Structure Analysis of Some Generalizations of Matchings and Matroids Under Algorithmic Aspects

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Vorwort (oder Kurzzusammenfassung)

Es ist nicht so einfach auf die Frage, was ich denn eigentlich in meiner Dissertation so gemacht hätte, eine kurze und allgemeinverständliche Antwort zu geben. "Ich habe einige diskrete Optimierungsprobleme durch effiziente Algorithmen charakterisiert" antworte ich häufig und erreiche, dass die meisten dann schon gar nicht mehr weiter nachfragen. Der ein oder andere möchte aber doch wissen, was diskrete Optimierungsprobleme eigentlich sind. Ich habe die Erfahrung gemacht, dass diese Frage am besten anhand eines Beispiels zu beantworten ist: Ein typisches diskretes Optimierungsproblem ist die Bestimmung eines kürzesten Weges von Köln nach Rom, oder zwischen zwei beliebigen anderen Standorten. Ein solches Kürzeste-Wege-Problem läßt sich abstrakt durch einen Graphen modellieren, der, ähnlich wie eine Straßenkarte, aus Knotenpunkten besteht, die durch Linien (Kanten) verbunden sind. Jeder Kante wird die Länge des Weges zwischen ihren beiden Endknoten zugewiesen. Das Problem lautet nun, eine kürzeste Verbindung zwischen zwei vorgegeben Knotenpunkten zu bestimmen.

Ganz allgemein läßt sich ein diskretes Optimierungsproblem als die Aufgabe modellieren, aus einer Menge zulässiger Lösungen, die oft als Teilmengen einer endlichen Menge beschrieben werden, eine optimale Lösung zu bestimmen. Wie im obigen Beispiel, bei dem die Lösungsmenge aus allen möglichen Wegen zwischen Köln und Rom besteht, ist es in den meisten Fällen bei weitem zu aufwendig, alle zulässen Lösungen miteinander zu vergleichen. Von daher wird versucht, je nach Struktur des Problems, effiziente Algorithmen zu entwickeln und diejenigen Problemklassen zu charakterisieren, für die mit diesen Algorithmen optimale Lösungen gefunden werden können.

Der bereits in den 50er Jahren entwickelte Ford-Bellman Algorithmus bestimmt zum Beispiel entweder alle kürzesten Wege von einem ausgezeichneten Knoten zu allen übrigen Knoten im Graphen, oder aber er entdeckt einen Kreis negativer Länge, der beweist, dass das Kürzeste-Wege Problem in dem betrachteten Graphen nicht lösbar ist. Das heißt, diejenigen Graphen, bei denen das Kürzeste-Wege Problem lösbar ist, sind zum einen durch die Optimalität des Ford-Bellman Algorithmus charakterisiert, und zum anderen durch den Ausschluß von Kreisen negativer Länge.

In meiner Dissertation konnte ich drei neue Klassen diskreter Optimierungsprobleme identifizieren, die mit sehr einfachen und schnellen Algorithmen gelöst werden können. Diese drei Klassen enthalten bekannte und bereits wohl untersuchte Probleme, wie z.B. die Optimierung über Matroiden, Gauss Greedoiden, Δ-Matroiden und Jump Systemen.

Neben Matroiden und deren Verallgemeinerungen habe ich mich auch mit Strukturen, die etwas mit sogenannten „Matchings" zu tun haben, auseinandergesetzt: Ein Matching in einem Graphen ist eine Menge von Kanten, in der keine zwei Kanten einen gemeinsamen Endknoten haben. Eng verwandt mit dem Matchingproblem ist das Überdeckungsproblem, bei dem eine möglichst kleine Teilmenge von Knoten gesucht wird, die von jeder Kante des Graphens mindestens einen Endknoten enthält. Bereits in den 60er Jahren hat Edmonds gezeigt, dass ein Matching maximaler Größe in jedem Graphen effizient bestimmt werden kann. Das Überdeckungsproblem ist allerdings sehr viel schwerer. Es ist sogar bekannt, dass es gar keinen effizienten Algorithmus geben kann, der in jedem Graphen eine optimale Überdeckung bestimmen könnte[1]. Beschränkt man sich aber auf die Klasse der König-Egerváry Graphen, die dadurch definiert sind, dass sie ein Matching und eine Überdeckung gleicher Größe enthalten, so wird auch dieses Problem handhabbar:

In meiner Dissertation zeige ich, dass König-Egerváry Graphen durch einen schnellen Algorithmus charakterisiert werden können, der ausgehend von einem maximalen Matching entweder eine minimale Überdeckung bestimmt, oder einen Untergraphen identifiziert, der beweist, dass der Graph kein König-Egerváry Graph sein kann. Diese Charakterisierung der König-Egerváry Graphen durch Ausschluß von Untergraphen, die übrigens die Gestalt einer Blume haben, ergab sich als Spezialfall der Charakterisierung einer allgemeineren Klassen von Graphen, die ich „Rot/Blau-Split Graphen“ gennant habe. Rot/Blau-Split Graphen modellieren lösbare 2–SAT Instanzen und verallgemeinern neben König-Egerváry Graphen auch klassische Split Graphen.

Wer jetzt immer noch genauer wissen möchte, was ich denn eigentlich in meiner Dissertation so gemacht habe, der muß wohl auch noch die nächsten 145 Seiten lesen...

Danksagung

Ich erinnere mich, dass Ulrich Faigle, der Betreuer dieser Dissertation, lauthals lachte, als ich auf seine Frage, ob ich als wissenschaftliche Mitarbeiterin am ZAIK anfangen wolle, erwiderte, ich könnte mir nichts Schöneres vorstellen. Meine Antwort mag sicherlich naiv gewesen sein, aber: Ich habe ein paar wunderschöne Jahre am ZAIK verbracht!

Dafür bedanken möchte ich mich an erster Stelle bei Prof. Dr. Ulrich Faigle und Prof. Dr. Rainer Schrader für all die Zeit, die sie sich für Fragen und Diskussionen nahmen, und für all die Anregungen und guten Ideen, die meine Interesse für verschiedenste Forschungsgebiete weckten und dadurch zum Entstehen dieser Arbeit beitrugen.

Dass ich so gerne ins Institut gegangen bin, lag natürlich vor allem an meinen tollen Kollegen, von denen mir viele zu echten Freunden geworden sind: Dirk, Christian, Bernhard, Dominique, Petra, ... ich hoffe, dass wir uns nicht aus den Augen verlieren werden.

Als ich anfing, die Inhalte meiner Forschung in dieser Dissertation zusammenzustellen, hatte ich die berechtigte Sorge, dass die Zeiten ja eigentlich nur noch schlechter werden könnten. Ich danke meiner Tochter Emma, denn sie hat mich eines Besseren belehrt!

Mein größter Dank gilt schliesslich meinen Eltern und Thorsten, die stets hinter all meinen Entscheidungen standen und mir dadurch die denkbar beste Unterstützung gewährten.

[1] Es sei denn, $\mathcal{P}=\mathcal{NP}$

Abstract

Combinatorial optimization problems whose underlying structures are matchings or matroids are well-known to be solvable with efficient algorithms. Matroids can even be characterized by a simple greedy algorithm.

In the first part of this thesis, some generalizations of matroids which allow the ground set to be partially ordered are considered. In particular, it will be shown that a special type of lattice polyhedra, for which Dietrich and Hoffman recently established a dual greedy algorithm, can be reduced to ordinary polymatroids. Moreover, strong exchange structures, Gauss greedoids and Δ-matroids will be extended from Boolean lattices to general distributive lattices, and the resulting structures will be characterized by certain greedy-type algorithms.

While a matching of maximal size can be determined by a polynomial algorithm, the dual problem of finding a vertex cover of minimal size in general graphs is one of the hardest problems in combinatorial optimization. However, in case the graph belongs to the class of König-Egerváry graphs, a maximum matching can be used to construct a minimum vertex cover. Lovász and Korach characterized König-Egerváry graphs by the exclusion of forbidden subgraphs.

In the second part of this dissertation, the structure of König-Egerváry graphs and the more general Red/Blue-split graphs will be analyzed. Red/Blue-split graphs have red and blue colored edges and the vertices of which can be split into two stable sets with respect to the red and blue edges, respectively. An algorithm that either determines a feasible partition of the vertices, or returns a red-blue colored subgraph (called "flower") characterizing non-Red/Blue-split graphs will be presented. This characterization allows the deduction of Lovász and Korach's characterizations of König-Egerváry graphs in case the red edges of the flower form a maximum matching. Furthermore, weighted Red/Blue-split graphs which model integrally solvable simple systems are introduced. A simple system is an inequality system where the sum of absolute values in each row of the integral matrix does not exceed the value two. A shortest-path algorithm and the presented Red/Blue-split algorithm will be used to find an integral solution of a simple system. These two algorithms lead to a characterization of weighted Red/Blue-split graphs by forbidden weighted subgraphs.

Contents

Chapter 1

Introduction

Combinatorial optimization is a relatively young, but lively area of applied mathematics having its roots in theoretical computer science, combinatorics, operation research and discrete mathematics.

A typical combinatorial optimization problem may be formulated as follows: Given a family $\mathcal{F} \subseteq 2^E$ of subsets of some finite set E together with a weight function $w : E \to \mathbb{R}$, determine a member X of $\mathcal{F}$ of maximal weight $w(X) = \sum_{e \in X} x(e)$ as fast as possible. That is, the task is to solve the problem

$$\max\{w(X) \mid X \in \mathcal{F}\}$$

efficiently. It is not hard to imagine that many real-life problems can be modeled this way.

We should note that, since maximizing w is equivalent to minimizing the negative weight $-w$, we could equally well formulate the problem as the minimization problem

$$\min\{-w(X) \mid X \in \mathcal{F}\}.$$

However, the family $\mathcal{F}$ (throughout, we call its members *feasible*) is usually too large to simply scan through all objects and search for the optimal one. Thus, one hopes to find special properties of $\mathcal{F}$ that guarantee the correctness of more efficient algorithms.

1.1 Greedy algorithm

If, for example, $\mathcal{F}$ contains the empty set, we might try the following *greedy algorithm*: we start with the empty set and, iteratively, as long as it is possible, increment the current solution by an element of maximal weight among those that can be added while keeping feasibility. More precisely, given a subset $X \subseteq E$, we define the set

$$\Gamma(X) = \{e \in E \setminus X \mid X \cup e \in \mathcal{F}\},$$

and apply the

GREEDY-ALGORITHM:
$X^* = \emptyset$;
while $\Gamma(X^*) \neq \emptyset$ **do**
Choose $i \in \Gamma(X^*)$ such that $w(i) \geq w(j)$ for all $j \in \Gamma(X^*)$;
$X^* = X^* \cup i$;
end while

We say that the greedy algorithm "works" if it determines a member of $\mathcal{F}$ of maximal weight given an arbitrary function $w : E \to \mathbb{R}$. Which properties should $\mathcal{F}$ satisfy such that this simple greedy algorithm works?

To make sure that all feasible elements can be reached by the greedy algorithm, let us first restrict our considerations to *monotone* families, i.e., non-empty families $\mathcal{F}$ satisfying

$$X \in \mathcal{F}, Y \subseteq X \quad \Rightarrow \quad Y \in \mathcal{F}.$$

Since $\mathcal{F}$ is monotone, an optimal member of $\mathcal{F}$ will never contain elements of negative weight. Thus, given a monotone family $\mathcal{F}$, it is sufficient to consider non-negative weight functions only.

Consider, for example, an undirected graph $G = (V, E)$ with vertices V and edges E. Then the family of *forests*

$$\mathcal{F} = \{F \subseteq E \mid F \text{ contains no cycle}\},$$

and the family of *matchings*

$$\mathcal{F} = \{M \subseteq E \mid \text{ no two edges in } M \text{ have an endpoint in common}\}$$

are easily seen to be monotone. It is well-known [Bor26] that the greedy algorithm determines a forest of maximal weight for any linear weight function, whereas the matching found by the greedy algorithm is not necessarily optimal. (Cf. the graph in Figure 1.1: the greedy algorithm returns the matching $M = \{(b, c)\}$ of weight 7, while the matching $M' = \{(a, b), (c, d)\}$ has weight 9.)

1.2 Matroids and the Monge algorithm

Monotone structures such that the greedy algorithm works are called *matroids*. Matroids play a central role in combinatorial theory and serve as a link between different areas of mathematics. Matroid theory goes back in the 1930's when van der Waerden in his "Modern Algebra" first approached linear and algebraic dependence axiomatically. Whitney [Whi35]

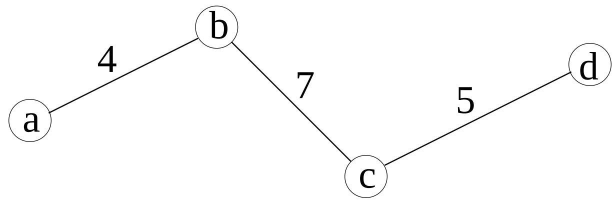


Figure 1.1: Greedy algorithm and the matching problem.

was the first to use the term “matroid”. (As the name suggests, a matroid can be conceived as an abstract generalization of a matrix.) For an overview about matroids, the reader is referred to [Wel76].

But how do we know whether a family $\mathcal{F} \subseteq 2^E$ is a matroid? What are the matroid-characterizing properties? We answer this question using some linear programming theory: Let us represent a subset $A \subseteq E$ via its incidence vector $x^A \in \{0,1\}^{|E|}$ with

$$x_e^A = 1 \quad \Leftrightarrow \quad e \in A.$$

Further, we define a *rank function* $r : 2^E \to \mathbb{N}$ such that

$$r(A) = \max\{|X| \mid X \in \mathcal{F}, X \subseteq A\}.$$

Then, if F is feasible, the inequality

$$x^F(A) = \sum_{e \in A} x_e^F = |F \cap A| \leq r(A)$$

follows from the monotony of $\mathcal{F}$. In fact, the incidence vectors of the elements in $\mathcal{F}$ are exactly the feasible integral solutions of the linear programming problem

$$(P) \quad \max_{x \geq 0}\{\sum_{e \in E} w_e x_e \mid x(A) \leq r(A), \forall A \subseteq E\}.$$

In particular, the incidence vector x^* of the greedy solution X^* is a feasible solution of problem (P).

We now construct a feasible solution y^* of the dual problem

$$(D) \quad \min_{y \geq 0}\{\sum_{A \subseteq E} r(A)y(A) \mid \sum_{A \ni e} y(A) \geq w_e, \forall e \in E\}$$

with the following procedure which goes back to Monge [Mon81].

```
MONGE-ALGORITHM:
  while E ≠ ∅ do
    Choose ē ∈ E of minimal weight;
    y*(E) = w(ē);
    for e ∈ E do
      w(e) ← w(e) − w(ē),
    end for
    E = E \ ē;
  end while
```

Since w is assumed to be non-negative, y^* is a feasible solution of problem (D) by construction. The feasibility of x^* and y^* implies the *weak duality*

$$\sum_{e\in E} w_e x_e^* \leq \sum_{e\in E}\sum_{A\subseteq E} y^*(A)x_e^* = \sum_{A\subseteq E}\sum_{e\in A} x_e^* y^*(A) \leq \sum_{A\subseteq E} r(A)y^*(A).$$

Now, if the rank function r is *submodular*, i.e., satisfies

$$r(X) + r(Y) \geq r(X\cap Y) + r(X\cup Y) \quad \forall X, Y \in \mathcal{F},$$

then it can be shown that x^* and y^* satisfy the weak duality with equality. Hence, the submodularity of r implies the optimality of x^* and y^*. Moreover, it is not hard to see that the submodularity of r is not only sufficient, but also necessary for the correctness of the greedy and the dual Monge algorithm. Thus, matroids can be characterized as exactly those monotone families, whose rank function is submodular.

Since $\mathcal{F}$ is monotone if and only if the rank function is normalized in the sense $r(\emptyset) = 0$ and unit-increasing, a function $r : 2^E \to \mathbb{N}$ is the rank function of a matroid if and only if for all $X, Y \subseteq E, e \in E$ holds

1. $r(\emptyset) = 0$,
2. $r(X) \leq r(X \cup e) \leq r(X) + 1$, and
3. $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$.

The rich structure of matroids admits several additional characterizations beside the algorithmic one or the rank function characterization above (Cf. [Whi00] for an overview of the different characterizations). For example, matroids turn out to be exactly those monotone set families $\mathcal{F} \subseteq 2^E$ that satisfy Steinitz' augmentation property

$$(AP) \quad X, Y \in \mathcal{F}, |X| < |Y| \quad \Rightarrow \quad \exists y \in Y \setminus X \text{ with } X \cup y \in \mathcal{F}.$$

Examples of matroids

Let us consider a few examples of matroids: Surely, the bases of forests of $G = (V, E)$ are all of cardinality $|V| - cc$, where cc denotes the number of connected components in G. Thus, the family of forests of G forms a matroid, called *graphical matroid.*

Another simple class of matroids are the *k-uniform matroids* for a given $k \in \mathbb{N}$ with feasible sets

$$\mathcal{F} = \{A \subseteq E \mid |A| \leq k\}.$$

Clearly, $\mathcal{F}$ is monotone and the bases are all of size k.

Slightly more complicated is the class of *partition matroids*: Given a partition $E = E_1 \dot\cup \ldots \dot\cup E_k$ of the ground set, the feasible subsets are those that meet each partition in at most one element, i.e.,

$$\mathcal{F} = \{A \subseteq E \mid |A \cap E_i| \leq 1, \forall i = 1, \ldots, k\}.$$

Again, $\mathcal{F}$ is monotone and the bases are all of size k.

If $\mathcal{M}_1 = (E, \mathcal{F}_1)$ and $\mathcal{M}_2 = (E, \mathcal{F}_2)$ are two matroids on the same ground set, it can be shown ([Wel76]) that also the union $\mathcal{M}_1 \cup \mathcal{M}_2 = (E, \mathcal{F}_1 \cup \mathcal{F}_2)$ with feasible sets

$$\mathcal{F}_1 \cup \mathcal{F}_2 = \{X_1 \cup X_2 \mid X_1 \in \mathcal{F}_1, X_2 \in \mathcal{F}_2\}$$

is a matroid again.

1.3 Matroid intersection and bipartite matching

Now consider the intersection $\mathcal{M}_1 \cap \mathcal{M}_2 = (E, \mathcal{F}_1 \cap \mathcal{F}_2)$ of two matroids with

$$\mathcal{F}_1 \cap \mathcal{F}_2 = \{X \subseteq E \mid X \in \mathcal{F}_1 \cap \mathcal{F}_2\}.$$

Surely, the intersection of two uniform matroids is a uniform matroid. But in general, the intersection of matroids need not be a matroid again. In particular, the intersection of two partition matroids is just another model for matchings in bipartite graphs and therefore no matroid:

A graph G is *bipartite* if it contains no cycle with an odd number of edges. The vertices of a bipartite graph $G = (S \dot\cup T, E)$ can be partitioned into two stable sets S and T, where a vertex set S is *stable* if no two elements in S are linked by an edge. Given a bipartite graph $G = (S \dot\cup T, E)$, the matchings of G are exactly the feasible elements in the intersection of the two partition matroids with corresponding partitions

$$E = \bigcup_{s \in S} \delta(s) \quad \text{and} \quad E = \bigcup_{t \in T} \delta(t),$$

where $\delta(v)$ denotes the set of edges incident to vertex $v \in V$. Thus, as we have seen that the greedy algorithm does not work for the matching problem in the bipartite graph shown in Figure 1.1, the intersection of two partition matroids cannot be a matroid.

Conversely, given two partition matroids $\mathcal{M}_1 = (E, \mathcal{F}_1)$ and $\mathcal{M}_2 = (E, \mathcal{F}_2)$ with corresponding partitions $E = S_1 \dot\cup \ \ldots \ \dot\cup S_r$ and $E = T_1 \dot\cup \ \ldots \ \dot\cup T_k$, the feasible sets in the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ are exactly the matchings in the bipartite graph $G = (S; T, E)$ with vertices $S = \{s_1, \ldots, s_r\}$, $T = \{t_1, \ldots, t_k\}$ and edges $E = \{(s_i, s_j) \mid s_i \in S_i, t_j \in T_j\}$.

1.4 König-Egerváry graphs

The greedy algorithm might even fail when applied in order to determine a matching of maximal cardinality in a bipartite graph. However, König [Kön31] found an efficient[1] algorithm to solve this problem.

Similar to the greedy algorithm, the bipartite matching algorithm starts with the empty set, and iteratively, as long it is possible, increases the size of the matching by one. The algorithm terminates with a matching and a vertex cover of identical size. (A vertex set C is a *cover*, if each edge is incident to at least one vertex in C.) Since a matching can never be larger than a vertex cover, the returned matching is of maximal size, and the returned vertex cover is of minimal size.

König's bipartite matching algorithm has been extended by Egerváry [Ege31] to solve the matching problem in weighted bipartite graphs.

Edmonds [Edm79] and Frank [Fra81] went even further and generalized the algorithm to solve the weighted matroid intersection problem.

Usually, the size of a maximum matching and the size of a minimum vertex cover of a graph G are denoted by $\nu(G)$ and $\tau(G)$, respectively. We observe that $\nu(G) = \tau(G) - 1$ holds for any odd cycle. Thus, if G is no longer bipartite, then $\tau(G)$ might be larger than $\nu(G)$, and the matching problem becomes more complicated. Nevertheless, Edmonds' augmenting path algorithm [Edm65] efficiently determines an optimal matching in general graphs.

In contrast to $\nu(G)$, the problem to calculate $\tau(G)$ in a general graph belongs to the hardest problems in combinatorial optimization (i.e., it is $\mathcal{NP}$-complete[2]) and cannot be solved efficiently, unless $\mathcal{P} = \mathcal{NP}$.

Of course, bipartite graphs are not the only graphs such that a matching and a vertex cover of the same cardinality exist. In general, graphs with the property $\nu(G) = \tau(G)$ are called *König-Egerváry graphs* [Dem79].

[1] We call an algorithm "efficient" if its running time is bounded by a polynomial in the input size. Combinatorial problems that can be solved efficiently form the class $\mathcal{P}$, where "$\mathcal{P}$" stands for "polynomial".

[2] "$\mathcal{NP}$" stands for "non-deterministic polynomial". For more details about computational complexity, the reader is referred to [GJ79]

König-Egerváry graphs can also be characterized in terms of linear programming: Consider the primal-dual pair of linear programs

$$(M) \quad \max_{x \geq 0} \{ \sum_{e \in E} x_e \mid x(\delta(v)) \leq 1, \forall v \in V \}, \quad \text{and}$$

$$(VC) \quad \min_{y \geq 0} \{ \sum_{v \in V} y_v \mid y_u + y_v \geq 1, \forall (u,v) \in E \}.$$

It is known (see, for example, [BP89]) that $G = (V, E)$ is a König-Egerváry graph if and only if (VC) has an integral optimal solution. In case the optimal solution of (VC) has no integral components at all, G is called *2-bicritical* [Pul79]. Bourjolly and Pulleyblank [BP89] proved that any graph can be decomposed into a König-Egerváry graph and a 2-bicritical graph.

But how do we know whether a graph is a König-Egerváry graph or not? To answer this question, we may restrict ourselves to graphs that admit a perfect matching, i.e., a matching covering all vertices (cf. Chapter 7). Deming [Dem79] and Sterboul [Ste79] presented an algorithm that uses a perfect matching M and either constructs a vertex cover of the same size, or returns a certain walk (called "M-handcuff"), whose edges alternate between matching and non-matching edges.

Lovász [Lov83] refined Deming-Sterboul's characterization of König-Egerváry graphs by forbidden M-handcuffs to a characterization by certain forbidden subgraphs with respect to a special perfect matching. An excluded subgraph characterization with respect to arbitrary perfect matchings was given by Korach [Kor82].

1.5 Red/Blue-split graphs

In this dissertation, we give an easier proof of Korach's result by characterizing the more general *Red/Blue-split graphs*: a Red/Blue-split graph is a graph $G = (V, R \cup B)$ with red and blue colored edges, whose vertex set can be split into a red and a blue stable set, where a *red [blue] stable set* denotes a stable set with respect to the red [blue] edges.

Why do Red/Blue-split graphs generalize König-Egerváry graphs? Let M be a perfect matching of $G = (V, E)$. A subset $C \subseteq V$ is a vertex cover in G if and only if the complement $V \setminus C$ is stable with respect to the edges in E. Moreover, C can only be of size $|M|$ if C is stable with respect to the edges in M. Hence, if we color the edges in E blue, add the edges of M and color them red, the resulting graph is a Red/Blue-split graph if and only if G is a König-Egerváry graph.

Red/Blue-split graphs have been introduced by Gavril [Gav93]. Beside König-Egerváry graphs they also generalize ordinary split graphs: Split graphs are (uncolored) graphs whose vertex set can be split into a stable set and a clique (a *clique* is a vertex set such that any two vertices are linked by an edge). Földes and Hammer [FH77] could characterize split graphs by the exclusion of three types of subgraphs, namely the $2K_2$ (two parallel edges), the C_4 and the C_5 (circuits with 4 respectively 5 edges).

The *Red/Blue-split problem*, i.e., the problem whether a red-blue colored graph is a Red/Blue-split graph, can be reduced to a 2-satisfiability problem. In fact, Red/Blue-split graphs are just another model for solvable 2-satisfiability instances (see Chapter 7).

Even more general, we introduce a weighted version of Red/Blue-split graphs which models integrally solvable simple systems $Ax \leq b$. (An inequality system $Ax \leq b$ is called *simple* if the matrix $A \in \mathbb{Z}^{m \times n}$ satisfies

$$\sum_{j=1}^{n} |a_{i,j}| \leq 2$$

in each row $i = 1, \ldots, m$.) Schrijver [Sch91] characterized integrally solvable simple systems by the exclusion of certain walks in bidirected graphs. We refine his result by an excluded subgraph characterization of so-called *weighted Red/Blue-split graphs*.

1.6 Outline

We have seen that matchings and matroids are strongly related: matchings in bipartite graphs are exactly the feasible sets in the intersection of two partition matroids. Moreover, matroids and matchings in bipartite graphs, or even in the more general König-Egerváry graphs, have in common that optimal integral solutions of the corresponding dual linear programming problems can be determined with efficient algorithms.

In its first part, this dissertation deals with some generalizations of matroids to ordered structures which are characterized by greedy-type algorithms.

The second part investigates the structure of König-Egerváry graphs and the more general Red/Blue-split graphs. We present algorithms that lead to characterizations by excluded subgraphs.

Outline of Part I

Since its introduction by Whitney [Whi35] matroids have been generalized in many different ways, and we will present some of these known generalizations in Chapter 2. Almost all of these structures are accompanied by appropriate greedy-type algorithms.

For example, integral polymatroids [Edm71], distributive supermatroids [DIW72], ordered matroids [Fai84] and submodular systems [Fuj91] extend the rank function characterization of matroids and several additional matroid characterizing properties. These structures generalize matroids by allowing the ground set to be partially ordered. Further on, primal greedy algorithms and dual Monge algorithms are known to solve the corresponding optimization problems.

A common framework for the just mentioned generalizations of matroids are modular clutter systems, which belong to the class of lattice polyhedra [Hof82]. Recently, Dietrich and Hoffman [DH03] proved the optimality of a Monge-type algorithm for modular clutter systems. In [FP06b], we complemented their result by establishing the corresponding primal greedy algorithm.

However, we will show in Chapter 3 that modular clutter systems can be reduced to submodular systems. This allows us to embed the problem of Dietrich and Hoffman into a framework where it can be solved with the generalized polymatroid greedy algorithm (see, e.g., [FK00a, FK96]).

Without allowing partial orders on the ground set, strong exchange structures [Goe86b], Gauss greedoids [Goe86a] and Δ-matroids [Bou87], [Bou89] extend certain matroid characterizing exchange properties. Strong exchange structures and Δ-matroids have been generalized to integral strong exchange structures [She04] and jump systems [BC95] by considering integral vectors instead of subsets. Again, for all of these generalizations of matroids greedy-type algorithms were shown to work optimally.

We generalize these structures from the Boolean lattice of all subsets of a finite set E to the distributive lattice of all ideals of some arbitrary partially ordered set $P = (P, \leq)$. (A subset $I \subseteq P$ is an *ideal* if $x \in I, y \leq x$ implies $y \in I$).

In Chapter 4 we introduce *distributive strong exchange structures* as a generalization of strong exchange structures and integral strong exchange structures. We characterize them by a certain exchange property and show that the greedy algorithm determines an optimal member for arbitrary admissible non-negative weight functions. (A weight function $w : P \to \mathbb{R}$ is *admissible* if $i \leq j$ implies $w_i \geq w_j$ for all $i, j \in P$.)

In Chapter 5, we define *distributive Gauss greedoids* as the collection of bases of distributive supermatroids in a strong map relation. They turn out to be a common generalization of Gauss greedoids and distributive supermatroids. We characterize distributive Gauss greedoids by some exchange property and prove that they are exactly those ideal systems for which a modified greedy algorithm works optimally for any admissible weight function.

In Chapter 6, we define *distributive Δ-matroids* by a certain "2-step axiom". We show that a greedy-type algorithm returns a set of optimal members for arbitrary admissible non-negative weight functions, and deduce the definitions of Δ-matroids and jump systems. (See Figures 2.10 and 2.11 at the end of Chapter 2 for an overview about the hierarchy of the mentioned generalizations of matroids.)

After having characterized several matroid-generalizing structures in Part I, we turn our considerations to the structure analysis, in particular the characterization, of König-Egerváry graphs and Red/Blue-split graphs in Part II:

Outline of Part II

In Chapter 7, we present the known characterizations of König-Egerváry graphs by Deming-Sterboul, Lovász and Korach in more detail. We then focus on Red/Blue-split graphs, the common generalization of König-Egerváry graphs and split graphs. We show that Red/Blue-split graphs model solvable 2-satisfiability instances, and that a weighted version of them, so-called *weighted Red/Blue-split graphs*, model integrally solvable simple systems $Ax \leq b$.

Furthermore, we introduce the *stable matroid basis problem* which is in some sense related to the Red/Blue-split problem: The stable matroid basis problem asks for a basis in a matroid $\mathcal{M} = (V, \mathcal{F})$ which is a stable set in a graph $G = (V, E)$ whose vertex set corresponds to the ground set of $\mathcal{M}$. It turns out that the problem is $\mathcal{NP}$-complete if $\mathcal{M}$ is a partition matroid, whereas it reduces to a Red/Blue-split problem in case $\mathcal{M}$ is the dual of a partition matroid.

We solve the Red/Blue-split problem algorithmically in Chapter 8. The algorithm will either determine a feasible partition of the vertices of $G = (V, R \cup B)$ into a red and a blue stable set, or return a certain red-blue alternating walk ("handcuff") proving that G is not a Red/Blue-split graph. These handcuffs generalize the M-handcuffs of Deming and Sterboul, which characterize König-Egerváry graphs.

Since a handcuff might not be edge or vertex disjoint, we normalize handcuffs in Chapter 9 such that the induced subgraphs are of a certain type, which we call "flower". The forbidden subgraphs of Lovász and Korach for König-Egerváry graphs, respectively of Földes and Hammer for split graphs, follow as consequences in the special cases, where the red edges form a perfect matching, respectively, where the red edges are the complement of the blue edges.

In Chapter 10, we solve the weighted Red/Blue-split problem (and therefore the problem whether a simple system $Ax \leq b$ is integrally solvable) by applying a shortest-path algorithm in an auxiliary directed bipartite graph, and our Red/Blue-split algorithm in an auxiliary red-blue colored graph. These two algorithms lead to a characterization of weighted Red/Blue-split graphs by the exclusion of certain weighted subgraphs. We call these excluded subgraphs "negative even circuits", "negative simple handcuffs", and "tight odd flowers".

Finally, in Chapter 11, we investigate the complexity status of the problem to determine the largest union of a red and a blue stable set in different graph classes.

Part I

Generalizations of matroids to ordered structures

Chapter 2

Posets, lattices and greedy algorithms

We introduced matroids as exactly those monotone set systems $\mathcal{F} \subseteq 2^E$ for which the greedy algorithm determines an optimal member for an arbitrary weight function $w : E \to \mathbb{R}$ (which we could assume to be non-negative). Beside several other characterizations, matroids can also be described via the submodularity of the rank function or the augmentation property (AP).

In the last decades, a lot of matroid-generalizing structures have been defined and investigated. Some of them were already mentioned, but we describe them in more detail in this Chapter. We distinguish between structures $\mathcal{F} \subseteq 2^E$ defined on the Boolean lattice 2^E, structures $\mathcal{F} \subseteq \mathbb{Z}^n$ defined on the lattice of integral vectors $\mathbb{Z}^n$, and those structures $\mathcal{F} \subseteq \mathcal{L}$ defined on a general distributive lattices $\mathcal{L}$.

But before, we recall some notions and results about partially ordered sets and lattices. In particular, we state Birkhoff's Theorem on the equivalence of distributive lattices $\mathcal{L}$ and the lattices of ideals

$$\mathcal{L}(P) = \{I \subseteq P \mid I \text{ ideal in } (P, \leq)\}$$

of some partially ordered set $(P, \leq)$. Thus, structures $\mathcal{F} \subseteq \mathcal{L}$ defined on a distributive lattice might as well be interpreted as families of ideals $\mathcal{F} \subseteq \mathcal{L}(P)$ of some poset $(P, \leq)$.

Matroid-generalizing structures defined on the Boolean lattice are, for example, strong exchange structures, Gauss greedoids and Δ-matroids. Matroid-generalizing families of integral vectors $\mathcal{F} \subseteq \mathbb{Z}^n$ are, e.g., integral polymatroids, integral strong exchange structures and jump systems. Finally, distributive supermatroids, ordered matroids and submodular systems are families of ideals which extend matroids by allowing the ground set E to be partially ordered. All these structures are accompanied by greedy-type algorithms.

A far reaching generalization of matroids are Hoffman and Schwartz' lattice polyhedra [HS78] where the underlying lattice need not be distributive. However, no greedy-type algorithm is known for lattice polyhedra in general. For modular clutter systems, which are special types of lattice polyhedra, a Monge-type algoritm was shown to work correctly [DH03]. We complemented this algorithm by a primal greedy algorithm in [FP06b].

However, we prove in Chapter 3 that modular clutter systems, though they seem to allow more general lattices than distributive lattices, can be reduced to submodular systems by showing that the underlying lattice is in fact a distributive lattice. Thus, the generalized polymatroid greedy algorithm (described in Section 3.3), which is known to work correctly for submodular systems can be applied to modular clutter systems as well.

In the subsequent Chapters of Part I, we introduce distributive strong exchange structures, distributive Gauss greedoids and distributive Δ-matroids and show that certain greedy-type algorithms work correctly. These families of ideals generalize (integral) strong exchange structures, Gauss greedoids, Δ-matroids and jump systems by considering distributive lattices instead of Boolean lattices or integral lattices.

2.1 Posets and (pseudo)lattices

A partially ordered set (poset) $P = (P, \leq)$ is a set P together with a binary operation $\leq: P \times P \to P$ satisfying for $x, y, z \in P$

1. $x \leq x$ "reflexivity",
2. $x \leq y,\, y \leq x \Rightarrow x = y$ "symmetry", and
3. $x \leq y,\, y \leq z \Rightarrow x \leq z$ "transitivity".

Two elements $x, y \in P$ with $x \leq y$ or $y \leq x$ are said to be *comparable*. A subset $C \subseteq P$ is a *chain* if any two elements in C are comparable. Dually, $A \subseteq P$ is an *antichain*, if no two elements in A are comparable. Equivalent to König's min-max result on matchings and vertex covers in bipartite graphs is Dilworth's Theorem [Dil50] on chains and antichains:

Theorem 2.1 (Dilworth's decomposition theorem). *Let $(P, \leq)$ be a partially ordered set. Then the minimum number of chains covering P is equal to the maximum size of an antichain.*

A subset $I \subseteq P$ is an *ideal* of $(P, \leq)$ if $y \in I$ and $x \leq y$ imply $x \in I$. We denote by $\mathcal{L}(P)$ the collection of all ideals of P.

We may visualize a poset $P = (P, \leq)$ via its *comparability graph*. The vertices of this comparability graph correspond to the elements of P, and two elements are linked by an edge if and only if they are comparable. Usually, x is located beneath y if $x \leq y$.

If we delete the transitive edges of a comparability graph, we obtain the *Hasse graph*. If $x \leq y$ and x and y are linked by an edge in the Hasse graph, x is called the *lower neighbor* of y and y is called the *upper neighbor* of x.

For example, the poset $P = (\{a, b, c, d\}, a \leq b, b \leq c, a \leq c, a \leq d)$ can be visualized by the Hasse graph shown in Figure 2.1.

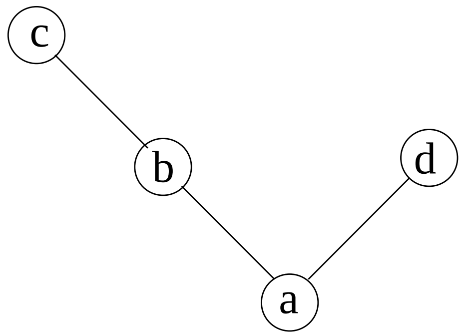


Figure 2.1: Hasse graph.

There are several classes of posets which are characterized via their Hasse graph: for example, a poset is a *tree-order*, if the Hasse graph is a tree (i.e., a connected forest) with unique minimal element (called the "root" of the tree). The Hasse graph in Figure 2.1 is such a tree-order with root a. Another type of posets are *series-parallel orders* (also called *N-free orders*), which are defined as those posets whose Hasse graph contains no N-subgraph as shown in Figure 2.2.

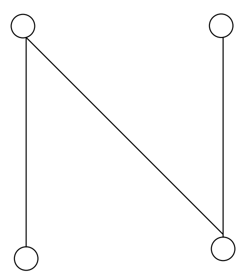


Figure 2.2: N-subgraph, excluded in series-parallel orders.

A poset $(L, \leq)$ is said to be a *lattice*, if for each pair of elements $x, y \in L$ there exists a unique minimal element $x \vee y$ of the set of upper bounds

$$\{z \in L \mid x, y \leq z\}.$$

Note that, in case there exists a unique minimal element in $(L, \leq)$, the existence of $x \vee y$ implies the existence of a unique maximal element $x \wedge y$ of the set of lower bounds

$$\{z \in L \mid x, y \geq z\}.$$

Usually, $x \vee y$ and $x \wedge y$ are called the *join*, resp. the *meet*, of x and y.

An element $p \in L$ is *join-irreducible* if p has exactly one lower neighbor in L, i.e., if

$$p = x \vee y \quad \text{implies} \quad p = x \text{ or } p = y.$$

Analogue, *meet-irreducible* elements are defined as the elements with exactly one upper neighbor. We now present two examples of lattices: the Boolean lattice and the lattice of integral vectors.

