ePS and $DePS$ the contributions from $\partial\mathfrak{a}_+$ are of codimension at least two.

Appendix B

Techniques needed in chapter two

A major tool of chapter two are generalised coherent states. These coherent states are built using group representations on fermionic Fock space. All groups representations we consider are subgroups of the large group of canonical transformations of fermionic Fock space. A second ingredient is a theorem by Howe on dual groups within the group of canonical transformations. This theorem leads us to the colour-flavour transformation.

Everything stated in this appendix is well known. Coherent states are discussed extensively in Perelomov [25]. Howe pairs are discussed in [27, 28]. The colour-flavour transformation is discussed in [29, 30]. Since the material is spread over different articles, and to introduce the notation we are using in chapter two, we develop the necessary constructions explicitly. However, some mathematical theorems are stated without proof.

B.1 Canonical transformations of fermionic Fock space

Our first aim is to construct group actions on fermionic Fock space. Denote the creation operators and annihilation operators by $c_n^\dagger$ and c_n. Then we can use the anticommutator to define a bilinear form by $\langle \cdot, \cdot \rangle := \{\cdot, \cdot\}$. Since $c_n^\dagger$ and c_n satisfy canonical anticommutation relations (CAR), the basis defined by

$$
\begin{aligned}
v_i &:= \frac{1}{\sqrt{2}}(c_i + c_i^\dagger) \\
v_{N+i} &:= \frac{i}{\sqrt{2}}(c_i - c_i^\dagger)
\end{aligned}
$$

is orthonormal with respect to $\langle\cdot,\cdot\rangle$. It is easy to check that

$$\langle [v_i v_j, v_k], v_l\rangle = -\langle v_k, [v_i v_j, v_l]\rangle$$

holds. Therefore $[v_i v_j, \cdot]$ generates infinitesimal isometries of $\langle\cdot,\cdot\rangle$ and thus canonical transformations. For an antisymmetric matrix Z with complex entries the following identities may also easily be checked:

$$\begin{aligned} Z^{ij}[v_i v_j, v_k] &= 2 v_i Z^{ik} \, , \\ [v_i Z^{ij} v_j, v_k \tilde{Z}^{kl} v_l] &= v_i \left([Z, \tilde{Z}]\right)^{ij} v_j \, . \end{aligned}$$

The last line gives a Lie-algebra isomorphism from $\mathfrak{so}(2N,\mathbb{C})$ to the vectorspace generated by $v_i v_j$ with the commutator as a Lie-bracket and the algebraic structure from the anticommutator $\{\cdot,\cdot\}$. Next we transform to the natural basis given directly by creation and annihilation operators. In order to do this we have to split up Z into $N\times N$ blocks

$$Z = \begin{pmatrix} A' & B' \\ -B'^t & D' \end{pmatrix} ,$$

where A' and D' are antisymmetric and B' is an arbitrary complex matrix. Then we have

$$\begin{aligned} v^t Z v =& \frac{1}{2}(c^\dagger c) \begin{pmatrix} A' + D' + iB' + iB'^t & A' - D' - iB' + iB'^t \\ A' - D' + iB' - iB'^t & A' + D' - iB' - iB'^t \end{pmatrix} \left(\begin{smallmatrix} c \\ c^\dagger \end{smallmatrix} \right) \\ =:& \frac{1}{2}(c^\dagger c) \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \left(\begin{smallmatrix} c \\ c^\dagger \end{smallmatrix} \right) =: \frac{1}{2}(c^\dagger c) M \left(\begin{smallmatrix} c \\ c^\dagger \end{smallmatrix} \right) =: \hat{M}, \end{aligned}$$

where A is an arbitrary complex matrix, whereas B and C are antisymmetric complex matrices. This is equivalent to the condition $M = -\Sigma_x M^t \Sigma_x$, with $\Sigma_x := \sigma_x \otimes \mathbb{1}_N$. The set of M matrices is a Lie algebra isomorphic to $\mathfrak{so}(2N,\mathbb{C})$ and the mapping $\hat{}: M \mapsto \hat{M}$ is a Lie algebra isomorphism.

