On weight multiplicities of complex simple Lie algebras

von Thomas Bliem

Kurzzusammenfassung. Der Autor führt den Begriff des tournierten und tranchierten Kegels ein und zeigt, dass die Gewichtsvielfachheiten halbeinfacher komplexer Liealgebren von diesem Begriff regiert werden. Daraus schließt er auf eine Darstellung der Gewichtsvielfachheitsfunktion als Verkettung einer Vektorpartitionsfunktion und einer linearen Abbildung. Mithilfe dieser Darstellung erhält er Strukturaussagen und asymptotische Aussagen über die Gewichtsvielfachheiten; zum Beispiel ergibt sich auf diese Weise ein Beweis von G. Heckmans Satz über das Duistermaat-Heckman-Maß. Der Autor beschreibt einen Algorithmus zur Berechnung allgemeiner Vektorpartitionsfunktionen und gibt eine Implementation als Maple-Prototyp an. Dieses Programm nutzt er, um die Gewichtsvielfachheitsfunktion der Liealgebra $\mathfrak{so}_5(\mathbf{C})$ vollständig zu berechnen und anzugeben.

Abstract. The author introduces the notion of a chopped and sliced cone and shows that the weight multiplicities of semisimple complex Lie algebras are governed by this notion. From this he derives a presentation of the weight multiplicity function as a composition of a vector partition function and a linear map. By virtue of this presentation he obtains structural and asymptotic properties of weight multiplicities; for example a proof of G. Heckman's theorem on the Duistermaat-Heckman measure is obtained in this way. The author describes an algorithm for computing general vector partition functions and gives an implementation as a Maple prototype. Using this program he determines and states the weight multiplicity function of the Lie algebra $\mathfrak{so}_5(\mathbf{C})$ completely.

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Preface

The project started when Peter Littelmann drew my attention to an article written by Sara Billey, Victor Guillemin and Etienne Rassart [**2**], where they study weight multiplicities of $\mathfrak{sl}_k(\mathbf{C})$ using the following method: They consider the Gelfand-Tsetlin decomposition of irreducible polynomial representations of $\mathrm{GL}_k(\mathbf{C})$ [**9**]. The one-dimensional subspaces are parameterised by diagrams obtained by filling out a triangular scheme according to certain rules. They observe that these rules can be stated by defining a polytope: Admissible fillings correspond to points with integral coordinates in this polytope, and the weight function corresponds to an affine function on the polytope. Studying the equations defining the polytope they obtain a presentation of the weight multiplicity function of $\mathfrak{sl}_k(\mathbf{C})$ in the following form [**2**, th. 2.1]:

$$\dim(V(\lambda)_\mu) = \Phi_{E_k}\left(B_k\begin{pmatrix}\lambda\\ \mu\end{pmatrix}\right). \tag{1}$$

Here Φ_{E_k} is a vector partition function, the weights λ and μ are represented as (not necessarily integral) vectors in $\mathbf{R}^k$ of sum 0, and B_k is a matrix. They draw several conclusions concerning the structure of the weight multiplicity function, and the relation to Duistermaat-Heckman theory.

The project reported on in this thesis consists in finding a similar presentation of the weight multiplicity function for arbitrary semisimple complex Lie algebras, and drawing similar conclusions. This has been achieved to the following extent: I define a new structure in elementary convex geometry, namely the *chopped and sliced cone* (definition 10). Using this abstract notion, I derive an abstract version of (1) (theorem 17). I also show an abstract version of Heckman's theorem on the Duistermaat-Heckman measure (theorem 12). Then I interpret previous work by Peter Littelmann [**21**] as saying that the weight multiplicities of semisimple complex Lie algebras are actually quantities associated with certain chopped and sliced cones (observation 25). This implies that a theorem of type (1) (corollary 18) holds in this generality and proves Heckman's theorem .

I continue with steps towards the exploitation of Sturmfels' structure theorem on vector partition functions [**23**], partially leaving the track indicated by [**2**] in favour of the one indicated by Welleda Baldoni, Matthias Beck, Charles Cochet and Michèle Vergne in [**1**]. Namely, by corollary 18 and [**23**], the weight multiplicity function is given as follows: Its domain, a real vector space of dimension twice the rank of the Lie algebra under consideration, decomposes into polyhedral cones such that on each cone the weight multiplicity function is given by a quasi-polynomial. I describe a general algorithm which, given an arbitrary vector partition function, calculates such a decomposition and the corresponding quasi-polynomials. This algorithm, based on the method of inverse Laplace transformation, relies on work by L. Jeffrey, F. Kirwan [**12**], A. Szenes, M. Vergne [**24**], C. De Concini and C. Procesi [**6**] and was used (in a special case) in [**1**] for computing the Kostant partition function of classical root systems. I give the, to my knowledge, first general implementation of this algorithm as a prototype Maple program (appendices). With this program, it is possible to carry out all computations for Lie algebras of small rank, as I demonstrate by doing so for $\mathfrak{so}_5(\mathbf{C})$ (§ 1.5).

The work is organised as follows: Chapter 1 contains a condensed version of the entire thesis, where the theory is applied to the Lie algebra $\mathfrak{so}_5(\mathbf{C})$. This was presented independently at a workshop on integer point enumeration in polyhedra [**4**]. Chapter 2 gathers some necessary prerequisites on Lie algebras and vector partition functions. I have taken care to include all definitions (the first equation is the Jacobi identity!) in a brief exposition. Chapter 3 introduces the notion of a chopped and sliced cone and contains the statements and proofs of the abstract theorems about them. Chapter 4 summarises the results leading to the presentation of the crystal associated with a Lie algebra as a chopped and sliced cone. We explicitly write down the spaces and maps defining these chopped and sliced cones for the classical simple Lie algebras. The final chapter 5 reports on the results leading to the above-mentioned algorithm for computing vector partition functions. The actual Maple code is enclosed in the appendices.

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Thomas Bliem, August 2008

Contents

CHAPTER 1

Illustrating example: Weight multiplicities for $\mathfrak{so}_5(\mathbf{C})$

We explicitly determine quasi-polynomials describing the weight multiplicities of the Lie algebra $\mathfrak{so}_5(\mathbf{C})$. This information entails immediate complete knowledge of the character of any simple representation as well as the asymptotic behaviour of characters.

1. Introduction

There have been investigations into the characters of simple Lie algebras initiated in the beginning of the discipline and going on to present days. From the early days it was possible to write down formulas explicitly describing the characters, as notably done by H. Weyl [**25**] and B. Kostant [**18**]. P. Littelmann's path model [**19, 20**] does not formally furnish a formula but rather an algorithm allowing to calculate the character performing finitely many combinatorial operations, so fits into the very same context.

Still, all these approaches, while allowing to calculate characters or individual weight multiplicities, at least in principle and for small instances, do not fully exhibit the rich structure underlying the characters. For example, from G. Heckman's thesis [**10**] it is known that considering a sequence of simple representations for a given simple Lie algebra such that the highest weights of the elements are the integral multiples of a given weight, the corresponding characters show a particular behaviour of convergence.

As pointed out by S. Billey, V. Guillemin and E. Rassart [**2**], for the case of $\mathfrak{sl}_k(\mathbf{C})$ Gelfand-Tsetin patterns [**9**] can be used as a key ingredient to develop descriptions of characters better reflecting their structure. In the following I will demonstrate that it is possible, substituting Gelfand-Tsetlin patterns by

P. Littelmann's patterns [**21**], using B. Sturmfels' structure theorem [**23**] on vector partition functions and Laplace transformation methods developed by L. Jeffrey, F. Kirwan [**12**], A. Szenes and M. Vergne [**24**] as well as work by C. De Concini and C. Procesi on the combinatorics of residues [**6**], to obtain indeed complete knowledge, structural and computational, of the characters of $\mathfrak{so}_5(\mathbf{C})$. In fact, I do not use any special properties of the Lie algebra $\mathfrak{so}_5(\mathbf{C})$. This is just a random example picked to demonstrate the power of the combination of the above-mentioned ideas, which, in principle, are applicable to any semisimple complex Lie algebra.

2. Preliminaries

Consider the Lie algebra $\mathfrak{so}_5(\mathbf{C})$ of complex (5×5)-matrices A such that $A^tM = -MA$ for a fixed nondegenerate complex (5×5)-matrix M. Choose a Cartan subalgebra $\mathfrak{h}$ and simple roots $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ such that α_2 is the long root. The Dynkin diagram associated with this enumeration of the simple roots is

$$1 \Longleftarrow 2.$$

Let $\omega_1, \omega_2 \in \mathfrak{h}^*$ be the corresponding fundamental weights, $\Lambda \subset Q \subset \mathfrak{h}^*$ the weight lattice respectively the root lattice. The irreducible $\mathfrak{so}_5(\mathbf{C})$-module of highest weight λ is denoted by $V(\lambda)$. For a weight μ denote by $V(\lambda)_\mu$ the space of vectors of weight μ in $V(\lambda)$. If $\lambda-\mu$ is not an element of the root lattice, then $V(\lambda)_\mu = 0$. For λ a dominant weight and β an element of the root lattice, define

$$K^\lambda_\beta := \dim V(\lambda)_{\lambda-\beta}. \tag{2}$$

If the weight λ is not dominant we define all K^λ_β to be 0. The character of $V(\lambda)$ is by definition the element

$$\sum_{\beta\in Q} K^\lambda_\beta \cdot [\lambda-\beta] \tag{3}$$

of the group ring $\mathbf{Z}[\Lambda]$. Hence knowledge of all the characters of $\mathfrak{so}_5(\mathbf{C})$ is equivalent to knowledge of the function $K : \Lambda \times Q \to \mathbf{Z}_{\geq 0}$.

For any dominant weight λ, let $B(\lambda)$ be the crystal associated with $V(\lambda)$ by M. Kashiwara [**15**] and $B(\infty) := \varinjlim_\lambda B(\lambda)\otimes T_{-\lambda}$ the direct limit of the crystals $B(\lambda)$, shifted to have highest weight 0. Consider the reduced decomposition $w_0 = s_1 s_2 s_1 s_2$ of the longest element of the Weyl group. To this decomposition, P. Littelmann [**21**] associates a convex polyhedral cone $\mathcal{C} \subset \mathbf{R}^4$, a family of polytopes $\mathcal{C}^\lambda \subset \mathcal{C}$ for dominant weights λ and a $\mathbf{Z}$-linear map $\psi : \mathbf{Z}^4 \to Q$ such that: (i) There is a canonical bijection $\sigma : B(\infty) \to \mathcal{S} := \mathcal{C} \cap \mathbf{Z}^4$. (ii) For each dominant weight λ, the bijection σ restricts to a bijection between $B(\lambda)$ and $\mathcal{S}^\lambda := \mathcal{C}^\lambda \cap \mathbf{Z}^4$. (iii) The weight of any element $b \in B(\lambda)$ is $\mathrm{wt}(b) = \lambda - \psi(\sigma(b))$.

Specifically, denote the standard coordinates by a_{22}, a_{11}, a_{12}, a_{13}. Then the cone $\mathcal{C}$ is given by the inequalities

$$2a_{11} \geq a_{12} \geq 2a_{13} \geq 0, \quad a_{22} \geq 0. \tag{4}$$

For a dominant weight $\lambda = \lambda_1 \omega_1 + \lambda_2 \omega_2$, the polytope $\mathcal{C}^\lambda$ is given inside $\mathcal{C}$ by the additional inequalities

$$\begin{aligned} a_{13} &\leq \lambda_2, & a_{11} &\leq \lambda_2 + a_{12} - 2a_{13}, \\ a_{12} &\leq \lambda_1 + 2a_{13}, & a_{22} &\leq \lambda_1 + 2a_{11} - 2a_{12} + 2a_{13}. \end{aligned} \tag{5}$$

The $\mathbf{Z}$-linear map is $\psi = (a_{22} + a_{12})\alpha_1 + (a_{11} + a_{13})\alpha_2$. For a given dominant weight λ and any element $\beta = \beta_1 \alpha_1 + \beta_2 \alpha_2$ of the root lattice, define

$$\mathcal{C}^\lambda_\beta := \{a \in \mathcal{C}^\lambda : a_{22} + a_{12} = \beta_1,\ a_{11} + a_{13} = \beta_2\} \tag{6}$$

and $\mathcal{S}^\lambda_\beta := \mathcal{C}^\lambda_\beta \cap \mathbf{Z}^4$. Then by [**21**] we have $K^\lambda_\beta = |\mathcal{S}^\lambda_\beta|$, that is: Determination of the weight multiplicities is reduced to counting the number of points in polytopes.

3. Reformulation using vector partition functions

For any positive integers n and N and any $(n \times N)$-matrix A with integral coefficients such that $\ker(A) \cap \mathbf{R}^n_{\geq 0} = 0$ define the vector partition function $\Phi_A : \mathbf{Z}^n \to \mathbf{Z}_{\geq 0}$ by $\Phi_A(v) := |\{a \in \mathbf{Z}^N_{\geq 0} : Aa = v\}|$. We will now reformulate Littelmann's result and explicitly determine matrices A and B such that $K^\lambda_\beta = \Phi_A(B \cdot (\lambda_1, \lambda_2, \beta_1, \beta_2)^t)$. Indeed, the inequalities (4) and (5) can be turned into equations using slack variables s_1, s_2 respectively t_1, t_2, t_3, t_4. Hence, the number of integral solutions $|\mathcal{S}^\lambda_\beta|$ of the system (4, 5, 6) is equal to the number of nonnegative integral solutions to the system

$$
\begin{array}{rl}
2a_{11} - a_{12} - s_1 &= 0, \\
a_{12} - 2a_{13} - s_2 &= 0, \\
a_{13} + t_1 &= \lambda_2, \\
a_{12} - 2a_{13} + t_2 &= \lambda_1, \\
a_{11} - a_{12} + 2a_{13} + t_3 &= \lambda_2, \\
a_{22} - 2a_{11} + 2a_{12} - 2a_{13} + t_4 &= \lambda_1, \\
a_{22} + a_{12} &= \beta_1, \\
a_{11} + a_{13} &= \beta_2.
\end{array}
\tag{7}
$$

In other words, $|\mathcal{S}^{\lambda}_{\beta}| = \Phi_A(B \cdot (\lambda_1, \lambda_2, \beta_1, \beta_2)^t)$ for matrices

$$
A = \left(\begin{array}{cccc|cc|cccc}
0 & 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & -1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\tag{8}
$$

and

$$
B = \left(\begin{array}{cc|cc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right).
\tag{9}
$$

4. Structure and calculation of vector partition functions

The presentation of K^{λ}_{β} in terms of a vector partition function gains its strength from the following structure theorem of B. Sturmfels [**23**]:

THEOREM 1. *Let $A \in \mathbf{Z}^{(n,N)}$ such that the vector partition function $\Phi_A : \mathbf{Z}^n \to \mathbf{Z}_{\geq 0}$ is defined. Then there is a homogeneous fan $F = \mathrm{fan}(A)$ in $\mathbf{R}^n$ and* *a family of quasi-polynomials* (f_C) *on* $\mathbf{Z}^n$*, indexed by the maximal cones in* F*, such that* Φ_A *coincides with* f_C *on* $C \cap \mathbf{Z}^n$ *and vanishes outside the support of* F*.*

Here, a *fan* is a finite set of convex polyhedral cones, closed under taking faces, and such that the intersection of any two cones is a face of both. The fan being *homogeneous* means that all maximal cones have the same dimension, which is in this case the rank of A. A function $f : \mathbf{Z}^n \to \mathbf{C}$ is called a *quasi-polynomial* if there is a lattice $L \subset \mathbf{Z}^n$ and a family $(f_{\bar{h}})_{\bar{h} \in \mathbf{Z}^n/L}$ of polynomials on $\mathbf{Z}^n$ such that $f(h) = f_{\bar{h}}(h)$ for all $h \in \mathbf{Z}^n$.

While the naive algorithm for computing individual values $\Phi_A(v)$ of a given vector partition functions has exponential execution time with respect to the *components* of v, this theorem allows the following strategy: Determine the maximal cones of F and for each maximal cone C determine the quasi-polynomial f_C. This task being accomplished, individual values $\Phi_A(v)$ of the vector partition function can be calculated by evaluating the corresponding quasi-polynomial at v, the execution time of which is of order of the *logarithm* of the components of v. In this sense, determination of the maximal cones and quasi-polynomials yields instant complete knowledge of the values of the given vector partition function. We will indeed perform these steps for the vector partition function given by (8). In this way we will determine all the characters of $\mathfrak{so}_5(\mathbf{C})$ at once.

Generally, the maximal cones and quasi-polynomials of a vector partition function can be determined as outlined in the sequel. For a more extensive treatment, refer to the study of the Kostant partition function for classical root systems [**1**] by W. Baldoni, M. Beck, Ch. Cochet and M. Vergne, which also served as a model for the following calculations.

Suppose that A has rank n. For a lattice $L \subset \mathbf{R}^n$ denote by $L^\perp \subset (\mathbf{R}^n)^*$ the corresponding dual lattice. If $L \subset \mathbf{Z}^n$ then $L^\perp \supset (\mathbf{Z}^n)^\perp$. Let $T := (\mathbf{R}^n)^*/(\mathbf{Z}^n)^\perp$, an n-dimensional torus. Denote the column vectors of A by $a_1, \dots, a_N$. A subset $\sigma \subset \{1, \dots, N\}$ is called a *basic subset* for A if $(a_i)_{i \in \sigma}$ is a basis of $\mathbf{R}^n$. For any basic subset $\sigma = \{i_1, \dots, i_n\}$ let $T(\sigma) := (\mathbf{Z}a_{i_1} + \cdots + \mathbf{Z}a_{i_n})^\perp$ mod $(\mathbf{Z}^n)^\perp \subset T$. The set $T(\sigma)$ contains $\operatorname{vol}(\sigma) := |\det(a_i : i \in \sigma)|$ elements. Let $\Gamma \subset T$ be the union of all $T(\sigma)$ for basic subsets σ for A.

The fan $F = \operatorname{fan}(A)$ associated with A can be described as follows: For any basic subset σ denote by $\operatorname{cone}(\sigma)$ the convex polyhedral cone generated by $\{a_i : i \in \sigma\}$. Cones of the form cone(σ) are called *basic cones*. Then the maximal cones of F are the minimal n-dimensional cones which can be written as an intersection of basic cones.

For $h \in \mathbf{Z}^n$ and $g \in T$ consider the meromorphic function on $(\mathbf{R}^n)^* \otimes_{\mathbf{R}} \mathbf{C} = (\mathbf{C}^n)^*$ given by

$$F_{g,h}(u) := \frac{e^{\langle u+2\pi i g, h\rangle}}{\prod_{k=1}^{N}\left(1-e^{-\langle u+2\pi i g, a_k\rangle}\right)}. \tag{10}$$

Note that $\langle g,h\rangle$ and $\langle g,a_k\rangle$ are determined modulo $\mathbf{Z}$, so the values of the exponential functions are unambiguous.[1]

A basic subset $\{i_1,\dots,i_n\}$ with $i_1 < \cdots < i_n$ is called *without broken circuits* if there are no $j \in \{1,\dots,n\}$ and $k \in \{i_j+1,\dots,N\}$ such that the family $(a_{i_1},\dots,a_{i_j},a_k)$ is linearly dependent.[2] For a maximal cone C of the fan F, let $B_{\mathrm{nb}}(C)$ denote the set of basic subsets σ without broken circuits such that cone$(\sigma) \supset C$.

For a meromorphic function $f(u)$ on $(\mathbf{R}^n)^* \otimes_{\mathbf{R}} \mathbf{C} = (\mathbf{C}^n)^*$ with poles along $a_i^{\perp}$ $(i = 1,\dots,N)$ and a basic subset σ, define the *iterated residue* of f with respect to σ to be

$$\operatorname{ires}_{\sigma} f(u) := \operatorname{res}_{a_{i_n}=0} \cdots \operatorname{res}_{a_{i_1}=0} f(u), \tag{11}$$

where a_{i_j} are interpreted as coordinates on $(\mathbf{C}^n)^*$.

The following theorem is a combination of A. Szenes and M. Vergne's expression [**24**, th. 3.1] of a vector partition function as a Jeffrey-Kirwan residue and C. De Concini and C. Procesi's work [**6**] on the Jeffrey-Kirwan residue:

THEOREM 2. *On any maximal cone C of* fan(A)*, the vector partition function associated with A is given by*

$$\Phi_A(h) = \sum_{\sigma \in B_{\mathrm{nb}}(C)} \frac{1}{\mathrm{vol}(\sigma)} \sum_{g \in \Gamma} \operatorname{ires}_{\sigma} F_{g,h}(u). \tag{12}$$

---

[1] The function $F_{g,h}(u)$ is called the *Kostant function* in [**1**, § 3]. I refrain from using this name because in the generality considered here, I do not see any connection to B. Kostant's work.

[2] Note that De Concini and Procesi [**6**] use the inverse order.

5. Computing the weight multiplicity function for $\mathfrak{so}_5(\mathbf{C})$

Recall that $K_\beta^\lambda = \Phi_A(B\cdot(\lambda_1,\lambda_2,\beta_1,\beta_2)^t)$ for $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$, $\beta = \beta_1\alpha_1 + \beta_2\alpha_2$ and A, B matrices as given in equations (8) respectively (9). We are now ready to perform explicit computations. All computations were performed using Maple 11 by Maplesoft[3] and the package Convex 1.1.2 by M. Franz[4].