The Boolean lattice

Consider a finite set E and the poset $(2^E, \subseteq)$ of all subsets of E ordered by inclusion. Since for any two subsets $X, Y \subseteq E$ the intersection $X \cap Y$ is the unique maximal lower bound, and the union $X \cup Y$ is the unique minimal upper bound, $\mathbb{B}_{|E|} := (2^E, \subseteq, \cap, \cup)$ is a lattice. $\mathbb{B}_{|E|}$ is called the *Boolean lattice* on E.

The unique join-irreducible elements of $\mathbb{B}_{|E|}$ are the one-element sets. Dually, the meet-irreducible elements consist of $|E| - 1$ elements.

The integral lattice

More general than the Boolean lattice is the lattice on integral vectors $\mathbb{Z}^n = (\mathbb{Z}^n, \leq, \vee, \wedge)$ with order relation

$$x \leq y \iff x_i \leq y_i \text{ for } i = 1, \ldots, n,$$

meet-operation

$$x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})^T,$$

and join-operation

$$x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})^T.$$

Representing a subset $A \subseteq E$ by its incidence vector $\chi^A : E \to \{0, 1\}$ with

$$\chi_i^A = 1 \iff i \in A$$

for $i = 1, \ldots, |E|$ leads to the observation that the Boolean lattice $\mathbb{B}_{|E|}$ is equivalent to the binary integral lattice $\{0, 1\}^{|E|}$.

Pseudolattices

Slightly more general than a lattice is a *pseudolattice* (cf. [DH03]). A poset $(L, \leq)$ is a pseudolattice if for each pair of elements $x, y \in L$ there exist a designated element $x \vee^* y$

among the upper bounds of x and y, and a designated element $x \wedge^* y$ among the lower bounds of x and y.

For example, the left Hasse graph in Figure 2.3 represents a lattice, while in the right one the uniqueness of minimal upper bounds, resp. of maximal lower bounds, is not assured. Nevertheless, the right Hasse graph might visualize a pseudolattice.

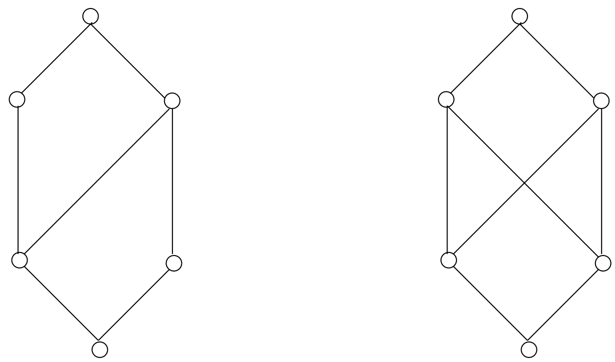


Figure 2.3: Lattice and pseudolattice.

Submodularity

We say that a function $r : L \to \mathbb{R}$ on a lattice L is *submodular* if for all $x, y \in L$ holds

$$r(x \vee y) + r(x \wedge y) \leq r(x) + r(y).$$

Accordingly, r is *supermodular* if

$$r(x \vee y) + r(x \wedge y) \geq r(x) + r(y)$$

is true. We call r *modular*, if r is both, sub- and supermodular.

For example, we have seen in Chapter 1 that a monotone set system $\mathcal{F} \subseteq \mathbb{B}_{|E|}$ is a matroid if and only if its rank function is submodular on $\mathbb{B}_{|E|}$.

2.2 Greedy algorithms on Boolean lattices

Consider for example the Boolean lattice $\mathbb{B}_3$ on $E = \{a, b, c\}$ shown in Figure 2.4, and the family $\mathcal{F} \subseteq E$ whose members are indicated by boxes.

We can easily check that $\mathcal{F}$ is monotone and the rank function r defined by

$$r(X) = \max\{|A| \mid A \subseteq X, A \in \mathcal{F}\}$$

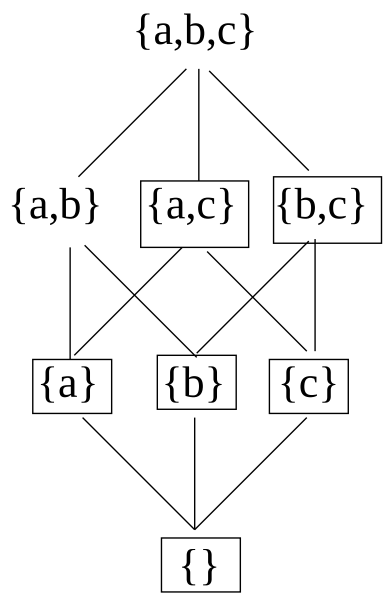


Figure 2.4: Set family $\mathcal{F} \subseteq 2^{\{a,b,c\}}$ in the Boolean lattice $\mathbb{B}_3$.

is submodular. Equivalently, we could observe that $\mathcal{F}$ satisfies Steinitz' augmentation property

$$(AP) \quad X, Y \in \mathcal{F}, |X| < |Y| \quad \Rightarrow \quad \exists y \in Y \setminus X \text{ with } X \cup y \in \mathcal{F}.$$

Thus, $\mathcal{F}$ is a matroid and we know that for any weight function $w : E \to \mathbb{R}_+$ the following greedy algorithm determines a feasible member of optimal weight $w(X) = \sum_{e \in X} w(e)$. Recall that we defined for each subset X the set

$$\Gamma(X) = \{i \in E \setminus X \mid X \cup i \in \mathcal{F}\}.$$

GREEDY-ALGORITHM:
  $X^* = \emptyset$;
  while $\Gamma(X^*) \neq \emptyset$ **do**
    Choose $i \in \Gamma(X^*)$ such that $w(i) \geq w(j)$ for all $j \in \Gamma(X^*)$;
    $X^* = X^* \cup i$;
  end while

We now consider the set system $\mathcal{F} \subseteq \mathbb{B}_3$ shown in Figure 2.5 which is not a matroid, since $\mathcal{F}$ is not monotone. Nevertheless, it can be checked that the greedy algorithm works optimally for any non-negative linear weight function.

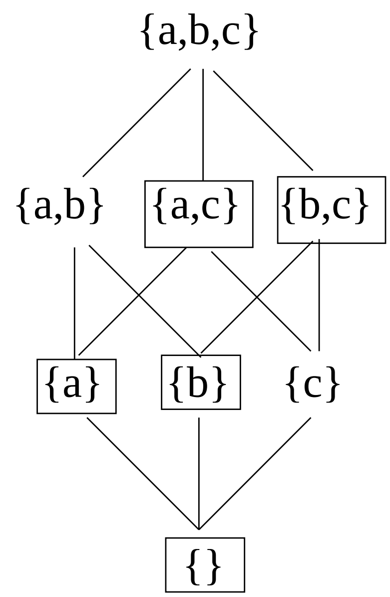


Figure 2.5: Non-monotone set family $\mathcal{F} \subseteq 2^{\{a,b,c\}}$.

Accessible systems and greedoids

Obviously, a necessary condition for the greedy algorithm to work optimally is that $\mathcal{F}$ is *accessible*, saying that for each $X \in \mathcal{F}$ there exists at least one lower neighbor of X which is feasible as well. Additionally, the augmentation property (AP) turns out to be necessary.

Korte, Lovász and Schrader [KLS91] generalized matroids to so-called *greedoids* as accessible set systems satisfying Steinitz' augmentation property (AP).

The system $\mathcal{F}$ shown in Figure 2.5 is an example of a greedoid. Still, there exist greedoids such that the greedy algorithm is not optimal for certain linear weight functions. For example, the greedy algorithm does not determine an optimal member of the greedoid shown in Figure 2.6 if $w(a) = 3$, $w(b) = 4$ and $w(c) = 2$.

2.2.1 Strong exchange structures

We observe that, given an accesible system $\mathcal{F} \subseteq 2^E$, the greedy algorithm above does always return a basis (i.e. an inclusion-wise maximal feasible subset of $\mathcal{F}$). What are the greedoids for which the basis found by the greedy algorithm is optimal for any linear weight function?

We call a greedoid $\mathcal{F} \subseteq \mathbb{B}_{|E|}$ a *strong exchange structure* if it satisfies the following *strong exchange property* which seems to go back to Brylawski (cf. [KLS91]).

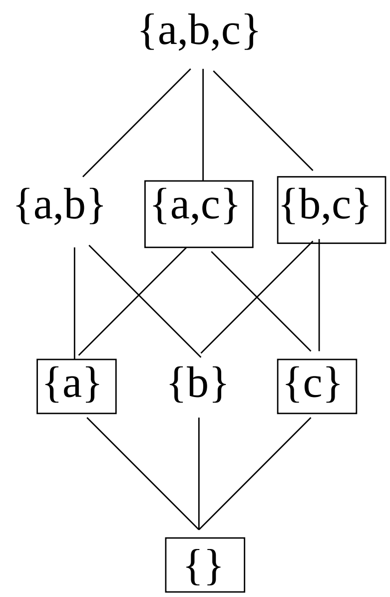


Figure 2.6: Greedoid for which the greedy is not necessarily optimal.

Definition 2.1 (Strong exchange property). *A set system $\mathcal{F} \subseteq \mathbb{B}_{|E|}$ has the* strong exchange property *if for $A \in \mathcal{F}$, $B \in \mathcal{B}(\mathcal{F})$, $A \subseteq B$ and $i \in E \setminus B$ with $A \cup i \in \mathcal{F}$ there exists $j \in B \setminus A$ such that $A \cup j \in \mathcal{F}$ and $B \setminus j \cup i \in \mathcal{F}$.*

For example, the greedoid shown in Figure 2.5 is such a strong exchange structure. Goetschel [Goe86b] proved that strong exchange structures are exactly those greedoids for which the greedy algorithm determines an optimal basis for any linear weight function.

2.2.2 Gauss greedoids

In case of non-negative weight functions, any optimal basis of a strong exchange structure is an optimal member as well. This is not necessarily true for arbitrary weight functions. If the set system is only accessible but not monotone, we cannot simply restrict to elements with non-negative weight. We therefore modify the greedy algorithm above such that the modified greedy algorithm remembers the current best solution in each step:

MODIFIED GREEDY-ALGORITHM:

$X^* = X = \emptyset$;
while $\Gamma(X) \neq \emptyset$ **do**
  Choose $i \in \Gamma(X)$ such that $w(i) \geq w(j)$ for all $j \in \Gamma(X)$;
  $X = X \cup i$;
  if $w(X) > w(X^*)$ **then**
    $X^* = X$;
  end if
end while

The question arises whether the set systems for which this modified greedy algorithm works optimally can be characterized. The answer is "Yes". Goecke's *Gauss greedoids*, a special type of strong exchange structures, can be shown to be exactly those set systems, for which the modified greedy algorithm determines an optimal member for any linear weight function.

For the definition of Gauss greedoids, we need to recall the notion of strong maps (cf. [Wel76]).

Definition 2.2 (Strong maps). *Given two matroids $\mathcal{M}_1 = (E, r_1)$ and $\mathcal{M}_2 = (E, r_2)$ on the same ground set E, $\mathcal{M}_1$ is a* strong map *of $\mathcal{M}_2$ if*

$$r_1(X \cup Y) + r_2(X \cap Y) \leq r_1(X) + r_2(Y)$$

is satisfied for all $X, Y \subseteq E$.

The two matroids $\mathcal{M}_1 = (E, r_1)$ and $\mathcal{M}_2 = (E, r_2)$ with feasible sets $\mathcal{M}_i = \{X \subseteq E \mid r_i(X) = |X|\}$ $(i = 1, 2)$ are *nested* if $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and the rank functions satisfy

$$r_1(E) = r_2(E) - 1.$$

We are now able to define Gauss greedoids.

Definition 2.3 (Gauss greedoids). *Let $\{(E, \mathcal{M}_i)\}_{i=1,\ldots,m}$ be a family of nested matroids with $r_1(E) = 1$ such that $\mathcal{M}_i$ is a strong map of $\mathcal{M}_{i+1}$ for $i = 1, \ldots, m-1$. Then the accessible system $\mathcal{F} \subseteq \mathbb{B}_{|E|}$ with*

$$\mathcal{F} = \{\{x_1, \ldots, x_k\} \subseteq E \mid \{x_1, \ldots, x_i\} \in \mathcal{M}_i \text{ for } 1 \leq i \leq k\} \cup \emptyset$$

is a Gauss greedoid.

Goecke [Goe86a] proved that Gauss greedoids are exactly those greedoids, for which the modified greedy algorithm determines an optimal solution for every linear weight function.

Remark 2.1. *The name* Gauss greedoid *is motivated by its relation to the Gaussian elimination algorithm. (Compare the example of Gaussian elimination greedoids in Section 5.4.1.)*

2.2.3 Δ-matroids

Even in case the set system $\mathcal{F}$ is not accessible it is sometimes possible to determine a member of $\mathcal{F}$ of maximal weight $w : E \to \mathbb{R}$ by the following greedy strategy:

GREEDY-ALGORITHM (for Δ-matroids):
Order $E = \{e_1, \ldots, e_n\}$ so that $|w(e_1)| \geq |w(e_2)| \geq \ldots \geq |w(e_n)|$;
for $i = 1, \ldots, n+1$ **do**
  $T_i = \{e_i, \ldots, e_n\}$;
end for
$J = \emptyset$;
for $i = 1, \ldots, n$ **do**
  if $w(e_i) \geq 0$ and there exists $F \in \mathcal{F}$ with $J \cup e_i \subseteq F \subseteq J \cup T_i$ **then**
    $J = J \cup e_i$;
  end if
  if $w(e_i) < 0$ and there does not exist $F \in \mathcal{F}$ with $J \cup e_i \subseteq F \subseteq J \cup T_{i+1}$ **then**
    $J = J \cup e_i$;
  end if
end for

The set systems for which this algorithm works optimally with respect to any linear weight function are Bouchet's Δ-*matroids* [Bou87], [Bou89] defined as follows:

A nonempty set system $\mathcal{F} \subseteq 2^E$ is called a Δ-*matroid* if it satisfies the *symmetric exchange axiom* saying

"For $A, B \in \mathcal{F}$ and $x \in A\Delta B$,
there exists $y \in A\Delta B$ such that $A\Delta\{x, y\} \in \mathcal{F}$."

Here and elsewhere Δ denotes the symmetric difference

$$A\Delta B = A \setminus B \cup B \setminus A.$$

We might observe that it is sufficient to consider non-negative weight functions, only: Given an arbitrary linear weight function $w : E \to \mathbb{R}$ and a set system $\mathcal{F} \subseteq 2^E$, let $N = \{e_i \in E \mid w(e_i) < 0\}$ denote the elements of negative weight, and $|w| : E \to \mathbb{R}_+$ denote the weight function whose components are the absolute values of w. Since

$$w(X) = w(N) - w(N \setminus X) + w(X \setminus N) = -|w|(N) + |w|(N \setminus X) + |w|(X \setminus N),$$

it follows that X is an optimal solution of $\max\{w(X) \mid X \in \mathcal{F}\}$ if and only if $\tilde{X} = X\Delta N$ is an optimal solution of $\max\{|w|(X) \mid X \in \mathcal{F}\Delta N\}$, where

$$\mathcal{F}\Delta N = \{X\Delta N \mid X \in \mathcal{F}\}.$$

Moreover, it can be shown that if $\mathcal{F}$ is a Δ-matroid, then the set system $\mathcal{F}\Delta N$ is a Δ-matroid as well. Hence, instead of solving the problem $\max\{w(X) \mid X \in \mathcal{F}\}$, we may solve the problem $\max\{|w|(X) \mid X \in \mathcal{F}\Delta N\}$ with non-negative weight function.

To see that Δ-matroids generalize matroids, we have to recall an additional characterization of matroids: A set system $\mathcal{F} \subseteq 2^E$ is the basis set of a matroid if and only if it satisfies the *exchange axiom* saying

"For $A, B \in \mathcal{F}$ and $x \in A \setminus B$,
there exists $y \in B \setminus A$ such that $A\Delta\{x, y\} \in \mathcal{F}$."

Hence, Δ-matroids generalize matroids. In fact, basis sets of matroids are precisely the Δ-matroids for which all members have the same cardinality.

Similar structures as Δ-matroids were defined by a number of authors, see Dunstan and Welsh [DW73], Dress and Havel [DH86], and Chandrasekaran and Kabadi [CK88].

2.3 Greedy algorithms on integral lattices

Some of the structures described above, namely matroids, strong exchange structures and Δ-matroids, have been generalized from Boolean lattices to integral lattices.

2.3.1 Integral polymatroids

Many combinatorial optimization problems whose constraints are presented by integer-valued submodular set functions fit into the framework of integral polymatroids, which are the integral vertices of polymatroids in the sense of Edmonds [Edm70] and form a generalization of matroids.

Let E be a finite set and r be a function from 2^E to $\mathbb{R}$. Suppose that $r : 2^E \to \mathbb{R}$ is submodular. Then (E, r) is called a *polymatroid* with rank function r and corresponding polyhedron

$$\{x \in \mathbb{R}_+^E \mid x(S) \leq r(S) \text{ for all } S \subseteq E\},$$

where $x(S) = \sum_{s \in S} x(s)$ is the sum of components of the vector x with index in S [Edm70].

With the polymatroid (E, r) we associate the *integral polymatroid*

$$\{x \in \mathbb{N}^E \mid x(S) \leq r(S) \text{ for all } S \subseteq E\}.$$

By adding a suitable modular function to the rank function of a polymatroid if necessary, we can always assume that r is normalized in the sense $r(\emptyset) = 0$ and monotone increasing in the sense

$$r(S) \leq r(T) \quad \text{for all } S \subseteq T \subseteq E.$$

It is known (cf.[KV01]) that the following polymatroid greedy algorithm solves the optimization problem

$$\max_{x \geq 0} \{w^T x \mid x(S) \leq r(S) \text{ for all } S \subseteq E\}$$

for any weight vector $w \in \mathbb{R}^E$ and normalized, monotone increasing rank function r.

POLYMATROID GREEDY ALGORITHM:
Order $E = \{e_1, \dots, e_n\}$ so that $w(e_1) \geq \dots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \dots \geq w(e_n)$;
if $k \geq 1$ **then**
  $x(e_1) = r(\{e_1\})$;
  for $i = 1, \dots, k$ **do**
    $x(e_i) = r(\{e_1, \dots, e_i\}) - r(\{e_1, \dots, e_{i-1}\})$;
  end for
  for $i = k+1, \dots, n$ **do**
    $x(e_i) = 0$;
  end for
end if

It follows by construction that the resulting vector x is integral if r is integral. Comparing the rank function characterization of matroids, we observe that integral polymatroids generalize matroids by lacking the unit-increase property.

2.3.2 Integral strong exchange structures

Strong exchange structures have recently been generalized to integral vectors by Shenmaier: Let $\mathcal{F} \subseteq \mathbb{N}^n$ be a family of non-negative integer vectors, and $\mathcal{B}(\mathcal{F})$ be the set of maximal vectors in $\mathcal{F}$. (A vector $x \in \mathcal{F}$ is *maximal* if for each $y \in \mathcal{F}$ we have that $x \leq y$ implies $x = y$.) In [She04], Shenmaier considers the optimization problem

$$\max\{f(x) \mid x \in \mathcal{B}(\mathcal{F})\},$$

where $f : \mathbb{N}^n \to \mathbb{R}$ is a separable concave function. Recall that f is (discrete) separable concave if $f(x) = \sum_{k=1}^n f_k(x_k)$ and each f_k satisfies

$$a \leq b \leq c \quad \text{implies} \quad f_k(a) + f_k(c) \leq 2 f_k(b).$$

Given a vector $x \geq 0$ and a family $\mathcal{F}$ of integer vectors, Shenmaier defines the sets

$$I(x) := \{k \mid x_k > 0\} \text{ and}$$

$$J(x) := \{k \mid x + e_k \in \mathcal{F}\},$$

where $e_k \in \{0,1\}^n$ is, as usual, the vector with a 1 exactly in component k. The collection

$$\mathcal{A}(\mathcal{F}) := \{e_{k_1} + \ldots + e_{k_i} \mid e_{k_1} + \ldots + e_{k_s} \in \mathcal{F} \text{ for } s \leq i\} \cup \{0\}$$

denotes the accessible part of $\mathcal{F}$.

Shenmaier seeks to characterize families $\mathcal{F} \subseteq \mathbb{N}^n$ such that the following greedy algorithm works correctly for any separabel concave function.

GREEDY-ALGORITHM (for integer vectors):
$x^* = 0$;
while $J(x^*) \neq \emptyset$ **do**
Choose $k \in J(x^*)$ such that $f(x^* + e_k) = \max_{j \in J(x^*)} f(x^* + e_j)$;
$x^* = x^* + e_k$;
end while

And he succeeds by proving

Theorem 2.2 ([She04]). *Let $\mathcal{F} \subseteq \mathbb{N}^n$ be a non-empty finite family of integer vectors. Then the greedy algorithm above finds an optimal solution for any separabel concave function $f : \mathbb{N}^n \to \mathbb{R}$ if and only if the following two conditions hold:*

$$\begin{array}{ll} (Sh1) & x \in \mathcal{A}(\mathcal{F}) \setminus \mathcal{B}(\mathcal{F}) \Rightarrow J(x) \neq \emptyset. \\ (Sh2) & x \leq y, x \in \mathcal{A}(\mathcal{F}), y \in \mathcal{B}(\mathcal{F}), i \in J(x) \setminus I(y-x) \\ & \Rightarrow \exists j \in J(x) \cap I(y-x) \textit{ with } y + e_i - e_j \in \mathcal{F}. \end{array}$$

For the special case of set systems $\mathcal{F} \subseteq \{0,1\}^n$, it is not hard to see that $(Sh1)$ and $(Sh2)$ imply $\mathcal{F}$ to be a greedoid, and that property $(Sh2)$ is equivalent to the strong exchange property for set systems (cf. Chapter 4). Accordingly, we call a non-empty family of integral vectors $\mathcal{F} \subseteq \mathbb{N}^n$ satisfying properties $(Sh1)$ and $(Sh2)$ an *integral strong exchange structure.*

2.3.3 Jump systems

As a generalization of Δ-matroids, Bouchet and Cunningham [BC95] introduced *jump systems* which have been popularized by results of Lovász [Lov97]. Jump systems are defined as follows (*cf.* [Gee96]):

Let $V = \{1, \ldots, n\}$. For $x, y \in \mathbb{Z}^V$ define

$$[x, y] := \{x' \in \mathbb{Z}^V \mid \min\{x_i, y_i\} \leq x'_i \leq \max\{x_i, y_i\}, \forall i \in V\}$$

and $d(x,y) := \sum_{i\in V} |x_i - y_i|$. Then x' is a step from x to y (or an (x,y)-step) if $x' \in [x,y]$ and $d(x,x') = 1$.

A non-empty subset $\mathcal{J} \subseteq \mathbb{Z}^V$ is called a *jump system* if it satisfies the *two-step axiom* saying

"Given $x, y \in \mathcal{J}$ and an (x,y)-step x',
either $x' \in \mathcal{J}$, or there exists an (x',y)-step x'' such that $x'' \in \mathcal{J}$.

We observe that, in case of set systems $\mathcal{F} \subseteq \{0,1\}^n$, jump systems reduce to Δ-matroids.

Beside many other properties, it has been shown that the reflection of a jump system $\mathcal{J} \subseteq \mathbb{Z}^V$ on coordinate $i \in V$, which is the set obtained by negating x_i in each $x \in \mathcal{J}$, is a jump system again.

Therefore, when searching for a vector $x \in \mathcal{J}$ maximizing a function $\sum_{i\in V} w_i x_i$, we can apply reflection on the coordinates with negative weight and consider the non-negative weight function $|w|$ instead of $w \in \mathbb{R}^n$. Thus, we may assume that w is non-negative and that the elements in V are ordered via $w_1 \geq \ldots \geq w_k = 0 = w_{k+1} \ldots = w_n$.

It is known (*cf.* [Gee96]) that the following greedy-type algorithm determines a set of members of maximal weight $\mathcal{J}^k \subseteq \mathcal{J}$ for any jump system $\mathcal{J} \subseteq \mathbb{Z}^V$ and any non-negative linear weight function $w \in \mathbb{R}^n_+$:

GREEDY-ALGORITHM for jump systems:
  $\mathcal{J}^0 = \mathcal{J}$;
  for $i = 1, \ldots, k$ **do**
    $\alpha_i = \max\{x_i \mid x \in \mathcal{J}^{i-1}\}$;
    $\mathcal{J}^i = \{x \in \mathcal{J}^{i-1} \mid x_i = \alpha_i\}$;
  end for

So far, we got to know matroid-generalizing structures defined on the Boolean lattice 2^E and on the integral lattice $\mathbb{Z}^n$. The Boolean and the integral lattice belong to the class of distributive lattices:

2.4 Distributive lattices and Birkhoff's Theorem

A lattice $L = (L, \leq, \wedge, \vee)$ is distributive if any three elements $x, y, z \in P$ satisfy the distributive law
$$x \wedge (y \vee z) = (x \vee y) \wedge (x \vee z),$$
or equivalently
$$x \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z).$$

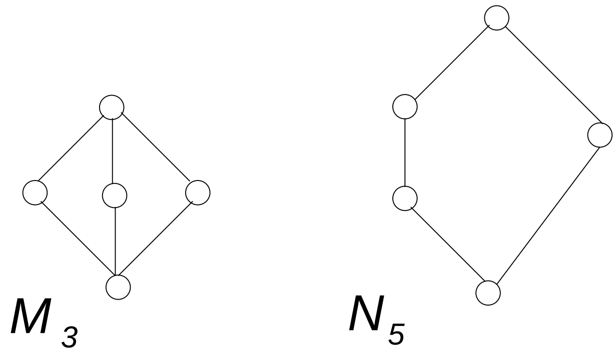


Figure 2.7: Subgraphs, forbidden in distributive lattices.

Interestingly, Hasse graphs of distributive lattices can be characterized by exlusion of the two subgraphs M_3 and N_5 shown in Figure 2.7. The proof of this characterization or any other result about lattices stated in this Chapter, can be found in Birkhoff's book "Lattice Theory" [Bir67].

Let $P = (P, \leq)$ denote the poset of join-irreducible elements of $L = (L, \leq, \wedge, \vee)$ induced by the ordering of L. By Birkhoff's Theorem [Bir67], a finite distributive lattice $L = (L, \leq, \wedge, \vee)$ is isomorphic to the lattice of ideals $\mathcal{L}(P) = (\mathcal{L}(P), \subseteq, \cap, \cup)$ via

$$\begin{aligned} a \in L &\simeq \{x \in P \mid x \leq a\} \in \mathcal{L}(P) \\ A \in \mathcal{L}(P) &\simeq \vee\{a \mid a \in A\}. \end{aligned}$$

See for example the two isomorphic distributive lattices L and $\mathcal{L}(P)$ shown in Figure 2.8.

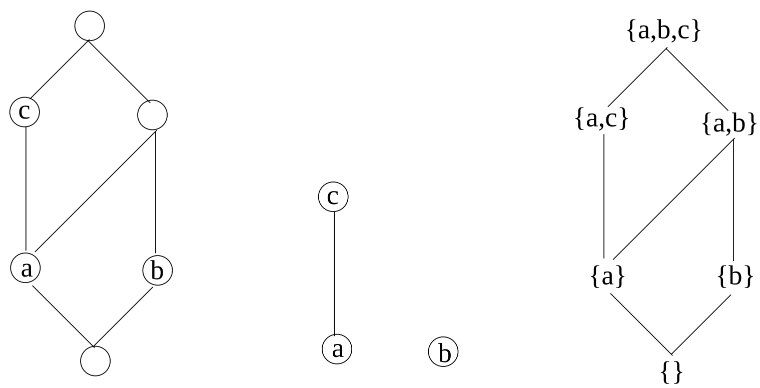


Figure 2.8: Distributive lattice $(L, \leq, \vee\wedge)$ with poset on join-irreducible elements $(P, \leq)$ and isomorphic lattice of ideals $\mathcal{L}(P) = (\mathcal{L}(P), \subseteq, \cap, \cup)$.

2.5 Ideal systems

Since distributive lattices allow a representation by subsets of their join-irreducible elements, it is a quite natural idea to extend results on Boolean lattices and integral lattices to distributive lattices in general. This way, we consider ideal systems $\mathcal{F} \subseteq \mathcal{L}(P)$ of some partially ordered sets $P = (P, \leq)$ instead of set systems $\mathcal{F} \subseteq 2^E$ of some unordered set E.

We note that ideal systems may be viewed as generalizations and, at the same time, as specializations of set systems: On the one hand, ideal systems specialize set systems, since ideals are subsets of P. On the other hand, ideal systems generalize set systems, since $\mathcal{L}(P)$ is just the Boolean lattice in case $(P, \leq)$ is an antichain. We interprete ideal systems as generalizations of set systems.

We further note that ideal systems generalize systems of integer vectors: In case $P = (P, \leq)$ consists of n disjoint chains $P = C_1 \ \dot\cup \ \ldots \ \dot\cup \ C_n$, the lattice $\mathcal{L}(P)$ is isomorphic to the lattice of integer vectors $\mathbb{N}^n$, since any non-negative integer vector $x = (x_1, \ldots, x_n)^T$ corresponds to the ideal consisting of the union of the first x_i elements of chain C_i for each $i = 1, \ldots, n$.

2.5.1 Notations

Given an ideal system $\mathcal{F} \subseteq \mathcal{L}(P)$ with rank function

$$r_{\mathcal{F}}(X) = \max\{|A| \mid A \subseteq X, A \in \mathcal{F}\},$$

we define a *(rank) closure operator* $\sigma_{\mathcal{F}} : \mathcal{L}(P) \to \mathcal{L}(P)$ via

$$\sigma_{\mathcal{F}}(X) := \bigcup\{A \in \mathcal{L}(P) \mid X \subseteq A, r(A) = r(X)\}.$$

An ideal is said to be *closed* if $X = \sigma_{\mathcal{F}}(X)$ holds. In case confusion is impossible, we simply write r and σ instead of $r_{\mathcal{F}}$ and $\sigma_{\mathcal{F}}$.

An ideal $B \in \mathcal{F}$ is a *basis* of $\mathcal{F}$ if B is maximal in $\mathcal{F}$, i.e., if

$$A \in \mathcal{F}, B \subseteq A \quad \text{implies} \quad A = B.$$

Accordingly, for each ideal $X \in \mathcal{L}(P)$ we call any maximal feasible ideal B with $B \subseteq X$ a basis of X. We denote by $\mathcal{B}(\mathcal{F})$ the collection of all bases of $\mathcal{F}$.

An ideal $C \in \mathcal{L}(P) \setminus \mathcal{F}$ is a *circuit* if

$$i \in C^+ \quad \text{implies} \quad C \setminus i \in \mathcal{F},$$

where C^+ denotes the set of maximal elements of $C \subseteq P$ with respect to the order $(P, \leq)$. In line, we define C^- to be the set of minimal elements of C.

For example, consider the ideal system $\mathcal{F}$ shown in Figure 2.9. The bases of $\mathcal{F}$ are $\{x, y\}$ and $\{z\}$ with $r(\{x, y\}) = r(\mathcal{F}) = 2$ and $r(\{z\}) = 1$. Ideal $\{x, z\}$ is closed, while $\{x\} \subset \sigma(\{x\}) = \{x, z\}$ is not. Moreover, $\{x, z\}$ is the unique circuit in $\mathcal{F}$.

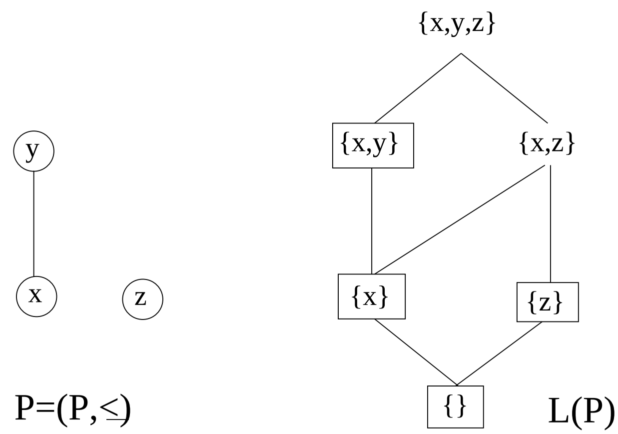


Figure 2.9: A system $\mathcal{F} \subseteq \mathcal{L}(P)$ of ideals. As before, feasible ideals are indicated by boxes.

2.5.2 Admissible weight functions

We call a weight function $w : P \to \mathbb{R}$ on P *admissible*, if

$$i \leq j \quad \text{implies} \quad w(i) \geq_{\mathbb{R}} w(j),$$

where the binary operation "$\geq_{\mathbb{R}}$" denotes the common ordering on the reals.

Note that w is always admissible in case P is an antichain. Moreover, in case P is the disjoint union of chains, the admissible weight functions correspond to separable discrete concave functions.

To see the latter, recall that the lattice of non-negative integer vectors $\mathbb{N}^n$ is isomorphic to the lattice $\mathcal{L}(P)$ of all ideals of the disjoint union of n chains $P = C_1 \dot{\cup} \ldots \dot{\cup} C_n$. I.e., a vector $x \in \mathbb{N}^n$ may be identified with an ideal $X \in \mathcal{L}(P)$ via

$$X = \{k_i \in P \mid k_i \in C_k, i \leq x_k, k = 1, \ldots, n\}.$$

Now consider a separable concave function $f : \mathbb{N}^n \to \mathbb{R}$, i.e., a function $f(x) = \sum_{k=1}^{n} f_k(x_k)$ such that each f_k satisfies

$$a \leq b \leq c \text{ implies } f_k(a) + f_k(c) \leq 2 f_k(b).$$

We assume f to be normalized such that $f(0) = 0$ and linearize f to a weight function $w : P \to \mathbb{R}$ on P via

$$w(k_i) := f_k(i) - f_k(i-1) \quad \text{for } k = 1, \ldots, n \text{ and } i \in \mathbb{N}.$$

Then we obtain for each vector $x \in \mathbb{N}^n$

$$f(x) = \sum_{k=1}^{n} f_k(x_k) = \sum_{k=1}^{n} \sum_{i=1}^{x_k} w(k_i) = \sum_{k_i \in X} w(k_i) = w(X).$$

To show that the weight function $w : P \to \mathbb{R}$ is admissible with respect to P, observe that each $k = 1, \ldots, n$ and $i \in \mathbb{N}$ satisfy

$$w(k_{i+1}) - w(k_i) = f_k(i+1) - 2f_k(i) + f_k(i-1) \leq 0.$$

In this dissertation, we introduce and investigate ideal systems for which the different greedy-type algorithms of Sections 2.2 and 2.3 work correctly for arbitrary admissible weight functions. As for set systems, a necessary condition for the optimality of the matroid greedy algorithm is the accessibility of the ideal system in question.

2.5.3 Distributive greedoids

Analogue to set systems, an ideal system $\mathcal{F} \subseteq \mathcal{L}(P)$ is *accessible* if with each feasible member $X \in \mathcal{F} \setminus \emptyset$, there exists at least one element $x \in X^+$ with $X \setminus x \in \mathcal{F}$. We call $\mathcal{F} \subseteq \mathcal{L}(P)$ a *(distributive) greedoid* if $\mathcal{F}$ is accessible and satisfies Steinitz' augmentation property

$$(AP) \quad X, Y \in \mathcal{F}, |X| < |Y| \quad \text{implies} \quad \exists y \in Y \setminus X \text{ with } X \cup y \in \mathcal{F}.$$

Let us observe that (AP) is a necessary and sufficient condition for the bases of an accessible ideal system to have constant cardinality:

Lemma 2.1. *An accessible ideal system $\mathcal{F} \subseteq \mathcal{L}(P)$ satisfies the augmentation property (AP) if and only if for each ideal $X \in \mathcal{L}(P)$ all bases of X have the same cardinality.*

Proof. Suppose that for some ideal $X \in \mathcal{L}(P)$ two bases of X have different cardinalities. Then (AP) implies that the smaller one can be augmented, which leads to a contradiction to the definition of a basis.

Conversely, let $A, B \in \mathcal{F}$ with $|A| < |B|$ and assume that A cannot be augmented from B. Then A is a basis of $A \cup B$ and B is contained in some basis of $A \cup B$ which is strictly larger than A, contradicting the assumption. □

2.6 Known generalizations of matroids to ordered sets

As in the unordered case, the property to be a distributive greedoid is only necessary but not sufficient for an ideal system to guarantee the optimality of the greedy algorithm.

We therefore need additional properties characterizing ideal systems which admit the different greedy algorithms. For example, adding the monotony-property leads to *distributive supermatroids* for which the matroid greedy algorithm determines a member of maximal weight for every admissible weight function.

Other generalizations of matroids to ordered sets, like *ordered matroids*, *submodular systems* or *lattice polyhedra* rather extend the rank function characterization. Let us recall the mentioned systems.

2.6.1 Distributive supermatroids

An ideal system $\mathcal{F} \subseteq \mathcal{L}(P)$ is called *monotone* if for each $A \in \mathcal{F}$ and all $i \in A^+$ we have $A \setminus i \in \mathcal{F}$.

In the special case of antichains P, the monotony-property is equivalent to the monotony-property of set systems saying that with each $X \subseteq \mathcal{F}$ all subsets of X must be feasible as well. Recall that matroids are monotone set systems satisfying Steinitz' augmentation property (AP).

As a generalization of matroids and integral polymatroids, Dunstan, Ingelton and Welsh [DIW72] introduced the concept of *distributive supermatroids*. In our terminology, a distributive supermatroid is a monotone ideal system $\mathcal{F} \subseteq \mathcal{L}(P)$ of some partial order $(P, \leq)$, which satisfies Steinitz' augmentation property (AP).

We observe that a distributive supermatroid is just a common matroid, in case no two elements in P are comparable. Hence, matroids have been generalized to distributive supermatroids by simply considering ideal systems instead of unordered set systems.

Many properties of matroids can be adapted to distributive supermatroids. For example, distributive supermatroids can be characterized by a rank function $r : \mathcal{L}(P) \to \mathbb{N}$ satisfying for all $X, Y \in \mathcal{L}(P)$, $y \in (P \setminus X)^-$, $z \in (P \setminus (X \cup y))^-$ with $x < y$ the properties

1. $0 \leq r(X) \leq |X|$,
2. $X \subseteq Y$ implies $r(X) \leq r(Y)$,
3. $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$, and
4. $r(X \cup y) - r(X) \geq r(X \cup \{y, z\}) - r(X \cup y)$.

We note that in case P is the disjoint union of chains, distributive supermatroids correspond to integral polymatroids.

Distributive supermatroids have been investigated in literature: For example, Faigle [**?**] could prove that the matroid greedy algorithm and its dual Monge algorithm work well for distributive supermatroids whenever the weight function on P is *admissible*. Tardos [Tar90] showed a matroid-type intersection theorem for distributive supermatroids, and Peled and Srinivasan [PS93] considered a generalization of the matroid independent matching problem for distributive supermatroids. Further, Barnabai, Nicoletti and Pezzoli [BNP98],[BNP93] studied distributive supermatroids in more detail. Fujishige, Koshevoy and Sano [FKS06] generalized distributive supermatroids to so-called *cg-matroids*, which are matroidal structures having convex geometries, instead of partially ordered sets, as underlying structure. A general framework containing distributive supermatroids is presented in [Fuj91].