The commutator action of $\hat{M}$ on the creation and annihilation operators is given by

$$[\hat{M}, (c^\dagger c)] = (c^\dagger c) \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

and thus we have

$$\exp(\hat{M})(c^\dagger c)\exp(-\hat{M}) = \exp([\hat{M},\cdot])(c^\dagger c) = (c^\dagger c)\exp(M) \, .$$

Demanding that the canonical transformations respect $(c^\dagger)^\dagger = c$ implies that $M = -M^\dagger$. Hence Z has to be real and the corresponding Lie algebra is $\mathfrak{so}(N,\mathbb{R})$. Exponentiating the Lie algebra representation $\hat{M}$ leads to a unitary representation of the spin group $Spin(N)$, which is a double covering of the special orthogonal group.

B.1.1 Matrix elements

Denote by $\langle\psi|$ and $|\psi\rangle$ standard fermionic coherent states. In the following we calculate the matrix element $\langle\psi|\exp(\hat{M})|\psi\rangle$. To proceed we decompose

$$g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \exp(M)$$

and then lift the decomposition to a decomposition of $\hat{g} := \exp(\hat{M})$. If D is invertible the Gaussian decomposition of g is given by

$$\begin{aligned} g &= \begin{pmatrix} \mathbb{1} & bd^{-1} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ d^{-1}c & \mathbb{1} \end{pmatrix} \\ &=: g_1\, g_2\, g_3\ . \end{aligned}$$

The single factors however live in the complex orthogonal group defined by $\{g \in GL(N,\mathbb{C}) | g^t \Sigma_x g = \Sigma_x\}$. Defining

$$M_1 := \begin{pmatrix} 0 & bd^{-1} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad M_2 := \begin{pmatrix} 0 & 0 \\ d^{-1}c & 0 \end{pmatrix}$$

we have $g_1 = \exp(M_1)$ and $g_3 = \exp(M_3)$. Hence, defining $\hat{g}_1 := \exp(\hat{M}_1)$, $\hat{g}_3 := \exp(\hat{M}_3)$ and $\hat{g}_2 := \hat{g}_1^{-1}\hat{g}\hat{g}_3^{-1}$, we obtain the decomposition $\hat{g} = \hat{g}_1\hat{g}_2\hat{g}_3$. Note that $\pi(\hat{g}_2) = g_2$, where π denotes the covering homomorphism. Assuming that there exists a matrix M_2 such that $\hat{g}_2 = \sigma \exp \hat{M}_2$, with σ being either plus or minus one, it is easy to show that

$$\begin{aligned} \langle\psi|\exp(\hat{M})|\psi\rangle &= \langle\psi|\hat{g}_1\hat{g}_2\hat{g}_3|\psi\rangle \\ &= \sigma \mathrm{Det}^{1/2}(d)\; e^{\frac{1}{2}\bar{\psi}bd^{-1}\bar{\psi}} e^{\frac{1}{2}\psi d^{-1}c\psi} e^{-\psi d^{-1}\bar{\psi}} \end{aligned} \tag{B.1}$$

holds. In addition we have

$$\langle\psi|\exp(\hat{M})|0\rangle = \sigma \mathrm{Det}^{1/2}(d) e^{\frac{1}{2}\bar{\psi}bd^{-1}\bar{\psi}}\ .$$

Using that $g^{-1} = g^\dagger$ we obtain

$$\langle 0|\exp(-\hat{M})|\psi\rangle = \sigma \mathrm{Det}^{1/2}(d^\dagger) e^{\frac{1}{2}\psi(bd^{-1})^\dagger\psi}\ ,$$

and $dd^\dagger = (1 + (bd^{-1})^\dagger bd^{-1})^{-1}$. Defining $Z := bd^{-1}$ we have

$$\langle\psi|\exp(\hat{M})|0\rangle\langle 0|\exp(-\hat{M})|\psi\rangle = \mathrm{Det}^{1/2}(1 + Z^\dagger Z) e^{\frac{1}{2}\bar{\psi}Z\bar{\psi}} e^{\frac{1}{2}\psi Z^\dagger\psi}\ . \tag{B.2}$$

It is important to note that the matrix elements (B.1) and (B.2) only depend on g and not on $\hat{g}$. Furthermore, there is no non-analyticity hidden in $\mathrm{Det}^{1/2}$ since Z is antisymmetric and

$$\mathrm{Det}^{1/2}(1 + Z^\dagger Z) = \mathrm{Pf}\begin{pmatrix} Z & -\mathbb{1} \\ \mathbb{1} & Z^\dagger \end{pmatrix}\ .$$

Pf denotes the Pfaffian.

