Digraph Coloring Games and Game-Perfectness

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Kurzzusammenfassung

In dieser Arbeit wird die spielchromatische Zahl eines Digraphen als spieltheoretische Variante der dichromatischen Zahl eingeführt. Dieser Begriff verallgemeinert die bekannte spielchromatische Zahl eines Graphen. Ein erweitertes Modell berücksichtigt ebenfalls relaxierte Färbungen sowie asymmetrische Zugfolgen. Spielperfektheit wird als spieltheoretische Variante der Perfektheit eines Graphen definiert und auf Digraphen verallgemeinert.

Obere und untere Schranken für die spielchromatische Zahl verschiedener Klassen von Digraphen werden untersucht. Im letzten Teil der Arbeit werden spielperfekte Digraphen mit kleiner Cliquenzahl charakterisiert, sowie allgemeine Resultate über Spielperfektheit gezeigt. Einige Resultate wurden mit Hilfe eines Computerprogramms verifiziert, welches im Anhang diskutiert wird.

Abstract

In this thesis the game chromatic number of a digraph is introduced as a game-theoretic variant of the dichromatic number. This notion generalizes the well-known game chromatic number of a graph. An extended model also takes into account relaxed colorings and asymmetric move sequences. Game-perfectness is defined as a game-theoretic variant of perfectness of a graph, and is generalized to digraphs.

We examine upper and lower bounds for the game chromatic number of several classes of digraphs. In the last part of the thesis, we characterize game-perfect digraphs with small clique number, and prove general results concerning game-perfectness. Some results are verified with the help of a computer program that is discussed in the appendix.

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About this thesis

In this thesis the game chromatic number of a digraph is introduced and examined. This parameter is defined by the following non-cooperative 2-person game g. We are given a finite digraph $D = (V, E)$, a finite color set C and two players, Alice and Bob. The players alternately color a vertex v of D with a color c of C which is selected in such a way that no in-neighbor of v is already colored with c. When no such move is possible any more, the game ends. If every vertex is colored at the end of the game, Alice wins, otherwise Bob. The game chromatic number of D is the smallest cardinality of a color set C for which Alice has a winning strategy for the game played with D and C. In this thesis graphs are always considered as digraphs with pairs of oppositely directed arcs. Then the game chromatic number of a digraph generalizes the well-known game chromatic number of a graph.

After an introduction into the notions, a motivation, and a historical review in Chapter 1, we examine a generalization of the game on undirected and directed forests in Chapter 2. This generalization takes into account defective colorings as well as asymmetric moving rules. The maximal game chromatic number of directed forests is exactly determined for all parameters of the generalized game, that of undirected forests up to an interval of the length 1. Chapter 3 is devoted to the study of a new digraph parameter, the lightness. With the aid of this parameter we determine upper bounds for the game chromatic number of graphs and simple digraphs with prescribed girth which are embeddable into certain surfaces. In Chapter 4 the game chromatic number of incidence graphs is narrowed down.

In Chapter 5 the notion of game perfectness is introduced. For these examinations the exact definition of the game is of primary importance, i.e. which player has the first move and whether a player is allowed to miss a turn. A digraph is called g-perfect if, for every induced subdigraph H, the game chromatic number of H equals the clique number of H. We characterize g-perfect graphs with clique number 2 for several variants g of the game. Let A be the variant where Alice is allowed to begin and to miss a turn, and B be the variant where Bob has these rights. We even characterize B-perfect graphs with clique number 3. In addition, it is proved that complements of bipartite graphs are A-perfect. Furthermore (in the directed case) every A-perfect semiorientations of paths and cycles are classified.

In Appendix A we reprove the fact that the digraph coloring problem which underlies the digraph coloring games is $\mathcal{NP}$-complete even for two colors. Appendix B describes a computer program that, by complete game-tree search, solves the problem considered in this thesis exactly.

Vorwort

Bei dieser Arbeit handelt es sich um eine von der Mathematisch-Naturwissenschaftlichen Fakultät der Universität zu Köln angenommene Dissertation. Die Referenten waren Prof. Dr. Ulrich Faigle und Prof. Dr. Rainer Schrader. Die Abschlussprüfung fand am 30.11.2007 statt.

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I would also like to thank Henry Glover and Thomas Bier for several attempts to bound $M(S)$ for an arbitrary surface. These attempts gave me some kind of insight into the nature of this topological problem and strengthened my opinion that the problem to determine $M(S)$ might be very difficult. Thanks to Artem Pyatkin for the hint that Proposition 53 is as easy to prove as it is. I acknowledge a very valuable suggestion concerning Theorem 46 by an anonymous referee of the paper [9].

Thanks to Käthe-Hack-Stiftung which gave me financial support to present some of the results of Chapter 3 at an international conference.

I am very grateful to Robert Nickel and to several of my colleagues and ex-colleagues at ZAIK for careful proofreading of chapters of this thesis or of preparative papers which later were included in the thesis. At this point I would like to mention Birgit Engels, Bernhard Fuchs, Stavros Kousidis, Martin Lätsch, Britta Peis, Alex Schönhut, and Anna Schulze. Thanks to Martin Olschewski as well for some practical advice concerning computers.

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Thanks to my family for all they have done for me.

Chapter 1

Introduction

Graph coloring games have been considered in the literature of discrete mathematics during the last two decades. Actually, the theory of these games seems to be an arising subject in this branch of science. Most of the published works in this field concerns coloring games of *undirected* graphs. The few proposed coloring games on directed graphs do not really generalize the undirected graph coloring game of Bodlaender [15] which forms the basis of all undirected graph coloring games. However, in this thesis we will show that there is a natural generalization of Bodlaender's game to *directed* graphs (digraphs for short). We will study this digraph coloring game and its extensions.

In order to explain what it means that a coloring game of digraphs is a generalization of a coloring game of undirected graphs we have to fix some notions. In Section 1.1 we will define digraphs in such a way that undirected graphs (graphs for short) are special digraphs. Due to this definition we are forced to use some notions from graph theory in a slightly different way than the usual. These notions are explained in the Sections 1.1 and 1.2. Since we will also consider digraphs with certain topological properties, Section 1.3 gives an introduction into the basics of topological graph theory.

A parameter defined by Bodlaender's graph coloring game is a game-theoretic version of the chromatic number of a graph. In the same sense, the analogous parameter of the digraph coloring game we develop is a game-theoretic version of the dichromatic number of a digraph which was introduced by Neumann-Lara [69]. Therefore, in Section 1.4 the dichromatic number and its basic coloring principle are discussed.

Our fundamental digraph coloring game is defined as follows. Two players, Alice and Bob, alternately color vertices of a given digraph D with a color from a given color set C. The choice of their colors is completely free, except for one rule: a vertex may not receive the same color that one of its in-neighbors has received before. The game ends if no move is possible any more. If every vertex is colored at the end, Alice wins. Otherwise Bob wins. (If Bob wins, then there is an uncolored vertex among the in-neighbors of which all colors of C occur.) The *game chromatic number* of a digraph is the minimum cardinality of a color set C such that Alice has a winning strategy for the game. In Chapter 4 and 5 we will study the game chromatic number defined like this.

As Bodlaender's game has been generalized to asymmetric and defective graph coloring games we will also generalize our digraph coloring game described above to asymmetric moving rules and the coloring with defect. The resulting game that extends all the games mentioned in the last two paragraphs is presented in Section 1.5. For Chapter 2 we need this generalized definition of the game. In Chapter 3 we need only a part of this generalization, namely asymmetric digraph coloring games. In this case, the parameters we study are bounded by other parameters defined by a marking game. This marking game is introduced in Section 1.6. Examples are given in Section 1.7 and an abstract of previous results concerning graph coloring games in Section 1.8.

1.1 Digraphs

A *digraph* is a pair $D = (V, E)$, where V is a finite set and $E \subseteq V \times V$. The elements of V are called *vertices*, the elements of E *arcs*. An arc (v, v) is called a *loop*. A set of two arcs of the form $\{(v, w), (w, v)\}$ with $v \neq w$ is called an *edge* and denoted by vw or wv. An arc that is not contained in any edge is called *single arc*.

In this thesis we will primarily consider two special types of digraphs. The first type are digraphs $D = (V, E)$ without loops, and with the property that whenever $(v, w) \in E$ then $(w, v) \in E$. Such a digraph is called a *graph* as usual. Hence a graph is uniquely determined by its vertex set and the set of its edges. The second important type of digraphs are *simple digraphs*. These are characterized by the property that whenever (v, w) is an arc then (w, v) is not an arc. This implies that a simple digraph has no loops. In other words, a graph is a digraph without single arcs (in particular without loops), and a simple digraph is a digraph without loops and edges. Examples are given in Fig. 1.1.

Sometimes a simple digraph D is also called *orientation* of a graph G. The latter notion is motivated by the idea of depicting a digraph in such a way that its vertices are distinct points and its arcs are arrows between corresponding points. Whenever an edge is depicted in this way, an arrow and its anti-arrow are often replaced by a single straight line. Thus reorienting a graph, i.e. replacing all its straight lines by arrows, gives the picture of a simple digraph. In Fig. 1.2 two ways of depicting a digraph are illustrated. Formally, D could be obtained from G by deleting exactly one arc of each edge, whereas G is obtained from D by adding the anti-arc (w, v) for each arc (v, w).

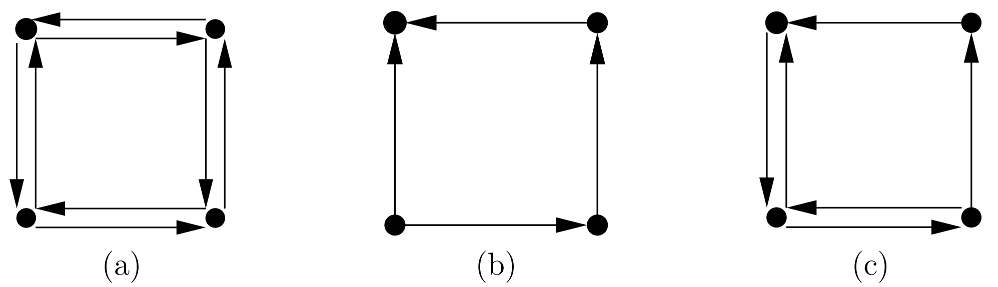


Figure 1.1: (a) A graph. (b) A simple digraph. (c) A digraph which is neither a graph nor a simple digraph.

Let $D = (V, E)$ be a digraph. If $e = (v, w) \in E$, then v (resp. w) is *incident* with e, and v and w are *adjacent* if e is not a loop. In the same way, if $e = vw$ is an edge, then v is *incident* with e. If $e = (v, w)$ and $f = (x, y)$ are two different arcs with $v = x$ or $v = y$ or $w = x$ or $w = y$, then e and f are called *adjacent*. Likewise, an edge $e = vw$ and an arc $f = (x, y)$ are *adjacent* if $f \notin e$, and $v = x$ or $v = y$ or $w = x$ or $w = y$. Two different edges $e = vw$ and $f = xy$ are *adjacent* if $v = x$ or $v = y$ or $w = x$ or $w = y$.

For a digraph $D = (V, E)$ we define $D^0 = (V, E^0)$ as the digraph with

$$E^0 = E - \{(v, v)|v \in V\}$$

as arc set. D^0 will be called *loop deletion digraph* of D. Let $v \in V$. An arc of type (w, v) is called *in-arc* of v, an arc of type (v, w) *out-arc* of v. The loop (v, v) is the only arc that is in-arc and as well out-arc of v. The *in-neighborhood* $N_D^+(v)$ of v is the set of all w for which there is an arc (w, v) in D^0. Its elements are called *in-neighbors* of v. The *out-neighborhood* $N_D^-(v)$ of v is the set of all w for which there is an arc (v, w) in D^0. Its elements are called *out-neighbors* of v. We further define the *in-degree* (or simply *degree*) of v as $d_D(v) = d_D^+(v) = \#N_D^+(v)$, and the *out-degree* of v as $d_D^-(v) = \#N_D^-(v)$. The *total degree* of v is $d_D^\pm(v) = d_D^+(v) + d_D^-(v)$. Whenever it is clear from the context which digraph we consider we omit the subscript D in the preceding

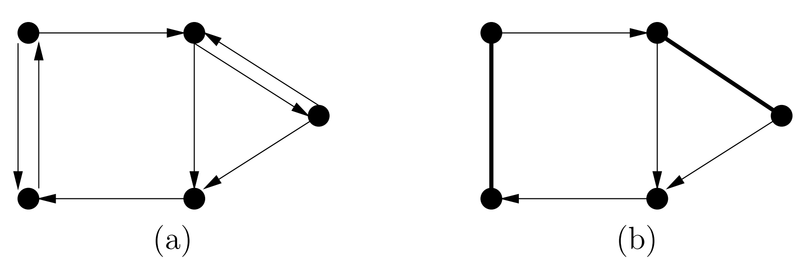


Figure 1.2: Two ways of depicting a digraph: (a) indicating each arc (b) considering oppositely directed arcs as a unit

notations. By

$$\delta(D) = \delta^+(D) = \min_{v \in V} d_D^+(v), \quad \delta^-(D) = \min_{v \in V} d_D^-(v), \quad \delta^\pm(D) = \min_{v \in V} d_D^\pm(v)$$

we denote the *minimum in-degree* (or simply *minimum degree*), *minimum out-degree*, *minimum total degree*, respectively. By

$$\Delta(D) = \Delta^+(D) = \max_{v \in V} d_D^+(v), \quad \Delta^-(D) = \max_{v \in V} d_D^-(v), \quad \Delta^\pm(D) = \max_{v \in V} d_D^\pm(v)$$

we denote the *maximum in-degree* (or simply *maximum degree*), *maximum out-degree*, *maximum total degree*, respectively. A *sink* of D is a vertex v with $d_D^-(v) = 0$.

Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be digraphs. D_1 is a *subdigraph* of D_2 if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. D_1 is an *induced subdigraph* of D_2 if $V_1 \subseteq V_2$ and $E_1 = E_2 \cap (V_1 \times V_1)$. From now on let D_2 be a graph. Then D_1 is a *subgraph* of D_2 if D_1 is a subdigraph of D_2 and D_1 is a graph. D_1 is an *induced subgraph* of D_2 if D_1 is an induced subdigraph of D_2. Note that an induced subdigraph of a graph is always a graph. D_1 is the *true complement* of D_2 if $V_1 = V_2$ and $E_1 = (V_1 \times V_1) - E_2$. The true complement of a digraph D is denoted by $\overline{D}$. The *complement* of a graph G is $\left(\overline{G}\right)^0$, which is a graph.

The graph $(V, V \times V)^0$ is called a *clique* (or *complete graph*), and the *size* of the clique is defined to be $\#V$. Let D be a digraph. The *clique number* $\omega(D)$ of D is the largest size of a clique which is an induced subdigraph of D^0. For example, the clique number of a simple digraph is 1.

1.2 Classes of graphs

The complete graph of size n is denoted by K_n. K_1 is also called *isolated vertex* or *trivial graph*.

The graph $G = (V, E)$ with two sets V_1 and V_2, $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$, $\#V_1 = m$, $\#V_2 = n$, and $E = (V_1 \times V_2) \cup (V_2 \times V_1)$ is called *complete bipartite graph* $K_{m,n}$. The graph which is formed by $K_{m,n}$ without a *matching* of cardinality k, i.e. the union of k pairwise nonadjacent edges, is denoted by $K_{m,n} - M_k$. A subdigraph of a complete bipartite graph is called *bipartite digraph*, a subgraph of a complete bipartite graph *bipartite graph*.

A *path* $v_1 v_2 \dots v_n$ (of *length* $n-1$), $n \geq 1$, is a digraph on the vertex set $\{v_1, v_2, \dots, v_n\}$, with the properties that for every $i = 1, 2, \dots, n-1$ there is an arc (v_i, v_{i+1}) or/and an arc (v_{i+1}, v_i), and there are no further arcs. A graph is an *undirected path* P_n if it is a path of length $n-1$. A path containing the vertices v and w is a *shortest path* between v and w if it has minimal length among all paths containing v and w. A digraph $D = (V, E)$ is *connected* if for every pair of vertices $v_1, v_2 \in V$ there is a path $P_{v_1,v_2} = (W, F)$ with

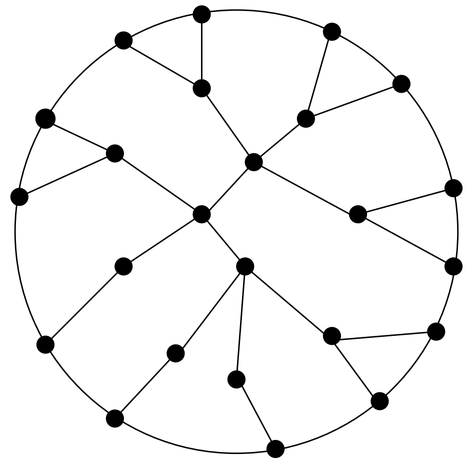


Figure 1.3: A Halin graph

$v_1, v_2 \in W$. Every digraph D decomposes into *connected components*. These are the maximal connected induced subdigraphs of D.

A digraph on the vertex set $\{v_1, v_2, \ldots, v_n\}$, $n \geq 3$, is a *cycle* or an *n-cycle* if for every $i = 1, 2, \ldots, n-1$ there is an arc (v_i, v_{i+1}) or/and an arc (v_{i+1}, v_i), and there is an arc (v_1, v_n) or/and an arc (v_n, v_1), and there are no further arcs. A graph is an *undirected cycle* C_n if it is an n-cycle. A digraph on the vertex set $\{v_1, v_2, \ldots, v_n\}$ is a *directed cycle* $\vec{C}_n$ if for every $i = 1, 2, \ldots, n-1$ there is an arc (v_i, v_{i+1}), and there is an arc (v_n, v_1), and there are no further arcs. The *girth* $g(D)$ of a digraph D is the length of its smallest induced n-cycle, $n \geq 3$, or infinity if there is no such cycle. Note that in our definitions of path, cycle, and girth we do not distinguish between arcs and edges.

A digraph is *acyclic* if it has no directed cycle $\vec{C}_n$, $n \geq 2$, as an induced subdigraph. Acyclic digraphs are always simple digraphs.

A *forest* is a digraph without n-cycles, for $n \geq 3$, as (induced) subdigraphs. Note that, by this definition, a forest needs *not* to be acyclic since any edge is a $\vec{C}_2$. An *undirected forest* is a forest that is a graph, and a *directed forest* is a forest that is a simple digraph. Directed forests are acyclic. We denote the class of all directed forests by $\vec{\mathcal{F}}$ and the class of all undirected forests by $\mathcal{F}$.

Connected forests (resp. connected undirected/directed forests) are called *trees* (resp. *undirected/directed trees*). We call an undirected tree with at most one vertex v of degree $d(v) > 1$ a *star*. An *in-star* is a directed tree with at most one vertex v of in-degree $d^+(v) > 0$.

We denote the disjoint union of two sets A and B by $A \uplus B$. Let $C = (W, F)$ be an undirected cycle drawn in the plane without edge crossings. Let $T = (V \uplus W, E)$ be an undirected tree drawn in the plane without edge crossings, so that W are exactly the vertices of degree 1 in T. Then $(V \cup W, E \cup F)$ is called a *Halin graph*. Such a Halin graph is called a *wheel* W_n if T is a star with $n+1$ vertices. An example of a Halin graph that is not a wheel is depicted in Fig. 1.3.

Sometimes it is useful to consider graphs with certain decomposition properties. The most important classes of graphs with such properties are k-degenerate graphs. A digraph D is *k-degenerate* if every induced subdigraph has a vertex v of degree $d^+_D(v) \leq k$.

1.3 Surfaces

A *surface* is a 2-dimensional manifold. We, however, will always additionally assume that a surface be closed and connected. *Closed* means that the surface is compact and and has no boundary. *Connected* means that for every pair (a, b) of points of the surface S there is a continuous map Φ from the unit interval $[0, 1]$ to S with $\Phi(0) = a$ and $\Phi(1) = b$. The simplest surface in our sense is the (2-dimensional) *sphere* S_0.

$$S_0 = \{x \in \mathbb{R}^3 \,|\, ||x||_2 = 1\}.$$

Here $||\cdot||_2$ denotes the Euclidean norm. Whenever we talk about surfaces we do not care about homeomorphic surfaces which are considered the same although they are not identical. Two surfaces S and T are *homeomorphic* if there is a bijective function $f:\ S \longrightarrow T$ which is continuous and whose inverse function f^{-1} is continuous, too.

There are two types of surfaces: orientable surfaces and nonorientable surfaces. It is well known that these can be obtained from the sphere by the following operations (see [45]). The *orientable surface* S_γ is obtained from the sphere by attaching γ handles to it. Formally, a handle is attached in the following way: first cut two circular holes in the sphere, and then stick together the two boundaries of the holes with the two boundaries of a compact cylinder. The *nonorientable surface* $N_{\overline{\gamma}}$ can be constructed by cutting $\overline{\gamma}$ holes in the sphere and replacing them by Möbius bands, i.e. glueing together the boundary of a hole and the boundary of a Möbius band.

Both numbers, either the number of handles or the number of inserted Möbius bands, are characteristic of a surface and yield a complete classification. γ is called the *genus* of the orientable surface S_γ, whereas $\overline{\gamma}$ is called the *crosscapnumber* of the nonorientable surface $N_{\overline{\gamma}}$. These notions can be extended to digraphs as we will see.

An *embedding* of a digraph in a surface S is a drawing of the digraph on S without arc crossings. By drawing a digraph on S, the surface is cut into different *regions* or *faces* which are the connected components of S minus the drawing of the digraph. The embedding is *cellular* if every face is homeomorphic to the plane $\mathbb{R}^2$.

Every digraph can be embedded in some surface S_γ and in some surface $N_{\overline{\gamma}}$. This is easy to see: if we start with a drawing of the graph on the sphere with arc crossings, then an arc crossing can be avoided by attaching a handle. In the same way an arc crossing can be avoided by inserting a Möbius band in the open neighborhood of the crossing point. Iteration gives a surface in which the digraph can be embedded. For a digraph D, the smallest γ such that D can be embedded in S_γ is called the *genus* $\gamma(D)$ of D. Likewise, the smallest $\overline{\gamma}$ such that D can be embedded in $N_{\overline{\gamma}}$ is called the *crosscapnumber* $\overline{\gamma}(D)$ of D.

Let $D = (V, E)$ be a connected digraph and F be the set of faces in a cellular embedding of D in a surface S. Then

$$\chi(S) = \#V - \#E + \#F$$

is called the *Euler characteristic* of S. It turns out that this is indeed well-defined, i.e. the Euler characteristic does not depend on the digraph or the embedding. It is well-known that $\chi(S_\gamma) = 2 - 2\gamma$ and $\chi(N_{\overline{\gamma}}) = 2 - \overline{\gamma}$. Details on these facts of topological graph theory can be found in [45].

The surfaces of nonnegative Euler characteristic are the sphere S_0, the *torus* S_1, the *projective plane* N_1, and the *Klein bottle* N_2. From a structural point of view the digraphs that can be embedded in the sphere are the most simple. These digraphs are called *planar digraphs* and their embeddings in the sphere are *planar embeddings*.

A graph without vertices of degree 2 is called an *irreducible graph* for the surface S or an *obstruction* for the surface S if the graph itself cannot be embedded in S but any of its subgraphs can be.

1.4 The dichromatic number of a digraph

Let $D = (V, E)$ be a digraph and $k \geq 0$ be an integer. A *k-coloring* of D is a color assignment $c : V \longrightarrow C$ with a color set of cardinality $\#C = k$ such that $c(V) = C$ and, for every color $i \in C$, the subdigraph induced by the vertices $c^{-1}(i)$ of color i is acyclic. Sometimes a k-coloring is also called *coloring* or *acyclic coloring*. The *dichromatic number* $\chi(D)$ of D is the smallest number k of colors, so that a k-coloring of D exists. In this way the dichromatic number was introduced by Neumann-Lara [69] in 1982.

Víctor Neumann-Lara was a Mexican mathematician. He was born in 1933 and died in 2004 (see [81]). His remarkable results on the dichromatic number [69, 74, 73, 70, 71, 42, 72] remained widely unstudied. This is a pity, since the dichromatic number of a digraph is the most fascinating and most natural generalization of the chromatic number of a graph. Indeed, for a graph G, $\chi(G)$ is exactly the *chromatic number* of G [69].

From a different point of view the dichromatic number could be defined in an algorithmic way. Let C be a color set. A *feasible coloring* of D is an ordering $v_1 < v_2 < \cdots < v_n$ of the elements of V together with a color assignment $c: V \longrightarrow C$ with a property (P_c). The ordering means that v_1 is colored first, then v_2, and so on, and v_n is the last vertex to be colored. Property (P_c) says that a vertex v_k may be colored with color i if it has no in-neighbor colored with i in the current digraph induced by $\{v_1, v_2, \ldots, v_k\}$, and after vertex v_k is colored every in-arc (w, v_k) is deleted. So at the end, if every vertex is colored, every arc will be deleted. The dichromatic number is then the minimum cardinality of a color set C such that D has a feasible coloring. Note that the color classes, i.e. the sets of vertices which have the same color, can be colored only in this way if starting with a sink in an acyclic digraph, deleting all in-arcs and iterating.

The main goal of this thesis is to examine a game-theoretic analogon of the dichromatic number of a digraph. This game-theoretic analogon can be seen as a generalization of the algorithmic point of view of the dichromatic number.

1.5 A digraph coloring game

We consider the following 2-person game which is played by the players Alice and Bob. We start with a digraph D that is uncolored, and a given color set C, and nonnegative integers d, a, and b. The players alternately color uncolored vertices of D with colors from C until one of the following conditions applies. Although a color can be used for several vertices, a vertex can have only one color. Alice colors a vertices in a turn, Bob b vertices, except if at the beginning of Alice's (resp. Bob's) last move only $x \leq a$ (resp. $x \leq b$) uncolored vertices are left, Alice (resp. Bob) colors only x vertices. Whenever a player colors a vertex v with color i, then every in-arc (w, v) is deleted in D except for those in-arcs (w, v) for which w has been already colored with i. In particular, this rule means that a loop (v, v) is always deleted when v is colored. This is the so-called *arc deletion rule.* For any color i, the remaining arcs in the subdigraph induced by the vertices of color i form the *defect digraph* D_i of color i. The main rule the players have to respect is that at any state of the game for any color i the defect digraph D_i must have maximum total degree of at most d. If they cannot respect this rule the game is over. Alice wins if every vertex is colored at the end of the game (or if $a = b = 0$), otherwise Bob is the winner. In order to make the game well-defined we have to specify which player has the first move and whether missing a turn is allowed for a player. We will consider four variants g of the game. The first variant $g = g_A$ is when Alice has the first move and passing is not allowed. The second variant $g = g_B$ means that Bob starts and passing is not allowed. In the variant $g = A$ Alice is allowed to move first and to miss one or several turns. In the variant $g = B$ Bob has these rights. Missing a turn includes the right to color less vertices in a move than usually required. Whenever we do not specify g the statements we prove for this game are true for all variants of the game. This game is called *(directed)* (a, b)*-coloring game with defect* d or *d-relaxed (directed)* (a, b)*-coloring game.* In the case $d = 0$ we may omit the term 0-relaxed.

Definition 1.1. A *strategy* of a player X is a function that assigns a feasible move of X to every possible (non-end) state of the game at the time when it is the turn of X. (A state of the game is *non-end* if further moves are possible.) A *winning strategy* of a player X is a strategy $\mathcal{S}_0$ of X, so that, for any strategy the other player uses, X wins when playing according to strategy $\mathcal{S}_0$.

Since the game described above is a noncooperative finite two player zero-sum game with perfect information in which no ties are possible, either Alice or Bob has a winning strategy.

Definition 1.2. The smallest cardinality $n = \#C$ of a color set C for which Alice has a winning strategy for the directed (a, b)-coloring game with defect d played on the digraph D is called *d-relaxed* (a, b)*-game chromatic number* ${}^{(a,b)}\chi_g^d(D)$ of D. For a nonempty class $\mathcal{C}$ of digraphs we define

$$ {}^{(a,b)}\chi_g^d(\mathcal{C}) = \sup_{D \in \mathcal{C}} {}^{(a,b)}\chi_g^d(D). \tag{1.1} $$

Observation 1. *Let D be a digraph and $a, b, d \geq 0$. Then*

$$ {}^{(a,b)}\chi_A^d(D) \leq {}^{(a,b)}\chi_{g_A}^d(D) \leq {}^{(a,b)}\chi_B^d(D), $$

$$ {}^{(a,b)}\chi_A^d(D) \leq {}^{(a,b)}\chi_{g_B}^d(D) \leq {}^{(a,b)}\chi_B^d(D). $$

Observation 2. *Let D be a digraph and $a, b, d \geq 0$. Then*

$$ {}^{(a,b)}\chi_g^d(D) = {}^{(a,b)}\chi_g^d(D^0). $$

Proof. This follows from the special case of the arc deletion rule concerning loops. $\square$

We further define:

$$ \begin{aligned} {}^{(a,b)}\chi_g(D) &= {}^{(a,b)}\chi_g^0(D), \\ \chi_g^d(D) &= {}^{(1,1)}\chi_g^d(D), \\ \chi_g(D) &= \chi_g^0(D) = {}^{(1,1)}\chi_g(D). \end{aligned} $$

Definition 1.3. ${}^{(a,b)}\chi_g(D)$ is called (a,b)*-game chromatic number* of D, $\chi_g^d(D)$ is called d*-relaxed game chromatic number* of D, and $\chi_g(D)$ is called *game chromatic number* or g*-game chromatic number* of D. These notations extend to nonempty classes of digraphs as in (1.1).

The parameters defined above for arbitrary digraphs generalize some well-known graph parameters. This is: let G be a graph, then $\chi_{g_A}(G)$ is the game chromatic number of G as it was introduced by Bodlaender [15]. Note that, if we assume $d = 0$ in our model, when the first vertex v of an edge vw is colored, then the arc (v, w) is not deleted, forcing w to be colored with a color different from v. The (a, b)-game chromatic number of G, which was introduced by Kierstead [53], is the same as ${}^{(a,b)}\chi_{g_A}(G)$. Finally, $\chi_{g_A}^d(G)$ is the d-relaxed game chromatic number of G introduced by Chou et al. [27]. In the game of Chou et al., which is defined on an undirected graph, the players have to color the vertices in such a way that the subgraphs induced by the color classes have maximum degree of at most d. Note that in our model exactly one of the two arcs of an edge between two vertices of the same color remains undeleted. Therefore the maximum total degree is the right measure to obtain the maximum degree of the subgraph of vertices of the same color in the original graph, which is considered in the model of Chou et al. For graphs, the game chromatic number of several classes of graphs has been determined, as well as the d-relaxed game chromatic number for several d, see Section 1.8.

We remark that even the noncompetitive parameters (where $a = 0$ or $b = 0$) are interesting or well-known parameters. If only Alice is playing, she tries to use as few colors as possible, and if only Bob is playing, he tries to use as many colors as possible. The results are given in the following observations.

Observation 3. ${}^{(0,b)}\chi_g(D) = \Delta^+(D) + 1$ *for* $b \geq 1$.

Proof. A winning strategy for Bob with $k \leq \Delta^+(D)$ colors is the following: Bob chooses a vertex v with $d^+(v) = \Delta^+(D)$ and colors k in-neighbors of v with k distinct colors. Since v cannot be colored any more, Bob wins. □

Observation 4. ${}^{(a,0)}\chi_g(D) = \chi(D)$*, for* $a \geq 1$.

In Observation 4, $\chi(D)$ is the *dichromatic number* of D which was introduced by Neumann-Lara [69]. In the case of graphs it is the *chromatic number* of a graph D.

A trivial upper bound for the d-relaxed (a, b)-game chromatic number is given in the next observation.

Observation 5. ${}^{(a,b)}\chi_g^d(D) \leq \Delta^+(D) + 1$ *for any* $a, b \geq 0$.

In most parts of this thesis we will only consider the simplified model of game chromatic numbers. In this model, the defection digraphs during the game are unions of isolated vertices. Thus, a vertex can be colored with color i only if it has no in-neighbors colored with i. In this way the color classes are acyclic (in the original digraph) as in the definition of the dichromatic number.

1.6 A digraph marking game

In order to bound the (a,b)-game chromatic number another game-theoretic model simplifies our considerations a lot. It is the idea of a marking game. Two players, Alice and Bob, are given an initially uncolored digraph D and a score n which is a nonnegative integer. The players alternately mark vertices of D, Alice marks a vertices in a turn, Bob b vertices. However, the last move may be incomplete if there are no vertices to mark any more. Again, we have an *arc deletion rule*: whenever a player marks a vertex v all in-arcs of v are deleted. The main rule of the game is that a vertex may be marked only if it has at most $n-1$ marked in-neighbors. Eventually the game ends when no vertex may be marked any more by this rule. If every vertex is marked at the end of the game, Alice wins. Otherwise Bob wins. This game is called *(a,b)-marking game*. To be precise, we consider a variant g of the game in which it is specified whether a player has the right to have the first move or the right to miss a turn. In line with the coloring game we have the variants A, B, g_A, and g_B.

Definition 1.4. The *(a,b)-game coloring number* ${}^{(a,b)}\mathrm{col}_g(D)$ of D is the smallest score n for which Alice has a winning strategy for the marking game played on D. We further define the *game coloring number* $\mathrm{col}_g(D)$ of D as the number ${}^{(1,1)}\mathrm{col}_g(D)$. The notations col_g and ${}^{(a,b)}\mathrm{col}_g$ extend to nonempty classes of digraphs as in (1.1).

If Bob wins the (a,b)-coloring game on a digraph D with n colors, then he can win the (a,b)-marking game on D with score n by marking instead of coloring vertices according to his winning strategy for the coloring game. In this way, eventually a vertex will have n marked in-neighbors. Thus we have the following very efficient tool to bound the game chromatic number.

Observation 6. *Let D be a digraph and $a,b \geq 0$. Then*

$$\chi(D) \leq {}^{(a,b)}\chi_g(D) \leq {}^{(a,b)}\mathrm{col}_g(D) \leq \Delta^+(D) + 1.$$

Proposition 7. *Let G be a graph and $\vec{G}$ be an orientation of G. Let $a,b \geq 0$. Then*

$${}^{(a,b)}\mathrm{col}_g(\vec{G}) \leq {}^{(a,b)}\mathrm{col}_g(G).$$

Proof. Assume that Alice has a winning strategy with score n for the (a,b)-marking game played on G. If she plays according to this strategy on $\vec{G}$, she will also win. Note that G and $\vec{G}$ have the same vertices, and $d^+_{\vec{G}}(v) \le d^+_G(v)$ for any vertex v. □

Remark. There is no analogon of Proposition 7 for the coloring game. Consider the complete bipartite graph $K_{2n,2n}$ with partite sets V_1 and V_2, and an orientation $\vec{K}_{2n,2n}$ of $K_{2n,2n}$ with arc set $V_1 \times V_2$. It is well-known that $K_{2n,2n}$ has B-game chromatic number 3 for $n \ge 2$ (see [52]). However, it is easy to see that $\chi_A(\vec{K}_{2n,2n}) = n+1$. A winning strategy for Bob with n colors is the following. In his first n moves he colors n vertices from V_1 with n distinct colors. Then there is an uncolored vertex in V_2 which cannot be colored any more. (Furthermore, a winning strategy for Alice with $n+1$ colors can be easily found.) So, for any variant g,

$$\chi_g(\vec{K}_{2n,2n}) \ge n+1 > 3 \ge \chi_g(K_{2n,2n})$$

for $n \ge 3$. This does not contradict Observation 6 and Proposition 7 since $\mathrm{col}_g(\vec{K}_{2n,2n}) = 2n$ and $\mathrm{col}_g(K_{2n,2n}) = 2n+1$.

1.7 Examples

In this section we will study some easy examples in order to get used to the games defined in the previous sections.

Directed cycles. The first digraphs we consider are some of the most simple ones: directed cycles. They are especially interesting in order to illustrate some of the features of relaxed digraph coloring games.

Example 1.1. $\chi_g(\vec{C}_n) = 2$ *for* $n \ge 2$.

This is obvious: $2 = \chi(\vec{C}_n) \le \chi_g(\vec{C}_n) \le \Delta^+(\vec{C}_n) + 1 = 2$ for $n \ge 2$.

Example 1.2. $\chi^1_{g_B}(\{\vec{C}_1, \vec{C}_2, \vec{C}_3, \vec{C}_4\}) = 1$.

For $\vec{C}_1$, $\vec{C}_2$ and $\vec{C}_3$ the statement is easy to see. So consider a $\vec{C}_4$ with vertices v_0, v_1, v_2, and v_3, and arcs (v_i, v_{i+1}) (index mod 4). Alice has the following winning strategy with one color. W.l.o.g. Bob, in his first move, colors v_0. Alice answers by coloring v_2. In the last moves the other two vertices can be colored. Note that, in spite of the fact that the vertices of color 1 induce the whole digraph, the defect digraph D_1 only contains (v_0, v_1) and (v_2, v_3) as arcs, since the other two arcs have been deleted during the process of the game.

Example 1.3. $\chi^1_A(\vec{C}_n) = 2$ *for* $n \ge 5$.

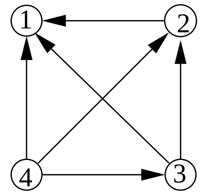


Figure 1.4: The transitive tournament T_4

We prove that Bob has a winning strategy with one color. Let $v_1, \ldots, v_n$ be the vertex set of $\vec{C}_n$, and arcs given by (v_i, v_{i+1}) (index mod n). If Alice uses her right to color a vertex v_k in the first move, Bob colors v_{k+1} and wins since v_{k+2} cannot be colored any more. If Alice misses her first turn, then Bob colors v_0. If Alice colors v_1 after that, Bob wins since v_2 cannot be colored any more. If Alice colors v_2, Bob colors v_3 and wins since v_4 is uncolored and cannot be colored any more (here we need $n \geq 5$.) If Alice colors neither v_1 nor v_2, Bob colors v_1, and v_2 cannot be colored any more. So in every case Bob wins.

The next examples are easy to see.

Example 1.4. $\chi_B^1(\{\vec{C}_1, \vec{C}_2\}) = 1.$

Example 1.5. $\chi_{g_A}^1(\vec{C}_n) = 2$ *for* $n \geq 3$.

From these results we can obtain the other values of 1-relaxed game chromatic numbers of directed cycles using Observation 1. We remark further

Example 1.6. $\chi_g^d(\vec{C}_n) = 1$ *for* $d \geq 2$.

Transitive tournaments. A *tournament* is an orientation of a complete graph. It is *transitive* if it is an acyclic digraph, i.e. if there is a linear order $>$ on its vertex set, so that there is an arc (v, w) if and only if $v > w$. Let T_n be the transitive tournament with n vertices. See Fig. 1.4.

Example 1.7. $\chi_{g_A}(T_n) = \left\lceil \frac{n}{2} \right\rceil.$

Note that $\chi(T_n) = 1$. This is a remarkably easy example where the difference between the dichromatic and the game chromatic number may be arbitrarily large.

In order to prove $\chi_{g_A}(T_n) \leq \left\lceil \frac{n}{2} \right\rceil = c$, consider the following winning strategy of Alice with c colors. Alice simply colors the smallest uncolored vertex with the smallest feasible color. In order to stop Alice using a color, Bob has to color another vertex with that color. So, in the first $2c-2$ moves, Bob will have made at most $c-1$ colors infeasible. The last color can be used for the remaining one or two vertices if the smaller one is colored first. This can be achieved by Alice, hence she is going to win. The other direction $\chi_{g_A}(T_n) \geq c$ is easy to see. Bob has the following strategy. In his first $c-1$ moves he colors the biggest $c-1$ vertices with distinct colors, starting with the biggest one. If there are at most $c-1$ colors this is a winning strategy for Bob.