2.6.2 Ordered matroids

Faigle [Fai79] described a greedy algorithm on posets $P = (P, \leq)$ and introduced certain *sequential families* such that the greedy algorithm works optimally for every admissible weight function. These sequential families then give rise to a rank function on the distributive lattice $\mathcal{L}(P)$ of all ideals of P, which determines *ordered matroids*.

An *ordered matroid* is an ideal system $\mathcal{F} \subseteq \mathcal{L}(P)$ together with a rank function $r : \mathcal{L}(P) \to \mathbb{N}$ satisfying

$$
\begin{array}{ll}
(R_0) & r(\emptyset) = 0, \\
(R_1) & A \subseteq B \text{ implies } 0 \leq r(B) - r(A) \leq |B \setminus A|, \\
(R_2) & r(A \cup B) + r(Y \cap B) \leq r(A) + r(B).
\end{array}
$$

The rank function characterization directly implies that ordered matroids generalize distributive supermatroids. Faigle proved that the matroid greedy algorithm determines a member of an ordered matroid for every admissible weight function.

Moreover, he has shown that the optimization problem of admissible weight functions on ordered matroids is in fact equivalent to the optimization problem of linear functions over polymatroids by extending r to a function $\bar{r} : 2^P \to \mathbb{N}$ defined on all subsets of P via

$$\bar{r}(A) = r(\tilde{A}) \quad \text{for all } A \subseteq P$$

where $\tilde{A} = \{p \in P \mid p \leq a \text{ for some } a \in A\}$ is the ideal generated by the subset A of P.

2.6.3 Submodular systems

We have seen that integral polymatroids generalize matroids by abstaining from the unit-increase property of the rank function. The axiomatic requirements can be still more relaxed:

Instead of all subsets of a set E, we consider the distributive lattice $\mathcal{L}(P) = (\mathcal{L}(P), \subseteq, \cup, \cap)$ of all ideals of a general poset $P = (P, \leq)$:

Let r be an integer-valued normalized submodular function on $\mathcal{L}(P)$, i.e., a function $r : \mathcal{L}(P) \to \mathbb{Z}$ satisfying

1. $r(\emptyset) = 0$, and

2. $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$ for all $X, Y \in \mathcal{L}(P)$.

Then the pair $(\mathcal{L}(P), r)$ is called a *submodular system* on $\mathcal{L}(P)$ with rank function r. Associated with the submodular system $(\mathcal{L}(P), r)$ is the *submodular polyhedron*

$$\{x \in \mathbb{R}^E \mid x(A) \leq r(A) \text{ für alle } A \in \mathcal{L}(P)\}.$$

For example, distributive supermatroids and ordered matroids are submodular systems.

It is known that a primal and dual greedy algorithm determine an optimal ideal of a submodular system for any admissible weight function (cf. Chapter 3). It should be noted that Faigle proved that the theory of submodular systems can be developed within the framework of integral polymatroids (cf. [Fai87]).

2.6.4 Lattice polyhedra and modular clutters

Analyzing the matroid greedy algorithm in the setting of linear programming, Edmonds [Edm70] showed that it can be generalized to integral polymatroids. Queyranne, Spieksma and Tardella [QST98] extended integral polymatroids to a model, which allows a common framework for the matroid greedy algorithm and the dual Monge algorithm. Their model is included by the model of Faigle and Kern [FK96] for optimizing linear functions under submodular constraints relative to antichains of rooted forests. The combinatorial models mentioned above are subsumed in Faigle and Kern's model of modular functions on posets [FK00b].

Even more general, Hoffman introduced *sub- and supermodular clutter systems* as a special type of *lattice polyhedra* by allowing an order structure on the feasible sets that need not coincide with the "natural" set-theoretic ordering by containment.

Lattice polyhedra

A function $f : L \to \{0, 1, -1\}$ defined on a pseudolattice $L = (L, \leq, \vee, \wedge)$ is *consecutive* if for $a, b, c \in L$ holds

$$a < b \quad \Rightarrow \quad f(a)f(b) \geq 0, \qquad \text{and}$$

$$a < b < c, f(a)f(c) > 0 \quad \Rightarrow \quad f(a) = f(b) = f(c).$$

Let A be a $(0, 1, -1)$-matrix whose rows are indexed by a pseudolattice L and interpret the columns of A as $(0, 1, -1)$-valued functions on L. If the columns of A are consecutive and supermodular and $r : L \to \mathbb{Z}$ is submodular, then the polyhedron

$$Q \equiv \{x \mid Ax \geq r\}$$

is called a *lattice polyhedron*.

If we interchange the terms sub- and supermodular, then

$$\bar{Q} = \{x \mid Ax \leq r\}$$

is the corresponding lattice polyhedron.

Though Hoffman [Hof82] could prove that lattice polyhedra are totally dual integral (i.e., that the dual problem $\min\{y^T r \mid y^T A = w^T, y \geq 0\}$ has an integral optimal solution for each weight function w) so far no greedy-type algorithm is known for lattice polyhedra in general.

But in case A is a $(0,1)$-matrix, we obtain the definition of lattice clutters which allow certain greedy algorithms to work optimally.

Modular clutter systems

If A is a $(0,1)$-matrix, we may regard the rows of A as incidence vectors of a pseudolattice $\mathcal{F} \subseteq 2^U$ whose elements are subsets of the column set U.

Hoffman called the family $\mathcal{F}$ a *sub-, resp. supermodular clutter*, if each column of A is consecutive and sub-, resp. supermodular. A *modular clutter* is both, sub- and supermodular.

Note that in this context, the consecutive property is equivalent to

$$(C1) \quad A < B < C \quad \Rightarrow \quad A \cap C \subseteq B \quad \text{for all } A, B \in \mathcal{F}.$$

Let $f_u : \mathcal{F} \to \{0,1\}$ be the incidence function with $f_u(A) = 1 \Leftrightarrow u \in A$. Then the modularity property is equivalent to

$$(M) \quad f_u(A) + f_u(B) = f_u(A \vee B) + f_u(A \wedge B) \quad \text{for all } A, B \in \mathcal{F}, u \in U.$$

The pair $(\mathcal{F}, r)$ is a *submodular clutter system* if r is submodular and $\mathcal{F}$ is a supermodular clutter satisfying the additional property

$$(C0) \quad A < B \quad \Rightarrow \quad B \setminus A \neq \emptyset.$$

Analogue, $(\mathcal{F}, r)$ is a *supermodular clutter system* if the terms sub- and supermodularity are interchanged. System $(\mathcal{F}, r)$ is a *modular clutter system* if r is sub- or supermodular and $\mathcal{F}$ is a modular clutter satisfying $(C0)$.

For example, the lattice of ideals $\mathcal{L}(P) = (\mathcal{L}(P), \subseteq, \cap, \cup)$ is a modular clutter satisfying $(C0)$. Hence, modular clutter systems generalize submodular systems.

Frank [Fra99] could provide a two-phase greedy algorithm for supermodular clutter systems and monotone increasing functions r. In [FP06a], we established a game theoretic interpretation of supermodular clutter systems where Frank's algorithm is applied.

Recently, Dietrich and Hoffman [DH03] proved that a Monge-type algorithm works optimally for modular clutter systems. We complemented their result by proving the optimality of a corresponding primal greedy algorithm in [FP06b]. We will see that modular clutter systems fit into the framework of submodular systems and thus of ordinary polymatroids.

2.7 Outline of Part I

In the following Chapter 3, we reduce modular clutter systems to submodular systems: We first show that a pseudolattice $\mathcal{F}$ of a modular clutter system $(\mathcal{F}, U, r)$ is in fact a distributive lattice which, by Birkhoff's Theorem, is isomorphic to the lattice of ideals $\mathcal{L}(P)$ of the poset $P = (P, \leq)$ on the join-irreducible elements of $\mathcal{F}$. Thereafter, given a weight function $w : U \to \mathbb{R}$ on U, we construct a weight function $c : P \to \mathbb{R}$ on P such that the optimization problem on the modular clutter system is equivalent to the optimization problem on the submodular system. Since submodular systems can be reduced to polymatroids (cf. [Fai87]), we therefore prove that modular clutter systems are essentially the same as polymatroids.

We got to know distributive supermatroids as a special type of submodular systems which can be defined as monotone ideal systems satisfying the augmentation property (AP). Moreover, distributive supermatroids can be characterized algorithmically as exactly those ideal systems $\mathcal{F} \subseteq \mathcal{L}(P)$ for which the matroid greedy algorithm determines an optimal member for each non-negative admissible weight function $w : P \to \mathbb{R}$. In case of negative components of w, the monotony and admissibility allows us to restrict to elements of P with non-negative weight.

Hence, matroids have been extended to distributive supermatroids by considering ideal systems instead of unordered set systems. Likewise, we generalize strong exchange structures, Gauss greedoids and Δ-matroids from set systems to ideal systems:

In Chapter 4 we introduce *distributive strong exchange structures* as distributive greedoids satisfying a certain distributive strong exchange property. We prove that distributive strong exchange structures characterize distributive greedoids for which the matroid greedy algorithm determines an optimal basis for every admissible weight function. We further reduce the results of Goetschel and Shenmaier on set systems and systems of integral vectors.

Since distributive strong exchange structures are not necessarily monotone, an optimal basis might not be an optimal member as long as negative weights are allowed. In order to handle arbitrary admissible weight functions, we introduce distributive Gauss greedoids as the collection of bases of nested distributive supermatroids in a certain strong map relation in Chapter 5. We show that they are a special type of distributive strong exchange structures and form a common generalization of Goecke's Gauss greedoids and distributive supermatroids. Further, we prove an exchange property for distributive Gauss greedoids, which yields an algorithmic characterization: distributive Gauss greedoids are exactly those ideal systems for which the modified greedy algorithm determines an optimal member for arbitrary admissible weight functions.

If the ideal system is not even accessible, it is still possible to determine a member of optimal weight. In Chapter 6 we introduce distributive Δ-matroids and prove that a certain Δ-*greedy algorithm* returns a collection of optimal members for each admissible non-negative weight function. We will see that distributive Δ-matroids generalize jump systems and ordinary Δ-matroids.

2.8 Inclusion charts

The following inclusion chart 2.10 provides an overview about the hierarchy of the considered structures. We note that, except for general lattice polyhedra, sub- and supermodular clutter systems, all structures are known to be accompanied by greedy-type algorithms.

In the subsequent inclusion chart 2.11, we take the structural equivalence of modular clutter systems and polymatroids into account.

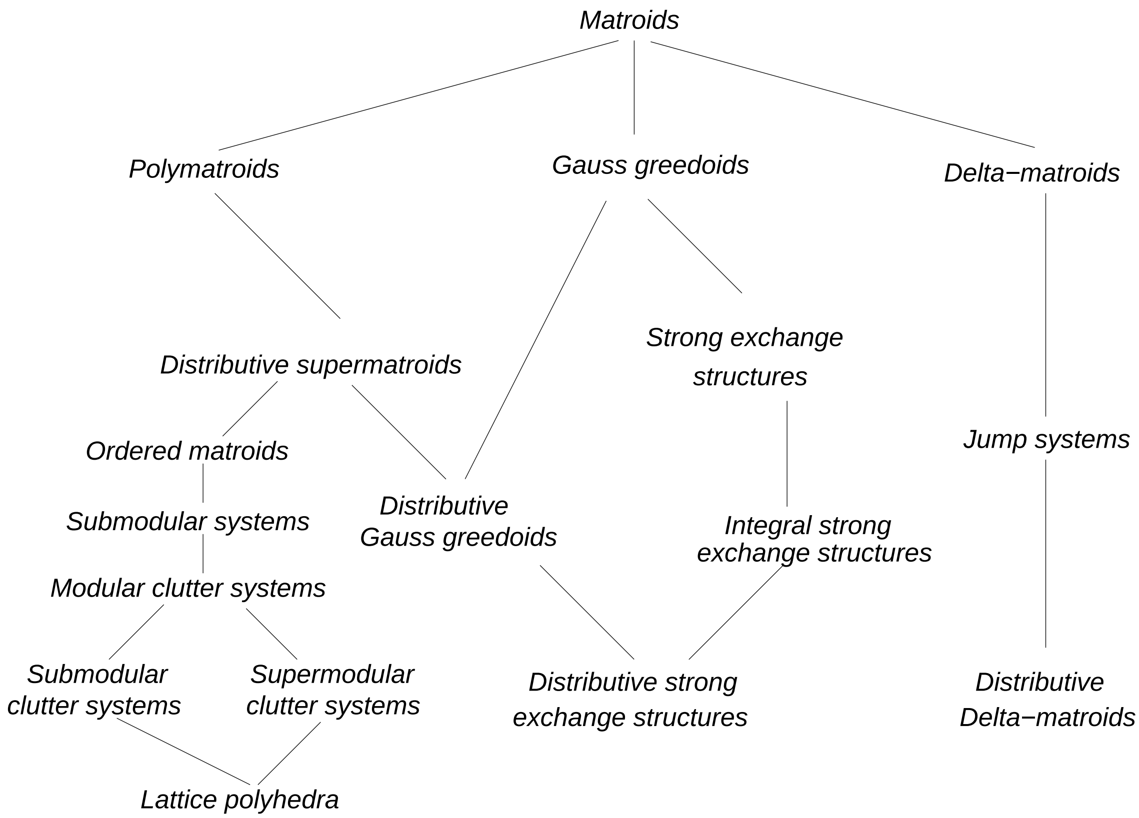


Figure 2.10: Inclusion chart of the considered generalizations of matroids. The structures become more general from the top to the bottom.

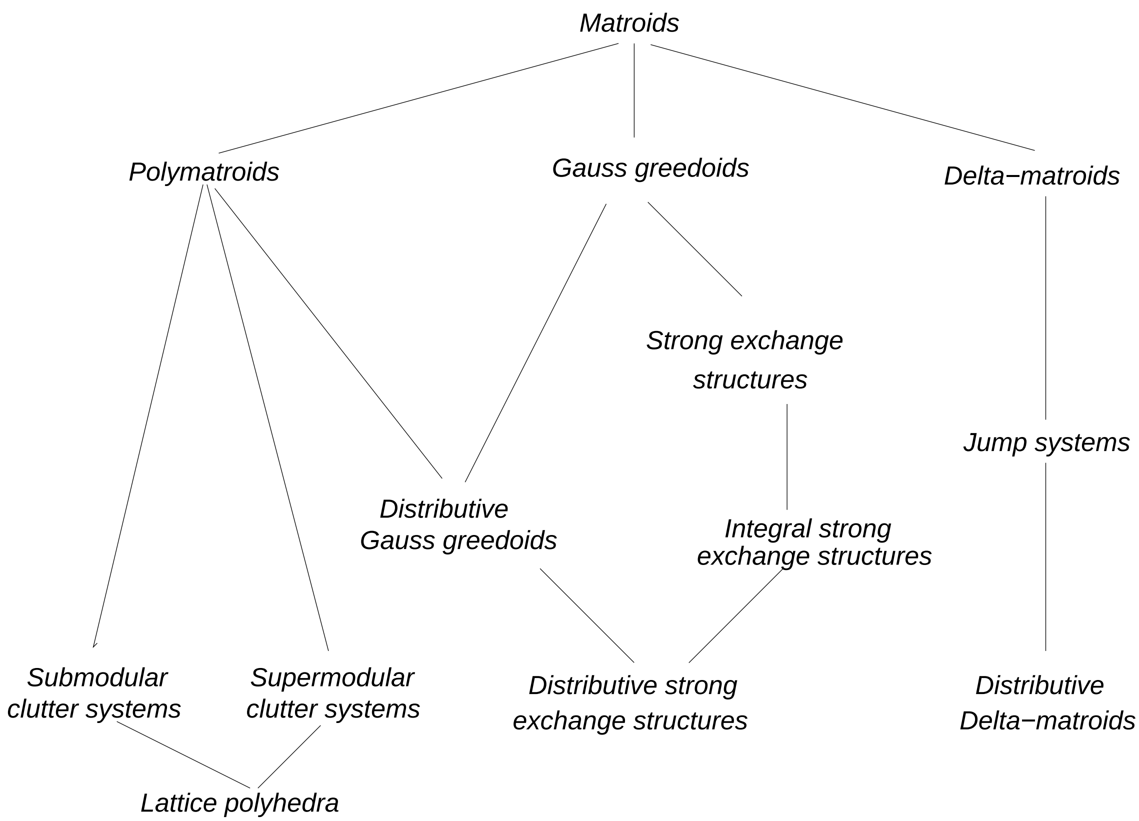


Figure 2.11: Inclusion chart with respect to the equivalence of modular clutters and polymatroids.

Chapter 3

Modular clutter systems and submodular systems

As a common generalization of submodular systems, the chain-product model of Queyranne, Spieksma, Tardella [QST98] and Faigle and Kern's model of modular functions on posets [FK00b], Hoffman and Dietrich proved the optimality of a Monge algorithm on modular clutter systems [DH03].

In this Chapter, we reduce modular clutter systems to submodular systems. Since submodular systems were shown to be structurally equivalent to integral polymatroids [Fai87] we therefore prove that Dietrich and Hoffman's model in fact fits into the framework of ordinary polymatroids.

Let us recall the definition of modular clutter systems. In order to avoid too much confusion in the subsequent proofs, we use a slightly different notation than we used before.

Recall that $L = (L, \leq, \vee, \wedge)$ is a pseudolattice if for any two elements $x, y \in L$ there exist two designated elements $x \vee y, x \wedge y \in L$ such that

$$x \wedge y \leq x, y \leq x \vee y.$$

Definition 3.1 (Modular clutter systems). *Let U be a finite set. A* modular clutter system (L, f, r) *consists of a pseudolattice $L = (L, \leq, \vee, \wedge)$, a sub- or supermodular function $r : L \to \mathbb{R}$, and a function $f : L \to 2^U$ such that*

$$f_u(a) = 1 \iff a \in f(u) \quad \textit{for all } u \in U, a \in L,$$

satisfying for all $u \in U, a, b \in L$

$$\begin{array}{ll} (C0) & a < b \quad \textit{implies} \quad f(b) \setminus f(a) \neq \emptyset, \\ (C1) & a < b < c, f_u(a) = f_u(c) = 1 \quad \textit{implies} \quad f_u(b) = 1, \\ (M) & f_u(a) + f_u(b) = f_u(a \vee b) + f_u(a \wedge b). \end{array}$$

We might assume that $f(m_0) = \emptyset$ for the minimal element $m_0 \in L$ (otherwise we simply add an empty set as minimal element to L), and that r is normalized in the sense $r(m_0) = 0$.

Throughout, we suppose that r is submodular and a non-negative weight function $w : U \to \mathbb{R}_+$ on U is given. Then $\{x \mid \sum_{u \in U} x_u f_u(a) \leq r(a), \forall a \in L\}$ is a lattice polyhedron which has been shown to be totally dual integral (cf.[Hof82]).

Hoffman and Dietrich [DH03] proved inductivly that a certain Monge-type algorithm determines an optimal solution of problem

$$(D') \qquad \min_{y \geq 0} \{\sum_{a \in L} r(a)y(a) \mid \sum_{a \in L} y(a) f_u(a) = w(u), \forall u \in U\},$$

provided (D') is feasible.

In our note [FP06b], we present a Monge-type algorithm together with a primal greedy algorithm that determine optimal solutions for the primal-dual pair of problems

$$\begin{aligned} (P) \qquad & \max_{x \geq 0} \{\sum_{u \in U} w_u x_u \mid \sum_{u \in U} x_u f_u(a) \leq r(a), \forall a \in L\}, \text{ and} \\ (D) \qquad & \min_{y \geq 0} \{\sum_{a \in L} r(a)y(a) \mid \sum_{a \in L} y(a) f_u(a) \geq w(u), \forall u \in U\} \end{aligned}$$

Since the optimal solution of (D) turns out to be an optimal solution of (D') in case (D') is feasible, we therefore complemented Dietrich and Hoffman's Monge algorithm by the corresponding primal greedy algorithm.

In the following Section, we prove that the pseudolattice L in a modular clutter system (L, f, r) is in fact a distributive lattice. Using this result, thereafter we reduce submodular clutter systems to submodular systems $(\mathcal{L}(P), r)$, where $P = (P, \leq)$ is the poset on the join-irreducible elements of L.

3.1 Equivalence to distributive lattices

Recall that a pseudolattice L is a proper lattice, if for each $a, b \in L$ there exists a unique minimal upper bound

$$\sup(a, b) = \min\{c \in L \mid c \geq a, b\}.$$

Together with $\sup(a, b)$, we know that in a lattice there exists a unique maximal lower bound

$$\inf(a, b) = \max\{c \in L \mid c \leq a, b\}.$$

We first prove that, if (L, f, r) is a modular clutter system, the pseudolattice L is a lattice with $a \vee b = \sup(a, b)$ and $a \wedge b \leq \inf(a, b)$. Thereafter, we show that $(L, \leq, \inf, \vee)$ is a distributive lattice.

Lemma 3.1. *Let $L = (L, \leq, \vee, \wedge)$ be a pseudolattice together with a map $f : L \to 2^U$ satisfying properties $(C0), (C1)$ and (M). Then L is a lattice with $a \vee b = \sup(a, b)$ for all $a, b \in L$.*

Proof. We claim that $\sup(a, b)$ exists and equals $a \vee b$ for all $a, b \in L$. So consider any $c \geq a, b$. We must show that $c \geq a \vee b$ is true. Suppose this is not the case and let $d = a \vee b$. Then we have

$$a, b \leq c \leq c \vee d.$$

By $(C0)$, there exists some $u \in f(c \vee d) \setminus f(c)$. The modularity (M) then implies $u \in f(d)$. In turn, $d = a \vee b$ implies $u \in f(a) \cup f(b)$. In view of $u \notin f(c)$, on the other hand, the consecutive property $(C1)$ yields $u \notin f(a) \cup f(b)$, which is a contradiction. □

Figure 3.1 is an example of a lattice $L = \{a, b, c, a \vee b, a \wedge b\}$, a set $U = \{u_1, u_2, u_3\}$ and a map $f : L \to 2^U$ such that $a \wedge b < \inf(a, b)$.

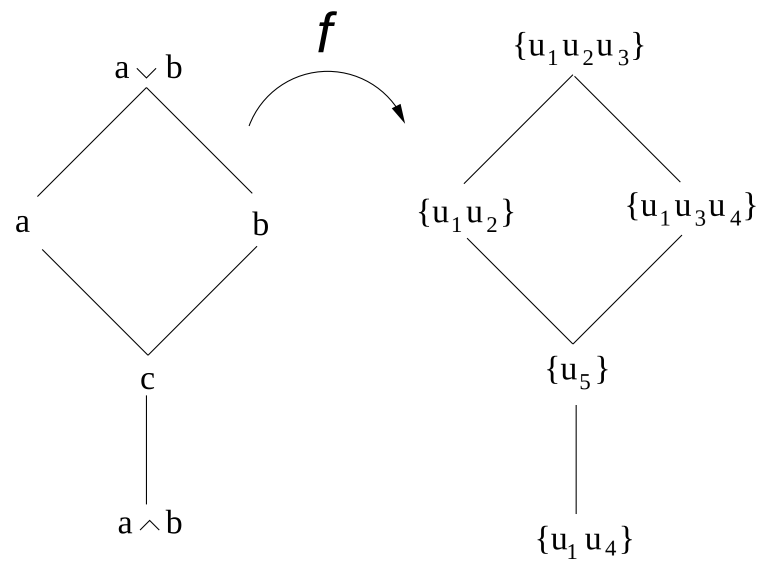


Figure 3.1: Lattice with $a \wedge b < c = \inf(a, b)$.

Lemma 3.2. *Let $L = (L, \leq, \vee, \wedge)$ be a lattice together with a map $f : L \to 2^U$ satisfying properties $(C0), (C1)$ and (M). Then L is a distributive lattice.*

Proof. It is well-known [Bir67] that a lattice is distributive if and only if it contains no sublattice isomorphic to an N_5 or an M_3 as shown in Figure 3.2.

Suppose for the contrary there exists a sublattice $N_5 = \{a, b, c, d, e\} \subseteq L$ such that $b < c, e = b \vee d = c \vee d$ and $a = \inf(b, d) = \inf(c, d)$.

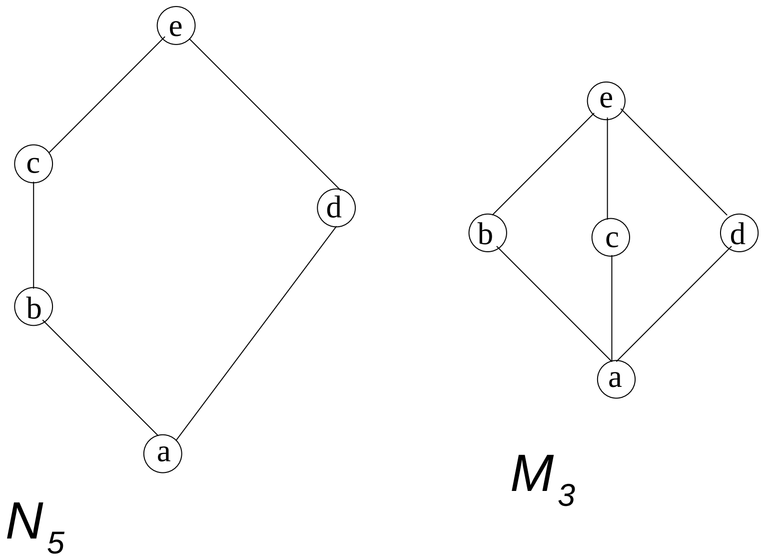


Figure 3.2: Sublattices forbidden in distributive lattices.

By $(C0)$, we may choose an element $u \in f(c) \setminus f(b)$. Property $(C1)$ implies $f_u(c \wedge d) = 0$. Hence, the modularity (M) implies $f_u(c \vee d) = 1$ and $f_u(d) = 0$. This way, $f_u(d) + f_u(b) = 0 < f_u(b \vee d) + f_u(c \vee d) = 1$ is a contradiction to (M).

Now suppose L contains a sublattice $M_3 = \{a, b, c, d, e\} \subseteq L$ such that $e = b \vee c = b \vee d = c \vee d$ and $a = \inf(b, c) = \inf(b, d) = \inf(c, d)$. Choose an element $u \in f(e) \setminus f(b)$. Property (M) implies $f_u(c) = f_u(d) = 1$ and $f_u(c \wedge d) = 1$. Therefore, we get a contradiction to property $(C1)$, since $c \wedge d \leq \inf(c, d) = a < b < e$. □

3.2 Reduction to submodular systems

Let (L, f, r) be a modular clutter system and $w : U \to \mathbb{R}_+$ be a non-negative weight function on U. W.l.o.g., let $U = \bigcup\{f(a) \mid a \in L\}$. Like in Dietrich and Hoffman's model [DH03], we consider the linear optimization problem

$$(D') \quad \min_{y \geq 0}\{\sum_{a \in L} r(a)y(a) \mid \sum_{f(a) \ni u} y(a) = w(u), \forall u \in U\}.$$

Let $P = (P, \leq)$ denote the poset on the join-irreducible elements of L and $\mathcal{L}(P)$ be the collection of ideals in P. Since $L = (L, \leq, \vee, \inf)$ is distributive, we know by Birkhoff's Theorem [Bir67] that $g : L \to \mathcal{L}(P)$ with

$$g(a) = \{i \in P \mid i \leq a\}$$

is an (order) isomorphism between L and $\mathcal{L}(P) = (\mathcal{L}(P), \subseteq, \cup, \cap)$.

Hence, the lattice L can be represented on the one hand as a lattice of subsets of U, and on the other hand, as a lattice of subsets of P (see Figure 3.3).

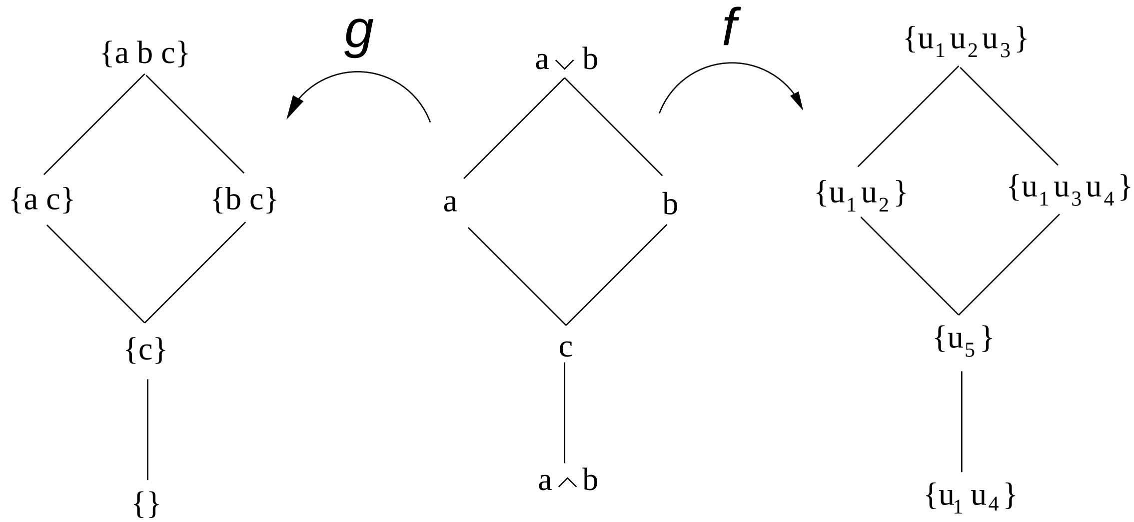


Figure 3.3: Representation of L by subsets of P resp. U.

We now reduce the modular clutter system (L, f, r) to the submodular system $(\mathcal{L}(P), r)$ by constructing a weight function $c : P \to \mathbb{R}_+$ such that

$$\{y \mid \sum_{f(a) \ni u} y(a) = w(u), \forall u \in U\} \equiv \{y \mid \sum_{g(a) \ni i} y(a) = c_i, \forall i \in P\}.$$

Recall that a listing $P = \{p_1, \ldots, p_n\}$ is a *linear extension* of $(P, \leq)$, if $i \leq j$ implies $p_i \leq p_j$ for all $i, j \in \{1, \ldots, n\}$. Since L is distributive, each linear extension $P = \{p_1, \ldots, p_n\}$ of $(P, \leq)$ corresponds to a maximal chain

$$m_0 < m_1 < \ldots < m_n \subseteq L \quad \text{with } m_i = m_{i-1} \vee i, \quad \forall i = 1, \ldots, n.$$

Let us choose such an arbitrary maximal chain $\mathcal{C} = m_0 < m_1 < \ldots < m_n$. As the modularity-property (M) is satisfied, each $u \in U$ occurs in the f-image of $\mathcal{C}$, i.e.,

$$\mathcal{C}_u = \{a \in \mathcal{C} \mid u \in f(a)\} \neq \emptyset.$$

Hence, each $u \in U$ determines a unique $i_u \in P$ such that

$$u \in f(m_{i_u}) \setminus f(m_{i_u - 1}).$$

Moreover, since f is consecutive, either $u_i \in f(m_k)$ for all $k \geq i_u$, or there exists a unique $j_u \in P$ such that

$$u \in f(m_{j_u - 1}) \setminus f(m_{j_u}).$$

Lemma 3.3. *Let L be a distributive lattice, $f : L \to 2^U$ satisfy $(C1)$ and (M), and $\mathcal{C} = m_0 < m_1 < \ldots < m_n$ be a maximal chain in L. Then $u \in f(m_{i_u}) \setminus f(m_{i_u-1})$ and $u \in f(m_{j_u-1}) \setminus f(m_{j_u})$ implies*

$$i_u < j_u.$$

Proof. For simplicity, let us replace i_u by i and j_u by j. Clearly, since $P = \{p_1, \ldots, p_n\}$ is a linear extension of P, $j \leq i$ is impossible.

We first observe that with $m_j = m_{j-1} \vee j$ and $m_i = m_{i-1} \vee i$, the modularity property (M) implies $f_u(j) = f_u(m_{i-1} \wedge i) = 0$ and $f_u(m_{j-1} \wedge j) = f_u(i) = 1$.

Now, let i' and j' denote the unique lower neighbors of i and j, respectively. Then $m_{i-1} \wedge i \leq i'$ and $m_{j-1} \wedge j \leq j'$, together with the consecutive property $(C1)$, implies $f_u(i') = 0$ and $f_u(j') = 1$.

Suppose i and j are not comparable. Then property (M) implies that either $f_u(i \wedge j) = 1$ or $f_u(i \vee j) = 1$. Since $j' < j \leq i \vee j$ is a chain with $f_u(j') = 1$ and $f_u(j) = 0$, the equality $f_u(i \vee j) = 0$ follows from $(C1)$. On the other hand, the chain $i \wedge j \leq i' < i$ with $f_u(i') = 0$ and $f_u(i) = 1$ implies $f_u(i \wedge j) = 0$. A contradiction. $\square$

Lemma 3.4. *Let L be a distributive lattice, $f : L \to 2^U$ satisfy $(C1)$ and (M), and $\mathcal{C} = m_0 < m_1 < \ldots < m_n$ be a maximal chain in L. Then*

$$u \in f(a) \quad \Leftrightarrow \quad \begin{cases} i \in g(a), j \notin g(a) & : \quad u \in f(m_k) \Leftrightarrow k = i, \ldots, j-1 \\ i \in g(a) & : \quad u \in f(m_k) \Leftrightarrow k = i, \ldots, n. \end{cases}$$

Proof. We already know from the previous proof that $f_u(j) = f_u(i') = 0$ and $f_u(i) = f_u(j') = 1$ holds for the unique lower neighbors i' and j' of i and j. Recall that for each $i \in P$ and $a \in L$, we have $i \in g(a)$ if and only if $i \leq a$.

Let $u \in f(a)$. Then, by property (M), $f_u(i) = f_u(a) = 1$ implies $f_u(a \wedge i) = 1$. Hence, $i \leq a$ (i.e., $i \wedge a = i$) follows, since otherwise the chain $a \wedge i \leq i' < i$ with $f_u(a \wedge i) = f_u(i) = 1$ and $f_u(i') = 0$ would contradict property $(C1)$.

Further, $u \in f(a)$ implies $j \not\leq a$, as otherwise the chain $j' < j \leq a$ with $f_u(j') = f_u(a) = 1$ and $f_u(j) = 0$ would contradict $(C1)$.

To prove the other direction, suppose there exists an element $a \in L$ with $i \leq a, j \not\leq a$ and $u \notin f(a)$. Choose such an element $a \in L$ minimal.

We first prove that i is maximal in the ideal $g(a)$ (i.e., that $i \in g(a)^+$). Suppose this is not true and there exists an element $k \in P$ with $k > j$ and $k \in g(a)^+$. Consider the ideal $g(a) \setminus k$ and $b \in L$ with $g(b) = g(a) \setminus k$. Since $i \leq b$ and $j \not\leq b$, it follows from the choice of a that $f_u(b) = 1$. Since $i \leq k, j \not\leq k$, it follows also from the choice of a that $f_u(k) = 1$. But this is a contradiction to the modularity property (M), as $a = k \vee b$ and $f_u(a) = 0$.

Therefore, $i \in g(a)^+$ and there exists $b \in L$ such that $g(b) = g(a) \setminus i$. Since $a = b \vee i$ with $f_u(a) = 0$ and $f_u(i) = 1$, the modularity (M) implies $f_u(b \wedge i) = 1$. But this is a contradiction to the consecutive property $(C1)$, as $b \wedge i \leq i' < i$ is a chain with $f_u(b \wedge i) = f_u(i) = 1$ and $f_u(i') = 0$. $\square$

Theorem 3.1. *Let L be a distributive lattice, $f : L \to 2^U$ satisfy $(C1)$ and (M), and $\mathcal{C} = m_0 < m_1 < \ldots < m_n$ be a maximal chain in L. Then*

$$\sum_{f(a)\ni u} y(a) \quad = \quad \begin{cases} \sum_{g(a)\ni i} y(a) - \sum_{g(a)\ni j} y(a) & : \quad u \in f(m_k) \Leftrightarrow k = i, \ldots, j-1 \\ \sum_{g(a)\ni i} y(a) & : \quad u \in f(m_k) \Leftrightarrow k = i, \ldots, n. \end{cases}$$

Proof. If $u \in f(m_k)$ for $k = i, \ldots, n$, the equality $\sum_{f(a)\ni u} y(a) = \sum_{g(a)\ni i} y(a)$ follows immediatly from Lemma 3.4.

If $u \in f(m_k)$ for $k = i, \ldots, j-1$, we know from Lemma 3.3 that $i < j$. Since the images of $g : L \to \mathcal{L}(P)$ are ideals, we observe $\{a \mid j \in g(a)\} \subseteq \{a \mid i \in g(a)\}$. Thus, $\sum_{f(a)\ni u} y(a) = \sum_{g(a)\ni i} y(a) - \sum_{g(a)\ni j} y(a)$ follows from Lemma 3.4. $\square$

By property $(C0)$, we may choose representatives $U^P = \{u_1, \ldots, u_n\} \subseteq U$ such that

$$u_i \in f(m_i) \setminus f(m_{i-1}) \quad \forall i = 1, \ldots, n.$$

Recursively, for $i = n, \ldots, 1$, we now construct a weight function $c : P \to \mathbb{R}$ on P via

$$c_i \quad = \quad \begin{cases} w(u_i) - c_j & : \quad u_i \in f(m_k) \Leftrightarrow k = i, \ldots, j-1 \\ w(u_i) & : \quad u_i \in f(m_k) \Leftrightarrow k = i, \ldots, n. \end{cases}$$

Then, by construction

$$\{y \mid \sum_{f(a)\ni u} y(a) = w(u) \forall u \in U^P\} \equiv \{y \mid \sum_{g(a)\ni i} y(a) = c_i \forall i \in P\}.$$

We further observe

Lemma 3.5. *If (D') is feasible, then $c : P \to \mathbb{R}$ is admissible.*

Proof. Consider $i, k \in P$ with $i < k$. Then $\{a \in L \mid k \in g(a)\} \subseteq \{a \in L \mid i \in g(a)\}$ implies for each non-negative $y : L \to \mathbb{R}$ that $c_i = \sum_{g(a)\ni i} y(a) \geq \sum_{g(a)\ni k} y(a) = c_k$.