B.2 Fermionic Howe pairs and colour-flavour transformation

A fermionic Howe dual pair (G, K) is a pair of two subgroups G and K of $\mathrm{SO}(2N)$ that centralise each other. In the bosonic case it becomes important to demand in addition that the pair is reductive. The mathematical theory of these pairs is developed in [28] and physical applications are given in [27]. We formulate a basic theorem on dual pairs in our context.

Theorem B.1. *Let (G, K) be a fermionic Howe dual pair and $\hat{H}$ be the Hamiltonian of a granular fermionic system, i.e., $\hat{H}$ is a polynomial in creation and annihilation operators and has a classical symmetry group K. Then $\hat{H}$ is a polynomial in the generators of G. In particular the quadratic invariants of K are the generators of G.*

The fermionic pairs are listed in table B.1. H denotes the subgroup of G

$\mathrm{SO}(2N)$	K	G	H
$N = (p+q)N_i$	$\mathrm{U}(N_i)$	$\mathrm{U}(p+q)$	$\mathrm{U}(p) \times \mathrm{U}(q)$
$N = N_e N_i$	$\mathrm{O}(N_i)$	$\mathrm{SO}(2N_e)$	$\mathrm{U}(N_e)$
$N = 2N_e N_i$	$\mathrm{USp}(2N_i)$	$\mathrm{USp}(2N_e)$	$\mathrm{U}(N_e)$

Table B.1: Fermionic Howe pairs

which leaves the vacuum state $|0\rangle$ invariant. It is known from [27, 28] that there are two different representations for the projector onto the subspace of states which are invariant under the K action, i.e. $\hat{k}|v\rangle = \rho(k)|v\rangle$ for $k \in K$. Here ρ is a one dimensional representation of G. The different representations for the projector P are

$$P = \int_K dk \; \rho(\hat{k}^{-1}) \; \hat{k} = \int_G dg \; \hat{g}|0\rangle\langle 0|\hat{g}^{-1} \; .$$

In the following we will evaluate $\langle\psi|P|\psi\rangle$ for the different representations for each of the three fermionic Howe dual pairs. Note that we will extend the notation of the previous section appropriately. Lie algebra elements of $Lie(G)$ are denoted by M_e and elements of $Lie(K)$ are denoted by M_i and so on. Note that $M_{i/e}$ inherits symmetry properties of $M \in \mathfrak{so}(2N)$, which we do not state explicitely in the following. Only the additional symmetries are discussed. Eventually an additional two by two substructure requires introducing a $\pm$ index.

B.2.1 $\mathrm{U}(p+q)\otimes\mathrm{U}(N_i)$

In the following we state how to realise $\mathfrak{u}(p+q)$ and $\mathfrak{u}(N_i)$ as commuting subalgebras of $\mathfrak{so}(2(p+q)N_i)$. $\mathfrak{u}(p+q)$ is given by matrices of the form

$$M_e := \begin{pmatrix} A_{e+} & 0 & 0 & B_{e+} \\ 0 & A_{e-} & -B_{e+}^t & 0 \\ 0 & \bar{B}_{e+} & -A_{e+}^t & 0 \\ -B_{e+}^\dagger & 0 & 0 & -A_{e-}^t \end{pmatrix} \otimes \mathbb{1}_{N_i} \ ,$$

where A_{e+} is a $p\times p$ matrix and A_{e-} is a $q\times q$ matrix. $\mathfrak{u}(N)$ is realised by

$$M_i := \begin{pmatrix} \mathbb{1}_p\otimes A_i & 0 & 0 & 0 \\ 0 & -\mathbb{1}_q\otimes A_i^t & 0 & 0 \\ 0 & 0 & -\mathbb{1}_p\otimes A_i^t & 0 \\ 0 & 0 & 0 & \mathbb{1}_q\otimes A_i \end{pmatrix} .$$