Example 1.8. $\chi_{g_B}(T_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

This is the same argument as in the previous example. As Bob has the first move an additional color has to be used if n is even.

1.8 Previous results

The game chromatic number of a graph was introduced in 1991 by Bodlaender [15] who considered the complexity of different graph coloring games. A special case of the coloring game considered in this thesis (with graphs as instances and $a = b = 1$ and $d = 0$), which defines the game chromatic number, was called "*coloring construction game*" by Bodlaender. He also considered a variant, the "*sequential coloring construction game*", where the order of the vertices to be colored is prespecified. Bodlaender proved that the sequential coloring construction game with three or more colors is $\mathcal{P}$**SPACE**-complete, and it is in $\mathcal{P}$ for two colors. However, the complexity of the coloring construction game with two or more colors is still an open problem. Another variant of the coloring construction game Bodlaender considers is the "*coloring game*". Here the winning rule is changed: a player wins if his opponent is unable to move. This game is $\mathcal{P}$**SPACE**-complete for one color [78] or two colors [15]. For three or more colors the complexity status is open. In a second article Bodlaender and Kratsch [16], amongst other results, consider the complexity of the sequential coloring construction game for special types of perfect graphs. Since then, complexity results for graph coloring games have not been published any more. Instead, by defining the game chromatic number of a graph, Bodlaender initiated a new field of mathematics, the determination of game chromatic numbers of classes of graphs.

The first results in this direction were those of Faigle et al. [40] who proved in 1993 that $\chi_g(\mathcal{F}) = 4$ for the class $\mathcal{F}$ of forests and determined an upper bound for the game chromatic number of interval graphs. After that, a competition began in order to give an upper bound for the game chromatic number of planar graphs, a bound which was reduced from 33 (Kierstead and Trotter [55], in 1994), 30 (Dinski and Zhu [30], in 1999), 19 (Zhu [86], in 1999), 18 (Kierstead [52], in 2000), to 17 (Zhu [88], recently).[1] Besides reducing the

[1] Wu [84] proved later that the game coloring number of the class of planar graphs is at least 11.

bound, this competition was very fruitful to obtain insights into the general nature of the game chromatic number as a graph parameter. As an upper bound of this graph parameter, Zhu [86] defined the game coloring number of a graph, which is easier to handle and which in a lot of good cases equals or nearly equals the game chromatic number of classes of graphs. By the methods of Kierstead [52], many upper bounds for the game coloring and thus for the game chromatic numbers of classes of graphs that were already known could be regained. One such class is, for example, the class of outerplanar graphs which has game chromatic number of at most 7 by a result obtained by Guan and Zhu [46] in 1999.

Not only planar graphs were considered, but also graphs embeddable in an orientable surface and planar graphs with special properties. By examining the game coloring number of the new class of pseudo partial k-trees, in 2000 Zhu [87] proved $\mathrm{col}_g(G) \leq \lfloor (3\sqrt{1+48\gamma}+23)/2 \rfloor$ for any graph embeddable into S_γ with $\gamma \geq 1$. Kierstead [52] improved this bound in the same year to the value $\lfloor (3\sqrt{73+96\gamma}+41)/4 \rfloor$. A paper of He et al. [48] of 2002 considers the game coloring number of planar graphs without 4-cycles and with given girth. A more detailed description of the results of He et al. is included in Chapter 3, as well as the generalization of these results to graphs embeddable in other surfaces. A generalization of the results of He et al. to surfaces of nonnegative Euler characteristic was recently given by Wang [80]. Another special type of planar graphs was analyzed by Wu [83] who, in 2001, showed that every Halin graph (with two exceptions) has g_A-game chromatic number 4. Here, finding a lower bound is the most difficult part of the work, whereas all previous authors concentrated on upper bounds.

Whenever a mathematical theory comes up, interesting variants of the theory are considered soon. This also applied to the game chromatic number. Chen et al. [26] introduced a "*new game chromatic number*" in 1997 where in the underlying game Bob is only allowed to take colors that are already used except if such a move is not possible any more. One of the results of Chen et al. is: the new game chromatic number of a tree is at most 3. Compared with the result of Faigle et al. [40], one little sharpening rule in the game gives a gain of one color.

Another variant of the game chromatic number is the *game chromatic index* of a graph G, which is the game chromatic number of the line graph $L(G)$ of G. In 2001 Cai and Zhu [24] introduced the game chromatic index and gave an upper bound for this parameter of k-degenerate graphs (in terms of k and the maximum degree Δ). In case of special types of trees with maximum degree 3 they tightened this bound. Continuing their work Erdös et al. [37] and the author [1, 3] proved that the game chromatic index of a forest of maximum degree $\Delta \neq 4$ is at most $\Delta + 1$. In the case $\Delta \geq 5$, these results were generalized by Marte [63] to the list game chromatic index.

The list game chromatic number was recently introduced independently by Borowiecki et al. [22]. They characterize all graphs that have list g_A-game chromatic number 2. A similar characterization of all graphs with g_A-game chromatic number 2 was given in 2007 by Borowiecki and Sidorowicz [21]. The latter result coincides with a result we develop in Chapter 5.

One should mention that there is a variant of the game chromatic number of oriented graphs which is not related to the model described in this thesis.[2] Some results on this "*oriented game chromatic number*" are given by Nešetřil and Sopena [68], Kierstead and Trotter [56], and Kierstead and Tuza [57]. Another variant of the game coloring number is the *complete game coloring number*. Yang [85] introduced this number and proved that the complete game coloring number of the line graph of a forest of maximum degree Δ is at most $\Delta + 1$. This number, as well, is not related to the model described in this thesis.

When in 2003 Chou et al. published their paper [27] on the "*relaxed game chromatic number*" of graphs, a new chapter in the theory of graph coloring games was established. They proved $\chi_g^d(F) \leq 3$ for $d \geq 1$ and a forest F, and $\chi_g^d(O) \leq 6$ for $d \geq 1$ and an outerplanar graph O. He et al. [49] continued their research in 2004 and proved that $\chi_g^d(F) \leq 2$ for $d \geq 2$, and $\chi_g^d(O) \leq 5$ for $d \geq 2$, and that, for any $d \leq 4$, there is an outerplanar graph O_d with $\chi_g^d(O_d) \geq 3$. The bounds for forests are tight. Dunn and Kierstead [32, 34] proved for every partial k-tree P_k that $\chi_g^d(P_k) \leq k+1$ for $d \geq 4k-1$. They also obtained a similar result for graphs with bounded k-admissibility. As a corollary they deduced $\chi_g^d(P) \leq 6$ for $d \geq 93$ and a planar graph P. For an (a, b)-pseudo partial k-tree G, Dunn and Kierstead [32, 35] obtained $\chi_g^d(G) \leq k+1$ if $d \geq 2k^2+3k+2ak+2kb+2ab+3b+2$. Furthermore, if G is a graph with a partial (k, D)-decomposition, then $\chi_g^d(G) \leq k+1$ if $d \geq k^2+3k+2kD+3D+1$. Combining this result with the results of He et al. [48], Dunn and Kierstead obtained the upper bound 2 for the d-relaxed game chromatic number of planar graphs with bounded girth and without 4-cycles, for sufficiently large d. Dunn and Kierstead [32, 36] also considered outerplanar graphs O and proved $\chi_g^d(O) \leq 2$ for $d \geq 8$. Dunn [32] also showed a remarkable irregularity of game chromatic numbers, namely that for every $k \in \mathbb{N}$ there exists a graph with game chromatic number k and 1-relaxed game chromatic number greater than k. In 2005 Wu [84] completed the examination of the d-relaxed game chromatic numbers of outerplanar graphs O by proving $\chi_g^d(O) \leq 7-t$ for $t = 2, 3, 4$ and $d \geq t$, and $\chi_g^d(O) \leq 2$ for $d \geq 6$.

Moreover, Wu [84] identified the game coloring number of the class of partial k-trees as $3k + 2$. Perhaps more importantly, Wu discovered that the game coloring number of a graph is a monotone parameter, which is not the case for the game chromatic number. *Monotone* means that the parameter is

[2]E.g., the oriented game chromatic number of the class of paths is 7, whereas the game chromatic number of an oriented path is at most 3.

at least as big for a graph as for any subgraph.

There are also a few results concerning the d-relaxed game chromatic number of line graphs of some graphs. Dunn [32, 33] examined line graphs $L(T_\Delta)$ of trees of maximum degree Δ. He proved $\chi_g^d(L(T_\Delta)) \leq \Delta + 1$ if $d \geq 1$ and $\chi_g^d(L(T_\Delta)) \leq \Delta$ if $d \geq 3$. In addition, he obtained $\chi_g^d(L(G_k)) \leq \Delta + k - 1$ if $d \geq 2k^2 + 4k$ for line graphs $L(G_k)$ of k-degenerate graphs. In 2006, Chang and Zhu [25] proved, for any k-degenerate graph G_k with maximum degree Δ, that $\chi_{g_A}^d(L(G)) \leq 2k + \frac{(\Delta+k-1)(k+1)}{d-2k^2-4k+2}$ if $d \geq 2k^2 + 5k - 1$. Remarkably, in this result we have d in the denominator of the upper bound, as we will have in the Theorems 12 and 17.

Asymmetric graph coloring games were introduced in 2005 by Kierstead [53] who solved these games completely for the class of forests. Kierstead and Yang [58] proved that, if a graph G has an orientation with maximum out-degree k, then ${}^{(k,1)}\chi_g(G) \leq 2k + 2$. In a further paper of 2006, Kierstead [54] examined the $(2, 1)$-coloring game on planar graphs.

A recent paper of Bohman et al. [17] considers the game chromatic number of random graphs. Its results suggest that for a random graph G the game chromatic number is high (at least $2\chi(G)$), whereas the trivial lower bound for the game chromatic number of G is $\chi(G)$.

Chapter 2

Forests

2.1 Coloring games on directed forests

Directed forests are a class of digraphs which is simple enough in order to examine the d-relaxed (a, b)-game chromatic number in its whole generality with the use of only one method. Before obtaining the general result, we will work out two special cases. The first special case concerns the game chromatic number of the class of directed forests. This number is already known from the author's diploma thesis [1]. Here we will obtain an upper bound for it as a corollary of the more general upper bound for the game coloring number of directed forests.

Let F be the orientation of a forest. We consider either the coloring game of Section 1.5 or the marking game of Section 1.6. During the game played on F we update F according to the arc deletion rule of the game. So, at every state of the game F has a decomposition into more and more connected components. Such a component will be called a *trunk*. Obviously, marking or coloring in different trunks does not depend on each other.

In the next theorems, we will distinguish between global and local sinks. A *global sink* or simply a *sink* in a digraph is a vertex with out-degree 0 and arbitrary in-degree. In contrast to that, a *sink in a path* or *local sink* is a vertex with out-degree 0 and in-degree 2 in the path, which may have higher out- and in-degree in the digraph of which the path is a subdigraph.

Theorem 8. $\mathrm{col}_g(\vec{\mathcal{F}}) \leq 3$.

Proof. Let F be a directed forest. Alice's winning strategy with score 3 guarantees that after each of her moves every trunk has at most one marked vertex. If after Bob's move every trunk has still at most one marked vertex, Alice simply marks a global sink, and her invariant is not destroyed. If Bob marks a vertex w in a (not completely unmarked) trunk T, then let v be the previously marked vertex of T. If the last arc on the path from v to w is directed towards w, then after Bob's move v and w are in different trunks and we are in the previous case. Otherwise, there is a (local) sink x in the path from v to w. Alice marks x which is markable since it has at most 2 marked in-neighbors (v and w). After her move v, w, and x are in three different trunks, and her invariant is reinstalled. By induction, Alice wins. $\square$

Corollary 9. **[1]** $\chi_g(\vec{\mathcal{F}}) = 3$.

Proof. The upper bound 3 follows from Theorem 8 and Observation 6. The lower bound 3 will be proven as a part of Proposition 14 (see page 27). $\square$

Now we consider the second special case, i.e. the d-relaxed graph coloring games on directed forests for $d \geq 1$ and prove

Theorem 10. $\chi_g^d(\vec{\mathcal{F}}) \leq 2$ *for* $d \geq 1$.

Proof. During the game we consider trunks. We recall that these are defined as follows: whenever a player colors a vertex v with color i, then all arcs (w, v) which point towards v are deleted, except in the case that w has been colored with i before. By this dynamic process the forest is subdivided into more and more trunks. The coloring of different trunks does not depend on each other.

Alice's winning strategy with 2 colors guarantees that after all of her moves every trunk has only colored vertices of at most one color and possibly several uncolored vertices. It also guarantees that the subdigraph induced by the colored vertices of a trunk is connected.

If Bob colors a vertex w in a (not completely uncolored) trunk T, then let v be the colored vertex of T with the shortest distance to w. If the last arc on the path from v to w is directed towards w, then, after Bob's move, either v and w are in different trunks or v and w are adjacent. In both cases Alice's invariant still holds. Otherwise, there is a local sink x in the path from v to w with out-degree 0 (in the path). Then, in general, Alice colors x with a color different from its colored neighbors. The only case when this is not possible is if the path has length 2 and v and w have different colors. Then Alice colors x with the same color as w. Note that the component of the defect digraph containing x and w does not contain other vertices, so that the defect of x and w is at most 1. The new trunk containing x and w does not contain colored vertices of the color of v. So Alice's invariant is reinstalled after her move.

In the cases where her invariant holds after Bob's move she just colors a global sink or a neighbor of a colored vertex v, with a color different from the color of v, without destroying her invariant. By induction, Alice wins. $\square$

The bound is tight as Lemma 13 states (see page 26). It is easy to see that Corollary 9 and Theorem 10 even hold for games where passing is allowed for Bob. In the same way, all results of this section are true for all variants

$g \in \{A, B, g_A, g_B\}$. Theorem 11 generalizes Theorem 8 to asymmetric marking games.

In Theorems 11 and 12 we distinguish between a move and a step of the player. A *move* of Alice (resp. Bob) consists of marking or coloring a (resp. b) vertices (or even less if the players miss a turn). On the other hand, the act of marking or coloring exactly one vertex is called a *step*. So a move consists of at most a (resp. b) steps.

Theorem 11. *${}^{(a,b)}\mathrm{col}_g(\vec{\mathcal{F}}) \leq b+2$ if $a \geq b \geq 1$.*

Proof. Let F be any directed forest. We will prove that Alice has the following winning strategy for the (a,b)-marking game with score $b+2$: at the end of each of her moves she guarantees that in every trunk there is at most one marked vertex.

Then, after Bob's next move, every trunk has at most $b+1$ marked vertices. Assume that Bob has marked $v_1, v_2, \ldots, v_b$. Alice will use b steps to reinstall her invariant. By induction, we may assume that, at the beginning of the k-th step, every trunk contains at most one marked vertex different from the vertices $v_k, v_{k+1}, \ldots, v_b$. This implies that every trunk has at most $b+1-(k-1)$ marked vertices. By Alice's invariant this asssumption is true at the beginning of the first step. We will show that it also holds at the beginning of the $(k+1)$-th step. In the k-th step, we call a marked vertex that is none of the vertices $v_k, v_{k+1}, \ldots, v_b$ *origin vertex*. Consider the k-th step.

If there is no origin vertex in the trunk that contains v_k, then this trunk has at most $b+1-k$ marked vertices ($v_k, v_{k+1}, \ldots, v_b$), and Alice's strategy is simply to mark a global sink (such a sink exists unless all vertices are marked.) By this type of move no unmarked vertex receives a higher number of marked in-neighbors. So every trunk will have at most $b+1-k$ marked vertices, and in every trunk there is at most one origin vertex for the $(k+1)$-th step, i.e. a vertex different from $v_{k+1}, v_{k+2}, \ldots, v_b$.

On the other hand, if there is an origin vertex v in the trunk that contains v_k, then Alice considers the path from v to v_k. As the first and the last arc of the path are directed towards the interior of the path (by the arc deletion rule for v and v_k) this path must have an unmarked local sink w (which is a sink in the path, not necessarily a global sink.) Alice marks w. Since now v, v_k, and w are in three distinct trunks, each of these three trunks has at most $b+1-k$ marked vertices, coming from a big trunk with formerly at most $b+2-k$ marked vertices. As a trunk, at the beginning of the k-th step, may have $b+2-k$ marked vertices only if it contains v_k, at the end of the k-th step every trunk has at most $b+1-k$ marked vertices. v_k and w are new origin vertices, however, they are in different trunks, and in trunks different from v and every other origin vertex. So, at the beginning of the $(k+1)$-th move, in every trunk there will be at most one origin vertex.

After b steps every trunk has at most $b+1-b=1$ marked vertex. In the next $a-b$ (≥ 0) steps Alice marks global sinks without destroying her invariant. Thus at the end of her move every trunk has at most one marked vertex. Since meanwhile no trunk had more than $b+1$ marked vertices, the score of $b+2$ is sufficient. $\square$

We are now ready to prove the main theorem of this chapter, a joint generalization of Corollary 9 and Theorem 10.

Theorem 12. ${}^{(a,b)}\chi_g^d(\vec{\mathcal{F}}) \leq \left\lfloor \frac{b}{d+1} \right\rfloor + 2$ *if* $a \geq b \geq 1$.

Proof. We show that Alice has a winning strategy with $c = \left\lfloor \frac{b}{d+1} \right\rfloor + 2$ colors. Because of Theorem 11 and Observation 6 this is true for $d=0$. So we might assume without loss of generality that $d>0$. However, we do not need this assumption.

During the game, the forest is split into trunks. As in Theorem 10, Alice's strategy guarantees that at the end of each of her moves in every trunk there are only vertices colored in one color (and uncolored vertices), and the colored vertices of a trunk induce a connected subdigraph. According to Theorem 10 this is possible for $(a,b)=(1,1)$. Now consider the more general case $a \geq b \geq 1$.

Alice's new winning strategy works as follows. In a certain way, Alice pretends to play the d-relaxed $(1,1)$-coloring game with Bob. More precisely, for each vertex Bob has colored, Alice, in reaction, colors exactly one vertex. Then her strategy will be reinstalled. This is possible since $a \geq b$. However, Alice will not react on the vertices in the same order as Bob has colored them.

Let $v_1, v_2, \ldots, v_b$ be the vertices Bob has colored during his last move. Alice will construct an order on these vertices, say $v_{i_1}, v_{i_2}, \ldots, v_{i_b}$. After Alice has colored the k-th vertex of her move we define any vertex Alice or Bob have colored during the game to be an *origin vertex*, except the vertices from the list $v_{i_{k+1}}, v_{i_{k+2}}, \ldots, v_{i_b}$ which are *non-origin vertices*. During the game, even more vertices will be deleted from the list of non-origin vertices and be considered as origin vertices. Alice's invariant will be that after each of her steps the subdigraph of the origin vertices of a trunk is connected (and therefore colored with only one color).

The proof will consist of two parts. First we will prove that Alice always finds a suitable vertex to reinstall her invariant. Later we prove that there is always a feasible color for such a vertex.

Consider the beginning of Alice's first step ($k=0$). If the trunk that contains v_1 does not contain origin vertices, Alice simply colors a global sink in some trunk. (If there is no sink any more, the digraph is completely colored.) By this type of move her invariant still holds after her first step since v_1 will be a new origin vertex (possibly connected to the vertex colored by Alice in the same color as v_1), and the vertex Alice has colored is either the only origin vertex in a new trunk or enlarges the set of origin vertices of a certain color in some trunk. Note that if there is an uncolored vertex or a vertex of a different color adjacent to the sink Alice colors, this vertex will be split off the sink by the rules of the game when Alice colors the sink.

Now, once more, consider the trunk that contains v_1 at the beginning of Alice's first step. If this trunk contains origin vertices, then there is a unique path Q between an origin vertex θ_0 and v_1. If every vertex of Q is colored, then Alice considers all vertices of Q (including v_1) as new origin vertices. Note that these vertices are colored in the same color, otherwise the path would be broken. So Alice's invariant still holds, and Alice colors a global sink. Consider the interesting case that not every vertex of Q is colored. Let $Q = \theta_0\theta_1\theta_2 \dots \theta_m v_1$ and let θ_{j_0} be the uncolored vertex with the smallest index in the path. Then Alice defines

$$o = \theta_{j_0-1},$$

and considers the (colored) vertices $\theta_1, \theta_2, \dots, \theta_{j_0-1}$ as new origin vertices. (In case $j_0 - 1 = 0$ the set of these vertices is empty and $o = \theta_0$.) Now let P be the subpath of Q between the (new) origin vertex o and v_1. Since the first and the last arc of this path are directed towards the interior of the path, P contains at least one local sink. (Here a *local sink* is not necessarily a global sink, only a vertex with two in-arcs in the path.) If every local sink of the path is colored, then necessarily every vertex of the path (including v_1) is colored in the same color. In this case Alice colors a global sink and her invariant holds after her step (when $k = 1$). Vertices on the path are considered as origin vertices from now on. Note that v_1 simply enlarges the set of origin vertices of the trunk to which v_1 belongs, and this set is connected.

The remaining case is that P contains at least one uncolored local sink. Let a *source* be a vertex with two out-arcs in the path, and a *transitive vertex* be a vertex with one in- and one out-arc in the path. P is of the form

$$oT_1^+s_1T_1^-w_1T_2^+s_2T_2^-w_2T_3^+s_3T_3^- \dots w_{j-1}T_j^+s_jT_j^- \dots w_{n-1}T_n^+s_nT_n^-v_1, \tag{2.1}$$

where s_i are local sinks, w_i are sources and T_j^+, T_j^- are sets of transitive vertices. In the worst case all sources and transitive vertices are colored by Bob, so that Alice, by coloring a local sink s_j, possibly does not split the trunk to which o, s_j, and v_1 belong. Therefore reordering Bob's vertices $v_1, \dots, v_b$ is necessary. Let $w_{j_1}, w_{j_2}, \dots, w_{j_{h-1}}, v_1$ be the sources (together with v_1) that Bob has colored, in the order of the path from left to right. Then Alice reorders Bob's vertices in the way that we have

$$w_{j_1}, w_{j_2}, \dots, w_{j_{h-1}}, v_1, v_{i_{h+1}}, \dots, v_{i_b},$$

where only the order of the first h vertices is important. Now Alice performs h steps. In the k-th step she considers the path from origin vertices to w_{j_k}, where $w_{j_h} = v_1$, and colors the left-most local sink in this path. (If every sink is colored she colors a global sink.) By such a move either the origin vertices are unified with the next source or Alice performs a split. Colored transitive vertices which connect sources and local sink in a trunk are considered as origin vertices from the time at which the respective local sink is colored. Therefore Alice's invariant holds after the k-th step for any $k \leq h$.

Now this procedure is iterated. Alice's invariant holds after step k_0, where $k_0 = 1$ or $k_0 = h$, depending on which case has been examined. We can forget the vertices $v_{i_1}, \ldots, v_{i_{k_0}}$. Then Alice's strategy works in the same way on the remaining vertices $v_{i_{k_0}+1}, \ldots, v_{i_b}$ Bob has colored. The next vertex to be considered is $v_{i_{k_0}+1}$. After b steps every colored vertex is origin vertex, and Alice's invariant holds, therefore every trunk has only vertices of one color, which induce a connected subdigraph. During the remaing $a - b$ steps, Alice colors global sinks without affecting her strategy.

We have seen that Alice can always choose a vertex in order to reinstall her strategy, but the question arises whether she can always find a feasible color. Therefore we consider the color weight of a vertex. A colored vertex v has *color weight* $g(v) = n$ if exactly $n - 1$ neighbors of v (in the trunk) are colored with the same color as v. The *weight* of an uncolored vertex w is then defined as

$$G(w) = \sum_{v \in N_{col}(w)} g(v),$$

where $N_{col}(w)$ is the set of colored neighbors of w in the trunk. So $G(w)$ is the number of colored neighbors and colored neighbors of colored neighbors of w in the trunk. Obviously (by the pigeon-hole principle) an uncolored vertex w can be colored feasibly if

$$G(w) \leq (d+1)c - 1,$$

see Fig. 2.1. This holds after Bob's move (before Alice's move) for every uncolored vertex w, as we shall see. Let i be chosen in such a way that

$$i(d+1) \leq b \leq (i+1)(d+1) - 1. \tag{2.2}$$

Thus $c = i + 2$. After Bob's move, an uncolored vertex w in a trunk may be neighbored with a colored vertex which was already colored before Bob's move and therefore has color weight at most $d+1$. In addition, the weight of w can increase by at most b when Bob colors b vertices. So, by the right-hand side of (2.2),

$$G(w) \leq d + 1 + b \leq (i+2)(d+1) - 1 = (d+1)c - 1$$

after Bob's move.

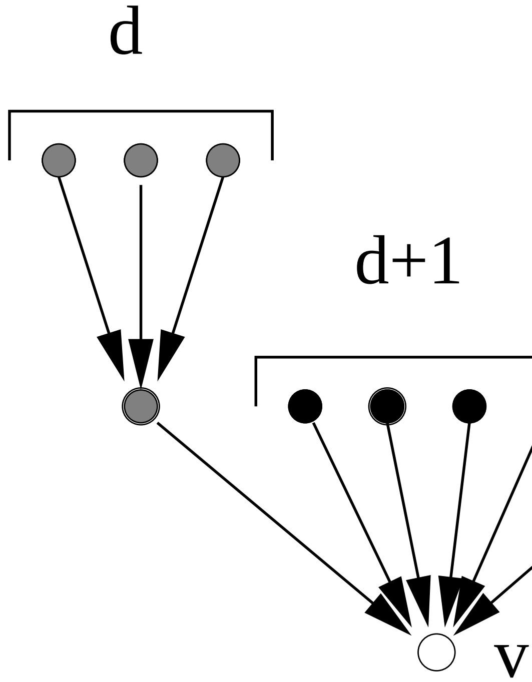


Figure 2.1: The pigeon-hole principle: Assume $d = 3$ and $c = 3$. Then the uncolored vertex v cannot be colored with dark grey and black, since the vertices colored with these colors contribute with $d + 1$ to the color weight. However, v can be colored with the third color light grey, which contributes to the weight of v only with d.

By this estimation, we see that we reserve one color completely for the origin vertices after Bob's move. Therefore w can be colored in a color different from the color of the origin.

Now we have to prove that $G(w) \leq (d + 1)c - 1$ holds during the whole of Alice's move. Assume that, in step k, Alice has colored a vertex v with color 1 in a trunk T. Then there might be other vertices u_i in T which are in-neighbors of v that are colored with color 1. By Alice's strategy, at most one of these, say u_1, is an origin vertex after Bob's move. However, if Alice always chooses a color different from the origin for the vertex v, then u_1 will be split off. As remarked above, she can choose a color in such a way. There might be uncolored out-neighbors of v. These out-neighbors (and the out-neighbors of colored in-neighbors of v) are the only vertices whose weight increases when Alice colors v. Let w be such an out-neighbor of v. (The argumentation for the out-neighbors of colored in-neighbors of v is symmetrical.) Let X be the set of colored in-neighbors of w without v. For $x \in X$ let Y_x^+ resp. Y_x^- be the set of in- resp. out-neighbors of x colored in the same color as x.

We observe that, by Alice's strategy, the vertices in X, Y_x^+, and Y_x^- (for

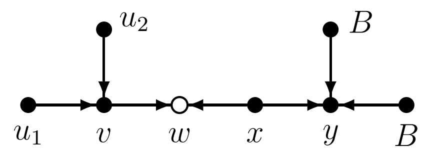


Figure 2.2: Possible configuration for w

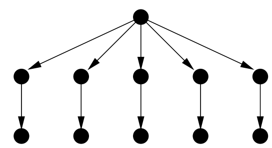


Figure 2.3: The digraph of Lemma 13 in case $(a, b, d) = (2, 1, 1)$

$x \in X$) are non-origin vertices, thus they have been colored by Bob in his last move. (For vertices from X and Y_x^+ it is clear that they are not origin vertices, since Alice would not color them as they are not sinks in a path starting at u_1. For vertices from Y_x^- we are in a case as shown in Fig. 2.2. Note that in this configuration the vertex y cannot be colored by Alice, since Alice would have to color w before y by her strategy of coloring the sinks of paths as the path in (2.1).)

Assume that $G(w) \geq (d+1)c$. That means

$$\#\left(X \cup \bigcup_{x \in X} (Y_x^+ \cup Y_x^-) \cup \{u_i\} \cup \{v\}\right) \geq (d+1)c.$$

We have $b \geq (d+1)c - 1$ since at most one vertex of the trunk at distance one or two from w, namely v, has not been colored by Bob in his last move. But as $c = i + 2$ and $d \geq 0$ we obtain

$$b \geq (d+1)c - 1 = (d+1)(i+2) - 1 > (d+1)(i+1) - 1 \geq b,$$

using (2.2), which is a contradiction. Thus our assumption is wrong, and the theorem is established. □

The bound of the previous theorem is tight as the next lemma and the succeeding proposition show.

Lemma 13. ${}^{(a,b)}\chi_g^d(\vec{\mathcal{F}}) \geq 2$ *for* $b \geq 1$.

Proof. Consider the following digraph D with vertex set

$$V = \{u, v_i, w_{i,j} | i = 1, \ldots, 2a+1; j = 1, \ldots, d\}$$

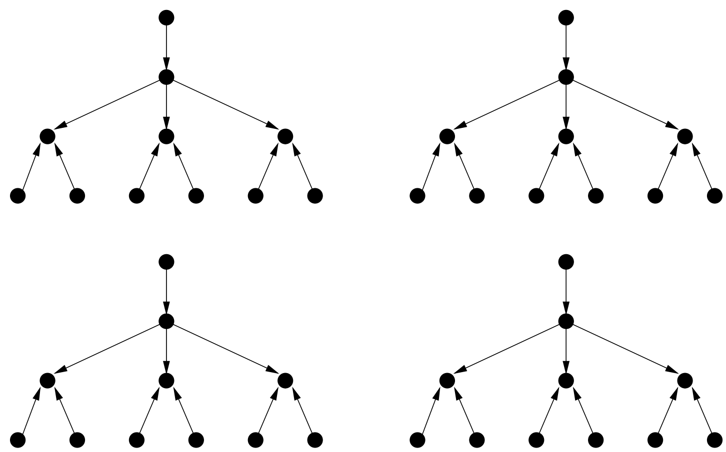


Figure 2.4: The digraph of Proposition 14 in case $(a, b, d) = (2, 2, 1)$

and arc set

$$E = \{(u, v_i), (v_i, w_{i,j}) | i = 1, \ldots, 2a + 1; j = 1, \ldots, d\}.$$

Fig. 2.3 depicts this tree in a special case. We have to prove that Bob has a winning strategy for the d-relaxed (a, b)-coloring game played on D with 1 color. We may assume that Alice, in her first two moves, colors vertices in the subtrees of vertices with index $i = 2, \ldots, 2a + 1$. If she does not color u, Bob colors u (and then possibly some other vertices, preferably v_1) in his first move. In case $d = 0$, v_1 cannot be colored any more. Otherwise, in his second move Bob colors v_1 (if it is not already colored). Now only $d - 1$ of the vertices $w_{1,j}$ can be colored, not the last vertex of $w_{1,j}$, so Bob will win. □

Proposition 14. ${}^{(a,b)}\chi_g^d(\vec{\mathcal{F}}) \geq \left\lfloor \frac{b}{d+1} \right\rfloor + 2$ *for* $b \geq 1$

Proof. For $b < d + 1$ the assertion is true by the Lemma 13. Thus consider the case $b \geq d + 1$. A digraph D (see Fig. 2.4 for an example) is defined by the vertex set

$$V = \{x_{i,n}, u_i, v_{i,j}, w_{i,j,k}\}_{i,j,k,n}$$

and the arc set

$$E = \{(x_{i,n}, u_i), (u_i, v_{i,j}), (w_{i,j,k}, v_{i,j})\}_{i,j,k,n}$$

where $i = 1, \ldots, a+b$, $j = 1, \ldots, a+1$, $k = 1, \ldots, b$, and $n = 1, \ldots, d$. We have to prove that Bob has a winning strategy for the d-relaxed (a, b)-coloring game played on D with $\left\lfloor \frac{b}{d+1} \right\rfloor + 1$ colors. We may assume that Alice, in her first move, colors vertices in subtrees with index $i = 2, \ldots, a + 1$. Then Bob colors all vertices $x_{1,n}$ with the first color. After that Bob colors u_1 with the first color. In this way, u_1 receives defect d. If Bob has to color further vertices, he chooses them with index $i \geq a+2$. Alice now colors w.l.o.g. vertices with index $i \geq 2$ or $j \geq 2$. Finally Bob colors $\lfloor \frac{b}{d+1} \rfloor (d+1)$ vertices $w_{1,1,k}$ in the colors $2, \dots, \lfloor \frac{b}{d+1} \rfloor + 1$, always $d+1$ vertices in the same color. Bob wins since $v_{1,1}$ cannot be colored any more. $\square$

Combining Theorem 12 and Proposition 14 we obtain

Corollary 15. ${}^{(a,b)}\chi_g^d(\vec{\mathcal{F}}) = \lfloor \frac{b}{d+1} \rfloor + 2$ *for* $a \geq b \geq 1$

The formulation of Theorem 12 is also best-possible in the sense that it cannot be extended from the case $a \geq b \geq 1$ to other values of (a,b). If $b = 0$ (and $a \neq 0$), the dichromatic number of a non-empty directed forest is 1, so also ${}^{(a,0)}\chi_g^d(\vec{\mathcal{F}}) = 1$ for any d. In the case $b > a$ we remark

Proposition 16. ${}^{(a,b)}\chi_g^d(\vec{\mathcal{F}}) = \infty$ *for any* $b > a$.

Proof. Let $k \geq 0$, $b > a$, and let F be the directed forest consisting of $b^{k(d+1)}$ components each of which is an in-star $S^{in}_{k(d+1)}$. Such an *in-star* has $k(d+1)+1$ vertices one of which has in-degree $k(d+1)$ and out-degree 0, and the other vertices have out-degree 1 and in-degree 0. We prove that Bob wins the d-relaxed (a,b)-coloring game on F with at most k colors. We consider the weakest variant A of the game.

The game is divided into $k(d+1)$ rounds. In the i-th round, there are $b^{k(d+1)-i}$ moves for each player, so Alice colors at most $b^{k(d+1)-i}a$ vertices, and Bob $b^{k(d+1)-i+1}$. In round i, Bob chooses a color c_i and $b^{k(d+1)-i+1}$ in-stars to which not any colors have been assigned by Alice in previous rounds, but which have been colored by Bob in all previous rounds, and Bob colors exactly one leaf vertex of these (in-stars) with color c_i. Obviously, in the first round, this is possible. If we assume it is possible in the i-th round, then there are at least

$$b^{k(d+1)-i}(b-a) \geq b^{k(d+1)-(i+1)+1}$$

in-stars left which have not been touched by Alice, so Bob can proceed as desired in round $i+1$. In particular, after the round $k(d+1)$, there is at least one in-star left that has not been colored by Alice. If Bob has chosen

$$(c_1, \dots, c_{k(d+1)}) = (\underbrace{1, \dots, 1}_{d+1}, \underbrace{2, \dots, 2}_{d+1}, \dots, \underbrace{k, \dots, k}_{d+1}),$$

then this in-star cannot be colored any more. Thus, Bob wins. Since k is arbitrarily chosen, ${}^{(a,b)}\chi_A^d(\vec{\mathcal{F}}) = \infty$ for $b > a$. By Observation 1, the assertion follows for any g. $\square$

2.2 Undirected forests

In this section we give an upper and a lower bound for ${}^{(a,b)}\chi_g^d(\mathcal{F})$. After that we determine the exact values for this parameter for special triples (a, b, d).

Recall the definition of (undirected) graphs: Every edge is considered as a pair of oppositely directed arcs.

In order to obtain upper bounds for the d-relaxed (a, b)-game chromatic number of undirected forests we need the notion of uncolored components. An *uncolored component* of a partially colored forest is a maximal connected component of uncolored vertices. Note that a trunk in such a forest may contain several colored vertices and several uncolored components (which are separated by colored vertices). An uncolored component C is *adjacent* to a vertex v if a vertex of C is adjacent to v.

Using the idea of uncolored components one can prove the results of Faigle et al. [40] and Chou et al. [27] which are $\chi_g^d(\mathcal{F}) \leq 3 + \delta_{0,d}$ where $\delta_{m,n}$ is the Kronecker Delta. Here, Alice's winning strategy guarantees that after each move of Alice every uncolored component is adjacent to at most two colored vertices. If Bob creates an uncolored component adjacent to 3 colored vertices it is immediately broken in Alice's next move. We will adapt this strategy in order to prove the following

Theorem 17. ${}^{(a,b)}\chi_g^d(\mathcal{F}) \leq \left\lfloor \frac{b}{d+1} \right\rfloor + 3$ *for* $a \geq b \geq 1$.

Proof. We describe a winning strategy for Alice with $c = \left\lfloor \frac{b}{d+1} \right\rfloor + 3$ colors. The invariant Alice maintains after each of her moves is that every uncolored component is adjacent to at most two colored vertices. At the beginning of the game this invariant obviously holds.

When it is Bob's turn, he colors at most b vertices and possibly destroys Alice's invariant. Alice lists the vertices $v_1, v_2, \ldots, v_b$ Bob has colored. Now Alice performs at most b steps. She keeps in mind an increasing set of *origin vertices*. At the beginning every colored vertex is an origin vertex, except the vertices $v_1, v_2, \ldots, v_b$. An *extended component* is a connected component of a trunk in which all origin vertices are deleted. So a trunk possibly contains several extended components and origin vertices, and an extended component possibly contains several uncolored components which are separated by colored non-origin vertices. An extended component is *adjacent* to an origin vertex x if one of its uncolored components is adjacent to x. Alice's winning strategy will guarantee that after each step every extended component will be adjacent to at most two origin vertices. In the k-th step, first, for any extended component C adjacent to two origin vertices, Alice considers the path between the two origin vertices and adds all members of $\{v_1, v_2, \ldots, v_b\}$ which are on this path to the set of origin vertices. This does not destroy her invariant, but has the consequence that all non-origin vertices on the path are uncolored. Then she considers the vertex v_{i_k} with the lowest index i_k among all non-origin vertices

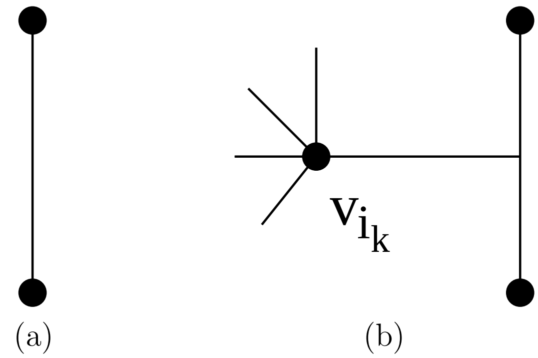


Figure 2.5: Vertex v_{i_k} is added to the set of origin vertices: (a) extended component before adding v_{i_k}. (b) extended components after adding v_{i_k}.

of the list $v_1, \ldots, v_b$. Alice adds v_{i_k} to the set of origin vertices. By Alice's strategy, all extended components which are adjacent to v_{i_k} are adjacent to at most two origin vertices except at most one extended component C_0 which is adjacent to three origin vertices, see Fig. 2.5.

If there is no such component C_0, Alice does not choose a vertex for coloring and goes to the $(k+1)$-th step. In this case every extended component is still adjacent to at most two origin vertices.

Otherwise, C_0 is adjacent to three origin vertices w_1, w_2, and v_{i_k}. The paths w_1w_2, $w_1v_{i_k}$, and $w_2v_{i_k}$ intersect at a single vertex w_3 (since a forest does not contain cycles with more than 2 vertices). The vertex w_3 is uncolored, as remarked above, because w_3 is a non-origin vertex on the path between w_1 and w_2. Alice chooses w_3 for coloring and adds w_3 to the set of origin vertices. She furthermore adds all members of $\{v_1, v_2, \ldots, v_b\}$ which are on the path from v_{i_k} to w_3 to the set of origin vertices. Then she continues with step $(k+1)$. By this type of move, C_0 is split into several (at least 3, possibly empty) extended components separated by w_3. Obviously, each of these extended components is adjacent to at most two origin vertices.