$\square$

It remains to prove

Lemma 3.6. *Each equation in system $\{\sum_{f(a)\ni u} y(a) = w(u) \mid u \in U\}$ is a linear combination of equations in $\{\sum_{f(a)\ni u} y(a) = w(u) \mid u \in U^P\}$.*

Proof. For each $i \in P$ there exists a chain $i = i^0 < i^1 < \ldots < i^{k_i} \subseteq P$ together with representants

$$U^i = \{u_{i^0}, \ldots, u_{i^{k_i}}\} \subseteq U^P$$

such that

$$u_{i^t} \quad \in \quad \begin{cases} f(m_{i^t}) \setminus f(m_{i^t-1}) & : \quad t = 0, \ldots, k_i \\ f(m_{i^{t+1}-1}) \setminus f(m_{i^{t+1}}) & : \quad t = 0, \ldots, k_i - 1, \\ f(m_n) & : \quad t = k_i. \end{cases}$$

By construction of $c : P \to \mathbb{R}$, we have

$$\sum_{g(a) \ni i} y(a) = \sum_{f(a) \ni u_{i^0}} y(a) + \ldots + \sum_{f(a) \ni u_{i^{k_i}}} y(a).$$

Theorem 3.1 then implies for each $u \in U$ with $u \in f(m_k)$ for $k = i, \ldots, j-1$

$$\sum_{f(a) \ni u} y(a) = \sum_{t=0}^{k_i} \sum_{f(a) \ni u_{i^t}} y(a) - \sum_{t=0}^{k_j} \sum_{f(a) \ni u_{j^t}} y(a),$$

resp. for each $u \in U$ with $u \in f(m_k)$ for $k = i, \ldots, n$

$$\sum_{f(a) \ni u} y(a) = \sum_{t=0}^{k_i} \sum_{f(a) \ni u_{i^t}} y(a).$$

$\square$

Hence, we proved

$$\{y \mid \sum_{f(a) \ni u} y(a) = w(u) \forall u \in U\} \equiv \{y \mid \sum_{g(a) \ni i} y(a) = c_i \forall i \in P\}.$$

Therefore, we may solve the problem

$$(D') \quad \min_{y \geq 0} \{\sum_{a \in L} r(a) y(a) \mid \sum_{f(a) \ni u} y(a) = w(u), \forall u \in U\}$$

by determining an optimal solution of problem

$$(D'') \quad \min_{y \geq 0} \{\sum_{a \in L} r(a) y(a) \mid \sum_{g(a) \ni i} y(a) = c_i, \forall i \in P\}.$$

Since (L, g, r) represents the submodular system $(\mathcal{L}(P), r)$, problem (D'') can be solved with the generalized polymatroid greedy algorithm which works for submodular systems and admissible weight functions (cf. [FK96, FK00a]).

Let us summarize how the modular clutter system (L, f, r) can be reduced to the submodular system $(\mathcal{L}(P), r)$, where P denotes the join-irreducible elements of L:

Given the weight function $w : U \rightarrow \mathbb{R}_+$ on U we choose an arbitrary maximal chain $\mathcal{C} = m_0 < m_1 < \ldots < m_n$ in L and construct the weight function $c : P \rightarrow \mathbb{R}_+$ on P via

$$c_i = \begin{cases} w(u_i) - c_j & : \quad u_i \in f(m_k) \Leftrightarrow k = i, \ldots, j-1 \\ w(u_i) & : \quad u_i \in f(m_k) \Leftrightarrow k = i, \ldots, n. \end{cases}$$

We then determine an optimal solution of problem

$$(D'') \quad \min_{y \geq 0} \{ \sum_{A \in \mathcal{L}(P)} r(A) y(A) \mid \sum_{A \ni i} y(A) = c_i, \forall i \in P \}$$

with the generalized polymatroid greedy algorithm. For the sake of completeness, we add the prove of the optimality of this algorithm for submodular systems with admissible weight functions in the following Section.

3.3 Generalized polymatroid greedy algorithm

Let $P = (P, \leq)$ be a poset and $\mathcal{L}(P)$ be the lattice of ideals in P. Given a submodular function $r : \mathcal{L}(P) \rightarrow \mathbb{R}$ and an admissible function $w : P \rightarrow \mathbb{R}$, we consider the primal and dual linear program

$$(P') \quad \max \{ \sum_{i \in P} w_i x_i \mid \sum_{i \in A} x_i \leq r(A), \forall A \in \mathcal{L}(P) \}, \quad \text{and}$$

$$(D') \quad \min_{y \geq 0} \{ \sum_{A \in \mathcal{L}(P)} r(A) y(A) \mid \sum_{A \ni i} y(A) = w_i, \forall i \in P \}.$$

Note that, since c is admissible, there exists a linear extension $P = \{1, \ldots, n\}$ of $(P, \leq)$ such that $w_1 \geq \ldots \geq w_n$. Moreover, the listing of P induces a maximal chain

$$\mathcal{C} = \emptyset = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = P$$

in $\mathcal{L}(P)$ with $M_i = \{1, \ldots, i\}$ for $i = 1, \ldots, n$.

The first phase of the generalized polymatroid greedy algorithm is called "Monge algorithm" as it goes back to a procedure of Monge. The Monge algorithm determines a solution y^* for problem (D') by assigning values to the elements on $\mathcal{C}$.

Monge algorithm:
Let $P = \{1, \ldots, n\}$ be a linear extension of $(P, \leq)$ with $w_1 \geq \ldots \geq w_n$;
Initialize $y^* \equiv 0$ and $M_n = P$;
for $i = n, \ldots, 1$ **do**
$y^*(M_i) = w_i$;
$M_{i-1} = M_i \setminus \{i\}$;
for $k = 1, \ldots, i-1$ **do**
$w_k = w_k - w_i$;
end for
end for

Clearly, the weight function needs to be admissible, since otherwise it is impossible to determine a linear extension of P with non-increasing weights. It is easy to see that, given an admissible weight function, y^* is a feasible dual solution.

Lemma 3.7. *Let $w : P \to \mathbb{R}$ be admissible. Then the vector y^*, returned by the Monge algorithm, is a feasible solution of problem (D').*

Proof. By the listing of P, y^* is non-negative. Moreover, the only non-zero values of y^* are $y^*(M_n) = w_n$ and $y^*(M_i) = w_i - w_{i+1}$ for $i = 1, .., n-1$. Therefore,

$$\sum_{A \ni i} y^*(A) = \sum_{j=i}^{n} y^*(M_j) = (w_i - w_{i+1}) + \ldots + (w_{n-1} - w_n) + w_n = w_i.$$

□

Given the chain $\mathcal{C}$, in the second phase of the generalized polymatroid greedy algorithm we construct a primal greedy vector $x^* : P \to \mathbb{R}$ as follows:

Generalized polymatroid greedy algorithm (phase II):
$x_1^* = r(M_1)$;
for $i = 2, .., n$ **do**
$x_i^* = r(M_i) - r(M_{i-1})$;
end for

If we require submodularity of r, we can show that x^* is a primal feasible solution.

Lemma 3.8. *If $r : \mathcal{L}(P) \to \mathbb{R}$ is submodular, then x^* is a feasible solution of problem (P').*

Proof. Consider the function $h : \mathcal{L}(P) \to \mathbb{R}$ with values

$$h(A) = r(A) - x^*(A).$$

Since r is submodular and x^* is modular, h itself is submodular and satisfies

$$h(M_i) = 0 \quad \text{for } i = 1, \ldots, n.$$

We need to show that h is non-negative. Suppose A is a minimal ideal such that $h(A) < 0$. We may choose a minimal element $M_i \in \mathcal{C}$ such that $M_i \geq A$. Hence, $M_i = M_{i-1} \cup A$ and the submodularity of h yields

$$0 > h(A) = h(A) + h(M_{i-1}) \geq h(A \cap M_{i-1}) + h(M_i) = h(A \cap M_{i-1}).$$

But, as $A \cap M_{i-1} \subset A$, this is a contradiction to the choice of A. □

Moreover, in case r is submodular, x^* and y^* are even optimal solutions:

Theorem 3.2. *If $r : \mathcal{L}(P) \to \mathbb{R}$ is submodular, then x^* and y^* are optimal solutions to (P') and (D'), respectively.*

Proof. By duality theory, it remains to prove that the objective values of (P') and (D') are identical.

$$\begin{aligned}
\sum_{A \in \mathcal{L}(P)} r(A) y^*(A) &= \sum_{i=1}^{n} r(M_i) y^*(M_i) \\
&= \sum_{i=1}^{n-1} r(M_i)(w_i - w_{i+1}) + r(M_n) w_n \\
&= w_1 r(M_1) + \sum_{i=2}^{n} w_i (r(M_i) - r(M_{i-1})) = \sum_{i=1}^{n} w_i x_i^*.
\end{aligned}$$

□

Chapter 4

Distributive strong exchange structures

In the previous Section 3.3, we proved the optimality of the generalized polymatroid greedy algorithm for submodular systems $(\mathcal{L}(P), r)$ and admissible weight functions $w : P \to \mathbb{R}$. Consider the special case where the submodular systems $(\mathcal{L}(P), r)$ is a distributive supermatroid with feasible elements

$$\mathcal{F} = \{A \in \mathcal{L}(P) \mid r(A) = |A|\}.$$

It can be shown that for distributive supermatroids, the greedy vectors x^* returned by the generalized polymatroid greedy algorithm are the incidence vectors of the solutions X^* of the following matroid greedy algorithm

GREEDY-ALGORITHM:
$X^* = \emptyset$;
while $\Gamma(X^*) \neq \emptyset$ **do**
Choose $i \in \Gamma(X^*)$ such that $w(i) \geq w(j)$ for all $j \in \Gamma(X^*)$;
$X^* = X^* \cup i$;
end while

Moreover, similar to matroids, distributive supermatroids can be characterized as exactly those monotone ideal systems such that the greedy algorithm above determines a basis of maximal weight for any admissible weight function $w : P \to \mathbb{R}$.

We already observed in Section 2.2 that the monotony is not necessary for the optimality of the greedy algorithm. Whereas the property to be a distributive greedoid, i.e., to be accessible and to satisfy the augmentation property

$$(AP) \quad X, Y \in \mathcal{F}, |X| < |Y| \quad \Rightarrow \quad \exists y \in Y \setminus X \text{ with } X \cup y \in \mathcal{F},$$

is a necessary, but not suffient condition.

For the case of set systems $\mathcal{F} \subseteq 2^E$, we saw that the greedy algorithm determines an optimal basis for any linear weight function $w : E \to \mathbb{R}$ if and only if $\mathcal{F}$ is a strong exchange structure, i.e., if $\mathcal{F}$ is a greedoid satisfying the *strong exchange property* saying

$$\text{for } A \in \mathcal{F},\ B \in \mathcal{B}(\mathcal{F}),\ A \subseteq B \text{ and } i \in E \setminus B \text{ with } A \cup i \in \mathcal{F}$$
$$\text{there exists } j \in B \setminus A \text{ such that } A \cup j \in \mathcal{F} \text{ and } B \setminus j \cup i \in \mathcal{F}.$$

The system in Figure 4.1 is an example of such a strong exchange structure.

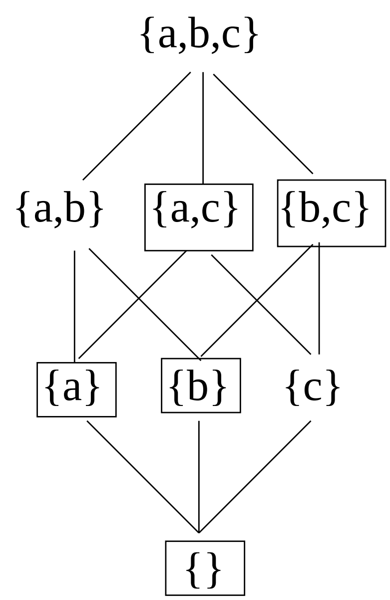


Figure 4.1: Non-monotone set family $\mathcal{F} \subseteq 2^{\{a,b,c\}}$.

Moreover, we saw in Section 2.3 how strong exchange structures have been generalized to integral strong exchange structures. Integral strong exchange structures are exactly those systems of integral vectors for which the greedy algorithm determines on optimal basis for any separabel concave weight function.

In this Chapter, we step even further and introduce *distributive strong exchange structures* as follows

Definition 4.1 (Distributive strong exchange structures). *A distributive greedoid $\mathcal{F} \subseteq \mathcal{L}(P)$ is a* distributive strong exchange structure *if it satifies the following* distributive strong exchange property

$$\textit{for } A \in \mathcal{F},\ B \in \mathcal{B}(\mathcal{F}),\ A \subseteq B \textit{ and } i \in P \setminus B \textit{ with } A \cup i \in \mathcal{F}$$
$$\textit{there exist } j \in (B \setminus A)^-,\ k \in (B \setminus A)^+ \textit{ with } j \leq k \textit{ such that } A \cup j \in \mathcal{F} \textit{ and } B \setminus k \cup i \in \mathcal{F}.$$

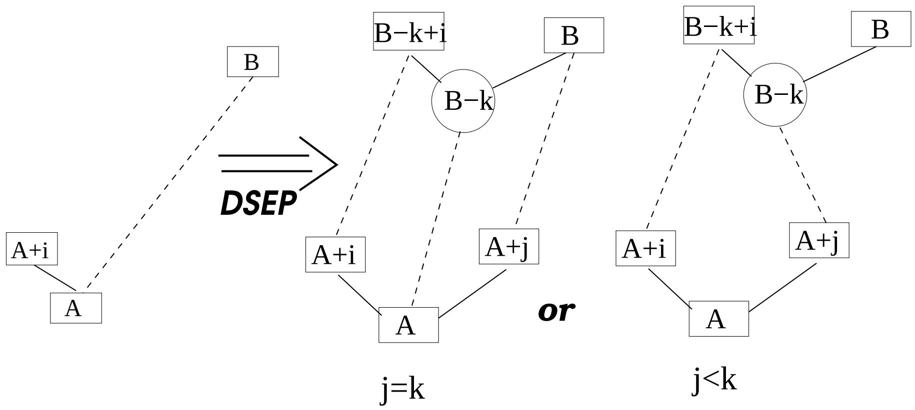


Figure 4.2: Distributive strong exchange property (DSEP)

Figure 4.2 demonstrates the distributive strong exchange property.

Note that in case of set systems $\mathcal{F} \subseteq 2^E$, this distributive strong exchange property reduces to the strong exchange property for set systems.

In the following Section, we prove that distributive strong exchange structures are exactly those ideal systems for which the greedy algorithm determines an optimal basis for any admissible weight function $w : P \to \mathbb{R}$. Thereafter, we reduce our model to Shenmaier's results on systems of integral vectors.

4.1 Algorithmic characterization

For the algorithmic chracterization of distributive strong exchange structures in Theorem 4.1 we need the following Lemma:

Lemma 4.1. *Let B be a basis of the distributive supermatroid $\mathcal{M} \subseteq \mathcal{L}(P)$, $x \in (P \setminus B)^-$ and C be a circuit in $B \cup x$. Then each $x \neq y \in C^+$ satisfies $y \in B^+$ and $B \setminus y \cup x \in \mathcal{M}$.*

Proof. Consider $C \setminus y \in \mathcal{M}$. Since $\mathcal{M}$ satisfies (AP), we can augment $C \setminus y$ from B to a basis $B' \in \mathcal{M}$. Then there exists some $z \in (B \setminus (C \setminus y))^+$ such that $B' = B \setminus z \cup x$. Hence, $y \neq z$ implies $y \in B'$ and therefore $C \subseteq B'$. But this is a contradiction to $C \notin \mathcal{M}$. ☐

Theorem 4.1. *Let $\mathcal{F} \subseteq \mathcal{L}(P)$ be a distributive greedoid. Then the following statements are equivalent:*

1. *For any admissible linear weight function, the greedy algorithm determines an optimal basis of $\mathcal{F}$.*

2. *$\mathcal{M} = \{X \in \mathcal{L}(P) \mid X \subseteq B$ and B is a basis of $\mathcal{F}\}$ is a distributive supermatroid, and every closed ideal in $\mathcal{F}$ is closed in $\mathcal{M}$.*

3. *$\mathcal{F}$ satisfies the distributive strong exchange property.*

Proof. "1$\Rightarrow$ 2": We first prove that $\mathcal{M}$ is a distributive supermatroid. Since $\mathcal{M}$ is monotone by definition, we have to show that Steinitz' augmentation property (AP) is satisfied. Consider two ideals $X, Y \in \mathcal{M}$ with $|X| < |Y|$ and define the weight function

$$c(x) = \begin{cases} 1 & : \quad x \in X \\ \frac{1}{2} & : \quad x \in Y \setminus X \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Since X and Y are ideals, it is easy to see that c is admissible.

As the weight of the elements in X is greater than the weight of the elements in Y, the greedy algorithm visits the ideal X at some step. Because $c(Y) > c(X)$, in the following step the greedy algorithm goes to some ideal $X \cup y \in \mathcal{F}$ with $y \in (Y \setminus X)^-$, before it terminates with some basis B such that $X \cup y \subseteq B$. Hence $X \cup \ y \in \mathcal{M}$, which implies that $\mathcal{M}$ is a distributive supermatroid.

It remains to prove that any closed ideal in $\mathcal{F}$ is also closed in $\mathcal{M}$. Let X be closed in $\mathcal{F}$. Choose $A \subseteq X$ with $r_{\mathcal{F}}(X) = |A|$ and A' with $A \subseteq A' \subseteq X$ and $r_{\mathcal{M}}(X) = |A'|$. Note that, since X is closed in $\mathcal{F}$, for any $y \in (P \setminus X)^-$ we must have $r_{\mathcal{F}}(X \cup y) > r_{\mathcal{F}}(X)$ and therefore $A \cup y \in \mathcal{F}$.

Suppose X is not closed in $\mathcal{M}$. Then there exists $y \in (P \setminus X)^-$ such that $r_{\mathcal{M}}(X \cup y) = r_{\mathcal{M}}(X)$, which implies $r_{\mathcal{M}}(A' \cup y) = r_{\mathcal{M}}(A')$ resp. $y \in \sigma_{\mathcal{M}}(A')$. We define the weight function

$$c(x) = \begin{cases} 1 & : \quad x \in A' \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Since A' is an ideal, c is admissible. By the definition of $\mathcal{M}$, A' is contained in some basis of $\mathcal{F}$. Hence, every solution of the greedy algorithm must contain A'. On the other hand, for any $y \in (P \setminus X)^-$ there is a greedy solution B starting with $A \cup y$. Hence

$$A \cup y \subseteq (A \cup y) \cup A' = A' \cup y \subseteq B,$$

which implies $A' \cup y \in \mathcal{M}$, i.e., $y \notin \sigma_{\mathcal{M}}(A')$. This contradiction proves that X is closed in $\mathcal{M}$.

"2$\Rightarrow$ 3": Let $A \in \mathcal{F}, B \in \mathcal{B}(\mathcal{F}), A \subseteq B$ and $i \notin B$ with $A \cup i \in \mathcal{F}$. We have to show that there exist $j \in (B \setminus A)^-$, $k \in (B \setminus A)^+$ with $j \leq k$ such that $A \cup j \in \mathcal{F}$ and $B \setminus k \cup i \in \mathcal{F}$.

The ideal $B \cup i$ contains a circuit C in the matroid $(P, \mathcal{M})$. Clearly, $i \in C \setminus \sigma_{\mathcal{F}}(A)$ and $i \in \sigma_{\mathcal{M}}(C \setminus i)$. Suppose $(C \setminus i) \setminus \sigma_{\mathcal{F}}(A) \neq \emptyset$. Since $\sigma_{\mathcal{F}}(A)$ is closed in $\mathcal{M}$ and $\sigma_{\mathcal{M}}$ is monotone increasing, it follows

$$\sigma_{\mathcal{M}}(C \setminus i) \subseteq \sigma_{\mathcal{M}}(\sigma_{\mathcal{F}}(A)) = \sigma_{\mathcal{F}}(A).$$

Hence, $i \in \sigma_{\mathcal{M}}(C \setminus i)$ implies $i \in \sigma_{\mathcal{F}}(A)$. But this is a contradiction to $A \cup i \in \mathcal{F}$.

Therefore, $(C \setminus i) \setminus \sigma_{\mathcal{F}}(A)$ is non-empty, and we may choose $j \in ((C \setminus i) \setminus \sigma_{\mathcal{F}}(A))^-$ and $k \in ((C \setminus i) \setminus \sigma_{\mathcal{F}}(A))^+$ with $j \leq k$. Then $j \notin \sigma_{\mathcal{F}}(A)$ implies $A \cup j \in \mathcal{F}$. Further on, $k \in ((C \setminus i) \setminus \sigma_{\mathcal{F}}(A))^+$ implies $k \in C^+$ and therefore $B \setminus k \cup i \in \mathcal{M}$ by Lemma 4.1. But $B \setminus k \cup i$ is a basis in $\mathcal{M}$, which implies $B \setminus k \cup i \in \mathcal{F}$.

"3⇒ 1": Let $c : P \to \mathbb{R}$ be any admissible weight function and $F^G = \{a_1, \dots, a_m\}$ be a greedy sequence, i.e., the greedy algorithm has gone from $\{a_1, \dots, a_{l-1}\} \in \mathcal{F}$ to $\{a_1, \dots, a_l\} \in \mathcal{F}$ in iteration $l \leq m$. Among all optimal bases, choose a basis B such that $A := \{a_1, \dots, a_k\} \subseteq B$ and k maximal. It follows from the choice of B that $a_{k+1} \in P \setminus B$.

Thus, by the distributive strong exchange property, there exist elements $j \in (B \setminus A)^-$ and $k \in (B \setminus A)^+$ with $j \leq k$ such that $A \cup j \in \mathcal{F}$ and $B' := B \setminus k \cup a_{k+1} \in \mathcal{F}$. Since a_{k+1} was chosen by the greedy principle, we have $c(j) \leq c(a_{k+1})$. Further, as c is admissible, we have $c(k) \leq c(j)$. Together we get

$$c(B') = c(B) - c(k) + c(a_{k+1}) \geq c(B).$$

Hence B' is an optimal basis with $\{a_1, \dots, a_{k+1}\} \subseteq B'$ in contradiction to the choice of B. □

Note that in case P is an antichain, Theorem 4.1 reduces to Theorem 2.2. in [KLS91], which is based on a result of Goetschel [Goe86b].

4.2 Reduction to integral strong exchange structures

In case P is the disjoint union of chains, Shenmaier's result about integral strong exchange structures [She04], which we already stated in Section 2.3 (cf. Theorem 2.2), can be deduced. We recall his model, before we prove that Theorem 2.2 is essentially a special case of Theorem 4.1.

Let $\mathcal{F} \subseteq \mathbb{N}^n$ be a family of non-negative integer vectors, $\mathcal{B}(\mathcal{F})$ be the collection of maximal vectors in $\mathcal{F}$, and $f : \mathbb{N}^n \to \mathbb{R}$ be a separable concave function such that $f(x) = \sum_{k=1}^{n} f_k(x_k)$. Defining the set $J(x) := \{k \mid x + e_k \in \mathcal{F}\}$, Shenmaier investigates the premises, under which the following greedy algorithm solves the problem

$$\max\{f(x) \mid x \in \mathcal{B}(\mathcal{F})\}$$

for arbitrary separable concave functions f.

GREEDY-ALGORITHM (for integer vectors):

$x^* = 0$;
while $J(x^*) \neq \emptyset$ **do**
  Choose $k \in J(x^*)$ such that $f(x^* + e_k) = \max_{j \in J(x^*)} f(x^* + e_j)$;
  $x^* = x^* + e_k$;
end while

Let $I(x) := \{k \mid x_k > 0\}$ and

$$\mathcal{A}(\mathcal{F}) := \{e_{k_1} + \ldots + e_{k_i} \mid e_{k_1} + \ldots + e_{k_s} \in \mathcal{F} \text{ for } s \leq i\} \cup \{0\}$$

denote the accessible part of $\mathcal{F}$. Then Shenmaier could prove

Theorem 3 ([She04]). *Let $\mathcal{F} \subseteq \mathbb{N}^n$ be a non-empty finite family of integer vectors. Then the greedy algorithm above finds an optimal solution for any separable concave function $f : \mathbb{N}^n \to \mathbb{R}$ if and only if the following two conditions hold:*

$$\begin{array}{ll} (Sh1) & x \in \mathcal{A}(\mathcal{F}) \setminus \mathcal{B}(\mathcal{F}) \Rightarrow J(x) \neq \emptyset. \\ (Sh2) & x \leq y, x \in \mathcal{A}(\mathcal{F}), y \in \mathcal{B}(\mathcal{F}), i \in J(x) \setminus I(y-x) \\ & \Rightarrow \exists j \in J(x) \cap I(y-x) \text{ with } y + e_i - e_j \in \mathcal{F}. \end{array}$$

We now translate Shenmaier's model of separable concave functions on the lattice of non-negative integer vectors into a model of admissible functions on ideal systems:

We already observed in Section 2.5 that the lattice of non-negative integer vectors $\mathbb{N}^n$ is isomorphic to the lattice $\mathcal{L}(P)$ of all ideals of the disjoint union of n chains $P = C_1 \dot\cup \ldots \dot\cup C_n$. I.e., a vector $x \in \mathbb{N}^n$ may be identified with an ideal $X \in \mathcal{L}(P)$ via

$$X = \{k_i \in P \mid k_i \in C_k, i \leq x_k, k = 1, \ldots, n\}.$$

Furthermore, assuming $f(0) = 0$, we saw that $f : \mathbb{N}^n \to \mathbb{R}$ may be linearized to an admissible weight function $w : P \to \mathbb{R}$ on P via

$$w(k_i) := f_k(i) - f_k(i-1) \quad \text{for } k = 1, \ldots, n \text{ and } i \in \mathbb{N}.$$

such that

$$f(x) = \sum_{k=1}^{n} f_k(x_k) = \sum_{k=1}^{n} \sum_{i=1}^{x_k} w(k_i) = \sum_{k_i \in X} w(k_i) = w(X).$$

It is easy to see that the greedy algorithm for integer vectors above is nothing else than our greedy algorithm for ideal systems. And, if $\mathcal{A}(\mathcal{F})$ denotes the accessible part of an ideal system $\mathcal{F}$ in general, Theorem 2.2 can be seen as a special case of the following

Theorem 4.2. *Let $\mathcal{F} \subseteq \mathcal{L}(P)$ be an ideal system. Then the greedy algorithm finds an optimal solution for any admissible weight function $w : P \to \mathbb{R}$ if and only if the following two conditions hold:*

1. *If $X \in \mathcal{A}(\mathcal{F}) \setminus \mathcal{B}(\mathcal{F})$, then $\Gamma(X) \neq \emptyset$.*
2. *If $X \subseteq Y, X \in \mathcal{A}(\mathcal{F}), Y \in \mathcal{B}(\mathcal{F})$ and $i \in P \setminus Y$ with $X \cup i \in \mathcal{F}$, then $X \cup j \in \mathcal{F}$ and $Y \setminus k \cup i \in \mathcal{F}$ for some $j \in (Y \setminus X)^-$, $k \in (Y \setminus X)^+$ and $j \leq k$.*

For the special case of a set system $\mathcal{F} \subseteq 2^P$, this result has been proved by Goecke, Korte and Lovàsz in [GKL89].

Observe that property 2 is just the distributive strong exchange property if $\mathcal{F}$ is accessible. We show that the two conditions 1 and 2 imply that $\mathcal{F}$ is a distributive greedoid. Then Theorem 4.2 (and therefore Theorem 2.2) follows as a direct consequence of Theorem 4.1.

Lemma 4.2. *If $\mathcal{F} \subseteq \mathcal{L}(P)$ satisfies properties 1 and 2, then $\mathcal{B}(\mathcal{F}) \subseteq \mathcal{A}(\mathcal{F})$.*

Proof. Suppose properties 1 and 2 hold, but there exists a basis $Y \in \mathcal{B}(\mathcal{F}) \setminus \mathcal{A}(\mathcal{F})$. Choose a maximal accessible ideal $X \in \mathcal{A}(\mathcal{F})$ with $X \subseteq Y$. Because of property 1, there exists an element $i \in (P \setminus X)^-$ such that $X \cup i \in \mathcal{A}(\mathcal{F})$ and $i \notin Y$. Hence, by property 2, there exists an element $j \in (Y \setminus X)^-$ with $X \cup j \in \mathcal{A}(\mathcal{F})$. But this is a contradiction to the choice of X. ☐

Since the greedy algorithm searches for an optimal basis of $\mathcal{F}$, we may therefore assume $\mathcal{F}$ to be accessible.

Lemma 4.3. *If $\mathcal{F} \subseteq \mathcal{L}(P)$ satisfies properties 1 and 2, then $\mathcal{F}$ is a greedoid.*

Proof. It remains to prove that $\mathcal{F}$ satisfies Steinitz' augmentation property (AP), or equivalently, that all bases of $\mathcal{F}$ are of constant cardinality. Suppose properties 1 and 2 hold, but there exist two bases $X, Y \in \mathcal{B}(\mathcal{F})$ with $|X| < |Y|$. By property 1, $X \not\subseteq Y$. Hence there exists an ideal $A \in \mathcal{A}(\mathcal{F})$ such that $A \subseteq X \cap Y$ and $A \cup i \in \mathcal{A}(\mathcal{F})$ for some $i \in (P \setminus A)^-, i \notin Y$. Choose a basis Y of maximal cardinality and such an ideal A, so that A is of maximal cardinality. By property 2, there exists a basis $Y' = Y \setminus k \cup i$ of maximal cardinality and an ideal $A' = A \cup j \subseteq Y' \in \mathcal{A}(\mathcal{F})$ with $|A'| > |A|$, in contradiction to the choice of A. ☐

Hence, an ideal system satisfying properties 1 and 2 is a distributive greedoid satisfying the distributive strong exchange property, and we may summarize:

Corollary 4.1. *If $\mathcal{F} \subseteq \mathcal{L}(P)$ satisfies properties 1 and 2, then $\mathcal{F}$ is a distributive strong exchange structure.*

Chapter 5

Distributive Gauss greedoids

In the previous Chapter, we characterized distributive strong exchange structures as those accessible ideal systems $\mathcal{F} \subseteq \mathcal{L}(P)$ such that the greedy algorithm determines a basis of maximal weight for any admissible weight function $w : P \to \mathbb{R}$.

Suppose now we are interested in an optimal member of $\mathcal{F}$. If w is non-negative, each optimal basis is an optimal member at the same time. This is not necessarly true for arbitrary weights.

As long as $\mathcal{F}$ is monotone and w is admissible, we may restrict to the elements of P with non-negative weight. We got to know distributive supermatroids as exactly those monotone ideal systems such that the greedy algorithm determines an optimal member for any admissible weight function.

The question arises whether for non-monotone but accessible ideal systems a member of maximal weight can be determined with a greedy-type algorithm for arbitrary admissible weight functions.

Recall for the special case of antichains P: Goecke's Gauss greedoids [Goe86a] characterize the set systems for which a modified greedy algorithm works correctly for any linear weight function. We generalize his result by introducing *distributive Gauss greedoids* as the collection of bases of a sequence of nested distributive supermatroids in strong map relation.

We first prove a basis exchange property for strong maps between distributive supermatroids. Thereafter, we prove an exchange property for distributive Gauss greedoids. This exchange property finally yields the algorithmic characterization of distributive Gauss greedoids as exactly those ideal systems for which the modified greedy algorithm determines an optimal member for any admissible weight function.

In Section 5.4, we present examples of distributive Gauss greedoids. Namely the *distributive Gaussian elemination greedoid* and the *distributive bipartite matching greedoid*. We further show that distributive Gauss greedoids generalize distributive supermatroids. Additionally, we prove that full distributive Gauss greedoids, i.e., distributive Gauss greedoids containing the ideal P, are closed under a certain duality operator.

5.1 Definition

In order to generalize Gauss greedoids to distributive Gauss greedoids, we need to generalize the notion of *strong maps* between matroids to *distributive strong maps* between distributive supermatroids. Note that we may also write $(P, \mathcal{F})$ for an ideal system $\mathcal{F} \in \mathcal{L}(P)$.

Definition 5.1 (Distributive strong maps). *Let $(P, \mathcal{M}_1)$ and $(P, \mathcal{M}_2)$ be two distributive supermatroids with rank functions r_1 and r_2. Then $\mathcal{M}_1$ is a* distributive strong map *of $\mathcal{M}_2$ if*

$$r_1(X \cup Y) + r_2(X \cap Y) \leq r_1(X) + r_2(Y)$$

is satisfied for all $X, Y \in \mathcal{L}(P)$.

Note that in case P is an antichain, distributive strong maps between distributive supermatroids reduce to strong maps between matroids.

Analogue to matroids, we call two distributive supermatroids $(P, \mathcal{M}_1)$ and $(P, \mathcal{M}_2)$ *nested*, if $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and the rank functions r_1 and r_2 satisfy

$$r_1(P) = r_2(P) - 1.$$

We now define *distributive Gauss greedoids* as follows:

Definition 5.2 (Distributive Gauss greedoid). *Let $\{(P, \mathcal{M}_i)\}_{i=1,\dots,m}$ be a family of nested distributive supermatroids with $r_1(P) = 1$ such that $\mathcal{M}_i$ is a distributive strong map of $\mathcal{M}_{i+1}$ for $i = 1, \dots, m-1$. Then the accessible system $(P, \mathcal{F})$ with*

$$\mathcal{F} = \{\{x_1, \dots, x_k\} \subseteq \mathcal{L}(P) \mid \{x_1, \dots, x_i\} \in \mathcal{M}_i \text{ for } 1 \leq i \leq k\} \cup \emptyset$$

is a distributive Gauss greedoid.

It follows that in case of an antichain P, a distributive Gauss greedoid is just a common Gauss greedoid. Let us convince ourselves that distributive Gauss greedoids are distributive greedoids:

Lemma 5.1. *Every distributive Gauss greedoid is a distributive greedoid.*

Proof. Let $(P, \mathcal{F})$ be a distributive Gauss greedoid defined by the sequence of distributive supermatroids $\mathcal{M}_i, i = 1, .., m$ and suppose $X = \{x_1, \dots, x_k\} \in \mathcal{F}$ and $Y = \{x_1, \dots, x_l\} \in \mathcal{F}$ with $l > k$. Then X is a basis of $\mathcal{M}_k$ and Y is a basis of $\mathcal{M}_l$. Since the distributive supermatroids are nested, X is also a member of $\mathcal{M}_l$ and can therefore be augmented by $y \in Y \setminus X$. □

It can also be seen from the proof that if $\mathcal{F}$ is a distributive Gauss greedoid defined by the sequence of distributive supermatroids $\mathcal{M}_i$ with basis sets $\mathcal{B}_i$, $i = 1, \ldots, m$, then

$$\mathcal{F} = \emptyset \cup \mathcal{B}_1 \cup \ldots . \cup \mathcal{B}_m.$$

5.2 Exchange properties

In this Section, we first prove a basis exchange property between distributive strong maps. This yields an exchange property for distributive Gauss greedoids, which we need for the algorithmic characterization in the subsequent Section. The basis exchange property for distributive strong maps is based on the following Theorem:

Theorem 5.1. *For two distributive supermatroids $(P, \mathcal{M}_1)$ and $(P, \mathcal{M}_2)$ with rank functions r_1 and r_2 and hull operators σ_1 and σ_2, the following properties are equivalent*

(1.) $r_1(X \cup Y) + r_2(X \cap Y) \leq r_1(X) + r_2(Y)$ *for all* $X, Y \in \mathcal{L}(P)$,

(2.) $r_1(Y) - r_1(X) \leq r_2(Y) - r_2(X)$ *for all* $X, Y \in \mathcal{L}(P)$ *with* $X \subseteq Y$,

(3.) $\sigma_1(X) = X \Rightarrow \sigma_2(X) = X$ *for all* $X \in \mathcal{L}(P)$.

Before we can prove Theorem 5.1, we need three additional lemmata. The first one is easily proved by induction:

Lemma 5.2. *For systems of ideals, property (2.) of Theorem 5.1 is equivalent to property*

(2.′) $r_1(X \cup e) - r_1(X) \leq r_2(X \cup e) - r_2(X)$ *for all* $X \in \mathcal{L}(P)$ *and* $e \in (P \setminus X)^-$.

Proof. Property (2.′) follows immediatly from property (2.), since $X \subseteq X \cup e$. We prove the sufficiency of (2.′) by induction on $|Y \setminus X|$: Consider two ideals $X \subseteq Y$, take $e \in (Y \setminus X)^+$ and define $Y' = Y \setminus e$. By induction, $r_1(Y') - r_1(X) \leq r_2(Y') - r_2(X)$ follows. Further on, property (2.) implies $r_1(Y' \cup e) - r_1(Y') \leq r_2(Y' \cup e) - r_2(Y')$. Adding these two inequalities, we get $r_1(Y) - r_1(X) \leq r_2(Y) - r_2(X)$. □

The following two lemmata state properties about distributive supermatroids.

Lemma 5.3. *Let $(P, \mathcal{F})$ be a distributive supermatroid, $A \in \mathcal{L}(P)$ and $e, f \in (P \setminus A)^-$. Then*

$$r(A \cup e) = r(A) \text{ implies } r(A \cup \{e, f\}) = r(A \cup f).$$

Proof. Suppose there exist $A \in \mathcal{L}(P)$ and $e, f \in (P \setminus A)^-$ with $r(A \cup e) = r(A)$, but $r(A \cup \{e, f\}) > r(A \cup f)$.

Let $B \cup e$ be a basis of $A \cup \{e, f\}$ and B be a basis of $A \cup f$. Then $f \notin B$, as otherwise B would be a basis of A, and $B \cup e$ would be basis of $A \cup e$, in contradiction to $r(A \cup e) = r(A)$.

Moreover $B \setminus f$ cannot be a basis of A, as otherwise $B \setminus f \cup e$ would be a basis of $A \cup e$, again in contradiction to $r(A \cup e) = r(A)$. Therefore, we can augment $B \setminus f$ by an element $a \in A$ to achieve a basis $B \setminus f \cup a$ of A.

Since $|B \setminus f \cup a| < |B \cup e|$, we can augment $B \setminus f \cup a$ by either f or e. It is impossible to augment by e, as otherwise $B \setminus f \cup \{a, e\}$ would be a basis of $A \cup e$. Hence $B \cup a \in \mathcal{F}$, in contradiction to our choice of B as a basis of A. □

Lemma 5.4. *Let $(P, \mathcal{F})$ be a distributive supermatroid, $A \in \mathcal{L}(P)$ and $e, f \in (P \setminus A)^-$. Then $r(A \cup e) = r(A) + 1$ and $r(A \cup f) = r(A)$ implies*

$$r(A \cup \{e, f\}) = r(A \cup f) + 1.$$

Proof. From $r(A \cup f) = r(A)$, it follows that any basis of A is a basis of $A \cup f$. Further on, $r(A \cup e) = r(A) + 1$ implies that any basis of A can be augmented by e to a basis of $A \cup e$. Summarizing, there exists a basis of $A \cup f$ which can be augmented by e. Thus $r(A \cup \{e, f\}) = r(A \cup f) + 1$. □

We are now able to prove Theorem 5.1.

Proof of Theorem 5.1.