There are no additional symmetry requirements beyond the ones the matrices inherit from $\mathfrak{so}(2(p+q)N)$. To evaluate the matrix elements (B.1) and (B.2), define $u := \exp(A_i)$ and

$$g := \begin{pmatrix} a_e & b_e \\ c_e & d_e \end{pmatrix} := \exp\begin{pmatrix} A_{e+} & B_{e+} \\ -B_{e+}^\dagger & -A_{e-}^t \end{pmatrix} .$$

Next we calculate $\exp(M_e)$. To this end it is useful that

$$\exp\begin{pmatrix} A_{e-} & -B_{e+}^t \\ \bar{B}_{e+} & -A_{e+}^t \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \bar{k} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

holds. Then we can read off

$$\exp(M_e) = \begin{pmatrix} a_e & 0 & 0 & b_e \\ 0 & \bar{d}_e & \bar{c}_e & 0 \\ 0 & \bar{b}_e & \bar{a}_e & 0 \\ c_e & 0 & 0 & d_e \end{pmatrix} \otimes \mathbb{1}_{N_i} .$$

Hence we have

$$Z = Z_e \otimes \mathbb{1}_{N_i} = \begin{pmatrix} 0 & b_e d_e^{-1} \\ \bar{c}_e \bar{a}_e^{-1} & 0 \end{pmatrix} \otimes \mathbb{1}_{N_i} .$$

In particular Z is antisymmetric, which implies that the right hand side only depends on $Z_+ := b_e d_e^{-1}$. Thus we have

$$\begin{aligned} \int_{\mathrm{U}(N_i)} du\, e^{\bar{\psi}\tilde{u}\psi} &= \int_{\mathrm{U}(p+q)} dk\, \mathrm{Det}^{N_i/2}(1+Z_e^\dagger Z_e)\, e^{\frac{1}{2}\bar{\psi}Z\bar{\psi}} e^{\frac{1}{2}\psi Z^\dagger\psi} \\ &= \int_{\mathrm{U}(p+q)} dk\, \mathrm{Det}^{N_i}(1+Z_+^\dagger Z_+)\, e^{\bar{\psi}_+ Z_+\otimes \mathbb{1}_{N_i}\bar{\psi}_-} e^{\psi_- Z_+^\dagger\otimes\mathbb{1}_{N_i}\psi_+} \ , \end{aligned} \tag{B.3}$$

where

$$\tilde{u} = \begin{pmatrix} \mathbb{1}_p \otimes u & 0 \\ 0 & \mathbb{1}_q \otimes \bar{u} \end{pmatrix} .$$

In fact the set of Z_+ matrices is a parametrisation of the quotient space $\mathrm{U}(p+q)/\mathrm{U}(p) \times \mathrm{U}(q)$. However for our purpose it is not necessary to know the exact form of the induced measure $d\mu(Z^\dagger, Z)$.

B.2.2 $\mathrm{SO}(2N_e) \times \mathrm{O}(N_i)$

In the following we state how to realise $\mathfrak{so}(2N_e)$ and $\mathfrak{o}(N_i)$ as commuting subalgebras of $\mathfrak{so}(2N_eN_i)$. $\mathfrak{so}(2N_e)$ is given by matrices

$$M_e := \begin{pmatrix} A_e & B_e \\ -\bar{B}_e & -A_e^t \end{pmatrix} \otimes \mathbb{1}_{N_i}$$

and $\mathfrak{o}(N_i)$ is given by

$$M_i := \mathbb{1}_{N_i} \otimes \begin{pmatrix} A_i & 0 \\ 0 & -A_i^t \end{pmatrix} ,$$

where the entries of A_i have to be real. Defining $o := \exp(A_i)$ and

$$O := \begin{pmatrix} a_e & b_e \\ c_e & d_e \end{pmatrix} := \exp \begin{pmatrix} A_e & B_e \\ -\bar{B}_e & -A_e^t \end{pmatrix}$$

the formulas for the matrix elements (B.1) and (B.2) yield

$$\int_{\mathrm{O}(N_i)} do\; e^{\bar{\psi}\mathbb{1}_{N_e} \otimes o \psi} = \int_{SO(2N_e)} dO\; \mathrm{Det}^{N_i/2}(1 + Z_e^\dagger Z_e) e^{\frac{1}{2}\bar{\psi} Z \bar{\psi}} e^{\frac{1}{2}\psi Z^\dagger \psi} , \quad \text{(B.4)}$$

where $Z = Z_e \otimes \mathbb{1}_{N_i}$ and $Z_e = b_e d_e^{-1} = -Z_e^t$ parametrises the coset space $\mathrm{SO}(2N_e)/\mathrm{U}(N_e)$.