First we have to determine $F = \mathrm{fan}(A)$. This is done as follows: Any maximal cone in F is the intersection of all the basic cones containing it. We can hence find the neighbours of a given maximal cone C as follows: For each facet f of C, the neighbouring maximal cone of C in direction f is the intersection of all basic cones $\mathrm{cone}(\sigma)$ such that $f \subset \mathrm{cone}(\sigma)$ and $(C \subset \mathrm{cone}(\sigma) \implies f \not\subset \partial(\mathrm{cone}(\sigma)))$. So we start with an arbitrary maximal cone and find the others by a standard algorithm for graph traversal using this description of the neighbour relation. There are 320 maximal cones altogether.

In order to determine Γ we proceed as follows: For any basic subset σ, let A_σ be the submatrix of A consisting of the columns with indices in σ. The subgroup $T(\sigma)$ of T is generated by the classes of the row vectors of A_σ^{-1}. So we start with the set of these classes and determine its closure under the operation of adding the class of any row vector of A_σ^{-1} by a standard algorithm of graph traversal.

The set of basic subsets without broken circuits is determined straightforwardly using the definition. In order to speed up the calculation, basic subsets are built up recursively, checking the additional prerequisites at every step of the recursion.

In fact we are only interested in maximal cones whose intersection with the image of B is 4-dimensional. There are 43 such intersections. For the calculation of the quasi-polynomials we now pick for each such intersection $\mathfrak{c}$ a maximal cone C of $\mathrm{fan}(A)$ such that $\mathfrak{c} = C \cap \mathrm{im}(B)$. Then we compute the quasi-polynomials for each of these maximal cones as described in section 4. The quasi-polynomials coincide for some of the neighbouring $\mathfrak{c}$, so we glue together the corresponding cones. The preimage under B of the resulting fan is given by the following maximal cones in $\mathbf{R}^4 = (\Lambda \otimes_{\mathbf{Z}} \mathbf{R}) \times (Q \otimes_{\mathbf{Z}} \mathbf{R})$.

[3] http://www.maplesoft.com/

[4] http://www-fourier.ujf-grenoble.fr/~franz/

$$
\begin{aligned}
\mathfrak{c}_1 &= \{\lambda_2-\beta_2\geq 0,\, \lambda_1-\beta_1\geq 0,\, \beta_1-\beta_2\geq 0,\, -\beta_1+2\beta_2\geq 0\},\\
\mathfrak{c}_2 &= \{\beta_1-2\beta_2\geq 0,\, \lambda_2-\beta_2\geq 0,\, -\lambda_1+\beta_1\geq 0,\, \lambda_1-\beta_1+\beta_2\geq 0\},\\
\mathfrak{c}_3 &= \{\lambda_2-\beta_2\geq 0,\, -\lambda_1+\beta_1\geq 0,\, -\beta_1+2\beta_2\geq 0,\, \beta_1-\beta_2\geq 0,\\
&\qquad \lambda_1-\beta_1+\beta_2\geq 0\},\\
\mathfrak{c}_4 &= \{\beta_1\geq 0,\, \lambda_2-\beta_2\geq 0,\, -\beta_1+\beta_2\geq 0,\, \lambda_1-\beta_1\geq 0\},\\
\mathfrak{c}_5 &= \{\beta_2\geq 0,\, \beta_1-2\beta_2\geq 0,\, \lambda_1-\beta_1\geq 0,\, \lambda_2-\beta_2\geq 0\},\\
\mathfrak{c}_6 &= \{\beta_1-\beta_2\geq 0,\, 2\lambda_2+\beta_1-2\beta_2\geq 0,\, -\beta_1+2\beta_2\geq 0,\\
&\qquad -\lambda_2+\beta_2\geq 0,\, \lambda_1-\beta_1\geq 0\},\\
\mathfrak{c}_7 &= \{\lambda_1-\beta_1+2\beta_2\geq 0,\, -\lambda_1+\beta_1-\beta_2\geq 0,\, \beta_1-2\beta_2\geq 0,\\
&\qquad \lambda_2-\beta_2\geq 0\},\\
\mathfrak{c}_8 &= \{-\lambda_1+\beta_1\geq 0,\, \lambda_1\geq 0,\, \lambda_2-\beta_2\geq 0,\, -\beta_1+\beta_2\geq 0\},\\
\mathfrak{c}_9 &= \{-\lambda_1+\beta_1\geq 0,\, \lambda_1-\beta_1+\beta_2\geq 0,\, \lambda_1+2\lambda_2-\beta_1\geq 0,\\
&\qquad -\lambda_2+\beta_2\geq 0,\, \beta_1-2\beta_2\geq 0\},\\
\mathfrak{c}_{10} &= \{\lambda_1+\lambda_2-\beta_2\geq 0,\, \lambda_1+2\lambda_2-\beta_1\geq 0,\, 2\lambda_2+\beta_1-2\beta_2\geq 0,\\
&\qquad \lambda_1-\beta_1+\beta_2\geq 0,\, -\lambda_1+\beta_1\geq 0\},\\
\mathfrak{c}_{11} &= \{2\lambda_2+\beta_1-2\beta_2\geq 0,\, -\lambda_2+\beta_2\geq 0,\, -\beta_1+\beta_2\geq 0,\\
&\qquad \lambda_1-\beta_1\geq 0\},\\
\mathfrak{c}_{12} &= \{-\lambda_2+\beta_2\geq 0,\, \lambda_2\geq 0,\, \beta_1-2\beta_2\geq 0,\, \lambda_1-\beta_1\geq 0\},\\
\mathfrak{c}_{13} &= \{-\beta_1+2\beta_2\geq 0,\, -\lambda_1+\beta_1-\beta_2\geq 0,\, \lambda_1\geq 0,\, \lambda_2-\beta_2\geq 0\},\\
\mathfrak{c}_{14} &= \{-\lambda_1+\beta_1-\beta_2\geq 0,\, \lambda_1+2\lambda_2-\beta_1\geq 0,\, -\lambda_2+\beta_2\geq 0,\\
&\qquad \beta_1-2\beta_2\geq 0\},\\
\mathfrak{c}_{15} &= \{-\beta_1+\beta_2\geq 0,\, -\lambda_2+\beta_2\geq 0,\, 2\lambda_2+\beta_1-2\beta_2\geq 0,\\
&\qquad \lambda_1+\lambda_2-\beta_2\geq 0,\, -\lambda_1+\beta_1\geq 0\},\\
\mathfrak{c}_{16} &= \{-\lambda_1-\lambda_2+\beta_2\geq 0,\, \lambda_1+2\lambda_2-\beta_1\geq 0,\, 2\lambda_2+\beta_1-2\beta_2\geq 0,\\
&\qquad \lambda_1-\beta_1+\beta_1\geq 0,\, \beta_1-\beta_2\geq 0\},\\
\mathfrak{c}_{17} &= \{\lambda_2+\beta_1-\beta_2\geq 0,\, -2\lambda_2-\beta_1+2\beta_2\geq 0,\, -\beta_1+\beta_2\geq 0,\\
&\qquad \lambda_1-\beta_1\geq 0\},\\
\mathfrak{c}_{18} &= \{\beta_1-\beta_2\geq 0,\, \lambda_2\geq 0,\, -2\lambda_2-\beta_1+2\beta_2\geq 0,\, \lambda_1-\beta_1\geq 0\},\\
\mathfrak{c}_{19} &= \{-\lambda_1+\beta_1-\beta_2\geq 0,\, -\lambda_2+\beta_2\geq 0,\, -\beta_1+2\beta_2\geq 0,\\
&\qquad \lambda_1+2\lambda_2-\beta_1\geq 0,\, \lambda_1+\lambda_2-\beta_2\geq 0\},
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{c}_{20} = \{&\lambda_1+\lambda_2-\beta_1+\beta_2 \geq 0,\ -\lambda_1-2\lambda_2+\beta_1 \geq 0,\\
&-\lambda_1+\beta_1-\beta_2 \geq 0,\ \beta_1-2\beta_2 \geq 0\},\\
\mathfrak{c}_{21} = \{&\lambda_1+\lambda_2-\beta_2 \geq 0,\ -\lambda_1+\beta_1 \geq 0,\ -2\lambda_2-\beta_1+2\beta_2 \geq 0,\\
&-\beta_1+\beta_2 \geq 0\},\\
\mathfrak{c}_{22} = \{&-\lambda_1-\lambda_2+\beta_2 \geq 0,\ \lambda_1 \geq 0,\ 2\lambda_2+\beta_1-2\beta_2 \geq 0,\\
&-\beta_1+\beta_2 \geq 0\},\\
\mathfrak{c}_{23} = \{&\lambda_2 \geq 0,\ -\lambda_1-2\lambda_2+\beta_1 \geq 0,\ \lambda_1-\beta_1+\beta_2 \geq 0,\ \beta_1-2\beta_2 \geq 0\},\\
\mathfrak{c}_{24} = \{&-2\lambda_2-\beta_1+2\beta_2 \geq 0,\ \beta_1-\beta_2 \geq 0,\ \lambda_1+\lambda_2-\beta_2 \geq 0,\\
&-\lambda_1+\beta_1 \geq 0,\ \lambda_1+2\lambda_2-\beta_1 \geq 0\},\\
\mathfrak{c}_{25} = \{&-\lambda_1-2\lambda_2+\beta_1 \geq 0,\ -\lambda_1+\beta_1-\beta_2 \geq 0,\ -\beta_1+2\beta_2 \geq 0,\\
&\lambda_1+\lambda_2-\beta_2 \geq 0\},\\
\mathfrak{c}_{26} = \{&\lambda_1 \geq 0,\ -\lambda_1-\lambda_2+\beta_2 \geq 0,\ -\lambda_1+\beta_1-\beta_2 \geq 0,\\
&\lambda_1+2\lambda_2-\beta_1 \geq 0\},\\
\mathfrak{c}_{27} = \{&\lambda_1+2\lambda_2+\beta_1-2\beta_2 \geq 0,\ -\lambda_1-\lambda_2+\beta_2 \geq 0,\\
&-2\lambda_2-\beta_1+2\beta_2 \geq 0,\ -\beta_1+\beta_2 \geq 0\},\\
\mathfrak{c}_{28} = \{&-\lambda_1-2\lambda_2+\beta_1 \geq 0,\ -\beta_1+2\beta_2 \geq 0,\ 2\lambda_2+\beta_1-2\beta_2 \geq 0,\\
&\lambda_1-\beta_1+\beta_2 \geq 0,\ \lambda_1+\lambda_2-\beta_2 \geq 0\},\\
\mathfrak{c}_{29} = \{&\lambda_2 \geq 0,\ -\lambda_1-2\lambda_2+\beta_1 \geq 0,\ -2\lambda_2-\beta_1+2\beta_2 \geq 0,\\
&\lambda_1+\lambda_2-\beta_2 \geq 0\},\\
\mathfrak{c}_{30} = \{&-\lambda_1-\lambda_2+\beta_2 \geq 0,\ \beta_1-\beta_2 \geq 0,\ \lambda_1+2\lambda_2-\beta_1 \geq 0,\\
&-2\lambda_2-\beta_1+2\beta_2 \geq 0\},\\
\mathfrak{c}_{31} = \{&-\lambda_1-2\lambda_2+\beta_1 \geq 0,\ -\lambda_1+\beta_1-\beta_2 \geq 0,\ 2\lambda_1+2\lambda_2-\beta_1 \geq 0,\\
&-\lambda_1-\lambda_2+\beta_2 \geq 0\},\\
\mathfrak{c}_{32} = \{&2\lambda_2+\beta_1-2\beta_2 \geq 0,\ -\lambda_1-\lambda_2+\beta_2 \geq 0,\ -\lambda_1-2\lambda_2+\beta_1 \geq 0,\\
&\lambda_1-\beta_1+\beta_2 \geq 0\},\\
\mathfrak{c}_{33} = \{&-\lambda_1-\lambda_2+\beta_2 \geq 0,\ \lambda_1+2\lambda_2-\beta_2 \geq 0,\ -\lambda_1-2\lambda_2+\beta_1 \geq 0,\\
&-2\lambda_2-\beta_1+2\beta_2 \geq 0\}.
\end{aligned}
$$

In order to get a feeling for this decomposition of $(\Lambda \otimes_{\mathbf{Z}} \mathbf{R}) \times (Q \otimes_{\mathbf{Z}} \mathbf{R})$, consider the intersection of the fan with the affine plane given by $\lambda = (1,2)$ as indicated in figure 1. In figure 2, you find a visualisation of the induced decomposition of $Q \otimes_{\mathbf{Z}} \mathbf{R}$ in Cartesian coordinates with respect to the Killing form. The highest weight $\omega_1 + 2\omega_2$ corresponds to the upper right corner. Note that this figure describes the structure of the weight multiplicity function of any module of highest weight $k\lambda$ for $k \in \mathbf{Z}_{>0}$.

The quasi-polynomials describing the weight multiplicity function on the above cones are

$$
\begin{aligned}
f_1 &= \tfrac{1}{2}\beta_1 + \tfrac{1}{2}\beta_2 + \beta_2\beta_1 - \tfrac{1}{2}\beta_2^2 + \tfrac{7}{8} - \tfrac{1}{4}\beta_1^2 + \tfrac{1}{8}(-1)^{\beta_1},\\
f_2 &= \tfrac{1}{2}\lambda_1 - \tfrac{1}{2}\beta_1 + \tfrac{3}{2}\beta_2 - \tfrac{1}{4}\lambda_1^2 + \tfrac{1}{2}\beta_2^2 + \tfrac{7}{8} - \tfrac{1}{4}\beta_1^2 + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} +\\
&\quad \tfrac{1}{2}\lambda_1\beta_1,\\
f_3 &= \tfrac{1}{8}(-1)^{\beta_1} + \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}\beta_2 - \tfrac{1}{4}\lambda_1^2 + \tfrac{3}{4} - \tfrac{1}{2}\beta_2^2 - \tfrac{1}{2}\beta_1^2 + \tfrac{1}{2}\lambda_1\beta_1 +\\
&\quad \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} + \beta_2\beta_1,\\
f_4 &= \tfrac{1}{8}(-1)^{\beta_1} + \beta_1 + \tfrac{1}{4}\beta_1^2 + \tfrac{7}{8},\\
f_5 &= 1 + \tfrac{3}{2}\beta_2 + \tfrac{1}{2}\beta_2^2,\\
f_6 &= \tfrac{1}{2}\lambda_2 + \tfrac{1}{2}\beta_1 + \beta_2\beta_1 - \beta_2^2 - \tfrac{1}{4}\beta_1^2 + \lambda_2\beta_2 - \tfrac{1}{2}\lambda_2{}^2 + \tfrac{1}{8}(-1)^{\beta_1} + \tfrac{7}{8},\\
f_7 &= \lambda_1 - \beta_1 + 2\beta_2 + \tfrac{1}{4}\lambda_1^2 + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} + \beta_2^2 + \tfrac{7}{8} - \tfrac{1}{2}\lambda_1\beta_1 +\\
&\quad \tfrac{1}{4}\beta_1^2 + \lambda_1\beta_2 - \beta_2\beta_1,\\
f_8 &= \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}\beta_1 - \tfrac{1}{4}\lambda_1^2 + \tfrac{3}{4} + \tfrac{1}{2}\lambda_1\beta_1 + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} + \tfrac{1}{8}(-1)^{\beta_1},\\
f_9 &= \tfrac{1}{2}\lambda_2 + \tfrac{1}{2}\lambda_1 - \tfrac{1}{2}\beta_1 + \beta_2 - \tfrac{1}{4}\lambda_1^2 - \tfrac{1}{4}\beta_1^2 + \lambda_2\beta_2 + \tfrac{1}{2}\lambda_1\beta_1 - \tfrac{1}{2}\lambda_2^2 +\\
&\quad \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} + \tfrac{7}{8},\\
f_{10} &= \tfrac{1}{2}\lambda_2 + \tfrac{1}{2}\lambda_1 - \tfrac{1}{4}\lambda_1^2 + \tfrac{3}{4} - \beta_2^2 - \tfrac{1}{2}\beta_1^2 + \lambda_2\beta_2 + \tfrac{1}{8}(-1)^{\beta_1} +\\
&\quad \tfrac{1}{2}\lambda_1\beta_1 - \tfrac{1}{2}\lambda_2^2 + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} + \beta_2\beta_1,\\
f_{11} &= \tfrac{1}{2}\lambda_2 + \beta_1 - \tfrac{1}{2}\beta_2 - \tfrac{1}{2}\beta_2^2 + \tfrac{1}{4}\beta_1^2 + \tfrac{7}{8} + \lambda_2\beta_2 + \tfrac{1}{8}(-1)^{\beta_1} - \tfrac{1}{2}\lambda_2^2,\\
f_{12} &= 1 + \tfrac{1}{2}\lambda_2 + \beta_2 + \lambda_2\beta_2 - \tfrac{1}{2}\lambda_2^2,\\
f_{13} &= \lambda_1 - \tfrac{1}{2}\beta_1 + \beta_2 + \tfrac{1}{4}\lambda_1^2 + \tfrac{3}{4} + \lambda_1\beta_2 + \tfrac{1}{8}(-1)^{\beta_1} + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} -\\
&\quad \tfrac{1}{2}\lambda_1\beta_1,\\
f_{14} &= \tfrac{1}{2}\lambda_2 + \lambda_1 - \beta_1 + \tfrac{3}{2}\beta_2 + \tfrac{1}{4}\lambda_1^2 + \tfrac{1}{2}\beta_2^2 + \lambda_1\beta_2 + \tfrac{1}{4}\beta_1^2 + \lambda_2\beta_2 -\\
&\quad \tfrac{1}{2}\lambda_2^2 - \beta_2\beta_1 - \tfrac{1}{2}\lambda_1\beta_1 + \tfrac{7}{8} + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1},\\
f_{15} &= \tfrac{1}{2}\lambda_2 + \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}\beta_1 - \tfrac{1}{2}\beta_2 + \tfrac{1}{8}(-1)^{\beta_1} - \tfrac{1}{4}\lambda_1^2 + \tfrac{3}{4} - \tfrac{1}{2}\beta_2^2 +\\
&\quad \lambda_2\beta_2 - \tfrac{1}{2}\lambda_2^2 + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} + \tfrac{1}{2}\lambda_1\beta_1,\\
f_{16} &= \lambda_2 + \lambda_1 - \tfrac{1}{2}\beta_2 + \tfrac{1}{8}(-1)^{\beta_1} + \tfrac{1}{4}\lambda_1^2 + \tfrac{3}{4} - \tfrac{1}{2}\beta_2^2 - \lambda_1\beta_2 - \tfrac{1}{2}\beta_1^2 +\\
&\quad \lambda_2\lambda_1 + \tfrac{1}{2}\lambda_1\beta_1 + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} + \beta_2\beta_1,
\end{aligned}
$$

$$f_{17} = 1 + \tfrac{3}{2}\lambda_2 + \tfrac{3}{2}\beta_1 - \tfrac{3}{2}\beta_2 + \tfrac{1}{2}\beta_2^2 + \tfrac{1}{2}\beta_1^2 - \lambda_2\beta_2 + \tfrac{1}{2}\lambda_2^2 + \lambda_2\beta_1 - \beta_2\beta_1,$$

$$f_{18} = 1 + \tfrac{3}{2}\lambda_2 + \beta_1 - \beta_2 - \lambda_2\beta_2 + \tfrac{1}{2}\lambda_2^2 + \lambda_2\beta_1,$$

$$f_{19} = \tfrac{1}{2}\lambda_2 + \lambda_1 - \tfrac{1}{2}\beta_1 + \tfrac{1}{2}\beta_2 + \tfrac{1}{4}\lambda_1^2 + \tfrac{3}{4} - \tfrac{1}{2}\beta_2^2 + \lambda_1\beta_2 + \lambda_2\beta_2 - \tfrac{1}{2}\lambda_2^2 - \tfrac{1}{2}\lambda_1\beta_1 + \tfrac{1}{8}(-1)^{\beta_1} + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1},$$

$$f_{20} = 1 + \tfrac{1}{2}\beta_2^2 + \tfrac{1}{2}\lambda_1^2 + \tfrac{1}{2}\lambda_2^2 + \lambda_2\lambda_1 + \lambda_2\beta_2 + \lambda_1\beta_2 + \tfrac{3}{2}\beta_2 + \tfrac{3}{2}\lambda_2 + \tfrac{3}{2}\lambda_1 - \tfrac{3}{2}\beta_1 + \tfrac{1}{2}\beta_1^2 - \lambda_2\beta_1 - \beta_2\beta_1 - \lambda_1\beta_1,$$

$$f_{21} = \tfrac{3}{2}\lambda_2 + \tfrac{1}{2}\lambda_1 + \beta_1 - \tfrac{3}{2}\beta_2 - \tfrac{1}{4}\lambda_1^2 + \tfrac{1}{2}\beta_2^2 + \tfrac{1}{4}\beta_1^2 - \lambda_2\beta_2 + \tfrac{7}{8} + \tfrac{1}{2}\lambda_2^2 + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} + \lambda_2\beta_1 + \tfrac{1}{2}\lambda_1\beta_1 - \beta_2\beta_1,$$

$$f_{22} = \lambda_2 + \lambda_1 + \tfrac{1}{2}\beta_1 - \beta_2 + \tfrac{1}{4}\lambda_1^2 + \tfrac{3}{4} - \lambda_1\beta_2 + \lambda_2\lambda_1 + \tfrac{1}{8}(-1)^{\beta_1} + \tfrac{1}{2}\lambda_1\beta_1 + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1},$$