After at most b steps all vertices of the list $v_1, \ldots, v_b$ will be origin vertices, therefore Alice's local invariant implies the global invariant that every uncolored component is adjacent to at most two colored vertices. After having reinstalled her strategy, Alice may simply color neighbors of colored vertices without affecting her invariant.

Now we have to prove that Alice finds a feasible color for each of her chosen vertices. For a colored vertex v, the *color weight* $G(v)$ is one plus the number of all neighbors of v which are colored with the same color as v. The *weight* $G(w)$ of an uncolored vertex w is

$$G(w) = \sum_{v \in N_{col}(w)} G(v),$$

where $N_{col}(w)$ is the set of colored neighbors of w. The *weight* $G(C)$ of an uncolored component C is defined as

$$G(C) = \sum_{w \in U(C)} G(w),$$

where $U(C)$ is the set of (uncolored) vertices of C. Obviously, by the pigeonhole principle, a chosen vertex w in the uncolored component C can be colored if $G(C) \leq (d+1)c - 1$. Let j be such that

$$(d+1)j \leq b \leq (d+1)(j+1) - 1,$$

i.e. $c = j + 3$. Then after Bob's move

$$G(C) \leq b + 2(d+1) \leq (d+1)(j+3) - 1 = (d+1)c - 1.$$

In particular, this implies that, after Bob's move, we find a feasible color for any uncolored vertex which is different from the colors of the origin vertices (each of which contributes at most $d+1$ to the estimation of $G(C)$).

Whenever Alice colors a vertex, in her model, she splits the actual uncolored component into several parts, so that the augmentation of the weight by one (because of her new colored vertex) is compensated by the splitting, except in one case. The exceptional case in which there is no splitting is if Alice colors a vertex w_3 adjacent to two or three vertices of the same color α in the k-th step, two of them are origin vertices before the k-th step and the third is v_{i_k}, and Alice has to color w_3 with α. But this case does not occur since, as argued above, Alice may always color w_3 different from the origin vertices. Thus, for every new uncolored component C', $G(C') \leq (d+1)c - 1$, so, by induction, Alice always finds a feasible color.

Since during Bob's move the weight of every uncolored component is always bounded by $b + 2(d+1)$, he can also always find a feasible color. □

We find that Theorem 17 is nearly tight.

Proposition 18. *${}^{(a,b)}\chi_g^d(\mathcal{F}) \geq \left\lfloor \frac{b}{d+1} \right\rfloor + 2$ for $b \geq 1$.*

Proof. This is very similar to the construction in the proof of Proposition 14, thus the details are omitted. □

Again, the precondition $a \geq b \geq 1$ of Theorem 17 is necessary because of

Proposition 19. *${}^{(a,b)}\chi_g^d(\vec{\mathcal{F}}) = \infty$ for $b > a$.*

Proof. This is the same proof as in Proposition 16 if we consider (undirected) stars instead of in-stars. □

Before formulating the next theorem, we start with the definition of an extended star. Imagine that during the game several vertices of a tree have been colored, others are uncolored. The tree is broken between two adjacent, differently colored vertices. Now consider one of the remaining subtrees, which are called *trunks* again. We construct the *structure tree* of a trunk as follows. Every connected component C of equally colored vertices is replaced by a single colored vertex connected with all former neighbors of C. A trunk is called *extended star* if in its structure tree all paths between colored vertices only intersect in a special colored vertex, the *center*. The union of these paths and the original connected components of colored vertices is then called *induced subtree* of the extended star. A connected component with possibly several colored vertices is called *big block*, while a connected component that contains a single vertex is called *small block*.

In [49] He et al. proved that ${}^{(1,1)}\chi_g^2(\mathcal{F}) = 2$. With a simpler construction we show the weaker general result

Theorem 20. *${}^{(a,1)}\chi_g^d(\mathcal{F}) = 2$ for $a \geq 2$, $d \geq 2$.*

Proof. We prove a winning strategy for Alice with 2 colors. Again we consider trunks. The induced subtree of a trunk is the union of all paths between colored vertices of a trunk. Alice maintains the invariant that after each of her moves the induced subtree of every trunk is an extended star with one big block of colored vertices in the center and small blocks as leaves which consist of one colored vertex only. Assume that Alice's invariant holds after her move. Then after Bob's move there are the 5 possibilities depicted in Fig. 2.6: First, Alice's invariant holds for every trunk. Second, there is one trunk with two big blocks, where one block only consists of two colored vertices. Third, there is a trunk in the induced subtree of which there is an uncolored vertex of degree 3 at distance one from the big block. Fourth, there is a trunk in the induced subtree of which there is an uncolored vertex of degree 3 at distance at least two from the big block. Fifth, there is a trunk with a path between the big block and a colored vertex on which another colored vertex lies. Now we describe Alice's reaction in these cases.

In the first case Alice colors the first uncolored vertex on the path from the big block to a small block in a color different from the color of the big block.

In the second case Alice colors the first uncolored vertex on the path from the old big block to the new big block in a color different from the color of the old big block. This is possible since $d \geq 2$ and by this move a vertex receives a defect of at most 2.

In the third case Alice colors the vertex of degree 3 in the induced subtree different from the color of the big block. This is possible as in the previous case.

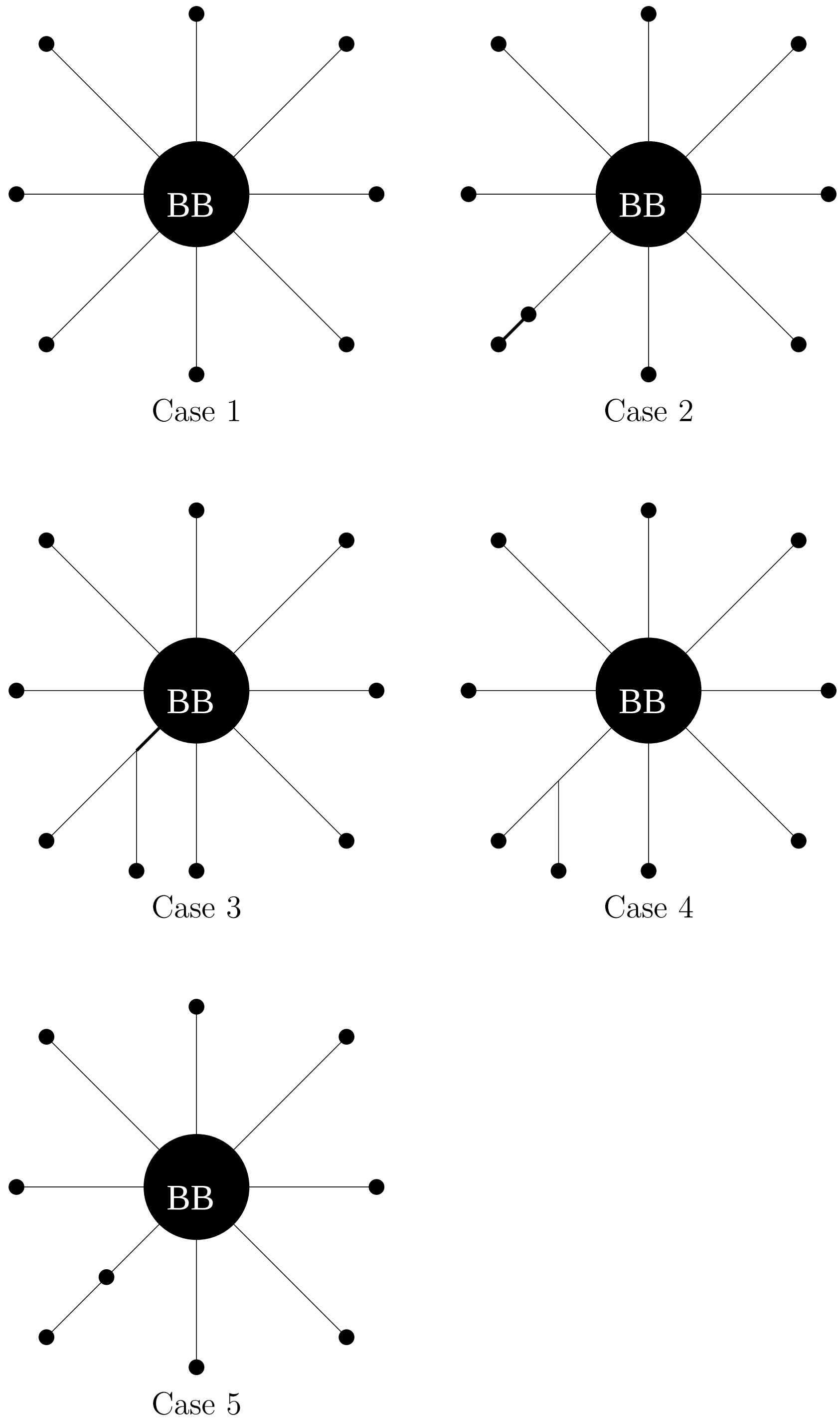


Figure 2.6: The five cases

In the fourth case assume that the big block is colored with color 1, and v is the vertex of degree 3 in the induced subtree. Then Alice colors v with color 1 and the first uncolored vertex on the path from v to the big block with color 2. This is the only case where Alice must use the fact that she can color at least 2 vertices in one move. Note that the coloring of v is feasible since its two neighbors colored with color 1, if there are any, are the new colored vertex and the single vertex of a small block.

In the fifth case assume that the big block is colored with 1 and v is the colored vertex on the path between the big block and a colored vertex w. If v and w are adjacent, Alice colors the first uncolored vertex on the path between the big block and v with color 2. (If there is no uncolored vertex on the path, nothing is to do.) If v and w are not adjacent, Alice colors the first uncolored vertex on the path between v and w different from the color of v, so that the trunk is split into two parts.

In most cases, Alice has to color additional vertices, since $a > b = 1$. But this is easy, since now every trunk is as described in the first case, and Alice can play as in the first case. Bob also has a legal move as long as there are uncolored vertices since he can imitate Alice's strategy for the first case. Thus Alice wins. □

2.3 Forest-like structures

Consider the following type of oriented graphs. It may contain directed 3-cycles and oriented tree structures which may be glued together at one of the vertices of a 3-cycle. However, two 3-cycles may share only one vertex, not two vertices. If every 3-cycle is replaced by its 3 vertices and an additional center vertex to which the 3 vertices are connected by arcs, then the resulting digraph must be a directed forest. We call this type of digraphs *directed 3-cycled forest*. An example of such a structure is depicted in Fig. 2.7. Let $\vec{\mathcal{FC}}_3$ be the class of all directed 3-cycled forests.

Theorem 21. $\chi_g(\vec{\mathcal{FC}}_3) \leq 4$

Proof. We play the 0-relaxed directed $(1, 1)$-coloring game on a directed 3-cycled forest with 4 colors and show that Alice has a winning strategy. Whenever a player colors a vertex v we multiply this vertex and break the directed 3-cycled tree (i.e. the connected component of the directed 3-cycled forest) at vertex v into pieces, so that v belongs to all these pieces which we call *independent 3-cycled subtrees*. The *induced subtree* of an independent 3-cycled subtree consists of the union of all shortest paths between its colored vertices. (Here a path needs not to be directed, it is only a sequence of adjacent arcs.) The splitting operation is useful since we consider a game with defect 0,

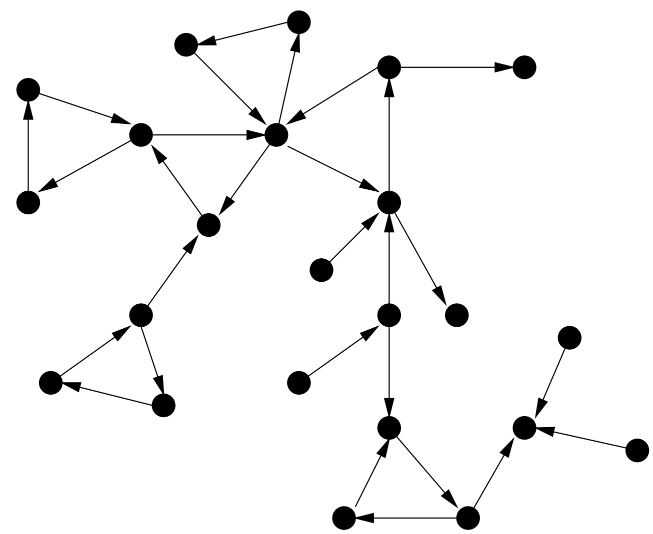


Figure 2.7: A directed 3-cycled forest

therefore the colored neighbors of colored neighbors of an uncolored vertex v do not affect the coloring of v.

A special class of independent 3-cycled subtrees are those with two colored vertices at most which are called *long lines*. An independent 3-cycled subtree with 3 colored vertices v_1, v_2 and v_3 whose induced subtree is depicted in Fig. 2.8 is called *T-component*. It consists of arcs (v_C, v_0) and (v_2, v_C), a (shortest) path of arcs between v_1 and v_C, and a (shortest) path of arcs between v_3 and v_0. The arcs on the latter paths may have arbitrary direction. Note that arcs on these paths may be part of a triangle. Alice's winning strategy consists in guaranteeing that after all her moves every independent 3-cycled subtree is either a long line or a T-component. At the beginning this is true since then every connected component is a long line with 0 colored vertices.

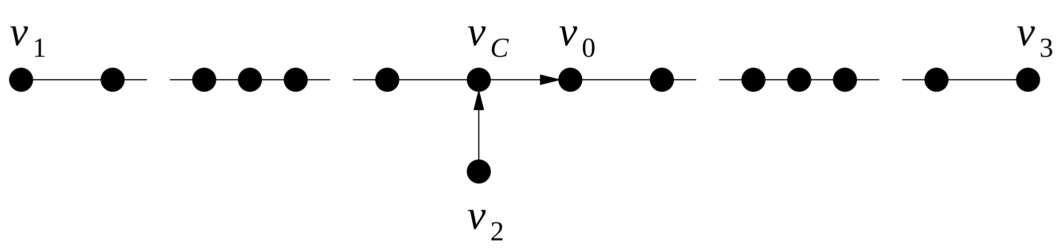


Figure 2.8: Structure of the induced subtree of a T-component with colored vertices v_1, v_2, and v_3. Note that there are other arcs and uncolored vertices which are not depicted.

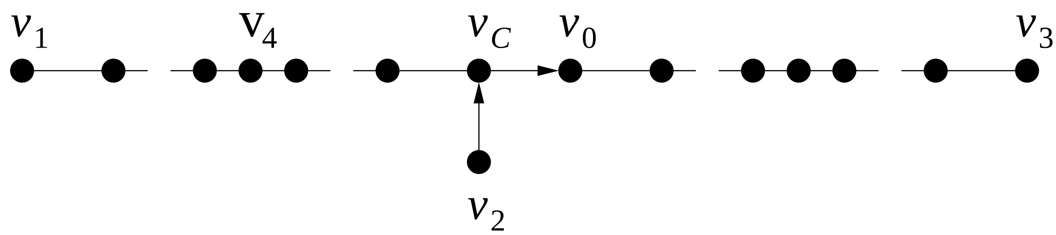


Figure 2.9: First case of splitting a T-component

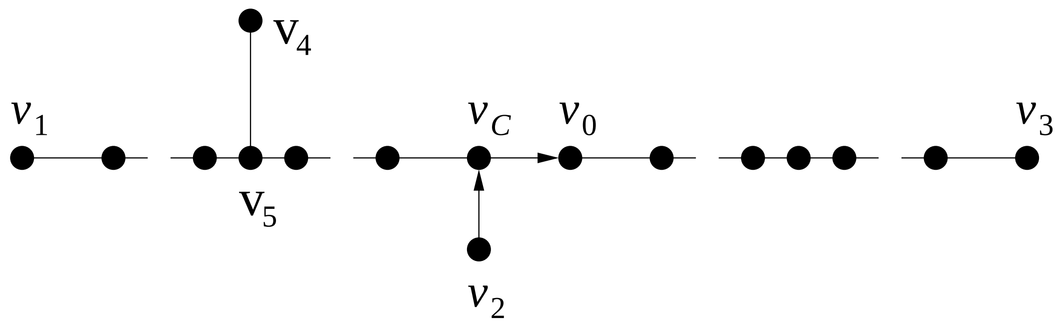


Figure 2.10: Second case of splitting a T-component

If Bob colors a third vertex v_3 in a long line with colored vertices v_1 and v_2, then there are three cases. First the induced subtree containing v_1, v_2 and v_3 is a path. Then Alice's invariant is not destroyed and she can easily find a feasible move (e.g. by coloring a neighbor of a colored vertex). Second the shortest paths between the colored vertices meet in exactly one vertex v_0. Then Alice colors v_0 (possibly in the fourth color) and breaks the independent 3-cycled subtree into at least three new long lines. Third there is a meeting triangle instead of a meeting point. Then Alice colors a vertex of the triangle and obtains a T-component (and several long lines).

If Bob colors a fourth vertex v_4 in a T-component, there are again three cases. First, if v_4 lies in the induced subtree of the former T-component, then Alice colors v_C or v_0 in order to break the independent 3-cycled subtree into long lines. See Fig. 2.9. Second, if there is only one shortest path from v_4 to the induced subtree of the former T-component which ends in v_5, then Alice colors v_5 and breaks the independent 3-cycled subtree into at most one T-component and several long lines. See Fig. 2.10. Note that, by the definition of a T-component, v_5 is adjacent to at most 3 colored vertices, thus there is a fourth color for Alice. Third, if there are two shortest paths from v_4 to the induced subtree of the former T-component, then these paths end in a meeting triangle. A special vertex v_x of the vertices of this triangle belongs to at least four of the six shortest paths between the colored vertices v_1, v_2, v_3, and v_4.

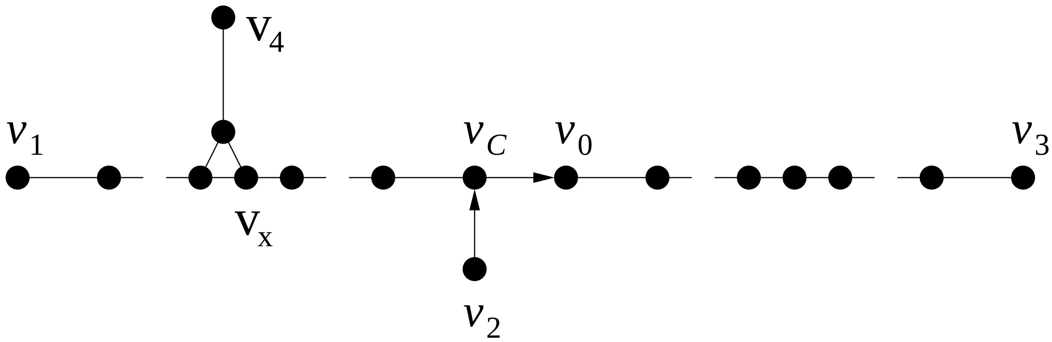


Figure 2.11: Third case of splitting a T-component

See Fig. 2.11. Alice colors v_x and breaks the T-component into at most two T-components and possibly several long lines. Note that the meeting triangle cannot contain v_C and v_0, since the old T-component must have been created by coloring v_2 and thus by deleting the arc (v_0, v_2) of a triangle (v_2, v_C, v_0). By the definition of directed 3-cycled forest this triangle may not be adjacent to the meeting triangle, thus the meeting triangle does neither contain the arc (v_C, v_0) nor the arc (v_2, v_C). Further note that $v_0 \neq v_3$ (otherwise the arc (v_C, v_0) would have been deleted when $v_0 = v_3$ was colored), i.e. v_0 is uncolored as long as it belongs to the old T-component. Therefore the vertex Alice colors is adjacent to at most 3 colored vertices, thus Alice can color it with the fourth color.

Bob can always move, since he can imitate Alice's strategy. Thus Alice will win. □

The bound is tight as the next theorem states.

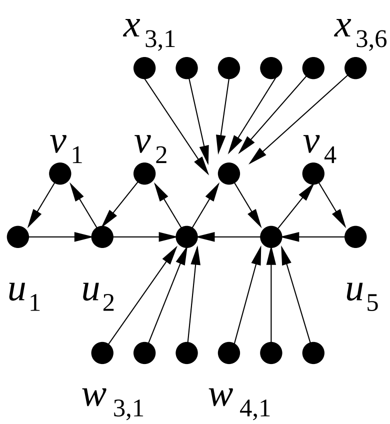


Figure 2.12: The main parts of the left half of D_1

Theorem 22. $\chi_g(\vec{\mathcal{FC}}_3) \geq 4$

Proof. Consider the following digraph D which consists of two identical components D_1 and D_2. $D_1 = (V, E)$ is defined by

$$\begin{aligned} V &= \{u_i | i = 1, \ldots, 9\} \cup \{v_i | i = 1, \ldots, 8\} \\ &\quad \cup \{w_{i,j}, x_{i,j} | i = 1, \ldots, 9; j = 1, \ldots, 6\} \end{aligned}$$

and

$$\begin{aligned} E &= \{(u_1, u_2), (u_2, u_3), (u_4, u_3), (u_5, u_4)\} \\ &\quad \cup \{(u_5, u_6), (u_6, u_7), (u_8, u_7), (u_9, u_8)\} \\ &\quad \cup \{(v_1, u_1), (u_2, v_1), (v_2, u_2), (u_3, v_2)\} \\ &\quad \cup \{(u_3, v_3), (v_3, u_4), (u_4, v_4), (v_4, u_5)\} \\ &\quad \cup \{(v_5, u_5), (u_6, v_5), (v_6, u_6), (u_7, v_6)\} \\ &\quad \cup \{(u_7, v_7), (v_7, u_8), (u_8, v_8), (v_8, u_9)\} \\ &\quad \cup \{(w_{i,j}, u_i), (x_{i,j}, v_i) | i = 1, \ldots, 8; j = 1, \ldots, 6\} \\ &\quad \cup \{(w_{9,j}, u_9) | j = 1, \ldots, 6\}. \end{aligned}$$

Some important structures of the left half of D_1 are depicted in Fig. 2.12. This half and an identical right half are glued together in vertex u_5 and form D_1. An important feature of such a half is that the triangles (u_1, u_2, v_1) and (u_2, u_3, v_2) are directed in a way opposite to the triangles (u_4, u_3, v_3) and (u_5, u_4, v_4).

We present a winning strategy for Bob with 3 colors for the game played on D. W.l.o.g. Alice, in her first move, colors a vertex of D_2. Then Bob colors u_5 with color 1. W.l.o.g. Alice answers by coloring a vertex of D_2 or with index $i \geq 5$. After that Bob colors u_2 with color 1. Now Alice's third move will bring the decision.

Case 1: If Alice colors u_4 (with color 2), then Bob colors $w_{3,1}$ with color 3, and u_3 cannot be colored any more.

Case 2: Alice colors v_3 with color c. If $c = 2$ ($c = 3$), then Bob colors $w_{4,1}$ with color 3 (2) and u_4 cannot be colored any more. On the other hand, if $c = 1$, then Bob colors $w_{4,1}$ with color 3. In order to prevent Bob from coloring $w_{4,2}$ or $w_{4,3}$ with color 2 and leaving u_4 uncolored, Alice has to color u_4 with color 2. But then we are in the same situation as in Case 1 and Bob has a winning strategy.

Case 3: Alice colors u_3 with color 2. Then Bob colors $x_{3,1}$ with color 1. In order to prevent Bob from coloring $x_{3,2}$ or $x_{3,3}$ with color 3 and leaving vertex v_3 uncolored, Alice has to color v_3 with color 3. Now Bob only has to color $w_{4,1}$ with color 2 in order to win since u_4 cannot be colored any more.

Case 4: Alice colors a vertex different from u_3, v_3 and u_4 in her third move. Then Bob colors one of the vertices $x_{3,j}$ with color 1. Alice is stuck now. If she colors u_3 (resp. v_3, resp. u_4) with color 2, then Bob colors a vertex $x_{3,j}$ (resp. $w_{4,j}$, resp. $w_{3,j}$) with color 3 and wins. Otherwise Bob colors another vertex $x_{3,j}$ with color 2. Again, Alice may not color u_3, v_3 or u_4. No matter what she does, Bob colors another vertex $x_{3,j}$ with color 3 and wins. □

2.4 Graph coloring games on orientations of planar graphs

While the game chromatic number of directed forests has been solved, the next interesting class of digraphs, i.e. orientations of planar graphs, until now resist a determination of its game chromatic number. We are not even able to give a reasonable upper bound for the game chromatic number of orientations of planar graphs. For an undirected planar graph P, Zhu [88] has proven that $\mathrm{col}_{g_A}(P) \leq 17$. So, by Proposition 7 we conclude for any orientation $\vec{P}$ of P that

$$\chi_{g_A}(\vec{P}) \leq \mathrm{col}_{g_A}(\vec{P}) \leq \mathrm{col}_{g_A}(P) \leq 17.$$

However, this bound seems to be too large. Let $\vec{\mathcal{P}}$ be the class of orientations of planar graphs and $\vec{\mathcal{P}}_{trans}$ be the class of orientations of planar graphs without directed cycles. In this section we will at least give a lower bound for $\chi_g(\vec{\mathcal{P}})$. The case of outerplanar graphs is studied in Section 3.7.

Theorem 23. $\chi_g(\vec{\mathcal{P}}) \geq \chi_g(\vec{\mathcal{P}}_{trans}) \geq 5$

Proof. Again, the first inequality is obvious, so we will only prove the second. Consider the graph of Fig. 2.13 for which Bob has the following winning strategy with 4 colors. W.l.o.g. Alice colors a vertex in the lower component. Bob colors 1 with color 1. W.l.o.g. Alice colors a vertex in the lower component or in the lower half of the upper component. Bob colors 2 with color 2. W.l.o.g. Alice colors a vertex in the lower component or in the lower or right half of the upper component. (This might be also the vertex adjacent to 1, 2, and 5.) Bob colors 3 with color 3. The next moves will bring the decision.

If Alice, in her fourth move, colors 5 (with color 4), then Bob colors 6 with color 1, and 8 cannot be colored any more.

If Alice, in her fourth move, colors 4, Bob colors 6 with color 1. Then we have to consider two subcases: if Alice, in her next move, colors 8 (with color 4) then 7 cannot be colored any more. Otherwise either Alice or Bob color 5 with color 4, and 8 cannot be colored any more.

If Alice proceeds differently in her fourth move, Bob colors 4 with color 4, and 5 cannot be colored any more. □

2.5 Open problems

In Section 2.1 the d-relaxed (a, b)-game chromatic number of the class of directed forests was completely determined for every value of a, b and d. However, for undirected forests there is a gap between the lower and the upper bound for the d-relaxed (a, b)-game chromatic number:

Open question. For which values of (a, b) and d, ${}^{(a,b)}\chi_g^d(\mathcal{F}) = \lfloor \frac{b}{d+1} \rfloor + 3$, resp. for which values, ${}^{(a,b)}\chi_g^d(\mathcal{F}) = \lfloor \frac{b}{d+1} \rfloor + 2$?

Both cases, i.e. attaining the upper resp. the lower bound, are possible. Kierstead [53] has shown that, in the 0-relaxed case, ${}^{(a,b)}\chi_g(\mathcal{F}) = b + 3$ if $b \leq a < 2b$ or $(a, b) = (2, 1)$, and ${}^{(a,b)}\chi_g(\mathcal{F}) = b+2$ if $a \geq 2b$ and $(a, b) \neq (2, 1)$. But not only the parameters a and b affect the switch between upper and lower bound, also the parameter d has some influence. In the symmetric case $(a, b) = (1, 1)$, Chou et al. [27] have proved that

$$\chi_g^1(\mathcal{F}) = 3 \quad \left(= \left\lfloor \frac{b}{d+1} \right\rfloor + 3\right),$$

whereas, by a result of He et al. [49],

$$\chi_g^2(\mathcal{F}) = 2 \quad \left(= \left\lfloor \frac{b}{d+1} \right\rfloor + 2\right).$$

Our results concerning 3-cycled forests might be generalized from the basic coloring game to the d-relaxed asymmetric coloring game:

Open question. Let $a, b, d \geq 0$. Determine ${}^{(a,b)}\chi_g^d(\mathcal{F}\vec{\mathcal{C}}_3)$.

The most demanding task arisen by this chapter is to determine the d-relaxed (a, b)-game chromatic number of planar graphs or orientations of planar graphs. Since the Four-Color-Theorem [10, 11] is a special case of this question it will be very hard to solve. More generally, one might consider graphs and orientations of graphs embeddable in an arbitrary surface instead of planar graphs. In the next chapter we will consider special cases of these digraphs, i.e. we examine digraphs embeddable in a surface with certain restrictions to their cycles.

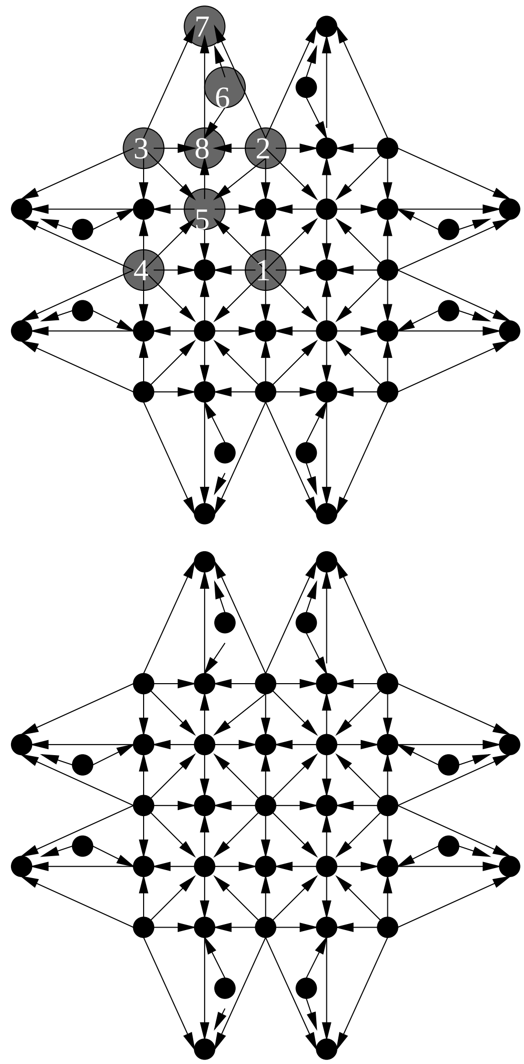


Figure 2.13: The digraph of Theorem 23

Chapter 3

Lightness of digraphs in surfaces

In order to examine the game chromatic number of planar graphs without 4-cycles and with prescribed girth, He et al. [48] proposed an approach subdivided into two parts. The first part concerns an edge-partition argument to bound the game coloring number of such a graph in terms of the game coloring number of a more simple subgraph (indeed, of a forest) and the maximum degree of the graph G' consisting of the remaining arcs. The second part is devoted to bound a parameter M^* of a related planar graph which results in an upper bound for the maximum degree of G'.

We will generalize the results of He et al. to graphs and simple digraphs embeddable in some surface with the same restrictions applying to cycles and girth. While the edge-partition argument still works in these cases (see Section 3.7), bounding M^* requires new methods developed in the following sections.

Recently, Wang [80] generalized the edge-partition argument from planar graphs to graphs with nonnegative Euler characteristic. With this approach he achieved better upper bounds for the game coloring number of graphs with prescribed girth embeddable in one of the two surfaces of Euler characteristic 0 than we will obtain. However, our bounds are still the best currently known bounds for the other surfaces we consider.

3.1 Lightness and weight

Following the ideas of He et al. [48] we define a more general parameter of a digraph. For a digraph $D = (V, E)$ and an arc $e = (v, w) \in E$, let $L^+_D(e) = \max\{d^+_D(v), d^+_D(w)\}$. We call

$$L(D) = L^+(D) = \min_{e \in E} L^+_D(e)$$

positive lightness or simply *lightness* of D. The *negative lightness* $L^-(D)$ of D could be defined in the same way by considering the out-degrees instead of

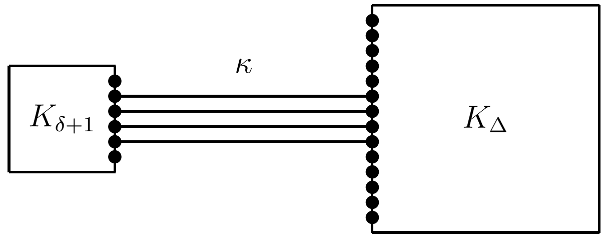


Figure 3.1: A non-regular graph for which (3.1) is trivial

the in-degrees.

For a graph G, $L(G)$ is exactly the parameter $M^*(G)$ of He et al. An edge vw with $L(G) = L_G^+((v,w))$ is called a *light edge*. The latter notion was already used by Zhu [86] and He et al. [48], which is our justification to call the related parameter 'lightness'.

The lightness of a digraph D seems to be closely related to another graph parameter, the *weight* $w(D)$. It is defined as the minimum arc weight, where the *weight* $w_D((u,v))$ of an arc (u,v) is the sum $d^+(u)+d^+(v)$. For a graph G, a *light edge* in the sense of weight could be defined as an edge uv with $w(G) = w_G((u,v))$. Obviously,

$$\frac{1}{2}w(D) \leq L^+(D) \leq w(D) - \delta^+(D). \tag{3.1}$$

The relation (3.1) between lightness and weight of a digraph D motivates the following nonnegative residue parameters

$$\begin{aligned} R_1(D) &= 2L^+(D) - w(D), \\ R_2(D) &= w(D) - \delta^+(D) - L^+(D). \end{aligned}$$

Obviously, $R_1(D) = R_2(D) = 0$ for *regular* digraphs, i.e., digraphs where each vertex has the same in-degree. But there are also non-regular digraphs with arbitrarily large maximum in-degree (or arbitrarily large clique number) Δ, arbitrarily large minimum in-degree $\delta < \Delta$, and arbitrarily large connectivity $\kappa < \delta$ that have the same property. E.g., consider the graph built by $K_{\delta+1}$ and K_Δ which are glued together by a matching of cardinality κ as in Figure 3.1. A general criterion to recognize those digraphs for which lightness and weight describes the same phenomenon is given by the following

Proposition 24. *Let $D = (V,E)$ be a digraph with $E \neq \emptyset$. Then the following statements are equivalent:*

(i) $R_1(D) = R_2(D) = 0$

(ii) $L^+(D) = \delta^+(D)$

(iii) D *contains an arc* (v, w) *with* $d^+(v) = d^+(w) = \delta^+(D)$

(iv) $w(D) = 2\delta^+(D)$

Proof. The system (i) is equivalent to $L^+(D) = \delta^+(D)$ and $w(D) = 2\delta^+(D)$, thus (ii) follows from (i). On the other hand, one of the conditions (ii) and (iv) is redundant as we shall see. Assume that $w(D) = 2\delta^+(D)$. Then we have

$$\begin{aligned} 0 \leq R_1(D) &= 2L^+(D) - w(D) = 2L^+(D) - 2\delta^+(D), \text{ and} \\ 0 \leq R_2(D) &= w(D) - L^+(D) - \delta^+(D) = \delta^+(D) - L^+(D), \end{aligned}$$

hence $\delta^+(D) = L^+(D)$. As a consequence, (iv) implies (i). Note that, if (iii) is not true, then, since $E \neq \emptyset$, each arc e has at least one end vertex v with $d^+(v) > \delta^+(D)$, and $L^+(D) = \min_e L^+(e) > \delta^+(D)$. This proves the implication (ii)⇒(iii). (iii)⇒(iv) follows from the definition of weight. □

Remark. In general, R_1 and R_2 may not be bounded, even when restricted to (undirected) trees. To see this, for given integers $n_1 \geq 1$ and $n_2 \geq 1$, we construct a rooted tree T. Its root v has $n_1 + n_2$ descendants, each one of which has $n_1 + 2n_2$ descendants again. To form the tree $T_{(n_1,n_2)}$ take T and a copy T' of T with root v' and connect v and v' by an edge. One can easily check that $L(T_{(n_1,n_2)}) = n_1 + n_2 + 1$, and $w(T_{(n_1,n_2)}) = n_1 + 2n_2 + 2$, therefore $R_1(T_{(n_1,n_2)}) = n_1$, and $R_2(T_{(n_1,n_2)}) = n_2$. Here, the difference between the concepts of lightness and weight is expressed by different light edges: vv' is the only 'light edge' in the sense of lightness (since $L(vv') = L(T_{(n_1,n_2)})$,) whereas only every leaf edge e is 'light' in the sense of weight (since $w(e) = w(T_{(n_1,n_2)})$.) Figure 3.2 depicts $T_{(1,1)}$.

In order to construct an example G with higher minimum degree $\delta(G) = \delta \leq n_1$, we start with $T_{(n_1,n_2)}$ and introduce $\delta - 1$ new vertices which are

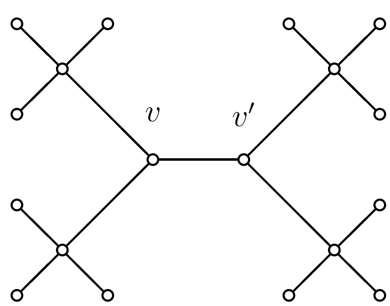


Figure 3.2: The tree $T_{(1,1)}$.

connected with every leaf of $T_{(n_1,n_2)}$. Then $L(G) = n_1 + n_2 + 1$ and $w(G) = n_1 + 2n_2 + 1 + \delta$, as careful case distinctions show. Thus $R_1(G) = n_1 + 1 - \delta$ and $R_2(G) = n_2$. By choosing n_1 large enough and δ and n_2 arbitrarily the residue parameters may take any value.

Determining the weight of certain kinds of planar graphs has been considered for some time. Let G_3 be a 3-connected planar graph, and G_2 be a planar graph with $\delta(G_2) \geq 2$. By the righthand side of (3.1), a result of Kotzig [59] concerning the weight of G_3 implies $L(G_3) \leq 10$. Similarly, if G_3 has no 4-cycles, then $L(G_3) \leq 7$, and if G_3 has girth 5, then $L(G_3) \leq 5$, both by a result of Borodin [20]. Planar graphs G_2 with minimum degree 2 and without a certain kind of "alternating" even cycles have $L(G_2) \leq 13$ by another result of Borodin [19] combined with (3.1).

He et al. [48] considered the case of planar graphs with minimum degree $\delta \geq 2$ and without 4-cycles. They determined upper bounds for the lightness of these graphs which depend on the girth k and are best-possible if $k \geq 5$. Let G_2^k be such a graph with girth k. Then, by [48], $L(G_2^3) \leq 8$, $L(G_2^k) \leq 5$ if $k \geq 5$, $L(G_2^k) \leq 3$ if $k \geq 7$, and $L(G_2^k) \leq 2$ if $k \geq 11$. The main aim of this chapter is to generalize these results to (planar) simple digraphs, and to graphs resp. simple digraphs which are embeddable in other surfaces. We apply the same restrictions on minimum degree, cycles, and girth in the case of graphs, and in the case of simple digraphs we consider those with minimum in-degree $\delta^+ \geq 1$, prescribed girth k, and without 4-cycles. In Section 3.3 we determine upper bounds for the lightness of such simple digraphs embeddable in a surface S, whereas Section 3.4 is devoted to the case of graphs in S. S may be either one of the orientable surfaces S_γ, $0 \leq \gamma \leq 6$, or one of the non-orientable surfaces $N_{\overline{\gamma}}$, $1 \leq \overline{\gamma} \leq 9$, possibly even some other surface.

Whenever $k \geq 5$, the bounds are tight for the surfaces of nonnegative Euler characteristic, i.e., for the sphere, the torus, the projective plane and the Klein bottle, as shown in Section 3.6. In the case of other surfaces, the bounds depend on a topological parameter which is not exactly known for any of these surfaces (exept the double torus). The next section concerns this topological parameter.