B.2.3 $\mathrm{USp}(2N_i) \times \mathrm{USp}(2N_e)$

In the following we state how to realise $\mathfrak{usp}(2N_i)$ and $\mathfrak{usp}(2N_e)$ as commuting subalgebras of $\mathfrak{so}(4N_iN_e)$. $\mathfrak{usp}(2N_i)$ is given by

$$M_i := \mathbb{1}_{N_e} \otimes \begin{pmatrix} A_i & B_i & 0 & 0 \\ -\bar{B}_i & -A_i^t & 0 & 0 \\ 0 & 0 & -A_i^t & \bar{B}_i \\ 0 & 0 & -B_i & A_i \end{pmatrix} ,$$

where B_i is symmetric. $\mathfrak{usp}(2N_e)$ is given by

$$M_e := \begin{pmatrix} A_e & 0 & 0 & B_e \\ 0 & A_e & -B_e & 0 \\ 0 & \bar{B}_e & -A_e^t & 0 \\ -\bar{B}_e & 0 & 0 & -A_e^t \end{pmatrix} \otimes \mathbb{1}_{N_i} ,$$

where B_e is symmetric. A short calculation shows that these two subalgebras indeed commute. Defining

$$s = \exp\begin{pmatrix} A_i & B_i \\ -\bar{B}_i & -A_i^t \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} a_e & b_e \\ c_e & d_e \end{pmatrix} = \exp\begin{pmatrix} A_e & B_e \\ C_e & -A_e^t \end{pmatrix}$$

and noting that

$$\exp\begin{pmatrix} A_e & -B_e \\ \bar{B}_e & -A_e^t \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \bar{S} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} ,$$

we can read off

$$\exp(M_e) = \begin{pmatrix} a_e & 0 & 0 & b_e \\ 0 & \bar{d}_e & \bar{c}_e & 0 \\ 0 & \bar{b}_e & \bar{a}_e & 0 \\ c_e & 0 & 0 & d_e \end{pmatrix} \otimes \mathbb{1}_{N_i} .$$

Hence we have

$$Z = Z_e \otimes \mathbb{1}_{N_i} = \begin{pmatrix} 0 & b_e d_e^{-1} \\ \bar{c}_e \bar{a}_e^{-1} & 0 \end{pmatrix} \otimes \mathbb{1}_{N_i} .$$

It can be easily checked that $J Z_e J^t = Z_e$ and $s Z_e s = -Z_e$. The formulas for the matrix elements (B.1) and (B.2) yield

$$\int_{\mathrm{USp}(2N_i)} ds\, e^{\bar{\psi}(\mathbb{1}_{N_e} \otimes s)\psi} = \int_{\mathrm{USp}(2N_e)} dS\, \mathrm{Det}^{N_i/2}(1 + Z_e^\dagger Z_e)\, e^{\frac{1}{2}\bar{\psi} Z \bar{\psi}} e^{\frac{1}{2}\psi Z^\dagger \psi} . \tag{B.5}$$

B.2.4 Invariant measure on G/H

In this subsection we want to simplify the right hand side of the colour-flavour transformation. To this end we want to use the following version of Fubini's theorem:

$$\int_G dg f(g) = \int_{G/H} d(gH) \int_H dh f(gh) . \tag{B.6}$$

Here, $d(gH)$ denotes the invariant measure on the coset space G/H, where H is a (closed and connected) subgroup of G. Since we are here dealing with a simpler situation than in the last section, we simplify our notation accordingly. For an element $g \in G$ we write

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

and we define $Z := bd^{-1}$. We will show that Z parametrises the coset space G/H up to a set of measure zero and construct a left invariant measure

$d\mu(Z^\dagger, Z)$. Here, Z differs from the matrix defined in the previous section. Note that left invariance and normalisation uniquely determine the measure $d(gH)$. In our case the H integration is trivial, since the integrand only depends on Z. In the following we treat the three different fermionic Howe pairs simultaneously. We define $s = \mathrm{Diag}(\mathbb{1}_k, -\mathbb{1}_l)$. For the first Howe pair set $k = p$ and $l = q$ and for the other two set $k = l = N_e$.