$$f_{23} = 1 + \beta_2 + \lambda_1 - \beta_1 + \tfrac{1}{2}\lambda_2^2 + \lambda_2\lambda_1 + \lambda_2\beta_2 + \tfrac{3}{2}\lambda_2 - \lambda_2\beta_1,$$

$$f_{24} = \tfrac{3}{2}\lambda_2 + \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}\beta_1 - \beta_2 - \tfrac{1}{4}\lambda_1^2 - \tfrac{1}{4}\beta_1^2 - \lambda_2\beta_2 + \tfrac{1}{2}\lambda_2^2 + \tfrac{1}{2}\lambda_1\beta_1 + \lambda_2\beta_1 + \tfrac{7}{8} + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1},$$

$$f_{25} = \tfrac{3}{2}\lambda_2 + \tfrac{3}{2}\lambda_1 - \beta_1 + \tfrac{1}{2}\beta_2 + \tfrac{1}{2}\lambda_1^2 - \tfrac{1}{2}\beta_2^2 + \lambda_1\beta_2 + \tfrac{1}{4}\beta_1^2 + \lambda_2\lambda_1 + \lambda_2\beta_2 + \tfrac{1}{2}\lambda_2^2 - \lambda_1\beta_1 + \tfrac{7}{8} + \tfrac{1}{8}(-1)^{\beta_1} - \lambda_2\beta_1,$$

$$f_{26} = \tfrac{3}{4} + \tfrac{3}{4}\lambda_1^2 + \lambda_2\lambda_1 + \lambda_2 + \tfrac{3}{2}\lambda_1 - \tfrac{1}{2}\lambda_1\beta_1 + \tfrac{1}{8}(-1)^{\beta_1} - \tfrac{1}{2}\beta_1 + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1},$$

$$f_{27} = 2\lambda_2 + \lambda_1 + \beta_1 - 2\beta_2 + \tfrac{1}{4}\lambda_1^2 + \beta_2^2 - \lambda_1\beta_2 + \tfrac{1}{4}\beta_1^2 + \lambda_2\lambda_1 - 2\lambda_2\beta_2 + \lambda_2^2 + \tfrac{1}{2}\lambda_1\beta_1 + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} - \beta_2\beta_1 + \lambda_2\beta_1 + \tfrac{7}{8},$$

$$f_{28} = \tfrac{3}{2}\lambda_2 + \lambda_1 - \tfrac{1}{2}\beta_1 - \beta_2^2 - \tfrac{1}{4}\beta_1^2 + \lambda_2\lambda_1 + \lambda_2\beta_2 + \tfrac{1}{2}\lambda_2^2 + \beta_2\beta_1 - \lambda_2\beta_1 + \tfrac{7}{8} + \tfrac{1}{8}(-1)^{\beta_1},$$

$$f_{29} = 1 + \tfrac{5}{2}\lambda_2 + \lambda_1 - \beta_2 + \lambda_2\lambda_1 - \lambda_2\beta_2 + \tfrac{3}{2}\lambda_2^2,$$

$$f_{30} = 2\lambda_2 + \lambda_1 + \tfrac{1}{2}\beta_1 - \tfrac{3}{2}\beta_2 + \tfrac{1}{4}\lambda_1^2 + \tfrac{1}{2}\beta_2^2 - \lambda_1\beta_2 - \tfrac{1}{4}\beta_1^2 + \lambda_2\lambda_1 - 2\lambda_2\beta_2 + \lambda_2^2 + \tfrac{1}{2}\lambda_1\beta_1 + \tfrac{7}{8} + \tfrac{1}{8}(-1)^{\beta_1+\lambda_1} + \lambda_2\beta_1,$$

$$f_{31} = \tfrac{7}{8} + \lambda_2^2 + \lambda_1^2 + 2\lambda_2\lambda_1 + 2\lambda_2 + 2\lambda_1 - \beta_1 + \tfrac{1}{4}\beta_1^2 - \lambda_2\beta_1 - \lambda_1\beta_1 + \tfrac{1}{8}(-1)^{\beta_1},$$

$$\begin{aligned}f_{32} &= 2\lambda_2 + \tfrac{3}{2}\lambda_1 - \tfrac{1}{2}\beta_1 - \tfrac{1}{2}\beta_2 + \tfrac{1}{2}\lambda_1^2 - \tfrac{1}{2}\beta_2^2 - \lambda_1\beta_2 - \tfrac{1}{4}\beta_1^2 + \\ &\quad 2\lambda_2\lambda_1 + \lambda_2^2 + \tfrac{1}{8}(-1)^{\beta_1} + \beta_2\beta_1 + \tfrac{7}{8} - \lambda_2\beta_1, \\ f_{33} &= 1 + 3\lambda_2 + \tfrac{3}{2}\lambda_1 - \tfrac{3}{2}\beta_2 + \tfrac{1}{2}\lambda_1^2 + \tfrac{1}{2}\beta_2^2 - \lambda_1\beta_2 + 2\lambda_2\lambda_1 - 2\lambda_2\beta_2 + \\ &\quad 2\lambda_2^2.\end{aligned}$$

An example on how to use these tables: In order to determine the character of $V(\lambda)$ for $\lambda = 4\omega_1 + 8\omega_2)$ we can observe that the tuples (λ, β) belong to $\mathfrak{c}_1$ for $\beta \in \{0,\, \alpha_1+\alpha_2,\, 2\alpha_1+\alpha_2,\, 2\alpha_1+2\alpha_2,\, 3\alpha_1+2\alpha_2,\, 4\alpha_1+2\alpha_2,\, 3\alpha_1+3\alpha_2, 4\alpha_1+3\alpha_2, 4\alpha_1+4\alpha_2\}$. So by evaluating the quasi-polynomial f_1, we immediately get the following weight multiplicities:

$$\begin{aligned} &K^{(4,8)}_{(0,0)} = 1, \quad K^{(4,8)}_{(1,1)} = 2, \quad K^{(4,8)}_{(2,1)} = 3, \\ &K^{(4,8)}_{(2,2)} = 4, \quad K^{(4,8)}_{(3,2)} = 5, \quad K^{(4,8)}_{(4,2)} = 6, \\ &K^{(4,8)}_{(3,3)} = 6, \quad K^{(4,8)}_{(4,3)} = 8, \quad K^{(4,8)}_{(4,4)} = 9. \end{aligned} \tag{13}$$

Note that if you want to compare the values e.g. using the LiE online calculator[5] by A. Cohen et al. you have to take into account that LiE uses the inverse parameterisation of the simple roots, and that LiE denotes weights absolutely, not with respect to the highest weight of the module under consideration. The necessary reparameterisation is

$$\begin{aligned} \tilde{\lambda}_1 &= \lambda_2, \quad \tilde{\mu}_1 = \lambda_2 + \beta_1 - 2\beta_2, \\ \tilde{\lambda}_2 &= \lambda_1, \quad \tilde{\mu}_2 = \lambda_1 - 2\beta_1 + 2\beta_2. \end{aligned} \tag{14}$$

6. Some conclusions

COROLLARY 3. *The weight* 0 *does not occur in* $V(\lambda)$ *unless* $\lambda = i\alpha_1 + j\alpha_2$ *for nonnegative integers* i, j *such that* $\frac{i}{2} \le j \le i$. *In this case*

$$\dim V(\lambda)_0 = \frac{i}{2} - i^2 + 3ij - 2j^2 + \frac{3 + (-1)^i}{4}. \tag{15}$$

PROOF. The weight 0 does not occur in $V(\lambda)$ unless $\lambda \in Q$, so suppose $\lambda = i\alpha_1 + j\alpha_2 = (2i-2j)\omega_1 + (-i+2j)\omega_2$. The inequalities imposed on i, j are equivalent to λ being dominant. We calculate $K^{(2i-2j,-i+2j)}_{(i,j)}$ using

[5] http://www-math.univ-poitiers.fr/~maavl/LiE/form.html

the above results: The vector $(2i-2j,-i+2j,i,j)$ is contained in $\mathfrak{c}_{10}$, so we get $\dim V(\lambda)_0$ by evaluating f_{10} at this vector. This yields the asserted formula. $\square$

It is well known that $\dim V(\lambda)_\lambda = 1$ for all dominant weights λ. But what is $\dim V(\lambda)_{\lambda-\varepsilon}$ for some fixed $\varepsilon \in Q$? See figure 3 for the picture of the Weyl polytope around the highest weight.

COROLLARY 4. *Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ be a dominant weight. Then the weight multiplicities in $V(\lambda)$ of weights close to λ are given by*

$$\begin{aligned} \dim V(\lambda)_{\lambda-\alpha_1} &= 1 \textit{ if } \lambda_1 \geq 1, \\ \dim V(\lambda)_{\lambda-2\alpha_1-\alpha_2} &= 3 \textit{ if } \lambda_1 \geq 2,\ \lambda_2 \geq 1, \\ \dim V(\lambda)_{\lambda-\alpha_1-\alpha_2} &= 2 \textit{ if } \lambda_1,\lambda_2 \geq 1, \\ \dim V(\lambda)_{\lambda-\alpha_2} &= 1 \textit{ if } \lambda_2 \geq 1. \end{aligned} \tag{16}$$

PROOF. The first equation can be seen as follows: $\dim V(\lambda)_{\lambda-\alpha_1} = K_{(1,0)}^{(\lambda_1,\lambda_2)}$. For $\lambda_1 \geq 1$, the vector $(\lambda_1,\lambda_2,1,0)$ is in $\mathfrak{c}_5$. The value of f_5 at this vector is 1.

The remaining equations can be shown similarly. Note that in order to show the second and third equation, one can use either f_5 or f_1 respectively either f_1 or f_4. $\square$

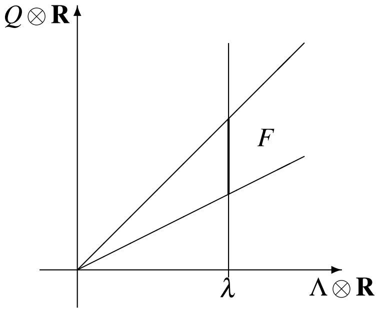

FIGURE 1. Intersecting F.

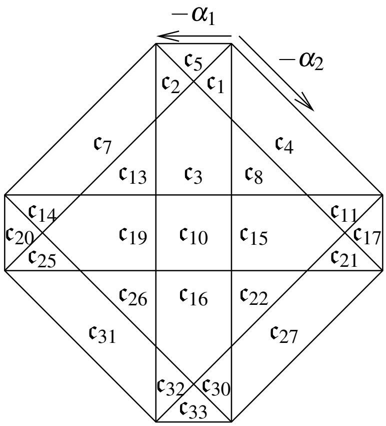

FIGURE 2. Induced decomposition of the Weyl polytope for the highest weight $\lambda = \omega_1 + 2\omega_2$.

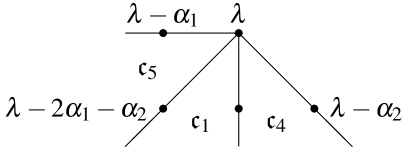

FIGURE 3. Around the highest weight.

CHAPTER 2

Preliminaries

This is an introduction to the theory of semisimple Lie algebras, as exposed e.g. in the textbooks by J. Humphreys [**11**] and V. Kac [**13**]. In particular, we define the basic object to be studied, the weight multiplicity function $K_{\lambda\mu}$ associated with a semisimple Lie algebra. The aim of this chapter is fixing the notations. It should generally be consulted only as the need occurs.

1. Lie algebras

Let K be a field. A *Lie algebra* over K is a vector space $\mathfrak{g}$ over K equipped with an alternating bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \tag{17}$$

such that the Jacobi identity

$$[\xi,[\psi,\eta]] + [\eta,[\xi,\psi]] + [\psi,[\eta,\xi]] = 0 \tag{18}$$

holds for all $\xi,\psi,\eta \in \mathfrak{g}$. A linear map $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ between Lie algebras is called a *homomorphism of Lie algebras*, if

$$\Phi([\xi,\psi]) = [\Phi(\xi),\Phi(\psi)] \tag{19}$$

for all $\xi,\psi \in \mathfrak{g}_1$. For any vector space V over K one can construct the Lie algebra $\mathfrak{aut}(V)$ by endowing the vector space $\mathrm{End}_K(V)$ of linear endomorphisms of V with the product given by

$$[A,B] := AB - BA \tag{20}$$

for $A,B \in \mathrm{End}_K(V)$.

Let $\mathfrak{g}$ be a Lie algebra and V a vector space. A *representation* of $\mathfrak{g}$ on V is a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{aut}(V)$. In this case, V is called a $\mathfrak{g}$-*module* and one abbreviates $\xi v := \rho(\xi)v$ for $\xi \in \mathfrak{g}$ and $v \in V$. For any

Lie algebra $\mathfrak{g}$ one considers in particular the *adjoint representation* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{aut}(\mathfrak{g})$, given by

$$(\mathrm{ad}\,\xi)\psi := [\xi, \psi] \tag{21}$$

for $\xi, \psi \in \mathfrak{g}$.

An $(\mathrm{ad}\,\mathfrak{g})$-invariant subspace I of a Lie algebra $\mathfrak{g}$ is called an *ideal*. In this case, I is in particular a Lie subalgebra of $\mathfrak{g}$. For example, for any Lie algebra $\mathfrak{g}$ the *derived Lie algebra* $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ is an ideal. A Lie algebra $\mathfrak{g}$ is called *Abelian*, if $\mathfrak{g}' = 0$. More generally, $\mathfrak{g}$ is called *solvable*, if the sequence $(\mathfrak{g}^{(i)})_{i \geq 0}$ of Lie algebras with $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} = (\mathfrak{g}^{(i)})'$ becomes 0 after a finite number of terms. A Lie algebra $\mathfrak{g} \neq 0$ is called *semisimple*, if it does not contain any nontrivial solvable ideals.

2. Semisimple Lie algebras

In the following, suppose the field K to be algebraically closed and of characteristic 0. By a theorem of von H. Weyl, in this case every finite dimensional representation of every semisimple Lie algebra is completely reducible. Let $\mathfrak{g}$ be a semisimple finite dimensional Lie algebra. In this case, the adjoint representation is faithful. For $\xi \in \mathfrak{g}$ consider the Jordan-Chevalley decomposition $\mathrm{ad}\,\xi = (\mathrm{ad}\,\xi)_{\mathrm{s}} + (\mathrm{ad}\,\xi)_{\mathrm{n}}$ of $\mathrm{ad}\,\xi$ into its semisimple and nilpotent part. Then $(\mathrm{ad}\,\xi)_{\mathrm{s}}$ and $(\mathrm{ad}\,\xi)_{\mathrm{n}}$ are in the image of ad and the preimages ξ_{s} respectively ξ_{n} fulfil $\xi = \xi_{\mathrm{s}} + \xi_{\mathrm{n}}$. This decomposition is called the *abstract Jordan decomposition* of ξ. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called *toral*, if the nilpotent part ξ_{n} in the abstract Jordan decomposition of any element $\xi \in \mathfrak{h}$ vanishes. Let $\mathfrak{h}$ be a maximal toral Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is nontrivial and Abelian. Its dimension $r = \dim_K \mathfrak{h}$ of $\mathfrak{h}$ is independent of $\mathfrak{h}$ and is called the *rank* of $\mathfrak{g}$.

Now let V be a finite dimensional $\mathfrak{g}$-module. Then V is also an $\mathfrak{h}$-module by restriction. As an $\mathfrak{h}$-module, V decomposes into a direct sum of simple representations. A complete system of representatives of the isomorphism classes of simple $\mathfrak{h}$-modules is given by $(K_\lambda)_{\lambda \in \mathfrak{h}^*}$, where each K_λ is a one-dimensional vector space over K endowed with the action

$$\xi v := \lambda(\xi) v \tag{22}$$

for $\xi \in \mathfrak{h}$ and $v \in K_\lambda$. The K_λ-isotypical component V_λ in the decomposition

$$V = \sum_{\lambda \in \mathfrak{h}^*} V_\lambda \tag{23}$$

of V as an $\mathfrak{h}$-module is called the *weight space of weight* λ. Its dimension $\dim_K V_\lambda$ is called the *multiplicity* of the weight λ in the representation V. A weight is said to *appear* in a given representation, if its multiplicity is positive. The set Λ of all weights appearing in any finite dimensional representation of $\mathfrak{g}$ is a lattice in $\mathfrak{h}^*$, i.e. they form a free Abelian subgroup and $\mathfrak{h}^* = K \otimes_{\mathbf{Z}} \Lambda$. The lattice Λ is called the *weight lattice* of $\mathfrak{g}$.

On $\mathfrak{g}$,

$$\kappa(\xi, \psi) := \operatorname{tr}(\operatorname{ad}\xi)(\operatorname{ad}\psi) \tag{24}$$

for $\xi, \psi \in \mathfrak{g}$ defines a nondegenerate symmetric bilinear form, the *Killing form*. The restriction of the Killing form to $\mathfrak{h} \times \mathfrak{h}$ ist nondegenerate and defines a canonical isomorphism $t : \mathfrak{h}^* \to \mathfrak{h}$. Let $(\ ,\ ) : \mathfrak{h}^* \times \mathfrak{h}^* \to K$ be the bilinear form on $\mathfrak{h}^*$ given by $(\lambda, \mu) := \kappa(t_\lambda, t_\mu)$ for $\lambda, \mu \in \mathfrak{h}^*$. Let $\mathfrak{h}^*_{\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} \Lambda \subset \mathfrak{h}^*$. The restriction of $(\ ,\ )$ to $\mathfrak{h}^*_{\mathbf{Q}} \times \mathfrak{h}^*_{\mathbf{Q}}$ has values in $\mathbf{Q}$ and defines a nondegenerate bilinear form on the vector space $\mathfrak{h}^*_{\mathbf{Q}}$ over $\mathbf{Q}$. Its extension to $\mathfrak{h}^*_{\mathbf{R}} := \mathbf{R} \otimes_{\mathbf{Q}} \Lambda_{\mathbf{Q}}$ is positive definite and defines the structure of a Euclidean vector space on $\mathfrak{h}^*_{\mathbf{R}}$. For $\lambda \in \mathfrak{h}^*_{\mathbf{R}} \setminus \{0\}$ let $s_\lambda \in \mathrm{O}(\mathfrak{h}^*_{\mathbf{R}})$ denote the reflection at $\lambda^\perp$ along λ, i.e.

$$s_\lambda(\mu) = \mu - 2\frac{(\mu, \lambda)}{\|\lambda\|^2}\lambda \tag{25}$$

for $\mu \in \mathfrak{h}^*_{\mathbf{R}}$.

The weight space of weight 0 in the adjoint representation of $\mathfrak{g}$ is $\mathfrak{h}$. The set $\Phi \subset \Lambda$ of the remaining weights appearing in the adjoint representation of $\mathfrak{g}$ is called the *root system* of $\mathfrak{g}$ with respect to $\mathfrak{h}$, its elements *roots*. The group $W \subset \mathrm{O}(\mathfrak{h}^*_{\mathbf{R}})$ generated by the reflections $\{s_\alpha : \alpha \in \Phi\}$ is called the *Weyl group* of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The connected components of $\mathfrak{h}^*_{\mathbf{R}} \setminus \bigcup_{\alpha\in\Phi} \alpha^\perp$ are called *Weyl chambers*. After choosing a fixed Weyl chamber C this one is called the *fundamental* Weyl chamber. Then $\bar{C}$ is a fundamental region and a complete set of coset representatives of the action of W on $\mathfrak{h}^*_{\mathbf{R}}$. Weights in $\Lambda_+ := \Lambda \cap \bar{C}$ are called *dominant*. A root $\alpha \in \Phi$ is called *positive* with respect to C, if $(\alpha, \lambda) > 0$ for all $\lambda \in C$. Let Φ_+ be the set of positive roots. For weights $\lambda, \mu \in \Lambda$ let $\lambda \preceq \mu$ if $\mu - \lambda$ can be expressed as a sum of positive roots. The partial order given on Λ by $\preceq$ is called the *dominance order*.

Let V be a finite dimensional simple $\mathfrak{g}$-module. Then the set of all weights appearing in V has a largest element with respect to the dominance order, the *highest weight* of V. The highest weight of any simple $\mathfrak{g}$-module V is dominant and V is uniquely determined up to isomorphism by its highest weight. Conversely, for every dominant weight $\lambda \in \Lambda_+$ there is a simple $\mathfrak{g}$-module $V(\lambda)$ with highest weight λ. This means that the family $(V(\lambda))_{\lambda \in \Lambda_+}$ is a complete set of representatives of isomorphism classes of simple $\mathfrak{g}$-modules.

The function $K : \Lambda_+ \times \Lambda \to \mathbf{Z}_{\geq 0}$ given by

$$K_{\lambda\mu} := \dim_K V(\lambda)_\mu \tag{26}$$

for $\lambda \in \Lambda_+$ and $\mu \in \Lambda$ is called the *weight multiplicity function*. It is the object to be studied in this work. We extend it by 0 to $\Lambda \times \Lambda$.

A positive root $\alpha \in \Phi_+$ is called *simple*, if it cannot be decomposed as a sum $\alpha = \alpha_1 + \alpha_2$ of positive roots $\alpha_1, \alpha_2 \in \Phi_+$. The set $\Delta \subset \Phi_+$ of simple roots is called the *root basis* associated with C. We write Δ as a family with index set I, i.e. $(\alpha_i)_{i \in I}$. The Abelian group $Q \subset \Lambda$ generated by Φ is called the *root lattice* of $\mathfrak{g}$ with respect to $\mathfrak{h}$. It is free of rank r with basis Δ. Let $(\omega_i)_{i \in I}$ be the basis of $\mathfrak{h}^*_{\mathbf{R}}$ dual to $(2\alpha_i/\|\alpha_i\|^2)_{i \in I}$ with respect to $(\ ,\ )$. Then $(\omega_i)_{i \in I}$ is a basis of Λ and all ω_i are dominant. The weights ω_i are called *fundamental*.