3.2 The parameter $M(S)$

For the orientable surface S_γ, the *minimum edge number* $M(S_\gamma)$ of S is the minimum number of arcs a digraph with genus γ can have. It is called minimum 'edge' number since it is clear that, if we replace 'digraph' by 'graph' and 'arc' by 'edge' in the definition, we will obtain the same number. In the same way, for the nonorientable surface $N_{\overline{\gamma}}$, the *minimum edge number* $M(N_{\overline{\gamma}})$ of S is the minimum number of arcs a digraph with crosscapnumber $\overline{\gamma}$ can have. (Here we define the crosscapnumber of planar graphs to be 0.)

Although it is easy to define $M(S)$ for any surface S, it seems to be very hard to determine the parameter exactly or even to give good lower bounds for it. Obviously, $M(S_0) = 0$. By Kuratowski's Theorem [60], $M(S_1) = M(N_1) = 9$. Indeed, for a surface S_γ we need only consider the irreducible graphs for the surface $S_{\gamma-1}$, one of them has the fewest number m of edges amongst all irreducible graphs, this number m will equal $M(S_\gamma)$. For S_0, there are only two irreducible graphs, K_5 (with 10 edges), and $K_{3,3}$ (with 9 edges), thus $M(S_1) = 9$. For nonorientable surfaces we have the same argument.

Unfortunately, the number of irreducible graphs explodes with growing genus resp. crosscapnumber. Apart from the sphere, only for the projective plane a complete classification of all irreducible graphs is known: in a series of papers Glover et al. [43] and Archdeacon [12] proved that there are exactly 103 irreducible graphs for the projective plane, and listed them. The smallest one has 15 edges. Thus we have

Theorem 25. (Glover, Huneke and Wang [43]; Archdeacon [12]) *$M(N_2) = 15$.*

Corollary 26. *$M(N_{\overline{\gamma}}) \geq 15$ for $\overline{\gamma} \geq 2$.*

This lower bound for $M(N_{\overline{\gamma}})$ is not satisfactory, but there seems to be no better bound known so far.

Myrvold [67] classified all irreducible graphs for the torus with at most 11 vertices. By her results there are irreducible graphs with 18 edges, and every irreducible graph has at least 18 edges. Irreducible graphs with 12 or more vertices must also contain at least 18 edges since irreducible graphs have minimum degree 3. Thus we have

Theorem 27. (Myrvold [67]) *$M(S_2) = 18$.*

Corollary 28. *For any orientable surface S_γ with $\gamma \geq 2$, $M(S_\gamma) \geq 16 + \gamma$.*

Proof. This is an obvious induction on γ. For $\gamma = 2$ the statement is true by Myrvold's theorem [67]. Note that the deletion of an edge in a graph reduces the genus by at most one, which implies the rest. □

$\overline{\gamma}$	1	2	3	4	5	6	7	8	9	10	11	12	...
$M(N_{\overline{\gamma}}) \geq$	9	15	15	15	15	15	15	15	15	15	15	15	
$M(N_{\overline{\gamma}}) \leq$	9	15	19	23	25	29	33	35	39	41	45	47	

Table 3.1: Bounds for M (nonorientable case)

Upper bounds for $M(S)$ of a surface S are easier to find. The upper bounds for $M(S)$ in the Tables 3.1 resp. 3.2 are given by examples of $K_{m,n} - M_k$ the crosscapnumber resp. the genus of which are well-known [66, 65].

γ	1	2	3	4	5	6	7	8	9	10	11	12	...
$M(S_\gamma) \geq$	9	18	19	20	21	22	23	24	25	26	27	28	
$M(S_\gamma) \leq$	9	18	25	33	39	45	49	55	61	67	71	77	

Table 3.2: Bounds for M (orientable case)

3.3 The structure of digraphs in surfaces

In this section we will generalize Theorems 2.1. and 2.2. in He et al. [48], which examine the lightness of planar graphs, to simple digraphs embeddable in surfaces. First we define for a nonnegative integer k and a surface S

$$F_S(k) = \frac{M(S)k + M(S)}{(2\chi(S) + M(S))\,k + 2\chi(S) - 3M(S)}, \tag{3.2}$$

$$H_S = \frac{5M(S)}{10\chi(S) + M(S)}. \tag{3.3}$$

These parameters $F_S(k)$ and H_S (whenever well-defined and positive) will be the main part of the upper bounds discussed in Theorems 29, 30, 31, and 32. In order to simplify the notation we will write M resp. χ instead of $M(S)$ resp. $\chi(S)$ when there is only one surface S. Clearly, $F_S(k)$ is non-increasing when $k \longrightarrow \infty$ for $2\chi + M > 0$ and $k \geq k_0 > \frac{3M-2\chi}{2\chi+M}$.

Theorem 29. *Let S be a surface with Euler characteristic $\chi(S)$ and D be a simple digraph embeddable in S with $\delta^+(D) \geq 1$ and $g(D) \geq k$ for odd $k \geq 5$.*

(a) If $\chi(S) > 0$, then

$$L^+(D) \leq \left\lceil \frac{4}{k-3} \right\rceil.$$

(b) If $\chi(S) \leq 0$, and $M(S) + 2\chi(S) > 0$, and $k > \frac{3M(S)-2\chi(S)}{2\chi(S)+M(S)}$, then

$$L^+(D) \leq \lfloor F_S(k) \rfloor.$$

Proof. Assume $D = (V, E)$ is a counterexample. W.l.o.g. D is connected. So $g(D) \geq k$ for odd $k \geq 5$ and $L^+(D) \geq c + 1$ where

$$c = \left\lceil \frac{4}{k-3} \right\rceil \quad \text{in case (a), resp.,} \tag{3.4}$$

$$c = \lfloor F_S(k) \rfloor \quad \text{in case (b),} \tag{3.5}$$

and there is a 2-cell embedding which embeds D in S. By deleting all vertices v with $d^+(v) + d^-(v) = 1$ succesively and subdividing each arc (v, w) with

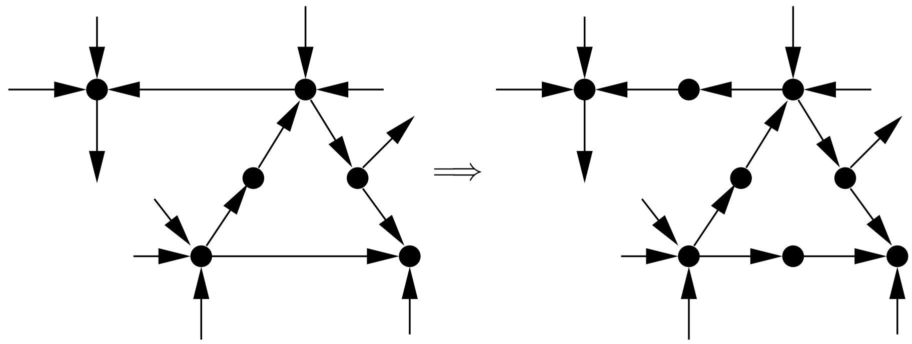


Figure 3.3: Subdividing the 'fat' arcs. A detail of the original digraph D is depicted on the left-hand side, a detail of the auxiliary digraph $\overline{D}$ on the right-hand side. (Here $c+1=3$.)

$d^+(v) \geq c+1$ and $d^+(w) \geq c+1$ once (maintaining the orientation) we obtain an auxiliary digraph $\overline{D} = (\overline{V}, \overline{E})$, see Fig. 3.3. Let

$$\begin{aligned} V_i &:= \left\{ v \in \overline{V} \,\middle|\, d^+(v) = i \right\}, \qquad\qquad i = 1, \ldots, c \\ V_{c+1} &:= \left\{ v \in \overline{V} \,\middle|\, d^+(v) \geq c+1 \right\}. \end{aligned}$$

Furthermore $n_i := \#V_i$, $m_i := \#\{(v,w) \in \overline{E} \,|\, w \in V_i\}$, $n := \#\overline{V}$, $m := \#\overline{E}$. $\overline{D}$ is bipartite with V_{c+1} forming one of the partite sets. Thus $g(\overline{D}) \geq k+1$ since k is odd. By the construction, $L^+(\overline{D}) \geq c+1$, $\delta^+(\overline{D}) \geq 1$, and

$$\delta^{\pm}(\overline{D}) \geq 2. \tag{3.6}$$

Like D, the auxiliary digraph $\overline{D}$ embeds in S, and for a fixed 2-cell embedding we have

$$f \leq \frac{2}{k+1} m$$

where f denotes the number of faces of $\overline{D}$.

Obviously,

$$\sum_{i=1}^{c+1} m_i = m. \tag{3.7}$$

In view of (3.6), $m_1 \leq m_{c+1}$, so that

$$-m_{c+1} \leq -m_1, \quad m_1 \leq \frac{m}{2} \tag{3.8}$$

Hence

$$\begin{aligned} n &= \sum_{i=1}^{c+1} n_i \leq m_1 + \sum_{i=2}^{c} \frac{m_i}{i} + \frac{m_{c+1}}{c+1} \\ &\leq \frac{m_1}{2} + \frac{m_1 + m_2 + \ldots + m_{c+1}}{2} + \frac{-c+1}{2c+2} m_{c+1} \\ &\overset{(3.7),(3.8)}{\leq} \frac{c+2}{2c+2} m = \frac{1}{2}\left(1 + \frac{1}{c+1}\right) m \end{aligned}$$

In case (a), by (3.4), $1/(c+1) \leq (k-3)/(k+1)$, which implies

$$n - m + f \leq \left(\frac{k-1}{k+1} - 1 + \frac{2}{k+1}\right) = 0,$$

and contradicts $\chi \geq 1$.

In case (b), by the preconditions, $F_S(k) > 0$. Then, by (3.5), $1/(c+1) < 1/F_S(k)$, hence

$$\begin{aligned} n - m + f &< \left(\frac{1}{2} + \frac{(2\chi + M)k + 2\chi - 3M}{2Mk + 2M} - 1 + \frac{2}{k+1}\right) m \\ &= \chi \frac{m}{M} \leq \chi. \end{aligned}$$

The last estimation holds since $\chi \leq 0$, and $0 < M \leq m$ as we may assume w.l.o.g. that D (and so $\overline{D}$) does not embed in a surface of lower genus resp. lower crosscapnumber than S. On the other hand, this is a contradiction, because by definition of Euler characteristics $n - m + f \geq \chi$. □

If we drop the prerequisite $\delta^+(D) \geq 1$, it is easy to see that the parameter $L^+(D)$ is not bounded by any constant. Think of an in-star, for example. The same problem occurs if we allow 4-cycles: for each $n \geq 1$, there is an orientation $\vec{K}_{2,2n}$ of the planar bipartite graph $K_{2,2n}$ with $\delta^+(\vec{K}_{2,2n}) \geq 1$ but $L^+(\vec{K}_{2,2n}) \geq n$. However, we may permit 3-cycles, as stated in the following theorem.

Theorem 30. *Let S be a surface with Euler characteristic $\chi(S)$ and D be a simple digraph embeddable in S with $\delta^+(D) \geq 1$ and $g(D) \geq 3$ which does not contain any 4-cycles.*

(a) If $\chi(S) > 0$, then $L^+(D) \leq 4$.

(b) If $\chi(S) \leq 0$ and $M(S) > -10\chi(S)$, then $L^+(D) \leq \lfloor H_S \rfloor$

Proof. Assume the theorem is false. W.l.o.g. we may assume that there is a connected counterexample $D = (V, E)$ with $\delta^{\pm}(D) \geq 2$ (cf. the proof of Theorem 29). Let $c = 4$ in case (a), and $c = \left\lfloor \frac{5M(S)}{10\chi(S)+M(S)} \right\rfloor$ in case (b). We define

$$\begin{aligned}
V_{c+1} &:= \{v \in V \mid d^+(v) \geq c+1\}, \\
T &:= \{(v,w) \in E \mid v \in V_{c+1} \wedge w \in V_{c+1}\}, \\
V_i &:= \{v \in V \mid \delta^+(v) = i\} \text{ for } 1 \leq i \leq c, \\
E_i &:= \{(v,w) \in E \mid w \in V_i\} \text{ for } 1 \leq i \leq c+1.
\end{aligned}$$

$n_i := \#V_i$, $m_i := \#E_i$, $n := \#V$, $m := \#E$, $t := \#T$. Let f_i be the number of i-faces, i.e., of faces bounded by exactly i arcs, and f the number of faces. Since $f_4 = 0$ we have

$$3f_3 + 5f_5 + 6f_6 + 7f_7 + \ldots = 2m,$$

and further

$$f = \frac{1}{5} \cdot \sum_{i \geq 3} 5f_i \leq \frac{1}{5}\left(2f_3 + \sum_{i\geq 3} if_i\right) = \frac{2}{5}m + \frac{2}{5}f_3. \tag{3.9}$$

As in the preceding proof we will also consider the digraph $\overline{D} = (\overline{V}, \overline{E})$ obtained from D by subdividing each arc from T once and maintaining the orientation. We define $\overline{n}_i$ resp. $\overline{m}_i$ by replacing V and E by $\overline{V}$ and $\overline{E}$, respectivly, in the definitions leading to n_i resp. m_i. Obviously, $\overline{n}_1 = n_1 + t$, $\overline{n}_i = n_i$ for $i \geq 2$, $\overline{m} = m + t$. As above we state that

$$\sum_{i=1}^{c+1} \overline{m}_i = \overline{m}, \tag{3.10}$$

$$-\overline{m}_{c+1} \leq -\overline{m}_1, \tag{3.11}$$

from which

$$\overline{m}_1 \leq \overline{m}/2 \tag{3.12}$$

follows. Thus we conclude

$$\begin{aligned}
n &= \sum n_i = \sum \overline{n}_i - t \leq \sum \frac{\overline{m}_i}{i} - t \\
&\leq \frac{\overline{m}_1}{2} + \frac{\overline{m}_1 + \overline{m}_2 + \ldots + \overline{m}_{c+1}}{2} + \left(\frac{1}{c+1} - \frac{1}{2}\right)\overline{m}_{c+1} - t \\
&\overset{(3.10)}{=} \frac{\overline{m}}{2} + \frac{\overline{m}_1}{2} - \frac{c-1}{2(c+1)}\overline{m}_{c+1} - t \\
&\overset{(3.11),(3.12)}{\leq} \frac{c+2}{2(c+1)}\overline{m} - t = \frac{1}{2}\left(1 + \frac{1}{c+1}\right)m - \frac{c}{2(c+1)}t
\end{aligned} \tag{3.13}$$

Note as well that $f_3 \leq t$ since every 3-face of D contains an arc from T and there are no adjacent 3-faces (otherwise there would be a 4-cycle). Combining this with (3.9) and (3.13) yields in case (a)

$$n - m + f \leq \frac{2}{5}(f_3 - t) \leq 0$$

since $1/(c+1) = 1/5$. In case (b), by the prerequisite $H_S > 0$, we therefore have $1/(c+1) < 1/H_S$. Furthermore, $c \geq 4$ because $M > 0$, which implies $-c/(2c+2) \leq -2/5$. W.l.o.g. we may assume again that D does not embed in a surface of lower genus resp. lower crosscapnumber, so $M \leq m$. We conclude

$$n - m + f < \left(\frac{1}{2} + \frac{10\chi + M}{10M} - 1 + \frac{2}{5}\right) m + \frac{2}{5}(f_3 - t) \leq \chi \frac{m}{M} \leq \chi.$$

In both cases we obtain a contradiction with respect to the definition of the Euler characteristic. $\square$

3.4 The structure of graphs in surfaces

Theorem 2.1. in He et al. [48], examines the lightness of planar graphs. The following theorem generalizes this to graphs embeddable in arbitrary surfaces.

Theorem 31. *Let S be a surface of Euler characteristic $\chi(S)$ and $G = (V, E)$ be a graph embeddable in S with $\delta(G) \geq 2$ and $g(G) \geq k$ for odd $k \geq 5$.*

(a) If $\chi(S) > 0$, then

$$L(G) \leq \left\lceil \frac{k+5}{k-3} \right\rceil.$$

(b) If $\chi(S) \leq 0$ and $M(S) + 2\chi(S) > 0$ and $k > \frac{3M(S) - 2\chi(S)}{2\chi(S) + M(S)}$, then

$$L(G) \leq \lfloor 2F_S(k) \rfloor.$$

Proof. (a) has been proven by He et al. [48] for planar graphs. The same proof holds for graphs embeddable in the projective plane. We are left to consider (b). Again we assume that a connected graph $G = (V, E)$ is a counterexample. Hence $g(G) \geq k \geq 5$ for odd k and $L(G) \geq c + 1$ where

$$c = \lfloor 2F_S(k) \rfloor, \tag{3.14}$$

and there is a 2-cell embedding which embeds G in S. By subdividing each edge vw with $d(v) \geq c+1$ and $d(w) \geq c+1$ once we obtain (as in the preceding proof) an auxiliary graph $\overline{G}$ with vertex set $\overline{V}$ and edge set $\overline{E}$ with

$g(\overline{G}) \geq k+1$ (since k is odd), $L(\overline{G}) \geq c+1$, and $\delta(\overline{G}) \geq 2$. Again this construction produces a bipartite graph with partite sets

$$\begin{aligned} V_1 &:= \left\{v \in \overline{V} \,\middle|\, d(v) \leq c\right\}, \text{ and} \\ V_2 &:= \left\{v \in \overline{V} \,\middle|\, d(v) \geq c+1\right\}. \end{aligned}$$

Further $n_i := \#V_i$, $n := \#\overline{V}$, $m := \#\overline{E}$. Note that here we count the edges, not the arcs. W.l.o.g. $\overline{G}$ does not embed in a surface of lower genus resp. lower crosscapnumber than S. Since $\overline{G}$ is homeomorphic to G, $\overline{G}$ embeds in S, and for a fixed 2-cell embedding we have

$$f \leq \frac{2}{k+1} m$$

where f denotes the number of faces of $\overline{G}$. The number of vertices is bounded by

$$n = n_1 + n_2 \leq \left(\frac{1}{2} + \frac{1}{c+1}\right) m,$$

c.f. He et al. [48]. By the preconditions and by (3.14), $1/(c+1) < 1/\left(2F_S(k)\right)$. As in the proof of Theorem 29 we obtain the contradiction $n - m + f < \chi$. $\square$

By the same refinement which extends the proof of Theorem 29 to a proof of Theorem 30 the proof of Theorem 31 may be modified to prove the following

Theorem 32. *Let S be a surface of Euler characteristic $\chi(S)$ and G be a graph embeddable in S with $\delta^+(G) \geq 2$ and $g(G) \geq 3$ which does not contain any 4-cycles.*

(a) If $\chi(S) > 0$, then $L(G) \leq 9$.

(b) If $\chi(S) \leq 0$ and $M(S) > -10\chi(S)$, then $L(G) \leq \lfloor 2H_S \rfloor$

Proof. Let G be a connected graph with vertex set V and edge set E which is a counterexample to the theorem. So $L(G) \geq c+1$ where $c = 9$ in the case of (a), and $c = \lfloor 2H_S \rfloor$ in the case of (b). Furthermore there is a 2-cell embedding which embeds G in S. Let

$$\begin{aligned} V_1 &= \left\{v \in V \,\middle|\, d(v) \leq c\right\}, \\ V_2 &= \left\{v \in V \,\middle|\, d(v) \geq c+1\right\}, \\ T &= \left\{vw \in E \,\middle|\, v \in V_2 \wedge w \in V_2\right\}, \end{aligned}$$

$n_i = \#V_i$ for $i = 1, 2$, $n = \#V$, $m = \#E$, and $t = \#T$.

Let f_i be the number of i-faces as in the proof of Theorem 30, and f be the number of faces. Again, since $f_4 = 0$, we have

$$3f_3 + 5f_5 + 6f_6 + 7f_7 + \ldots = 2m,$$

from which

$$f \leq \frac{2}{5}m + \frac{2}{5}f_3 \tag{3.15}$$

follows, cf. the proof of Theorem 30.

We will consider the graph $\overline{G}$ with vertex set $\overline{V}$ and edge set $\overline{E}$, obtained from G by subdividing each edge from T once. $\overline{G}$ is bipartite. By construction, $L(\overline{G}) \geq c+1$. We define $\overline{n}$, $\overline{n}_i$ resp. $\overline{m}$ by replacing V, V_i resp. E by $\overline{V}$, $\overline{V}_i$ resp. $\overline{E}$ in the definitions leading to n, n_i resp. m. Then we have $\overline{n}_1 = n_1 + t$, $\overline{n}_2 = n_2$, and $\overline{m} = m + t$. Since $\delta(\overline{G}) = \delta(G) \geq 2$ we have

$$\overline{n}_1 \leq \frac{1}{2}\overline{m}.$$

Therefore we calculate

$$\begin{aligned} n &= n_1 + n_2 = \overline{n}_1 + \overline{n}_2 - t \leq \left(\frac{1}{2} + \frac{1}{c+1}\right)\overline{m} - t \\ &= \left(\frac{1}{2} + \frac{1}{c+1}\right) m - \left(\frac{1}{2} - \frac{1}{c+1}\right) t. \end{aligned} \tag{3.16}$$

Also note that

$$f_3 \leq t \tag{3.17}$$

since every 3-face of G contains an edge of T, and there are no adjacent 3-faces (otherwise there would be a 4-cycle).

In case (a) we obtain

$$n - m + f \overset{(3.15),(3.16)}{\leq} \frac{3}{5}m - \frac{2}{5}t - m + \frac{2}{5}m + \frac{2}{5}f_3 = \frac{2}{5}(f_3 - t) \overset{(3.17)}{\leq} 0$$

since $1/(c+1) = 1/10$. In case (b), by the prerequisite $H_S > 0$, therefore

$$\frac{1}{c+1} < \frac{1}{2H_S}. \tag{3.18}$$

Furthermore $c \geq 9$ because $M > 0$, which implies

$$-\left(\frac{1}{2} - \frac{1}{c+1}\right) \leq -\frac{2}{5}. \tag{3.19}$$

We may assume again that G does not embed in a surface of lower genus resp. lower crosscapnumber, so $M \leq m$. By (3.15), (3.16), (3.17), (3.18), and (3.19) we conclude

$$\begin{aligned} n - m + f &< \left(\frac{1}{2} + \frac{10\chi + M}{10M}\right) m - \frac{2}{5}t - m + \frac{2}{5}m + \frac{2}{5}f_3 \\ &\leq \frac{2}{5}(f_3 - t) + \chi\frac{m}{M} \leq \chi. \end{aligned}$$

As in case (a) this is also a contradiction with respect to the definition of Euler characteristic. □

Note that, in the context of Theorem 32 (a), He et al. [48] achieved the tighter bound 8 for planar graphs using special properties of cycles in planar embeddings.

3.5 Application range of Theorems 29, 30, 31, and 32

In Theorems 29 (b) and 31 (b) we assume $M(S)+2\chi(S) > 0$. It might be true that this inequality is valid for any nonorientable or any orientable surface S. However, from the lower bounds for $M(S)$ in Section 3.2 we may only deduce that it is valid for the surfaces $N_{\overline{\gamma}}$, $2 \le \overline{\gamma} \le 9$, since then, by Corollary 26,

$$M(N_{\overline{\gamma}}) + 2\chi(N_{\overline{\gamma}}) \ge 19 - 2\overline{\gamma} > 0,$$

and that it is valid for the surfaces S_γ, $1 \le \gamma \le 6$, since then, by Corollary 28,

$$M(S_\gamma) + 2\chi(S_\gamma) \ge 20 - 3\gamma > 0.$$

Better lower bounds for $M(S)$ not only tighten the bounds of the theorems, but also enlarge the application range of the theorems.

The situation is a bit different for Theorems 30 (b) and 32 (b). Here we have the precondition $M + 10\chi > 0$. The upper bounds of Tables 3.1 and 3.2 suggest that for the surfaces $N_{\overline{\gamma}}$, $\overline{\gamma} \ge 5$, and for the surfaces S_γ, $\gamma \ge 2$, the condition does not hold. On the other hand the condition is true for the surfaces $N_{\overline{\gamma}}$, $2 \le \overline{\gamma} \le 3$, since then, by Corollary 26,

$$M(N_{\overline{\gamma}}) + 10\chi(N_{\overline{\gamma}}) \ge 35 - 10\overline{\gamma} > 0,$$

and for the surface S_1 since, by Kuratowski's Theorem,

$$M(S_1) + 10\chi(S_1) = 9 > 0.$$

So improving the lower bounds for $M(N_4)$ might include N_4 into the application range of Theorems 30 (b) and 32 (b), other improvements do not have any effect in this case.

In Tables 3.3 resp. 3.4 the lightness-values are given for graphs embeddable in a surface of genus γ resp. crosscapnumber $\overline{\gamma}$ with girth k, minimum degree 2, and without 4-cycles. In order to obtain the lightness-values for simple digraphs embeddable in the respective surface with girth k and without 4-cycles and with minimum in-degree 1 the floor of the values in the tables divided by 2 has to be taken.

S / k	$s_k(0)$	$s_k(1)$	$s_k(2)$	$s_k(3)$	$s_k(4)$	$s_k(5)$
3	(8)	10	∞	∞	∞	∞
5	(5)	6	18	∞	∞	∞
7	(3)	4	7	25		
9		3	5	11		
11	(2)		4	8	30	
13		2		6	17	
15			3		13	
17				5	11	126
19					10	52
21					9	35
23				4	8	28
25						23
27					7	21
29						19
31						17
33						16
35					6	15
37			2			
39						14
43						13
47						12
51				3		
57						11
61					5	
71						10
105						9
253						8

Table 3.3: Upper bounds $s_k(\gamma)$ for the lightness of graphs with girth at least k and without 4-cycles and minimum degree at least 2 in the orientable surface S_γ. The bounds in brackets were already obtained by He et al. [48].

3.6 Tightness of the bounds

There is a series of corollaries from Theorems 29 and 31. Note that, if $\chi = 0$, the parameter M is canceled out in the expression $F_S(k)$.

Corollary 33. *Let D resp. G be a simple digraph with $\delta^+(D) \geq 1$ resp. a graph with $\delta(G) \geq 2$ embeddable in the sphere or the projective plane. Then*

a) $L(G) \leq 5$ *if* $g(G) \geq 5$,

b) $L(G) \leq 3$ *if* $g(G) \geq 7$,

c) $L(G) \leq 2$ *if* $g(G) \geq 11$,

a') $L^+(D) \leq 2$ *if* $g(D) \geq 5$,

b') $L^+(D) \leq 1$ *if* $g(D) \geq 7$.

Corollary 33 is tight in the projective plane case: For a) consider the complete graph K_6 and subdivide each edge once. By the result of Ringel

S k	$n_k(1)$	$n_k(2)$	$n_k(3)$	$n_k(4)$	$n_k(5)$	$n_k(6)$	$n_k(7)$	$n_k(8)$
3	9	10	30	∞	∞	∞	∞	∞
5	5	6	10	30	∞	∞	∞	∞
7	3	4	5	8	20			
9		3	4	6	10	30		
11	2		3	5	7	15		
13		2		4	6	11	42	
15					5	9	24	
17				3		8	18	
19						7	15	
21			2		4		13	110
23						6	12	60
25							11	43
27							10	35
29								30
31						5	9	26
33								24
35								22
37							8	21
39								20
41					3			19
43								18
45								17
49							7	16
53								15
61				2		4		14
71								13
85							6	
87								12
121								11
221								10

Table 3.4: Upper bounds $n_k(\overline{\gamma})$ for the lightness of graphs with girth at least k and without 4-cycles and minimum degree at least 2 in the nonorientable surface $N_{\overline{\gamma}}$.

and Youngs [76] K_6, and thus the resulting graph, can be embedded in the projective plane. It has girth 6, minimum degree 2, and lightness 5. For b) consider the graph depicted in Fig. 3.4. For a') take the example from a) and orient the edges in such a way that $d^+(v) \geq 1$ for every vertex v and every vertex of degree 5 has in-degree at least 2. The tightness in the undirected sphere case was already proven in [48], in the directed sphere case a') edges are oriented suitably as above.

Corollary 34. *Let D resp. G be a simple digraph with $\delta^+(D) \geq 1$ resp. a graph with $\delta(G) \geq 2$ embeddable in the torus or the Klein bottle. Then*

a) $L(G) \leq 6$ *if* $g(G) \geq 5$,

b) $L(G) \leq 4$ *if* $g(G) \geq 7$,

c) $L(G) \leq 3$ *if* $g(G) \geq 9$,

d) $L(G) \leq 2$ *if* $g(G) \geq 13$,

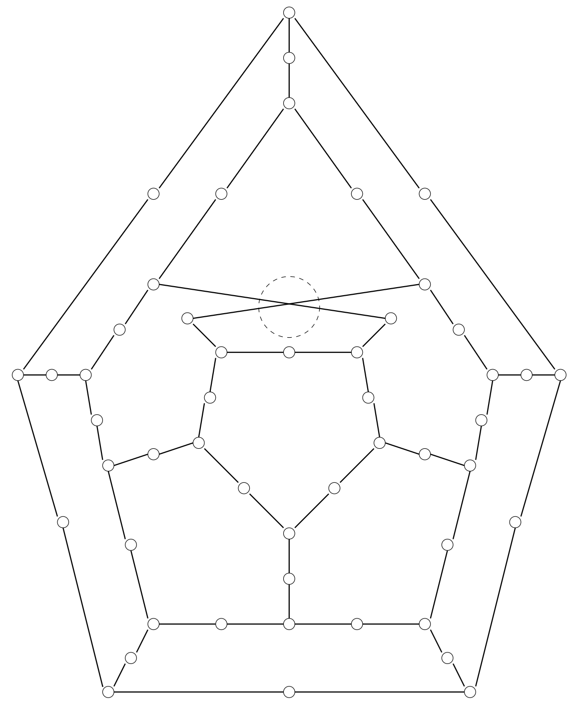


Figure 3.4: Projective twisted dodekahedron

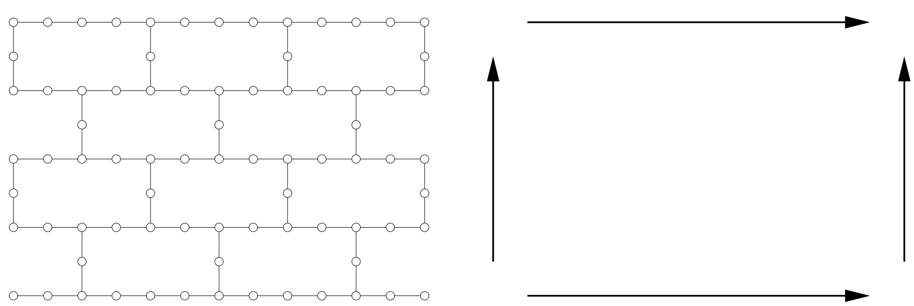


Figure 3.5: Graph with torus identification

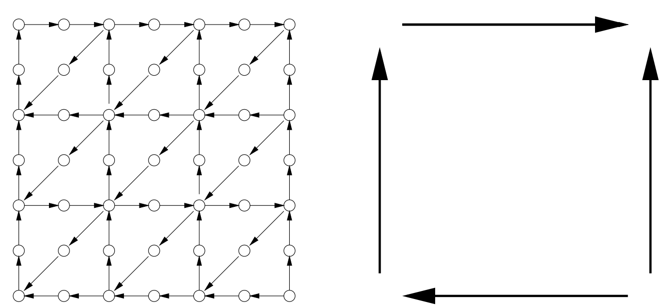


Figure 3.6: Digraph with Klein bottle identification

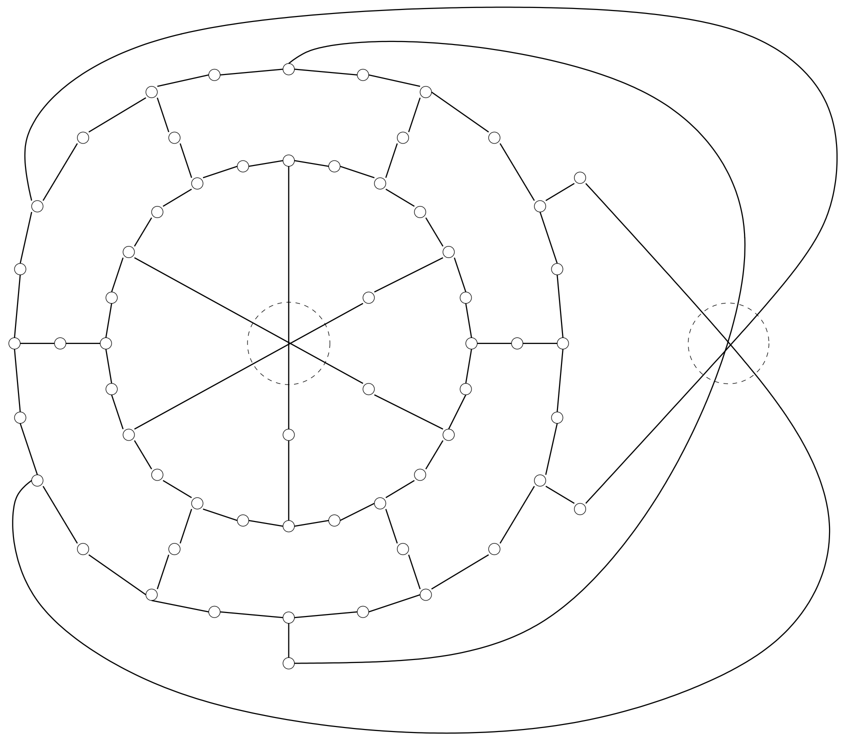


Figure 3.7: Double-twisted double-clock

a') $L^+(D) \leq 3$ *if* $g(D) \geq 5$,

b') $L^+(D) \leq 2$ *if* $g(D) \geq 7$,

c') $L^+(D) \leq 1$ *if* $g(D) \geq 9$.

Corollary 34 is tight in the torus case: For a) consider the complete graph K_7 and subdivide each edge once. The resulting graph can be embedded in the torus and has girth 6, minimum degree 2, and lightness 6. For b) consider the graph obtained from subdividing each edge once in either the complete bipartite graph $K_{4,4}$ or the 4-dimensional hypercube. Both examples have genus 1, girth 8, minimum degree 2 and lightness 4. Furthermore, c) is tight since the graph G of Fig. 3.5 with $g(G) = 12$, $\delta(G) = 2$, and $L(G) = 3$ can be embedded in the torus. For a') resp. b') we may take the same examples as for a) resp. b) and orient the edges in such a way that the minimum in-degree is 1 and the positive lightness 3 resp. 2. Such orientations can be found easily.

Corollary 34 is tight in the Klein bottle case: For a) consider the graph G of Fig. 3.6 (without the orientation) with $g(G) = 6$, $\delta(G) = 2$, and

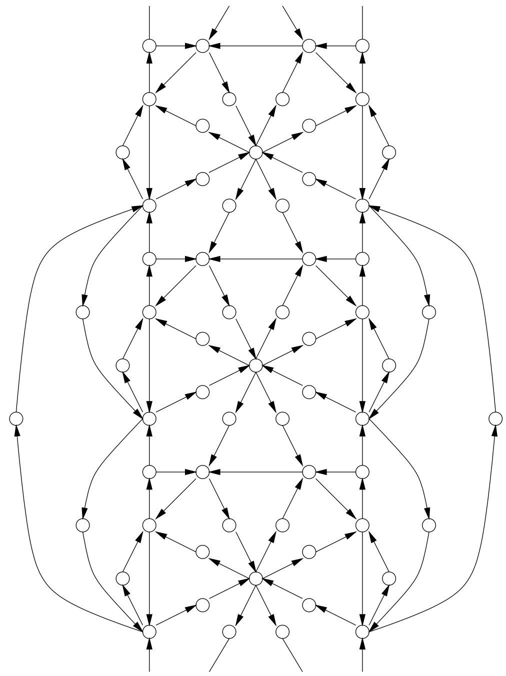


Figure 3.8: Digraph with 63 vertices and 105 arcs. The upper and the lower border of this illustration have to be glued together, so that a planar digraph is obtained. (Imagine rolling the rectangle on a cylinder.)

$L(G) = 6$. An example for the tightness of b) is obtained from $K_{4,4}$ by subdividing each edge once. This graph has crosscapnumber 2, girth 8, minimum degree 2, and lightness 4. For c) consider the double-twisted doubleclock which is depicted in Fig. 3.7. For a') consider the graph of Fig. 3.6 again, for b') the subdivision of $K_{4,4}$ with an obvious orientation.

We do not know whether the result of Theorem 30 is tight, not even in the case of planar digraphs, i.e. whether there exists a planar digraph D with $\delta^+(D) \geq 1$ and which does not contain 4-cycles, but having $L^+(D) = 4$. Figure 3.8 depicts a planar digraph D meeting the preconditions of Theorem 30 with $L^+(D) = 3$.

3.7 Application to graph coloring and marking games

The application of our results to game coloring numbers is based on a simple but important observation of Zhu [86] on edge partitions. Let $G = (V, E)$, $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be graphs with the same vertex set. $G_1|G_2$ is an *edge partition* of G if $E = E_1 \dot\cup E_2$.

Observation 35. (Zhu [86]; Guan and Zhu [46])
If a graph G has an edge partition $G_1|G_2$, then $\mathrm{col}_g(G) \leq \mathrm{col}_g(G_1) + \Delta(G_2)$ for any version g of the marking game.

We may define an *arc partition* $D_1|D_2$ of a digraph $D = (V, E)$ in line, i.e. if $D_1 = (V, E_1)$ and $D_2 = (V, E_2)$ are digraphs, and $E = E_1 \dot{\cup} E_2$.

Observation 36. *If a digraph D has an arc partition $D_1|D_2$, then*

$$\mathrm{col}_g(D) \leq \mathrm{col}_g(D_1) + \Delta^+(D_2)$$

for any version g of the directed marking game.

Indeed we may generalize these observations to the (a, b)-marking game.

Observation 37. *If a graph G has an edge partition $G_1|G_2$, then*

$$^{(a,b)}\mathrm{col}_g(G) \leq {}^{(a,b)}\mathrm{col}_g(G_1) + \Delta(G_2)$$

for any version g of the (a, b)-marking game.

Observation 38. *If a digraph D has an arc partition $D_1|D_2$, then*

$$^{(a,b)}\mathrm{col}_g(D) \leq {}^{(a,b)}\mathrm{col}_g(D_1) + \Delta^+(D_2)$$

for any version g of the directed marking game.

A graph G is called *i-hereditary* if, for every subgraph H of G,

$$\delta(H) \leq 1 \quad \text{or} \quad L(H) \leq i.$$

Let $u(S, k)$ be an upper bound for the lightness of graphs embeddable in a surface S with girth at least k, minimum degree at least 2, and without 4-cycles. Possibly, $u(S, k) = \infty$. For example, $u(S_\gamma, k) = s_k(\gamma)$ can be taken from Table 3.3, and $u(N_{\overline{\gamma}}, k) = n_k(\overline{\gamma})$ can be taken from Table 3.4. In general, the values $u(S, k)$ can be calculated with Theorems 31 and 32. Since every subgraph of a graph G embeddable in S with girth at least k and without 4-cycles embeds in S and has girth at least k and no 4-cycles, too, G is $u(S, k)$-hereditary.

He et al. proved the following

Lemma 39. (He, Hou, Lih, Shao, Wang and Zhu [48])
If a graph G is i-hereditary, G has an edge partition $G_1|G_2$, so that G_1 is an undirected forest and $\Delta(G_2) \leq i - 1$.

By a result of Faigle et al. [40], the game coloring number of a forest is at most 4. Kierstead [53] obtained the following generalization of this result.