Symmetry class	V_1	V_2
unitary	$W_{p,q}$	$W_{q,p}$
orthogonal	$\{Z \in W_{N_e,N_e} \vert Z^t = -Z\}$	$\{Z \in W_{N_e,N_e} \vert Z^t = -Z\}$
unitary symplectic	$\{Z \in W_{N_e,N_e} \vert Z^t = Z\}$	$\{Z \in W_{N_e,N_e} \vert Z^t = Z\}$

Table B.2: $W_{k,l} := \mathrm{Hom}(\mathbb{C}^l, \mathbb{C}^k)$ is used. The different symmetries of $Z = bd^{-1}$ and the corresponding vectorspace for each of the three settings is defined.

Then it is clear that $\mathrm{Ad}(G)s$ is isomorphic to G/H. Note that for $g \in G$ we have

$$\begin{aligned}\mathrm{Ad}(g)s &\equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} s \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & bd^{-1} \\ ca^{-1} & 1 \end{pmatrix} s \begin{pmatrix} 1 & bd^{-1} \\ ca^{-1} & 1 \end{pmatrix}^{-1} .\end{aligned}$$

It can be checked that $bd^{-1} = -(ca^{-1})^\dagger$. Hence $Z = bd^{-1}$ parametrises $\mathrm{Ad}(G)s$ and thus G/H. For the construction of the left invariant measure on G/H we have to consider the adjoint action of G on $\mathrm{Ad}(G)s$. Here it is convenient to consider first the adjoint action on

$$f(Z, \tilde{Z}) := \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} s \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1}$$

with $Z \in V_1$ and $\tilde{Z} \in V_2$. It induces an action on V_1 and V_2. We have

$$\mathrm{Ad}(g) f(Z, \tilde{Z}) = f(\rho_g^1(Z), \rho_g^2(\tilde{Z})) ,$$

where ρ^1 and ρ^2 are given by

$$\begin{aligned}\rho_g^1(Z) &:= (aZ + b)(cZ + d)^{-1} \\ \rho_g^2(\tilde{Z}) &:= -(c - d\tilde{Z})(a - b\tilde{Z})^{-1} .\end{aligned}$$

Note that ρ^1 gives the left action on G/H in the parametrisation given by the coordinates $Z = bd^{-1}$. Moreover we can define a bilinear form $\mathrm{Tr}(df \otimes df)$[1],

[1] It is convenient to show invariance of the unsymmetrised bilinear form and to postpone (anti)symmetrisation.

where d denote the outer derivative. It is invariant since

$$\begin{aligned}\rho_g^* \operatorname{Tr}(df \otimes df) &= \operatorname{Tr}(d(f \circ \rho_g) \otimes d(f \circ \rho_g)) = \operatorname{Tr}(\operatorname{Ad}(g)[df \otimes df]) \\ &= \operatorname{Tr}(df \otimes df) \ .\end{aligned}$$

Defining $v := (1 - Z\tilde{Z})^{-1}$ and $w = (1 - \tilde{Z}Z)^{-1}$ a simple calculation shows that

$$\operatorname{Tr}(df \otimes df)/4 = -\operatorname{Tr}(wd\tilde{Z} \otimes vdZ) - \operatorname{Tr}(vdZ \otimes wd\tilde{Z}) \ .$$

Since s commutes with the differential of ρ_g, we can define an invariant two form as

$$\begin{aligned}\tilde{\omega}_{(Z,\tilde{Z})}(X,Y) &:= \frac{1}{4}\operatorname{Tr}(df \otimes df)_{(Z,\tilde{Z})}(sX,Y) \\ &= \operatorname{Tr}(vd\tilde{Z} \otimes wdZ) - \operatorname{Tr}(wdZ \otimes v\tilde{Z})(X,Y) \\ &= \operatorname{Tr}(vd\tilde{Z} \wedge wdZ)(X,Y) \ .\end{aligned}$$