We have to prove that for two distributive supermatroids $\mathcal{M}_1 = (P, \mathcal{F}_1)$ and $\mathcal{M}_2 = (P, \mathcal{F}_2)$ with rank functions r_i and hull operators σ_i $(i = 1, 2)$ the following three properties are equivalent.

(1.) $r_1(X \cup Y) + r_2(X \cap Y) \leq r_1(X) + r_2(Y)$ for all $X, Y \in \mathcal{L}(P)$,

(2.) $r_1(Y) - r_1(X) \leq r_2(Y) - r_2(X)$ for all $X, Y \in \mathcal{L}(P)$ with $X \subseteq Y$,

(3.) $\sigma_1(X) = X \Rightarrow \sigma_2(X) = X$ for all $X \in \mathcal{L}(P)$.

"(1.) $\Rightarrow$ (2.)": Suppose property (1.) holds and consider two ideals $X \subseteq Y$, i.e., $X = X \cap Y$ and $Y = X \cup Y$. Then (1.) is equivalent to $r_1(Y) + r_2(X) \leq r_1(X) + r_2(Y)$, which is equivalent to (2.).

"(2.) $\Rightarrow$ (1.)": Suppose (2.) holds and there exist two ideals X and Y such that

$$r_1(X \cup Y) + r_2(X \cap Y) > r_1(X) + r_2(Y).$$

Since r_1 is submodular, $r_1(X) - r_1(X \cup Y) \geq r_1(X \cap Y) - r_1(Y)$ holds. It follows

$$r_1(X \cap Y) - r_1(Y) < r_2(X \cap Y) - r_2(Y).$$

Since $X \cap Y \subseteq Y$, this is a contradiction to (2.).

"(2.) $\Rightarrow$ (3.)": Suppose (2.) holds and there exist an ideal A and an element $e \in (P \setminus A)^-$ such that $\sigma_1(A) = A$ but $r_2(A \cup e) = r_2(A)$. Then $r_1(A \cup e) - r_1(A) > r_2(A \cup e) - r_2(A)$ follows, in contradiction to (2.).

"(3.) $\Rightarrow$ (2.)": Suppose (3.) holds, but (2.), resp. (2.'), is not satisfied. Take an ideal A and an element $e \in (P \setminus A)^-$ such that $r_1(A \cup e) - r_1(A) > r_2(A \cup e) - r_2(A)$ with $|A|$ maximal.

Clearly, $r_1(A \cup e) = r_1(A) + 1$ and $r_2(A \cup e) = r_2(A)$. In particular, A is not closed in $\mathcal{M}_2$. To obtain a contradiction, we prove that A is closed in $\mathcal{M}_1$: Suppose for the contrary, there exists an element $f \in (P \setminus A)^-$ with $r_1(A \cup f) = r_1(A)$. By the choice of A, we have

$$r_1((A \cup f) \cup e) - r_1((A \cup f)) \leq r_2((A \cup f) \cup e) - r_2((A \cup f)).$$

Then $r_2(A \cup e) = r_2(A)$ implies $r_2((A \cup f) \cup e) = r_2((A \cup f))$ by Lemma 5.3. Hence $r_1((A \cup f) \cup e) = r_1((A \cup f))$ follows. On the other hand, $r_1(A \cup f) = r_1(A)$ and $r_1(A \cup e) = r_1(A) + 1$ imply $r_1((A \cup f) \cup e) = r_1((A \cup f)) + 1$ by Lemma 5.4. A contradiction. □

5.2.1 A basis exchange property for distributive strong maps

For the case of antichains P, Goecke [Goe86a] tried to characterize strong maps between nested matroids by the following basis exchange property:

Theorem 5.2 (Basis exchange property of strong maps [Goe86a]). *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two nested matroids with basis sets $\mathcal{B}_1$ and $\mathcal{B}_2$. Then $\mathcal{M}_1$ is a strong map of $\mathcal{M}_2$ if and only if the following property holds:*

If $B \in \mathcal{B}_1$ and $B \setminus e \cup \{x, y\} \in \mathcal{B}_2$ with $e \in B$ and $x, y \notin B$ then either
$B \cup y \in \mathcal{B}_2$ or $B \setminus e \cup y \in \mathcal{M}_1$.

But this Theorem is not true: the following counterexample proves that the basis exchange property above is not sufficient for nested matroids to be strong maps.

Example 5.1 (Counterexample). *(Compare Figure 5.1). Let $\mathcal{M}_1 = \{\emptyset, \{z\}\}$ and $\mathcal{M}_2 = 2^P \setminus \{x, y, z\}$ be two set systems of the antichain $P = \{x, y, z\}$. Obviously, $\mathcal{M}_1$ and $\mathcal{M}_2$ are matroids with $\mathcal{M}_1 \subseteq \mathcal{M}_2$ and $r_1(P) = r_2(P) - 1$. Moreover, as $B = \{z\}$ is the only basis in $\mathcal{M}_1$ and $B \setminus z \cup \{x, y\} = \{x, y\}$ is the only basis in $\mathcal{M}_2$ not containing z, Goecke's basis exchange property is satisfied. But $\mathcal{M}_1$ is not a strong map of $\mathcal{M}_2$: Though $\{x, y\}$ is closed in $\mathcal{M}_1$, $\{x, y\}$ is not closed in $\mathcal{M}_2$.*

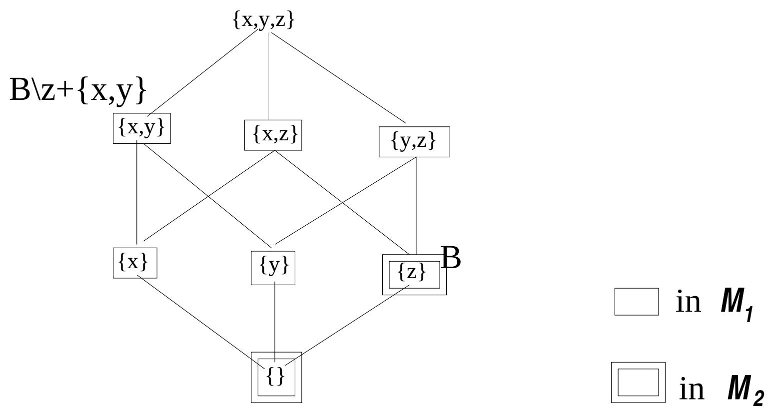


Figure 5.1: Matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ are in no strong map relation.

However, if we add the condition that the bases of $\mathcal{M}_2$ are accessible from the bases of $\mathcal{M}_1$, we can correct Goecke's basis exchange property and, at the same time, generalize it from matroids to distributive supermatroids:

Theorem 5.3 (Basis exchange property of distributive strong maps). *Let $(P, \mathcal{M}_1)$ and $(P, \mathcal{M}_2)$ be two nested distributive supermatroids with basis sets $\mathcal{B}_1$ and $\mathcal{B}_2$ such that for any $B \in \mathcal{B}_2$ there exists $b \in B^+$ with $B \setminus b \in \mathcal{B}_1$. Then $(P, \mathcal{M}_1)$ is a distributive strong map of $(P, \mathcal{M}_2)$ if and only if the following property $(*)$ holds:*

$(*)$ *If $B \in \mathcal{B}_1$ and $B \setminus e \cup \{x, y\} \in \mathcal{B}_2$ with $e \in B, x \not\leq y$ and $x, y \notin B$, then either $B \cup y \in \mathcal{B}_2$ or $B \setminus e \cup y \in \mathcal{M}_1$.*

Proof. Let $\mathcal{M}_1$ be a distributive strong map of $\mathcal{M}_2$, B a basis in $\mathcal{M}_1$, and $B \setminus e \cup \{x, y\}$ a basis in $\mathcal{M}_2$. Then, by property (2.'),

$$r_1(B \setminus e \cup y) + r_2(B \cup y) \geq r_1(B \cup y) + r_2(B \setminus e \cup y) = 2|B|.$$

Hence, if $B \cup y$ is not a basis of $\mathcal{M}_2$, then $B \setminus e \cup y$ is a basis of $\mathcal{M}_1$.

Conversely, assume $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy the basis exchange property of distributive strong maps, but property (2.') is not satisfied. I.e., there exist an ideal $A \in \mathcal{L}(P)$ and an element $e \in (P \setminus A)^-$ with

$$r_1(A \cup e) - r_1(A) > r_2(A \cup e) - r_2(A).$$

Because of $M_1 \subseteq \mathcal{M}_2$, we observe $A \notin \mathcal{M}_1$.

Let us convince ourselves that we may assume $A \in \mathcal{M}_2$: If A is not in $\mathcal{M}_2$, we can choose an ideal $\tilde{A} \in \mathcal{M}_2$ with $\tilde{A} \subseteq A$ and $|\tilde{A}| = r_2(A)$. Then $r_2(A \cup e) = r_2(A)$ implies $\tilde{A} \cup e \notin M_2$. Therefore, since any subideal of A in $\mathcal{M}_1$ can be augmented by e, it follows that $\tilde{A} \in \mathcal{M}_2 \setminus \mathcal{M}_1$. Hence we obtain $r_1(\tilde{A} \cup e) - r_1(\tilde{A}) > r_2(\tilde{A} \cup e) - r_2(\tilde{A})$.

Now, choose an ideal $A \in \mathcal{M}_2 \setminus \mathcal{M}_1$ with $r_1(A \cup e) - r_1(A) > r_2(A \cup e) - r_2(A)$ of maximal cardinality.

Consider any $C \in \mathcal{B}_2$ with $A \subseteq C$ and choose $y \in C^+$ such that $C \setminus y \in \mathcal{B}_1$. Since $A \notin \mathcal{M}_1$, we obtain $A \setminus y \in \mathcal{M}_1$ and $A \setminus y \cup e \in \mathcal{M}_1$. Further on, $|A| = |A \setminus y \cup e|$ and $r_1(P) < r_2(P)$ imply $A \notin \mathcal{B}_2$. We prove $A \setminus y \cup e \in \mathcal{B}_1$:

Suppose for the contrary $A \setminus y \cup e \notin \mathcal{B}_1$. Then $|C \setminus e| > |A \setminus y \cup e|$. By Steinitz' augmentation property (AP), there must exist an element $z \in (C \setminus A)^-$ such that $A \setminus y \cup \{e, z\} \in \mathcal{M}_1$, $A \cup z \in \mathcal{M}_2 \setminus \mathcal{M}_1$ and $A \cup \{e, z\} \notin \mathcal{M}_2$. But this implies

$$r_1((A \cup z) \cup e) - r_1((A \cup z)) > r_2((A \cup z) \cup e) - r_2((A \cup z)),$$

in contradiction to the choice of A.

Now, $A \setminus y \cup e \in \mathcal{B}_1$ and $A \in \mathcal{M}_2$ imply that there exists an element $x \in (P \setminus A)^-$ with $A \cup x \in \mathcal{B}_2$. Summarizing, we have $B := A \setminus y \cup e \in \mathcal{B}_1$ and $B \setminus e \cup \{x, y\} = A \cup x \in \mathcal{B}_2$, but neither $B \cup y = A \cup e \in \mathcal{B}_2$ nor $B \setminus e \cup y = A \in \mathcal{B}_1$. This contradicts property $(*)$. $\square$

5.2.2 An exchange property for distributive Gauss greedoids

The basis exchange property of distributive strong maps yields an exchange property for distributive Gauss greedoids, which in turn will be used for the algorithmic characterization of distributive Gauss greedoids in Section 5.3.

We need the following characterization of distributive supermatroids by a basis exchange property.

Lemma 5.5. *An ideal system $\mathcal{B} \subseteq \mathcal{L}(P)$ is the basis set of a distributive supermatroid $(P, \mathcal{F})$ if and only if $\mathcal{B}$ satisfies the* (distributive) basis exchange property:

$$B_1, B_2 \in \mathcal{B} \text{ and } x \in B_1^+ \setminus B_2 \quad \text{implies} \quad \exists y \in B_2 \setminus B_1 \text{ with } B_1 \setminus x \cup y \in \mathcal{B}.$$

Proof. The necessity of the basis exchange property follows immediatly from Steinitz' augmentation property (AP).

To prove sufficiency, let $\mathcal{B}$ be an ideal system satisfying the basis exchange property. Obviously, $\mathcal{B}$ is the basis set of the induced ideal system

$$\mathcal{F} := \{X \in \mathcal{L}(P) \mid X \subseteq B \text{ for some } B \in \mathcal{B}\}.$$

Suppose there exist two members in $\mathcal{B}$ of different cardinality. Choose $B_1, B_2 \in \mathcal{B}$ such that $|B_1| > |B_2|$ with $|B_1 \cap B_2|$ maximal. By the basis exchange property, there exist $x \in B_1^+ \setminus B_2$ and $y \in B_2 \setminus B_1$ such that $B' = B_1 \setminus x \cup y \in \mathcal{B}$. Since $|B' \cap B_2| > |B_1 \cap B_2|$ and $|B'| > |B_2|$, this contradicts our choice of B_1 and B_2.

So all bases of $\mathcal{F}$ are of constant cardinality. We now prove that for each ideal $X \in \mathcal{L}(P)$ all bases of $\mathcal{F}(X) = \{A \in \mathcal{F} \mid A \subseteq X\}$ are of constant cardinality as well: Suppose this is not true. Choose two bases B_1' and B_2' of $\mathcal{F}(X)$ with $|B_2'| > |B_1'|$. We know that there exist bases B_1, B_2 of $\mathcal{F}$ with $B_1' \subset B_1$ and $B_2' \subseteq B_2$. Choose such bases B_1 and B_2 with $|B_1 \cap B_2|$ maximal. There exists an element $x \in B_1 \setminus (B_2 \cup B_1')$. By the basis exchange property, there exists a basis $B_3 = B_1 \setminus x \cup y$ for some $y \in B_2 \setminus B_1$. Now, $B_1' \subset B_3$ and $|B_1 \cap B_2| < |B_3 \cup B_2|$ in contradiction to the choice of B_1. Thus, all bases of $\mathcal{F}(X)$ must be of constant cardinality.

By Lemma 2.1, $\mathcal{F}$ satisfies Steinitz' augmentation property (AP). Therefore, since $\mathcal{F}$ is monotone by definition, $\mathcal{B}$ is the basis set of the distributive supermatroid $\mathcal{F}$. □

Theorem 5.4 (Exchange property of distributive Gauss greedoids). *A distributive greedoid $(P, \mathcal{F})$ is a distributive Gauss greedoid if and only if it satisfies the following property:*

$$\text{For any } B \in \mathcal{F}, z \in B, x \not\leq y \text{ and } x, y \notin B,$$
$$B \setminus \{z\} \cup \{x, y\} \in \mathcal{F} \text{ implies } B \cup \{y\} \in \mathcal{F} \text{ or } B \setminus \{z\} \cup \{y\} \in \mathcal{F}.$$

Proof. "Necessity:" The necessity is more or less immediate from the definition of distributive strong maps: Let $(P, \mathcal{F})$ be a distributive Gauss greedoid and $\mathcal{M}_1, \ldots, \mathcal{M}_m$ be the distributive supermatroids defining $\mathcal{F}$. Given $B \in \mathcal{F}, z \in B, x \not\leq y$, $x, y \notin B$ and $B \setminus \{z\} \cup \{x, y\} \in \mathcal{F}$ with $|B| = k$, we know that B is a basis in $\mathcal{M}_k$ and $B \setminus \{z\} \cup \{x, y\}$ is a basis in $\mathcal{M}_{k+1}$.

Since $\mathcal{M}_k$ is a distributive strong map of $\mathcal{M}_{k+1}$, we know by Theorem 5.3 that there either exist a basis $B \cup \{y\}$ in $\mathcal{M}_{k+1}$ or a basis $B \setminus \{z\} \cup \{y\}$ in $\mathcal{M}_k$. Hence $B \cup \{y\} \in \mathcal{F}$ or $B \setminus \{z\} \cup \{y\} \in \mathcal{F}$.

"Sufficiency:" Let $(P, \mathcal{F})$ be a distributive greedoid satisfying the exchange property of distributive Gauss greedoids. Consider for each $k = 1, \ldots, r(P)$ the ideal system

$$\mathcal{B}_k = \{X \in \mathcal{F} \mid |X| = k\}.$$

If we can show that each $\mathcal{B}_k$ is the basis set of a distributive supermatroid, then Theorem 5.3 implies that $\mathcal{F}$ is a distributive Gauss greedoid.

Consider two ideals $B_1, B_2 \in \mathcal{B}_k$ and an element $x \in B_1 \setminus B_2$. According to Lemma 5.5, we need to prove that there exists an element $y \in B_2 \setminus B_1$ with $B_1 \setminus x \cup y \in \mathcal{F}$:

Since $\mathcal{F}$ is accessible, there exists a largest ideal $X \subseteq B_1$ with $x \in X$ such that $X \in \mathcal{F}$ and $X \setminus x \in \mathcal{F}$.

If $X = B_1$, we augment $B_1 \setminus x$ from B_2 and we are done. So we may assume that X is a proper subset of B_1. Choose an ideal $Y \subseteq B_2$ with $l = |Y| = |X|$ and $Y \in \mathcal{F}$. By induction on k, $\mathcal{B}_l$ is the basis set of a distributive supermatroid. Hence there exists an element $y \in Y \setminus X$ such that $X \setminus x \cup y \in \mathcal{F}$.

By the choice of X, we have $y \notin B_1$. Otherwise we could augment $X \setminus x \cup y$ from B_1 and reach an $X' \in \mathcal{F}$ with $X' \setminus x \in \mathcal{F}$, in contradiction to the maximality of $|X|$.

Let $B_1 = X \cup \{x_1, \ldots, x_l\}$, where $X_i := X \cup \{x_1, \ldots, x_i\} \in \mathcal{F}, 1 \leq i \leq l$. Successively augment $X \setminus x \cup y$ from X_i.

We prove that we can choose in each augmentation step an element $x_j \neq x$: Suppose this is not true and there exists a step j with $B := X \setminus x \cup \{y, x_1, \ldots, x_{j-1}\} \in \mathcal{F}$, $B \cup x_j \notin \mathcal{F}$, $B \cup x \in \mathcal{F}$ and $X_j = B \setminus y \cup \{x, x_j\} \in \mathcal{F}$. By the exchange property of distributive Gauss greedoids, there exist either $B \cup x_j \in \mathcal{F}$ or $B \setminus y \cup x_j \in \mathcal{F}$.

The first case contradicts our assumption on step j, whereas the latter case contradicts our choice of X. (Otherwise we could agument $B \setminus y \cup x_j$ from B_1 and get an ideal $X' \in \mathcal{F}$ with $X' \setminus x \in \mathcal{F}$, which is larger than X.)

Thus, for $1 \leq k \leq r(P)$, the sets $\mathcal{B}_k$ are the sets of bases of nested distributive supermatroids

$$\mathcal{M}_k = \{X \in \mathcal{L}(P) \mid X \subseteq B \text{ for some } B \in \mathcal{B}_k\},$$

which are in distributive strong map relation. Therefore, by Theorem 5.3, $\mathcal{F}$ is a distributive Gauss greedoid. $\square$

5.3 Algorithmic characterization

We now return to the problem of optimizing admissible weight functions $w : P \to \mathbb{R}$ over ideal systems $(P, \mathcal{F})$. I.e., we consider the problem

$$\max\{w(X) \mid X \in \mathcal{F}\}.$$

In order to determine an optimal member of $\mathcal{F}$, we try the modified greedy algorithm from Section 2.2:

MODIFIED GREEDY-ALGORITHM:
$X^* = X = \emptyset$;
while $\Gamma(X) \neq \emptyset$ **do**
  Choose $i \in \Gamma(X)$ such that $w(i) \geq w(j)$ for all $j \in \Gamma(X)$;
  $X = X \cup i$;
  if $w(X) > w(X^*)$ **then**
    $X^* = X$;
  end if
end while

We prove that distributive Gauss greedoids can be characterized as those non-empty ideal systems for which the modified Greedy algorithm works correctly. The following Theorem generalizes Theorem 2.4. in [KLS91] from set systems to ideal systems.

Given an ideal system $\mathcal{F} \subseteq \mathcal{L}(P)$, let us define the *k-truncation* of $\mathcal{F}$ as the ideal system

$$\mathcal{F}^{(k)} := \{X \in \mathcal{F} \mid |X| \leq k\} \quad \forall k = 1, \ldots, r(\mathcal{F}).$$

Theorem 5.5. *Let $(P, \mathcal{F})$ be an ideal system with $\emptyset \in \mathcal{F}$. Then the following statements are equivalent:*

(G1) For any admissible linear weight function $c : P \to \mathbb{R}$ the modified greedy algorithm determines an optimal member of $\mathcal{F}$.

(G2) $(P, \mathcal{F}^{(k)})$ is a distributive strong exchange structure for every $k = 0, \ldots, r(P)$.

(G3) $(P, \mathcal{F})$ is a distributive Gauss greedoid.

Proof. "(G2) $\Rightarrow$ (G1)": By Theorem 4.1, we know that property (G2) implies that the modified Greedy algorithm visits an optimal basis of $\mathcal{F}^{(k)}$ for every $k = 0, \ldots, r(P)$. Hence the algorithm determines a member of $\mathcal{F}$ of maximal weight.

"(G1) $\Rightarrow$ (G3)": We first prove that $(P, \mathcal{F})$ is a distributive greedoid. Given $A \in \mathcal{L}(P)$ we define the weight function

$$c_A(i) = \begin{cases} 1 & : \ i \in A \\ -1 & : \ \text{otherwise.} \end{cases}$$

Since A is an ideal, c_A is admissible. If $A \in \mathcal{F}$, then A is the unique optimal solution for problem $\max\{c_A(X) \mid X \in \mathcal{F}\}$.

By assumption, the modified greedy algorithm must produce A, and therefore there exists an ordering $\{a_1, \ldots, a_k\}$ of the elements in A such that $\{a_1, \ldots, a_l\} \in \mathcal{F}$ for each $l \leq k$. Hence $(P, \mathcal{F})$ is accessible. We now have to show that Steinitz' augmentation property (AP) is satisfied. By Lemma 2.1, it is enough to prove that for any ideal $A \in \mathcal{L}(P)$ all bases of A have the same cardinality. This is easy to see, since for the weight function c_A defined above the solutions of the modified greedy algorithm correspond to bases of A. Hence the bases must have the same cardinality, which implies that $(P, \mathcal{F})$ is a distributive greedoid.

Slightly more difficult is to prove that $(P, \mathcal{F})$ is a distributive Gauss greedoid. Consider $B \in \mathcal{F}, z \in B^+, x \not\leq y$ and $x, y \in (P \setminus B)^-$ with $B \setminus z \cup \{x, y\} \in \mathcal{F}$. By Theorem 5.4 it remains to show that either $B \cup y \in \mathcal{F}$ or $B \setminus z \cup y \in \mathcal{F}$.

Suppose this is not true, i.e., neither $B \cup y \in \mathcal{F}$ nor $B \setminus z \cup y \in \mathcal{F}$. Define the weight function

$$c(i) = \begin{cases} 1 & : \quad i \in B \setminus z \\ c_z & : \quad i = z \\ c_y & : \quad i = y \\ c_x & : \quad i = x \\ -|P| & : \quad \text{otherwise,} \end{cases}$$

where $-|P| < c_x < c_z < c_y < 1$. Since $x \not\leq y$ and both, B and $B \setminus z \cup \{x, y\}$, are ideals, c is admissible.

Observe that the set of sequences, which are possibly generated by the modified greedy algorithm, do not depend on the choice of the values of c_x, c_y and c_z, as long as the above inequalities hold. Moreover, B must be part of any possible greedy-sequence. To see this, let $c_x = 1 - |P|, c_z = \frac{1}{3}$ and $c_y = \frac{1}{2}$. Then B is the unique optimal solution for $\max\{c(X) \mid X \in \mathcal{F}\}$.

Now let $c_x = -\frac{1}{2}, c_z = -\frac{1}{3}$ and $c_y = \frac{2}{3}$. Since $B \cup y \notin \mathcal{F}$ and $B \setminus z \cup y \notin \mathcal{F}$, the ideal $B \setminus z \cup \{x, y\}$ is the unique optimal solution for this weight function. However, $B \not\subseteq B \setminus z \cup \{x, y\}$, contradicting the optimality of the modified greedy algorithm. Hence either $B \cup y \in \mathcal{F}$ or $B \setminus z \cup y \in \mathcal{F}$, which proves that $(P, \mathcal{F})$ is a distributive Gauss greedoid.

"(G3) $\Rightarrow$ (G2)": Let $(P, \mathcal{F})$ be a distributive Gauss greedoid with defining distributive supermatroids $\mathcal{M}_1, \ldots, \mathcal{M}_{r(P)}$. Consider a basis B of $\mathcal{F}^{(k)}$ and $A \subseteq B$ such that $A \in \mathcal{F}^{(k)}$ and $i \in P \setminus B$ with $A \cup\ i \in \mathcal{F}^{(k)}$. Let C be a circuit in $B \cup i$ with respect to the distributive supermatroid $(P, \mathcal{M}_k)$, where $|A| = t$.

Suppose $C \setminus i \subseteq \sigma_{\mathcal{M}_{t+1}}(A)$. Clearly, $\sigma_{\mathcal{M}_{t+1}}(A)$ is closed in $\mathcal{M}_{t+1}$. Since $t + 1 \leq k$, it follows from the characterization of distibutive strong maps in Theorem 5.1 that $\sigma_{\mathcal{M}_{t+1}}(A)$ is also closed in $\mathcal{M}_k$.

Therefore, $i \in \sigma_{\mathcal{M}_{t+1}}(A)$, as otherwise (remember we assumed $C \setminus i \subseteq \sigma_{\mathcal{M}_{t+1}}(A)$) the ideal $C \setminus i \in \mathcal{M}_k$ could be augmented by i, in contradiction to C being a circuit of $\mathcal{M}_k$. On the other hand, $i \in \sigma_{\mathcal{M}_{t+1}}(A)$ implies $A \cup i \notin \mathcal{F}^{(k)}$, in contradiction to our assumption.

Hence $(C \setminus i) \setminus \sigma_{\mathcal{M}_{t+1}}(A)$ is non-empty and we may choose elements $j \in ((C \setminus i) \setminus \sigma_{\mathcal{M}_{t+1}}(A))^-$ and $k \in ((C \setminus i) \setminus \sigma_{\mathcal{M}_{t+1}}(A))^+$ with $j \leq k$. Then $j \notin \sigma_{\mathcal{M}_{t+1}}(A)$ implies $A \cup j \in \mathcal{F}^{(k)}$ and, by Lemma 4.1, $i \neq k \in C^+$ implies that $B \setminus k \cup i$ is a basis of $\mathcal{M}_k$. Thus $B \setminus k \cup i$ belongs to $\mathcal{F}^{(k)}$.

$\square$

5.4 Examples

In this Section, we describe three examples of distributive Gauss greedoids: distributive Gaussian elimination greedoids, distributive bipartite matching greedoids and distributive supermatroids.

The distributive Gaussian elimination greedoid arises from the Gaussian elimination algorithm to solve an equality system $Ax = b$, where $(P, \leq)$ is a partial order on the set of columns of A.

The optimization problem corresponding to bipartite matching greedoids is called "ordered marriage problem". It can be interpreted as the problem of two families to find a wedding between the sons of one family and the daughters of the other family such that the sons are married in the order of their ages, and the daughters are married with respect to a possibly more complicated order.

The example of distributive supermatroids simply shows that distributive Gauss greedoids generalize distributive supermatroids.

Moreover, we show that distributive Gauss greedoids are closed under a certain duality operator.

5.4.1 Distributive Gaussian elimination greedoids

Let $A = (a_{ij})$ be an (m, n)-matrix with full row rank and $P = (P, \leq)$ denote a partial order of the set of column indixes. As a generalization of the common Gaussian elimination algorithm, we indroduce the *distributive Gaussian elimination algorithm* as follows:

Iteratively, for $k = 1, \ldots, m$, the algorithm constructs a sequence $\{j_1, \ldots, j_k\} \in \mathcal{L}(P)$ such that it selects for row k a column index $j_k \in (P \setminus \{j_1, \ldots, j_{k-1}\})^-$ and performs a pivot operation. It can be seen that a sequence $\{j_1, \ldots, j_k\} \in \mathcal{L}(P)$ can be chosen as a pivot sequence if and only if for all $l \leq k$ the submatrices $(a_{ij_i})_{1 \leq i \leq l}$ are non-singular.

The family of pivot sequences

$$\mathcal{F} := \{\{j_1, \ldots, j_k\} \in \mathcal{L}(P) \mid (a_{ij_i})_{1 \leq i \leq l} \text{ is non-singular for any } l \leq k\}.$$

is called the *distributive Gaussian elimination greedoid*.

Consider now for any $k = 1, \ldots, m$ the distributive supermatroid $(P, \mathcal{M}_k)$ with

$$\mathcal{M}_k := \{\{j_1, \ldots, j_l\} \in \mathcal{L}(P) \mid \{(a_{ij_1})_{1 \leq i \leq k}, \ldots, (a_{ij_l})_{1 \leq i \leq k}\} \text{ linearly independent}\}.$$

Then the sequence $\{j_1, \ldots, j_k\}$ is a pivot sequence if and only if $\{j_1, \ldots, j_k\}$ is a basis in $\mathcal{M}_k$. Hence the distributive Gaussian elimination greedoid is a distributive Gauss greedoid.

5.4.2 The ordered marriage problem

Let $G = (S \cup P, E)$ be a bipartite graph such that $s_1, \ldots, s_m$ is an ordering of the vertices in S, and $(P, \leq)$ is a partial order of the vertices in P.

For an ideal $A \in \mathcal{L}(P)$, let $G(A)$ be the subgraph induced by the vertices in $\{s_1, \dots, s_{|A|}\} \cup A$. Then we define

$$\mathcal{F} := \{A \in \mathcal{L}(P) \mid \text{there exists a perfect matching in } G(A)\}$$

to be a *distributive bipartite matching greedoid*.

If $w : P \to \mathbb{R}$ is an admissible weighting of the elements in P, then the *ordered marriage problem*

$$\max\{w(F) \mid F \in \mathcal{F}\}$$

can be described as to find an optimal matching with respect to the total order on S and the partial order on P. The ordered marriage problem generalizes the *medieval marriage problem.* [1]

We show that the modified greedy algorithm solves the ordered marriage problem by proving that $\mathcal{F}$ is a distributive Gauss greedoid: For each $k = 1, \dots, m$ let G_k be the subgraph induced by the vertices in $\{s_1, \dots, s_k\} \cup P$. If we consider the distributive supermatroid $(P, \mathcal{M}_k)$ with

$$\mathcal{M}_k := \{A \in \mathcal{L}(P) \mid \text{there exists a matching in } G_k \text{ covering } A\},$$

then we have

$$A \text{ is a basis in } \mathcal{M}_k \Leftrightarrow A \in \mathcal{F}.$$

Further on, we can show that for each $k = 1, \dots, m-1$, $(P, \mathcal{M}_k)$ is a distributive strong map of $(P, \mathcal{M}_{k+1})$: By Theorem 5.1 and Lemma 5.2 we have to prove that for any $A \in \mathcal{L}(P)$ and $p \in (P \setminus A)^-$

$$r_k(A \cup p) > r_k(A) \Rightarrow r_{k+1}(A \cup p) > r_{k+1}(A)$$

holds. Choose $A \in \mathcal{L}(P)$ and $p \in (P \setminus A)^-$ such that $r_k(A \cup p) > r_k(A)$. If $r_k(A) = r_{k+1}(A)$, there is nothing to prove. In case $r_k(A) < r_{k+1}(A)$, there exists a path from s_{k+1} to a vertex in A, whose edges alternate between a matching M in G_{k+1} and a matching N in G_k. Because of $r_k(A \cup p) > r_k(A)$, there exists an edge between p and some vertex in $\{s_1, \dots, s_k\}$, which is not on the augmenting path. Hence this edge can be added to M, implying $r_{k+1}(A \cup p) > r_{k+1}(A)$.

5.4.3 Distributive supermatroids

In case $\mathcal{F} \subseteq \mathcal{L}(P)$ is a distributive supermatroid, it is easy to see that for each $k \le r(\mathcal{F})$ the k-truncation $\mathcal{F}^{(k)} = \{X \in \mathcal{F} \mid |X| \le k\}$ is a distributive supermatroid itself. Moreover, the distributive supermatroids $\{\mathcal{F}^{(k)}\}_{k \le r(\mathcal{F})}$ are in distributive strong map relation. Therefore,

$$\mathcal{F} = \mathcal{B}(\mathcal{F}^{(1)}) \cup \ldots \cup \mathcal{B}(\mathcal{F}^{(r(\mathcal{F}))})$$

is a distributive Gauss greedoid.

[1] This name was given by J. Edmonds. In medieval times fathers used to marry their daughters in the order of their ages. I am pretty sure that also the sons were married with respect to some order.

5.4.4 Duality

For arbitrary ideal systems $(P, \mathcal{F})$, a dual ideal system $(P^*, \mathcal{F}^*)$ is defined on the poset $P^* = (P, \leq^*)$ with order relation

$$i \leq^* j \quad \Leftrightarrow \quad j \leq i$$

via

$$\mathcal{F}^* = \{X \in \mathcal{L}(P) \mid B \subseteq X \text{ for some basis } B \text{in } \mathcal{F}\}.$$

For example, the dual of a distributive supermatroid is a distributive supermatroids again.

This is not necessarily true for distributive Gauss greedoids. However, full distributive Gauss greedoids, i.e., those that contain the maximal ideal P, are closed under duality. Generalizing a result of Goecke about the dual of a Gauss greedoid ([Goe86a], Theorem 2.3), we can prove:

Theorem 5.6. *If $(P, \mathcal{F})$ is a full distributive Gauss greedoid, then its dual $(P^*, \mathcal{F}^*)$ is again a full distributive Gauss greedoid.*

Proof. Let $\mathcal{M}_1^*, \ldots, \mathcal{M}_m^*$ be the distributive supermatroids defining $(P^*, \mathcal{F}^*)$. We need to show that for any $k = 1, \ldots, m-1$, M_{k+1}^* is a distributive strong map of M_k^*.

It is known (cf. [DIW72]) that, given the rank functions r_k of the distributive supermatroids $(P, \mathcal{M}_k)$, the rank functions r_k^* of the distributive supermatroids $(P^*, \mathcal{M}_k^*)$ satisfy

$$r_k^*(X) = |P| - |X| - r_k(P) + r_k(X) \quad (\forall X \in \mathcal{L}(P)).$$

As $\mathcal{M}_k^*$ is a distributive supermatroid with respect to the order P^*, we obtain

$$X \subseteq^* Y \iff Y \subseteq X$$

for all $X, Y \in \mathcal{L}(P)$. By Lemma 5.2, we therefore need to prove

$$r_k^*(X \setminus e) - r_k^*(X) \leq r_{k+1}^*(X \setminus e) - r_{k+1}^*(X)$$

for all $X \in \mathcal{L}(P)$ and $e \in (P \setminus X)^-$.

Since $\mathcal{M}_k$ is a distributive strong map of M_{k+1}, we observe:

$$\begin{aligned} r_k^*(X \setminus e) - r_k^*(X) &= |X| - |X \setminus e| + r_k(X \setminus e) - r_k(X) \\ &= 1 + r_k(X \setminus e) - r_k(X) \\ &\geq 1 + r_{k+1}(X \setminus e) - r_{k+1}(X) \\ &= r_{k+1}^*(X \setminus e) - r_{k+1}^*(X) \end{aligned}$$

Thus, the distributive supermatroids $\mathcal{M}_1^*, \ldots, \mathcal{M}_m^*$ are nested and in a distributive strong map relation, i.e.,

$$\mathcal{F}^* = \mathcal{B}(\mathcal{M}_1^*) \cup \ldots \cup \mathcal{B}(\mathcal{M}_m^*) \cup P$$

is a distributive Gauss greedoid. $\square$

Chapter 6

Distributive Δ-matroids

So far, we got to know three different types of ideal systems which can be characterized by appropriate greedy-type algorithms:

Distributive supermatroids are the monotone ideal systems $\mathcal{F} \subseteq \mathcal{L}(P)$ such that the matroid greedy algorithm determines an optimal member for arbitrary admissible weight functions $w : P \to \mathbb{R}$.

If we abstain from the monotony, but require $\mathcal{F}$ to be accessible, we saw that the same greedy algorithm determines an optimal basis if $\mathcal{F}$ is a distributive strong exchange structure.

Such an optimal basis is as well an optimal member as long as w is non-negative. If negative components are allowed, we need to apply a modication of the greedy algorithm. This modified greedy algorithm determines an optimal member in case $\mathcal{F}$ is a distributive Gauss greedoid.

But what happens if $\mathcal{F}$ is not even accessible? Is there still a greedy strategy which solves the optimization problem? We saw in Chapter 2 that Δ-matroids and jump systems are families of subsets, resp. integral vectors, for which certain greedy-type algorithms work well for linear and separable concave weight functions respectively.

In this Chapter, we extend Δ-matroids and jump systems to *distributive* Δ*-matroids* and prove that a certain Δ-*greedy algorithm* determines an optimal member of such a distributive Δ-matroid for any non-negative admissible weight function.

6.1 Definition of distributive Δ-matroids

For the definition of distributive Δ-matroids, we simply adapt the definition of jump systems to the case where integral vectors are regarded as ideals of a disjoint union of chains.

This way, we extend the definition of jump systems from integral vectors to ideals of arbitrary posets $P = (P, \leq)$.

For two ideals $X, Y \in \mathcal{L}(P)$ we define

$$\mathcal{B}(X,Y) := \{Z \in \mathcal{L}(P) \mid X \cap Y \subseteq Z \subseteq X \cup Y\}$$

and call $\mathcal{B}(X,Y)$ a *box*. Further, we dentote by

$$d(X,Y) = |X \Delta Y| = |X \setminus Y| + |Y \setminus X|$$

the *distance* between two ideals $X, Y \in \mathcal{L}(P)$. Accordingly, for two ideal systems $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{L}(P)$, let

$$d(\mathcal{F}_1, \mathcal{F}_2) = \min\{d(X,Y) \mid X \in \mathcal{F}_1, Y \in \mathcal{F}_2\}.$$

We say that an ideal $X' \in \mathcal{L}(P)$ is a *step* from X to Y (or an (X,Y)-step) if $X' \in \mathcal{B}(X,Y)$ and $d(X,X') = 1$. Therefore, for any $i \in (Y \setminus X)^-$ and any $j \in (X \setminus Y)^+$ the ideals $X \cup i$ and $X \setminus j$ are (X,Y)-steps. We are now able to define distributive Δ-matroids:

Definition 6.1 (Distributive Δ-matroid). *A nonempty ideal system $\mathcal{F} \subseteq \mathcal{L}(P)$ is a* distributive Δ-matroid *if $\mathcal{F}$ satisfies the* two-step axiom *saying*

"Given $A, B \in \mathcal{F}$ and an (A,B)-step A',
either $A' \in \mathcal{F}$, or there exists an (A', B)-step A'' such that $A'' \in \mathcal{F}$."