Theorem 40. (Kierstead [53])
For any variant of the (a,b)-marking game,

$$\begin{aligned} {}^{(a,b)}\mathrm{col}_g(\mathcal{F}) &= \infty \text{ if } a < b \\ {}^{(a,b)}\mathrm{col}_g(\mathcal{F}) &= b+3 \text{ if } b \le a < 3b \\ {}^{(a,b)}\mathrm{col}_g(\mathcal{F}) &= b+2 \text{ if } 3b \le a \end{aligned}$$

Theorem 41. *For a graph G embeddable in a surface S with girth at least k and without 4-cycles,*

$$\begin{aligned} {}^{(a,b)}\mathrm{col}_g(G) &\le u(S,k)+b+2 \text{ if } b \le a < 3b \\ {}^{(a,b)}\mathrm{col}_g(G) &\le u(S,k)+b+1 \text{ if } 3b \le a \end{aligned}$$

for any version g of the (a,b)-marking game.

Proof. By Lemma 39, since G is $u(S,k)$-hereditary, G has an edge partition $G_1|G_2$, so that G_1 is a forest and $\Delta(G_2) \le u(S,k)-1$. By Theorem 40, ${}^{(a,b)}\mathrm{col}_g(G_1) \le b+3$ if $b \le a < 3b$, and ${}^{(a,b)}\mathrm{col}_g(G_1) \le b+2$ if $3b \le a$. The combination of these facts with Observation 37 yields the theorem. □

For $a = b = 1$, in Table 3.5 resp. Table 3.6 these bounds which result from Theorems 31 and 32 and Theorem 41 are given explicitly for the surfaces S_γ, $0 \le \gamma \le 5$, resp., $N_{\overline{\gamma}}$, $1 \le \overline{\gamma} \le 8$. For reasons of clarity and space, the bounds for S_6 and N_9 are omitted. By Observation 6, these numbers are bounds for the respective game chromatic numbers, too.

One method to obtain bounds for the (a,b)-game coloring number of simple digraphs is to use Lemma 39 again in conjunction with Observation 38 and Theorem 11. So we obtain the main result of this section:

Theorem 42. *For a digraph D which is the orientation of a graph G embeddable in a surface S with girth at least k and without 4-cycles,*

$${}^{(a,b)}\mathrm{col}_g(D) \le u(S,k)+b+1$$

for any version g of the (a,b)-marking game with $1 \le b \le a$.

Proof. By Lemma 39, since G is $u(S,k)$-hereditary, G has an edge partition $G_1|G_2$, so that G_1 is an undirected forest and $\Delta(G_2) \le u(S,k)-1$. Let V be the vertex set of D, E the arc set of D, E_1 the arc set of G_1, and E_2 the arc set of G_2. Define $D_1 = (V, E \cap E_1)$ and $D_2 = (V, E \cap E_2)$. Then $D_1|D_2$ is an arc partition of D, so that D_1 is a directed forest and $\Delta^+(D_2) \le \Delta(G_2) \le u(S,k)-1$. By Theorem 11, ${}^{(a,b)}\mathrm{col}_g(D_1) \le b+2$ if $1 \le b \le a$. By Observation 38,

$${}^{(a,b)}\mathrm{col}_g(D) \le {}^{(a,b)}\mathrm{col}_g(D_1) + \Delta^+(D_2) \le u(S,k)+b+1$$

if $1 \le b \le a$. □

Guan and Zhu [46] proved that the game chromatic number of outerplanar graphs is at most 7, a bound which they obtained by analyzing that outerplanar graphs have an edge partition $G_1|G_2$, where G_1 is an undirected forest and $\Delta(G_2) \le 3$. By this result, in the same way as Theorem 42, we obtain

Theorem 43. *Let $\vec{\mathcal{O}}$ be the class of orientations of outerplanar graphs, $a \ge b \ge 1$, and g be any version of the (a,b)-marking game. Then*

$$^{(a,b)}\mathrm{col}_g(\vec{\mathcal{O}}) \le b + 5.$$

Corollary 44. *Let $\vec{\mathcal{O}}$ be the class of orientations of outerplanar graphs and $\vec{\mathcal{O}}_{acy}$ be the subclass of acyclic orientations of outerplanar graphs. Then*

$$4 \le \chi_g(\vec{\mathcal{O}}_{acy}) \le \chi_g(\vec{\mathcal{O}}) \le 6$$

Proof. The upper bound follows from Theorem 43.

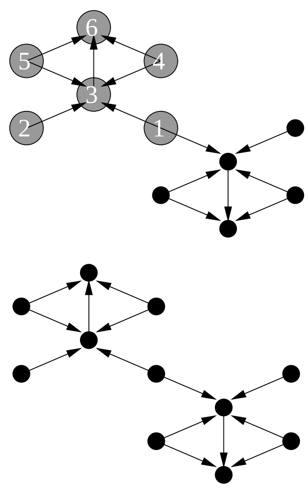


Figure 3.9: The digraph of Theorem 43

For the proof of the lower bound, consider the digraph of Fig. 3.9, which is an acyclic orientation of an outerplanar graph. We illustrate a winning strategy for Bob with 3 colors. In her first move, Alice colors a vertex in the lower component. Then Bob colors vertex 1 with color 1. W.l.o.g. Alice colors a vertex different from the numbered vertices. Then Bob colors vertex 4 with color 2. If Alice colors vertex 3 (with color 3), then Bob colors vertex 5 with color 1, and vertex 6 cannot be colored any more. Otherwise, Bob colors either vertex 2 or vertex 5 with color 3, and vertex 3 cannot be colored any more. Thus Bob wins. □

$k \geq$	$s_k(0)$	$s_k(1)$	$s_k(2)$	$s_k(3)$	$s_k(4)$	$s_k(5)$
3	(10)[80]	(10)[80]	(22)[52]	(24)[52]	(26)[52]	(27)[52]
5	(8)[48]	(8)[80]	21	(24)[52]	(26)[52]	(27)[52]
7	(6)[48]	(6)[80]	10			
9			8	14		
11	(5)[48]	(5)[80]	7	8		
13				9	20	
15			6		16	
17				8	14	
19					13	
21					12	
23				7	11	
25						26
27					10	24
29						22
31						20
33						19
35					9	18
37			5			
39						17
43						16
47						15
51				6		
57						14
61					8	
71						13
105						12
253						11
	$s(0)$	$s(1)$	$s(2)$	$s(3)$	$s(4)$	$s(5)$
	(17)[88]	(20)[52]	(22)[52]	(24)[52]	(26)[52]	(27)[52]

Table 3.5: Upper bounds $s_k(\gamma)$ for the game coloring number of graphs embeddable in the orientable surface S_γ with girth at least k and without 4-cycles. Also the best-known upper bounds $s(\gamma)$ for the game coloring number of graphs embeddable in S_γ in general. Previously known bounds are in brackets. The superscript numbers refer to the bibliography. For the given surfaces, our results do not provide better bounds if the girth is augmented, without improving the lower bounds for $M(S)$.

3.8 Another application idea

In Section 3.7 we determined upper bounds for directed coloring numbers of a simple digraph by using our result concerning the lightness of a graph. The conjecture that these bounds can be improved by applying the results concerning the positive lightness of a simple digraph seems to suggest itself. A first step towards this conjecture is the following theorem which makes use of a refined definition of i-hereditary. A digraph D is called *i-$^+$hereditary* if, for every subdigraph H of D, $\delta^+(H) = 0$ or $L^+(H) \leq i$.

Theorem 45. *Let $i \geq 0$. An i-$^+$hereditary digraph D has an arc partition $D_1|D_2$, so that D_1 is acyclic, i.e., does not contain a directed cycle, and $\Delta^+(D_2) \leq i$.*

$k \geq$	$n_k(1)$	$n_k(2)$	$n_k(3)$	$n_k(4)$	$n_k(5)$	$n_k(6)$	$n_k(7)$	$n_k(8)$
3	(10)[80]	(10)[80]	33	???	???	???	???	???
5	8	(8)[80]	13	33	???	???	???	???
7	6	(6)[80]	8	11	23	???	???	???
9			7	9	13	33	???	???
11	5	(5)[80]	6	8	10	18	???	???
13				7	9	14	45	???
15					8	12	27	???
17				6		11	21	???
19						10	18	???
21			5		7		16	113
23						9	15	63
25							14	46
27							13	38
29								33
31						8	12	29
33								27
35								25
37							11	24
39								23
41					6			22
43								21
45								20
49							10	19
53								18
61				5		7		17
71								16
85							9	
87								15
121								14
221								13
	$n(1)$	$n(2)$	$n(3)$	$n(4)$	$n(5)$	$n(6)$	$n(7)$	$n(8)$
	(19)[86]	???	???	???	???	???	???	???

Table 3.6: Upper bounds $n_k(\overline{\gamma})$ for the game coloring number of graphs embeddable in the nonorientable surface $N_{\overline{\gamma}}$ with girth at least k and without 4-cycles. Also upper bounds $n(\overline{\gamma})$ for the game coloring number of graphs embeddable in $N_{\overline{\gamma}}$. Previously known bounds are in brackets. The superscript numbers refer to the bibliography. For the entries with question marks, it is not known whether bounds exist.

Proof. We proceed by induction on the number of arcs. If there is no arc the statement is trivial. If $\delta^+(D) = 0$, there is an arc (v, w) with $d^+(v) = 0$, and by induction hypothesis an arc partition $D_1'|D_2'$ of $D' = D - (v, w)$ exists with the desired properties for D'. Set $D_2 = D_2'$ and $D_1 = D_1' + (v, w)$. D_1 is acyclic since D_1' contains no directed cycle and $d^+(v) = 0$. On the other hand, in case $\delta^+(D) > 0$, there is an arc $e = (v, w)$ with $L^+(e) \leq i$, and by induction hypothesis an arc partition $D_1'|D_2'$ of $D' = D - e$ with the desired properties for D'. Let $D_2 = D_2' + e$ and $D_1 = D_1'$. We have $d^+_{D_2}(v) \leq d^+_D(v) \leq i$, and $d^+_{D_2}(w) \leq i$. So $\Delta^+(D_2) \leq i$. In both cases, $D_1|D_2$ is the required edge partition. $\square$

In order to apply Observation 38, however, we have to determine the directed coloring number of acyclic simple digraphs embeddable in a given surface with a given girth and without 4-cycles. Maybe, this problem is as difficult as the general (not necessarily acyclic) case. Thus we have the following

Open question. Determine good upper bounds for ${}^{(a,b)}\mathrm{col}_g(D)$ for acyclic digraphs D which are embeddable in a fixed surface, and have given girth and do not contain 4-cycles.

Future work. Our discussion of lightness (and weight) leaves some interesting open problems, too. Here we mention some of them.

Open question. Characterize the class of digraphs D with $L^-(D) = L^+(D)$.

Open question. For fixed $n_1, n_2 \geq 0$, characterize the class of digraphs D with $R_1(D) = n_1$ and $R_2(D) = n_2$.

Open question. Discuss the tightness of the bounds of Theorems 29, 30, 31 and 32 in the case of surfaces of negative Euler characteristic.

The most demanding task for future research on the subjects introduced in this chapter will be to examine the topological parameter $M(S)$.

Open question. For a surface S, determine M(S). Or, at least, determine good lower bounds for $M(S)$.

Chapter 4

The incidence game chromatic number

This chapter is devoted to the game chromatic number of a special type of graphs: incidence graphs. The *incidence graph* of a graph $G = (V, E)$ with edge set $\tilde{E}$ is the digraph $G^I = (V^I, E^I)$ the vertex set V^I of which is the set of all incidences of G. An *incidence* of G is a pair (v, e) with $v \in V$ and $e \in \tilde{E}$, so that v is incident with e. Two distinct incidences (v, e) and (w, f) are *adjacent* if $v = w$ or $e = f$ or v is incident with f or w is incident with e. (In this definition, the conditions $v = w$ and $e = f$ are redundant.) See Fig. 4.1 for examples and counterexamples for the adjacency of incidences. $((v, e), (w, f))$ is defined to be in E^I if, and only if, (v, e) and (w, f) are adjacent. Because of the symmetry of the adjacency relation of incidences in this case $((w, f), (v, e)) \in E^I$, too. Thus G^I is a graph.

Later we will also consider incidences of the orientation $\vec{G}$ of a graph G. These incidences will be denoted in the form $(v, \vec{e})$, where v is a vertex and $\vec{e}$ is an arc contained in an edge e of G. However, by $(v, \vec{e})$ we mean the same object as (v, e). The orientation of the graph will only help us to simplify the

(a) (b)

Figure 4.1: (a) pairs of adjacent incidences (b) pairs of non-adjacent incidences. A white dot denotes a vertex, a black dot on an edge e next to a vertex v denotes the incidence (v, e).

proof of Theorem 46.

The *incidence game chromatic number* $\iota_g(G)$ is the game chromatic number $\chi_g(G^I)$ of G^I. Whenever we consider the incidence game chromatic number of a graph, with the exception of the proof of Theorem 48, we will not explicitly consider the coloring game on the vertices of its incidence graph since this graph is very dense, but, in order to simplify the arguments, we will consider a coloring game on the incidences which can be regarded as a half of an edge. For an edge vw we say that *vw has the incidences* (v, vw) and (w, vw), and that these *are incidences* of vw.

Throughout this chapter we assume that the version g of the coloring game we consider is g_A, the version where Alice begins and passing is not allowed.

The incidence game chromatic number of a graph is a competitive version of the incidence coloring number of a graph introduced by Brualdi and Massey [23]. Upper bounds for the incidence coloring number have been determined for several classes of graphs, e.g. for k-degenerate graphs, K_4-minor free graphs and planar graphs [50], and graphs with maximum degree 3 [64]. These bounds depend on the maximum degree of the graphs. Guiduli [47] found the tight asymptotic upper bound $\Delta + O(\log \Delta)$ for the incidence coloring number of graphs with maximum degree Δ. For forests [23], Halin graphs of maximum degree $\Delta \geq 5$ and outerplanar graphs of maximum degree $\Delta \geq 4$ [79], certain types of meshes [51], and complete k-partite graphs [61], the exact values of the incidence coloring numbers are known.

A trivial upper bound for the incidence game chromatic number of graphs G_Δ with maximum degree $\Delta \geq 1$ is

$$\iota_g(G_\Delta) \leq 3\Delta - 1. \tag{4.1}$$

This is simply the maximum number of adjacent incidences an incidence can have plus one. (Indeed, consider two adjacent vertices v and w of degree Δ in G_Δ and an uncolored incidence (v, vw). Then all $2\Delta - 2$ incidences of the edges incident with v (except vw) are adjacent to (v, vw). The incidence (w, vw) is also adjacent to (v, vw). As well, the $\Delta - 1$ incidences (w, e) of the edges incident with w (except vw) are adjacent to (v, vw). The incidence (v, vw) cannot have more than these $3\Delta - 2$ adjacent incidences. In the worst case, if all incidences are colored distinctly, we need color $3\Delta - 1$ in order to color (v, vw).)

The main theorem of this chapter improves the trivial upper bound for k-degenerate graphs (Section 4.1.) As a corollary, we obtain bounds for forests, outerplanar graphs, and planar graphs. The trivial upper bound is attained by K_2 and K_3. It is attained by large cycles, too, whose incidence game chromatic number is determined in Section 4.2. Furthermore, in that section, we determine the exact incidence game chromatic number of stars and wheels. By using the ideas of these proofs we obtain a lower bound for the incidence game chromatic number of graphs with maximum degree Δ, a bound which is half of the trivial upper bound. This phenomenon also occurs for the game chromatic index, the trivial lower bound of which is half of its trivial upper bound.

4.1 k-degenerate graphs

In this section we will consider the incidence game chromatic number of k-degenerate graphs. The proof uses an activation strategy. The idea of such an activation strategy was already used by Cai and Zhu [24] in order to bound the game chromatic index of k-degenerate graphs.

We start with a reformulation of the definition of k-degeneracy. A graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ is *k-degenerate* if there is a linear order

$$L(v_1) < L(v_2) < \cdots < L(v_n)$$

on its vertex set such that, for every $1 \leq i \leq n$, the vertex v_i has degree at most k in the induced subgraph on the vertex set $\{v_1, v_2, \ldots, v_i\}$. We will prove the following

Theorem 46. *For a k-degenerate graph G with maximum degree Δ we have*

(a) $\iota_g(G) \leq 2\Delta + 4k - 2$,

(b) $\iota_g(G) \leq 2\Delta + 3k - 1$ *if* $\Delta \geq 5k - 1$,

(c) $\iota_g(G) \leq \Delta + 8k - 2$ *if* $\Delta \leq 5k - 1$.

Proof. We will describe a winning strategy for Alice with

$$\max\{\Delta + 8k - 2, 2\Delta + 3k - 1\}$$

colors. First, we need some preparations. The vertex set $\{v_1, v_2, \ldots, v_n\}$ of G is considered as ordered according to a linear order L, i.e.

$$L(v_1) < L(v_2) < \cdots < L(v_n),$$

so that, for every $1 \leq i \leq n$, v_i has vertex degree at most k in the induced subgraph on $\{v_1, v_2, \ldots, v_i\}$. (Such an order exists, since G is k-degenerate.) To every edge v_iv_j an orientation is assigned in such a way that, if $L(v_i) > L(v_j)$, then (v_i, v_j) is an arc. We call the digraph created this way $\vec{G}$. Every time we color an incidence, we color it in both G and $\vec{G}$. Recall that $d^-(v)$ is the number of *out-arcs* (v, w) of a vertex v. By construction, $d^-(v) \leq k$ for every vertex v. The *level* of vertex v_i is the number i. The *level* of an arc is the level of its starting vertex.

During the game certain arcs are considered as *active*, all other arcs as *inactive*. A subset of the active arcs is represented by the *half-selected* resp. *selected* arcs, i.e. those arcs one incidence resp. both incidences of which are colored. For short, by an *arc of minimum level* we mean an uncolored or half-selected arc the level of which is minimal among all such arcs. For an arc (v, w), we call the incidence $(v,(v, w))$ *top incidence* of (v, w), and the incidence $(w,(v, w))$ is called *bottom incidence* of (v, w). An arc is called *top-half-selected* if its top incidence is colored and its bottom incidence is uncolored.

At the beginning Alice colors a bottom incidence of an arc incident with a sink and activates its arc. After that, for each of Bob's moves, where he colors an incidence of an arc $e_0 = (v, w)$, she answers as follows:

(Step 0) activate e_0
(Step 1) **while** w has inactive out-arcs **do**
choose an inactive out-arc (w, w_1), activate it,
$w := w_1$
end do
(Step 2a) **if** w has an unselected or top-half-selected active out-arc e
color the bottom incidence of e
(Step 2b) **else if** w has an half-selected active out-arc e
color the uncolored incidence of e,
(Step 2c) **else** color an incidence of an arc e of minimum level, activate e
(in Step 2c color a bottom incidence if possible)

We will show that as a result of this strategy, after every move of Alice, every uncolored incidence of an unselected or half-selected arc (v, w) has at most $\max\{2\Delta + 3k - 3, \Delta + 8k - 4\}$ colored adjacent incidences. Note that, in Step 2c, no arc (x, v) can be (half-)selected before (v, w) by Alice as (v, w) lies on a lower level. Therefore arcs (x, v) can only show up in Step 1 or Step 2a or Step 2b of Alice's answer before both incidences of (v, w) are colored. We

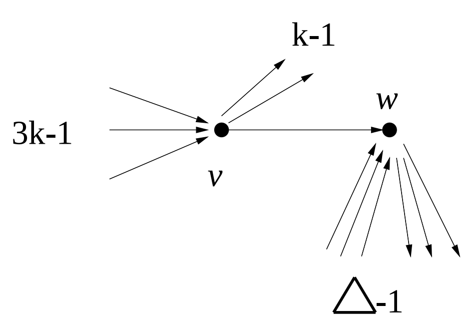


Figure 4.2: Arcs adjacent to (v, w) which may be activated

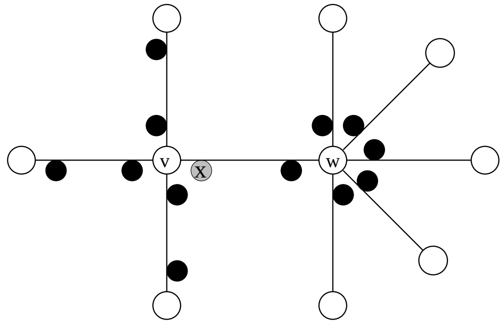


Figure 4.3: The incidences adjacent to the incidence $x = (v, vw)$ are indicated by black dots and the incidence (v, vw) is depicted by a grey dot with 'x'. Here a black dot on an edge f next to a vertex z means the incidence (z, f).

will now count the number of activated arcs of type (x, v). There are at most $\#v_{out} = k - 1$ arcs (v, y) different from (v, w). So at most k arcs (x, v) can have been activated in Step 1 right before activating arcs of the type (v, y). After that at most $2k - 1$ arcs (x, v) can have been activated right before coloring $2k - 1$ of the $2k$ incidences of the arcs (v, y) (Steps 1, 2a, 2b). Thus there are at most $\#v_{in} = 3k - 1$ active arcs of the form (x, v). Note that, if Bob colors an incidence of an arc of type (x, v), Alice will continue activating or coloring arcs of type (v, y) by Step 1, 2a, or 2b. Therefore $\#v_{in}$ is really an upper bound for the number of active arcs of type (x, v). In the worst case there are $\#w = \Delta - 1$ active arcs incident with w. See Fig. 4.2.

So, in the worst case the top incidence $(v, (v, w))$ has at most

$$\#w + 1 + 2\#v_{out} + 2\#v_{in} = \Delta + 8k - 4$$

adjacent incidences on active arcs. See Fig. 4.3 for an example. On the other hand, the bottom incidence $(w, (v, w))$ has at most

$$2\#w + 1 + \#v_{out} + \#v_{in} = 2\Delta + 4k - 3$$

adjacent incidences on active arcs after a move by Alice.

However, we need not count the number of adjacent incidences on active arcs, but only the number of *colored* adjacent incidences. The latter number is smaller than the previous since the set of selected and half-selected arcs is a subset of the set of active arcs, and since Alice, by her strategy, prefers to color bottom incidences. Therefore, when the incidence $(w, (v, w))$ is to be colored, the top incidence of any arc of type (v, y) is either uncolored or colored by Bob. If $\#v'_{out} = m$ top incidences of the arcs of type (v, y) (including (v, w)) are colored by Bob, then, by the same reasons as before, there are at most $\#v'_{in} = 3k - 1 - m$ active arcs of the form (x, v). Thus the bottom incidence

$(w, (v, w))$ has at most

$$2\#w + \#v'_{out} + \#v'_{in} = 2\Delta + 3k - 3$$

colored adjacent incidences.

Summarizing, we state that after Alice's move an incidence can have at most

$$\max\{\Delta + 8k - 4, 2\Delta + 3k - 3\}$$

colored adjacent incidences. After Bob's next move an incidence can have at most

$$\max\{\Delta + 8k - 3, 2\Delta + 3k - 2\}$$

colored adjacent incidences. We obtain the conclusion that

$$\max\{\Delta + 8k - 2, 2\Delta + 3k - 1\}$$

colors are sufficient to color every incidence during the game.

Further we have $\Delta + 8k - 2 \leq 2\Delta + 3k - 1$ if, and only if, $\Delta \geq 5k - 1$, from which (b) and (c) follows. Note that, for $\Delta \geq 4k$, $\Delta + 8k - 2 \leq 2\Delta + 4k - 2$, and for $\Delta \leq 4k - 1$, the trivial upper bound $3\Delta - 1$ is better than the bound $b_0 = 2\Delta + 4k - 2$. Thus b_0 is an upper bound for all Δ, as stated in (a). $\square$

As we have seen in the structure of the proof, upper bounds for the incidence game chromatic number are more related to the game chromatic index than to the game chromatic number. The *game chromatic index* $\chi'_g(G)$ of a graph G is the game chromatic number of the line graph $L(G)$ of G. The *line graph* $L(G)$ has the edge set $\tilde{E}$ of G as vertex set, and two vertices $e_1, e_2 \in \tilde{E}$ are connected by an edge in $L(G)$ if, and only if, they are adjacent in G. Mainly two methods are known in order to deal with the game chromatic index: activation strategies and splitting strategies.

The game chromatic index of a k-degenerate graph of maximum degree Δ is at most $\Delta + 3k - 1$ [24]. Our bound is better than simply doubling this bound.

Corollary 47. *Let P be a planar graph, O be an outerplanar graph, and F be a forest. Then*

$$\iota_g(P) \leq 2\Delta(P) + 18, \qquad \iota_g(O) \leq 2\Delta(O) + 6, \qquad \iota_g(F) \leq 2\Delta(F) + 2.$$

Proof. This follows, by Theorem 46 (a), from the fact that planar graphs are 5-degenerate, outerplanar graphs are 2-degenerate and forests are 1-degenerate. $\square$

Note that the upper bound $2\Delta+2$ for the incidence game chromatic number of a forest of maximum degree Δ could be obtained in a different way, namely by the use of a splitting strategy instead of an activation strategy. Splitting strategies have been successfully used by several authors [24, 37, 1, 3] in order to prove that the game chromatic index of forests of maximum degree $\Delta \neq 4$ is at most $\Delta + 1$. In this case the activation strategy of Cai and Zhu [24] only led to the weaker upper bound $\Delta + 2$. However, a second proof of the third assertion of Corollary 47 by means of a splitting strategy is not described here since this proof is not easier than the proof of the more general Theorem 46.

The results of this section have an interesting noncompetitive analogue: Hosseini Dolama et al. [50] found $\Delta+2k-1$ as an upper bound for the incidence coloring number of k-degenerate graphs and $\Delta + 7$ as an upper bound for the incidence coloring number of planar graphs of maximum degree Δ.

4.2 Some simple classes of graphs

In this section we consider the incidence game chromatic number of cycles, stars and wheels. While the incidence game chromatic number of cycles attains the trivial upper bound (4.1), its counterparts for stars and wheels will make us discover a trivial lower bound that cannot be improved for graphs in general. All following theorems depend on the assumption that $g = g_A$ is the game where Alice has the first move and missing a turn is not allowed.

Theorem 48. *$\iota_g(C_k) = 5$ for $k \geq 7$.*

Proof. Let C_k be a cycle with $k \geq 7$ edges. Then $C_k^I = (V^I, E^I)$ is the graph with vertex set $V^I = \{v_1, v_2, \ldots, v_{2k}\}$ and arc set

$$E^I = \{(v_i, v_j) \mid i - j \equiv \pm 1, \pm 2 \mod 2k\}.$$

$\iota_g(C_k) \leq 5$ follows from the trivial upper bound (4.1). So, we only have to prove $\iota_g(C_k) = \chi_{g_A}(C_k^I) \geq 5$, i.e. we have to explain a winning strategy for Bob for the coloring game with 4 or less colors on C_k^I. It is easy to see that Bob wins with 3 or less colors. Assume for the following that the players have 4 colors at their disposal. The general winning idea of Bob is to construct a configuration as in Fig. 4.4.

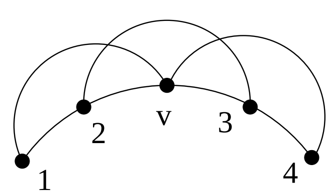


Figure 4.4: A winning configuration for Bob on a cycle. If there are only 4 colors in the game, the uncolored vertex v cannot be colored any more.

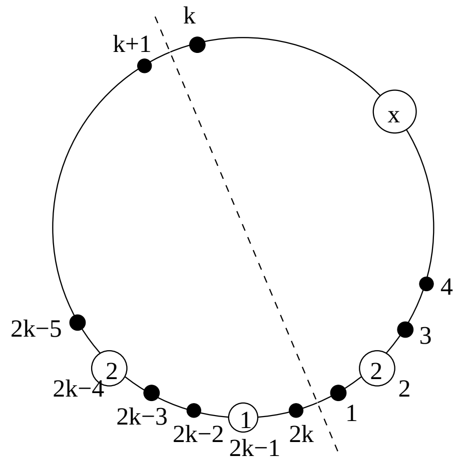


Figure 4.5: State of the game after Bob's second move. Uncolored vertices of of C_k^I are black circles, colored vertices are white circles with the number of the color in it.

W.l.o.g. Alice starts the game by coloring v_{2k-1} with color 1. Bob then colors a vertex at distance 3, namely v_2 with color 2. By reasons of symmetry we may assume that Alice, in her next move, colors one of the vertices $v_3, \ldots, v_k$, and if she colors v_3, then she uses color 1. Indeed if Alice colored v_{2k} or v_1 (with color 3), Bob would color v_3 or v_{2k-2} at distance 3 from Alice's vertex with color 4, so that v_1 or v_{2k} could not be colored any more. In the same way, if Alice colored v_3 with color 3, Bob would color v_{2k} with color 4, which would result in a win for him since he has constructed a winning configuration around the uncolored vertex v_1. Since the vertex Alice has colored in her second move is far enough away from v_{2k-4}, Bob can color v_{2k-4} with color 2 in his second move. (Here we need the precondition $k \geq 7$, from which $(2k-4) - k \geq 3$ follows.) The situation of the game after Bob's second move is depicted in Fig. 4.5.

Now Alice is stuck. By the reason discussed above, Alice may not color v_{2k-5} or v_3 with a color different from 1. Furthermore, if she colors v_{2k-3}, v_{2k-2}, v_{2k} or v_1, Bob can force a win in his next move. The next moves until either there is a winning configuration for Bob or there are only 5 uncolored vertices left is called a *round*. Everything Alice can do is to color the remaining of the vertices $v_3, \ldots, v_{2k-5}$ in the round. Bob will then color only vertices $v_4, \ldots, v_{2k-6}$ in the round. Assume that the round ends if there are only 5 uncolored vertices left. We will show that in this case Bob wins, too.

If Alice, in her last move, has colored v_3 with a color different from color 1, Bob can force a win as described above. Thus we may assume without loss of generality that v_3 is colored with color 1. We distinguish two cases.

The first case is that one of the vertices $v_4, \ldots, v_{2k-6}$ is uncolored or v_{2k-5}

is uncolored but can be colored with color 1. Then Bob colors this vertex, preferably with color 1. Now the only uncolored vertices are v_{2k-3}, v_{2k-2}, v_{2k}, and v_1. Alice colors one of these vertices with color 3 or 4. Then Bob colors the uncolored vertex at distance 3 with a different color, and one of the remaining uncolored vertices cannot be colored any more. (The colors 3 and 4 are allowed for the vertex Bob has colored since v_{2k-5} and v_3 are colored with color 1.)

In the second case v_{2k-5} is uncolored and cannot be colored with color 1. We may assume that v_{2k-5} can be colored with color 3. Then Bob colors v_{2k-2} with color 4. In order to avoid the situation that v_{2k-3} cannot be colored any more, Alice must either color v_{2k-3} with color 3 or v_{2k-5} with color 4. Then Bob colors v_1 with color 3 and wins since v_{2k} cannot be colored any. His move is feasible as we have assumed that v_3 is colored with color 1. $\square$

The reason why the same argumentation does not work for paths with $k \geq 8$ edges is that in the incidence graph of a cycle, because of its symmetry, we can assume without loss of generality that Alice colors v_{2k-1} in her first move. However, in the incidence graph of a path we cannot assume that Alice starts coloring a vertex in the middle of a path. So we may formulate

Open question. Let P be a large path. Is $\iota_g(P) = 4$ or $\iota_g(P) = 5$?

Let S_k be the star with k edges which are incident with the same center vertex. Further let W_k be the wheel with $2k$ edges. Let v_0 be the center vertex of a star or a wheel. We call an incidence (v_0, e) *inner incidence*, and an incidence (v, e) with $v \neq v_0$, but where v_0 is incident with e, *outer incidence*. Incidences on the rim of a wheel, which are neither inner nor outer incidences, are called *border incidences*. In a star or a wheel, inner incidences are adjacent to all inner and outer incidences, but two different outer incidences are not adjacent.

Theorem 49. *For $k \geq 1$, $\iota_g(S_{2k}) = 3k$.*

Proof. First, we describe a winning strategy for Alice playing on S_{2k} with $3k$ colors. At the beginning, Alice colors inner incidences. If Bob colors an outer incidence with color i, then Alice also colors an outer incidence with color i. In this way, Bob can use at most k colors for outer incidences. If Alice is forced to color an outer incidence (this is only the case when all inner incidences are colored), then she colors an outer incidence with a color already assigned to an outer incidence, except if there is no such color. In the latter case there are k remaining colors for the outer incidences. Even if Bob always uses a new color for his outer incidences, in his last move he will be forced to take a color already used. In every case, $2k$ colors are used for inner incidences, and at most k colors for outer incidences, so Alice wins.

Now we discuss a winning strategy for Bob with $3k-1$ or fewer colors. In his k first moves, Bob colors k outer incidences with k distinct colors. Then there are at most $2k-1$ colors left to color the inner incidences. Thus it is impossible to color all of them, and Bob wins. □

Theorem 50. *For $k \geq 0$, $\iota_g(S_{2k+1}) = 3k+2$.*

Proof. A winning strategy for Alice playing on S_{2k+1} with $3k+2$ colors is given as follows. Alice, in her first $k+1$ moves, colors outer incidences with a fixed color i. Meanwhile, Bob can use at most k distinct colors for outer incidences. So $2k+1$ colors are left for the inner incidences, and Alice wins.

We still have to exhibit a winning strategy for Bob with $3k+1$ or fewer colors. If Alice, in her first move, colors an inner incidence, Bob, in his first $k+1$ moves, colors outer incidences with $k+1$ distinct colors. On the other hand, if Alice, in her first move, colors an outer incidence with color i, then Bob, in his next k moves, colors outer incidences with k distinct colors different from i. In both cases, at most $2k$ colors are left for the inner incidences, which therefore cannot be colored any more, i.e. Bob wins. □

Theorem 51. *For $k \geq 7$, $\iota_g(W_{2k}) = 3k$.*

Proof. First, we describe a winning strategy for Alice with $3k$ colors which is very similar to the case of stars. Alice wants to keep 7 colors which are not used for inner incidences for coloring border incidences. Since a border incidence has exactly 7 adjacent incidences one of which is an inner incidence then every border incidence can be colored by using one of these 7 colors. Alice chooses 7 colors. During the game, if one of her chosen colors is used for an inner incidence, she exchanges that color with one of the colors not yet used for an inner incidence. This is possible since there are $3k \geq 2k+7$ colors. As long as Bob colors inner incidences and border incidences, Alice colors inner incidences (or, if she is forced to, border incidences). If Bob colors an outer incidence, Alice replies by coloring an outer incidence with one of her chosen 7 colors, preferably with the same color as Bob (if necessary Alice exchanges a color not used before among her 7 colors with Bob's color). There is always a feasible color among these because an outer incidence is affected by at most 4 border incidences. The same strategy will make Alice win if she is forced to color outer incidences first (at the end Bob can also use one of the 7 colors). Note that border incidences are affected by at most 4 border incidences, at most 2 outer incidences, and one inner incidence. Thus a border incidence can always be colored by one of Alice's 7 colors.

The winning strategy for Bob with $3k-1$ or fewer colors is just the same as for a star: Bob first colors outer incidences with k distinct colors, then he will win. □

Theorem 52. *For $k \geq 6$, $\iota_g(W_{2k+1}) = 3k + 2$.*

Proof. First, we prove a winning strategy for Alice with $3k+2$ colors. Alice chooses 7 colors. Every time one of these colors is used for inner incidences she exchanges this color with a color not used so far. This is possible since there are $3k + 2 \geq (2k + 1) + 7$ colors. In her first move, Alice colors an outer incidence with one of the chosen colors. As long as Bob colors inner or border incidences, Alice colors inner or border incidences. If Bob colors an outer incidence, she colors an outer incidence, preferably with the same color (and in this case she exchanges this color with one of the seven colors which has not been used before if such a color exists), or else with one of the seven colors, preferably with a color already used. Playing this way Alice can guarantee that the final set of seven colors will be used for outer incidences and that the outer incidences are colored by at most $k + 1$ colors. Note that there is no problem coloring the border incidences with these $k+1 \geq 7$ colors. So Alice wins.

A winning strategy for Bob with $3k+1$ or fewer colors is the same as with stars. In his first $k + 1$ moves Bob can guarantee that $k + 1$ outer incidences are colored distinctly. Then he will win as the inner incidences cannot be colored any more. □

Stars and wheels are classes of graphs for which the incidence game chromatic number is asymptotically half of the value of the trivial upper bound. There is no class of graphs where this fraction is lower because of the following proposition.

Proposition 53. *For any graph G with maximum degree Δ, $\iota_g(G) > \frac{3\Delta-1}{2}$*

Proof. A winning strategy for Bob with $\lceil\frac{3\Delta}{2}\rceil - 1$ or fewer colors is given as follows. Bob chooses a vertex v of degree Δ. An *outer incidence of* v is an incidence (w, e) such that $e = vw$. In his first $\lceil\frac{\Delta}{2}\rceil$ moves, Bob (possibly with Alice's help) colors $\lceil\frac{\Delta}{2}\rceil$ outer incidences of v with $\lceil\frac{\Delta}{2}\rceil$ distinct colors. Then there are only less than Δ colors left for the incidences of type (v, e). Thus Bob wins. □

Let G_Δ be a graph of maximum degree Δ. We have the analogon between the lower and upper bounds for the incidence game chromatic number

$$\frac{3\Delta - 1}{2} < \iota_g(G_\Delta) \leq 3\Delta - 1 \tag{4.2}$$

and the trivial lower and upper bounds for the game chromatic index

$$\frac{2\Delta - 1}{2} < \chi'_g(G_\Delta) \leq 2\Delta - 1. \tag{4.3}$$

By the example of stars, the lower bounds are tight for every Δ in the sense that no positive integer may be added to them. The upper bound (4.2) is tight for $\Delta = 2$, as we have seen in Theorem 48 for large cycles, the upper bound (4.3) is also tight for $\Delta = 2$, consider paths P_k with $k \geq 5$.

Open question. For fixed $\Delta > 2$, are the upper bounds in (4.2) and (4.3) tight?

In Appendix B we will see that for $\Delta = 3$ the bound (4.3) is also tight, by the example of the Petersen graph (Theorem 101).

Open question. Is the incidence game chromatic number a monotonic parameter (i.e. for a graph G is it as least as big as for any subgraph of G?)

The game chromatic number is known to be non-monotonic (consider $K_{m,m}$ and its subgraph where a perfect matching is deleted.) The game chromatic index, too, is non-monotonic (consider C_6 together with an isolated edge, and its subgraph P_6.) Hence, it would be no surprise if the answer of the last question was not affirmative.

Future work. If $\vec{G^I}$ is an orientation of the incidence graph of a k-degenerate graph G, it would be interesting to determine an upper bound for $\chi_g(\vec{G^I})$. This construction would be one of the possibilities to generalize the incidence game chromatic number of a graph in a certain way to digraphs. Other possibilities would be to define an adjacency relation for incidences of digraphs. There are several ways to define an incidence of a digraph, e.g. the pair of arcs (v, w) and (w, v) could consist of the four incidences $(v, (v, w))$, $(w, (v, w))$, $(v, (w, v))$, and $(w, (w, v))$, or only of the two incidences (v, vw) and (w, vw) as in the case of graphs.

In order to tighten the upper bounds of Theorem 46 and Corollary 47 one could try to obtain non-trivial lower bounds for the incidence game chromatic number of the class of k-degenerate graphs resp. forests. Another approach to generalize the results of this chapter would be to consider the parameter ${}^{(a,b)}\chi_g^d(G^I)$ for certain graphs G, and $a, b, d \geq 0$.

Chapter 5

Game-perfect graphs

5.1 Perfectness and game-perfectness

A graph G is called *nice* if its chromatic number $\chi(G)$ equals its clique number $\omega(G)$. There are many nice graphs: for example, the graph formed by the two components C_5 and K_3 is nice. However, the example also illustrates a problem: C_5 itself is not nice. One might prefer a graph property that holds for all induced subgraphs as well. So, a graph G is called *perfect* if every induced subgraph of G is nice, i.e. if for every induced subgraph H of G, $\chi(H) = \omega(H)$. Motivated by an application in coding theory, Berge was the first to examine the structure of perfect graphs, cf. [13]. Since then, there have been several hundreds of contributions to the theory of perfect graphs. By the famous Strong Perfect Graph Theorem [29] a graph is perfect if, and only if, it contains neither induced cycles C_l of odd length $l \geq 5$ nor their complements.