Next we restrict ourselves to the subspace $\tilde{Z} = -Z^\dagger$. This is compatible with the group action since $\rho_g^2(-Z^\dagger) = -[\rho_g^1(Z)]^\dagger$. Hence we have an invariant (symplectic) two form given by:

$$\omega_Z := \tilde{\omega}_{Z,-Z^\dagger}|_{V\times V}$$

with $V := \{(Z, -Z^\dagger) | Z \in V_1\}$. The two form ω leads to the (nontrivial) invariant volume form

$$d\mu(Z^\dagger, Z) := c\,\omega^n = \operatorname{Det}^{-2N_e}(1 + Z^\dagger Z) \bigwedge_{i,j} dZ_{ij}^\dagger \bigwedge_{i,j} dZ_{ij} \ , \tag{B.7}$$

where c is a normalisation constant and $n = \dim_{\mathbb{C}} V_1$.

Bibliography

[1] P.A. Lee and T.V. Ramakrishnan, Disordered electronic systems, *Rev. Mod. Phys.* **57**, 287 (1985).

[2] K.B. Efetov, Supersymmetry and the theory of disordered metals, *Adv. Phys.* **32**, 53 (1983).

[3] F. Wegner, The mobility edge problem: continuous symmetry and a conjecture, *Z. Phys B* **36**, 207 (1979).

[4] L. Schäfer and F. Wegner, Disordered systems with n orbitals per site: Lagrange formulation, hyperbolic symmetry, and Goldstone modes, *Z. Phys. B* **38**, 113 (1980).

[5] A.M.M. Pruisken and L. Schäfer, The Anderson model for electron localization nonlinear sigma model, asymptotic gauge invariance, *Nucl. Phys. B* **200**, 20 (1982).

[6] J.J.M. Verbaarschot, H.A. Weidenmüller, and M.R. Zirnbauer, Grassmann integration in stochastic quantum physics - the case of compound nucleus scattering, *Phys. Rep.* **129**, 367 (1985).

[7] M.R. Zirnbauer, Riemannian symmetric superspaces and their origin in random-matrix theory, *J. Math. Phys.* **37**, 4986 (1996).

[8] M.R. Zirnbauer, Supersymmetry methods in random matrix theory, Encyclop. of Math. Phys. vol. 5, p.151-160, Elsevier: Amsterdam (2006).

[9] J.E. Bunder, K.B. Efetov, V.E. Kravtsov, O.M. Yevtushenko, and M.R. Zirnbauer, Superbosonization formula and its applications to random matrix theory, *J. Stat. Phys.* **129**, 809 (2007).

[10] Y.V. Fyodorov, On Hubbard-Stratonovich transformations over hyperbolic domains, *J. Phys.: Condensed Matter* **17**, S1915 (2005).

[11] Yi Wei and Yan V. Fyodorov, A conjecture on Hubbard-Stratonovich transformations for the Pruisken-Schäfer parametrizations of real hyperbolic domains, *J Phys. A* **40**, 13587 (2007).

[12] Y.V. Fyodorov, Y. Wei, and M.R. Zirnbauer, Hyperbolic Hubbard-Stratonovich Transformation Made Rigorous, *J. Math. Phys* **9**, 53507 (2008).

[13] M. Beck and S. Robins, Computing the Continuous Discretely, Springer (2007).

[14] Anthony W. Knapp, Lie Groups Beyond an Introduction, Birkhäuser (2005, second edition 3rd printing).

[15] K.B. Efetov, Supersymmetry and the theory of disordered metals, Cambridge University Press (1997).

[16] A.D. Mirlin, Statistics of energy levels and eigenfuctions in disordered systems, *Phys. Rep.* **326**, 260 (2000).

[17] F. Evers, A. D. Mirlin, Anderson transitions, *Rev. mod. Phys. 80*, 1355 (2008).

[18] L.R. Mead and N. Papanicolaou, Holstein-Primakoff theory for many-body systems, *Phys. Rev. B* **28**, 1633 (1983).