See Figure 6.1 for an example of a distributive Δ-matroid.

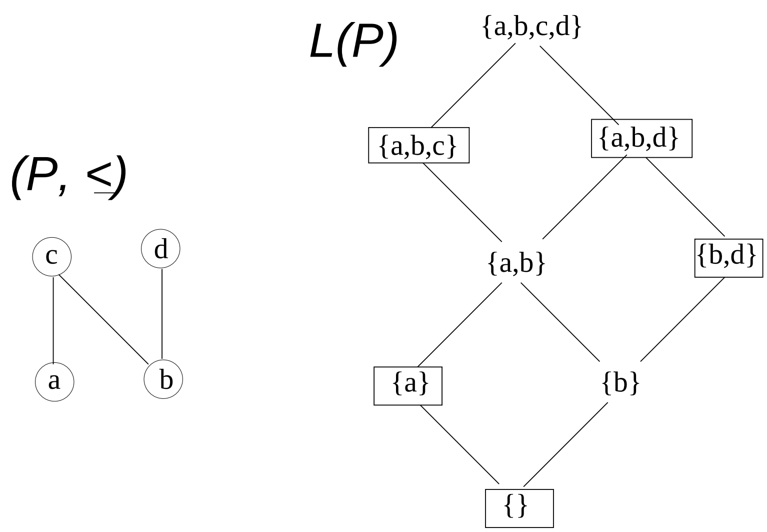


Figure 6.1: Distributive Δ-matroid.

Let us recall the definitions of ordinary Δ-matroids and jump systems from Sections 2.2 and 2.3 , and observe that they are special instances of distributive Δ-matroids:

6.2 Special case: Δ-matroids

Bouchet [Bou89], [Bou87] introduced Δ-matroids as follows: A nonempty set system $\mathcal{F} \subseteq 2^E$ is called a Δ-*matroid* if it satisfies the *symmetric exchange axiom* saying

"For $A, B \in \mathcal{F}$ and $x \in A\Delta B$,
there exists $y \in A\Delta B$ such that $A\Delta\{x, y\} \in \mathcal{F}$."

We now consider distributive Δ-matroids $\mathcal{F} \subseteq \mathcal{L}(P)$ in the special case, where P is an antichain. Then $A' = A\Delta x$ is an (A, B)-step and either $A'' = A\Delta\{x, y\}$ is an (A', B)-step, or $A' = A''$ (if $x = y$). Hence, distributive Δ-matroids generalize Δ-matroids. Since Δ-matroids generalize matroids, distributive Δ-matroids are therefore an extension of matroids as well.

6.3 Special case: jump systems

Representing subsets via binary vectors and generalizing the definitions of Δ-matroids to arbitrary integral vectors, Bouchet and Cunningham [BC95] introduced jump systems. We recall their definition from Section 2.3:

Let $V = \{1, \dots, n\}$. For $x, y \in \mathbb{Z}^V$ define

$$[x, y] := \{x' \in \mathbb{Z}^V \mid \min\{x_i, y_i\} \leq x'_i \leq \max\{x_i, y_i\}, \forall i \in V\}$$

and $d(x, y) := \sum_{i \in V} |x_i - y_i|$. Then x' is a step from x to y (or an (x, y)-step) if $x' \in [x, y]$ and $d(x, x') = 1$.

We now consider distributive Δ-matroids $\mathcal{F} \subseteq \mathcal{L}(P)$ in the special case, where P is the disjoint union of n chains $P = C_1 \ \dot\cup \ \dots \ \dot\cup \ C_n$. Then we may identify each vector $x \in \mathbb{Z}^n$ with the ideal $X \in \mathcal{L}(P)$ consisting of the first x_i elements of chain C_i for each $i = 1, .., n$. Hence, $[x, y]$ is just the box $\mathcal{B}(X, Y)$, $d(x, y) = d(X, Y)$, and x' is an (x, y)-step if and only if X' is an (X, Y)-step.

A nonempty subset $\mathcal{J} \subseteq \mathbb{Z}^V$ is called a *jump system* if it satisfies the *two-step axiom* which says in this context

"Given $x, y \in \mathcal{J}$ and an (x, y)-step x',
either $x' \in \mathcal{J}$, or there exists an (x', y)-step x'' such that $x'' \in \mathcal{J}$.

Hence, distributive Δ-matroids reduce to jump systems in case P is the disjoint union of chains.

It is known (*cf.* [Gee96]) that the following greedy-type algorithm determines a member of maximal weight for any jump system $\mathcal{J} \subseteq \mathbb{Z}^V$:

GREEDY-ALGORITHM for jump systems:
$\mathcal{J}^0 = \mathcal{J}$;
for $i = 1, \ldots, n$ **do**
  $\alpha_i = \max\{x_i \mid x \in \mathcal{J}^{i-1}\}$;
  $\mathcal{J}^i = \{x \in \mathcal{J}^{i-1} \mid x_i = \alpha_i\}$;
end for

Further, note that jump systems have a very nice polyhedral characterization: It has been shown by Bouchet and Cunningham [BC95] that a polyhedra with integral vertices is bisubmodular if and only if the integral points in it form a jump system. Unfortunately, we were not able to prove a similiar result for the more general distributive Δ-matroids.

6.4 The Δ-greedy algorithm

We prove in the rest of this Chapter that the following Δ-*greedy algorithm* (which generalizes the greedy algorithm for jump systems) determines an optimal member of a distributive Δ-matroid for any nonnegative admissible weight function $w : P \to \mathbb{R}_+$. Since we consider only admissible weight functions, we assume $P = \{1, \ldots, n\}$ to be ordered such that

$$w_1 \geq \ldots \geq w_k > 0 = w_{k+1} = \ldots = w_n \quad \text{and} \quad P_i = \{1, \ldots, i\} \in \mathcal{L}(P)$$

Δ-GREEDY-ALGORITHM:
$\mathcal{F}^0 = \mathcal{F}$;
for $i = 1, \ldots, k$ **do**
  if $\{A \in \mathcal{F}^{i-1} \mid i \in A\} \neq \emptyset$ **then**
    $\mathcal{F}^i = \{A \in \mathcal{F}^{i-1} \mid i \in A\}$;
  end if
end for

Unlike jump systems, the reflection of a distributive Δ-matroid is not defined. Hence, we need to require the weight function to be non-negative.

The following observation about the distance between an ideal and a box is fundamental for subsequent results:

Lemma 6.1. *Let $A, X, Y \in \mathcal{L}(P)$ and $\mathcal{B}(X, Y)$ be a box. Then*

$$d(A, \mathcal{B}(X, Y)) \;=\; |A \setminus (X \cup Y)| + |(X \cap Y) \setminus A|.$$

Proof. Choose an ideal $B \in \mathcal{B}(X,Y)$ such that $d(A,B) = d(A,\mathcal{B}(X,Y))$. Then $X \cap Y \subseteq B \subseteq X \cup Y$ implies

$$|B \setminus A| \geq |(X \cap Y) \setminus A| \quad \text{and} \quad |A \setminus B| \geq |A \setminus (X \cup Y)|.$$

Suppose $|B \setminus A| > |(X \cap Y) \setminus A|$. Then we may choose an element $b \in ((B \setminus (X \cap Y)) \setminus A)^+$ such that $B \setminus b \in \mathcal{L}(P)$. In particular, $B \setminus b \in \mathcal{B}(X,Y)$. Since $|(B \setminus b) \setminus A| < |B \setminus A|$ and $|A \setminus (B \setminus b)| = |A \setminus B|$, we get a contradiction to $d(A,B) = d(A,\mathcal{B}(X,Y))$.

Suppose $|A \setminus B| > |A \setminus (X \cup Y)|$. Then we may choose an element $a \in (((X \cup Y) \cap A) \setminus B)^-$ such that $B \cup a \in \mathcal{L}(P)$. In particular, $B \cup a \in \mathcal{B}(X,Y)$. Since $|(B \cup a) \setminus A| = |B \setminus A|$ and $|A \setminus (B \cup a)| < |A \setminus B|$, we obtain again a contradiction to $d(A,B) = d(A,\mathcal{B}(X,Y))$.

Summarizing we obtain $|B \setminus A| = |(X \cap Y) \setminus A|$ and $|A \setminus B| = |A \setminus (X \cup Y)|$.

□

Given a box $\mathcal{B}$, we denote the set of feasible ideals with minimal distance to $\mathcal{B}$ as

$$\mathcal{F}_{\mathcal{B}} := \{A \in \mathcal{F} \mid d(A,\mathcal{B}) = d(\mathcal{F},\mathcal{B})\}.$$

The following characterization of distributive Δ-matroids generalizes a result, due to Lovász [Lov97], about jump systems:

Theorem 6.1. *For $\mathcal{F} \subseteq \mathcal{L}(P)$ are equivalent*

1. *$\mathcal{F}$ is a distributive Δ-matroid.*
2. *Given boxes $\mathcal{B}^1 \subseteq \ldots \subseteq \mathcal{B}^r$, then $\mathcal{F}_{\mathcal{B}^1} \cap \ldots \cap \mathcal{F}_{\mathcal{B}^r} \neq \emptyset$.*

Proof. (2) ⇒ (1):
Suppose $\mathcal{F}$ satisfies property (2) and we are given $A, B \in \mathcal{F}$ and an (A,B)-step A'. Since $A' = \mathcal{B}(A',A') \subseteq \mathcal{B}(A',B)$, by property (2), there exists an ideal $A'' \in \mathcal{F}_{A'} \cap \mathcal{F}_{\mathcal{B}(A',B)}$. The box $\mathcal{B}(A',B)$ contains the ideal $B \in \mathcal{F}$. Hence $A'' \in \mathcal{B}(A',B)$. Furthermore, $d(A',A) = 1$ implies $d(A'',A') \leq 1$. Therefore, either $A' \in \mathcal{F}$ or $A'' \in \mathcal{F}$, *i.e.* $\mathcal{F}$ satisfies the two-step axiom, as required.

(1) ⇒ (2):
Suppose $\mathcal{F}$ is a distributive Δ-matroid and we are given boxes $\mathcal{B}^1 \subseteq \ldots \subseteq \mathcal{B}^r$. We prove (2) by induction on r. Since $\mathcal{F}$ is non-empty, the set $\mathcal{F}_{\mathcal{B}}$ is non-empty for any box $\mathcal{B}$. Therefore, (2) holds in case $r = 1$.

Suppose $r > 1$ and (2) holds for all smaller cases. Take $B \in \mathcal{F}_{\mathcal{B}^r}$ and choose $A \in \mathcal{F}_{\mathcal{B}^1} \cap \ldots \cap \mathcal{F}_{\mathcal{B}^{r-1}}$ minimizing $d(A,B)$. In case $A \in \mathcal{F}_{\mathcal{B}^r}$, there is nothing to prove. We first show that in case $A \notin \mathcal{F}_{\mathcal{B}^r}$, there exists an (A,B)-step A' with $d(A',\mathcal{B}^r) < d(A,B)$:

Let $\mathcal{B}^r = \mathcal{B}(X, Y)$ for two ideals $X, Y \in \mathcal{L}(P)$. By Lemma 6.1, the inequality $d(B, \mathcal{B}^r) < d(A, \mathcal{B}^r)$ is equivalent to

$$|B \setminus (X \cup Y)| + |(X \cap Y) \setminus B| < |A \setminus (X \cup Y)| + |(X \cap Y) \setminus A|.$$

If $|B \setminus (X \cup Y)| < |A \setminus (X \cup Y)|$, choose $a \in (A \setminus (X \cup Y \cup B))^+$ and set $A' = A \setminus a$. Then A' is an (A, B)-step with

$$|A' \setminus (X \cup Y)| < |A \setminus (X \cup Y)| \text{ and } |(X \cap Y) \setminus A'| = |(X \cap Y) \setminus A|.$$

Otherwise, we have $|(X \cap Y) \setminus B| < |(X \cap Y) \setminus A|$. Choose $b \in ((B \cap X \cap Y) \setminus A)^-$ and set $A' = A \cup b$. Then A' is an (A, B)-step with

$$|A' \setminus (X \cup Y)| = |A \setminus (X \cup Y)| \text{ and } |(X \cap Y) \setminus A'| < |(X \cap Y) \setminus A|.$$

In both cases, we get $d(A', \mathcal{B}^r) < d(A, \mathcal{B}^r)$.

Since the boxes are nested, $d(A', \mathcal{B}^l) < d(A, \mathcal{B}^l)$ holds for any $l \leq r - 1$. Thus $A' \notin \mathcal{F}$. By the two-step axiom, there exists an (A', B)-step $A'' \in \mathcal{F}$. Let $d(A', \mathcal{B}^l) = d(A', C^l)$ for $l = 1, \ldots, r - 1$. Then $d(A', A'') = 1$ implies

$$d(A'', \mathcal{B}^l) \leq |A'' \setminus C^l| + |C^l \setminus A''| \leq |A' \setminus C^l| + |C^l \setminus A'| + 1 = d(A', \mathcal{B}^l) + 1.$$

Together with $d(A', \mathcal{B}^l) < d(A, \mathcal{B}^l)$, it follows $d(A'', \mathcal{B}^l) \leq d(A, \mathcal{B}^l)$ for any $l \leq r - 1$. Therefore, we obtain $A'' \in \mathcal{F}_{\mathcal{B}^1} \cap \ldots \cap \mathcal{F}_{\mathcal{B}^{r-1}}$ with $d(A'', B) < d(A, B)$, in contradiction to the choice of A.

□

We now prove that, given any admissible nonnegative weight function and any distributive Δ-matroid $\mathcal{F}$, each member of the set $\mathcal{F}^k$, returned by the Δ-greedy algorithm, has maximal weight. But before, we prove that the algorithm works optimally for elementary weight functions.

For $l = 1, \ldots, n$ define the elementary weight functions $c^l : P \to \{0, 1\}$ with components

$$c^l_1 = \ldots = c^l_l = 1 \text{ and } c^l_{l+1} = \ldots = c^l_n = 0.$$

In other words, $c^l(X) = |X \cap \{1, \ldots, l\}|$.

Lemma 6.2. *If $\mathcal{F}$ is a distributive Δ-matroid, then each ideal $A \in \mathcal{F}^k$ simultaneously maximizes $c^l(X)$ over $\mathcal{F}$ for each $l = 1, \ldots, n$.*

Proof. For $i = 0, \ldots, k$ define the boxes

$$\mathcal{B}^i = \mathcal{B}(P_i, P).$$

I.e. $\mathcal{B}^0 = \mathcal{L}(P)$ and $\mathcal{B}^i = \{X \in \mathcal{L}(P) \mid \{1, \ldots, i\} \subseteq X\}$ for $i = 1, \ldots, k$. For each ideal $X \in \mathcal{L}(P)$, it follows from the definition that

$$d(X, \mathcal{B}^0) = 0 \text{ and } d(X, \mathcal{B}^i) = i - |X \cap \{1, .., i\}| \text{ for } i = 1, .., k.$$

Hence, $A \in \mathcal{F}_{\mathcal{B}^i}$ implies that A maximizes $c^i(X)$ over $\mathcal{F}$, and vice versa. It remains to show that $\mathcal{F}^k = \mathcal{F}_{\mathcal{B}^0} \cap \ldots \cap \mathcal{F}_{\mathcal{B}^k}$.

Since $\mathcal{B}^k \subseteq \ldots \subseteq \mathcal{B}^0$, we know by Theorem 6.1 that $\mathcal{F}_{\mathcal{B}^0} \cap \ldots \cap \mathcal{F}_{\mathcal{B}^k}$ is non-empty. Clearly, $\mathcal{F}_{\mathcal{B}^0} = \mathcal{F} = \mathcal{F}^0$.

Inductively, assume $\mathcal{F}^{k-1} = \mathcal{F}_{\mathcal{B}^0} \cap \ldots \cap \mathcal{F}_{\mathcal{B}^{k-1}}$ and choose $A \in \mathcal{F}_{\mathcal{B}^0} \cap \ldots \cap \mathcal{F}_{\mathcal{B}^k}$. By induction hypothesis, we have $A \in \mathcal{F}^{k-1}$. Moreover, $A \in \mathcal{F}_{\mathcal{B}^k}$ is equivalent to A being a maximizer of $c^k(X)$ over $\mathcal{F}$. Hence, A must be a maximizer of $|k \cap X|$ over $\mathcal{F}^{k-1}$. By the proceeding of the Δ-greedy algorithm, this implies $A \in \mathcal{F}^k$. □

We are now able to prove the main Theorem of this Chapter.

Theorem 6.2. *If $\mathcal{F} \subseteq \mathcal{L}(P)$ is a distributive Δ-matroid and $w : P \to \mathbb{R}_+$ is admissible, then each ideal $A \in \mathcal{F}^k$ maximizes $w(X)$ over $\mathcal{F}$.*

Proof. Define the weight function $w' : P \to \mathbb{R}$ with

$$w'_n = w_n \text{ and } w'_i = w_i - w_{i+1} \text{ for } i = 1, \ldots, n-1.$$

Since w is non-increasing, w' is non-negative. Moreover

$$w = \sum_{i \in P} w'_i c^i.$$

By Lemma 6.2, each $A \in \mathcal{F}^k$ simultaneously maximizes $c^l(X)$ over $\mathcal{F}$ for $l = 1, \ldots, k$. Therefore, since w is a non-negative linear combination of these c^l, each $A \in \mathcal{F}^k$ maximizes $w(X)$ over $\mathcal{F}$. □

Summarizing, in the first part of this thesis, we proved the optimality of the greedy algorithm, the modified greedy algorithm and the Δ-greedy algorithm for distributive strong exchange structures, distributive Gauss greedoids and distributive Δ-matroids, respectively.

Part II

Characterization of colored split graphs

Chapter 7

König-Egerváry graphs and Red/Blue-split graphs

After we investigated and characterized different generalizations of matroids to ordered structures in the first part, the second part is used to analyze the structure of König-Egerváry graphs and Red/Blue-split graphs. In particular, we characterize Red/Blue-split graphs (and a weighted version of them) by excluded subgraphs such that the characterization of König-Egerváry graphs can be deduced.

7.1 König-Egerváry graphs

Recall from Chapter 1 that are defined as those undirected graphs $G = (V, E)$ such that the maximal size of a matching equals the minimal size of a vertex cover. Thus, König-Egerváry graphs generalize bipartite graphs.

While a matching of maximal size can be calculated efficiently (e.g. with Edmonds' augmenting path algorithm [Edm65]), it is $\mathcal{NP}$-hard to determine a vertex cover of minimal size in general graphs [Kar72]. However, in case G is a König-Egerváry graph, a maximum matching can be used to find a vertex cover of minimal size in polynomial time (cf. [Dem79], [Ste79] or Chapter 8). But how do we know whether a graph is a König-Egerváry graph, or not?

To answer this question, we first observe that we may restrict our considerations to graphs with *perfect* matchings, i.e. matchings covering all vertices.

7.1.1 Restriction to graphs admitting perfect matchings

Consider a graph $G = (V, E)$ with a maximum matching M such that the vertices in $V_0 \subseteq V$ are not covered by M. We now extend G to a graph G' with perfect matching M'

such that G is a König-Egerváry graph if and only if G' is a König-Egerváry graph:

To obtain the expanded graph G', we replace each vertex $v \in V_0$ by two copies v', v'', and each edge $(u, v) \in E$ with $v \in V_0$ by two edges $(u, v'), (u, v'')$. Further, we add a new matching edge (v', v''). Clearly, $M' = M \cup \{(v', v'') \mid v \in V_0\}$ is a perfect matching in the resulting graph G' (See Figure 7.1).

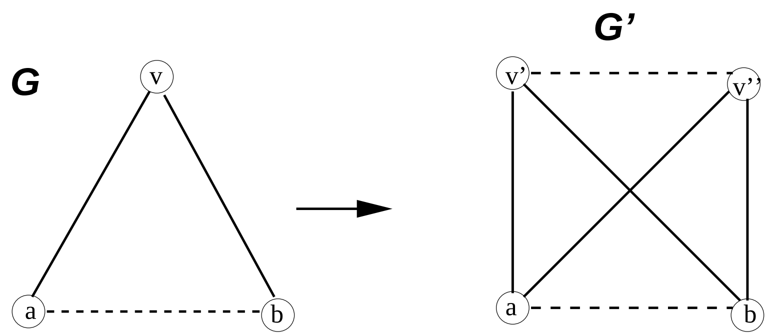


Figure 7.1: Graph G and its extension G': dashed edges are matching edges.

Lemma 7.1. *G is a König-Egerváry graph if and only if the expanded graph G' is a König-Egerváry graph.*

Proof. Suppose G is a König-Egerváry graph and let $C \subseteq V$ be a vertex cover of size $|M|$. Then each vertex in C covers exactly one edge in M and vice versa. Therefore, $C \cap V_0 = \emptyset$. Extending C by the set $\{v' \mid v \in V_0\}$, we result in a vertex cover $C' = C \cup \{v' \mid v \in V_0\}$ in G' of size $|M'|$. Hence, G' is a König-Egerváry graph.

Conversely, suppose the expanded graph G' is a König-Egerváry graph and let C' be a vertex cover of size $|M'|$. Since each vertex in C' covers exactly one edge in M', the restriction of C' to the vertices in V is a cover of size $|M|$. Thus, G is a König-Egerváry graph as well. $\square$

König-Egerváry graphs have been studied and characterized before. But before we state some of the known characterizations of König-Egerváry graphs, let us agree on certain notations:

Let $G = (V, E)$ be an undirected graph and $M \subseteq E$ be a matching. A *walk* is a sequence

$$W = \{v_1, e_1, v_2, e_2, \ldots, v_{k-1}, e_{k-1}, v_k\}$$

with vertices $v_1, \ldots, v_k \subseteq V$ and edges $e_1 = (v_1, v_2), \ldots, e_{k-1} = (v_{k-1}, v_k)$. If matching edges alternate with non-matching edges, W is an *M-alternating walk*. The walk is a *cycle*, if $v_1 = v_k$. The starting- and ending vertex of a cycle is called the *base* of the cycle. Depending on the number of traversed edges, a walk is either *even* (i.e. k is odd), or *odd* (i.e. k is even).

We note that a walk might neither be vertex-, nor edge disjoint. If a walk W is vertex disjoint, W is also called a *path*. A cycle which is vertex disjoint (except for the base) is a *circuit*.

7.1.2 Characterization of Deming-Sterboul

An odd M-alternating circuit is called a *blossom*. Two blossoms, whose bases are linked by an odd M-alternating path starting and ending with matching edges form an *M-handcuff*. (see Figure 7.2).

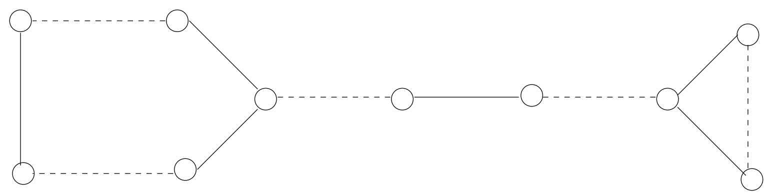


Figure 7.2: M-handcuff: dashed edges are matching edges.

Deming and Sterboul proved

Theorem 7.1 ([Dem79], [Ste79]). *Let M be a perfect matching in G. Then G is a König-Egerváry graph if and only if there exists no M-handcuff in G.*

We will generalize this result in Chapter 8. But, this characterization is not really sufficient when we are interested in a characterization by excluded subgraphs: the two blossoms are not necessarly disjoint, and we do not know the concrete structure of the subgraph induced by an M-handcuff. However, an excluded subgraph characterization was given by Lovász:

7.1.3 Characterization of Lovász

An *even subdivison* of an edge is the replacement of a matching edge by an odd M-alternating path starting and ending with matching edges, or the replacement of a non-matching edge by an odd M-alternating path starting and ending with non-matching edges. Lovász proved

Theorem 7.2 ([Lov83]). *Let G be an undirected graph. Then G is not a König-Egerváry graph if and only if there exists a perfect matching $\hat{M}$ such that G contains a subgraph resulting of even subdivisions of one of the configurations in Figure 7.3.*

We will deduce Lovász characterization from ours in Chapter 9. The problem of his characterization is that he needs a very special perfect matching, not an arbitrary given one. Korach was able to characterize König-Egerváry graphs by exclusion of subgraphs with respect to an arbitrary perfect matching:

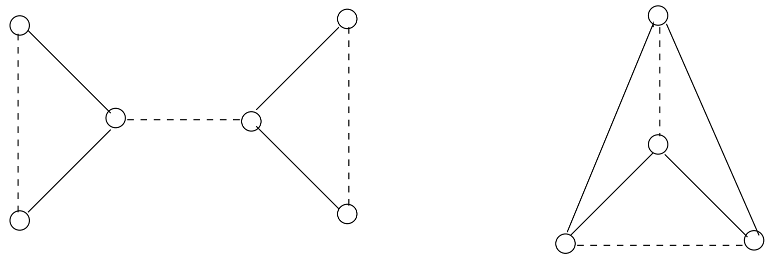


Figure 7.3: Lovász forbidden subgraphs: dashed edges are matching edges.

7.1.4 Characterization of Korach

Korach proved in his dissertation:

Theorem 7.3 ([Kor82]). *Let M be a perfect matching in G. Then G is a König-Egerváry graph if and only if no subgraph results of even subdivisions of one of the configurations in Figure 7.4.*

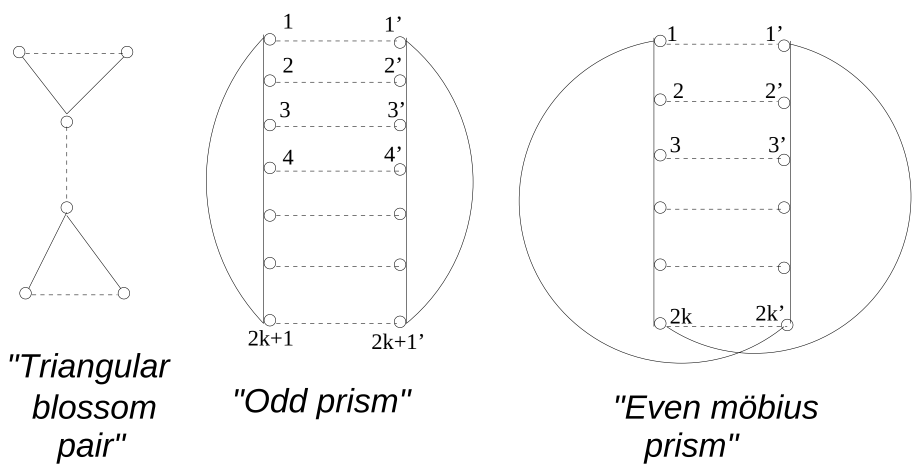


Figure 7.4: Korach's forbidden configurations: dashed edges are the matching edges.

Additionally, he proved for the case of arbitrary maximal matchings, which are not necessarly perfect:

Theorem 7.4 ([Kor82]). *Let M be a maximum matching in G. Then G is a König-Egerváry graph if and only if no subgraph results of even subdivisions of one of the configurations in Figure 7.4 or Figure 7.5.*

In Chapter 9, we present an easier prove of Korach's result by characterizing the more general *Red/Blue-split graphs*.

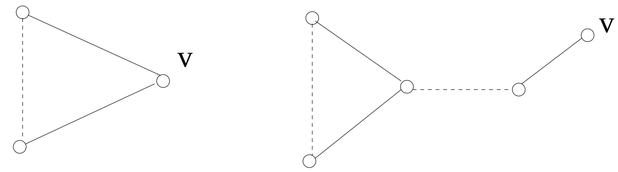


Figure 7.5: Korach's additional forbidden configurations: vertex v is uncovered.

7.2 Red/Blue-split graphs

Let us recall the definition of Red/Blue-split graphs: A *red-blue graph* is a graph $G = (V, R \cup B)$ whose edge set consists of red and blue colored edges (R denotes the red, and B denotes the blue edges)[1].

Definition 7.1. *A red-blue graph $G = (V, R \cup B)$ is a* Red/Blue-split graph *(or* R/B-split graph *for short), if there exists a partition*

$$V = S_R \ \dot{\cup}\ S_B$$

of the vertices into a "red" stable set S_R in the "red" graph $G_R = (V, R)$ and a "blue" stable set S_B in the "blue" graph $G_B = (V, B)$.

We assume the two subgraphs $G_R = (V, R)$ and $G_B = (V, B)$ to contain neither loops nor multiple edges, each. Nevertheless, a red and a blue edge might be parallel. See Figure 7.6 for an example of an R/B-split graph.

Given a feasible partition of $V = S_R \ \dot{\cup}\ S_B$ into a red and a blue stable set, we can always modify the partition to a feasible partition $V = S'_R \ \dot{\cup}\ S'_B$ by iteratively moving a vertex v from S_B to S_R if $S_R \cup \{v\}$ is stable in G_R. Hence, we may assume the red stable set to be inclusionwise maximal.

We already observed in the Introduction that Red/Blue-split graphs generalize König-Egerváry graphs. Beside König-Egerváry graphs, the notion of R/B-split graphs also generalizes ordinary split graphs: Recall that an (uncolored) graph $G = (V, E)$ is a split graph if and only if the vertices can be split into a stable set and a clique. Therefore, in case the red edges form the complement of the blue edges (i.e. $R = \bar{B}$), a red-blue graph $G = (V, R \cup B)$ is an R/B-split graph if and only if G_B as well as G_R are split graphs.

The problem to decide whether a red-blue graph is an R/B-split graph or not is called R/B-SPLIT PROBLEM. We prove that it is equivalent to the 2-SATISFIABILITY PROBLEM and therefore solvable in polynomial time:

[1] red-blue graphs are more famous under the name "signed graphs", which were introduced by Harary [Har54] and studied intensively by Zaslavsky [Zas82]

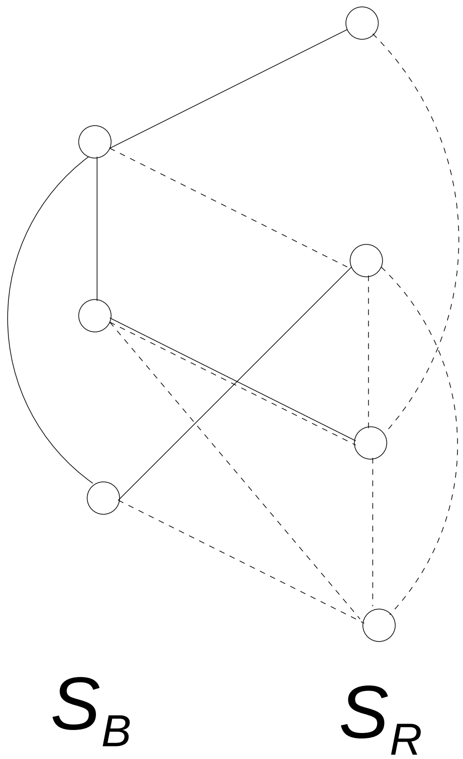


Figure 7.6: R/B-split graph: dashed edges are blue, solid edges are red.

7.2.1 Equivalence to the 2-satisfiability problem

The 2-SATISFIABILITY PROBLEM can be formulated as follows: Suppose we are given a function $f : \{0,1\}^n \to \{0,1\}$ such that

$$f(x) = \min_{k=1,\dots,m} \max\{\alpha_k, \beta_k\},$$

where $\alpha_k, \beta_k \in \{x_i, \neg x_i := |1 - x_i|\}$ for $i = 1, \dots, n$ and $k = 1, \dots, m$. Does there exist $x^* \in \{0,1\}^n$ such that $f(x^*) = 1$?

Note that the formula above is equivalent to the (perhaps more familiar) formulation

$$f(x) = \bigwedge_{k=1,\dots,m} (\alpha_k \vee \beta_k).$$

Terms of the form $\max\{\alpha_k, \beta_k\}$, resp. $\alpha_k \vee \beta_k$, are called *clauses*. In a 3-SATISFIABILITY PROBLEM, clauses contain three elements instead of two. The 2-SATISFIABILITY PROBLEM is efficiently solvable [EIS76], whereas the 3-SATISFIABILITY PROBLEM is a classical $\mathcal{NP}$-complete problem [GJ79].

Theorem 7.5. *The* R/B-SPLIT PROBLEM *can be efficiently reduced to a* 2-SATISFIABILITY PROBLEM*.*

Proof. Given a red-blue graph $G = (V, R \cup B)$ with $|V| = n$, we show that the R/B-SPLIT PROBLEM is equivalent to the 2-SATISFIABILITY PROBLEM either to determine a vector $(x, y) \in \{0,1\}^{2n}$ that satisfies the 2-satisfiability formula

$$\bigwedge_{(i,j)\in R} (\neg x_i \vee \neg x_j) \wedge \bigwedge_{(i,j)\in B} (\neg y_i \vee \neg y_j) \wedge \bigwedge_{i\in V} (x_i \vee y_i) \tag{7.1}$$

or to prove that no such vector exists:

If a vector $(\hat{x}, \hat{y}) \in \{0,1\}^{2n}$ is to satisfy formula (7.1), each clause must be satisfied individually. Clauses of the form $\neg x_i \vee \neg x_j$ (resp. $\neg y_i \vee \neg y_j$) guarantee for each pair of vertices $i, j \in V$ that not both i and j are part of

$$S_R := \{i \in V \mid x_i \neq 0\} \quad (\text{resp. } S_B := \{i \in V \mid y_i \neq 0\})$$

whenever i and j are joined by an edge in the red (resp. blue) graph. Therefore, S_R is a red stable set and S_B is a blue stable set. Moreover, each pair of clauses of the form $x_i \vee y_i$ guarantees each node $i \in V$ to lie in S_R or S_B (or possibly in both sets). So $V = S_R \,\dot{\cup}\, (S_B \setminus S_R)$ yields a partition of V into a red and a blue stable set.

On the other hand, if V can be partitioned into a red stable set S_R and a blue stable set S_B, the incidence vectors x and y of S_R and S_B, respectively, satisfy the formula above. □

Example 7.1. *Consider the red-blue graph shown in Figure 7.7 with corresponding 2-satisfiability formula* $f(x,y) = (\neg x_1 \vee \neg x_2) \wedge (\neg x_2 \vee \neg x_3) \wedge (\neg y_1 \vee \neg y_3) \wedge (\neg y_2 \vee \neg y_3) \wedge (x_1 \vee y_1) \wedge (x_2 \vee y_2) \wedge (x_3 \vee y_3)$*. Then the satisfying vector* $x = (0,0,1)^T, y = (1,1,0)^T$ *induces the red stable set* $S_R = \{v_3\}$ *and the blue stable set* $S_B = \{v_1, v_2\}$*.*

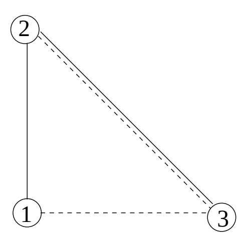


Figure 7.7: Dashed edges are blue, solid edges are red.

Theorem 7.5 exhibits the R/B-SPLIT PROBLEM to be not more difficult than the 2-SATISFIABILITY PROBLEM. In fact, the two problems are equivalent as any 2-SATISFIABILITY PROBLEM can be solved by solving a corresponding R/B-SPLIT PROBLEM:

Theorem 7.6. *The* 2-SATISFIABILITY PROBLEM *can be efficiently reduced to an* R/B-SPLIT PROBLEM.

Proof. Consider an arbitrary 2-satisfiability instance with k clauses on n variables x_i, i.e.

$$f(x) = \bigwedge_{j=1}^{k} (\alpha_j \vee \beta_j),$$

where $\alpha_j, \beta_j \in \{x_1, \dots, x_n, \neg x_1, \dots, \neg x_n\}$ for all $j \in \{1, \dots, k\}$.

Now construct the graph G^f with red and blue edges on the vertex set

$$V^f := \{x_1, \dots, x_n, \neg x_1, \dots, \neg x_n\},$$

where each vertex x_i is joined with its complement $\neg x_i$ by both a red and a blue edge. Furthermore, join a pair of vertices corresponding to $\neg\alpha_j$ and $\neg\beta_j$ by a red edge whenever $\alpha_j \vee \beta_j$ forms a clause in f.

Observe that S_R and $V^f \setminus S_R$ are stable in the red graph G^f_R, resp. blue graph G^f_B, of G^f if and only if the elements in S_R correspond to the true literals in a satisfying variable assignment. $\square$

Example 7.2. *The 2-satisfiability formula* $f(x,y,z) = (x \vee y) \wedge (\neg y \vee z)$ *is satisfiable if and only if the red-blue graph* G^f *shown in Figure 7.8 is an R/B-split graph.*

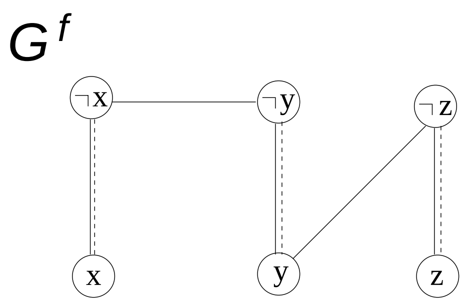


Figure 7.8: $f(x,y,z) = (x \vee y) \wedge (\neg y \vee z)$.

One may wonder if the generalized R/B/G-SPLIT PROBLEM of splitting a graph with red, blue and green edges into stable sets is also polynomial. We note

Lemma 7.2. *The* R/B/G-SPLIT PROBLEM *is* $\mathcal{NP}$*-complete.*

Proof. The special case $G_R = G_B = G_G$ of the generalized R/B/G-SPLIT PROBLEM is the well-known $\mathcal{NP}$-complete 3-COLORING PROBLEM [GJ79]. $\square$

Note that the 3-COLORING PROBLEM is polynomial relative to the class of comparability graphs: it is the problem to decide whether a partially ordered set can be covered by three antichains. Since the minimal number of antichains covering all elements equals the maximal size of a chain ([Sch91], Thm. 14.1), the problem can be solved by simply calculating a longest chain in the partial order and checking if it has not more than three elements. It is an open problem to determine the complexity status of the R/B/G-SPLIT PROBLEM relative to the class of comparability graphs.