We generalize perfectness of a graph in a natural way to perfectness of a digraph. A digraph D is called *perfect* if for every induced subdigraph H of D, $\chi(H) = \omega(H)$, where $\chi(D)$ is the dichromatic number of D. It is obvious that, for graphs, our two definitions of perfectness are the same. Astonishingly, this generalization of perfectness seems to be unknown in literature. There are three other definitions of perfectness of digraphs known, namely *diperfect* digraphs [14], *kernel-perfect* digraphs [31, 41, 42], and the perfect digraphs of Fachini and Körner [39]. But by these definitions, every graph is diperfect and kernel-perfect, and the transitive tournament T_3 is not perfect in the sense of Fachini and Körner. (In our sense T_3 is perfect since $\chi(T_3) = \omega(T_3) = 1$.) Thus these parameters are different from the perfectness we consider.

It is evident that a digraph with clique number 1 is perfect if, and only if, it is acyclic. That means that simple digraphs are perfect if, and only if, they are acyclic. A graph with clique number 2 is perfect if, and only if, it is bipartite. However, it seems to be a big problem to decide whether a given digraph with clique number 2 is perfect or not. This could be related to the fact that the decision problem whether a digraph has dichromatic number at most 2 is $\mathcal{NP}$-complete as we will prove in Appendix A and as was already shown by Bokal et al. [18]. In contrast, the same decision problem for a graph is easy, as remarked above. Graph coloring becomes hard for 3 colors. In spite of that it is possible to recognize perfect graphs in polynomial time. Two results of ingenious work led to the solution of this recognition problem: first, the proof that recognizing Berge graphs is in $\mathcal{P}$ by Chudnovsky et al. [28], and second, the proof of the Strong Perfect Graph Theorem [29] which means that a graph is perfect if, and only if, it is a Berge graph. The latter theorem was formerly known as Berge's Strong Perfect Graph Conjecture.

In the spirit of Berge we formulate the following *Strong Perfect Digraph Conjecture*. First we start with two definitions. An *extended odd hole* of size n is a digraph $D = (V, E \cup E')$ with an odd number $\#V = n$ of vertices, so that (V, E) is an undirected cycle C_n and (V, E') is a simple digraph. An *extended odd anti-hole* of size n is a digraph $D = (V, E \cup E')$ with an odd number $\#V = n$ of vertices, so that (V, E) is the complement of an undirected cycle C_n and (V, E') is a simple digraph.

Conjecture 54. *A digraph D is perfect if, and only if,*

(1) D contains no induced directed cycles $\vec{C}_n$ with length $n \geq 3$,

(2) D contains no induced extended odd holes of size $n \geq 5$, and

(3) D contains no induced extended odd anti-holes of size $n \geq 5$.

Since this conjecture is a generalization of the Strong Perfect Graph Theorem (SPGT) it might be difficult to solve. The only help to simplify it could be to use the SPGT essentially.

This was only a motivation for the main definition of this section. Let g be a variant of the coloring game. In this chapter we assume $a = b = 1$ and $d = 0$. A digraph D is *g-nice* if $\chi_g(D) = \omega(D)$. A digraph D is *g-perfect* if every induced subdigraph of D is g-nice. Thus g-perfectness is a competitive version of perfectness.

While most results in the literature concerning game chromatic numbers deal with Bodlaender's original version g_A of the game where Alice has the first move and missing a turn is not allowed, considering the versions A and B has some advantages.

The first advantage is given by Observation 1: any upper bound for the B-game chromatic number is also an upper bound for the g-game chromatic number for any g, and any lower bound for the A-game chromatic number is also a lower bound for the g-game chromatic number for any g. This does not seem to help a lot besides giving a simple Sandwich structure, since for a lot of single digraphs the A-game chromatic and the B-game chromatic numbers differ. However, for interesting classes of digraphs these numbers often equal, as we have seen in Chapter 2 for directed forests. So for classes instead of single digraphs sometimes the examination of the versions g_A and g_B is superfluous after an examination of the versions A and B of the game.

Furthermore, as a second advantage, the versions A and B are more stable. There are graphs for which Alice wins the version g_A, but if an isolated vertex is added, then Bob wins the version g_A. The versions A and B are more 'stable' since such a simple trick does not work for these versions where the outcome of the game depends on the global structure of the digraph and does not depend so much on local details as in Bodlaender's version.

This motivates us to focus on the more stable versions A and B. We have

$$\omega(G) \leq \chi(G) \leq \chi_A(G) \leq \chi_B(G).$$

The first two inequalities are obvious, the third is part of Observation 1. Thus, B-perfect graphs are A-perfect, and A-perfect graphs are perfect.

In this chapter we determine all the A-nice and B-nice graphs with clique number 2 and thus the A-perfect and B-perfect graphs with clique number 2. As a corollary we obtain the game-perfect graphs with clique number 2 for Bodlaender's original version g_A of the game and for its dual g_B. We also determine the B-perfect graphs with clique number 3. The only class of perfect graphs which could be recognized as A-perfect in general are complements of bipartite graphs, see Section 5.5. These are first steps towards results analogous to the Strong Perfect Graph Theorem for game-perfectness. However, the sets of forbidden induced subgraphs occurring in Sections 5.2, 5.3, and 5.6 are far from being complete.

Unlike perfectness of graphs, game-perfectness of graphs does not have the Weak Perfect Graph Theorem's property, i.e. a graph is perfect if, and only if, its complement is perfect [62], but there are A-perfect (B-perfect) graphs whose complements are not A-perfect (B-perfect), cf. Section 5.6. One should mention that for digraphs in general there is no Weak Perfect Digraph Theorem, even in the non-competitive case. E.g., the digraph $\vec{C}_4$ is not perfect, whereas its complement is perfect.

In Section 5.8 a classification of some A-perfect digraphs with clique number 2 is given. In particular, we determine the A-perfect paths, cycles, and semiorientations of complete graphs with clique number 2.

The cycle C_4 The path P_4

Figure 5.1: Two forbidden configurations for B-perfect graphs

5.2 B-perfect graphs with clique number 2

In Sections 5.2, 5.3, 5.4, 5.5, and 5.6 we consider (undirected) graphs. Recall that a graph G is B-nice if $\chi_B(G) = \omega(G)$. G is B-perfect if every induced subgraph of G is B-nice. The classification of B-nice (and B-perfect) graphs with clique number 2 is given in this section. In particular, we will see that B-perfect graphs are a subclass of the class of trivially perfect graphs. A graph is *trivially perfect* if it does not contain a cycle C_4 or a path P_4 as an induced subgraph. C_4 and P_4 are depicted in Fig. 5.1. The notion 'trivially perfect graphs' was introduced by Golumbic [44], however this type of graphs has been examined already earlier, e.g. by Wolk [82].

Lemma 55. *Let G be a graph with $\omega(G) = 2$ containing an induced C_4. Then G is not B-nice.*

Proof. Let $v_1v_2v_3v_4$ be an induced C_4. Bob has the following winning strategy with 2 colors: He misses his turns until Alice either colors a vertex v_i of the C_4 or one of its neighbors n with – say – color 1. In case Alice has colored v_i he replies by coloring v_{i+2} (index mod 4) with color 2. This is possible since neither v_{i+1}, v_{i+2}, v_{i+3} nor any other neighbor of v_{i+2} has been colored before. Bob wins, since v_{i+1} cannot be colored any more. In case Alice has colored a neighbor n of v_i outside the C_4, Bob answers by coloring v_{i+1} with color 2. This is possible, since no neighbor of v_{i+1} has been colored. Note that n is not a neighbor of v_{i+1}, otherwise there would be a triangle nv_iv_{i+1} contradicting $\omega(G) = 2$. Here again, since v_i cannot be colored any more, Bob wins. □

Lemma 56. *Let G be a graph with $\omega(G) = 2$ containing an induced P_4. Then G is not B-nice.*

Proof. This is quite similar to the previous lemma. □

Theorem 57. *A graph G with $\omega(G) \leq 2$ is B-nice if, and only if, it contains neither an induced C_4 nor an induced P_4 (i.e. if it is trivially perfect). This is the case if, and only if, G is a forest of stars.*

Proof. The graphs without induced C_3, C_4 and P_4 are obviously forests whose components have diameter of at most 2, i.e. forests of stars. If G contains a C_4 or P_4, G is not B-nice, by Lemma 55 resp. 56. We are left to prove a winning strategy for Alice with 2 colors in case the graph is a forest of stars: in order to fix the coloring, whenever possible, Alice colors centers of stars, preferably the center of the star in which Bob has colored a vertex in his previous move. □

Since every induced subgraph of a forest of stars is a forest of stars we obtain

Corollary 58. *A graph G with $\omega(G) \leq 2$ is B-perfect if, and only if, it is trivially perfect. This is the case if, and only if, G is a forest of stars.*

5.3 A-perfect graphs with clique number 2

Recall that a graph G is A-nice if $\chi_A(G) = \omega(G)$. G is A-perfect if every induced subgraph of G is A-nice. It is very easy to decide whether a graph with clique number 2 and diameter $\neq 3$ has A-game chromatic number 2, see the Propositions 59 and 60. But there are graphs with diameter 3 and clique number 2 with A-game chromatic number 2 as well as with larger A-game chromatic number. E.g., $\chi_A(C_6) = 2$ and $\chi_A(\Pi) = 3$, where Π is the graph formed by the path $P_6 = v_1v_2v_3v_4v_5v_6$ with an additional edge connecting v_2 and v_5, see Fig. 5.2. This section contains a classification of A-nice graphs with clique number 2 which begins with Lemma 61 and does not make use of the notion diameter.

Proposition 59. *Let G be bipartite and diam$(G_i) \leq 2$ for each component G_i of G. Then G is A-nice.*

Proof. We prove a winning strategy with 2 colors for Alice. Alice forces Bob to be the first to color any component. That means, if Bob does not color a vertex of a new component, she misses her turn. Assume Bob has colored a first vertex x in some component G_i. If every remaining vertex of G_i is a neighbor of x, Alice continues passing since the coloring of G_i is fixed by the color of x. On the other hand, if there is a vertex y with distance $d(x, y) = 2$, then Alice colors the middle vertex z on a shortest path xzy from x to y. We state that now the coloring is fixed since every vertex of G_i has either distance at most 1 from x, or distance at most 1 from z. Assume that there is a vertex a with $d(a, x) = 2$ and $d(a, z) = 2$. Then there are paths abx and acz. In case $b = c$ we have a triangle bxz, otherwise a C_5 $abxzc$, both contradicting the

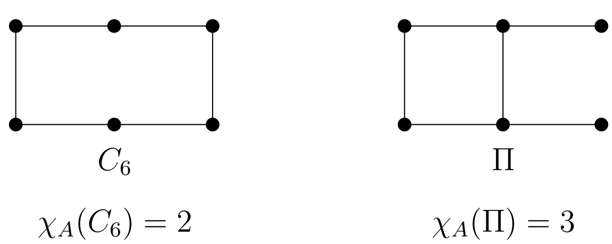


Figure 5.2: Two graphs with diameter 3

fact that G is bipartite. Alice uses this strategy for each component that Bob begins to color, so there is no need for more than two colors. $\square$

Proposition 60. *Let G be bipartite and $diam(G_i) \geq 4$ for some component G_i of G. Then G is not A-nice.*

Proof. We prove a winning strategy for Bob with 2 colors. In G_i there is a shortest path $v_1v_2v_3v_4v_5$ from a vertex v_1 to another vertex v_5. If Alice colors v_j or some neighbor of v_j, Bob replies by coloring a vertex at distance 2 with the remaining color, making it impossible to color the graph completely. Otherwise, Bob colors v_3. No matter what Alice does, since there are no common neighbors of v_1 and v_5, Bob may color either v_1 or v_5 different from v_3 in his next move. Again, Bob wins. $\square$

For the following, we need the definition of $K_{m,n} - M_k$, which is given in Section 1.2 on page 4.

Lemma 61. *Let $1 \leq k + 1 \leq m \leq n$. Then $K_{m,n} - M_k$ is A-nice.*

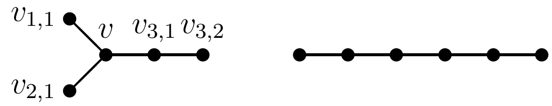
Proof. Let P resp. Q be the partite sets with m resp. n vertices. Since $k \leq m - 1 \leq n - 1$, P and Q each contain a vertex p resp. q that is adjacent to all vertices of the other side. Alice's winning strategy with two colors is the following. In her first move she uses her right to miss a turn. W.l.o.g. Bob colors a vertex of P. If he colors a vertex that is adjacent to every vertex of Q, Alice colors q with the second color. If Bob colors a vertex that is adjacent to every vertex of Q except one vertex q', Alice colors q' with the second color. In both cases the coloring is fixed after Alice's move. $\square$

Lemma 62. *For $m \geq 2$, $K_{m,m} - M_m$ is A-nice.*

Proof. This is the same as the second case in the proof of the previous lemma. $\square$

Lemma 63. *For $2 \leq m < n$, $K_{m,n} - M_m$ is not A-nice.*

Proof. Let P resp. Q be the partite sets with m resp. n vertices. Let $W \subseteq Q$ be the vertices which are adjacent to all vertices of P. As $m < n$, $W \neq \emptyset$. We prove a winning strategy for Bob with two colors in the game A. If Alice uses her first move to color a vertex with the first color, then Bob colors a vertex of the same partite set with the second color. As the graph is connected, Bob will win. If Alice misses her turn, Bob colors a vertex of W with the first color. No matter what Alice does in her second move, since $m \geq 2$, Bob may always color a vertex of $Q \setminus W$ with the second color in order to win. $\square$



The multistar $S_{1,1,2}$ The path P_6

Figure 5.3: Two forbidden configurations for A-perfect graphs

Lemma 64. *Let G be a graph with $\omega(G) = 2$ and let H be an induced subgraph of G which does not contain any isolated vertices. Assume that, in her first move of the game A played on G with 2 colors, Alice colors a neighbor $v \in V(G \setminus H)$ of $w \in V(H)$. Then Bob wins.*

Proof. Since there are no isolated vertices in H, there is an edge $wz \in E(H)$. There is no edge $vz \in V(G)$, otherwise there would be a triangle vwz, contradicting $\omega(G) = 2$. So Bob may color z different from v. During the game, w cannot be colored feasibly any more, i.e., Bob wins. □

The proof of the following lemma is straightforward.

Lemma 65. *Let G be a graph with $\omega(G) = 2$ and let H be an induced subgraph of G, so that every vertex of H lies on an induced $P_4 \subseteq H$. Assume that, in her first move of the game A played on G with 2 colors, Alice colors a vertex of H. Then Bob wins.*

For the next lemma we use the notation from Fig. 5.3.

Lemma 66. *Let G be a graph with $\omega(G) = 2$ that contains an induced multistar $S_{1,1,2}$. Then G is not A-nice.*

Proof. We prove a winning strategy for Bob with 2 colors. If Alice colors a neighbor of v, $v_{1,1}$, $v_{2,1}$, $v_{3,1}$ or $v_{3,2}$ in her first move, Bob will win by Lemma 64. If Alice colors v, $v_{1,1}$, $v_{2,1}$, $v_{3,1}$ or $v_{3,2}$, then Bob will also win, by Lemma 65. So we are restricted to the case that Alice passes or colors a vertex which is neither one of $S_{1,1,2}$ nor one of its neighbor vertices. In this case Bob colors $v_{1,1}$ with the first color. Now, Alice may neither color $v_{3,2}$ with the first color nor $v_{2,1}$ or $v_{3,1}$ with the second color, otherwise she will loose. If Alice colors a vertex with the first color, Bob answers by coloring either $v_{2,1}$ or $v_{3,1}$ with the second color. On the other hand, if Alice colors a vertex with the second color or if she misses her second turn, Bob colors either $v_{3,2}$ with the first color or $v_{2,1}$ with the second color. In either case, Bob wins. □

Lemma 67. *Let G be a graph with $\omega(G) = 2$ that contains an induced path P_6. Then G is not A-nice.*

Proof. Let $v_1v_2v_3v_4v_5v_6$ be an induced P_6 in G. We prove a winning strategy for Bob with 2 colors. If Alice, in her first move, colors a neighbor of some v_i, Bob has a winning strategy by Lemma 64, if she colors some v_i, Bob wins by Lemma 65. We are left with the case that Alice colors some other vertex or misses her turn. In this case, Bob will respond by coloring v_3 with the first color. Then, if Alice colors some vertex with the first color or if she passes, Bob may color either v_1 or v_5 with the second color in order to win. In case Alice colors some vertex $v \neq v_6$ with the second color, Bob colors v_6 with the first color and wins. If Alice colors v_6 with the second color, Bob wins by coloring v_1 with the second color. □

Note that in Lemma 67 we may not replace P_6 by P_5. The undirected cycle C_6 ($= K_{3,3} - M_1$) is A-nice by Lemma 61 but contains an induced P_5.

Lemma 68. *Let G be a connected bipartite graph that does neither contain an induced $S_{1,1,2}$ nor an induced P_6. Then G is a $K_{m,n} - M_k$, where $k \leq \min\{m, n\}$.*

Proof. Since G is bipartite, there exist integers m and n, so that G is a subgraph of $K_{m,n}$. Assume that in G there are two different vertices v and w on the same side and a vertex z on the other side which are pairwise non-adjacent. As G is connected, there are shortest paths $v \ldots z$ and $w \ldots z$ in G. Assume that one of them has length ≥ 5. Then this path contains an induced P_6, contrary to the precondition. So both paths have length exactly 3. Let $vxyz$ and $wabz$ be these paths which w.l.o.g. have a maximal number of common edges among all such pairs of paths. Let z_0 be the first common vertex of both paths. So the vertices after z_0 are equal in both paths.

Case 1: $z_0 = z$

There is no edge ya, otherwise y would be the first common vertex of the paths $vxyz$ and $wayz$, but the existence of the path $wayz$ would be a contradiction to the choice of $vxyz$ and $wabz$ as paths with a maximal number of common edges. By the same argument, va, wx, and bx do not exist in G, since otherwise a, x, or b would be the first common vertex, respectively. Thus $vxyzbaw$ is an induced P_7, which contradicts the precondition.

Case 2: $z_0 = y = b$

There is no edge va (resp. wx), because otherwise a (resp. x) would be the first common vertex of the paths. Thus $\{v, x, y, a, z\}$ induces an $S_{1,1,2}$, contrary to the precondition.

Case 3: $z_0 = x = a$, $y = b$

In this case $\{v, w, x, y, z\}$ induces an $S_{1,1,2}$.

In every case, we obtain a contradiction, hence our assumption was wrong, and in $K_{m,n}$ misses a matching at most. □

Theorem 69. *Let G be a graph with $\omega(G) = 2$. Then G is A-nice if, and only if, each component H of G is K_1 or $K_{m,n} - M_k$, for $k < m \leq n$, or $K_{m,m} - M_m$.*

Proof. If every component H of G is K_1 or $K_{m,n} - M_k$, for $k < m \leq n$, or $K_{m,m} - M_m$, then by Lemma 61 resp. Lemma 62 Alice has a winning strategy with two colors for every component, which gives her a global winning strategy if she always plays in the component where Bob has played just before. Note that her local winning strategies always guarantee that she may miss her first turn. So G is A-nice.

Now consider the case that a component H of G is different from K_1 and $K_{m,n} - M_k$, for any $k < m \leq n$, and $K_{m,m} - M_m$. Then either H is a $K_{m,n} - M_m$ with $2 \leq m < n$, in which case Bob has a winning strategy with two colors by Lemma 63, or H is not bipartite, in which case Bob obviously wins with two colors, or H is bipartite but not of the form $K_{m,n} - M_k$. In the latter case, by Lemma 68, H contains an induced $S_{1,1,2}$ or an induced P_6, therefore Bob has a winning strategy with two colors by playing only in H, according to Lemma 66 resp. Lemma 67. So G is not A-nice since $\omega(G) = 2$. □

Corollary 70. *A graph G with $\omega(G) \leq 2$ is A-perfect if, and only if, every component of G is either K_1 or $K_{m,n}$ or $K_{m,n} - M_1$ for some m, n.*

Proof. The A-nice configurations $K_{m,n} - M_k$ for $m, n \geq 3$, $k \geq 2$ are excluded from being A-perfect, since they contain an induced subgraph isomorphic to $K_{3,2} - M_2$, which is not A-nice. □

5.4 Bodlaender's original version

We consider two other variants of the game. The first one is Bodlaender's original game, which we denote by g_A, where Alice has the first move, but missing a turn is not allowed for any player. Its dual version is g_B, where Bob has the first move and missing a turn is not allowed. For the associated game chromatic numbers by [3] we have for any graph G

$$\chi_A(G) \leq \chi_{g_A}(G) \leq \chi_B(G),$$

and

$$\chi_A(G) \leq \chi_{g_B}(G) \leq \chi_B(G).$$

Thus g_A-nice and g_B-nice graphs with clique number 2 are contained in the set of A-nice graphs. As in the proof of Lemma 61 resp. 62 it is easy to see that the connected g_A-nice graphs with clique number 2 are exactly nontrivial stars, i.e. the connected B-nice graphs with clique number 2, and the connected g_B-nice graphs with clique number 2 are the connected A-nice graphs with clique number 2.

In the following, *odd* resp. *even stars* are stars with an odd resp. even number of vertices. Denote by *A-component* a connected $K_{m,m} - M_m$ or a connected $K_{m,n} - M_k$ for $k < m \leq n$ that is not a star. An *odd* resp. *even A-component* is an A-component with an odd resp. even number of vertices. Then we may formulate

Theorem 71. *Let G be a graph with clique number 2. Then G is g_A-nice if, and only if, either*

(i) *every component of G is a star, not all of which are trivial, or*

(ii) *G consists of an odd number of odd stars and an arbitrary number of even stars and exactly one odd A-component, or*

(iii) *G consists of an odd number of odd stars and an arbitrary number of even stars and an arbitrary number of even A-components.*

Theorem 72. *Let G be a graph with clique number 2. Then G is g_B-nice if, and only if, either*

(i) *every component of G is a star, not all of which are trivial, or*

(ii) *G consists of an even number of odd stars and an arbitrary number of even stars and exactly one odd A-component, or*

(iii) *G consists of an even number of odd stars and an arbitrary number of even stars and an arbitrary number of even A-components.*

Proof. (Theorems 71 and 72) In both variants, every A-component has to be colored by Bob first, otherwise Alice will loose. The possibilities for Alice to force Bob to do this are given in the theorems. $\square$

Independently from the author, Borowiecki and Sidorowicz gave an analogue characterization of graphs with g_A-game chromatic number 2. In their recent paper [21], their Theorem 2 corresponds to our Theorem 71.

Corollary 73. *A graph G with clique number 2 is g_A-perfect if, and only if, it is a forest of stars.*

Proof. The graphs in case (i) of Theorem 71 are obviously g_A-perfect. Consider the case that G contains an odd number of odd stars as in case (ii) and (iii). Then the subgraph of G in which one of the odd stars is deleted has an even number of odd stars and at least one A-component, hence it is not g_A-nice. $\square$

Corollary 74. *A graph G with clique number 2 is g_B-perfect if, and only if, it is either a forest of stars or a single A-component of type $K_{m,n}$ or $K_{m,n} - M_1$.*

Proof. Case (i) of Theorem 72 obviously describes g_B-perfect graphs. Now consider case (ii) and (iii) of Theorem 72. If G contains an even number ≥ 2 of odd stars, then the subgraph where one of these odd stars is deleted has an odd number of odd stars and is not g_B-nice. Also, if G contains an arbitrary number ≥ 1 of even stars, then the subgraph where one vertex in an even star is deleted has an odd star, thus it is not g_B-nice. So G has no stars and a single odd A-component in case (ii) or some even A-components in case (iii). In the latter case, if G has more than one even A-component then the subgraph H which is obtained by deleting one vertex has either one odd A-component and at least one even A-component or a single odd star and at least one even A-component, thus H is not g_B-nice. So in case (ii) and (iii) G is a single A-component which is of the form $K_{m,n} - M_0$ or $K_{m,n} - M_1$ by Corollary 70 using $\chi_{g_B}(G) \geq \chi_A(G)$. It is easy to see that such a graph is always g_B-perfect since every induced subgraph is of the form $K_{m',n'} - M_k$ with $k \in \{0, 1\}$. □

5.5 The general case

Theorem 75. *A graph G is B-nice if, and only if, for each component H of G Alice has a winning strategy in the game B played on H with $\omega(G)$ colors.*

Proof. Assume that there is a component H_0 with the property that Bob has a winning strategy in the game B played on H_0 with $\omega(G)$ colors. We have to prove that Bob has a winning strategy on G with $\omega(G)$ colors. Consider the game played on G. Then Bob only plays on H_0 according to his winning strategy for H_0. (If Alice plays in a component different from H_0, Bob misses his next turn.) So Bob will win globally, i.e. on G.

Now assume that Alice has a winning strategy for every component H of G in the game B with $\omega(G)$ colors. We shall prove a winning strategy for Alice on G with $\omega(G)$ colors. Alice always answers a move of Bob by playing in the same component where Bob has just colored a vertex according to her winning strategy for this component. If the component is completely colored or if Bob misses his turn, she arbitrarily chooses a component and thinks that Bob has missed his turn playing in that particular component. Playing like that Alice wins on G with $\omega(G)$ colors. □

Theorem 76. *If a graph G is A-nice, then for each component H of G Alice has a winning strategy in the game A played on H with $\omega(G)$ colors.*

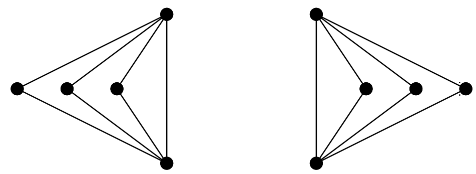


Figure 5.4: The graph G_2

Proof. Assume that there is a component H_0, so that Bob wins the game A on H_0 with $\omega(G)$ colors. Then Bob has a global winning strategy on G with $\omega(G)$ colors if he only plays on H_0. If Alice colors a vertex in a different component, Bob imagines that she has missed a turn. So Bob wins on G. $\square$

Remarkably, the inverse implication of Theorem 76 does not hold for $\omega(G) \geq 3$. It is easy to see that each component of the graph G_2 of Fig. 5.4 has A-game chromatic number $3 = \omega(G_2)$, but G_2 itself has A-game chromatic number 4.

Special classes of perfect graphs are bipartite graphs, comparability graphs and triangulated graphs, and their complements, cf. [13]. Interval graphs are special triangulated graphs. Bipartite graphs are special comparability graphs. None of these classes, except the class of complements of bipartite graphs, is contained completely in the class of B-perfect or A-perfect graphs. P_4 is a bipartite interval graph and the complement of a bipartite interval graph, as it is self-complementary, but not B-perfect. P_5 is an example of a bipartite interval graph which is not A-perfect. $S_{1,1,2}$ is not A-perfect but the complement of an interval and comparability graph, namely the graph in Fig. 5.10 (a). Some non-trivial examples for interval graphs are given in Figs. 5.5 and 5.6.

Theorem 77. *Complements of bipartite graphs are A-perfect.*

Proof. Let $G = (A \cup B, E)$ be a bipartite graph and G' be its complement. So A and B are the vertex sets of cliques in G', but not necessarily of maximum cliques. Let $\omega(G')$ be the clique number of G'. Since G' is perfect, there is a coloring $c : A \cup B \longrightarrow \{1, \ldots, \omega(G')\}$ with $\omega(G')$ colors. We call a vertex which is the only vertex in c of a certain color a *single* vertex. All other vertices are

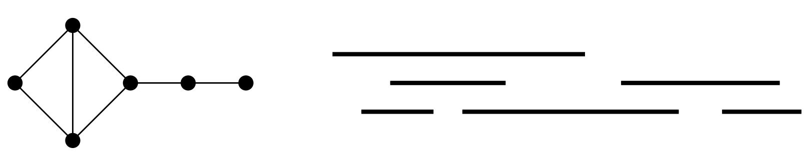


Figure 5.5: An interval graph which is not B-perfect

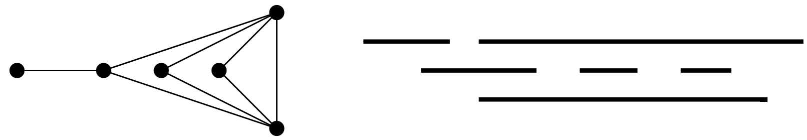


Figure 5.6: An interval graph which is not A-perfect

called *double* vertices as every color class in c has at most two vertices. If v is a vertex and w is a vertex of the same color in c, then w is called the *companion* of v. We will prove that for the variant A, Alice has a winning strategy with $\omega(G')$ colors. (This will prove the theorem since induced subgraphs of G' are also complements of bipartite graphs.)

Alice misses her first turn. If Bob colors a single vertex, Alice misses her turn. If Bob colors a double vertex v with a color not used so far and the companion w of v is uncolored, then Alice colors w with the same color as Bob has colored v. The last case is that Bob colors a double vertex v with a color already used for another vertex and the companion w of v is uncolored. In this case Alice colors w with a new color. Note that the number of colors in the partial coloring Alice and Bob produce is never greater than the number of colors in the partial coloring of c induced by the same vertices. This is true because the last case may only occur if at a certain point of the game Bob has used the color he already assigned to a single vertex for a double vertex. (And after that Bob may iteratively have used the color of a double vertex for a double vertex which is not its companion.) There are no further cases since after Alice's moves, if a double vertex is colored, its companion is also colored. So at the end the players will have used only $\omega(G')$ colors. $\square$

5.6 Towards a Strong Perfect Graph Theorem for B-perfect graphs

We define a *broken wheel* as a graph G with a center vertex v_0 and n sets $A_1, \ldots, A_n$ of vertices and possibly an additional set B of vertices with the following properties. Between vertices of different sets A_i and A_j or B there are no edges. The subgraph induced by $A_i \cup \{v_0\}$ is a complete graph. The subgraph induced by $B \cup \{v_0\}$ is a complete graph without one edge b_1b_2 between two vertices $b_1, b_2 \in B$. So the maximum cardinality of $A_i \cup \{v_0\}$ resp. B determines the clique number $\omega(G)$. See Fig. 5.7 for an example of a broken wheel with clique number 3.

Theorem 78. *A graph each component of which is a broken wheel is B-perfect.*

Proof. First, we prove that such a graph G is B-nice. By Theorem 75 we may assume that G is a broken wheel. We describe a winning strategy for Alice with $\omega(G) \geq 3$ colors. Alice has to do two things: to color the center vertex as early as possible, but if Bob colors b_i, $i \in \{1,2\}$, then she has to color b_{3-i} with the same color. So the center vertex will be colored after Alice's second move (possibly with the third color) and the remaining $\omega(G) - 1$ or fewer vertices of a set A_i can always be colored. The same holds for B if Alice colors b_{3-i} immediately after Bob has colored b_i. If Bob forces Alice to color the first vertex of $\{b_1, b_2\}$, then this will be at the end of the game when every vertex of B except b_1 and b_2 is colored. But then there is no danger for Alice when she colors a vertex b_i. So Alice will win in every case. For $\omega(G) \leq 2$ a broken wheel is simply a star.

Now we have to prove that every subgraph of a graph the components of which are broken wheels is a graph all components of which are broken wheels. But this is obvious: if vertices in a set of type A_i are missing then we obtain again a clique, thus a set of type A_i. On the other hand, if vertices in a set of type B are missing we either obtain a set of type A_i or of type B. Hence, in every component there is at most one set of type B. If v_0 is missing, then every set A_i and B forms a trivial broken wheel where any vertex except b_1 and b_2 can be considered as a new center vertex. We conclude that a graph of broken wheels is B-perfect. □

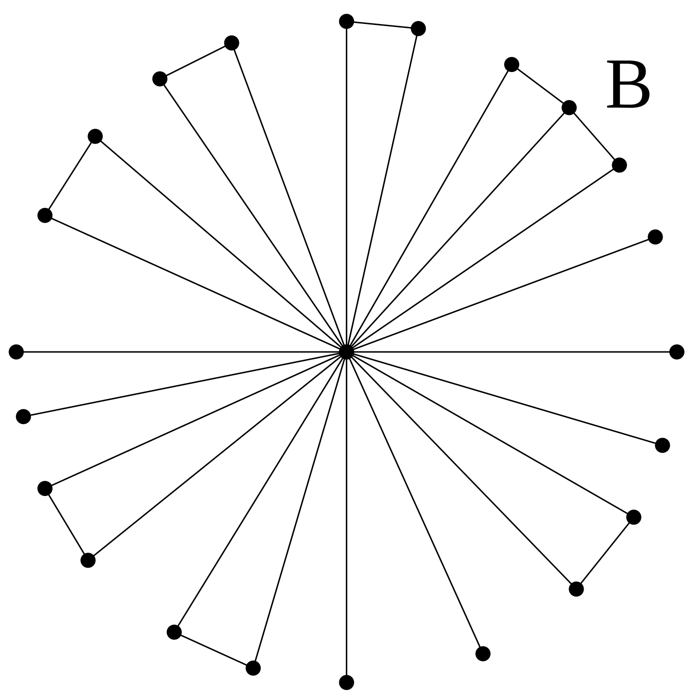


Figure 5.7: A broken wheel with clique number 3

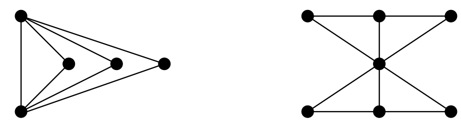


Figure 5.8: Two more forbidden configurations for B-perfect graphs

One may conjecture that these are mainly all B-perfect graphs.

Conjecture 79. *Every B-perfect graph is a graph all components of which are some* generalized *broken wheels.*

It is left open how to generalize broken wheels appropriately. The structure of A-perfect graphs seems to be a lot richer and thus more complicated.

We will prove that for graphs with clique number 3 the B-perfect graphs are exactly the broken wheels. In order to prove this result we need a lemma which was shown by Wolk [82]. We start with a definition.

Definition 5.1. Let $G = (V, E)$ be a graph. A *universal vertex* of G is a vertex $v \in V$ that is adjacent to all vertices $w \in V$, $v \neq w$.

Lemma 80. (Wolk [82])
Let G be a connected trivially perfect graph. Then G has a universal vertex.

Theorem 81. *A graph G with $\omega(G) = 3$ is B-perfect if, and only if, every component of G is a broken wheel.*

Proof. By Theorem 78 a graph each component of which is a broken wheel is B-perfect. Now consider the case that G with $\omega(G) = 3$ is B-perfect. By Lemmata 55 and 56 G is trivially perfect. Then by Lemma 80 every component H of G contains a universal vertex of H. Consider such a component H_0 with universal vertex v_0. Let S be a 2-connected block of this component. Since $S \setminus \{v_0\}$ has clique number of at most 2 and does not contain an induced P_4 or C_4, $S \setminus \{v_0\}$ is a star. If it is a star with three or more leaves, then Bob has a winning strategy with three colors: in his first move he colors a leaf, in his second move a leaf with a different color. So $S \setminus \{v_0\}$ is either K_1, K_2 or P_3. If two different blocks of H_0 without the universal vertex are P_3, then Bob has the following winning strategy: in his first move he colors the first leaf v_1 of the first P_3 with the first color. If Alice colors one of the neighbors v_2 or v_0 of v_1 with the second color or a vertex of the second P_3 or of another block with an arbitrary color, then Bob colors the second leaf of the first P_3 with the third color, so that eventually either v_2 or v_0 will be surrounded by all three colors. So the only possibility for Alice to play safely is to color the second leaf of the first P_3 with the first color. But then Bob colors the first

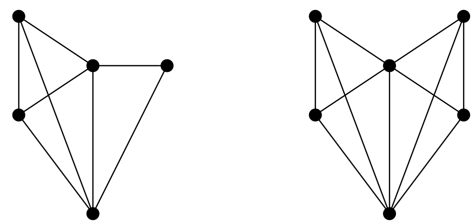


Figure 5.9: Two B-perfect graphs with clique number 4

leaf of the second P_3 with the second color. By the same argument as above, the only chance for Alice to play safely is to color the second leaf of the second P_3 with the second color. However, then Bob colors v_2, the third vertex of the first P_3, with the third color, and Alice has lost as v_0 cannot be colored any more. Thus the respective component of G is a broken wheel. □

From Theorem 81 we can deduce that the two graphs with clique number 3 in Fig. 5.8 are forbidden configurations for B-perfectness since they are no broken wheels.

There are connected graphs with clique number of at least 4 which are B-perfect but no broken wheels, e.g. the graphs in Figure 5.9. Thus, Conjecture 79 is not correct without the word 'generalized'. So the definition of broken wheels has to be refined.

In order to formulate the next conjecture we need the following definitions. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Then $G_1 \cup G_2$ denotes the *disjoint union* of G_1 and G_2, i.e. the union of disjoint copies of G_1 and G_2. $G_1 \vee G_2$ denotes the *join* of G_1 and G_2, i.e. the graph consisting of $G_1 \cup G_2$ and the additional arcs of $(V_1 \times V_2) \cup (V_2 \times V_1)$. For a graph G and $n \geq 0$, nG denotes the graph

$$\underbrace{G \cup G \cup \ldots \cup G}_{n}.$$

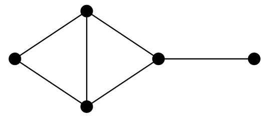
Here $0G$ denotes the empty graph.

Conjecture 82. *A graph G with $\omega(G) = 4$ is B-perfect if, and only if, G is of the form*

$$\begin{aligned} K_1 \quad \vee \quad & (n_1 K_1 \cup n_2 K_2 \cup n_3 P_3 \\ & \cup n_4 K_3 \cup n_5 (K_1 \vee 2K_2) \\ & \cup n_6 (K_1 \vee (K_1 \cup K_2)) \cup n_0 (K_2 \vee \overline{K_2}^0)) \end{aligned} \tag{5.1}$$

with $n_i \geq 0$, and $n_0 + n_3 + n_5 + n_6 \in \{0, 1\}$.

The two graphs of Fig. 5.9 are of the type described in Conjecture 82. For the left-hand graph, we have to choose $n_6 = 1$ and $n_i = 0$ for $i \neq 6$, for the right-hand graph, $n_5 = 1$ and $n_i = 0$ for $i \neq 5$.


(a)

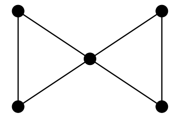

(b)

Figure 5.10: An (a) A-perfect resp. a (b) B-perfect graph the complement of which is not A-perfect resp. B-perfect

Final remark. The Weak Perfect Graph Theorem [62] states that a graph is perfect if, and only if, its complement is perfect. There is no game theoretic analogon of this theorem. An example with 5 vertices is the graph in Figure 5.10 (b) which is B-perfect but its complement $C_4 \cup K_1$ is not B-perfect. The complement of the A-perfect graph in Figure 5.10 (a) is $S_{1,1,2}$ which is not A-perfect.

Open question. Find a characterization of game-perfectness in terms of forbidden induced subgraphs as an analogon to the Strong Perfect Graph Theorem.

5.7 B-perfect digraphs

The characterization of B-perfect digraphs is very easy, provided the B-perfect graphs are known. Since the digraph consisting of two vertices connected by a single arc has clique number 1 but B-game chromatic number 2 and, therefore, is *not* B-perfect, we have the following

Observation 83. *The class of B-perfect digraphs is exactly the same as the class of B-perfect graphs.*

5.8 A-perfect digraphs with clique number 2

In spite of the afore-mentioned characterization of B-perfect digraphs, the class of A-perfect digraphs is richer than the class of A-perfect graphs. In the following subsections we will examine some of the internal structures of A-perfect digraphs with clique number 2. The fact that DIRECTED 2-COLORING is $\mathcal{NP}$-complete (see Appendix A) might indicate that a complete characterization of all A-perfect digraphs with clique number of at most 2 is very difficult.