[19] M. Stone, Bosonization, (World Scientific, 1994).

[20] M. Stone, Coherent-state path integrals for loop groups and non-abelian bosonization, *Nucl. Phys. B* **327**, 399 (1989).

[21] H.J. Lipkin, N. Meshkov, and A.J. Glick, Validity of many-body approximation methods for a solvable model, *Nucl. Phys.* **62**, 188 (1965).

[22] D.J. Gross and A. Neveu, Dynamical symmetry breaking in asymptotically free field theories, *Phys. Rev. D* **10**, 3235 (1974).

[23] I.L. Aleiner, P.W. Brouwer, and L.I. Glazman, Quantum Effects in Coulomb Blockade, *Phys. Rep.* **358**, 309-440 (2002).

[24] I.S. Beloborodov, A.V. Lopatin, V. M. Vinokur, and K. B. Efetov, Granular electronic systems, *Rev. Mod. Phys.* **79**, 469 (2007).

[25] A. Perelomov, Generalized Coherent States and their Applications, (Springer, 1985).

[26] D. Bar-Moshe and M.S. Marinov, Berzin quantization and unitary representations of Lie groups (Contribution to Berezin Memorial volume), **arXiv:** 9407093.

[27] R. Howe, Dual Pairs in Physics: Harmonic Oscillators, Photons, Electrons, and Singletons, *Lectures in Applied Mathematics* **21**, 179 (1985).

[28] R. Howe, Remarks on Classical Invariant Theory, *Trans. Amer. Math. Soc.* **313**, 539 (1989).

[29] M.R. Zirnbauer, Supersymmetry for systems with unitary disorder: circular ensembles, *J. Phys. A* **29**, 7113 (1996).

[30] M.R. Zirnbauer, The Color-Flavor Transformation and a new approach to quantum chaotic maps, Proceedings of the XIIth International Congress of Mathematical Physics, Brisbane 1997, (Cambridge, 1999).

[31] P. Littelmann, H.-J. Sommers, and M.R. Zirnbauer, Superbosonization of invariant random matrix ensembles, **arXiv:** 07072929.

[32] R. Forman, Determinants Finite-Difference Operators and Boundary value Problems, *Commun. Math. Phys.* **147**, 485 (1992).

[33] F. A. Berezin, Models of Gross-Neveu Type are Quantization of a Classical Mechanics with Nonlinear Phase Space, *Commun. math. Phys* **63**, 131 (1978).

[34] G. Yaffe, Large N limits as classical mechanics, *Rev. Mod. Phys.* **54**, 407 (1982).

[35] J.-P. Blaizot and H. Orland, Path integrals for the nuclear many-body Problem, *Phys. Rev C* **24**, 1740 (1981).

[36] H. Kuratsuji and T. Suzuki, *Supp. Prog. Theor. Phys.* **74**, 209 (1983).

[37] J. R. Klauder, Path integrals and stationary-phase approximations, *Phys. Rev. D* **19**, 2349 (1979).

[38] L. D. Faddeev, in Methods in Field Theory, Les Houches 1975, edited by R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976).

[39] H. G. Solari, Semiclassical treatment of spin systems by means of coherent states, *J. Math. Phys.* **28**, 1097 (1987).

[40] E. Kochetov, Quasiclassical path integral in coherent-state manifolds, *J. Phys. A* **31**, 4473 (1998).

[41] E. Kochetov, SU(2) coherent-state path integral, *J. Math. Phys.* 36, 4667 (1995).

[42] M. Stone, Kee-Su Park, and A. Garg, The Semiclassical Propagator for Spin Coherent States, *J. Math. Phys.* **41**, 8025 (2000).

[43] M. Baranger, M. A. M. de Aguiar, F. Keck, H. J. Korsch and B. Schellhaaß, Semiclassical approximations in phase space with coherent states, *J. Phys. A* **34**, 7227 (2001).

[44] A. D. Ribeiro, M. A. M. de Aguiar, A. F. R. de Toledo Piza, Semiclassical coherent state propagator for systems with spin, *J. Phys. A* **39**, 3085 (2006).

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Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Herrn Prof. Dr. Martin Zirnbauer betreut worden.

Jakob Müller-Hill

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