7.2.2 Stable matroid bases

We now introduce and investigate the STABLE (MATROID) BASIS PROBLEM, which is in some sense related to our R/B-SPLIT PROBLEM. Interestingly, the STABLE (MATROID) BASIS PROBLEM turns out to be $\mathcal{NP}$-hard relative to a partition matroid, while it is polynomially solvable relative to the dual of a partition matroid:

Let $\mathcal{M} = (V, \mathcal{F})$ be a matroid with ground set V, independent sets $\mathcal{F}$ and basis set $\mathcal{B}(\mathcal{F})$. It is known that the complements of the bases

$$\mathcal{B}^*(\mathcal{F}^*) = \{V \setminus B \mid B \in \mathcal{B}(\mathcal{F})\}$$

form the collection of bases of the dual matroid $\mathcal{M}^* = (V, \mathcal{F}^*)$ whose independent sets are

$$\mathcal{F}^* = \{X \subseteq V \mid V \setminus X \subseteq B \text{ for some } B \in \mathcal{B}(\mathcal{F})\}.$$

We observe that the class of *partition matroids* are strongly related to R/B-split graphs: In case the red edge set consists of a union of disjoint cliques, i.e.

$$R = C_1 \ \dot\cup \ \dots \ \dot\cup \ C_k,$$

the red-blue graph $G = (V, R \cup B)$ is an R/B-split graph if and only if V can be split into a stable set in $G_B = (V, B)$ and an independent set in the partition matroid $\mathcal{M} = (V, \mathcal{F})$ with

$$S \in \mathcal{F} \Leftrightarrow |S \cap C_i| \leq 1, \quad i = 1, \dots, k.$$

As we assume the red stable set to be maximal, the corresponding R/B-SPLIT PROBLEM is equivalent to the question

"Can V be split into a basis in $\mathcal{M} = (V, \mathcal{F})$ and a stable set in $G_B = (V, B)$?"

Moreover, if we consider the dual matroid M^* instead, this decision problem is equivalent to the STABLE MATROID BASIS PROBLEM which we define for uncolored graphs and matroids in general:

Definition 7.2 (Stable basis problem). *Given a matroid $\mathcal{M} = (V, \mathcal{F})$ and a graph $G_B = (V, B)$, the* STABLE BASIS PROBLEM *is the problem to decide whether there exists a* stable basis, *i.e. a subset $S \subseteq V$ which is stable in G_B as well as a basis of $\mathcal{M}$.*

Corollary 7.1. *If M^* is a partition matroid, the* STABLE BASIS PROBLEM *is polynomially solvable.*

Proof. Consider the red-blue graph $G = (V, R \cup B)$ whose red edge set consists of disjoint cliques corresponding to the partitions in M^*. Given a partition $V = S'_R \ \dot{\cup} \ S'_B$ into a red and a blue stable set, we may extend S'_R to a maximal red stable set S_R. Now S_R is a basis of the partition matroid M^*, implying that its complement $S_B = V \setminus S_R$ is a basis of its dual matroid M. Hence, G is an R/B-split graph if and only if there exists a blue stable set S_B which is a basis in the dual matroid M. □

It is easy to see that the problem is $\mathcal{NP}$-hard in general:

Lemma 7.3. *If M is a k-uniform matroid, the* STABLE BASIS PROBLEM *is $\mathcal{NP}$-complete.*

Proof. The bases of a k-uniform matroid $M = (V, \mathcal{F})$ are, by definition, all subsets of V of cardinality k. Therefore, in this special case, a stable basis is a blue stable set of cardinality k, which is $\mathcal{NP}$-hard to compute. □

Of course, if we restrict the blue graph to a special class of graphs, the problem becomes polynomial for general matroids: Recall that a partially ordered set $P = (P, \leq)$ is a *tree-order*, if its Hasse-diagramm forms a rooted tree (we assume the root to be at the bottom).

A *co-comparability graph* is the complement graph of a comparability graph. Therefore, a stable set in a co-comparability graph is a chain in the corresponding order and vice versa.

Lemma 7.4. *If G_B is the co-comparability graph of a tree order, the* STABLE BASIS PROBLEM *is polynomially solvable.*

Proof. Since G_B is the co-comparability graph of a tree order P, a stable basis is a basis whose elements form a chain in the corresponding tree-order. Let k be the rank of matroid M, i.e. the cardinality of each basis. As P is a tree-order, each leave i corresponds to a unique maximal chain C_i. Hence, a stable basis is a basis of cardinality k of one of the restricted matroids $M_i = (C_i, \mathcal{I}_i)$ where $\mathcal{I}_i = \{I \cap C_i \mid I \in \mathcal{I}\}$. This way, we only have to calculate a basis (e.g. with the matroid greedy algorithm) for each restricted matroid M_i corresponding to leave i. □

Although the STABLE BASIS PROBLEM is polynomial in case the dual matroid M^* is a partition matroid, the problem becomes $\mathcal{NP}$-complete if M is a partition matroid. Even if the blue graph is the comparabiliy graph of a series-parallel orders. We prove in Chapter 11:

Theorem 7.7. *If M is a partition matroid, and G_B the comparability graph of a series-parallel order, then the* STABLE BASIS PROBLEM *is $\mathcal{NP}$-complete.*

7.3 Weighted Red/Blue-split graphs

After our excursion to the STABLE BASIS PROBLEM, we now return to red-blue graphs $G = (V, R \cup B)$ and consider integral weights $b : R \cup B \to \mathbb{Z}$ on the edge set.

Note that in case two vertices i and j are linked by both, a red and a blue edge, the weight function b might assign two different values to the pair $\{i, j\}$.

We generalize R/B-split graphs to *weighted R/B-split graphs* and show that they model integrally solvable inequality systems $Ax \leq b$ where A is *simple*, i.e. where $A = (a_{i,j}) \in \mathbb{Z}^{n \times n}$ satisfies $\sum_{i=1}^{n} |a_{i,j}| \leq 2$ in each row $j = 1, \ldots, m$.

Definition 7.3 (Feasible potential). *A vector $x : V \to \mathbb{R}$ is a* feasible potential *of instance $(G, R \cup B, b)$ if*

$$\begin{array}{rcll} x_i + x_j & \leq & b(i,j) & \textit{if } (i,j) \in R \quad \textit{and} \\ -x_i - x_j & \leq & b(i,j) & \textit{if } (i,j) \in B. \end{array}$$

Graph G_1 in Figure 7.9 proves that not every weighted red-blue graph admits a feasible potential. Wheras graph G_2 in Figure 7.9 is an example of a graph that admits a feasible potential, but not an integral one. (Here, and in the rest of this Chapter, blue edges are indicated through dashed lines.)

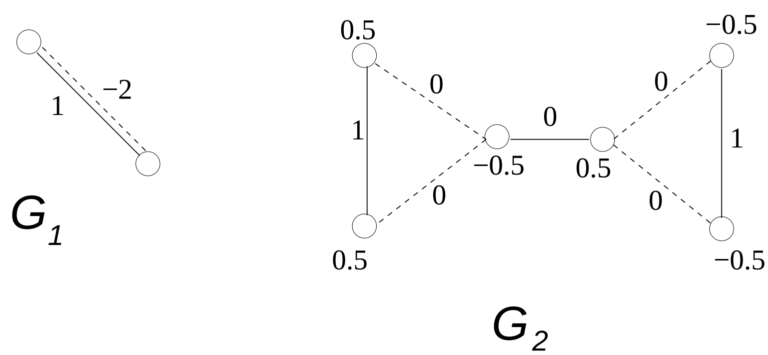


Figure 7.9: G_1 admits no feasible potential; G_2 has a feasible potential, but no integral one exists.

The question whether there exists an integral feasible potential for some instance (G, b) turns out to be a generalization of the R/B-SPLIT PROBLEM.

Definition 7.4 (Weighted R/B-split problem). *The* WEIGHTED R/B-SPLIT PROBLEM *is the problem to decide whether a red-blue graph G with weight function $b : R \cup B \to \mathbb{Z}$ admits an integral feasible potential.*

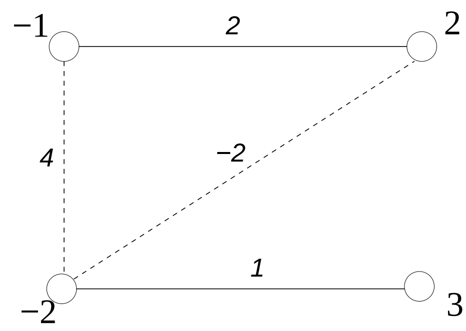


Figure 7.10: Weighted red-blue graph with integral feasible potential.

See Figure 7.10 for an example of a solution of some WEIGHTED R/B-SPLIT PROBLEM.

Note that the WEIGHTED R/B-SPLIT PROBLEM reduces to the ordinary R/B-SPLIT PROBLEM in case each red edge has weight 1, and each blue edge has weight -1: Given a feasible integral potential x, we simply set $S_R := \{i \in V \mid x_i \geq 1\}$ to be the red stable set, and $S_B := \{i \in V \mid x_i \leq 0\}$ to be the blue stable set.

We have shown in Section 7.2 that the R/B-SPLIT PROBLEM is equivalent to a 2–SATISFIABILITY PROBLEM. Likewise, the WEIGHTED R/B-SPLIT PROBLEM is equivalent to an extension of the 2–SATISFIABILITY PROBLEM, which we call the SIMPLE SYSTEM PROBLEM:

7.3.1 Equivalence to the simple system problem

Given a matrix $A \in \mathbb{Z}^{m \times n}$ and a vector $b \in \mathbb{Z}^m$, consider the system

$$Ax \leq b$$

of linear inequalities. In general, it is $\mathcal{NP}$-hard to decide whether there exists an integer solution $x \in \mathbb{Z}^n$ of $Ax \leq b$. It therefore makes sense to restrict to "simple" matrices, i.e. matrices with the property that the sum of absolute values in each row does not exceed the value 2.

We show that the WEIGHTED R/B-SPLIT PROBLEM is equivalent to the SIMPLE SYSTEM PROBLEM, defined as follows.

Definition 7.5 (Simple system problem). *Given a simple matrix A and an integer vector b, the* SIMPLE SYSTEM PROBLEM *is the problem to decide whether the inequality system $Ax \leq b$ is integrally solvable.*

We observe that the 2-SATSIFYABLITY PROBLEM can be viewed as a special SIMPLE SYSTEM PROBLEM: Given a 2-satisfiablity instance $f : \{0,1\}^n \to \{0,1\}$ with

$$f(x) = \bigwedge(\alpha_k \vee \beta_k),$$

for each clause $\alpha_k \vee \beta_k$ we set $b_k = -1$ and

$$a_{i,k} \quad = \quad \begin{cases} -1 & : \quad \alpha_k = x_i, \\ 1 & : \quad \alpha_k = \neg x_i, \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Further, we add $2n$ rows to $A = (a_{i,j})$ such that the inequalities $x_i \leq 1$ and $x_i \geq 0$ are required for all $i = 1, \ldots, n$. Then any integral solution of the simple system $Ax \leq b$ satisfies the 2-satisfiability formula f.

In the following Lemma 7.5, we model simple systems via a weighted red-blue graph such that the simple system has an integral solution if and only if the graph is a weighted R/B-split graph.

Lemma 7.5. *The* Simple system problem *can be reduced to a* weighted R/B-split problem*, and vice versa.*

Proof. Let $A \in \mathbb{Z}^{m \times n}$ be a simple matrix and $b \in \mathbb{Z}^m$. Observe that each row of A contains either one or two non-zero entries. Hence, the possible rows are of type $\{-1\}$, $\{1\}$, $\{-2\}$ or $\{2\}$ in the first case, or of type $\{1,1\}, \{-1,1\}, \{1,-1\}$ or $\{-1,-1\}$ in the second case.

Construct an (uncolored) auxiliary graph H as follows: Vertices correspond to columns, and edges correspond to rows of A in such a way that vertices corresponding to non-zero entries of the same row are linked by an edge. Therefore, the loops of H correspond to rows of type $\{-1\}$, $\{1\}$, $\{-2\}$ or $\{2\}$, while the remaining edges correspond to rows of type $\{1,1\}, \{-1,1\}, \{1,-1\}$ or $\{-1,-1\}$.

Given H, color and weight its edges as follows (compare Figure 7.11):

1. Color the edges of type $\{1,1\}$ red and the edges of type $\{-1,-1\}$ blue. Assign each edge the weight of the corresponding row.

2. Divide any edge (u,v) of type $\{1,-1\}$ [resp. $\{-1,1\}$] into two edges (u,a) and (a,v) by adding a dummy node a. Color (u,a) red [resp. blue] and (a,v) blue [resp. red]. Assign the blue edge the weight zero and the red edge the weight of the row corresponding to (u,v).

3. Divide any loop (v,v) of type $\{2\}$ [resp. $\{-2\}$] into three edges $(v,c), (c,d)$ and (d,v) by adding two dummy vertices c and d for each loop. Color (v,c) and (d,v) red [resp. blue] and (c,d) blue [resp. red]. Assign (v,c) the weight of the row corresponding to v and the two remaining edges the weight zero.

4. Divide any loop (v,v) of type $\{1\}$ [resp. $\{-1\}$] as before into three edges $(v,c), (c,d)$ and (d,v) by adding two dummy vertices $\{c,d\}$ for each loop. Color (v,c) and (d,v) red [resp. blue] and (c,d) blue [resp. red]. In contrast to the previous case, assign both (v,c) and (d,v) the weight of the row corresponding to v and the two remaining edges the weight zero.

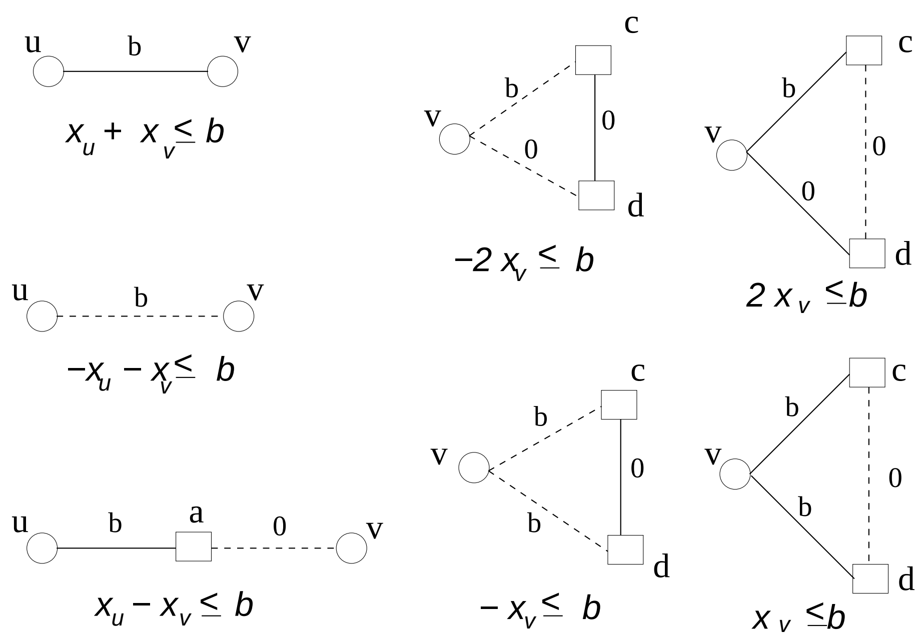


Figure 7.11: Construction of the graph corresponding to system $Ax \le b$

Let (H, b) be the resulting weighted red-blue graph. Obviously, for any feasible potential x of (H, b), by deleting the components of x that correspond to dummy vertices of H, we obtain a solution of $Ax \le b$.

On the other hand, any solution x of $Ax \le b$ can be expanded to a feasible potential of (H, b) as follows: For any inequality of type $x_u - x_v \le b(u, v)$, assign $x_a = -x_v$ to the dummy vertex a. Further, for any inequality of type $2x_v \le b_v$ [resp. $-2x_v \le b_v$], assign $x_c = x_v$ and $x_d = -x_v$ to the dummy vertices c and d. Finally, for any inequality of type $x_v \le b_v$ [resp. $-x_v \le b_v$], assign $x_c = x_d = 0$ to the dummy vertices c and d.

To prove that the WEIGHTED R/B-SPLIT PROBLEM of any weighted red-blue graph (G, b) can be reduced to a SIMPLE SYSTEM PROBLEM, let us construct matrix $A_G \in \{-1, 1\}^{|R \cup B| \times |V|}$ with entries

$$a_{(u,v),u} = \begin{cases} 1 & : \quad (u, v) \in R \\ -1 & : \quad (u, v) \in B \end{cases}$$

Obviously, A_G is simple and any feasible potential x^* of (G, b) is a solution of the inequality system $A_G x \le b$. □

A polynomial-time algorithm to solve the SIMPLE SYSTEM PROBLEM can be found in Schrijver [Sch91]. In his paper, Schrijver also provides a characterization of solvable instances by exclusion of special types of walks in bidirected graphs. Since these walks are not necessarily vertex- or edge-disjoint, his characterization is not sufficient when we are interested in excluded subgraphs.

In Chapter 10, we solve the WEIGHTED R/B-SPLIT PROBLEM and characterize weighted R/B-split graphs by exclusion of weighted subgraphs. The main idea to solve the WEIGHTED R/B-SPLIT PROBLEM (similar to Schrijver's algorithm), can be desribed as follows:

1. In a first step, we use a Shortest-Path algorithm to either determine a feasible potential $x : V \to \mathbb{R}$ of (G, b), or to return a "negative even circuit" resp. "negative simple handcuff", proving that no feasible solution exists. Any feasible potential x turns out to be half-integral.

2. In a second step, we solve an R/B-SPLIT PROBLEM in an auxiliary red-blue graph G^x. This way, we are able to either modify x into an integral feasible potential x^*, or to provide a subgraph ("tight odd flower", to be defined later) proving that no integral solution exists.

7.4 Outline of Part II

In the following Chapter 8, we present an algorithm that either determines a partition of the vertices of a red-blue graph $G = (V, R \cup B)$ into a red stable set S_R and a blue stable set S_B, or returns a *handcuff* which proves that G is not an R/B-split graph. This handcuff turns out to be a generalization of Deming and Sterboul's M-handcuffs, which characterize König-Egerváry graphs.

Since handcuffs are neither vertex-, nor edge disjoint, we normalize the handcuffs in Chapter 9 in such a way that the induced subgraphs are of the type shown in Figure 7.12, which we call "flower". Such a flower consists of two red-blue alternating circuits that overlap as indicated in Figure 7.12. We prove how Lovász and Korach's characterization of König-Egerváry graphs, resp. Földes and Hammer's characterization of split graphs, can be deduced in the special cases where the red edges form a maximum matching, resp. where the red edges are the complement of the blue edges.

In Chapter 10, we show how the WEIGHTED R/B-SPLIT PROBLEM can be solved with a shortest-path algorithm in an auxiliary directed bipartite graph, together with an R/B-split algorithm in an auxiliary (unweighted) red-blue graph. These two algorithms lead to a characterization of weighted R/B-split graphs by exclusion of so-called "negative even circuits", "negative simple handcuffs" and "tight odd flowers". Thus, we obtain an excluded-subgraph characterization of integrally solvable simple systems, which can be seen as a refinement of the result of Schrijver [Sch91].

If a red-blue graph is not an R/B-split graph, one might ask for a best possible covering of the vertices by a red and a blue stable set. This optimization problem is easily seen to be $\mathcal{NP}$-hard, since it includes the $\mathcal{NP}$-hard MAXIMUM STABLE SET problem as a special case. However, there are some polynomially solvable instances of this MAXIMUM R/B-SPLIT PROBLEM. If, for example, the red and blue edgeset are identical, and the graph is a

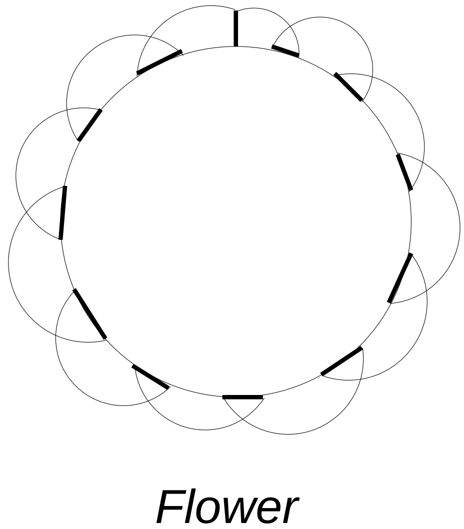


Figure 7.12: Flower, forbidden in R/B-split graphs

comparability graph, the problem amounts to determine the largest union of two antichains in the corresponding partial order. This maximization problem is well known to be solvable in polynomial time even for the union of k antichains [Fra80]. In Chapter 11, we discuss some polynomial solvable instances of the MAXIMUM R/B-SPLIT PROBLEM. Interestingly, the problem of determining a maximal union of a red and a blue antichain turns out to be $\mathcal{NP}$-hard already for the class of series-parallel orders. Moreover, we characterize posets containing a disjoint union of two chains, two antichains, or a chain and an antichain.

Chapter 8

Red/Blue-split algorithm

In this Chapter, we solve the R/B-SPLIT PROBLEM algorithmically: If the red-blue graph $G = (V, R \cup B)$ is an R/B-split graph, the algorithm will return a partition of the vertex set into a stable set in the red graph and a stable set in the blue graph. Otherwise, it will return a handcuff proving that G is not an R/B-split graph.

In fact, this characterization generalizes the characterization of Deming [Dem79] and Sterboul [Ste79] of König-Egerváry graphs by exclusion of M-handcuffs. We are going to use the handcuff characterization of R/B-split graphs to prove our characterization by excluded subgraphs in Chapter 9.

We first need some additional notations: Generalizing M-alternating walks, we call

$$W = \{v_1, e_1, v_2, e_2, \dots, e_{k-1}, v_k\}$$

with $v_i \in V$ $(i = 1, \dots, k)$ and $e_i = (v_i, v_{i+1}) \in R \cup B$ $(i = 1, \dots, k-1)$ an *alternating walk* if for each $i = 1, \dots, k-1$ the edges e_i and e_{i+1} are of different color. (As we consider only alternating walks in this dissertation, we simply write "walk" instead of "alternating walk" for most of the time.) Clearly, if W is a walk, the reverse sequence

$$\bar{W} = \{v_k, e_{k-1}, v_{k-1}, \dots, e_1, v_1\}$$

is a walk, too. Given two walks W_1 and W_2 such that the last vertex of W_1 and the first vertex of W_2 are identical, and the last edge of W_1 and the first edge of W_2 are of different color, we write

$$W = W_1 + W_2$$

to indicate that W_1 is traversed before W_2.

A walk might traverse vertices and edges more than once. We define the number of traversed edges $|W| = k - 1$ to be the *size* of W. Note that the size $|W|$ might be greater than the size of the edge set used in W.

If the walk $C = \{v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k\}$ is an odd cycle (i.e., if $v_1 = v_k$ and k is even), then the starting and ending edges e_1 and e_{k-1} are of the same color. Depending on the color of the starting- and ending edges of an odd cycle, we talk about a *red cycle* or a *blue cycle*.

8.1 Determining a red and a blue stable set

Note that G is an R/B-split graph if and only if the vertices can be colored red and blue in such a way that the red vertices form a red stable set, and the blue vertices form a blue stable set. We denote such a coloring a *feasible coloring* of G.

The subsequent Theorem 8.1 characterizes R/B-split graphs by exclusion of a pair of a red and a blue cycle sharing a base. Its proof describes an algorithm to obtain either a feasible coloring or the forbidden pair of odd cycles.

Theorem 8.1. *$G = (V, R \cup B)$ is an R/B-split graph if and only if there does not exist a red and a blue cycle having the same base.*

Proof. "Necessity": Given a feasible coloring consider a walk $W = \{v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k\}$ in G where we assume e_1 to be red. If v_1 is red, for each even index $i \leq k$ vertex v_i has to be colored blue, and for each odd index $j \leq k$ vertex v_j has to be colored red. That is, the vertices of W need to be colored alternatingly. Therefore, the base of each red cycle needs to be colored blue, and the base of each blue cycle needs to be colored red. Thus, it is impossible for a graph containing a red and blue cycle sharing the same base to allow a feasible coloring.

"Sufficiency": To prove sufficiency, we describe a procedure of coloring a graph without a red and blue cycle sharing a base feasibly:

Pick an arbitrary vertex p. We may assume that there are no red cycles with base p (otherwise interchange colors). Color p red. In the next step color all vertices, which are reachable from p on an alternating walk starting with a red edge, with the proper color. This can be done by constructing an *alternating breath first search* tree with root p.

Since there exist no red cycles with base p, there will never be two red vertices connected by a red edge, nor two blue vertices connected by a blue edge.

In case not every vertex can be reached from p, pick an uncolored vertex and apply the above algorithm again. Repeat this procedure until every vertex is colored.

Observe that whenever an uncolored vertex is picked, after this vertex and all the vertices reachable from it are colored with proper colors, the new colored vertices can never hurt the coloring we obtained before. Thus, at the end, we get a feasible coloring of G. $\square$

8.2 Characterization by forbidden handcuffs

The procedure described in the proof of Theorem 8.1 shows that, by applying a breath first search at most twice, we either achieve a feasible coloring of G, or we determine a red and a blue cycle sharing the same base (cf. Figure 8.1).

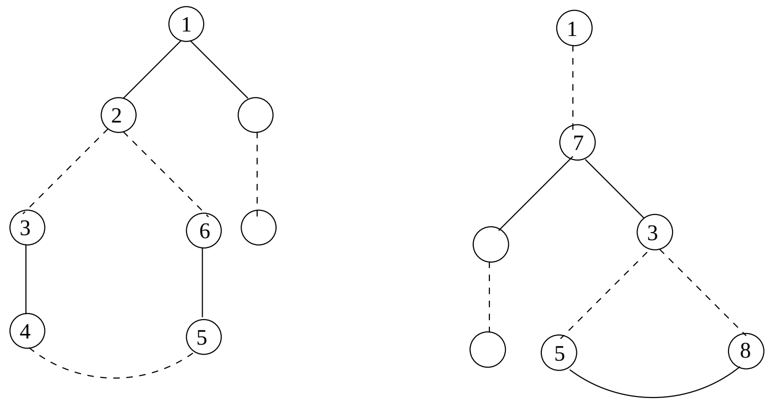


Figure 8.1: Two BFS-trees with odd cycles, determined by the algorithm.

This pair of a red and a blue cycle with identical base describe an even cycle of a special structure, which we define as "handcuff" (cf. Figure 8.2):

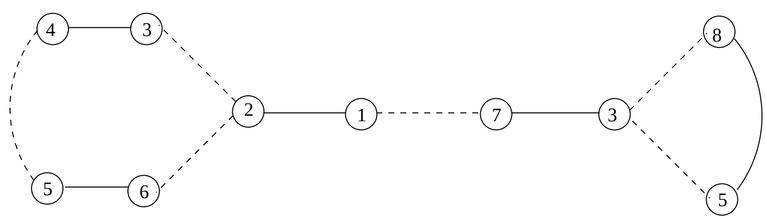


Figure 8.2: Handcuff, described by the two odd cycles.

Definition 8.1 (Handcuff). *A* handcuff *consists of two odd circuits C_1 and C_2 whose bases are linked by a path P (possibly of size 0) such that*

$$C_1 + P + C_2 + \bar{P}$$

is an even cycle.

It follows from the proof of Theorem 8.1 that handcuffs are forbidden in R/B-split graphs and that every non-R/B-split graph contains a handcuff. Hence, we may characterize R/B-split graphs as follows:

Corollary 8.1. *G is an R/B-split graph if and only if G contains no handcuff.*

It is easy to see that, in case the red edges of $G = (V, R \cup B)$ form a perfect matching M in the blue graph $G_B = (V, B)$, the constructed handcuff reduces to an M-handcuff which Deming [Dem79] and Sterboul [Ste79] used to characterize König-Egerváry graphs.

Again, the two circuits C_1 and C_2 of a handcuff may have vertices in common. Hence, we cannot describe the subgraphs induced by forbidden handcuffs sufficiently. (See for instance the subgraph $G(H)$ induced by handcuff H in Figure 8.3.)

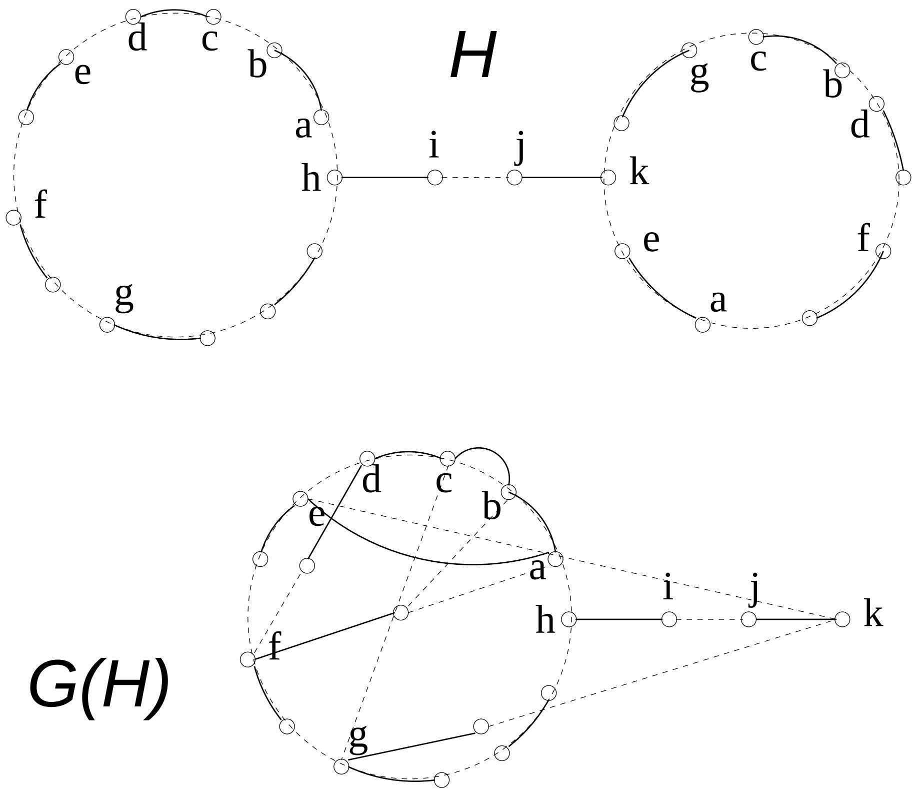


Figure 8.3: Handcuff H with induced subgraph $G(H)$.

In the following Chapter 9, we normalize handcuffs in such a way that the induced subgraphs form a special configuration which we call “flower”.

Chapter 9

Characterization of Red/Blue-split graphs

The goal of this Chapter is to normalize handcuffs $H = C_1 + P + C_2 + \bar{P}$ in such a way that the two circuits C_1 and C_2 overlap in intervals located as in Figure 9.1. We call the induced subgraph of such a normalized handcuff a "flower".

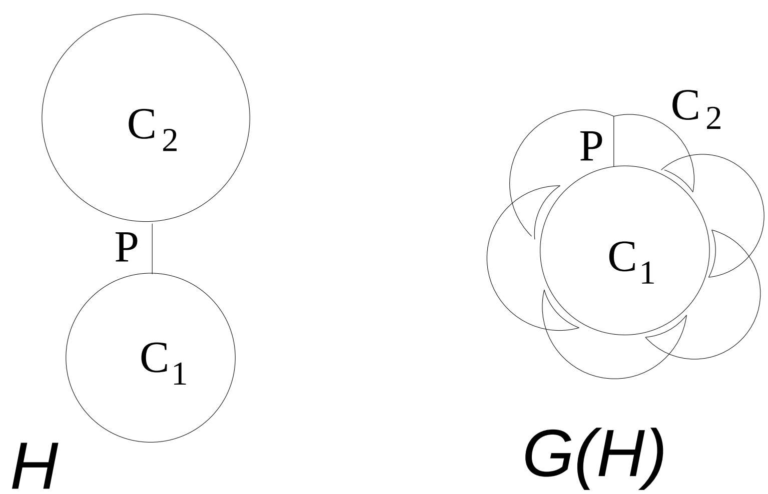


Figure 9.1: Normalized handcuff and induced flower with 5 intervals.

Note that C_1 and C_2 may intersect in arbitrary many intervals. Moreover, some of the intervals may be of size 0, i.e., consist of only one vertex. In case C_1 and C_2 intersect in 0 intervals, the flower looks like a handcuff. But in this case, the handcuff is *simple*, i.e., is vertex disjoint.

9.1 Normalized handcuffs and forbidden cycles

To prove the characterization of R/B-split graph by the exclusion of flowers, we need some additional definitions and notations. For a technical reason, we always imagine a cycle to be drawn on the plane.

This way, we observe: if C is an even cycle, and p a vertex occuring twice in C, then C can be cut along an imaginary cutline c_p into two subcycles each having base p. These two subcycles are either both even or both odd. Accordingly, we talk about an *even cut* c_p or an *odd cut* c_p (cf. Figure 9.2). Using this notations, we are able to describe handcuffs as

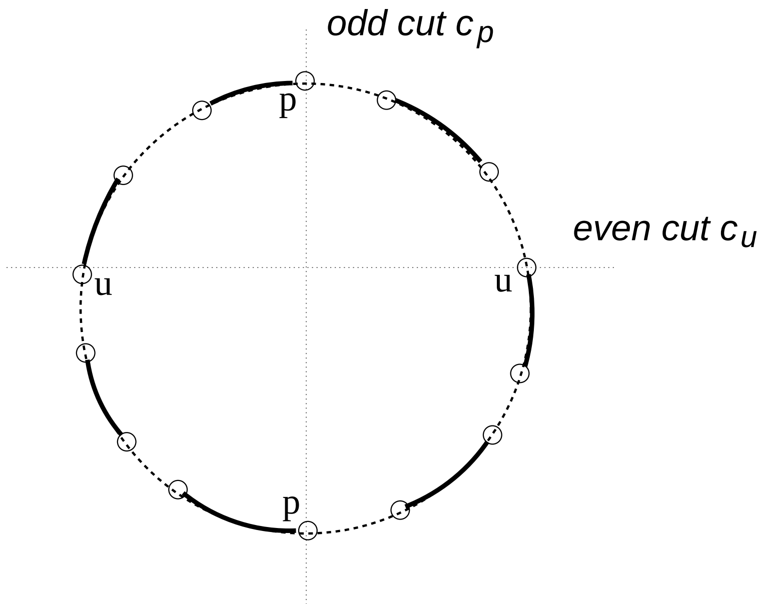


Figure 9.2: Even cycle with odd cut c_p and even cut c_u.

so-called "forbidden cycles":

Definition 9.1 (Forbidden cycle). *A* forbidden cycle *is an even cycle with at least one odd cut.*

Since in any feasible coloring, the base of a red cycle has to be colored blue, and the base of a blue cycle has to be colored red, a forbidden cycle can never be colored feasibly. Moreover, since each handcuff corresponds to a forbidden cycle, it follows from Corollary 8.1:

Corollary 9.1. *G is an R/B-split graph if and only if G contains no forbidden cycle.*

In case there are two vertices u and p, each occuring twice in cycle C, we distinguish between "parallel cuts" and "crossing cuts":
If the two imaginary cutlines c_u and c_p cross inside C, we say c_u and c_p *cross*. Otherwise we say c_u and c_p lie *parallel* in C (see Figure 9.3).

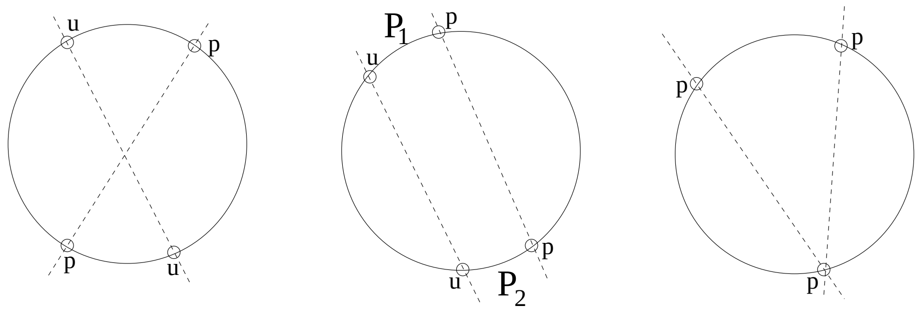


Figure 9.3: Cycle with crossing cuts, and two cycles with parallel cuts.

Definition 9.2 (Parallel intervals). *Let c_u and c_p be two parallel odd cuts in cycle C. We call the two subwalks between the two cutlines c_u and c_p* parallel intervals*.*

For example, the two subwalks P_1 and P_2 of Figure 9.3 are parallel intervals. Clearly, in a forbidden cycle, if we interchange two parallel intervals P_1 and P_2, or replace one by the other, we obtain a forbidden cycle again.

We are now able to describe flowers and simple handcuffs by a different term: "normalized forbidden cycles":

Definition 9.3 (Normalized forbidden cycle). *A* normalized forbidden cycle *is a forbidden cycle without even cuts and with identical parallel intervals.*

Given a normalized forbidden cycle C, let us call those alternating paths of C corresponding to identical parallel intervals as *double paths*, and the remaining (alternating) paths of C as *single paths*.

It is not hard to see that the induced subgraph of a normalized forbidden cycle is a flower. See Figure 9.4 for an example. The thick lines correspond to double paths.

9.2 Proof of the excluded subgraph characterization

To prove the charaterization of R/B-split graph by the exclusion of flowers, it remains to show that the existence of a forbidden cycle implies the existence of a normalized forbidden cycle. In a first step, we show that normalized forbidden cycles are "minimal", in the following sense:

Definition 9.4 (Minimal forbidden cycle). *A forbidden cycle C is* minimal *if there does not exist another forbidden cycle of smaller size in the subgraph consisting of the edges of C.*

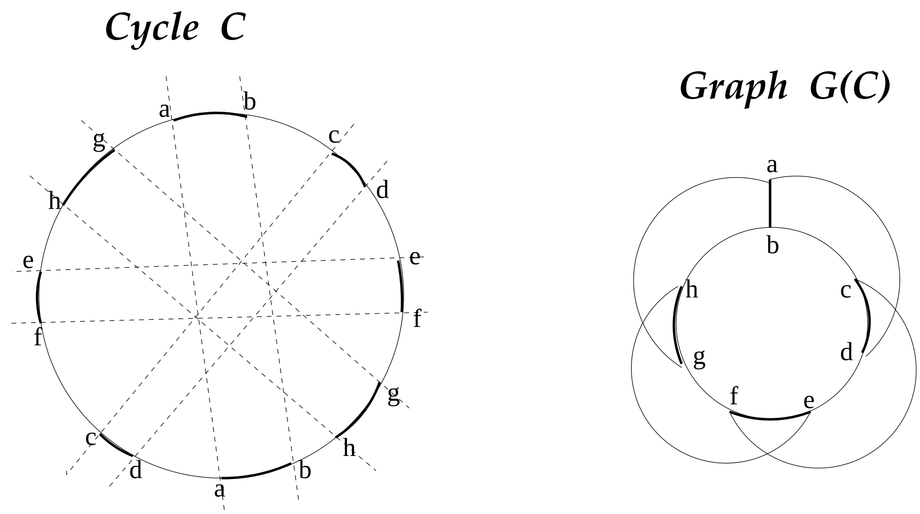


Figure 9.4: Normalized forbidden cycle C with induced flower $G(C)$.