We start with the following remark on A-perfect digraphs with clique number 1.

Proposition 84. *The A-perfect digraphs with clique number 1 are exactly the in-stars.*

Proof. Obviously, in-stars are A-perfect. Let D be a simple digraph (i.e., a digraph with clique number 1) which is not an in-star. We will prove that D is not A-nice (and therefore, not A-perfect).

Since D is not an in-star, D has at least two vertices with in-degree of at least 1. Let v and w be such vertices. A winning strategy for Bob with 1 color is the following. If Alice, in her first move, colors a vertex z with non-zero out-degree, Bob wins, since an out-neighbor of z cannot be colored any more. Otherwise, we may assume w.l.o.g. that w has not been colored by Alice. Then Bob colors an in-neighbor of w, so that w cannot be colored any more. Thus he wins in any case. $\square$

5.8.1 Semiorientations of complete graphs

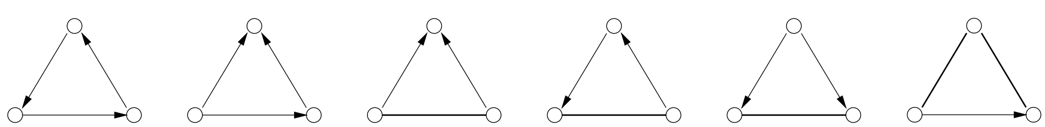
A *semiorientation* of a graph $G = (V, E)$ is a digraph $D = (V, \vec{E} \cup F)$ consisting of an orientation $(V, \vec{E})$ of G and a possibly nonempty set $F \subseteq E$ of additional arcs. E.g., by our definition, the paths are the semiorientations of undirected paths. Fig. 5.11 depicts all semiorientations of the complete graph K_3 with clique number of at most 2.

Theorem 85. *The only A-perfect semiorientations of K_3 with clique number of at most 2 are $K_3^{1,++}$ and $K_3^{1,+-}$.*

Proof. This is a simple case analysis on the configurations of Fig. 5.11. $\square$

Theorem 86. *The only A-perfect semiorientation of K_4 with clique number of at most 2 is $\overline{\vec{C}_4}^{\,0}$, the loop deletion digraph of the true complement of the directed 4-cycle.*

Proof. Let D be a semiorientation of K_4 with vertices v_1, v_2, v_3, v_4. In case D has at most one edge, D contains an orientation of a K_3 which is not A-perfect. If D has two adjacent edges, either D contains a K_3^2 which is not A-perfect or D has clique number at least 3. So we may assume that D contains the



$K_3^{0,c}$ $K_3^{0,t}$ $K_3^{1,++}$ $K_3^{1,+-}$ $K_3^{1,--}$ K_3^{2}

Figure 5.11: Semiorientations of K_3

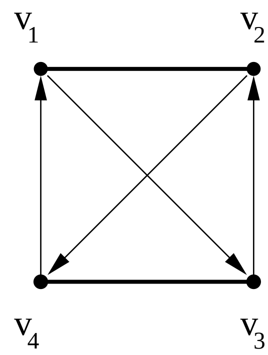


Figure 5.12: $\overline{\vec{C}_4}^0$

edges v_1v_2 and v_3v_4 and no further edges. W.l.o.g. the arc between v_1 and v_3 is directed as (v_1, v_3). Since the subdigraph on the vertices v_1, v_3, v_4 may not be $K_3^{1,--}$ which is not A-perfect the arc between v_1 and v_4 is directed as (v_4, v_1). With the same arguments concerning the sets of vertices $\{v_1, v_2, v_4\}$ resp. $\{v_2, v_3, v_4\}$ one finds the orientation of the other arcs that are (v_2, v_4) resp. (v_3, v_2). □

$\overline{\vec{C}_4}^0$ is depicted in Fig. 5.12.

Theorem 87. *There is no semiorientation of K_n, $n \geq 5$, with clique number 2 that is A-perfect.*

Proof. Obviously, it is sufficient to prove the theorem for $n = 5$. Let D be a semiorientation of K_5 with vertices v_1, v_2, v_3, v_4, v_5. Assume that D is A-perfect. By Theorem 86, the subdigraph on the vertices v_1, v_2, v_3, v_4 must be the digraph of Fig. 5.12. Either there is an edge v_1v_5 or an edge v_3v_5, otherwise the digraph on $\{v_1, v_3, v_5\}$ would be an orientation of K_3. W.l.o.g. there is an edge v_1v_5. Then there is no edge v_3v_5 (otherwise $\{v_1, v_3, v_5\}$ would induce a K_3^2.) Since $\omega(D) = 2$, there is no edge v_2v_5. But then, as $\{v_2, v_4, v_5\}$ may not induce an orientation of a K_3, there is an edge v_4v_5. Hence $\{v_1, v_4, v_5\}$ induces a K_3^2, which is not A-perfect, a contradiction. □

5.8.2 Paths

Paths are a simple class of digraphs since every subdigraph of a path is a forest of paths. Therefore we consider the hereditary class $\vec{\mathcal{PF}}$ of forests of paths, i.e. of those digraphs each component of which is a path. For questions of A-perfectness we do not need to consider forbidden configurations that are not in $\vec{\mathcal{PF}}$.

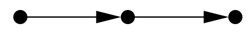


Figure 5.13: The forbidden configuration $F_{3,1}$

Figure 5.14: The forbidden configuration $F_{3,2}$

Figure 5.15: The forbidden configuration F_4

Figure 5.16: The forbidden configuration $F_{5,1}$

Figure 5.17: The forbidden configuration $F_{5,2}$

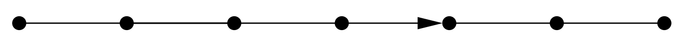


Figure 5.18: The forbidden configuration $F_{7,1}$

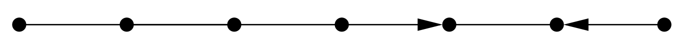


Figure 5.19: The forbidden configuration $F_{7,2}$

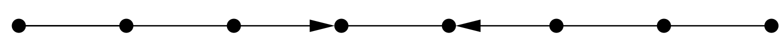


Figure 5.20: The forbidden configuration F_8

Lemma 88. *If a digraph D contains any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, $F_{5,1}$, $F_{5,2}$, $F_{7,1}$, $F_{7,2}$, or F_8 depicted in Figs. 5.13–5.20 as induced subdigraph, then D is not A-perfect.*

Proof. It is easy to see that the forbidden configurations have A-game chromatic number 2 if they are simple digraphs, and 3 otherwise, thus they are not A-perfect. Then, by the definition of A-perfectness, D is not A-perfect. $\square$

Recall that an arc which is not contained in any edge is called *single arc*.

Lemma 89. *Let P be a path with $n \geq 10$ vertices. Then P is not A-perfect. Moreover, P contains a forbidden configuration as an induced subdigraph.*

Proof. Assume P is A-perfect. If P contains 3 single arcs, P has an induced $F_{3,1}$, $F_{3,2}$ or F_4. So P has at most 2 single arcs. If there are 2 single arcs then these are adjacent or at distance 1, otherwise P has an induced F_4. Since the length of the path is $n-1 \geq 9$, P contains either an induced $P_5 = F_{5,2}$, which is a forbidden configuration, or P is of the form $v_1v_2v_3v_4v_5v_6v_7v_8v_9v_{10}$, where $v_1v_2v_3v_4$ and $v_7v_8v_9v_{10}$ are (undirected) P_4's, v_5v_6 is an edge, and between v_4 and v_5 resp. between v_6 and v_7 there are single arcs. If there was an arc (v_5, v_4) or an arc (v_6, v_7), P would contain $F_{5,1}$. So there are arcs (v_4, v_5) and (v_7, v_6). But then P contains $F_{7,2}$, which is a contradiction. $\square$

Theorem 90. *Let F be a forest of paths with components $D_1, D_2, \ldots, D_k$. Then the following statements are equivalent:*

(a) F is A-perfect.

(b) F does not contain any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, $F_{5,1}$, $F_{5,2}$, $F_{7,1}$, $F_{7,2}$, or F_8 depicted in Figs. 5.13–5.20 as an induced subdigraph.

(c) Every component of F, except at most one, is either an undirected path P_1, P_2, P_3, or P_4, and the remaining component is one of the 47 configurations depicted in Fig. 5.21.

In particular, the only A-perfect paths are those depicted in Fig. 5.21.

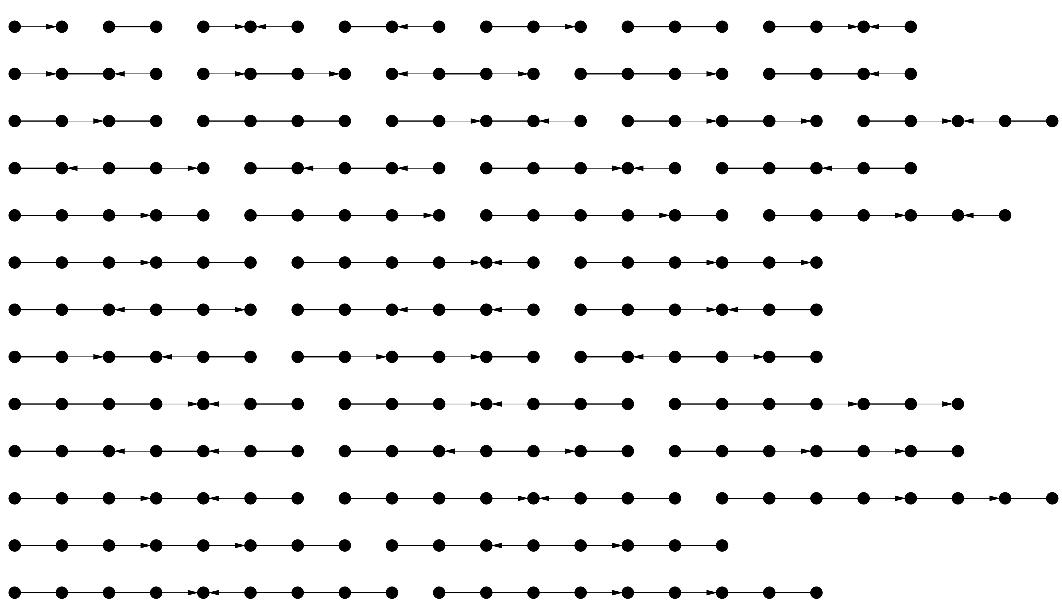


Figure 5.21: The 47 A-perfect paths

Proof. By Lemma 88 we have (a) $\Longrightarrow$ (b).

Consider (b) $\Longrightarrow$ (c). Assume that F does not contain any forbidden configuration. As F_4 is forbidden, every component D_i (with at most one exception, say D_1) is a graph, i.e. an undirected path P_{n_i}. Since $P_5 = F_{5,2}$ is forbidden, $n_i \leq 4$ for all $i \geq 2$. By Lemma 89 the remaining component has at most 9 vertices. The configurations of Fig. 5.21 are exactly those paths with at most 9 vertices which do not contain any of the forbidden configurations as induced subdigraphs (list all paths with at most 9 vertices and delete all enlargements of forbidden configurations). Thus F is of the desired form.

Finally we prove (c) $\Longrightarrow$ (a). Assume that F is of the form as in (c). By case analysis or the use of the program given in Appendix B it is easy to see that the 47 configurations of Fig. 5.21 are A-nice. Every digraph consisting of an arbitrary component C' which is one of the digraphs of Fig. 5.21 and some components which are undirected paths P_1, P_2, P_3, or P_4 is A-nice as well, as we shall see. Indeed, a winning strategy for Alice is the following: in her first move she plays on C', after that she always plays in the component on which Bob has played in his last move, in both cases according to her winning strategy for the respective components. Playing on a component possibly includes the use of Alice's right to miss a turn if this is necessary according to her winning strategy for C' or if a component is completely colored. Note that her winning strategies for P_j, $1 \leq j \leq 4$, always allow her to make Bob color the first vertex, therefore the strategy described above is feasible. Since every subdigraph of F is also of the type of digraphs described in (c), F is not only A-nice, but A-perfect. $\square$

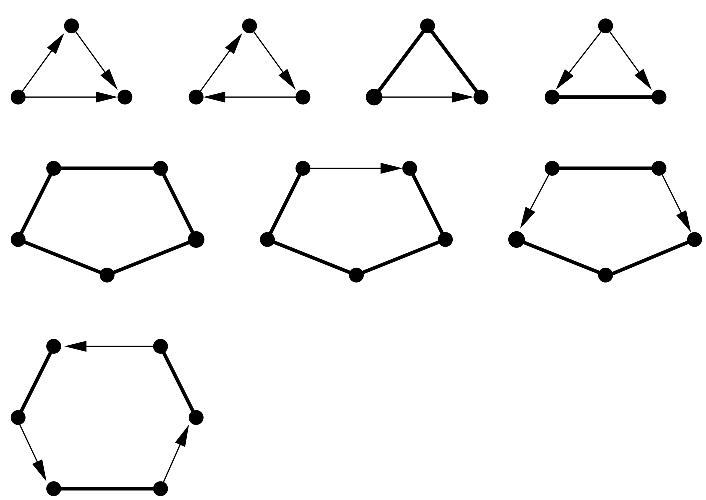


Figure 5.22: 8 forbidden cycles

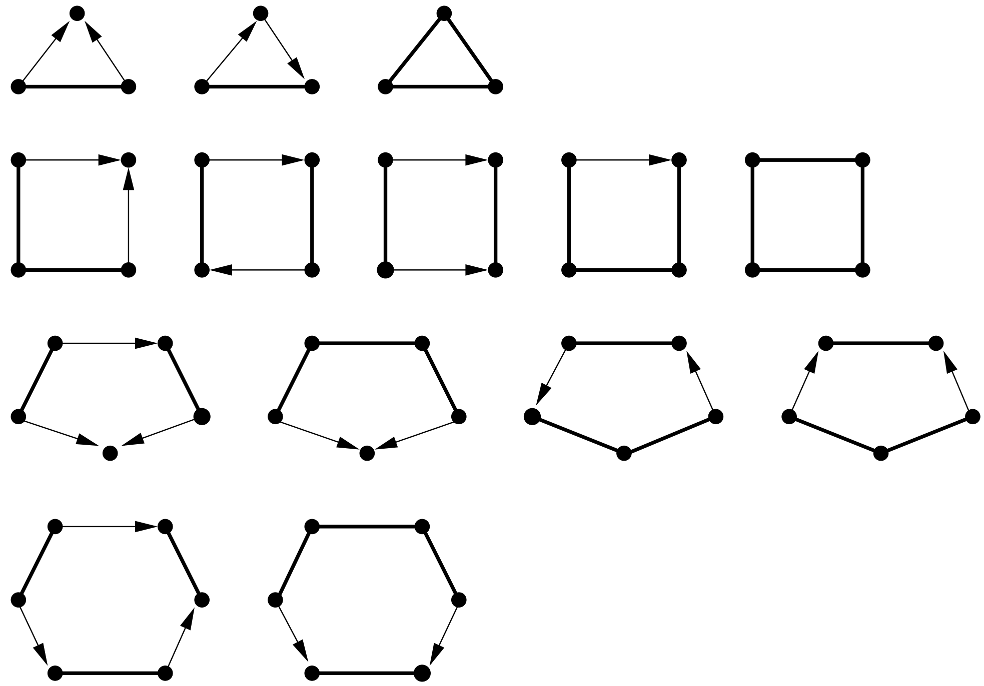


Figure 5.23: The 14 A-perfect semiorientations of cycles

5.8.3 Cycles

Lemma 91. *Let C be a cycle with $n \geq 7$ vertices. Then C is not A-perfect.*

Proof. Assume C is A-perfect. If C has three single arcs, then it contains a forbidden configuration $F_{3,1}$, $F_{3,2}$, or F_4 as induced subdigraph. So C has at most 2 single arcs, and if there are two, then these are adjacent or at distance 1. There are remaining $m \geq n-3 \geq 4$ edges, which form a (forbidden) $P_5 = F_{5,2}$, a contradiction. □

Theorem 92. *Let C be a cycle. C is A-perfect if, and only if, C is one of the 14 configurations of Fig. 5.23.*

Proof. Proper subdigraphs of cycles are forests of paths. By case analysis or the use of the program in Appendix B it is easy to see that among all 22 cycles with at most 6 vertices which do not contain any of the forbidden configurations $F_{3,1}$, $F_{3,2}$, F_4, $F_{5,1}$, or $F_{5,2}$ as induced subdigraphs there are exactly the 14 configurations of Fig. 5.23 which are A-nice. Thus, as they do not contain the forbidden configurations, they are A-perfect. By Lemma 91, cycles with more than 6 vertices are not A-perfect. □

In Fig. 5.22, 8 forbidden cycles are depicted. These are minimal forbidden configurations, i.e. they do not contain other forbidden configurations as proper induced subdigraphs. Together with the 7 forbidden paths $F_{3,1}$, $F_{3,2}$, $F_{5,1}$, $F_{5,2}$, $F_{7,1}$, $F_{7,2}$, and F_8, and the non-connected forbidden configuration F_4, so far we have found 16 minimal forbidden configurations for A-perfectness of digraphs. There might be many more minimal forbidden configurations. E.g., if we consider forests, we have already seen that the multistar $S_{1,1,2}$ of Fig. 5.3 is such a minimal forbidden configuration. But also the digraphs formed by $S_{1,1,2}$ in which some leaf edges are replaced by single arcs directed towards the interior are minimal forbidden configurations.

The next step in order to complete the list of minimal forbidden configurations for A-perfectness would be to consider forests in general, instead of forests of paths. By Lemma 89 we have that a tree of diameter $d \geq 9$ is not A-perfect. However, a lot of trees would have to be examined in order to determine the forbidden configurations. Note that the number of A-perfect trees is infinite since, for example, every in-star is A-perfect. Whether or not the number of minimal forbidden trees is finite remains open.

The last step for the classification of A-perfect digraphs with clique number 2 would consist in considering semiorientations of arbitrary graphs. It is clear that every component but one of an A-perfect digraph with clique number 2 must be a bipartite graph of the form as discussed in Section 5.3. However, the remaining exceptional component will cause a lot of work.

A classification of all A-perfect digraphs (without restriction to the clique number) seems to be a demanding task for the future, as well as a description of A-perfect digraphs by minimal forbidden induced subdigraphs. Such a classification would be of equal weight as the Strong Perfect Graph Theorem.

Final remark. Another interesting question is the following: Let g be a variant of the coloring game and k be a nonnegative integer. Let $N(g,k)$ be the number of isomorphism classes of connected g-perfect digraphs D with maximum in-degree $\Delta^+(D) = k$. Let $N'(g,k)$ be the number of isomorphism classes of connected g-perfect graphs G with maximum degree $\Delta(G) = k$.

Open question. Is it true that $N(g,k) < \infty$? If yes, determine the exact value of $N(g,k)$.

A weaker formulation of this question concerns only graphs:

Open question. Is it true that $N'(g,k) < \infty$? If yes, determine the exact value of $N'(g,k)$.

Note that we need the precondition 'connected', otherwise the numbers would be always infinite. For example, if a digraph is A- or B-perfect, then it is still A- resp. B-perfect when an arbitrary number of isolated vertices is added.

Obviously, $N'(g,0) = N'(g,1) = 1$ for any g. Connected graphs with maximum degree 2 are cycles C_n or paths P_n with $n \geq 3$. C_3 is B-perfect (and thus A-perfect) since it is a complete graph. By the results of this chapter, P_3 is B-perfect, and P_n and C_n, $n \geq 4$, are not B-perfect. Therefore $N'(B, 2) = 2$. Furthermore P_4 and C_4 are A-perfect, but P_n and C_n, $n \geq 5$, are not A-perfect. Thus $N'(A, 2) = 4$. For $k \geq 3$, $N'(g, k)$ is not known.

Counting the numbers $N(g, k)$ and $N'(g, k)$, if they are finite, might give new insights into the structure of g-perfect digraphs resp. graphs. Here we come from the parameter 'maximum in-degree' instead of the parameter 'clique number' as in the main parts of this chapter. Research combining these two parameters might lead to better characterizations of g-perfect digraphs than the use of only one parameter.

Appendix

Appendix A

Complexity results

Let $k \geq 0$ be a fixed integer. Recall that a *k-coloring* of a digraph D is a color assignment from the set $\{1, 2, \ldots, k\}$ to the vertices of D, so that the preimage of every color induces an acyclic digraph. The problem to decide whether there exists a k-coloring of a given graph is known as k-COLORING. We consider the more general problem of DIRECTED k-COLORING where any digraph is allowed as an instance of the problem. The interesting special case of SIMPLE DIRECTED k-COLORING is defined by restricting the instances to simple digraphs.

It is well-known that 1-COLORING and 2-COLORING are in $\mathcal{P}$, but 3-COLORING is $\mathcal{NP}$-complete (see Papadimitriou [75]). In the more general case of digraphs this situation changes. DIRECTED 1-COLORING is in $\mathcal{P}$ since Depth-First-Search detects directed cycles in linear time. Bokal et al. [18] proved that SIMPLE DIRECTED 2-COLORING (and thus DIRECTED 2-COLORING) is $\mathcal{NP}$-complete by reducing 2-COLORABILITY OF 3-UNIFORM HYPERGRAPHS to SIMPLE DIRECTED 2-COLORING. They reinvent the dichromatic number in [18], the work of Neumann-Lara [69] seems to have been unknown to them.

The main result of Appendix A will be another proof of the $\mathcal{NP}$-completeness of SIMPLE DIRECTED 2-COLORING, which was discovered independently. We will use a reduction from NAE3SAT (which is defined in the following paragraph) to SIMPLE DIRECTED 2-COLORING which is similar to the well-known reduction from NAE3SAT to 3-COLORING [75].

The Not-All-Equal 3-Satisfiability Problem (NAE3SAT) is defined as follows. An instance of the problem is a boolean formula $f(x)$ in conjunctive normal form

$$\begin{aligned} f(x) &= \bigwedge_{j=1}^{m} C_j(x) \qquad \text{with} \\ C_j(x) &= (\ell_{j1}(x) \vee \ell_{j2}(x) \vee \ell_{j3}(x)) \wedge (\neg\ell_{j1}(x) \vee \neg\ell_{j2}(x) \vee \neg\ell_{j3}(x)) \end{aligned}$$

with pairs of clauses with 3 literals in each clause as displayed above. The

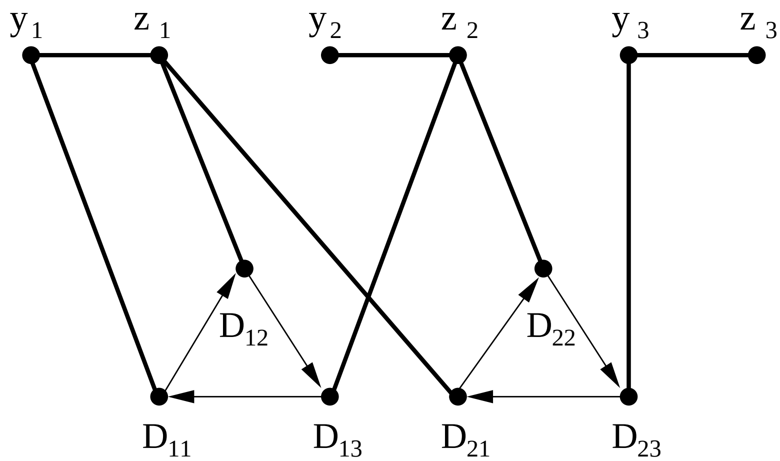


Figure A.1: Digraph for the formula $f(x) = C_1(x) \wedge C_2(x)$ with $C_1(x) = (x_1 \vee \overline{x}_1 \vee \overline{x}_2) \wedge (\overline{x}_1 \vee x_1 \vee x_2)$ and $C_2(x) = (\overline{x}_1 \vee \overline{x}_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x}_3)$

literals $\ell_{jk}(x)$ are variables x_i or negated variables $\overline{x}_i$. The problem consists in deciding whether there is a truth assignment which satisfies the formula. It is well-known that NAE3SAT is $\mathcal{NP}$-complete, see Papadimitriou [75].

Theorem 93. *DIRECTED 2-COLORING is $\mathcal{NP}$-complete.*

Proof. We will reduce NAE3SAT to 2-DIRECTED COLORING. Let

$$\begin{aligned} f(x) &= \bigwedge_{j=1}^{m} C_j(x) \qquad \text{with} \\ C_j(x) &= (\ell_{j1}(x) \vee \ell_{j2}(x) \vee \ell_{j3}(x)) \wedge (\neg\ell_{j1}(x) \vee \neg\ell_{j2}(x) \vee \neg\ell_{j3}(x)) \end{aligned}$$

be a NAE3SAT-formula with literals $\ell_{jk}(x) \in \{x_1, \ldots, x_n\} \cup \{\overline{x}_1, \ldots, \overline{x}_n\}$. Then we construct a digraph $G = (V, E)$ with $2n + 3m$ vertices and $2n + 9m$ arcs, as in Fig. A.1:

$$V = \{y_i, z_i : \ i = 1, \ldots, n\} \cup \{D_{jk} : \ j = 1, \ldots, m; \ k = 1, 2, 3\}$$

So for each variable x_i there are two vertices y_i and z_i, and for each pair $C_j(x)$ of clauses there are three vertices D_{j1}, D_{j2}, and D_{j3}. For all i we connect y_i and z_i by an edge, i.e. by the two arcs $(y_i, z_i), (z_i, y_i) \in E$. We further connect y_i and D_{jk} by an edge if $\ell_{jk}(x) = x_i$, we do the same with z_i and D_{jk} if $\ell_{jk}(x) = \overline{x}_i$. Up to this moment we have an (undirected) graph but now we add directed triangles, namely the arcs (D_{j1}, D_{j2}), (D_{j2}, D_{j3}), (D_{j3}, D_{j1}) for each j, which are the essential part of the reduction. Note that a directed triangle may always be colored with two colors if we do not take the rest of the digraph into account.

We will prove that there is a truth assignment $a \in \{0,1\}^n$ with $f(a) = 1$ if, and only if, there is a 2-coloring of G.

If a satisfies f we define a color assignment $c: V \longrightarrow \{0,1\}$ by setting $c(y_i) := 1 - a_i$; $c(z_i) := a_i$; $c(D_{jk}) := a_i$ if there is an edge $y_i D_{jk}$, and $c(D_{jk}) := 1 - a_i$ if there is an edge $z_i D_{jk}$. So $c(D_{jk}) = \ell_{jk}(a)$. Because of the pairwise opposite clauses in $C_j(x)$ and because a is a satisfying truth assignment it is impossible that $c(D_{j1}) = c(D_{j2}) = c(D_{j3})$. Thus each triangle is colored in both colors. Thus c is a 2-coloring.

On the other hand, if a directed coloring $c: V \longrightarrow \{0,1\}$ exists, define $a_i := c(z_i)$ and observe again that $c(D_{jk}) = \ell_{jk}(a)$. Since every triangle is colored in two colors every double-clause C_j is satisfied by a. $\square$

DIRECTED k-COLORING may be reduced to DIRECTED $(k+1)$-COLORING: for a digraph D, construct a digraph D' with an additional vertex v_0 adjacent (by an edge, not only by a single arc!) to every other vertex of the original digraph. If D' can be colored with $k+1$ colors, then, in every coloring of D', v_0 is colored differently from every other vertex. Hence the coloring restricted to the vertices of D only uses k colors. On the other hand, if D can be colored with k colors, D' can obviously be colored with $k+1$ colors. Thus we have:

Corollary 94. *DIRECTED k-COLORING is $\mathcal{NP}$-complete for $k \geq 2$.*

We will now discuss the complexity of coloring simple digraphs and obtain a stricter version of Theorem 93. All we need to do is to replace the edges in the reductions of Theorem 93 and Corollary 94 by appropriate configurations, such that the arising digraph is simple and can be colored with k colors if, and only if, the original digraph can be colored like this.

We start with the following definitions. Recall that a *tournament* is the orientation of a complete graph. A digraph D is *n-dichromatic* if $\chi(D) = n$. D is *minimal n-dichromatic* if D is n-dichromatic and, for every digraph H with less vertices than D, $\chi(H) \leq n-1$.

A series $(AT(n))_n$ of tournaments with $\chi(AT(n)) = n$ for each nonnegative integer n is easily constructed. Let $AT(1)$ be the trivial graph with one vertex. For $n \geq 1$ let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two disjoint copies of $AT(n)$ and z be an additional special vertex. Define $AT(n+1) = (V, E)$ by $V = V_1 \cup V_2 \cup \{z\}$ and

$$E = E_1 \cup E_2 \cup (V_1 \times V_2) \cup (V_2 \times \{z\}) \cup (\{z\} \times V_1).$$

Observation 95. $\chi(AT(n)) = n$.

We omit the proof which runs by induction. The first reason for introducing the tournaments $AT(n)$ is only to prove the existence of a class of

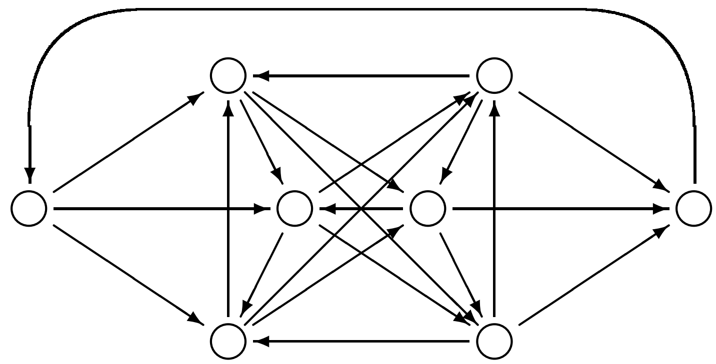


Figure A.2: Simple digraph replacing an edge

tournaments with arbitrarily large dichromatic number. Later we will come back to the tournaments $AT(n)$.

However, these tournaments are not minimal n-dichromatic for $n \geq 4$. Erdös et al. [38] have proved that the minimal order of an n-dichromatic tournament is $\theta(n^2 \log^2 n)$, whereas the order of $AT(n)$ is $2^n - 1$.

Let $T(n) = (V, E)$ be a tournament of minimal order that has dichromatic number $n+1$. To find such a tournament can be regarded as diffficult because we only know such tournaments for $1 \leq n+1 \leq 4$ (cf. [70]). Nevertheless such a tournament exists since the tournaments $AT(n+1)$ have dichromatic number $n+1$. Let $n \geq 2$. We pick a vertex z of $T(n)$ and define a new digraph, the *tournament line* $TL(n) = (V', E')$ with vertex set $V' = V \setminus \{z\} \cup \{x, y\}$ that contains all arcs of $T(n) - z$. In addition, for each arc of type (v, z), $TL(n)$ has an arc (v, x) and, for each arc of type (z, v), an arc (y, v). At last we add the arc (x, y). Thus $TL(n)$ can be constructed by splitting the vertex z into a *target* and a *source vertex* and by adding an arc from the target to the source.

Lemma 96. *For $n \geq 2$, $TL(n)$ may be colored with n colors, so that x and y have arbitrary, different, prescribed colors. For each n-coloring c of $TL(n)$ we have $c(x) \neq c(y)$.*

Proof. In order to prove the first statement, let $[n] = \{1, 2, \ldots, n\}$ be a set of n colors. Assign distinct colors $c(x), c(y) \in [n]$ to the vertices x resp. y. $TL(n) - \{x, y\}$ is a proper subtournament of $T(n)$, thus it can be colored with n colors, among them are $c(x)$ and $c(y)$. Since x and y are a sink resp. a source in the digraphs induced by the vertices colored with $c(x)$ and $c(y)$ these digraphs are acyclic. Therefore $TL(n)$ can be colored with n colors.

Assume that there is an n-coloring c with $c(x) = c(y)$. Then, if we identified x and y (which would not create new directed cycles!), there would be a directed n-coloring of $T(n)$, which is a contradiction to $\chi(T(n)) = n+1$. This proves the second assertion. $\square$

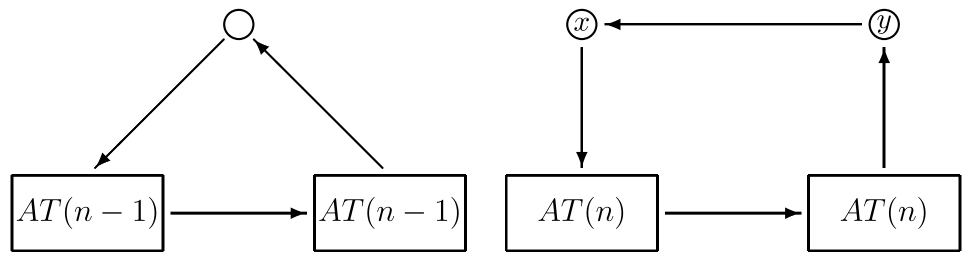


Figure A.3: The configurations $AT(n)$ (left-hand side) and $AL(n)$ (right-hand side)

Theorem 97. *SIMPLE DIRECTED 2-COLORING is $\mathcal{NP}$-complete.*

Proof. In the reduction of Theorem 93 we replace each edge ab by a tournament line $TL(2)$ identifying a with x and b with y (see Fig. A.2) and obtain a simple digraph G'.

By the proof of Theorem 93 the NAE3SAT-formula is satisfied if, and only if, the digraph G from Theorem 93 can be colored with 2 colors. By Lemma 96, G can be colored with 2 colors if, and only if, G' can be colored with 2 colors. □

Corollary 98. *The problem SIMPLE DIRECTED k-COLORING is $\mathcal{NP}$-complete for $k \geq 2$.*

Proof. In the reduction of Corollary 94 we replace each edge ab by a $TL(k)$-digraph, identifying x with a and y with b for $k \geq 3$. Then argue as in the preceding theorem using Lemma 96. □

We do not need the minimality of $T(n)$ in order to construct a tournament with a functionality similar to $TL(n)$. Alternatively, we might have replaced the edges by configurations $AL(n)$ which appear from $AT(n+1)$ when we split the special vertex of $AT(n+1)$ into a source x and a target vertex y and connecting the target to the source (cf. Figure A.3). Note that splitting a vertex different from the special vertex would not always work. It is easy to see that these configurations have the same property as $TL(n)$. (Indeed, since $\chi(AT(n)) = n$ we also have $\chi(AL(n)) = n$. Assume that in an n-coloring of $AL(n)$, x and y are colored with the same color. Then, if we identify x and y, we obtain an n-coloring of $AT(n+1)$, which contradicts $\chi(AT(n+1)) = n+1$. Therefore we can replace $TL(n)$ in Lemma 96 by $AL(n)$.)

While the complexity of noncompetitive digraph coloring has been solved, the complexity status of the most interesting competitive digraph coloring problem still remains open: given a digraph D and a number k, decide whether $\chi_g(D) \leq k$. Even the complexity of the graph relaxation of this problem is still open. However, Bodlaender [15] proved for three related problems that they are $\mathcal{P}$**SPACE**-complete if k is large enough. In Chapter 5 we saw that, if we restrict ourselves to graphs and $k = 2$, the problem is in $\mathcal{P}$, indeed we gave an explicit solution to this problem.

Appendix B

Game-tree search

In this chapter a computer program is presented. With this program, in principle, the task of determining the game chromatic number of any digraph D can be solved. However, there are natural bounds concerning the time and space complexity of this algorithm. Since the program is mainly a game-tree search, its running time is exponential in the size of the input. Thus only very small instances can be solved within reasonable time. For these instances, on modern computers, there is no problem with the space complexity, which depends linearly on the size of the input.

A *game-tree* is a tree with root r. In order to distinguish between vertices of the digraph D and vertices of the game-tree, we will call a vertex of the game-tree *node*, whereas a *vertex* denotes a vertex of the digraph. The distance of a node v to r is the *level* of v. In case Alice begins the game, the nodes of even level belong to Alice, the nodes of odd level to Bob. If w is the last inner node on the path from r to a node v, then w is the *parent* of v, and v is the *child* of w. The nodes of level $2k-2$, $2k-1$ represent the decision of the k-th move of a player. For each possibility a player has in node w, there is exactly one child v of w in the next level. A possibility means that a player chooses a vertex (of D) and a color for this vertex. If we consider a node v (in the game-tree), then we assume that the state of the game is given by the chosen vertices (of D) and the chosen colors on the path from r to v. Since the game is finite, this game-tree will have leaves, and each path beginning at r will end in a leaf. These leaves are labeled with 1 if Alice wins at this stage, or labeled with 0 if Bob wins there.

In order to calculate whether Alice or Bob has a winning strategy for the game we proceed recursively as follows. A node of even level (Alice's move) obtains the maximum of the labels of its children. On the other hand, a node of odd level (Bob's move) receives the minimum of the labels of its children. The *value of the game* is the label of the root r. If the value is 1, Alice has a winning strategy, otherwise, if the value is 0, Bob has a winning strategy. This is the so-called *minimax-principle*.

An algorithm that calculates the value of the game via the minimax-principle is called *complete game-tree search*. Usually it is not necessary to calculate the labels of every node of the game-tree in order to determine the label of the root. Often a lot of branches of the tree can be cut off and need not be considered. In this case we speak of a *(nearly complete) game-tree search*.

In Section B.1 we present a basic game-tree search algorithm. The running time of this program is examined in Section B.2. Some empirical results are given in Section B.3. In Section B.4 we obtain a faster program by introducing another cutting operation. The section is concluded with empirical results concerning the improved program.

B.1 The program

The program we describe tests whether for a given digraph D and a given number of colors Alice wins the coloring game. For simplicity, we have assumed that the coloring game is 0-relaxed, and that every player colors at most one vertex in a move. Three functions were developped: the first for a game where missing a turn is not allowed, the other two for games where Alice resp. Bob have the right to miss a turn.