Let $\mathfrak{h}_{\mathbf{R}}$ be the dual space of $\mathfrak{h}^*_{\mathbf{R}}$ and $\langle\ ,\ \rangle : \mathfrak{h}_{\mathbf{R}} \times \mathfrak{h}^*_{\mathbf{R}} \to \mathbf{R}$ the canonical pairing. Let $(h_i)_{i \in I}$ be the basis of $\mathfrak{h}_{\mathbf{R}}$ dual to $(\omega_i)_{i \in I}$ with respect to $\langle\ ,\ \rangle$.

Let $F := \bigcup_{p \in \mathbf{Z}_{\geq 0}} I^p$ be the free monoid over I. For $i \in I$ let $s_i := s_{\alpha_i}$. The Weyl group W is generated by the simple reflections $\{s_i : i \in I\}$ and $s_i = s_i^{-1}$ for all $i \in I$. Hence, the canonical homomorphism of monoids $F \to W$ defined by $i \mapsto s_i$ is surjective. A preimage $\underline{w} \in F$ of $w \in W$ is called a *decomposition* of w. A decomposition is called *reduced*, if its length is minimal. The *length* $l(w)$ of an element $w \in W$ of the Weyl group is the length of a reduced decomposition. The Weyl group contains exactly one element $w_0 \in W$ of maximal length, the so-called *longest element* of the Weyl group.[6]

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[6]This is not stated explicitly in [**11**], but follows e.g. from the fact, that the fundamental Weyl chamber C is a fundamental region for the action of W on $\mathfrak{h}^*_{\mathbf{R}}$: The longest element w_0 is simply the element of W corresponding to $-C$.

3. Vector partition functions

In this section we introduce the notion of a vector partition function as well as the related notion of a vector partition measure and its continuous analogon. We also state B. Sturmfels' structure theorem [**23**] on vector partition functions.

Let $r, s \in \mathbf{Z}_{>0}$ be positive integers and $M \in \mathbf{Z}^{(s,r)}$ a matrix. Consider M as a linear map $\mathbf{R}^r \to \mathbf{R}^s$. Suppose

$$\ker(M) \cap \mathbf{R}^r_{\geq 0} = \{0\}. \tag{27}$$

Then $\Phi_M(w) := |\{v \in \mathbf{Z}^r_{\geq 0} : Mv = w\}|$ for $w \in \mathbf{Z}^s$ defines a function $\Phi_M : \mathbf{Z}^s \to \mathbf{Z}_{\geq 0}$, the so-called *vector partition function* associated with M.

EXAMPLE 5. For

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{28}$$

we get the vector partition function

$$\Phi_M \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{cases} h_2 & \text{if } h_1 \geq h_2 \geq 0, \\ h_1 & \text{if } h_2 \geq h_1 \geq 0, \\ 0 & \text{if } h_1 \text{ or } h2 < 0. \end{cases} \tag{29}$$

REMARK 6. Actually, Φ_M is invariant under permutation of the columns of M. Hence, one could more intrinsically define the vector partition function associated with a finite multiset of vectors which do not fulfil any nontrivial linear relations with nonnegative coefficients.

For $w \in \mathbf{R}^s$ let δ_w denote the Dirac measure concentrated in w. Instead of Φ_M one can equivalently consider the following measure:

DEFINITION 7. The measure

$$\mu_M := \sum_{w \in \mathbf{Z}^s} \Phi_M(w) \delta_w \tag{30}$$

on $\mathbf{R}^s$ is called the *vector partition measure* associated with M.

For a measurable map $f : X \to Y$ between measurable spaces and a measure μ on X let $f_*\mu$ denote the image measure of μ under f. Let $\delta_+^r := \sum_{v \in \mathbf{Z}_{\geq 0}^r} \delta_v$ be the counting measure of $\mathbf{Z}_{\geq 0}^r$ on $\mathbf{R}^r$. Then

$$\begin{aligned} \mu_M = \sum_{w \in \mathbf{Z}^s} \Phi_M(w)\delta_w &= \sum_{w \in \mathbf{Z}^s} |\{v \in \mathbf{Z}_{\geq 0}^r : Mv = w\}|\delta_w \\ &= \sum_{v \in \mathbf{Z}_{\geq 0}^r} \delta_{Mv} = M_*\delta_+^r. \end{aligned} \tag{31}$$

In this presentation, the vector partition measure has the following continuous analogon:

DEFINITION 8. Let λ_+^r be the Lebesgue measure on $\mathbf{R}^r$, restricted to $\mathbf{R}_{\geq 0}^r$. Then

$$\tilde{\mu} := M_*\lambda_+^r \tag{32}$$

defines an absolutely continuous measure on $\mathbf{R}^s$, the *continuous vector partition measure*.

While the vector partition function respectively the vector partition measure express the number of ways a given vector $w \in R^s$ can be written as a linear combination of the column vectors of M with nonnegative integral coefficients, the continuous vector partition measure expresses the amount of linear combinations with nonnegative real coefficients.

In order to be able to state Sturmfels' structure theorem for vector partition functions, we need some notions from combinatorial geometry. Let V be a real finite dimensional vector space. A subset $C \subset V$ is called a *salient pointed convex polyhedral cone* or just *cone* for short, if it can be written in the form

$$C = \operatorname{pos}\{v_1, \dots, v_k\} := \mathbf{R}_{\geq 0}v_1 + \cdots + \mathbf{R}_{\geq 0}v_k, \tag{33}$$

where $v_1, \dots, v_k$ are vectors contained in an open half-space of V. If A is the matrix with columns $v_1, \dots, v_k$, we also write $\operatorname{pos}(A) := \operatorname{pos}\{v_1, \dots, v_k\} = A \cdot \mathbf{R}_{\geq 0}^k$. By the *dimension* of a cone we mean the dimension of the vector space it generates. A subset $C' \subset C$ of a cone is called a *face*, if $C' = C$ or if there is a hyperplane $H \subset V$ such that C is contained in one of the closed half-spaces delimited by H and that C' has the form $C' = C \cap H$. A *fan* in V is a finite set F of cones in V such that:

(1) For any cone $C \in F$, all faces of C are contained in F, too.

(2) For any two $C, C' \in F$, the intersection $C \cap C'$ is a face of C and C'.

The union $\operatorname{supp}(F) := \bigcup_{C \in F} C$ of all cones of a fan F is called the *support* of F. The cones contained in a given fan are partially ordered by inclusion. A fan is called *homogeneous*, if all its maximal cones share the same dimension. In this case, the maximal cones are also called the *chambers* of the fan.

A function $f : \mathbf{Z}^s \to \mathbf{R}$ is called a *quasi-polynomial*, if there is a subgroup $\Gamma \subset \mathbf{Z}^s$ of finite index and a family $\left(f_{[w]}\right)_{[w] \in \mathbf{Z}^s/\Gamma}$ of polynomial functions, such that $f(w) = f_{[w]}(w)$ for all $w \in \mathbf{Z}^s$.

THEOREM 9 ([**23**, th. 1][7]). *Let $M \in \mathbf{Z}^{(s,r)}$ be a matrix subject to* (27). *Then there is a homogeneous fan F in $\mathbf{R}^s$ and for any chamber C of F there is a quasi-polynomial f_C on $\mathbf{Z}^s$ such that:*

$$\Phi_M = f_C \text{ on } C \cap \mathbf{Z}^s \tag{34}$$

and $\Phi_M = 0$ *beyond* $\operatorname{supp}(F)$.

Indeed, one can explicitly describe the fan F associated with M and state some properties of the quasi-polynomials f_C: For $\sigma \subset \{1, \ldots, r\}$ let M_σ denote the submatrix of M consisting of the column vectors indexed by σ. For $w \in \operatorname{pos}(M)$ let

$$\Delta(w) := \left\{\sigma \subset \{1, \ldots, r\} : w \in \operatorname{pos}(M_\sigma)\right\}. \tag{35}$$

Then the fan F from theorem 9 consists of the cones of the form

$$\bigcap_{\sigma \in \Delta(w)} \operatorname{pos}(M_\sigma) \tag{36}$$

for $w \in \operatorname{pos}(M)$. If C is a chamber in F and w_1 and w_2 contained in the interior of C, then $\Delta(w_1) = \Delta(w_2)$. Hence one can define $\Delta(C) := \Delta(w)$ for an arbitrary w contained in the interior of C. The quasi-polynomials f_C have the form

$$f_C = \sum_{\sigma \in \Delta(C)} f_{C,\sigma} g_\sigma \tag{37}$$

for polynomial functions $f_{C,\sigma}$ of degree $|\sigma| - s$ and $\operatorname{im}_{\mathbf{Z}}(M_\sigma)$-periodic functions g_σ.

[7] Also stated in a less precise form by G. Blakley, [**3**, p. 339].

CHAPTER 3

Chopped and sliced cones

We define a new structure in combinatorial geometry, the chopped and sliced cone. To each chopped and slices cone we associate a family of sequences of discrete measures. We show that each such sequence converges weakly to an absolutely continuous measure. Finally, to each chopped and sliced cone we associate a vector partition function such that important numerical quantities can be calculated by evaluating that vector partition function.

1. Introduction

Contemplating weight diagrams of simple Lie algebras of rank 2, as depicted e.g. in figure 4, a few observations are immediate. For example, weight multiplicities tend to be large at the centre of the Weyl polytope and small around the corners, the weight multiplicity function seems to be continuous (in an imprecise sense) and governed by a uniform behaviour on certain pieces of the Weyl polytope. Giving precise statements about the structure of weight diagrams is less obvious: Suppose, say, we would like to make the last observation precise by stating that the Weyl polytope of a given representation decomposes into a finite number of pieces such that the weight multiplicity function is piecewise polynomial. This statement is vacuous: As there are only finitely many weights in the Weyl polytope, one could e.g. choose a decomposition such that any piece contains only one weight. Or, the other extreme, take a trivial decomposition into one single piece with a polynomial of high degree interpolating all the weight multiplicities. None of these solutions is satisfactory in the sense that it reflects the intuitive observations.

It turns out that one *can* in fact get (and prove) precise statements by considering the family of weight diagrams of all simple representations of a given Lie algebra at once. The here-proposed notion of a chopped and sliced cone is an abstract way to define families of weight diagrams. In fact, P. Littelmann's work [**21**] on generalisations of Gelfand-Tsetlin patterns can be reformulated as saying that for each complex simple Lie algebra there is a chopped and sliced cone giving rise to the weight diagrams of its simple representations.

Having this application in mind, we investigate here abstractly the properties of families of weight diagrams associated with a chopped and sliced cone. The actual translation of Littelmann's results into this language and the corresponding conclusions will follow in a different chapter.

2. Definition

Let K, Λ and Q be free Abelian groups of finite rank. Let $\tilde{\Lambda}_+$ and R_+ be free Abelian monoids of finite rank. The free Abelian groups generated by a basis of $\tilde{\Lambda}_+$ respectively R_+ are denoted by $\tilde{\Lambda}$ and R. On $\tilde{\Lambda}$, consider the partial order defined by $x \leq y$ if $y - x \in \tilde{\Lambda}_+$ and similarly for R. Let $p : K \to \tilde{\Lambda}$, $q : K \to Q$, $r : K \to R$ and $s : \Lambda \to \tilde{\Lambda}$ be homomorphisms of Abelian groups. Let $K_{\mathbf{R}} := K \otimes_{\mathbf{Z}} \mathbf{R}$ and similarly for the other groups. The extensions of p, q, r and s to these real vector spaces are still denoted by the same symbol.

DEFINITION 10. A 9-tuple $(K, \Lambda, Q, \tilde{\Lambda}_+, R_+, p, q, r, s)$ as above is called a *chopped and sliced cone* if the set

$$\mathcal{C}^{\lambda} := \{x \in K_{\mathbf{R}} : r(x) \geq 0,\ p(x) \leq s(\lambda)\} \tag{38}$$

is bounded for all $\lambda \in \Lambda$.

If the extra structure is clear from the context, we may simply speak of the cone

$$\mathcal{C} := \{x \in K_{\mathbf{R}} : r(x) \geq 0\} \tag{39}$$

as being chopped and sliced. For $\lambda \in \Lambda$ and $\beta \in Q$, define

$$\mathcal{C}^{\lambda}_{\beta} := \{x \in \mathcal{C}^{\lambda} : q(x) = \beta\}. \tag{40}$$

EXAMPLE 11. Let $K = \mathbf{Z}^2$, $\Lambda = Q = \mathbf{Z}$, $\tilde{\Lambda}_+ = \mathbf{Z}_{\geq 0}$ and $R_+ = \mathbf{Z}^2_{\geq 0}$. Consider the morphisms given by $r_1(x) = x_1 - x_2$, $r_2(x) = x_2$, $p(x) = x_1 + x_2$, $q(x) = x_1$, $s = \mathrm{id}_{\mathbf{Z}}$. Then the polyhedrons defined by this chopped and sliced cones are

$$\begin{aligned} \mathcal{C} &:= \{x \in \mathbf{R}^2 : x_1 \geq x_2 \geq 0\}, \\ \mathcal{C}^\lambda &:= \{x \in \mathbf{R}^2 : x_1 \geq x_2 \geq 0,\ x_1 + x_2 \leq \lambda\}, \\ \mathcal{C}^\lambda_\beta &:= \{x \in \mathbf{R}^2 : x_1 \geq x_2 \geq 0,\ x_1 + x_2 \leq \lambda,\ x_1 = \beta\}. \end{aligned} \tag{41}$$

for any $\lambda, \beta \in \mathbf{Z}$. This chopped and sliced cone is depicted in figure 5.

The sets of integral points in any of the polyhedrons associated with a chopped and sliced cone are denoted by $\mathcal{S} := \mathcal{C} \cap K$, $\mathcal{S}^\lambda := \mathcal{C}^\lambda \cap K$ and $\mathcal{S}^\lambda_\beta := \mathcal{C}^\lambda_\beta \cap K$. In order to simplify notation, for the rest of the chapter we assume without loss of generality that $\Lambda = \tilde{\Lambda}$ and $s = \mathrm{id}$.

3. Measures associated with chopped and sliced cones

For a fixed chopped and sliced cone $\mathcal{C}$ we define some measures on $Q_{\mathbf{R}}$. All appearing real vector spaces are topological spaces with the natural topology and as such equipped with the σ-algebra of Borel sets. For any set A let δ_A denote the counting measure associated with A. If $A = \{\beta\}$ contains only one element, we abbreviate $\delta_\beta := \delta_{\{\beta\}}$ for the corresponding Dirac measure concentrated in β. For any measurable map ϕ let ϕ_* denote the induced map on measures, which maps any measure to its image measure under ϕ. For $\lambda \in \Lambda$ and $n \in \mathbf{Z}_{>0}$ let

$$\mu_\lambda := \sum_{\beta \in Q} |\mathcal{S}^\lambda_\beta| \cdot \delta_\beta = q_* \delta_{\mathcal{S}^\lambda} \tag{42}$$

and

$$\mu^{(n)}_\lambda := \frac{1}{n^{\mathrm{rk}(K)}} \left(\tfrac{1}{n} \mathrm{id}_{Q_{\mathbf{R}}}\right)_* \mu_{n\lambda}\,. \tag{43}$$

Let $\lambda_{K_{\mathbf{R}}}$ be the Lebesgue measure on $K_{\mathbf{R}}$, normalised such that K has covolume 1. For a measurable subset $M \subset K_{\mathbf{R}}$ let λ_M be the $\lambda_{K_{\mathbf{R}}}$-absolutely continuous measure with density 1_M, the characteristic function of M. For $\lambda \in \Lambda$ we define

$$\tilde{\mu}_\lambda := q_* \lambda_{\mathcal{C}^\lambda}\,. \tag{44}$$

These measures are finite for any chopped and sliced cone $\mathcal{C}$.

THEOREM 12. *Let $\mathcal{C}$ be a chopped and sliced cone and $\lambda \in \Lambda$. Then $\mu^{(n)}_\lambda$ converges weakly towards $\tilde{\mu}_\lambda$ for $n \to \infty$.*

For the proof, we provide two lemmas. The first one should be obvious to anybody who knows about B. Riemann's definition of the integral:

LEMMA 13. *Let $M \subset K_{\mathbf{R}}$ be a bounded set such that the characteristic function 1_M of M is Riemann integrable. Then $\frac{1}{n^{\mathrm{rk}(K)}}\delta_{M\cap\frac{1}{n}K}$ converges weakly towards λ_M for $n \to \infty$.*

PROOF. We denote the canonical pairing between functions and measures by $(\ ,\ )$, i.e. $(f,\mu) = \int f\mu = \int f(x)\mu(dx)$. Let $f \in C_b(K_{\mathbf{R}},\mathbf{R})$ be an arbitrary continuous bounded function on $K_{\mathbf{R}}$. We have to show that $(f, \frac{1}{n^{\mathrm{rk}(K)}}\delta_{M\cap\frac{1}{n}K}) \to (f,\lambda_M)$ for $n \to \infty$. For any $n \in \mathbf{Z}_{>0}$ we have

$$\left(f, \frac{1}{n^{\mathrm{rk}(K)}}\delta_{M\cap\frac{1}{n}K}\right) = \sum_{a\in M\cap\frac{1}{n}K} \frac{f(a)}{n^{\mathrm{rk}(K)}} = \sum_{a\in\frac{1}{n}K} \frac{1_M(a)f(a)}{n^{\mathrm{rk}(K)}}. \tag{45}$$

As 1_M is Riemann integrable and f is continuous, the product $1_M f$ is Riemann integrable and the above expression is a Riemann sum which converges for $n \to \infty$ towards $\int_{K_{\mathbf{R}}} 1_M(a)f(a)\,da$. As the Riemann integral and the Lebesgue integral coincide for Riemann integrable functions, we get

$$\begin{aligned}\left(f, \frac{1}{n^{\mathrm{rk}(K)}}\delta_{M\cap\frac{1}{n}K}\right) &\to \int_{K_{\mathbf{R}}} 1_M(a)f(a)\,da \\ &= \int_{K_{\mathbf{R}}} 1_M(a)f(a)\lambda_{K_{\mathbf{R}}}(da) \\ &= \int_{K_{\mathbf{R}}} f(a)\lambda_M(da) = (f,\lambda_M)\end{aligned} \tag{46}$$

for $n \to \infty$ as required. □

The second lemma says that weak convergence and push-forward of measures commute:

LEMMA 14. *Let $\phi : X \to Y$ be a continuous map between metric spaces and $(\mu_n)_{n\in\mathbf{Z}_{>0}}$ a sequence of finite measures on X which converges weakly towards μ. Then the sequence $(\phi_*\mu_n)_{n\in\mathbf{Z}_{>0}}$ converges weakly towards $\phi_*\mu$.*

PROOF. Let $f \in C_b(Y,\mathbf{R})$ be an arbitrary continuous bounded function on Y. We have to show that $(f,\phi_*\mu_n) \to (f,\phi_*\mu)$ for $n \to \infty$. Indeed,

$$(f,\phi_*\mu_n) = (f\circ\phi,\mu_n) \to (f\circ\phi,\mu) = (f,\phi_*\mu), \tag{47}$$

as $f \circ \phi$ is continuous and bounded, too. □

Given these lemmas, we can directly verify theorem 12:

PROOF OF THEOREM 12. By definition

$$\mu_\lambda^{(n)} = \frac{1}{n^{\mathrm{rk}(K)}} \left(\tfrac{1}{n}\mathrm{id}_{Q_{\mathbf{R}}}\right)_* \mu_{n\lambda} = \frac{1}{n^{\mathrm{rk}(K)}} \left(\tfrac{1}{n}\mathrm{id}_{Q_{\mathbf{R}}}\right)_* q_* \delta_{\mathcal{S}^{n\lambda}}. \tag{48}$$

As q is linear

$$\left(\tfrac{1}{n}\mathrm{id}_{Q_{\mathbf{R}}}\right)_* q_* = q_* \left(\tfrac{1}{n}\mathrm{id}_{K_{\mathbf{R}}}\right)_* \tag{49}$$

and furthermore

$$\left(\tfrac{1}{n}\mathrm{id}_{K_{\mathbf{R}}}\right)_* \delta_{\mathcal{S}^{n\lambda}} = \delta_{\frac{1}{n}\mathcal{S}^{n\lambda}} = \delta_{\mathcal{C}^\lambda \cap \frac{1}{n}K}, \tag{50}$$

so altogether

$$\mu_\lambda^{(n)} = q_* \Big(\frac{1}{n^{\mathrm{rk}(K)}} \delta_{\mathcal{C}^\lambda \cap \frac{1}{n}K} \Big). \tag{51}$$

By lemma 13, $\frac{1}{n^{\mathrm{rk}(K)}} \delta_{\mathcal{C}^\lambda \cap \frac{1}{n}K}$ converges weakly towards $\lambda_{\mathcal{C}^\lambda}$ and we conclude by lemma 14. □

Next, investigate the limit measure $\tilde{\mu}_\lambda$ a bit more closely.