These functions return 1 if Alice wins the game and 0 if Bob wins the game. They use a nearly complete game tree search with only two cutting off rules (in the first function marked by (3) and (3.1.2)). The program was implemented in C++. For the input and output `stdio.h` has to be included. The first function `gametree` calculates the game-tree recursively at a certain level.

```
int gametree
 (int ncol,  // number of colors
  int nvert, // number of vertices
  int move,  // number of the actual move
  int alice, // 1 if it is Alice's turn,
             // 0 if it is Bob's turn
  int*C,     // colors of vertices,
             // 0 for uncolored vertices
  int**N,    // adjacency list
  int*a)     // number of in-neighbors of vertices

{ int max,    // saves the actual maximum label
              // of the children in the game-tree
      min,    // saves the actual minimum label
              // of the children in the game-tree
```

```
    h,      // auxiliary variable
    i,      // vertex to be colored
    j,      // color to be used for i
    k,      // label of a child in the game-tree
    value,  // 1 if Alice wins in the current
            // level, 0 if Bob wins
    xbreak, // 1 if a vertex cannot be
            // colored feasibly any more
    test;   // auxiliary variable
int*CCC;    // new color array

// The new color array CCC is initialized
// by the old color array C
CCC=new int[nvert+1];
for(i=0;i<=nvert;i++)CCC[i]=C[i];

// max and min are initialized
max=0;
min=1;

// Alice wins
// if there is no uncolored vertex any more.
if(move>nvert)value=1;

// Otherwise explore the game-tree's next level
else
 { // (1) i is the vertex selected to be colored
   //     in this step.
   // It is initialized by the first vertex.
   i=1;

   // (2) xbreak is 1 if there is a vertex that
   //     cannot be colored feasibly any more
   // It is initialized by 0
   // since no vertex has been examined
   xbreak=0;

   // (3) This loop considers all vertices i.
   //     It stops if xbreak is 1 since
   // then we can cut off the game-tree
   // at the current level
   // as Bob will win at this level.
   while((i<=nvert)&&(xbreak==0))
```

```
// (*) This while-loop can be refined
//     see Section B.4

 { // (3.1) We only consider
   //       uncolored vertices
   if(C[i]==0)

    { // (3.1.1) We suspect that vertex i
      //         cannot be colored feasibly
      xbreak=1;

      // (3.1.2) We test every color j
      //         on vertex i.
      // We need not use a color
      // greater than the move number
      // since the use of these colors
      // does not give new value results
      for(j=1;(j<=ncol)&&(j<=move);j++)
      // (*) This for-loop can be refined
      //     see Section B.4

       { // (3.1.2.1) We assume that vertex i
         //           can be colored
         // with color j
         test=1;

         // (3.1.2.2) If i has an in-neighbor
         //           h colored with j, then
         // our assumption was wrong
         for(h=0;h<a[i];h++)
           if(j==C[N[i][h]])test=0;

         // (3.1.2.3) If our assumption was
         //           true, then we can color
         // i with j. We do that [2] and
         // calculate the label of the game-
         // tree at the next level [3].
         // Our suspicion was wrong [1].
         // We uncolor i [4] and we calculate
         // the actual min and max of the
         // game-tree labels [5]
         if(test==1)
          { xbreak=0; // [1]
```

```
                CCC[i]=j; // [2]

                // [3] (recursion)
                k=gametree
                   ( ncol, nvert, (move+1),
                     (1-alice), CCC, N, a );

                CCC[i]=0; // [4]

                // [5]
                if(k>max)max=k;
                if(k<min)min=k;
              }
           }
        }

        // (3.2) The next vertex is considered
        i++;
      }

     // (4) minimax-principle
     if(alice==1)value=max;else value=min;

     // (5) If the game tree was cut off,
     //     then the value is 0 (Bob wins)
     if(xbreak==1)value=0;
   }

  // Technical necessity in order to keep
  // memory space polynomial
  delete[] CCC;

  // Now 1 is returned if Alice wins at this level
  // otherwise 0
  return (value);
}
```

The other two functions are very similar to the first one. Thus we omit comments at similar passages. `gametreeAA` calculates the value of the game tree for the game where Alice is allowed to miss one or several turns, the function `gametreeBB` calculates the corresponding value for the game where this is true for Bob.

Note that the variable `move` has a different meaning in the functions `gametreeAA` and `gametreeBB` compared with the function `gametree` studied above. In general, `move` counts the number of colored vertices minus one. This number equals the number of a move only in a game where passing is not allowed and we count all moves of Alice and Bob from the beginning. However, in case missing a turn is allowed for a player, the number n_m of the move is in the range $\texttt{move} \le n_m \le 2 \cdot \texttt{move}$.

```
int gametreeAA
 (int ncol, int nvert, int move,
  int alice, int*C, int**N, int*a)
{ int max,min,i,j,k,value,xbreak,test;
  int*CCC;
  CCC=new int[nvert+1];
  for(i=0;i<=nvert;i++)CCC[i]=C[i];
  max=0;
  min=1;
  if(move>nvert)value=1;
  else
   { i=1;
     xbreak=0;

     // If it is Alice's move she has the right
     // to miss the turn. This possibility
     // is another child of the game tree.
     // It is examined here
     if(alice==1)
      { k=gametreeAA(ncol,nvert,move,0,CCC,N,a);
        if(k>max)max=k;
        if(k<min)min=k;
      }

     while((i<=nvert)&&(xbreak==0))
     // (*) This while-loop can be refined
     //     see Section B.4
      { if(C[i]==0)
         { xbreak=1;
           for(j=1;(j<=ncol)&&(j<=move);j++)
           // (*) This for-loop can be refined
           //      see Section B.4
            { test=1;
              for(k=0;k<a[i];k++)
```

```
                if(j==C[N[i][k]])test=0;
              if(test==1)
               { xbreak=0;
                 CCC[i]=j;
                 k=gametreeAA
                    ( ncol, nvert, (move+1),
                      (1-alice), CCC, N, a );
                 CCC[i]=0;
                 if(k>max)max=k;
                 if(k<min)min=k;
               }
            }
         }
        i++;
      }
     if(alice==1)value=max;else value=min;
     if(xbreak==1)value=0;
   }
  delete[] CCC;
  return (value);
}



int gametreeBB
 (int ncol, int nvert, int move,
  int alice, int*C, int**N, int*a)
{ int max,min,i,j,k,value,xbreak,test;
  int*CCC;
  CCC=new int[nvert+1];
  for(i=0;i<=nvert;i++)CCC[i]=C[i];
  max=0;min=1;
  if(move>nvert)value=1;
  else
   { i=1;
     xbreak=0;

     // If it is Bob's move he has the right
     // to miss the turn. This possibility
     // is another child of the game tree.
     // It is examined here
     if(alice==0)
      { k=gametreeBB(ncol,nvert,move,1,CCC,N,a);
```

```
        if(k>max)max=k;
        if(k<min)min=k;
      }

     while((i<=nvert)&&(xbreak==0))
     // (*) This while-loop can be refined
     //     see Section B.4
      { if(C[i]==0)
          { xbreak=1;
            for(j=1;(j<=ncol)&&(j<=move);j++)
            // (*) This for-loop can be refined
            //      see Section B.4
            { test=1;
               for(k=0;k<a[i];k++)
                 if(j==C[N[i][k]])test=0;
               if(test==1)
                { xbreak=0;
                  CCC[i]=j;
                  k=gametreeBB
                     ( ncol, nvert, (move+1),
                       (1-alice), CCC, N, a );
                  CCC[i]=0;
                  if(k>max)max=k;
                  if(k<min)min=k;
                }
             }
          }
        i++;
      }
     if(alice==1)value=max;else value=min;
     if(xbreak==1)value=0;
   }
  delete[] CCC;
  return (value);
}
```

We consider two functions to simplify the input of graphs (`graphinput`) and digraphs (`digraphinput`), respectively. In the latter function, for an edge vw of a graph, both arcs (v, w) and (w, v) have to be read. The array `N` saves the in-neighborhood of all vertices. The array `a` saves $d^+(v)$ for all vertices v.

```
int graphinput(int nedges,int**N,int*a)
{ int counter,knota,knotb;
  printf("\nInput of the edges:\n");
  for(counter=0;counter<nedges;counter++)
   { printf("Edge %i:\n",counter+1);
     scanf("%i%i",&knota,&knotb);
     N[knotb][a[knotb]]=knota;
     N[knota][a[knota]]=knotb;
     a[knotb]++;
     a[knota]++;
   }
  return 0;
}


int digraphinput(int narcs,int**N,int*a)
{ int counter,knota,knotb;
  printf("\nInput of the arcs:\n");
  for(counter=0;counter<narcs;counter++)
   { printf("Arc %i:\n",counter+1);
     scanf("%i%i",&knota,&knotb);
     N[knotb][a[knotb]]=knota;
     a[knotb]++;
   }
  return 0;
}
```

The main program only reads the data and calls the different game-tree functions. The version below considers the game g_A, where Alice has the first move and missing a turn is not allowed.

```
int main()
{ // (1) variables
  int maximumdegree, // maximum (in-)degree
                     // of input (di)graph D
      nvert,         // number of vertices of D
      narcs,         // number of arcs/edges of D
      ncol,          // number of colors
      ii,jj,         // counter variables
      isgraph;       // indicates whether we con-
                     // sider graphs or digraphs
  int result;        // variable for the result
```

```
                   // (win of Alice or Bob)
int*a;     // a[i] is number of (in-)neighbors
           // of vertex i
int*Color; // Color[i] actual color of vertex i
int**N;    // adjacency list: N[i][j] j-th
           // (in-)neighbor vertex of vertex i

// (2) Input of general parameters
printf("\nInput graph (1) or digraph (0)?");
scanf("%i",&isgraph);
if(isgraph==1)
 printf("\nInput graph\nMaximum degree: ");
 else
 printf("\nInput digraph\nMaximum in-degree: ");
scanf("%i",&maximumdegree);
printf("\nNumber of vertices: ");
scanf("%i",&nvert);
if(isgraph==1)printf("\nNumber of edges: ");
 else printf("\nNumber of arcs: ");
scanf("%i",&narcs);
printf("\nNumber of colors: ");
scanf("%i",&ncol);

// (3) Initialization of the three arrays
a=new int[nvert+1];
Color=new int[nvert+1];
N=new int*[nvert+1];
for(ii=0;ii<=nvert;ii++)
  N[ii]=new int[maximumdegree];
for(ii=0;ii<=nvert;ii++)Color[ii]=0;
for(ii=0;ii<=nvert;ii++)a[ii]=0;
for(ii=0;ii<=nvert;ii++)
  for(jj=0;jj<maximumdegree;jj++)N[ii][jj]=0;

// (4) Input of the edges or arcs
if(isgraph==1)graphinput(narcs,N,a);
 else digraphinput(narcs,N,a);

// (5) Calculation
result=gametree(ncol,nvert,1,1,Color,N,a);

// (6) Output of results
printf("\n\nwith %i Colors: %3i\n",ncol,result);
```

```
  return 0;
}
```

Sometimes we use a modified main program where the command `(5)` is replaced by another command. In case we consider the game A, the command is

```
  // (5) Calculation
  result=gametreeAA(ncol,nvert,1,1,Color,N,a);
```

In case we consider the game g_B, the command is

```
  // (5) Calculation
  result=gametree(ncol,nvert,1,0,Color,N,a);
```

For the game B we have to use the function `gametreeBB` instead.

B.2 Running time of the program

Not surprisingly, the CPU-time of the program increases rather rapidly when the number of vertices or colors is increased.

Proposition 99. *For one call of the function*

`gametree(ncol,nvert,1,1,Color,N,a)`

we need (in the worst case) $O(n!c!c^{n-c})$ time where $n =$ `nvert` and $c =$ `ncol`. Moreover, the worst case occurs, if the digraph we consider is I_n, the graph with n isolated vertices.

Proof. In the naive approach of a game-tree search, the game-tree has at most $n!c^n$ leaves since in the first move we can choose one of the n vertices and assign one of the c colors to it, in the second move we can choose one of the remaining $n-1$ vertices and assign again one of the c colors to it, in general, in the k-th move, we can choose between $n-k+1$ vertices and are free to assign any of the c colors to it.

However, we can do better, since in our program, in step `(3.1.2)`, we assume that in the k-th move a player uses only colors from the set $\{1, 2, \ldots, k\}$. This assumption has no influence on the outcome of the game (all new colors are equivalent with the k-th color), but cuts off a lot of unnecessary branches of the tree. So, our game-tree has at most

$$n \cdot 1 \cdot (n-1) \cdot 2 \cdots (n-c+1) \cdot c \cdot (n-c) \cdot c \cdots 2 \cdot c \cdot 1 \cdot c$$

leaves, i.e. $O(n!c!c^{n-c})$ leaves in case $n \geq c$ (which we can assume). Since the total number of elements of the game-tree is dominated by the number of its leaves, and running time is directly proportional to number of elements, we obtain the first assertion.

Furthermore, if the digraph we consider is I_n, and $c \geq 1$, there will be no cutting of branches of the game-tree. Note that cutting only occurs if at a certain point Bob wins, but in I_n Alice will win for every leaf, and so for every element of the game-tree. Therefore the worst case really occurs. □

Now we will analyze the running time of the more complex function where Alice is allowed to miss one or several turns. For this purpose let ϕ be the *golden ratio*:

$$\phi = \frac{\sqrt{5}+1}{2}$$

Proposition 100. *For one call of the function*

`gametreeAA(ncol,nvert,1,1,Color,N,a)`

we need (in the worst case) $O(\phi^n n! c! c^{n-c})$ time where $n =$ `nvert` and $c =$ `ncol`. Moreover, the worst case occurs for the graph I_n.

Proof. If we do not count the moves in which Alice misses her turn, then in the k-th move a player chooses between $n-k+1$ vertices and between $\min\{k, c\}$ colors. Furthermore (maybe at the beginning of the game) Alice chooses a move sequence $X \in \{A, B\}^n$, where A resp. B in the k-th place means that Alice resp. Bob colors in move k. Since the rules of the game do not permit Alice to have two moves without Bob moving inbetween, no subsequence AA is allowed in X. There are no further restrictions on X. We want to count all possible move sequences. Let $R_A(n)$ be the number of all possible move sequences of length n starting with A, and $R_B(n)$ be the number of all possible move sequences of length n starting with B. Then

$$\begin{aligned}
R_A(1) &= 1 \\
R_B(1) &= 1 \\
R_A(n+1) &= R_B(n) \text{ for } n \geq 1 \\
R_B(n+1) &= R_A(n) + R_B(n) \text{ for } n \geq 1
\end{aligned}$$

This leads to the recursive formula

$$R_B(n+1) = R_B(n-1) + R_B(n) \text{ for } n \geq 2.$$

$\#V$	$\#C$	G	time(s)	G	time(s)	G	time(s)	G	time(s)
6	2	I_6	0.008	M_6	0.000	P_6	0.000	C_6	0.004
7	2	I_7	0.112	M_7	0.028	P_7	0.008	C_7	0.008
8	2	I_8	1.840	M_8	0.212	P_8	0.048	C_8	0.036
9	2	I_9	33.826	M_9	3.576	P_9	0.420	C_9	0.176
10	2	I_{10}	681.239	M_{10}	39.774	P_{10}	4.100	C_{10}	3.128
11	2	I_{11}	>1000	M_{11}	856.029	P_{11}	48.267	C_{11}	18.233

Table B.1: Running time versus density: The table depicts the running time of the check whether Alice has a winning strategy with 2 colors in the game g_A played on instance G. Here the instances in a row are of increasing density. Therefore the running time decreases in most cases.

So $R_B(n) = F(n)$, $R_A(n) = F(n-1)$, where $F(n)$ denotes the n-th *Fibonacci number*, and the number $R(n)$ of possible move sequences of length n is $R(n) = R_A(n) + R_B(n) = F(n+1)$. It is well-known that the number $F(n+1)$ tends asymptotically to the value $\frac{1}{\sqrt{5}}\phi^{n+1}$. As in the proof of the preceding proposition we can multiply the numbers of choices (which are independent) and obtain that the game-tree has at most $O(\phi^n n! c! c^{n-c})$ leaves (which dominate the number of elements of the game-tree). $\square$

B.3 Computational results

All tests of the program were run on `fireball.mi.uni-koeln.de`, a Siemens Scenic W600 with a 3 GHz Pentium 4 processor and 512 kB cache and 2 GB RAM, run by Debian Linux.

The running time decreases significantly with increasing density of a digraph (if the number of vertices is fixed). This phenomenon is illustrated in the Tables B.1, B.2 and B.3. Table B.1 and B.2 consider the case that a game is played with 2 colors. Table B.1 summarizes running time data for the game g_A, Bodlaender's original variant, and in Table B.2 the game A, where Alice is allowed to miss a turn, is examined. Table B.3 considers the game g_A played with 3 colors. For each instance, the running time of the program is given. The running time is measured in seconds, the smallest displayed unit is one millisecond.

As instances we study the graph I_n for some n, which consists of n isolated vertices, and the graph M_n for some n, which has n vertices and a (nearly perfect) matching of $\lfloor \frac{n}{2} \rfloor$ edges. On these instances, Alice obviously wins any game g with 2 colors. We further study undirected paths P_n and undirected cycles C_n for some $n \geq 5$, on which Bob wins any game g with 2 colors, with

$\#V$	$\#C$	G	time(s)	G	time(s)	G	time(s)	G	time(s)
5	2	I_5	0.012	M_5	0.004	P_5	0.000	C_5	0.004
6	2	I_6	0.156	M_6	0.036	P_6	0.012	C_6	0.008
7	2	I_7	3.692	M_7	0.672	P_7	0.156	C_7	0.052
8	2	I_8	99.570	M_8	9.705	P_8	1.860	C_8	1.396
9	2	I_9	—	M_9	—	P_9	27.622	C_9	9.177

Table B.2: Running time versus density: The table depicts the running time of the check whether Alice has a winning strategy with 2 colors in the game A played on instance G.

$\#V$	$\#C$	D	time(s)	D	time(s)	D	time(s)
6	3	C_6	0.008	C_6^2	0.000	$C_6^{2,3}$	0.000
7	3	C_7	0.068	C_7^2	0.020	$C_7^{2,3}$	0.000
8	3	C_8	1.184	C_8^2	0.200	$C_8^{2,3}$	0.028
9	3	C_9	22.401	C_9^2	2.760	$C_9^{2,3}$	0.224
10	3	C_{10}	489.439	C_{10}^2	38.390	$C_{10}^{2,3}$	2.428

Table B.3: Running time versus density: The table depicts the running time of the check whether Alice has a winning strategy with 3 colors in the game g_A played on instance D.

one exceptional case: Alice wins on C_6 if the rules of the game allow her to force Bob to color first, for example in the game A. In spite of the fact that undirected cycles C_n are very sparse graphs and are only a little denser than I_n, the decrease of the running time for C_n compared with I_n is enormous (see Tables B.1 and B.2).

In Table B.3 we examine digraphs of increasing density again. Here we start with undirected cycles C_n and proceed with digraphs C_n^2 and $C_n^{2,3}$, which are defined as follows:

In C_n^2 there are n vertices $1, 2, \ldots, n$, and arcs (i, j) if

$$j \equiv i - 1 \mod n \quad \text{or} \quad j \equiv i + 1 \mod n \quad \text{or} \quad j \equiv i + 2 \mod n.$$

In $C_n^{2,3}$ there are the same vertices and arcs as in C_2^n, and additional arcs (i, j) if $j \equiv i + 3 \mod n$.

As Table B.4 suggests, the program is indeed useful to calculate the g_A-game chromatic number of digraphs with at most 7 vertices and to calculate the A-game chromatic number of digraphs with at most 6 vertices. Depending on the density of the digraph, the number of colors needed, and on the game,

G	$\#V$	time(s) game g_A	time(s) game A
I_1	1	0.008	0.000
I_2	2	0.000	0.004
I_3	3	0.000	0.000
I_4	4	0.004	0.000
I_5	5	0.004	0.036
I_6	6	0.100	2.120
I_7	7	5.184	172.935
I_8	8	>200	—

Table B.4: Running time in a sparse case with the maximal number of colors

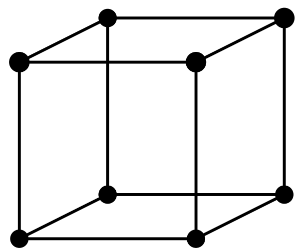


Figure B.1: The cube

the program can be useful for digraphs with at most 11 vertices, see the Tables B.1 and B.5. Exploiting symmetries in the input digraphs the program could easily be refined to deal with digraphs of more than 11 vertices (see Section B.4).

The program was applied to various instances. We want to mention 3 interesting results. The first one concerns the graph *Cube*, see Fig. B.1. From

G	$\#V$	time(s) game g_A	time(s) game A
K_7	6	0.008	0.152
K_8	7	0.060	2.492
K_9	8	0.636	42.719
K_{10}	9	7.712	—
K_{11}	10	100.346	—

Table B.5: Running time in the densest case with the maximal number of colors

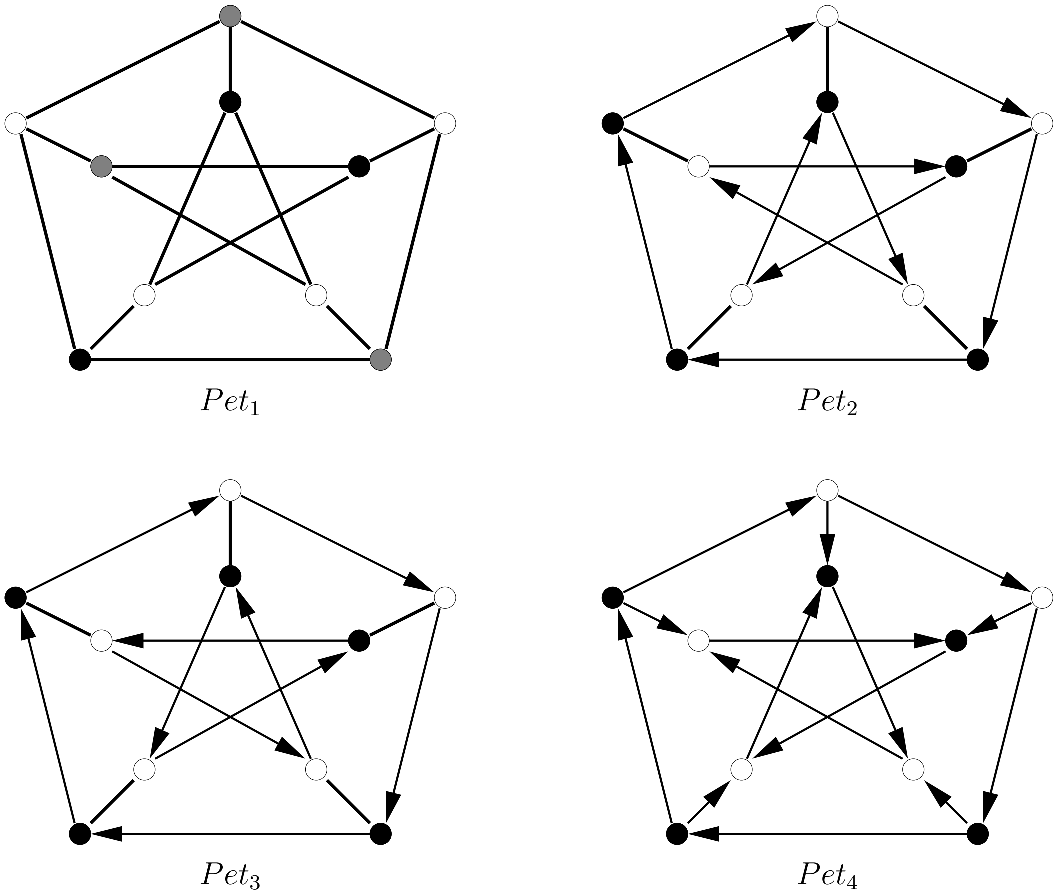


Figure B.2: Semiorientations of the Petersen graph

the results of Table B.6 one can deduce that the g_A-game chromatic number of the *Cube* is 4, whereas the A-game chromatic number of the *Cube* is 2. Once more, this illustrates the fact that the game chromatic number can jump by more than 1 between two ‘adjacent’ variants of the considered game. The second application of the program was to prove that the g_A-game chromatic number of the incidence graphs P_k^I of small paths P_k, $k \leq 6$, is at most 4, see Table B.6. This result was already mentioned in Chapter 4. Finally, semiorientations of the Petersen graph were examined. The Petersen graph Pet_1, which has (di)chromatic number 3, and three semiorientations, which have dichromatic number 2, are depicted in Fig. B.2. Not surprisingly, the game chromatic numbers of the three latter semiorientations have game chromatic number 3, which is bigger than their dichromatic number, for any variant of the game, see Table B.6. One should expect the same phenomenon for the undirected Petersen graph, i.e. its game chromatic numbers should also exceed its chromatic number by 1. However, this is not the case, as can be deduced from Table B.6. The g_A-game chromatic number of Pet_1 is 4, but the g_B-game

Digraph	Game	$\#V$	$\#C$	A wins	B wins	time(s)
$Cube$	g_A	8	2		$\times$	0.032
$Cube$	g_A	8	3		$\times$	0.584
$Cube$	A	8	2	$\times$		1.508
P_2^I	g_A	2	4	$\times$		0.000
P_3^I	g_A	4	4	$\times$		0.000
P_4^I	g_A	6	4	$\times$		0.004
P_5^I	g_A	8	4	$\times$		1.676
P_6^I	g_A	10	4	$\times$		695.667
Pet_1	g_A	10	3		$\times$	81.269
Pet_1	A	10	3	$\times$		10098.735
Pet_1	g_B	10	3	$\times$		81.841
Pet_2	g_A	10	2		$\times$	2.280
Pet_2	A	10	2		$\times$	222.558
Pet_3	g_A	10	2		$\times$	2.304
Pet_3	A	10	2		$\times$	230.046
Pet_4	g_A	10	2		$\times$	14.861
Pet_4	A	10	2		$\times$	1786.976

Table B.6: Computer results and CPU time for various instances

chromatic number of Pet_1 is 3, equal to its chromatic number. Again the Petersen graph has been proven as a counterexample to the mathematician's expectations.

B.4 Accelerating the algorithm

The program described so far can be improved by a further cutting operation on the game-tree. It is done by replacing the code (number `(3)` in the listing of the function `gametree`)

```
while((i<=nvert)&&(kaputt==0))
// (*) This while-loop can be refined
//     see Section B.3
```

in the functions `gametree`, `gametreeAA` and `gametreeBB` by the following code

```
while (    (i<=nvert)
        && (kaputt==0)
        && (    ( (alice==1) && (max<1) )
             || ( (alice==0) && (min>0) )
           )
      )
```

further by replacing the code (number `(3.1.2)` in the listing of the function `gametree`)

```
for(j=1;(j<=ncol)&&(j<=move);j++)
// (*) This for-loop can be refined
//      see Section B.3
```

in the functions `gametree`, `gametreeAA` and `gametreeBB` by the code

```
for ( j=1;
              (j<=ncol)
           && (j<=move)
           && (    ( (alice==1) && (max<1) )
                || ( (alice==0) && (min>0) )
              );
      j++ )
```

The inserted condition has the following meaning. If it is Alice's move (`alice==1`) and `max` $= 1$, then the program has observed that Alice has a winning strategy for one of the branches of the game-tree rooted in a child of the actual node of the game-tree. Therefore, by the minimax-principle, Alice has a winning strategy for the actual node. So it is unnecessary to consider other children of the actual node in this case. The same argument holds for the case that it is Bob's move (`alice==0`) and `min` $= 0$. In this case the program has observed that Bob has a winning strategy for one of the branches of the game-tree rooted in a child of the actual node. Thus Bob has a winning strategy for the actual node, and it is not necessary to consider the other children of the actual node in this case. Hence we can terminate the `while`-loop and the `for`-loop if one of the cases discussed above occurs, which is guaranteed by the inserted condition.

These modifications of the algorithm have a significant effect on the running time. While the old program needs 681.239 seconds to confirm that Alice has a winning strategy on the instance I_{10} in the game g_A with 2 colors (see

$\#V$	$\#C$	G	time(s)	G	time(s)	G	time(s)	G	time(s)
9	2	I_9	0.004	M_9	0.004	P_9	0.012	C_9	0.000
10	2	I_{10}	0.024	M_{10}	0.008	P_{10}	0.008	C_{10}	0.004
11	2	I_{11}	0.076	M_{11}	0.044	P_{11}	0.040	C_{11}	0.008
12	2	I_{12}	0.352	M_{12}	0.140	P_{12}	0.124	C_{12}	0.100
13	2	I_{13}	1.988	M_{13}	0.860	P_{13}	0.744	C_{13}	0.128
14	2	I_{14}	10.289	M_{14}	3.124	P_{14}	2.504	C_{14}	1.700
15	2	I_{15}	59.668	M_{15}	20.317	P_{15}	14.521	C_{15}	1.968
16	2	I_{16}	320.496	M_{16}	79.109	P_{16}	49.343	C_{16}	33.694

Table B.7: Running time versus density: The table depicts the running time of the check whether Alice has a winning strategy with 2 colors in the game g_A played on instance G. Here the improved algorithm is used.

$\#V$	$\#C$	G	time(s)	G	time(s)	G	time(s)	G	time(s)
6	2	I_6	0.012	M_6	0.008	P_6	0.004	C_6	0.000
7	2	I_7	0.156	M_7	0.036	P_7	0.000	C_7	0.000
8	2	I_8	2.552	M_8	0.340	P_8	0.012	C_8	0.012
9	2	I_9	54.963	M_9	5.820	P_9	0.048	C_9	0.008
10	2	I_{10}	964.352	M_{10}	64.044	P_{10}	0.156	C_{10}	0.172
11	2	I_{11}	—	M_{11}	1375.046	P_{11}	1.128	C_{11}	0.084

Table B.8: Running time versus density: The table depicts the running time of the check whether Alice has a winning strategy with 2 colors in the game A played on instance G. The improved algorithm is used.

Table B.1), the new program needs only 0.024 seconds for the same test (see Table B.7). For the games in which passing is allowed the improvement is less remarkable. However, even there is an improvement. E.g., the old program needs 99.570 seconds to confirm that Alice has a winning strategy on the instance I_8 in the game A with 2 colors (see Table B.2), the new program only needs 2.552 seconds (see Table B.8). In some cases the comparatively bad performance of the function `gametreeAA` might be improved by reordering the two main parts of the function. Instead of first checking the possibility that Alice misses her turn one might check this last.

Tables B.7 resp. B.8 contain the same type of instances as Tables B.1 resp. B.2 for the accelerated algorithm instead of the basic algorithm. In Tables B.7 and B.8 one can see that for a fixed number of colors and increasing density of the graph under consideration the CPU time decreases. However, there are exceptions from a strict decrease. E.g., while the general tendency in Table B.8 from left to right shows a decrease, between P_{10} and C_{10} the

G	$\#V$	time(s) game g_A	time(s) game A
I_4	4	—	0.000
I_5	5	—	0.008
I_6	6	—	0.156
I_7	7	0.000	6.500
I_8	8	0.008	408.978
I_9	9	0.072	—
I_{10}	10	0.904	—
I_{11}	11	7.284	—
I_{12}	12	125.660	—

Table B.9: Running time in a sparse case with the maximal number of colors (improved algorithm).

running time increases.

In Table B.7 the decrease between M_i and P_i is very small compared to the decrease in Table B.1. The explanation is that paths are structurally complicated (as far as the coloring game with 2 colors is concerned), whereas matchings are easy. The accelerated algorithm seems to be fast on easy instances and slow on instances where it is more complicated to determine the exact game chromatic number.

The same phenomenon can be observed if one compares the running time of P_i and C_i in Table B.7. For odd i there is a decrease of the running time by a large factor, for even i there is hardly any decrease. The reason is that C_i is not bipartite for odd i, so the program does not have to check whether $\chi_{g_A}(C_i) > 2$, only whether $\chi(C_i) > 2$, which is a less complex task: for the instance C_i with odd i, the label of every vertex of the game-tree which belongs to Bob is 0, therefore each of Bob's vertices will have at most one child, thus a branching of the game-tree will only occur in Alice's vertices.

Table B.9 illustrates that, with the improved program, it is possible to determine the g_A-game chromatic number of digraphs with up to 12 vertices and the A-game chromatic number of digraphs with up to 8 vertices. Since usually the maximum number of colors is not needed, digraphs with even a few more vertices can be examined. Table B.11 compared with Tables B.10 and B.9 suggests that the most difficult task for the program is to determine the game chromatic number of graphs which are *semi-dense*, i.e. which have approximately half of the number of edges as the complete graph with the same number of vertices.

Tables B.10 and B.11 illustrate that, if the same digraph D is under consideration and we increase the number of colors, the running time of the program

$\#C$	time(s) K_{11}	A wins	time(s) K_{12}	A wins	time(s) K_{13}	A wins	time(s) K_{14}	A wins
6	0.000		0.004		0.004		0.004	
7	0.004		0.012		0.016		0.024	
8	0.012		0.020		0.024		0.040	
9	0.028		0.056		0.104		0.188	
10	0.036		0.100		0.164		0.292	
11	0.016	×	0.164		0.428		0.968	
12	0.016	×	0.048	×	0.620		1.448	
13	0.016	×	0.044	×	0.196	×	2.892	
14	0.012	×	0.044	×	0.200	×	0.732	×

Table B.10: Running time of the check whether Alice wins the game g_A with $\#C$ colors on the complete graphs K_{11}, K_{12}, K_{13}, and K_{14} (improved algorithm).

is not strictly increasing. Moreover, there is a local maximum of running time (often followed by a local minimum) if the number of colors equals the game chromatic number $\chi_{g_A}(D)$ or to $\chi_{g_A}(D) - 1$. When we prove results concerning the game chromatic number often most of the difficulty also lies in the threshold between $\chi_{g_A}(D) - 1$ and $\chi_{g_A}(D)$. Further a local maximum can be explained since a winning strategy of Bob with $\chi_{g_A}(D) - 1$ colors usually has to check a lot of winning branches of Alice in every game-tree node of Bob which leads to a higher running time since sometimes cut operations can be executed only very late.

$\#C$	time(s) $G_{11,27}$	A wins	time(s) $G_{12,33}$	A wins	time(s) $G_{13,39}$	A wins	time(s) $G_{14,45}$	A wins
2	0.000		0.000		0.000		0.000	
3	0.016		0.020		0.024		0.024	
4	0.544		0.412		0.904		1.944	
5	0.436	×	23.813	×	388.940		1104.209	
6	0.480	×	2.840	×	18.453	×	110.171	×
7	0.744	×	4.156	×	29.430	×	200.977	×

Table B.11: Running time of the check with the improved algorithm whether Alice wins the game g_A with $\#C$ colors on certain semi-dense graphs $G_{n,m}$ with n vertices and m edges

$\#C$	time(s)	A wins	B wins
1	0.000		$\times$
2	0.004		$\times$
3	0.120		$\times$
4	398.965		$\times$
5	681.395	$\times$	

Table B.12: Results and running time of the check whether Alice has a winning strategy for the game g_A played on the line graph $L(Pet_1)$ of the Petersen graph with $\#C$ colors. (Improved algorithm with precoloring)

Line graph of the Petersen graph. We conclude this chapter with a remarkable result which was enabled by accelerating the program. Let E' be the edge set of the undirected Petersen graph Pet_1. For the structure of Pet_1 we refer to Fig. B.2. The *line graph* $L(Pet_1)$ of the Petersen graph is defined to be the graph with vertex set E', and $e_1, e_2 \in E'$ are connected by an edge if, and only if, e_1 is adjacent to e_2 in Pet_1. In order to calculate the g_A- and g_B-game chromatic number of $L(Pet_1)$, we use a modification of the program, where the first vertex is precolored with the first color. This does not change the value of the game because of the symmetry of the Petersen graph. (Mapping any 5-cycle to some other 5-cycle of the Petersen graph induces an automorphism of the Petersen graph. Therefore all edges are 'equivalent'.)

In the program the precoloring is done, e.g. for the game g_A, by replacing the command `(5)` of the main program by

```
// (5) Calculation
Color[1]=1;
result=gametree(ncol,nvert,2,0,Color,N,a);
```

Here, in the call of the game-tree search function, the variable `move` is set to 2, which means that the first move is skipped, and the variable `alice` is set to 0, which means that it is Bob's turn. The fixed first move of Alice is given by setting the first entry of the `Color`-array to the (fixed) value 1.

With this modification of the program the results of Table B.12 were obtained. With an anologue modification for the game g_B we obtain the results of Table B.13. On the basis of these computational results we conclude with the following theorems:

Theorem 101. $\chi_{g_A}(L(Pet_1)) = 5.$

Theorem 102. $\chi_{g_B}(L(Pet_1)) = 5.$

$\#C$	time(s)	A wins	B wins
1	0.004		$\times$
2	0.000		$\times$
3	0.480		$\times$
4	1703.014		$\times$
5	160.522	$\times$	

Table B.13: Results and running time of the check whether Alice has a winning strategy for the game g_B played on the line graph $L(Pet_1)$ of the Petersen graph with $\#C$ colors. (Improved algorithm with precoloring)

Theorem 101 answers a long-time open question. Since the articles by Cai and Zhu [24], Erdös et al. [37] and the author [3] it has been conjectured that the game chromatic index of a graph with maximum degree Δ might be bounded above by $\Delta + 1$, which would be a competitive analogon of Vizing's Theorem (see Berge [13]). Theorem 101 disproves this conjecture.

Appendix C

Minor open questions

- Chapter 1

Problem C.1. *Is there a generalization of* $col_g(D)$ *to some* d*-relaxed parameter* $\mathrm{col}_g^d(D)$ *with* $\chi_g^d(D) \leq \mathrm{col}_g^d(D)$*?*

- Chapter 2

Problem C.2. *For which* $a \geq b \geq 1$, $d \geq 0$ *is* ${}^{(a,b)}\chi_g^d(\mathcal{F}) = \left\lfloor \frac{b}{d+1} \right\rfloor + 2$ *resp.* ${}^{(a,b)}\chi_g^d(\mathcal{F}) = \left\lfloor \frac{b}{d+1} \right\rfloor + 3$*?*

Problem C.2 has been solved completely for $d = 0$ by Kierstead [53], and for $b = 1$ by the results of Kierstead [53], He et al. [49], and Theorem 20.

Problem C.3. *Can the results concerning 3-cycled forests be generalized if the 3-cycles are replaced by* n*-cycles for* $n \geq 4$*?*

Problem C.4. *For* $a, b, d \geq 0$*, determine the* d*-relaxed* (a, b)*-game chromatic number of the class of 3-cycled forests.*

- Chapter 2/3

Problem C.5. *Is there an upper bound for the game chromatic number of orientations of outerplanar graphs which is smaller than 6?*

Problem C.6. *Is there an upper bound for the game chromatic number of orientations of planar graphs which is smaller than 17?*

- Chapter 3

Problem C.7. *Give a good lower bound for* $M(S)$ *for any surface* S*.*

Problem C.8. *Find a good upper bound for* ${}^{(a,b)}\mathrm{col}_g(D)$ *for acyclic digraphs* D *which are embeddable in a fixed surface, have given girth, but no 4-cycles.*

Problem C.9. *Characterize the class of digraphs D with $L^-(D) = L^+(D)$.*

Problem C.10. *For fixed $n_1, n_2 \geq 0$, characterize the class of digraphs D with residue parameters $R_1(D) = n_1$ and $R_2(D) = n_2$.*

In Proposition 24 the case $R_1(D) = R_2(D) = 0$ was solved.

Problem C.11. *Discuss the tightness of the bounds of Theorems 29 and 31 in the case of surfaces of negative Euler characteristic.*

Problem C.12. *Discuss the tightness of the bounds of Theorems 30 and 32.*

- Chapter 4

Problem C.13. *Discuss the tightness of the upper bounds for the incidence game chromatic number given in Chapter 4. In particular, what is a lower bound for the class of forests?*

In (4.2) the upper bound $3\Delta - 1$ for the incidence game chromatic number of a graph of maximum degree Δ was given.

Problem C.14. *For fixed $\Delta > 2$, is the upper bound (4.2) tight?*

For $\Delta = 2$ the tightness has been proven by Theorem 48.

Problem C.15. *Is $\iota_g(G)$ a monotone parameter?*

Problem C.16. *Is $\iota_g(P_k) \leq 4$ for all paths P_k of length $k-1$ and for a certain version g of the game?*

Problem C.17. *How must the incidence game chromatic number be generalized to digraphs in such a way that we obtain 'interesting' results?*

- Chapter 5

Problem C.18. *In which way must broken wheels be generalized, so that every B-perfect graph is a graph all components of which are some generalized broken wheels?*

Problem C.19. *Characterize the B-perfect graphs with clique number 4.*

Problem C.20. *Characterize the A-perfect digraphs with clique number 2.*

Problem C.21. *For which k and g is the number of isomorphism classes of connected g-perfect (di)graphs with maximum (in-)degree k finite?*

Appendix D

Major open questions

Problem D.1. *For any $a, b, d \geq 0$, determine the d-relaxed (a, b)-game chromatic number of the classes of planar graphs and of graphs embeddable in some other surface and of the classes of their orientations.*

For the class $\mathcal{P}$ of planar graphs, we have ${}^{(1,0)}\chi_g^0(\mathcal{P}) = 4$ by the Four-Color-Theorem [10, 11].

Problem D.2. *For any surface S, determine the exact value of $M(S)$.*

$M(S)$ is only known for the surfaces S_0, S_1, S_2, N_1, and N_2. [60, 43, 12, 67]

Problem D.3. *Is the Strong Perfect Digraph Conjecture true?*

Problem D.4. *Find Strong Perfect Graph Theorems for A- and B-perfectness.*

Problem D.5. *Find Strong Perfect Digraph Theorems for A- and B-perfectness.*

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