THEOREM 15. *For any $\lambda \in \Lambda$, the measure $\tilde{\mu}_\lambda$ is absolutely continuous with respect to Lebesgue measure on $Q_{\mathbf{R}}$.*

PROOF. By definition, $\lambda_{\mathcal{C}^\lambda}$ is $\lambda_{K_{\mathbf{R}}}$-absolutely continuous on $K_{\mathbf{R}}$ with density $1_{\mathcal{C}^\lambda}$, the characteristic function of $\mathcal{C}^\lambda$. We choose a section of the linear map $q : K_{\mathbf{R}} \to Q_{\mathbf{R}}$. This yields an isomorphism $K_{\mathbf{R}} \cong Q_{\mathbf{R}} \times \ker(q)$ and all the fibres $q^{-1}(\{\beta\})$ of q are identified canonically with $\ker(q)$. Let the Lebesgue measure on $Q_{\mathbf{R}}$ be normalised such that Q has covolume 1. Let the Lebesgue measure on $\ker(q)$ be normalised such that $\lambda_{K_{\mathbf{R}}}$ is the product measure of both. By Fubini's theorem it follows that for any measurable set $A \subset Q_{\mathbf{R}}$ we have

$$\begin{aligned}
\int_A \tilde{\mu}_\lambda &= \int_{q^{-1}(A)} 1_{\mathcal{C}^\lambda} \lambda_{K_{\mathbf{R}}} \\
&= \int_A \Big(\int_{q^{-1}(\{\beta\})} 1_{\mathcal{C}^\lambda_\beta} \lambda_{q^{-1}(\{\beta\})} \Big) \lambda_{K_{\mathbf{R}}}(d\beta) \\
&= \int_A \Big(\int_{\mathcal{C}^\lambda_\beta} \lambda_{q^{-1}(\{\beta\})} \Big) \lambda_{K_{\mathbf{R}}}(d\beta),
\end{aligned} \tag{52}$$

i.e. $f_\lambda(\beta) := \int_{\mathcal{C}^\lambda_\beta} \lambda_{q^{-1}(\{\beta\})}$ is the density of $\tilde{\mu}_\lambda$ with respect to $\lambda_{Q_\mathbf{R}}$. □

4. Cones and vector partition functions

We show that the numbers of points $|\mathcal{S}^\lambda_\beta|$ can be written as a composition of a vector partition function with a linear map for any chopped and sliced cone. Combined with the presentation of the weight multiplicities as numbers of points an a chopped and sliced cone, we get an expression for the weight multiplicity function of any semisimple complex Lie algebra using a single vector partition function. In particular, in contrast do the classical formula [**18**] by B. Kostant, there is no summation over the Weyl group. This generalises the expression [**2**, th. 2.1] obtained by S. Billey et al. for the Lie algebras $\mathfrak{sl}_{r+1}(\mathbf{C})$.

Let $\mathcal{C}$ be a chopped and sliced cone. We use the symbols R, Λ, $\tilde{\Lambda}$, Q, p, q, r, s as in section 2. As of the free Abelian group K, fix a free Abelian monoid K_+ of finite rank and let K be the free Abelian group generated by a basis of K_+. Suppose that

$$\mathcal{C} \subset K_+. \tag{53}$$

In this case, the map $r : K \to R$ defining $\mathcal{C}$ can be replaced by a map $\tilde{r} : K \to \tilde{R}$ to a free Abelian group of potentially lower rank by omitting the inequalities defining K_+.

EXAMPLE 16. The map $r : \mathbf{Z}^2 \to \mathbf{Z}^2$ from example 11 is replaced by $\tilde{r} : \mathbf{Z}^2 \to \mathbf{Z}$ with $\tilde{r}(x) = x_1 + x_2$.

From the present presentation of a chopped and sliced cone, one can get the original one e.g. by imposing $R := \tilde{R} \times K$ and $r := (\tilde{R}, \mathrm{id}_K)$.

For $\lambda \in \Lambda$, $\beta \in Q$ we have

$$\begin{aligned} |\mathcal{S}^\lambda_\beta| &= |\{x \in K_+ : \tilde{r}(x) \geq 0,\ p(x) \leq s(\lambda),\ q(x) = \beta\}| \\ &= |\{x \in K_+ : \exists y \in \tilde{R}_+, z \in \tilde{\Lambda}_+ : \tilde{r}(x) - y = 0, \\ &\qquad\qquad p(x) + z = s(\lambda),\ q(x) = \beta\}| \\ &= \Phi_E \begin{pmatrix} 0_{\tilde{R}} \\ s(\lambda) \\ \beta \end{pmatrix} \end{aligned} \tag{54}$$

for

$$E = \begin{pmatrix} \tilde{r} & -\mathrm{id}_{\tilde{R}} & 0 \\ p & 0 & \mathrm{id}_{\tilde{\Lambda}} \\ q & 0 & 0 \end{pmatrix} : K \times \tilde{R} \times \tilde{\Lambda} \to \tilde{R} \times \tilde{\Lambda} \times Q. \tag{55}$$

If we define

$$\begin{pmatrix} 0_{\tilde{R}} \\ s(\lambda) \\ \beta \end{pmatrix} = B \begin{pmatrix} \lambda \\ \beta \end{pmatrix} \tag{56}$$

for

$$B = \begin{pmatrix} 0 & 0 \\ s & 0 \\ 0 & \mathrm{id}_Q \end{pmatrix} : \Lambda \times Q \to \tilde{R} \times \tilde{\Lambda} \times Q \tag{57}$$

and $X := K \times \tilde{R} \times \tilde{\Lambda}$, $Y := \tilde{R} \times \tilde{\Lambda} \times Q$, we get:

THEOREM 17. *Let $\mathcal{C}$ be a chopped and sliced cone fulfilling property* (53). *Then there are free Abelian groups X and Y and morphisms $E : X \to Y$ and $B : \Lambda \times Q \to Y$ such that*

$$|\mathcal{S}^{\lambda}_{\beta}| = \Phi_E \left(B \begin{pmatrix} \lambda \\ \beta \end{pmatrix} \right). \tag{58}$$

Applying this theorem to the chopped and sliced cones defined in chapter 4 we get:

COROLLARY 18. *Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra. Then there are free Abelian groups X and Y and morphisms $E : X \to Y$ and $B : \Lambda \times Q \to Y$ such that*

$$K^{\lambda}_{\beta} = \Phi_E \left(B \begin{pmatrix} \lambda \\ \beta \end{pmatrix} \right) \tag{59}$$

for all $\lambda \in \Lambda$ and $\beta \in Q$.

By the construction of the free Abelian groups X and Y in the proof of theorem 17 one can explicitly determine their ranks. To this aim let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra of rank r and $k := l(w_0) = |\Phi_+|$. For example using the explicit calculations in [**11**, § 12.2, tab. 1] one sees that

$$k = O(r^2). \tag{60}$$

PROPOSITION 19. *The free Abelian groups X and Y in corollary 18 can be chosen of rank $O(r^2)$.*

PROOF. We have $X = K \times \tilde{R} \times \tilde{\Lambda}$ and $Y = \tilde{R} \times \tilde{\Lambda} \times Q$. So we have to estimate the ranks of the individual groups. By definition $\mathrm{rk}(K) = k$. The equations defining $\mathcal{C}^\lambda$ (38) are also parameterised by a set of cardinal $|\Phi_+| = k$, so $\mathrm{rk}(\tilde{\Lambda}) = k$. Obviously $\mathrm{rk}(Q) = r$. Using the systems of inequalities explicitly given in examples 26 to 29 one sees that $\mathrm{rk}(\tilde{R}) = O(k)$. This implies the proposition. □

A quasi-polynomial $(f_{\bar{x}})_{\bar{x}\in\Gamma/\Gamma'}$ is said to be *of degree* $d \in \mathbf{Z}_{\geq 0}$, if all polynomials $f_{\bar{x}}$ are of degree d. The null polynomial is not of degree d for any $d \in \mathbf{Z}_{\geq 0}$.

PROPOSITION 20. *The map $K : \Lambda \times Q \longrightarrow \mathbf{Z}_{\geq 0}$ is piecewise quasi-polynomial. The individual quasi-polynomials $(f_{\bar{x}})$ are of degree $|\Phi_+| - r$, and the homogeneous part of highest degree coincides for all $f_{\bar{x}}$.*

PROOF. This is a direct application of B. Sturmfels' structure theorem [**23**, th. 1]. The assumptions we have to check is that the morphism $E : X \longrightarrow Y$ from corollary 18 is surjective, and that $\mathrm{rk}(X) - \mathrm{rk}(Y) = |\Phi_+| - r$.

For the second statement, consider again the construction of X and Y in the proof of theorem 17. As $X = K \times \tilde{R} \times \tilde{\Lambda}$ and $Y = \tilde{R} \times \tilde{\Lambda} \times Q$, the difference between the ranks is simply $\mathrm{rk}(K) - \mathrm{rk}(Q) = |\Phi_+| - r$ as claimed.

The first statement can be seen as follows: Consider the decomposition $\underline{w}_0 = (i(k))_{k\in J} \in I^J$ of the longest element w_0 of the Weyl group W underlying the definition of K. Every vertex $i \in I$ of the Dynkin diagram appears in $\underline{w}_0$. The map $q : K = \mathbf{Z}^J \longrightarrow Q$, given on elements e_k $(k \in J)$ of the standard basis of K by $e_k \mapsto \alpha_{i(k)}$, is hence surjective. By the structure of the matrix E given in (55), $\tilde{\Lambda} \times Q$ is contained in the image of E. Together with the surjectivity of q this implies the proposition. □

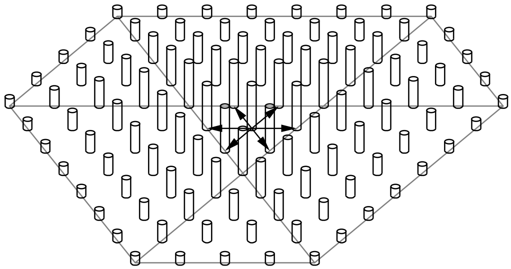


FIGURE 4. Weight multiplicity diagram for type A_2 and highest weight $7\omega_1 + 4\omega_2$.

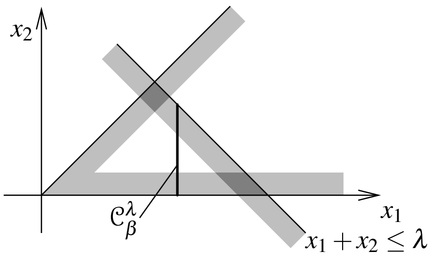


FIGURE 5. The chopped and sliced cone from example 11

CHAPTER 4

Patterns

This chapter summarises the results leading to the presentation of the crystal associated with a Lie algebra as a chopped and sliced cone. We explicitly write down the spaces and maps defining these chopped and sliced cones for the classical simple Lie algebras.

1. Introduction

As an introduction to patterns, it is probably best start by considering the classical Gelfand-Tsetlin patterns for $\mathrm{GL}_n(\mathbf{C})$ [**9**], as done in [**2**, § 1.2]. Note that the Schur-Weyl dual case concerning representations of symmetric groups is also interesting, for an exposition see e.g. [**17**, ch. 2].

Let $\lambda = (a_{1,1}, \ldots, a_{1,n}) \in \mathbf{Z}^n$ be monotonically decreasing. Let $V(\lambda)$ denote the irreducible $\mathrm{GL}_n(\mathbf{C})$-module of highest weight λ. Consider the embedding of $\mathrm{GL}_{n-1}(\mathbf{C})$ in $\mathrm{GL}_n(\mathbf{C})$ in the upper left corner, i.e. by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $V(\lambda)$ becomes a $\mathrm{GL}_{n-1}(\mathbf{C})$-module and Weyl's branching rule describes its decomposition into simple modules:

THEOREM 21. *As a* $\mathrm{GL}_{n-1}(\mathbf{C})$*-module,* $V(\lambda)$ *decomposes into a direct sum*

$$V(\lambda) = \sum_{\lambda'} V(\lambda'), \tag{61}$$

where the sum is over all $\lambda' = (a_{2,1}, \ldots, a_{2,n-1}) \in \mathbf{Z}^{n-1}$ *such that the zipped sequence* $(a_{1,1}, a_{2,1}, a_{1,2}, a_{2,2}, \ldots, a_{1,n-1}, a_{2,n-2}, a_{1,n})$ *is monotonically decreasing.*

Iteratively we decompose the individual components as $\mathrm{GL}_{n-2}(\mathbf{C})$-modules etc. and get:

COROLLARY 22. *As a* $\mathrm{GL}_1(\mathbf{C})$*-module,* $V(\lambda)$ *decomposes into a direct sum*

$$V(\lambda) = \sum_{a \in \mathcal{S}^\lambda} V(a), \tag{62}$$

where the sum is over the set $\mathcal{S}^\lambda$ *of all families* $(a_{i,j})$, $i = 2, \ldots, n$, $j = 1, \ldots, n - i + 1$, *such that for any* $i = 1, \ldots, n-1$, *the families* $(a_{i,j})_j$ *and* $(a_{i+1,j})_j$ *fulfil the above condition.*

Several observations are to be made: All simple $\mathrm{GL}_1(\mathbf{C})$-module are one-dimensional, so $\dim V(\lambda) = |\mathcal{S}^\lambda|$. For each $a \in \mathcal{S}^\lambda$, the nonzero elements of $V(a)$ are weight vectors of weight

$$\Big(a_{n,1}, \sum_j a_{n-1,j} - \sum_j a_{n,j}, \ldots, \sum_j a_{1,j} - \sum_j a_{2,j}\Big). \tag{63}$$

So defining $\tilde{\mathcal{S}}^\lambda_\mu \subset \mathcal{S}^\lambda$ as the preimage of μ under this affine function, we get that $\dim V(\lambda)_\mu = |\tilde{\mathcal{S}}^\lambda_\mu|$.

As the notation suggests, the Gelfand-Tsetlin patterns for $\mathrm{GL}_n(\mathbf{C})$ for a fixed n form a chopped and sliced cone, more explicitly described as follows: Define $K := \mathbf{Z}^J$ where $J = \{(i,j) : i = 2, \ldots, n,\ j = 1, \ldots, n-i+1\}$. So $K_\mathbf{R} = \mathbf{R}^J$ is an $\frac{n(n-1)}{2}$-dimensional vector space. The cone $\mathcal{C}$ is given by the equations $a_{i,j} \geq a_{i+1,j} \geq a_{i,j+1}$ for $i = 2, \ldots n-1$, $j = 1, \ldots, n-i$. This can be encoded in a map $r : K \to \mathbf{Z}^{(n-1)(n-2)}$, namely

$$r = \big(a_{i,j} - a_{i+1,j}, a_{i+1,j} - a_{i,j+1}\big)_{\substack{i=2,\ldots,n-1 \\ j=1,\ldots,n-i}}. \tag{64}$$

Any of the sets $\mathcal{C}^\lambda$ is given inside $\mathcal{C}$ by the additional zipping condition $a_{1,j} \geq a_{2,j} \geq a_{1,j+1}$ for $j = 1, \ldots, n-1$. This can be encoded in maps $p : K_\mathbf{R} \to \mathbf{R}^{2(n-1)}$ and $s : \mathbf{R}^n \to \mathbf{R}^{2(n-1)}$ as follows: Let

$$p = (a_{2,1}, -a_{2,1}, \ldots, a_{2,n-1}, -a_{2,n-1}) \tag{65}$$

and

$$s = (a_{1,1}, -a_{1,2}, a_{1,2}, \ldots, -a_{1,n-1}, a_{1,n-1}, a_{1,2}), \tag{66}$$

where $a_{1,j}$ are the canonical coordinates on $\mathbf{R}^n$. For the slices, note that (63) does not define a linear function on $K_\mathbf{R}$ because of the dependence on $\lambda =$

$(a_{1,1},\dots,a_{1,n})$ in the last component. So we just define

$$q = \Big(a_{n,1}, \sum_j a_{n-1,j} - \sum_j a_{n,j}, \dots, \sum_j a_{2,j} - \sum_j a_{3,j}, -\sum_j a_{2,j}\Big) \tag{67}$$

and let $(\mathcal{C}^\lambda_\beta)$ be the associated chopped and sliced cone. Then $\tilde{\mathcal{S}}^\lambda_\mu = \mathcal{S}^\lambda_\beta$ for $\beta = \mu - \big(\sum_{j=1}^n a_{1,j}\big)\varepsilon_n$, where ε_n is the n-th element of the canonical basis of $\mathbf{R}^n$. Note that this is the same simple trick we use in the general case (with the transformation $\beta = \lambda - \mu$) to define the weight multiplicity function K^λ_β in a way adapted to our language of chopped and sliced cones.

2. Crystals

In order to introduce P. Littelmann's patterns [**21**] for any semisimple Lie algebra, we need M. Kashiwara's notion of a crystal. Crystals themselves have appeared in [**14**]; the following definition can be found in [**15**, § 7.2] and [**16**, § 4.2].

Let $\mathfrak{g}$ be a semisimple finite dimensional complex Lie algebra. We fix a maximal toral Lie subalgebra $\mathfrak{h}$, a dominant Weyl chamber $C \subset \mathfrak{h}^*_{\mathbf{R}}$ and use the notations from chapter 2. A *crystal* for $\mathfrak{g}$ is a set B endowed with a map $\mathrm{wt} : B \to \Lambda$, families of functions $\varepsilon_i, \phi_i : B \to \mathbf{Z} \cup \{-\infty\}$ for $i \in I$ and families of partial maps $\tilde{e}_i, \tilde{f}_i : B \to B$ for $i \in I$, subject to the following axioms for all $b, b' \in B$ and $i \in I$:

$$\phi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle, \tag{68a}$$

$$\left.\begin{aligned} \varepsilon_i(\tilde{e}_i b) &= \varepsilon_i(b) - 1 \\ \phi_i(\tilde{e}_i b) &= \phi_i(b) + 1 \\ \mathrm{wt}(\tilde{e}_i b) &= \mathrm{wt}(b) + \alpha_i \end{aligned}\right\} \text{ if } \tilde{e}_i b \text{ is defined,} \tag{68b}$$

$$\left.\begin{aligned} \varepsilon_i(\tilde{f}_i b) &= \varepsilon_i(b) + 1 \\ \phi_i(\tilde{f}_i b) &= \phi_i(b) - 1 \\ \mathrm{wt}(\tilde{f}_i b) &= \mathrm{wt}(b) - \alpha_i \end{aligned}\right\} \text{ if } \tilde{f}_i b \text{ is defined,} \tag{68c}$$

$$b' = \tilde{f}_i b \iff b = \tilde{e}_i b', \tag{68d}$$

$$\phi_i(b) = -\infty \implies \tilde{e}_i b, \tilde{f}_i b \text{ undefined.} \tag{68e}$$

The substructure $(B, \mathrm{wt}, (\tilde{e}_i)_{i\in I}, (\tilde{f}_i)_{i\in I})$ can be visualised as a directed graph (B, D) with labelled vertices and edges, where $(b, b') \in B \times B$ is an edge of label i if $b' = \tilde{f}_i b$, and where the labelling of the vertices is given by the map wt. This graph is called the *crystal graph* associated with the crystal under consideration.[8]

In [**14**], M. Kashiwara associates a crystal to every simple representation of a semisimple Lie algebra. Instead of reporting the definition, it is more convenient to use an equivalent construction described in the following paragraph.

3. The path model of a representation

We report the definition and a result about the path model as defined in P. Littelmann's articles [**19**] and [**20**]. The necessary results can already be found in [**19**]; because of the slightly nicer definition of the root operators we refer nevertheless to [**20**].

Let Π be the set of all piecewise affine continuous maps $\pi : [0, l(\pi)] \to \mathfrak{h}^*_{\mathbf{R}}$, parameterised with arc length,[9] such that $\pi(0) = 0$. Elements of Π are called *paths*. The *concatenation product* $* : \Pi \times \Pi \to \Pi$ is given by

$$(\pi_1 * \pi_2)(t) := \begin{cases} \pi_1(t) & t \le l(\pi_1), \\ \pi_1(l(\pi_1)) + \pi_2(t - l(\pi_1)) & t \ge l(\pi_1) \end{cases} \tag{69}$$

for $\pi_1, \pi_2 \in \Pi$. For a path $\pi \in \Pi$ and $0 \le a < b \le l(\pi)$ define the restriction $\pi|_{[a,b]}$ by

$$\pi|_{[a,b]}(t) := \pi(a+t) \tag{70}$$

for $0 \le t \le b - a$. The Weyl group W acts value-wise on Π, i.e. by $(w\pi)(t) := w\pi(t)$.

For any simple root α_i ($i \in I$) we define a partial map $\tilde{f}_i : \Pi \to \Pi$ as follows: For a path $\pi \in \Pi$ let $h_{\pi,i} : [0, l(\pi)] \to \mathbf{R}$ be the function given by

$$h_{\pi,i}(t) := \langle h_i, \pi(t) \rangle. \tag{71}$$

[8] Some people use the term "crystal graph" to refer to the structure obtained by further neglecting the map wt.

[9] The original normalisation is to use parameterisation proportional to arc length, such that the path is defined on the interval $[0, 1]$.

Let $m_{\pi,i} := \min\,\{h_{\pi,i}(t) : 0 \le t \le l(\pi)\}$. If $h_{\pi,i}(l(\pi)) < m_{\pi,i}+1$, then $\tilde{f}_i$ is undefined on π. Otherwise consider the function $d : [0,1] \to [0,l(\pi)]$ with $d(x) = \max\{t : h_{\pi,i}(t) = m_{\pi,i}+x\}$ and let $T := d([0,1])$. Then T decomposes into a union of intervals $I_1,\dots,I_r$. The complement $[1,l(\pi)] \setminus T$ decomposes correspondingly into a union of intervals $J_0,\dots,J_r$. With these definitions,

$$\tilde{f}_i\pi = \pi|_{J_0} * \prod_{k=1}^{r} \big(s_i(\pi|_{I_k}) * \pi|_{J_k}\big). \tag{72}$$

Let $\lambda \in \Lambda_+$ be a dominant weight. We construct in the following a directed graph $B(\lambda) = (B,D)$ with a vertex labelling $\mathrm{wt} : B \to \Lambda$ and an edge labelling $D \to I$. Let $\pi_\lambda : [0,\|\lambda\|] \to \mathfrak{h}^*_{\mathbf{R}}$ with $\pi_\lambda(t) := t\lambda/\|\lambda\|$ be the linear path connecting 0 with λ. The set of vertices B is the closure of $\{\pi_\lambda\} \subset \Pi$ under the operators $\tilde{f}_i$ for $i \in I$. A pair $(\pi,\pi') \in B \times B$ is an edge labelled $i \in I$, if $\tilde{f}_i\pi = \pi'$. The labelling of the vertices is given by $\mathrm{wt}(\pi) := \pi(l(\pi)) \in \Lambda$. The labelled directed graph $B(\lambda)$ is called the crystal graph associated with λ.

This construction gains its importance by the following theorem [**19**] [**20**]:

THEOREM 23. *Let $\lambda \in \Lambda_+$ and $\mu \in \Lambda$. Then*

$$K_{\lambda\mu} = |\{\pi \in B(\lambda) : \mathrm{wt}(\pi) = \mu\}|. \tag{73}$$

For every $i \in I$ and every vertex $\pi' \in B$ there is at most one edge $(\pi,\pi') \in D$ of label i. Let $\tilde{e}_i : B \to B$ be the partial map given by $\pi' \mapsto \pi$.

4. Littelmann's patterns

We gather some important notions and properties about P. Littelmann's patterns [**21**].

Let $\mathfrak{g}$ be a semisimple finite dimensional complex Lie algebra. Fix a maximal toral subalgebra $\mathfrak{h}$ and a dominant Weyl chamber $C \subset \mathfrak{h}^*_{\mathbf{R}}$.

Let $w_0 \in W$ be the longest element of the Weyl group. Fix a reduced decomposition $\underline{w}_0$ of w_0. In order to simplify notation in the examples, the decomposition is not necessarily supposed to be parameterised by $\{1,\dots,l(w_0)\}$, but by an arbitrary totally ordered set $(J,\le)$ of cardinal $l(w_0)$: $\underline{w}_0 = (i(k))_{k\in J} \in I^J$.[10]

[10] In the examples, J is a subset of $\mathbf{Z}_{>0} \times \mathbf{Z}_{>0}$, interpreted as the set of boxes of a generalised Gelfand-Tsetlin diagram.

Let $\lambda \in \Lambda_+$ be a dominant weight. Define a map

$$\sigma_\lambda : \begin{cases} B(\lambda) \to \mathbf{Z}^J_{\geq 0} \\ \qquad \pi \mapsto (a_k)_{k \in J} \end{cases} \tag{74}$$

as follows: For $\pi \in B(\lambda)$ let $\pi_{\min(J)} := \pi$. For $k \in J$ define recursively

$$a_k := \max\{a : \tilde{e}^a_{\alpha_{i(k)}} \pi_k \text{ defined}\}, \tag{75a}$$

$$\pi_{\mathrm{succ}(k)} := \tilde{e}^{a_k}_{\alpha_{i(k)}} \pi_k. \tag{75b}$$

For $\pi \in B(\lambda)$, we call $\sigma_\lambda(\pi) \in \mathbf{Z}^J_{\geq 0}$ the *adapted string* of π, adapted meaning adapted to the reduced decomposition of w_0. The map σ_λ ist injective. Let $\mathcal{S}^\lambda \subset \mathbf{Z}^J_{\geq 0}$ be its image and $\mathcal{S} := \bigcup_{\lambda \in \Lambda_+} \mathcal{S}^\lambda$.

For $\lambda \in \Lambda_+$ denote by $B(\lambda) \otimes T_{-\lambda}$ the labelled directed graph obtained from $B(\lambda)$ by subtracting λ from all labels of the vertices. On Λ_+ consider the partial order given by $\lambda \leq \mu$ if $\mu - \lambda \in \Lambda_+$. The family $(B(\lambda) \otimes T_{-\lambda})_{\lambda \in \Lambda_+}$ with the canonical maps is an inductive system with respect to this partial order. Let

$$B(\infty) := \varinjlim_{\lambda \in \Lambda_+} B(\lambda) \otimes T_{-\lambda} \tag{76}$$

its inductive limit.

The cone $\mathcal{C}$ spanned by $\mathcal{S}$ in $\mathbf{R}^J$ is called the *string cone* with respect to the reduced decomposition $(i(k))_{k \in J}$. Let $(a_k)_{k \in J}$ be the canonical coordinates on $\mathbf{R}^J$. For $\lambda \in \Lambda_+$ let $\mathcal{C}^\lambda \subset \mathcal{C}$ denote the set of common solutions of

$$a_k \leq \left\langle h_{\alpha_{i(k)}}, \lambda - \sum_{k' > k} a_{k'} \alpha_{i(k')} \right\rangle \tag{77}$$

for $k \in J$. Hence we have:

PROPOSITION 24. *Let $\lambda \in \Lambda_+$.*

(1) *$\mathcal{C}$ is a rational cone and fulfils $\mathcal{S} = \mathcal{C} \cap \mathbf{Z}^J$.*
(2) *$\mathcal{C}^\lambda$ is a rational polytope and fulfils $\mathcal{S}^\lambda = \mathcal{C}^\lambda \cap \mathbf{Z}^J$.*
(3) *For $a = (a_k)_{k \in J} \in \mathcal{S}^\lambda$ we have*

$$\mathrm{wt}(\sigma_\lambda^{-1}(a)) = \lambda - \sum_{k \in J} a_k \alpha_{i(k)}. \tag{78}$$

PROOF. Properties (i) and (ii) are taken from [**21**, pr. 1.5]. The remaining property (iii) can be seen as follows: First, consider $a = 0$. Then $\sigma_\lambda^{-1}(a) = \pi_\lambda$ and $\mathrm{wt}(\pi_\lambda) = \pi_\lambda(l(\pi_\lambda)) = \lambda$ as desired. The general case follows directly from $\mathrm{wt}(\tilde{f}_i\pi) = \mathrm{wt}(\pi) - \alpha_i$ for any $\pi \in B(\lambda)$ and $i \in I$ such that $\tilde{f}_i\pi$ is defined [**20**, le. 2.1, a)]. $\square$

For $\lambda \in \Lambda$ and $\beta \in Q$ define $\mathcal{C}_\beta^\lambda := \{a \in \mathcal{C}^\lambda : \sum_{k\in J} a_k \alpha_{i(k)} = \beta\}$ and $\mathcal{S}_\beta^\lambda := \mathcal{C}_\beta^\lambda \cap \mathbf{Z}^J$. Let $K_\beta^\lambda := K_{\lambda,\lambda-\beta}$. By proposition 24 it follows that $K_\beta^\lambda = |\mathcal{S}_\beta^\lambda|$ for all $\lambda \in \Lambda$ and $\beta \in Q$. The following fact drops so smoothly out of the exposition that it seems almost not worth mentioning – let us nevertheless record it for the sake of reference:

OBSERVATION 25. *For every semisimple complex Lie algebra, the family* $(\mathcal{C}_\beta^\lambda)$ *as defined above is a chopped and sliced cone.*

5. Examples

P. Littelmann has explicitly determined the polytopes $\mathcal{C}^\lambda$ for all simple complex Lie algebras in [**21**]. For the classical families, they are given by the inequalities stated in the following examples.

EXAMPLE 26 ($\mathfrak{g} = \mathfrak{sl}_{r+1}(\mathbf{C}) = \mathfrak{g}(A_r)$, [**21**, § 5]). Let $\mathfrak{g} = \mathfrak{sl}_{r+1}(\mathbf{C})$ be the Lie algebra with Dynkin diagram

.

Here, $I = \{1, \dots, r\}$. we have $l(w_0) = \frac{1}{2}r(r+1)$. As a set of boxes, we use $J = \{(i, j) \in \mathbf{Z}_{>0}^2 : i + j \leq r+1\}$ with the order inverse to the lexicographic order. Then $(j)_{(i,j)\in J}$ is a reduced decomposition of w_0. For example, for $r = 3$ we get

$$\begin{array}{rrrr} J = (& & & (3,1), \\ & & (2,2), & (2,1), \\ & (1,3), & (1,2), & (1,1)) \end{array} \tag{79}$$

and the reduced decomposition $(1,2,1,3,2,1)$ of w_0. The string cone $\mathcal{C}$ is given by the inequalities

$$0 \leq a_{i,1} \leq \cdots \leq a_{i,n-i+1} \tag{80}$$

for $i = 1, \dots, r$ [**21**, th. 5.1]. By substituting the special values in (77) we obtain that the polytopes $\mathcal{C}^\lambda$ are given by

$$\begin{aligned} a_{(i,j)} &\leq \Big\langle h_j, \lambda - \sum_{(i',j')>(i,j)} a_{(i',j')} \alpha_{j'} \Big\rangle \\ &= \langle h_j, \lambda \rangle - \sum_{(i',j')>(i,j)} a_{(i',j')} \langle h_j, \alpha_{j'} \rangle \\ &= \langle h_j, \lambda \rangle + \sum_{i' \leq i} a_{(i',j-1)} + \sum_{i'<i} \big(-2a_{(i',j)} + a_{(i',j+1)}\big), \end{aligned} \tag{81}$$

where we define $a_{i,j} = 0$ for $(i,j) \notin J$ [**21**, co. 4].

EXAMPLE 27 ($\mathfrak{g} = \mathfrak{so}_{2r+1}(\mathbf{C}) = \mathfrak{g}(B_r)$, [**21**, § 6]). Let $\mathfrak{g} = \mathfrak{so}_{2r+1}(\mathbf{C})$ be the Lie algebra with Dynkin diagram

Here $I = \{1, \dots, r\}$. We have $l(w_0) = r^2$. As a set of boxes, we use $J = \{(i,j) \in \mathbf{Z}^2_{>0} : i \leq j,\, i+j \leq 2r\}$ with the order given by $(i,j) \leq (i',j')$ if $i > i'$ or $i = i'$ and $j \leq j'$. Then $(|j-r|+1)_{(i,j)\in J}$ is a reduced decomposition of w_0. For example, for $r = 3$ we get

$$\begin{array}{rccccc} J = (& & & (3,3), & & \\ & & (2,2), & (2,3), & (2,4), & \\ & (1,1), & (1,2), & (1,3), & (1,4), & (1,5)) \end{array} \tag{82}$$

and the reduced decomposition $(1,2,1,2,3,2,1,2,3)$ of w_0. The string cone $\mathcal{C}$ is given by the inequalities

$$\begin{aligned} 2a_{(i,i)} \geq \cdots \geq 2a_{(i,r-1)} \geq a_{(i,r)} \geq & \\ & 2a_{(i,r+1)} \geq \cdots \geq 2a_{(i,2r-i)} \geq 0 \end{aligned} \tag{83}$$

for $i = 1, \dots, r$ [**21**, th. 6.1]. By substituting the special values in (77) we obtain [**21**, co. 6] that the polytopes $\mathcal{C}^\lambda$ are given by

$$\begin{aligned} a_{(i,j)} \leq \langle h_{r-j+1}, \lambda \rangle + \sum_{i'<i} \big(a_{(i',j-1)} - 2a_{(i',j)}\big) + & \\ \sum_{i' \leq i} \big(a_{(i',j+1)} + a_{(i',2r-j-1)} - 2a_{(i',2r-j)} + & \\ a_{(i',2r-j+1)}\big) \text{ for } j \leq r-2, & \end{aligned} \tag{84a}$$

$$a_{(i,r-1)} \leq \langle h_2, \lambda \rangle + \sum_{i'<i} \big(a_{(i',r-2)} - 2a_{(i',r-1)} \big) + \tag{84b}$$
$$\sum_{i' \leq i} \big(a_{(i',r)} - 2a_{(i',r+1)} + a_{(i',r+2)} \big),$$
$$a_{(i,r)} \leq \langle h_1, \lambda \rangle + \sum_{i'<i} \big(2a_{(i',r-1)} - 2a_{(i',r)} \big) + \sum_{i' \leq i} 2a_{(i',r+1)}, \tag{84c}$$
$$a_{(i,r+1)} \leq \langle h_2, \lambda \rangle + \sum_{i'<i} \big(a_{(i',r-2)} - 2a_{(i',r-1)} + \tag{84d}$$
$$a_{(i',r)} - 2a_{(i',r+1)} \big) + \sum_{i' \leq i} a_{(i',r+2)},$$
$$a_{(i,j)} \leq \langle h_{j-r+1}, \lambda \rangle + \sum_{i'<i} \big(a_{(i',2r-j-1)} - 2a_{(i',2r-j)} + \tag{84e}$$
$$a_{(i',2r-j+1)} + a_{(i',j-1)} - 2a_{(i',j)} \big) +$$
$$\sum_{i' \leq i} a_{(i',j+1)} \text{ for } j \geq r+2.$$

EXAMPLE 28 ($\mathfrak{g} = \mathfrak{sp}_{2r}(\mathbf{C}) = \mathfrak{g}(C_r)$, [**21**, § 6]). Let $\mathfrak{g} = \mathfrak{sp}_{2r}(\mathbf{C})$ be the Lie algebra with Dynkin diagram

Here $I = \{1, \dots, r\}$. We have $l(w_0) = r^2$. As the set of boxes we use $J = \{(i,j) \in \mathbf{Z}^2_{>0} : i \leq j, i+j \leq 2r\}$ with the order given by $(i,j) \leq (i',j')$ if $i > i'$ or $i = i'$ and $j \leq j'$. Then $(|j-r|+1)_{(i,j)\in J}$ is a reduced decomposition of w_0. For example, for $r = 3$ we obtain

$$\begin{aligned} J = (\qquad\qquad & (3,3), \\ & (2,2),\ (2,3),\ (2,4), \\ & (1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5)) \end{aligned} \tag{85}$$

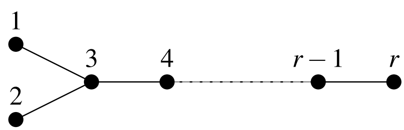
and the reduced decomposition $(1,2,1,2,3,2,1,2,3)$ of w_0. The string cone $\mathcal{C}$ is given by the inequalities

$$a_{(i,i)} \geq \cdots \geq a_{(i,2r-i)} \geq 0 \tag{86}$$

for $i = 1, \ldots, r$ [**21**, th. 6.1]. By substituting the special values in (77) we get [**21**, co. 6], that the polytopes $\mathcal{C}^\lambda$ are given by

$$\begin{aligned}
a_{(i,j)} \leq & \langle h_{r-j+1}, \lambda \rangle + \sum_{i'<i} \big(a_{(i',j-1)} - 2a_{(i',j)} \big) + \\
& \sum_{i' \leq i} \big(a_{(i',j+1)} + a_{(i',2r-j-1)} - 2a_{(i',2r-j)} + \\
& a_{(i',2r-j+1)} \big) \text{ für } j \leq r-2, && \text{(87a)}\\
a_{(i,r-1)} \leq & \langle h_2, \lambda \rangle + \sum_{i'<i} \big(a_{(i',r-2)} - 2a_{(i',r-1)} \big) + \\
& \sum_{i' \leq i} \big(2a_{(i',r)} - 2a_{(i',r+1)} + a_{(i',r+2)} \big), && \text{(87b)}\\
a_{(i,r)} \leq & \langle h_1, \lambda \rangle + \sum_{i'<i} \big(a_{(i',r-1)} - 2a_{(i',r)} \big) + \sum_{i' \leq i} a_{(i',r+1)}, && \text{(87c)}\\
a_{(i,r+1)} \leq & \langle h_2, \lambda \rangle + \sum_{i'<i} \big(a_{(i',r-2)} - 2a_{(i',r-1)} + \\
& 2a_{(i',r)} - 2a_{(i',r+1)} \big) + \sum_{i' \leq i} a_{(i',r+2)}, && \text{(87d)}\\
a_{(i,j)} \leq & \langle h_{j-r+1}, \lambda \rangle + \sum_{i'<i} \big(a_{(i',2r-j-1)} - 2a_{(i',2r-j)} + \\
& a_{(i',2r-j+1)} + a_{(i',j-1)} - 2a_{(i',j)} \big) + \sum_{i' \leq i} a_{(i',j+1)} \\
& \text{für } j \geq r+2. && \text{(87e)}
\end{aligned}$$

EXAMPLE 29 ($\mathfrak{g} = \mathfrak{so}_{2r}(\mathbf{C}) = \mathfrak{g}(D_r)$, [**21**, § 7]). Let $\mathfrak{g} = \mathfrak{so}_{2r}(\mathbf{C})$ be the Lie algebra with Dynkin diagram



Here $I = \{1, \ldots, r\}$. We have $l(w_0) = r^2 - r$. As the set of boxed we use $J = \{(i,j) \in \mathbf{Z}^2_{>0} : i \leq j, i+j \leq 2r-1\}$ with the order given by $(i,j) \leq (i',j')$

if $i > i'$ or $i = i'$ and $j \leq j'$. Then the map

$$(i,j) \mapsto \begin{cases} r-j+1 & : j < r-1 \\ 1 & : j = r-1 \\ 2 & : j = r \\ j-r+2 & : j > r \end{cases} \tag{88}$$

is a reduced decomposition of w_0. For example, for $r = 4$ we get

$$\begin{aligned} J = (\qquad\qquad\qquad\qquad & (3,3),\ (3,4), \\ (2,2),\ & (2,3),\ (2,4),\ (2,5), \\ (1,1),\ (1,2),\ & (1,3),\ (1,4),\ (1,5),\ (1,6)) \end{aligned} \tag{89}$$

and the reduced decomposition $(1,2,3,1,2,3,4,3,1,2,3,4)$ of w_0. The string cone $\mathcal{C}$ is given by the inequalities

$$\begin{array}{ccccccc} a_{(i,i)} \geq \cdots \geq & a_{(i,r-2)} & \geq & a_{(i,r-1)} & & & \\ & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & & \\ & a_{(i,r)} & \geq & a_{(i,r+1)} & \geq \cdots \geq & a_{(i,2r-i-1)} & \end{array} \tag{90}$$

for $i = 1, \dots, r$ [**21**, th. 7.1]. By substituting the special values in (77), we get [**21**, co. 8], that the polytopes $\mathcal{C}^\lambda$ are given by

$$\begin{aligned} a_{(i,j)} \leq & \langle h_{r-j+1}, \lambda \rangle + \sum_{i'<i} \big(a_{(i',j-1)} - 2a_{(i',j)} \big) + \\ & \sum_{i' \leq i} \big(a_{(i',j+1)} + a_{(i',2r-j-2)} - 2a_{(i',2r-j-1)} + \\ & a_{(i',2r-j)} \big) \text{ für } j \leq r-3, \end{aligned} \tag{91a}$$

$$\begin{aligned} a_{(i,r-2)} \leq & \langle h_3, \lambda \rangle + \sum_{i'<i} \big(a_{(i',r-3)} - 2a_{(i',r-2)} \big) + \\ & \sum_{i' \leq i} \big(a_{(i',r-1)} + a_{(i',r)} - 2a_{(i',r+1)} + a_{(i',r+2)} \big), \end{aligned} \tag{91b}$$

$$a_{(i,r-1)} \leq \langle h_1, \lambda \rangle + \sum_{i'<i} (-2) a_{(i',r-1)} + \sum_{i' \leq i} a_{(i',r)}, \tag{91c}$$

$$a_{(i,r)} \leq \langle h_2, \lambda \rangle + \sum_{i'<i} \big(a_{(i',r-2)} - 2a_{(i',r)} \big) + \sum_{i' \leq i} a_{(i',r+1)}, \tag{91d}$$

$$\begin{aligned}
a_{(i,r+1)} \leq \langle h_3, \lambda \rangle + \sum_{i'<i} \big(& a_{(i',r-3)} - 2a_{(i',r-2)} + \\
& a_{(i',r-1)} + a_{(i',r)} - 2a_{(i',r+1)} \big) + \sum_{i' \leq i} a_{(i',r+2)},
\end{aligned} \tag{91e}$$

$$\begin{aligned}
a_{(i,j)} \leq \langle h_{j-r+2}, \lambda \rangle + \sum_{i'<i} \big(& a_{(i',2r-j-2)} - 2a_{(i',2r-j-1)} + \\
& a_{(i',2r-j)} + a_{(i',j-1)} - 2a_{(i',j)} \big) + \sum_{i' \leq i} a_{(i',j+1)} \\
& \text{für } j \geq r+2.
\end{aligned} \tag{91f}$$

CHAPTER 5

Calculating vector partition functions

1. Introduction: Calculating partition functions

Before reporting the result for general vector partition functions, we explicitly derive the corresponding formula for partition functions. This is elementary, but already allows some insight into the general formula.

Let $a_1, \dots, a_N \in \mathbf{Z}_{\geq 0}$. For $h \in \mathbf{Z}$ let

$$\Phi_{a_1,\dots,a_N}(h) := \left| \left\{ (c_1, \dots, c_r) \in \mathbf{Z}^N_{\geq 0} : \sum_{i=1}^{N} c_i a_i = h \right\} \right|. \tag{92}$$

This defines a function $\Phi = \Phi_{a_1,\dots,a_N} : \mathbf{Z} \to \mathbf{Z}_{\geq 0}$, the *partition function* associated with the tuple $(a_1, \dots, a_N)$. Note that the partition function $\Phi_{a_1,\dots,a_N}$ associated with the tuple $(a_1, \dots, a_N)$ coincides with the vector partition function Φ_A associated with the $(1 \times N)$-matrix $A = (a_1 \ \cdots \ a_N)$, so a partition function is a special case of a vector partition function.

EXAMPLE 30. It is clear that $\Phi(h) = 0$ for $h < 0$ and all partition functions. For $h \geq 0$, some simple examples are

$$\Phi_1(h) = 1, \tag{93}$$

$$\Phi_2(h) = \begin{cases} 1 & \text{if } h \equiv 0 \quad (2), \\ 0 & \text{if } h \equiv 1 \quad (2), \end{cases} \tag{94}$$

$$\Phi_{1,1}(h) = h + 1. \tag{95}$$

For these small examples, it is easy to guess the given formulas for the partition functions. Note that for more difficult $(a_1, \dots, a_N)$, it is still obvious how to calculate individual values $\Phi(h)$ for given $h \in \mathbf{Z}$: Just determine a suitable set of candidates $(c_1, \dots, c_N)$ as in (92) and check for each of them whether the defining equation $\sum_{i=1}^N c_i a_i = h$ holds or not. But, e.g. as a special case of Sturmfels' theorem 9, we know that a nice formula always exists, namely each partition function $\Phi(h)$ is quasi-polynomial in h. Our goal is hence to find an algorithm which computes this quasi-polynomial for given $(a_1, \ldots, a_N)$.

As a first idea, consider the generating function $f(x) := \sum_{h=0}^{\infty} \Phi(h) x^h$ of Φ. This formal power series converges absolutely for $|x| < 1$, and there

$$f(x) = \frac{1}{\prod_{i=1}^N (1 - x^{a_i})}. \tag{96}$$

Hence, f extends to a rational function on $\mathbf{C}$ which we denote by the same symbol f. By definition, $\Phi(h)$ is the x^h-coefficient of f, in other words $\Phi(h) = \mathrm{res}_{x=0}\, x^{-h-1} f(x)$. If we want to calculate h by this formula, we have to develop $x^{-h-1} f(x)$ to the order $-\mathrm{ord}_{x=0}(x^{-h-1} f(x)) - 1$, the pole order reduced by one. As the pole order is not bounded for $h \to \infty$, this approach is not efficient for calculating individual $\Phi(h)$ for $h \gg 0$ and unsuitable for calculations in a formal variable h.

This can be bypassed by the following trick: The function f is rational, in particular meromorphic, on the Riemann sphere $\hat{\mathbf{C}}$. Hence

$$\sum_{x_0 \in \hat{\mathbf{C}}} \mathrm{res}_{x=x_0}\, x^{-h-1} f(x) = 0. \tag{97}$$

The set of poles besides 0 is the set $\tilde{\Gamma} := \bigcup_{i=1}^N \mu_{a_i}$ of all a_i-th roots of unity for $i = 1, \ldots, N$. Using (97), we get the important formula

$$\Phi(h) = -\sum_{\zeta \in \tilde{\Gamma}} \mathrm{res}_{x=\zeta}\, x^{-h-1} f(x). \tag{98}$$

EXAMPLE 31. We use formula (98) to derive systematically the expression for $\Phi_{1,1}(h)$ given in (95) above. In this case, $f(x) = 1/(1-x)^2$ and $\tilde{\Gamma} = \{1\}$. By changing variable $x = 1 + \tilde{x}$ we get

$$\begin{aligned} x^{-h-1} f(x) &= (1+\tilde{x})^{-h-1} \tilde{x}^{-2} \\ &= \big(1 - \tilde{x} + O(\tilde{x}^2)\big)^{h+1} \tilde{x}^{-2} \\ &= \big(1 - (h+1)\tilde{x} + O(\tilde{x}^2)\big) \tilde{x}^{-2}. \end{aligned} \tag{99}$$

Hence $\Phi(h) = -\mathrm{res}_{x=1}\, x^{-h-1} f(x) = -(-(h+1)) = h+1$ as stated in (95).

Any partition function can be calculated as in the example, so although its description is still a bit vague, it is clear that we have obtained an algorithm which calculates for every tuple $(a_1, \ldots, a_N)$ the quasi-polynomial describing the associated partition function.

We continue by introducing some changes of variable which are superfluous in the present case but will facilitate the comprehension of the general case concerning vector partition functions. In order to apply formula (98), it is sufficient to consider the restriction of $x^{-h-1} f(x)$ to an open set containing $\tilde{\Gamma}$, in particular to $\mathbf{C}^*$. So we can consider the composition of this function with $e^{-u} : \mathbf{C} \to \mathbf{C}^*$ and get $(e^{-u})^{-h-1} / \prod_{i=1}^N (1-(e^{-u})^{a_i}) = e^{u(h+1)} / \prod_{i=1}^N (1-e^{-ua_i})$. The Jacobian determinant of the transformation $x = e^{-u}$ is $-e^{-u}$, so

$$\begin{aligned} -\operatorname{res}_{x=\zeta} x^{-h-1} f(x) &= -\operatorname{res}_{u=u_0} -e^{-u} \frac{e^{u(h+1)}}{\prod_{i=1}^N (1-e^{-ua_i})} \\ &= \operatorname{res}_{u=u_0} \frac{e^{uh}}{\prod_{i=1}^N (1-e^{-ua_i})}, \end{aligned} \tag{100}$$

where $u_0 \in \mathbf{C}$ is chosen such that $e^{-u_0} = \zeta$. Let $\Gamma = \bigcup_{i=1}^N \{\frac{1}{a_i}, \ldots, \frac{a_i-1}{a_i}\}$. Then $e^{-2\pi i g}$ runs through $\tilde{\Gamma}$ for $g \in \Gamma$, so

$$\Phi(h) = \sum_{g \in \Gamma} \operatorname{res}_{u=2\pi i g} \frac{e^{uh}}{\prod_{i=1}^N (1-e^{-ua_i})}. \tag{101}$$

In order to simplify notation, we can translate the function for the individual residues to be calculated and have shown the following theorem:

THEOREM 32. *Let $a_1, \ldots, a_N \in \mathbf{Z}_{>0}$, $h \in \mathbf{Z}_{\geq 0}$ and $\Gamma \subset \mathbf{R}$ as above. Then the value of the partition function associated with (a_i) is given by*

$$\Phi(h) = \sum_{g \in \Gamma} \operatorname{res}_{u=0} F_{g,h}(u) \tag{102}$$

for

$$F_{g,h}(u) := \frac{e^{(u+2\pi i g)h}}{\prod_{i=1}^N (1-e^{-(u+2\pi i g)a_i})}.^{11} \tag{103}$$

[11] Cf. footnote 1.

In this form, the formula generalises to vector partition functions. In fact, the notation was chosen carefully to match [**1**, th. 3.3, 2.], (10) and theorem 2.

2. Multidimensional Laplace transformation

Let μ be a measure on $\mathbf{R}^n$. For $u \in (\mathbf{R}^n)^*$ let

$$(\mathcal{L}\mu)(u) := \int_{\mathbf{R}^n} e^{-\langle u,v\rangle}\mu(dv) \tag{104}$$

if the integral exists and is finite. This defines a partial function $\mathcal{L}\mu : (\mathbf{R}^n)^* \to \mathbf{R}$, the *Laplace transform* of μ.

EXAMPLE 33. (1) Let $n = 1$ and suppose that μ is absolutely continuous with respect to Lebesgue measure with density f, i.e. $\mu(dv) = f(v)\lambda(dv)$. We identify $\mathbf{R}^*$ with $\mathbf{R}$ by $\langle u,v\rangle := uv$ for $u,v \in \mathbf{R}$. Then $\mathcal{L}\mu$ coincides with the usual one-dimensional two-sided Laplace transform $\mathcal{L}f$ of f, given by

$$(\mathcal{L}f)(u) = \int_{-\infty}^{\infty} e^{-uv} f(v)\lambda(dv). \tag{105}$$

(2) A famous one-dimensional example is

$$\Big(\mathcal{L}\sum_{n=1}^{\infty}\delta_{\log(n)}\Big)(u) = \zeta(u) \tag{106}$$

for $u > 1$, where ζ denotes the Riemann zeta function. Other one-dimensional examples known from number theory are $\big(\mathcal{L}e^{-e^{-v}}\big)(u) = \Gamma(u)$ and $(\mathcal{L}\delta_{\frac{1}{2}\log(\pi)})(u) = \pi^{-\frac{u}{2}}$, culminating in the expression

$$\big(\mathcal{L}(\theta(ie^{-2v}) - 1)\big)(u) = \pi^{-\frac{u}{2}}\Gamma\big(\tfrac{u}{2}\big)\zeta(u) \tag{107}$$

for the complete zeta function, where $\theta(z) := \sum_{n\in\mathbf{Z}} e^{i\pi n^2 z}$ for $\mathrm{Im}(z) > 0$ denotes the classical Jacobi theta function.[12]

[12]Note that in the context of number theory it is natural to perform a change of variable $y = e^{-u}$ between $\mathbf{R}$ and $\mathbf{R}_{>0}$. The composition of the Laplace transformation with this change of variable is called the *Mellin transformation*. Up to this, the announced formulas can be found e.g. in [**22**, § VII.1].

If $\mathcal{L}\mu$ is defined on an open subset of $(\mathbf{R}^n)^*$ and has a meromorphic extension on $(\mathbf{R}^n)^*$ we refer to this extension by $\mathcal{L}\mu$, too. Note that if we want to compute the inverse Laplace transformation it is important to keep track of the domain of definition. For example the Laplace transforms of $1_{[0,\infty)}$ and $-1_{(-\infty,0]}$ are both $\frac{1}{u}$, only distinguished by the fact that the former is defined for $u > 0$ while the latter is defined for $u < 0$.

Let $A \in \mathbf{Z}^{(n,N)}$ such that the vector partition function $\Phi_A : \mathbf{Z^n} \to \mathbf{Z}_{\geq 0}$ is defined. Consider the vector partition measure μ_A and the continuous vector partition measure $\tilde{\mu}_A$ on $\mathbf{R}^n$ as defined in definitions 7 respectively 8. Then their Laplace transforms are defined on the interior of cone$(A)^\perp \subset (\mathbf{R}^n)^*$, the dual cone of the cone spanned by the column vectors of A. On this cone, they are given by

$$(\mathcal{L}\mu_A)(u) = \frac{1}{\prod_{k=1}^N \left(1 - e^{-\langle u, a_k \rangle}\right)}, \tag{108}$$

$$(\mathcal{L}\tilde{\mu}_A)(u) = \frac{1}{\prod_{k=1}^N \langle u, a_k \rangle}. \tag{109}$$

3. Residues

As in the one-dimensional case explained in § 1, residues will play a key role in the calculation of general vector partition functions. This section introduces the necessary notions to speak about residues of functions of several variables. We follow A. Szenes and M. Vergne [**24**] for the definition of the Jeffrey-Kirwan residue and C. De Concini and C. Procesi [**6**] for the connection to iterated residues.

Recall that for a meromorphic function $f(z)$ on a neighbourhood of $0 \in \mathbf{C}$ one can define the residue at $z = 0$ either as the coefficient of z^{-1} in the Laurent expansion of $f(z)$ or (up to multiplication by a constant) as the value of the integral $\int_c f(z)dz$ over a small circle c around 0, oriented clockwise. Some observations about this definition: (i) This does not intrinsically define the residue of a function at the point 0 of the Riemann surface $\mathbf{C}$. The coordinate is decisive, so one should not talk about the residue "at 0" but "at $z = 0$." (ii) Consider the integral definition of the residue. The fact that there is essentially only one notion of residue is a consequence of the fact that $H_1(\mathbf{C} \setminus \{0\})$ is a free Abelian group of rank 1: If we know the integral of a function over the generator $[c]$, we know the integral over every cycle. This becomes more involved in the multidimensional case.

Let U and V be n-dimensional real vector spaces with a pairing $\langle\ ,\ \rangle$. Let $U_{\mathbf{C}} := U \otimes_{\mathbf{R}} \mathbf{C}$ and similarly for V. Let $\Delta \subset V$ be a finite set of vectors generating V and contained in an open half-space in V. Consider the algebra $R_\Delta := \Delta^{-1}S(V_{\mathbf{C}})$ of rational functions on $U_{\mathbf{C}}$ with poles along $\{\ker(\alpha) : \alpha \in \Delta\}$. For $u \in U_{\mathbf{C}}$ and $f \in R_\Delta$ denote by $\partial_u f$ the derivative of f in direction u. Let $D_\Delta \subset R_\Delta$ be the vector space generated by all derivatives $\partial_u f$. Let $S_\Delta \subset R_\Delta$ be the vector space generated by all elements $f_\sigma := \prod_{\alpha\in\sigma}\alpha^{-1}$ such that $\sigma \subset \Delta$ is a basis of V. Then R_Δ decomposes as a direct sum of D_Δ and S_Δ.

DEFINITION 34. Let $\operatorname{tres} : R_\Delta \to S_\Delta$ be the projection along D_Δ. Denote its canonical extension to the completion $\hat{R}_\Delta := \Delta^{-1}\hat{S}(V_{\mathbf{C}})$ by the same symbol. For $f \in \hat{R}_\Delta$, the function $\operatorname{tres}(f) \in S_\Delta$ is called the *total residue* of f.

Fix a normalisation of the Lebesgue measure on V. For a basis $\sigma \subset \Delta$ of V, denote the volume of $\sum_{\alpha\in\sigma}[0,1]\alpha$ by $\operatorname{vol}(\sigma)$. Let F be the fan in V associated with Δ as in § 2.3.

DEFINITION 35. For a maximal cone $C \in F$, the association

$$f_\sigma \mapsto \begin{cases} \operatorname{vol}(\sigma)^{-1} & \text{if } C \subset \operatorname{cone}(\sigma) \\ 0 & \text{otherwise.} \end{cases} \tag{110}$$

defines a linear functional on S_Δ (i.e. respects the linear relations between the f_σ). Let $\mathrm{JK}_C : \hat{R}_\Delta \to \mathbf{C}$ be the linear functional obtained by composition with the total residue map. For $f \in \hat{R}_\Delta$, the number $\mathrm{JK}_C(f)$ is called the *Jeffrey-Kirwan residue* of f with respect to the cone C.

The usual residue of a function of one variable is a special case of the Jeffrey-Kirwan residue:

EXAMPLE 36. Let $U = V = \mathbf{R}$ with the canonical pairing $\langle x, y\rangle := xy$. Fix the canonical coordinate z on U, i.e. $z = 1 \in V$. Consider $\Delta = \{z\}$. Then $R_\Delta = \mathbf{C}[z^{\pm 1}]$ and $\hat{R}_\Delta = \mathbf{C}((z))$. Normalise the Lebesgue measure on V such that $\operatorname{vol}([0,1]) = 1$. There is only one maximal cone $C = [0,\infty)$ and only one basis $\sigma = \{z\} \subset \Delta$. In this case $f_\sigma = \frac{1}{z}$ and $\operatorname{vol}(\sigma) = 1$, so $\mathrm{JK}_C(f) = \operatorname{res}_{z=0}(f)$ for all $f \in \hat{R}_\Delta$.

The first nontrivial example is the following:

EXAMPLE 37. Let $U = \mathbf{R}^2$ and $V := U^*$. Let $z, w \in V$ be the canonical coordinates on U. Normalise the Lebesgue measure on V such that $\mathrm{vol}([0,1]z + [0,1]w) = 1$. Consider $\Delta = \{z, w, z+w\}$. The fan associated with Δ has maximal cones $C_1 := \mathrm{cone}(z, z+w)$ and $C_2 := \mathrm{cone}(w, z+w)$. Then the Jeffrey-Kirwan residue is given on S_Δ by the following table:

$$\begin{array}{c|ccc} f & \frac{1}{zw} & \frac{1}{z(z+w)} & \frac{1}{w(z+w)} \\ \hline \mathrm{JK}_{C_1}(f) & 1 & 1 & 0 \\ \mathrm{JK}_{C_2}(f) & 1 & 0 & 1 \end{array} \tag{111}$$

There is a more naive generalisation of the one-dimensional notion of a residue as follows: Namely, for $u \in V_{\mathbf{C}}$, interpreted as a coordinate on $U_{\mathbf{C}}$, and a rational function f on U, define $\mathrm{res}_{u=0}(f)$ as a rational function on $u^\perp \subset U_{\mathbf{C}}$ by choosing additional coordinates on $U_{\mathbf{C}}$ and considering them as constants.

EXAMPLE 38. Let $U_{\mathbf{C}} = \mathbf{C}^2$ with standard coordinates z, w. Let $\tilde{w} := z + w$. Then $\frac{1}{zw} = \frac{1}{z(\tilde{w}-z)}$. The residue at $z = 0$ is $\mathrm{res}_{z=0}\left(\frac{1}{zw}\right) = \frac{1}{w}$, respectively $\mathrm{res}_{z=0}\left(\frac{1}{z(\tilde{w}-z)}\right) = \frac{1}{\tilde{w}} = \frac{1}{w+z} = \frac{1}{w}$ on $\{z = 0\}$. Note that the two choices of the additional coordinate show that the residue is in fact only well defined as a function on $\{z = 0\}$, as opposed to the full space $\mathbf{C}^2$.

For a basis $u_1, \ldots, u_n \in \Delta$ of V define the *iterated residue* of a function f to be

$$\mathrm{ires}_{u_1,\ldots,u_n}(f) := \mathrm{res}_{u_n=0} \cdots \mathrm{res}_{u_1=0} f. \tag{112}$$

Note that this is a function on $\bigcap u_i^\perp = \{0\}$, i.e. a complex number. Hence, $\mathrm{ires}_{u_1,\ldots,u_n}$ is a function $R_\Delta \to \mathbf{C}$, or similarly $\hat{R}_\Delta \to \mathbf{C}$.

The Jeffrey-Kirwan residue can be expressed in terms of iterated residues by a theorem of C. De Concini and C. Procesi [**6**] as follows:

Fix a total order on Δ, so $\Delta = \{\alpha_1, \ldots, \alpha_N\}$ with $\alpha_1 < \cdots < \alpha_N$. A basis $\{\alpha_{i_1}, \ldots, \alpha_{i_n}\} \subset \Delta$ with $i_1 < \cdots < i_n$ is called *without broken circuits* if there are no $j \in \{1, \ldots, n\}$ and $k \in \{i_j + 1, \ldots, N\}$ such that the set $\{\alpha_{i_1}, \ldots, \alpha_{i_j}, a_k\}$ is linearly dependent. For a maximal cone C of the fan F, let $B_{\mathrm{nb}}(C)$ denote the set of bases σ without broken circuits such that the cone spanned by σ contains C. Then by [**6**, note after def. 5; (4); th. 3.1] we have:

THEOREM 39. *Let $f \in \hat{R}_\Delta$ be the germ of a meromorphic function with poles along* $\{\ker(\alpha) : \alpha \in \Delta\}$ *and C a maximal cone in F. Then*

$$\mathrm{JK}_C(f) = \sum_{\sigma \in B_{\mathrm{nb}}(C)} \frac{1}{\mathrm{vol}(\sigma)} \mathrm{ires}_\sigma f. \tag{113}$$

4. A residue formula for vector partition functions

We return to the catulation of vector partition functions. Let $A \in \mathbf{Z}^{(n,N)}$ be a matrix subject to (27), i.e. such that the vector partition function $\Phi_A : \mathbf{Z}^n \to \mathbf{Z}_{\geq 0}$ is defined. Suppose that $\mathrm{rk}(A) = n$. Let F be the fan in $\mathbf{R}^n$ associated with A as in § 2.3. Consider the set Δ of column vectors of A. Then for every maximal cone $C \in F$ by § 3 we have the notion of the Jeffrey-Kirwan residue of a function with respect to C. Recall the definitions of Γ and $F_{g,h}$ from § 1.4. With these definitions we have the following theorem [**24**, th. 3.1]:

THEOREM 40. *On any maximal cone C of* $\mathrm{fan}(A)$*, the vector partition function associated with A is given by*

$$\Phi_A(h) = \sum_{g \in \Gamma} \mathrm{JK}_C(F_{g,h}(u)). \tag{114}$$

Note that in § 1.4 we have defined basic subsets of $\{1, \ldots, N\}$ while in § 3 we have consideres bases which are subsets of Δ. Although these notions do not coincide in general (because the matrix A can contain a column repeatedly), the corresponding notions of basic subsets without broken circuits and bases without broken circuits coincide. Now we can combine theorems 39 and 40 and obtain theorem 2 as reported in chapter 1:

THEOREM 41. *On any maximal cone C of* $\mathrm{fan}(A)$*, the vector partition function associated with A is given by*

$$\Phi_A(h) = \sum_{\sigma \in B_{\mathrm{nb}}(C)} \frac{1}{\mathrm{vol}(\sigma)} \sum_{g \in \Gamma} \mathrm{ires}_\sigma F_{g,h}(u). \tag{115}$$

This formula is suitable for algorithmic implementation: The sets $B_{\mathrm{nb}}(C)$ and Γ can be computed explicitly and the calculation of residues is implemented in any formal calculation software. A Maple program fulfilling this task is included as appendix A. Note that the maximal cone C of $\mathrm{fan}(A)$ is specified by a vector contained in its interior. An algorithm computing the maximal cones is explained in the following section.

5. Computing fan(A)

Let $A \in \mathbf{Z}^{(n,N)}$ be a matrix subject to (27), i.e. such that the vector partition function $\Phi_A : \mathbf{Z}^n \to \mathbf{Z}_{\geq 0}$ is defined. Suppose that $\mathrm{rk}(A) = n$. This section describes the relevant ideas necessary for the computation of a fan $F = \mathrm{fan}(A)$ such that Φ_A is quasi-polynomial on each maximal cone of F as in Sturmfels' theorem (theorem 9).

We start by recalling some definitions. A subset $\sigma \subset \{1, \dots, N\}$ is called *basic* if the corresponding column vectors $(a_i)_{i\in\sigma}$ of the matrix A form a basis of $\mathbf{R}^n$. In this case we write $\mathrm{cone}(\sigma)$ for the cone generated by $(a_i)_{i\in\sigma}$. Cones of the form $\mathrm{cone}(\sigma)$ are called *basic cones*. The fan considered by B. Sturmfels is the coarsest fan such that every basic cone is a union of cones in F (i.e. the "common refinement" of the basic cones, [**23**, p. 304]). Hence the maximal cones in F are the minimal n-dimensional cones which can be written as an intersection of basic cones. By a *regular vector* for A we mean a vector contained in the interior of a maximal cone of F.

The idea of the algorithm is to traverse the graph whose vertices are the maximal cones of F, two vertices being adjacent if they share a common facet. A maximal cone C in F can be represented by the set of basic sets σ such that $\mathrm{cone}(\sigma) \supset C$. A maximal cone C_0 can be found by choosing an arbitrary regular vector h_0 and taking

$$C_0 = \bigcap_{\sigma:\mathrm{cone}(\sigma)\ni h_0} \mathrm{cone}(\sigma). \tag{116}$$

The remaining step is to describe the adjacency relation explicitly.

LEMMA 42. *Let C be a maximal cone in F and f a facet of f. Let $\Sigma_f(C) := \{\sigma \text{ basic} : \mathrm{cone}(\sigma) \supset f,\ \mathrm{cone}(\sigma) \supset C \implies \partial\mathrm{cone}(\sigma) \not\supset f\}$.*

1. *C has a neighbour in direction f if and only if $\Sigma_f(C) \neq \emptyset$.*
2. *In this case the neighbour of C in direction f is $\bigcap_{\sigma\in\Sigma_f(C)} \mathrm{cone}(\sigma)$.*

PROOF. First, suppose that C has a neighbour $\tilde{C}$ in direction f. Let σ be a basic set such that $\mathrm{cone}(\sigma) \supset \tilde{C}$. As f is also a facet of $\tilde{C}$, it follows that $\mathrm{cone}(\sigma) \supset f$. If $\mathrm{cone}(\sigma) \supset C$, then $\mathrm{cone}(\sigma) \supset C \cup \tilde{C}$. As f intersects the interior of $C \cup \tilde{C}$ nontrivially, it follows that $\partial\mathrm{cone}(\sigma) \not\supset f$. This shows that $\sigma \in \Sigma_f(C)$.

On the other hand, let $\sigma \in \Sigma_f(C)$. Similarly one sees that $\mathrm{cone}(\sigma) \supset \tilde{C}$, so (2) is shown.

Next, suppose that C has no neighbour in direction f, i.e. that $f \subset \partial\mathrm{supp}(F)$. Assume that $\Sigma_f(C) \ni \sigma$. Then $\mathrm{cone}(\sigma) \supset f$, and as $f \subset \partial\mathrm{supp}(F)$ it follows that $\mathrm{cone}(\sigma) \supset C$. By the second defining property of $\Sigma_f(C)$ it follows that $\partial\mathrm{cone}(\sigma) \not\supset f$. On the other hand $f \subset \mathrm{cone}(\sigma) \cap \partial\mathrm{supp}(F) \subset \partial\mathrm{cone}(\sigma)$, a contradiction. $\square$

Given this lemma, the implementation of the algorithm is straightforward. The corresponding Maple code can be found in appendix B.

APPENDIX A

Maple program for calculating vector partition functions

The following Maple program has been written and used for the explicit calculations of the quasi-polynomials describing the weight multiplicity function of $\mathfrak{so}_5(\mathbf{C})$.

The first part of the program contains procedures for calculating non broken circuit bases.

is_linearly_dependent. Let s be a list of integers and v be a list of vectors. The following procedure returns whether the sublist $(v_i)_{i \in s}$ is linearly dependent or not.

```
is_linearly_dependent := proc(s, v)
  local i;

  return evalb(LinearAlgebra[Rank](
    Matrix([seq(v[i], i in s)])) <
    nops(s));
end proc:
```

non_broken_circuit_bases. Let v be a list of vectors. The following procedure returns the list of all non broken circuit bases in v in the sense e.g. of [**6**]. Each non broken circuit base returned is represented as a list of indices with respect to v.

```
non_broken_circuit_bases := proc(v)
  local nbcb_starting, N, n;

  N := nops(v);
```

```
  n := LinearAlgebra[Dimension](v[1]);

  nbcb_starting := proc(s)
    local i, j;

    if ormap(is_linearly_dependent,
      [seq([op(s), j], j = op(-1, s) + 1
      .. N)], v) then
      return NULL;
    end if;

    if nops(s) = n then return s; end if;

    seq(nbcb_starting([op(s), i]),
      i = op(-1, s) + 1 ..
      N - n + nops(s) + 1);
  end proc;

  [seq(nbcb_starting([i]), i = 1 ..
    N - n + 1)];
end proc:
```

is_in_cone. Let v be a list of vectors, assumed to be a basis. Let a be a vector. Returns whether a is an element of the cone spanned by v.

```
is_in_cone := proc(a, v)
  local n, w, i;

  n := nops(a); # the dimension of the
    # vector space by assumption

  w := Matrix(a)^(-1) . v;

  return andmap(type, w, 'nonnegative');
end proc:
```

is_in_cone_indexed. Let v be a list of vectors. Let i be a list of integers, such that $(v_k)_{k \in i}$ is a basis. Let h be a vector. Returns whether h is an element of the cone spanned by $(v_k)_{k \in i}$.

```
is_in_cone_indexed := proc(i, v, h)
  local j;

  return is_in_cone([seq(v[j],
    j in i)], h);
end proc:
```

nbcb. Let A be a matrix and h a column vector. Returns the list of all non broken circuit bases M which are subsets of the list of column vectors of A such that h is contained in the cone spanned by M. Each non broken circuit basis is returned as a list of indices with respect to the list of column vectors of A.

```
nbcb := proc(A, h)
  local v;

  v := [LinearAlgebra[Column](A, 1 ..
    LinearAlgebra[ColumnDimension](A))];

  select(is_in_cone_indexed,
    non_broken_circuit_bases(v), v, h);
end proc:
```

The second part of the program contains procedures for computing Γ.

BFS. This is a generic implementation of the breadth first search algorithm. Let v_0 be any expression, interpreted as the initial vertex of the graph to be traversed. Let N be a procedure mapping expressions to lists of expressions, encoding the neighbour relation of the graph. Let eq be a predicate on pairs of expressions, determining whether they should be considered as equal or not.

```
BFS := proc(v0, N, eq)
  local V, i, v;
  V := [v0];
  i := 0;
```

```
  while i < nops(V) do
    i := i + 1;

    for v in N(V[i]) do;
      if not ormap(eq, V, v) then
        V := [op(V), v];
      end if;
    end do;
  end do;

  return V;
end proc:
```

T. Let A be a matrix and s be a list of integers, representing the list σ of clumn vectors of A with the corresponding indices. Returns the set $T(\sigma)$ as defined in [**1**, p. 559].

```
T := proc(A, s)
  local n, As, Asi, det, N, j;

  n := LinearAlgebra[RowDimension](A);
  As := LinearAlgebra[SubMatrix](A,
    1 .. n, s);
  det := LinearAlgebra[Determinant](As);

  if det = 0 then return []; end if;

  if abs(det) = 1 then
    return [Vector[row](n, 0)];
  end if;

  Asi := As^(-1);

  N := proc(v)
    [seq(map(x->x-floor(x),
      v + LinearAlgebra[Row](Asi, j)),
```

```
      j = 1 .. n)];
  end proc;

  BFS(Vector[row](n, 0), N,
    LinearAlgebra[Equal]);
end proc:

# T(<<1, 0> | <0, 1> | <1, 1>>, [1, 2]);
# T(<<1, 0> | <0, 1> | <1, 1>>, [1, 2]);
```

Gamma. Let A be a matrix. Returns the set $\Gamma := \bigcup_\sigma T(\sigma)$, where the union is over all subsets of the column vectors of A forming a basis.

```
Gamma := proc(A)
  local N, n, s;

  N := LinearAlgebra[ColumnDimension](A);
  n := LinearAlgebra[RowDimension](A);
  ListTools[MakeUnique]([seq(op(T(A, s)),
    s in combinat[choose](N, n))], 1,
    LinearAlgebra[Equal]);
end proc:
```

The third part of the program actually computes the vector partition function by means of the method of inverse Laplace transformation.

ires. Let f be a function, represented as a formal expression. Let u be list of variables. Returns the iterated residue $\mathrm{ires}_u(f) = \mathrm{res}_{u_n=0} \cdots \mathrm{res}_{u_1=0} f$.

```
ires := proc(f, u)
  foldl((f, u) -> residue(f, u=0),
    f, op(u));
end proc:
```

kostant. Let A be a matrix, g and u row vectors and h a column vector. Returns the function $F_{g,h}(u)$ as defined in (10).

```
kostant := proc(A, g, h, u)
  local i;
```

```
  return exp((2*Pi*I*g + u) . h) /
    mul(1 - exp(-(2*Pi*I*g + u) .
    LinearAlgebra[Column](A, i)), i = 1 ..
    LinearAlgebra[ColumnDimension](A));
end proc:
```

transformed_kostant. Let A, g, u, h be as before. Let s be a list of integers, interpreted as the corresponding list of column vectors of A. Returns the function $F_{g,h}(u)$ in coordinates given by s.

```
transformed_kostant := proc(A, g, h, u, s)
  local n, As, Asi;

  n := LinearAlgebra[RowDimension](A);
  As := LinearAlgebra[SubMatrix](A,
    1 .. n, s);
  Asi := As^(-1);
  return kostant(Asi.A, g.As, Asi.h, u);
end proc:
```

vpf_summand. Let A be a matrix, g a row vector, h a column vector, s a list of integers. Returns the summand of the vector partition function associated with A, evaluated at h, corresponding to the non broken circuit basis s and to $g \in \Gamma$.

```
vpf_summand := proc(A, g, h, s)
  local n, us, u, i, vol, As;

  n := LinearAlgebra[RowDimension](A);
  us := [seq(u[i], i = 1 .. n)];
  As := LinearAlgebra[SubMatrix](A,
    1 .. n, s);

  ires(transformed_kostant(
    A, g, h, Vector[row](us), s), us) /
    abs(LinearAlgebra[Determinant](As));
end proc:
```

vpf. Let A be a matrix, Γ the associated list of row vectors as above, nbcb the list of relevant non broken circuit bases, represented as their indices w.r.t. the column vectors of A. Let h be a column vector. The following procedure returns the value $\Phi_A(h)$ of the vector partition function associated with A at h.

```
vpf := proc(A, Gamma, nbcb, h)
  local g, s;

  add(add(vpf_summand(A, g, h, s),
    g in Gamma), s in nbcb);
end proc:
```

vpf_for_chamber. Let A be a matrix, h_0 a regular column vector, used to determine a maximal cone of fan(A). Let h be a column vector, possibly singular of formal. Returns the value $\Phi_A(h)$ of the vector partition function associated with A at h, supposing that h is an element of the maximal cone determined by h_0.

```
vpf_for_chamber := proc(A, h0, h)
  local my_Gamma, my_nbcb;

  my_Gamma := Gamma(A);
  my_nbcb := nbcb(A, h0);

  vpf(A, my_Gamma, my_nbcb, h);
end proc:
```

This ends the actual program. We conclude with some examples. First, we use the program to calculate the vector partition function associated with the matrix $\left(\begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix}\right)$, as in example 5.

```
vpf_for_chamber(<<1, 0> | <0, 1> | <1, 1>>,
  <2, 1>, <h1, h2>);
```

$$1+h_2$$

```
vpf_for_chamber(<<1, 0> | <0, 1> | <1, 1>>,
  <1, 2>, <h1, h2>);
```

$$1+h_1$$

```
vpf_for_chamber(<<1, 0> | <0, 1> | <1, 1>>,
  <-1, 2>, <h1, h2>);
```

$$0$$

Of course, the vector partition function is independent of the order of the vectors defining it. Nevertheless, the non broken circuit bases used in the computation are different ones for different orders.

```
vpf_for_chamber(<<1, 0> | <1, 1> | <0, 1>>,
  <2, 1>, <h1, h2>);
```

$$1+h_2$$

The program can also be used to compute partition functions:

```
vpf_for_chamber(<<1>>, Vector[column]([1]),
  Vector[column]([h]));
```

$$1$$

```
vpf_for_chamber(<<1 | 1>>,
  Vector[column]([1]),
  Vector[column]([h]));
```

$$1+h$$

Some care has been taken to ensure that the supporting cone of the vector partition function does not have to lie in the positive orthant.

```
vpf_for_chamber(<<-1 | -1>>,
  Vector[column]([-1]),
  Vector[column]([h]));
```

$$1-h$$

In the interesting cases, the computed quasi-polynomials have a nontrivial period, i.e. they are not actual polynomials.

```
vpf_for_chamber(<<1 | 2>>,
  Vector[column]([1]),
  Vector[column]([h]));
```

$$\frac{1}{2}h+\frac{3}{4}+\frac{1}{4}e^{h\pi i}$$

```
vpf_for_chamber(<<-1 | -2>>,
  Vector[column]([-1]),
  Vector[column]([h]));
```

$$-\frac{1}{2}h+\frac{3}{4}+\frac{1}{4}e^{h\pi i}$$

The actual computations for $\mathfrak{so}_5(\mathbf{C})$ have been done as follows:

```
E := << 0| 2|-1| 0|-1| 0| 0| 0| 0| 0>,
      < 0| 0| 1|-2| 0|-1| 0| 0| 0| 0>,
      < 0| 0| 0| 1| 0| 0| 1| 0| 0| 0>,
      < 0| 0| 1|-2| 0| 0| 0| 1| 0| 0>,
      < 0| 1|-1| 2| 0| 0| 0| 0| 1| 0>,
      < 1|-2| 2|-2| 0| 0| 0| 0| 0| 1>,
      < 1| 0| 1| 0| 0| 0| 0| 0| 0| 0>,
      < 0| 1| 0| 1| 0| 0| 0| 0| 0| 0>>:

regular_vectors := [
  < -2, -4, 5,  9,  8,  7,  8,  5>,
  < -2, -4, 5,  9,  8,  7,  9,  5>,
  < -4, -6, 7,  8, 11,  9, 11,  7>,
  < -1, -2, 3,  3,  5, -1,  2,  3>,
  <  0, -2, 3,  7,  5,  8,  7,  3>,
  < -5, -1, 5, 12,  4, 13, 11,  5>,
  < -4, -6, 7,  8, 11,  9, 12,  7>,
  < -1, -2, 3,  3,  5, -1,  3,  3>,
  < -3, -2, 4,  2,  6,  1,  4,  4>,
  < -5, -1, 5, 12,  4, 13, 12,  5>,
  <-10, -4, 9, 11,  8, 17, 17,  9>,
  < -3,  0, 3,  5,  2,  3,  4,  3>,
  < -2,  0, 3,  9,  2, 12,  9,  3>,
  < -3, -2, 4,  2,  6,  1,  5,  4>,
  < -2, -2, 2,  7,  2,  7,  6,  3>,
  <-10, -4, 9, 11,  8, 17, 18,  9>,
  < -3,  0, 3,  5,  2,  3,  5,  3>,
  < -6, -1, 5,  4,  4,  5,  7,  5>,
  < -4, -4, 6, -1,  6,  3,  7,  6>,
  < -1, -1, 2,  3,  1,  1,  2,  2>,
  < -2, -2, 2,  7,  2,  7,  7,  3>,
  < -7, -6, 6,  6,  6, 11, 12,  7>,
```

```
  < -2, -1, 1,  4,  1,  3,  3,  2>,
  < -1, -1, 1,  6,  1,  8,  6,  2>,
  < -6, -1, 5,  4,  4,  5,  8,  5>,
  < -4, -4, 6, -1,  6,  3,  8,  6>,
  < -1, -1, 2,  3,  1,  1,  3,  2>,
  < -2, -2, 3,  2,  2,  1,  3,  3>,
  < -3, -2, 4, -1,  4,  1,  4,  4>,
  < -7, -6, 6,  6,  6, 11, 13,  7>,
  < -2, -1, 1,  4,  1,  3,  4,  2>,
  < -5, -3, 3,  3,  3,  5,  6,  4>,
  < -4, -6, 5, -1,  6,  3,  7,  6>,
  < -2, -2, 3,  2,  2,  1,  4,  3>,
  < -3, -2, 4, -1,  4,  1,  5,  4>,
  < -1, -3, 3, -1,  3, -1,  2,  3>,
  < -5, -3, 3,  3,  3,  5,  7,  4>,
  < -4, -6, 5, -1,  6,  3,  8,  6>,
  < -3, -1, 1,  2,  1,  6,  5,  2>,
  < -3, -4, 3, -1,  4,  1,  4,  4>,
  < -1, -3, 3, -1,  3, -1,  3,  3>,
  < -3, -4, 3, -1,  4,  1,  5,  4>,
  < -2, -3, 2, -1,  3,  2,  4,  3>]:

for v in regular_vectors do
  lprint(simplify(vpf_for_chamber(E, v,
  <0, 0, lambda2, lambda1, lambda2,
  lambda1, beta1,  beta2>)));
end do:
```

APPENDIX B

Maple program for calculating the fan of a matrix

The following program computes, given a matrix A subject to condition (27), the fan associated with A, represented as a list of its maximal cones. The individual cones are objects of type convex[CONE] as defined in the package Convex by M. Franz[13]. I used version 1.1.2 of this package and Maple 11. Note that the program computing the quasi-polynomials (appendix A) represents a maximal cone by a vector contained in its interior. Having calculated a cone as an object of type convex[CONE] one can immediately determine such a vector, e.g. the sum of vectors representing the rays.

basiccones. Let A be a matrix. Returns the list of basic cones for the matrix A.

```
basiccones := proc(A)
  local N, n, cones;

  N := LinearAlgebra[ColumnDimension](A);
  n := LinearAlgebra[RowDimension](A);

  cones := [seq(convex[poshull](
    LinearAlgebra[Column](A, s)),
    s in combinat[choose](N, n))];
  cones := select(convex[isfulldim],
    cones);
  ListTools[MakeUnique](cones, 1,
    CONE[`&=`])
```

[13] http://www-fourier.ujf-grenoble.fr/~franz/

```
end proc:
```

chamber. Let A be a matrix and h a regular vector for A, i.e. a vector which is contained in the interior of a maximal cone of $\text{fan}(A)$. Returns the maximal cone containing h as an object of type convex[CONE].

```
chamber := proc(A, h)
  local basic_cones_containing_h;

  basic_cones_containing_h := select(
    convex[contains], basiccones(A), h);
  convex[intersection](op(
    basic_cones_containing_h));
end proc:
```

BFS. This is a generic implementation of the breadth first search algorithm. Let v_0 be any expression, interpreted as the initial vertex of the graph to be traversed. Let N be a procedure mapping expressions to lists of expressions, encoding the neighbour relation of the graph. Let eq be a predicate on pairs of expressions, determining whether they should be considered as equal or not.

```
BFS := proc(v0, N, eq)
  local V, i, v;

  V := [v0];
  i := 0;

  while i < nops(V) do
    i := i + 1;

    for v in N(V[i]) do;
      if not ormap(eq, V, v) then
        V := [op(V), v];
      end if;
    end do;
  end do;

  return V;
```

```
end proc:
```

chambers. Let A be a matrix and h a regular vector for A. Returns the list of maximal cones of fan(A).

```
chambers := proc(A, h)
  local basic_cones, c0, otherside,
    inborder, N;

  basic_cones := basiccones(A);
  c0 := convex[intersection](op(
    select(convex[contains],
    basiccones(A), h)));

  inborder := proc(C, c)
    ormap(`&>=`, map(convert,
      convex[facets](C), CONE), c);
  end proc;

  otherside := proc(f, c)
    local list;

    list := select(b -> (b &>= f and not
      (b &>= c and inborder(b, f))),
      basic_cones);

    if list = [] then
      return NULL;
    end if;

    convex[intersection](op(list));
  end proc;

  N := proc(c)
    local fs;

    fs := map(convert, convex[facets](c),
```

```
      CONE);
    map(otherside, fs, c);
  end proc;

  BFS(c0, N, `&=`);
end proc:
```

For example, the fans of the Kostant partition function of a root system of type A_2 and A_3 can be calculated as follows:

```
chambers(<<1|0|1>,
          <0|1|1>>, <2,1>);

chambers(<<1|0|0|1|0|1>,
          <0|1|0|1|1|1>,
          <0|0|1|0|1|1>>, <1,1/10,1>);
```

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