Clearly, whenever a graph contains a forbidden cycle at all, it also contains a minimal one. We observe the following properties of minimal forbidden cycles:

Lemma 9.1. *In a minimal forbidden cycle an even cut can never lie parallel to an odd cut. As a consequence, in a minimal forbidden cycle each vertex occurs at most twice.*

Proof. Suppose c_u is an even cut parallel to an odd cut c_p in a minimal forbidden cycle C. From the two subcycles defined by the even cut c_u, we cut off the one not containing c_p. This way, we get a smaller forbidden cycle contradicting the minimality of C. (Figure 9.5)

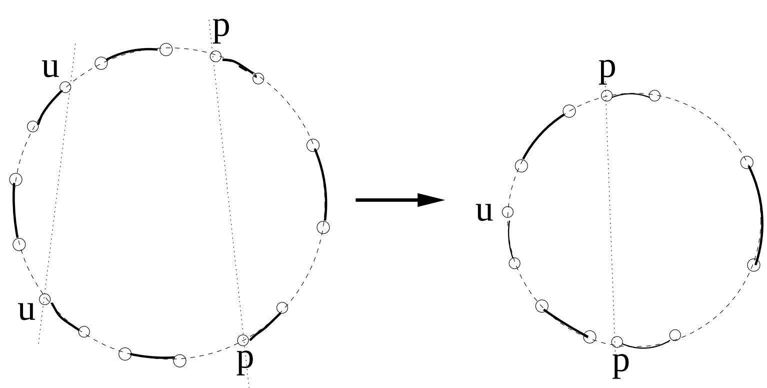


Figure 9.5: Forbidden cycle and the shorter one.

Now, suppose a vertex u occurs three times in C. Then there exist three parallel u-cuts in C. In case one of these cuts is odd, exactly one of the two other cuts has to be even.

Thus, we have an odd and an even parallel cut. In case all u-cuts are even, consider any odd cut c_p in C. At most two u-cuts cross c_p, and at least one u-cut lies parallel to c_p. In both cases we get a situation where an odd cut lies parallel to an even cut. Due to the observation above, C cannot be a minimal forbidden cycle. □

A minimal forbidden cycle has some of the properties required for normalized forbidden cycles:

Lemma 9.2. *For any pair of parallel intervals P_1 and P_2 of a minimal forbidden cycle C holds: Each interval corresponds to a path and the inner vertices of P_1 and P_2 do not appear elsewhere outside P_1 and P_2.*

Proof. Let P_1 and P_2 be intervals between parallel odd cuts c_u and c_p. Obviously, P_1 and P_2 must be of the same size, since otherwise the bigger interval could be replaced by the smaller one to obtain a shorter forbidden cycle.

Suppose P_1 is not simple. Replacing P_2 by P_1, we obtain a forbidden cycle where at least one vertex occurs four times. Similarly, suppose an inner vertex of P_1 appears outside P_1 and P_2: Replacing P_2 by P_1, we obtain a forbidden cycle with a vertex appearing more than twice. Both cases lead to a contradiction according to Lemma 9.1. □

Now we prove that any non-R/B-split graph contains a minimal forbidden cycle without even cuts.

Lemma 9.3. *If G is not an R/B-split graph, there exists a minimal forbidden cycle without even cuts.*

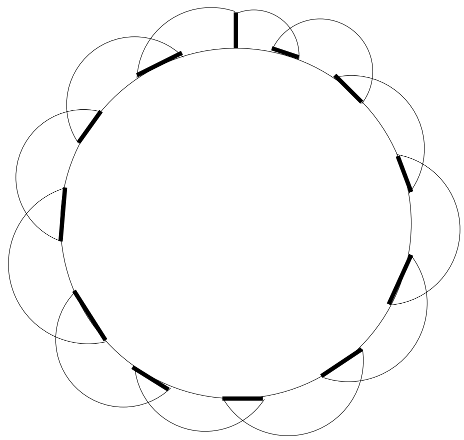
Proof. Let C be a minimal forbidden cycle with odd cut c_p. Hence $C = C_1(p) + C_2(p)$, where $C_1(p)$ and $C_2(p)$ are the odd cycles defined by the cut-line c_p. Consider the cycle $\tilde{C} = C_1(p) + \bar{C}_2(p)$ obtained by reversing the order of the odd cycle C_2 on C. Again, $\tilde{C}$ is a minimal forbidden cycle with odd cut c_p. We may observe that for any cut c_u of C the property to parallel or cross c_p keeps the same in $\tilde{C}$. Moreover, any cut parallel to c_p in C has the same parity in C as it has in $\tilde{C}$. But any cut crossing c_p has a different parity in C as it has in $\tilde{C}$.

Let C be a minimal forbidden cycle in G with odd cut c_p and even cut c_u. In case c_p is the only odd cut, we know by Lemma 9.1 that all even cuts cross c_p. Due to the observation above, all cuts in the minimal forbidden cycle $\tilde{C} = C_1(p) + \bar{C}_2(p)$ are odd.

In case there exists a second odd cut c_q in C, Lemma 9.1 implies that the even cut c_u crosses c_p as well as c_q. Then c_p and c_q have to lie parallel in C, as otherwise c_q would be an even cut parallel to the odd cut c_u in $\tilde{C}$ contradicting Lemma 9.1. Thus, every odd cut lies parallel to c_p implying all cuts in $\tilde{C}$ are odd. □

We are now able to prove the main Theorem of this Chapter:

Theorem 9.1. *A red-blue graph G is an R/B-split graph if and only if it contains no subgraph of type "flower" (see Figure 9.6).*



Flower

Figure 9.6: Forbidden flower: thick lines correspond to double paths.

Proof. Due to Corollary 9.1, a graph is not an R/B-split graph if and only if it contains a forbidden cycle. By Lemma 9.3, we may choose a minimal forbidden cycle without even cuts. Let $(P_1, P_1'), (P_2, P_2') \ldots, (P_k, P_k')$ be the pairs of maximal parallel intervals. By Lemma 9.2, we know that they correspond to (alternating) paths whose inner vertices are disjoint from the rest of the cycle. Replacing P_i' by P_i for $i = 1, \ldots, k$, we get a normalized forbidden cycle, whose induced graph is a flower. $\square$

Note that the above proof also gives an efficient algorithm for finding a normalized forbidden cycle.

9.3 Applications

Theorem 9.1 has some interesting applications: The characterizations of König-Egerváry graphs and split graphs. Note that we already deduced Deming and Sterboul's characterization in Chapter 8. In this Section, we prove Korach's characterization of König-Egerváry graphs (cf. Theorems 7.3 and 7.4), and show that Lovász characterization (cf. Theorem 7.2) is a corollary of Korach's Theorems. Furthermore, we prove that the split graph characterization of Földes and Hammer [FH77] can be deduced from our model.

9.3.1 Characterization of König-Egerváry graphs

For the sake of completeness, we repeat Korach's Theorems (stated already in Chapter 7), before proving them as consequences of Theorem 9.1:

Theorem ([Kor82]) *Let R be a perfect matching in $G_B = (V, B)$. Then G_B is a König-Egerváry graph if and only if no subgraph results of even subdivisions of one of the configurations in Figure 9.7.*

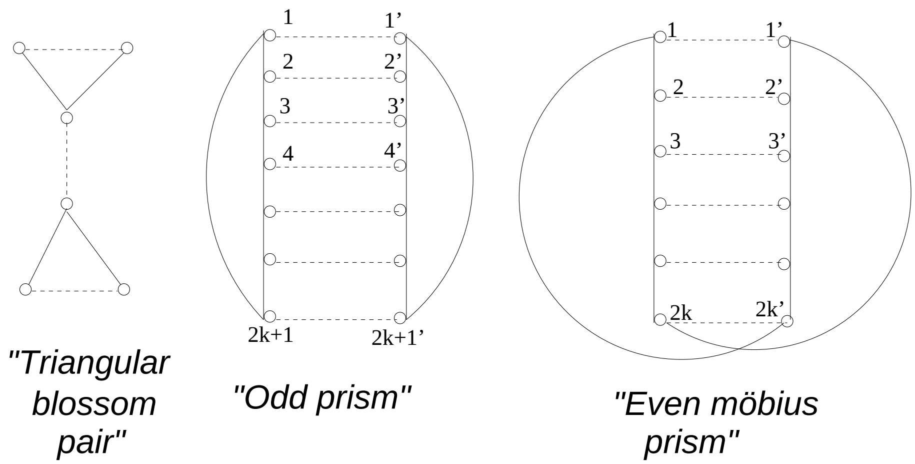


Figure 9.7: Korach's forbidden configurations: dashed edges are the matching edges.

Proof. Let H be a flower or a simple handcuff in $G = (V, R \cup B)$. Then no vertex of H is incident to four edges of H, (i.e., there exists no double interval that consists of a single vertex in H), as otherwise two matching-edges would have to be adjacent.

Moreover, since vertices of degree three in H correspond to end vertices of double intervals, double intervals have to be odd alternating paths with matching edges at the end. For the same reason, single intervals have to be odd alternating paths with non-matching edges at the end.

Hence, the subgraph induced by H results of a sequence of even subdivisions of either a simple handcuff, or the flower shown in Figure 9.8. It can be observed that a simple handcuff corresponds to Korach's triangular blossom pair, while a flower corresponds to Korach's odd [resp. even möbius] prism in case of an odd [resp. even] number of matching edges.

The converse implications are obvious. $\square$

Theorem ([Kor82]) *Let R be a maximum matching in $G_B = (V, B)$. Then G_B is a König-Egerváry graph if and only if no subgraph results of even subdivisions of one of the configurations in Figure 9.7 or Figure 9.9.*

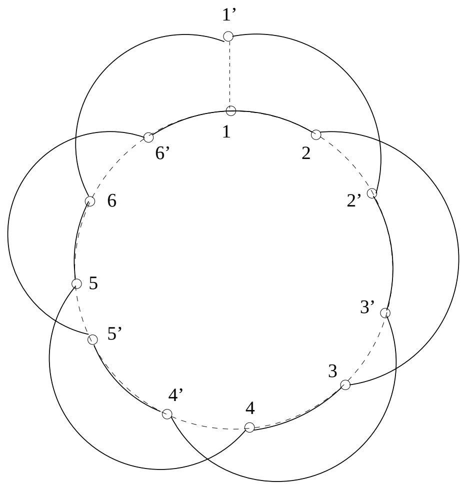


Figure 9.8: Forbidden flower in König-Egerváry graphs: Dashed edges are matching edges.

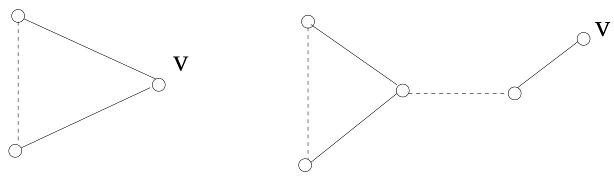


Figure 9.9: Korach's additional forbidden configurations: vertex v is uncovered.

Proof. If R is a perfect matching, we are done. Otherwise, consider the expanded graph $G' = (V', R' \cup B')$ as constructed in Lemma 7.1. By the same Lemma, G_B is a König-Egerváry graph if and only if G' is an R/B-split graph. Let H be a flower or a simple handcuff in G'. If H does not contain any new edge $e \in (R' \cup B') \setminus (R \cup B)$, we have seen in the previous proof that the subgraph induced by H results of even subdivisions of one of the types of Figure 7.4.

We claim H cannot contain more than one new edge $e \in (R' \cup B') \setminus (R \cup B)$: Otherwise we could find an alternating path in G between two uncovered vertices, contradicting the maximality of matching R.

Assume H contains one edge $e \in (R' \cup B') \setminus (R \cup B)$. Since H corresponds to an even alternating cycle in G', e must be an edge of type $e = (v', v'')$ for some vertex $v \in V$ not covered by R. Consider H as a normalized forbidden cycle with odd cut c_p. The cut c_p divides H into two odd subcycles $H = C_1(p) + C_2(p)$, each having base p. Assume (v', v'') is an edge of $C_1(p)$ and contract edge (v', v'') back to vertex v. (Figure 9.10). Then there exists a path P from v to p such that $P + C_2(p)$ results of an even subdivision of one of the configurations of Figure 7.5.

Conversely, suppose G contains a subgraph H which is the result of even subdivisions of

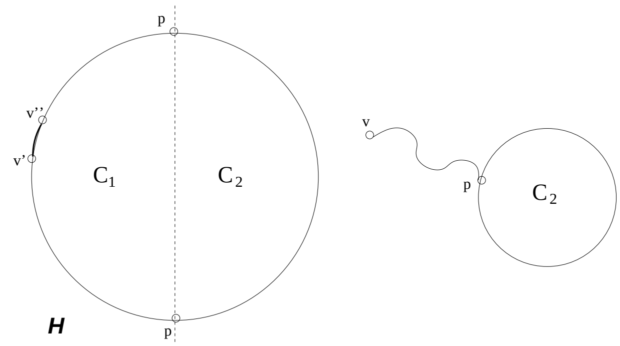


Figure 9.10: Forbidden cycle H and one of Korach's forbidden configurations.

one of the configurations in Figure 7.5. In this case, we can find forbidden cycles in G', implying G' cannot be an R/B-split graph. □

Remark 9.1. *Figure 9.11 indicates that flowers or simple handcuffs (in particular Korach's forbidden subgraphs) are not necessarly induced subgraphs: The only forbidden subgraph of* G_1 *is* G_1 *itself. Whereas the forbidden subgraph of* G_2 *is the subgraph isomorphic to* G_1*, which is not induced.*

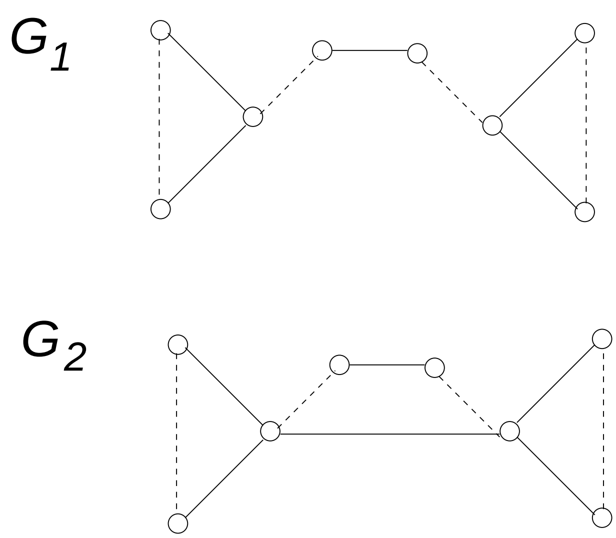


Figure 9.11: The forbidden subgraph of G_1 is induced, while the one of G_2 is not induced.

9.3.2 Deduction of Lovász' characterization

Lovász [Lov83] used the theory of matching cover graphs to give a characterization of König-Egerváry graphs by excluded subgraphs. However, his characterization is based on a *particular* perfect matching, not on an arbitrary given one. Let us recall his characterization which we already stated in Chapter 7:

Theorem ([Lov83]) *Let $G_B = (V, B)$ be an undirected graph. Then G_B is not a König-Egerváry graph if and only if there exists a perfect matching $\hat{R}$ such that a subgraph results of even subdivisions of one of the configurations in Figure 9.12.*

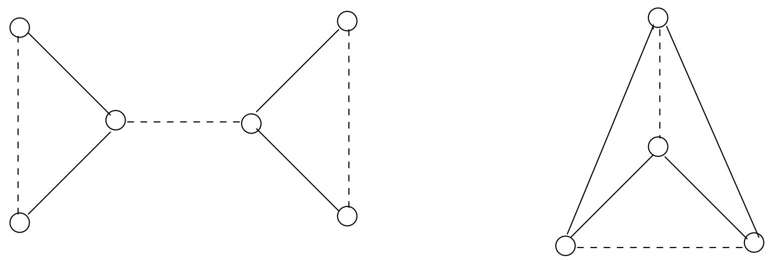


Figure 9.12: Lovász forbidden configurations: Dashed edges are matching edges, vertex v is R-exposed.

Proof. Observe that Lovász' Theorem works only for a particular perfect matching, while Korach's Theorem works for an arbitrary one. But, given one of the forbidden configurations of Korach's Theorem, we may modify the matching to obtain one of Lovász' configurations:

Suppose G_B is not a König-Egerváry graph and let R be any perfect matching of G_B. By Korach's Theorem, there exists a subgraph which is a result of even subdivisions of one of the forbidden configurations G' in Figures 9.7.

If G' is an odd prism shown in Figure 9.7, let us interchange matching edges and non-matching edges in the following collection of disjoint cycles:

$$\{2, 2', 3', 3, 2\}, \{4, 4', 5', 5, 4\}, \ldots, \{2k, 2k', 2k+1', 2k+1, 2k\}.$$

This way we result in a new perfect matching $\hat{R}$. In a second step, we remove edges $(2, 2'), (3, 3'), .., (2k+1, 2k+1')$ to obtain the configuration in Figure 9.12(a).

A similar operation can be applied if G' is an even möbius prism shown in Figure 9.7: Interchanging matching and non-matching edges of the disjoint cycles

$$\{3, 3', 4', 4, 3\}, \{5, 5', 6', 6, 5\}, \ldots, \{2k-1, 2k-1', 2k', 2k, 2k-1\},$$

and deleting the edges $(3, 3'), .., (2k+1, 2k+1')$, we get the configuration in Figure 9.12(b).
$\square$

9.3.3 Characterization of split graphs

After having seen that an R/B-split graph corresponds to a König-Egerváry graph in the special case, where the red edges form a maximum matching of the blue edges, we now deal with the case, where the red edges form the complement of the blue edges, i.e., where $R = \bar{B}$

We observed in Chapter 7 that in case $R = \bar{B}$, the red-blue graph $G = (V, R \cup B)$ is an R/B-split graph if and only if G_B as well as G_R are split graphs. That is, G is an R/B-split graph if and only if G_B and G_R can be split into a clique and a stable set, each.

Földes and Hammer [FH77] proved that an uncolored graph is a split graph if and only if it contains no subgraph isomorphic to a $2K_2, C_4$ or C_5. In our context, graph $G = (V, R \cup B)$ with $R = \bar{B}$ is therefore an R/B-split graph if and only if G contains no subgraph isomorphic to one of the three types shown in Figure 9.13.

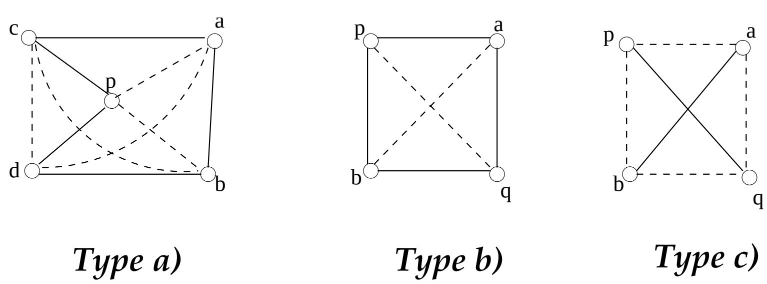


Figure 9.13: The three induced forbidden subgraphs in case $R = \bar{B}$.

We show how Földes' and Hammer's characterization of split graphs can be deduced from our more general characterization of R/B-split graphs.

We have seen in Chapter 9 that any flower corresponds to a normalized forbidden cycle C. Further, C can be divided into subwalks

$$C = C_p + P + C_q + P^-$$

such that C_p and C_q are odd circuits with bases p and q, and P is a simple path starting at p and ending at q. (Figure 9.14). Note that p and q are identical, in case P is empty.

Accordingly, we define for any normalized forbidden cycle (resp. flower) $C = C_p + P + C_q + P^-$ the *length*

$$l(C) = |C_p| + |P| + |C_q| + 2|V(C)|,$$

where $V(C)$ denotes the set of vertices occuring in C.

We prove that any flower of minimal length is isomorphic to one of the three subgraphs shown in Figure 9.13.

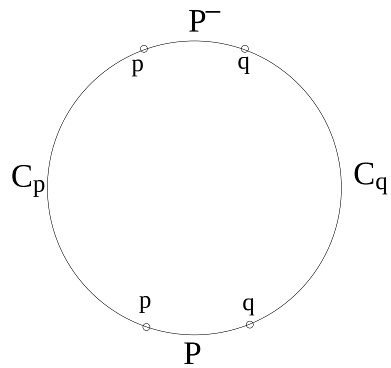


Figure 9.14: $C = C_p + P + C_q + P^-$.

Theorem 9.2 (cf. [FH77]). *Let $G = (V, R \cup B)$ be a red-blue graph with $R = \bar{B}$. Then G is an R/B-split graph if and only if G contains no subgraph isomorphic to one of the three subgraphs shown in Figure 9.13.*

Proof. By Theorem 9.1, G is an R/B-split graph if and only if G contains no flower. If G is not an R/B-split graph, we may therefore choose a flower, whose normalized forbidden cycle $C = C_p + P + C_q + P^-$ minimizes $l(C)$.

Since G_R and G_B are loopless, the circuits C_p and C_q are each of size at least three. Let us convince that C_p and C_q each consist of exact three edges, and that P consists of at most one edge:

Suppose C_p consists of more than three edges, i.e., let $C_p = \{p, 1, 2, \ldots., k, k+1, p\}$ with $k > 3$ denote the vertices of C_p. W.l.o.g., we assume $(p, 1)$ and $(k+1, p)$ to be red edges. Since $R = \bar{B}$, any two vertices of C_p must be joined by either a red or (exclusivly) a blue edge. If $(p, 2) \in R$ [resp. $(k+1, p) \in R$], we can replace C_p by the odd alternating circuit $C'_p = \{p, 1, 2, p\}$ [resp. $C'_p = \{p, k, k+1, p\}$] to achieve a normalized forbidden cycle of smaller length than C. Therefore $(p, 2), (k+1, p) \in B$, implying that $\{2, p, 1, 2\} + \{2, .., k\} + \{k, k+1, p, k\} + \{2, .., k\}^-$ is a normalized forbidden cycle of smaller length than C. A contradtiction.

Now suppose P consists of more than one edge. Let $P = \{p = 1, 2, \ldots, k = q\}$ with $k > 2$ denote the vertices in P and assume $(p, 2) \in R$. If $(p, 3) \in R$, the normalized forbidden cycle $\tilde{C} = C_p + \{p, 2, 3, p\}$ is shorter than C. Similarly, if $(p, 3) \in B$, the normalized forbidden cycle $\tilde{C} = \{3, 2, p, 3\} + \{3, \ldots, q\} + C_q + \{3, \ldots, q\}^-$ is shorter than C.

We distinguish the cases, where P consists of one or zero edges. In the first case, our normalized forbidden cycle can be written as

$$C = \{p, a, b, p\} + (p, q) + \{q, c, d, q\} + (q, p)$$

with length $l(C) = 3 + 1 + 3 + 2|V(C)|$ (cf. Figure 9.15). Clearly, $V(C)$ contains at

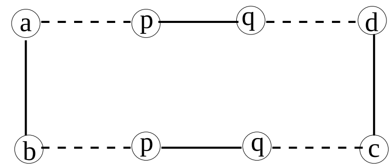


Figure 9.15: $p \neq q$.

least four vertices. Suppose $|V(C)| > 4$. If (p, c) [resp. (p, d)] is of the same color as (p, q), cycle $\tilde{C} = \{p, a, b, p\} + \{p, c, q, p\}$ [resp. $\tilde{C} = \{p, a, b, p\} + \{p, d, q, p\}$] is shorter than C. Hence (p, c) and (p, d) are both of another color than (p, q), which implies that $\tilde{C} = \{p, a, b, p\} + \{p, c, d, p\}$ is shorter than C. Therefore, $|V(C)| = 4$, and the subgraph induced by C has to be of Type b) or c).

In the latter case, where $p = q$, our normalized forbidden cycle can be written as

$$C = \{p, a, b, p\} + \{p, c, d, p\},$$

and the only common vertex of the two circuits is p. Thus, the length of C is $l(C) = 3 + 3 + 2 * 5 = 16$ (cf. Figure 9.16).

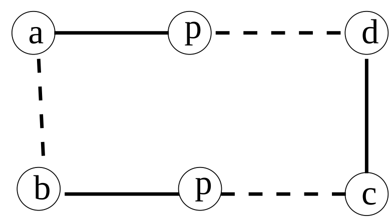


Figure 9.16: $p = q$.

W.l.o.g., we assume that $(p, a), (b, p)$ and (c, d) are red, and $(a, b), (p, c)$ and (d, p) are blue. By symmetry, we may also assume that (c, a) is red.

Then (b, c) has to be blue, as otherwise $\tilde{C} = \{c, a, b, c\} + (c, p) + \{p, a, b, p\} + (p, c)$ with $l(\tilde{C}) = 3+3+1+2*4 = 15$ is shorter than C. Therefore, (b, d) has to be red, as otherwise $\{p, c, d, p\} + (p, b) + \{b, c, d, b\} + (b, p)$ is shorter than C. Finally, (a, d) has to be blue, as otherwise $\{p, a, b, p\} + (p, b) + \{b, p, d, b\} + (b, p)$ is shorter than C.

We conclude that in case $p = q$, the subgraph induced by C has to be of Type a). □

In the following Chapter 10, we solve the WEIGHTED R/B-SPLIT PROBLEM and characterize weighted R/B-split graphs by exclusion of certain weighted subgraphs. As an application, we refine Schrivjer's characterization of solvable instances $Ax \leq b$, where b is an integer vector and $A \in \mathbb{Z}^{m \times n}$ satisfies $\sum_{j=1}^{n} |a_{ij}| \leq 2$ for each row i.

Chapter 10

Weighted Red/Blue-split graphs

By considering an integral weight function $b : R \cup B \to \mathbb{Z}$ on the edge set of a red-blue graph $G = (V, R \cup B)$, we extended the notion of R/B-split graph to weighted R/B-split graphs in Chapter 7. Moreover, we have seen that weighted R/B-split graphs model integrally solvable inequality systems $Ax \leq b$ where the matrix $A = (a_{i,j}) \in \mathbb{Z}^{m \times n}$ satisfies

$$\sum_{i=1}^{n} |a_{i,j}| \leq 2$$

in each row $j = 1, \ldots, m$.

In this Chapter, we solve the WEIGHTED R/B-SPLIT PROBLEM, i.e., the problem to decide whether a weighted instance (G, b) is an R/B-split graph by solving a SHORTEST-PATH PROBLEM and an R/B-SPLIT PROBLEM. Additionally, we characterize weighted R/B-split graphs by exclusion of certain weighted subgraphs.

Let us recall the definition of feasible potentials and weighted R/B-split graphs: Given a red-blue graph $G = (V, R \cup B)$ with weights $b : R \cup B \to \mathbb{Z}$, we call a vector $x : V \to \mathbb{R}$ a *feasible potential* of instance (G, b) if

$$\begin{array}{rcll} x_i + x_j & \leq & b(i,j) & \text{if } (i,j) \in R \quad \text{and} \\ -x_i - x_j & \leq & b(i,j) & \text{if } (i,j) \in B. \end{array}$$

If (G, b) admits an integral feasible potential, G is called a weighted R/B-split graph.

In the following Section 10.1, we show how to determine a feasible potential (if possible) by applying a shortest-path algorithm in an auxiliary directed bipartite graph. It turns out that such a feasible potential is at least half-integral.

In Section 10.2, we show how the R/B-split algorithm, applied in an auxiliary unweighted red-blue graph, helps to modify the half-integral potential to an integral one.

The two algorithms then lead to our characterization of weighted R/B-split graphs by excluded subgraphs as shown in Section 10.3.

In order to determine a feasible potential of (G, b), we need a few additional notations: Let $W = \{e_1, e_2, \dots, e_k\}$ be a walk in $G = (V, R \cup B)$ with $e_i = (v_i, v_{i+1})$ for $i = 1, \dots, k$. (Recall that we only consider alternating walks when not stated otherwise.) We define the *weight* of W to be

$$b(W) := \sum_{e \in W} b(e)$$

and talk about a positive resp. negative walk W depending on its weight to be positive, resp. negative.

Given a walk W, we call $x : V \to \mathbb{R}$ with

$$x(W) := \sum_{e_i \in W \cap R} (x_i + x_{i+1}) - \sum_{e_j \in W \cap B} (x_j + x_{j+1})$$

a *potential* of W. It follows from the definition that for each cycle C in G holds:

$$x(C) = \begin{cases} 0 & : \ C \text{ is even,} \\ 2x_v & : \ C \text{ is a red cycle with base } v, \\ -2x_v & : \ C \text{ is a blue cycle with base } v. \end{cases}$$

Hence, $x : V \to \mathbb{R}$ is a feasible potential of the weighted instance (G, b) if and only if for each walk W in G holds:

$$x(W) \leq b(W).$$

A cycle C is said to be *tight* in (G, b) if C is of even size and of weight zero. We observe that any feasible potential satisfies each edge of a tight cycle with equality:

Lemma 10.1. *An even cycle C is tight if and only if for each edge $e \in C$, each feasible potential x satisfies $x(e) = b(e)$.*

Proof. “Necessity”: Suppose C is tight, i.e., $|C|$ is even and $b(C) = 0$. Given $e \in C$, we observe $x(C \setminus e) + x(e) = x(C) = 0 = b(C) = b(C \setminus e) + b(e)$. Hence, $x(e) \neq 0$ would imply that x is not feasible.

“Sufficiency”: If $x(e) = b(e)$ holds for each edge e of an even cycle C, then $b(C) = x(C) = 0$, i.e., C is tight. □

10.1 Shortest paths and half-integral solutions

Since we use a shortest-path algorithm to determine a feasible potential for the weighted graph (G, b), we need to recall a few facts about shortest paths in directed graphs:

Let $D = (V, A)$ be a directed graph with weight $w : A \to \mathbb{Z}$ on the arc set A. We may assume that there exists a source $s \in V$ such that each vertex $v \in V$ can be reached by a directed path P. (Otherwise, we simply add a source s and edges $\{(s, v) \mid v \in V\}$ of weight zero.)

For each vertex $v \in V$, let

$$\pi(v) = \min\{\sum_{e \in P} w(e) \mid P \text{ is } (s, v) - \text{path}\}$$

denote the length of a shortest path from s to v. It is well known that $\pi : V \to \mathbb{Z}$ can be computed in polynomial time if and only if D contains no directed cycle C of negative weight $w(C) = \sum_{e \in C} w(e)$. (See for example [CCPS98].) Further, $\pi : V \to \mathbb{Z}$ satisfies

$$\pi_j - \pi_i \quad \leq \quad c(i, j) \text{ for each arc } (i, j) \in A.$$

We show how to determine a feasible potential of (G, b) by calculating shortest paths in an auxiliary directed graph D. We will see that a negative directed cycle in D corresponds to an even negative cycle in (G, b).

Lemma 10.2. *A (halfintegral) feasible potential $x : V \to \mathbb{R}$ of (G, b) exists if and only if (G, b) contains no even negative cycle.*

Proof. The necessity of the exclusion of even negative cycles is obvious, since for any feasible potential x and any even cycle C we have

$$0 = x(C) \leq b(C).$$

To prove sufficiency, let V' and V'' be two copies of V and construct the weighted bipartite directed graph $D(G) = (V' \cup V'', A, b)$ such that each edge $(u, v) \in R \cup B$ corresponds to two arcs in $D(G)$ via

$$A = \left\{ \begin{array}{lcl} (u', v''), (v', u'') & : & (u, v) \in R, \\ (u'', v'), (v'', u') & : & (u, v) \in B \end{array} \right\}$$

and the weight of each arc in A is the same as the weight of the corresponding edge in G (cf. Figure 10.1).

Since $D(G)$ is bipartite, any cycle in $D(G)$ is even and corresponds to an even cycle in G of the same weight. Thus, as we assume (G, b) to contain no negative even cycle, we know that there cannot exist any negative cycle in $D(G)$ at all, and we can compute shortest paths $\pi : V \to \mathbb{Z}$ in $D(G)$.

Given π, let us construct the vector $x : V \to \mathbb{R}$ with components

$$x(v) := \frac{\pi(v'') - \pi(v')}{2}.$$

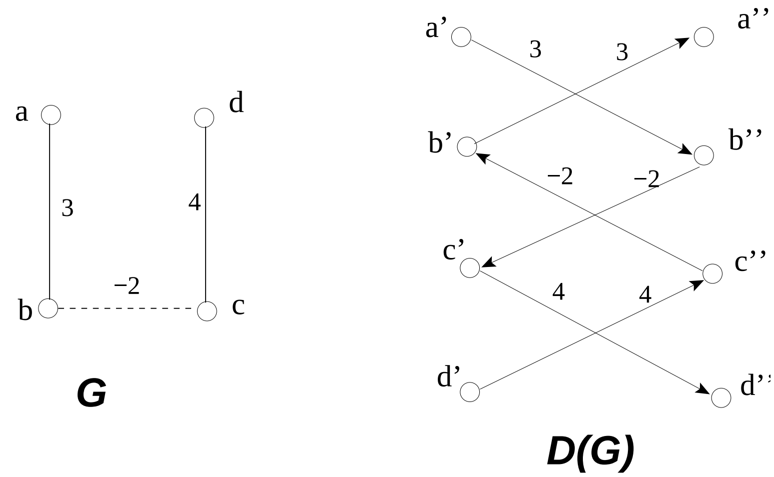


Figure 10.1: Weighted red-blue graph (G, b) and weighted directed bipartite graph $D(G)$.

It is not hard to see that x is a feasible potential of (G, b): In case (u, v) is red, observe that $(u', v''), (v', u'') \in A$ implies

$$x_u + x_v = \frac{1}{2}(\pi(v'') - \pi(u')) + \frac{1}{2}(\pi(u'') - \pi(v')) \leq b(u, v).$$

Analogue, in case (u, v) is blue, $(u'', v'), (v'', u') \in A$ implies

$$-x_u - x_v = \frac{1}{2}(\pi(u') - \pi(v'')) + \frac{1}{2}(\pi(v') - \pi(u'')) \leq b(u, v).$$

Hence, x is a halfintegral feasible potential of G. $\square$

Still, a negative even cycle is not necessarily vertex-disjoint. However, a negative even cycle always implies the existence of a negative even circuit resp. negative simple handcuff. (A *negative simple handcuff* is a vertex-disjoint handcuff $H = C_1 + P + C_2 + \bar{P}$ of negative weight $w(H) = w(C_1) + w(C_2) + 2w(P)$):

Lemma 10.3. (G, b) *contains a negative even cycle if and only if* (G, b) *contains a negative even circuit or negative simple handcuff.*

Proof. Choose a negative even cycle C of minimal size. If C is a circuit, we are done. Otherwise, there exists at least one vertex p occuring twice in C.

Let c_p be the corresponding cut. I.e. C can be cut into two subcycles $C = C_1(p) + C_2(p)$. By the minimality of the size, C contains no even cuts.

Suppose there exists a second odd cut c_u, which crosses the odd cut c_p. Then we could divide C into subwalks

$$C = P_1 + P_2 + P_3 + P_4,$$

such that $C_1(p) = P_1 + P_2, C_2(p) = P_3 + P_4, C_1(u) = P_2 + P_3$ and $C_2(u) = P_1 + P_4$ (see Figure 10.2). Since $C' = P_1 + P_3^-$ as well as $C'' = P_2 + P_4^-$ are even subcycles of C,

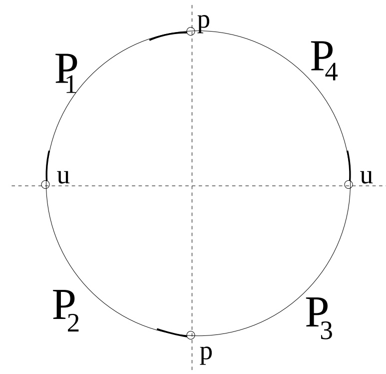


Figure 10.2: $C = P_1 + P_2 + P_3 + P_4$

$b(C') + b(C'') = b(C)$ implies that either C' or C'' have to be negative. Again, this is a contradiction to the minimality of C.

Therefore, all cuts in C are parallel. Let P and P' be the two maximal parallel intervals of C. Interchanging P' by P, we obtain a simple handcuff. $\square$

Summarizing, we are able to characterize weighted red-blue graphs that admit a feasible potential by exclusion of forbidden subgraphs:

Corollary 10.1. *A feasible potential of (G, b) exists if and only if (G, b) contains no subgraph of negative weight, which is an even subdivision of one of the subgraphs of Figure 10.3.*

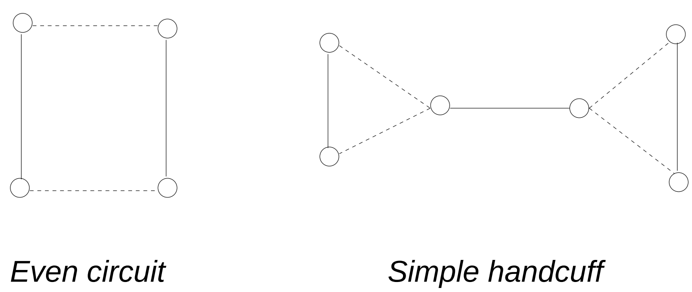


Figure 10.3: Even circuit and simple handcuff.

10.2 The R/B-split algorithm and integral solutions

Suppose the weighted red-blue graph $G = (V, R \cup B, b)$ does not contain any negative even cycle. Using a shortest-path algorithm, we saw in the proof of Lemma 10.2 how to compute a half-integral feasible potential $x : V \to \mathbb{R}$ of G.

In order to modify x into an integral feasible potential x^*, or prove that no integral solution exists, we construct an auxiliary red-blue graph $G^x = (V^x, R^x \cup B^x)$ with vertex set

$$V^x = \{v \in V \mid x_v \notin \mathbb{Z}\},$$

red edges

$$R^x = \{(u, v) \in R \mid u, v \in V^x, x_u + x_v = b(u, v)\}$$

and blue edges

$$B^x = \{(u, v) \in B \mid u, v \in V^x, -x_u - x_v = b(u, v)\}.$$

We can now solve the WEIGHTED R/B-SPLIT PROBLEM of (G, b) by solving the R/B-SPLIT PROBLEM of G^x:

Theorem 10.1. *There exists an integral feasible potential $x^* : V \to \mathbb{Z}$ of (G, b) if and only if G^x is an R/B-split graph.*

Proof. "Sufficiency": Let G^x be an R/B-split graph and $V^x = S_R \dot\cup S_B$ be a partition of V^x into a stable set S_R in $G^x_R = (V^x, R^x)$ and a stable set S_B in $G^x_B = (V^x, B^x)$. Let us construct the vector $x^* : V \to \mathbb{Z}$ with components

$$x^*_v = \begin{cases} x_v + \frac{1}{2} & : \quad v \in S_R, \\ x_v - \frac{1}{2} & : \quad v \in S_B. \end{cases}$$

Since x is half-integral, x^* is an integral potential. It remains to show that x^* is feasible:

We proof that $(u, v) \in R$ implies $x^*_u + x^*_v \leq b(u, v)$. (The proof that $(u, v) \in B$ implies $-x^*_u - x^*_v \leq b(u, v)$ is analogue.)

If both vertices, u and v, are in $V \setminus V^x$, there is nothing to show. If only one of the two vertices u and v is in V^x, the inequality $x_u + x_v \leq b(u, v) - \frac{1}{2}$ implies $x^*_u + x^*_v \leq b(u, v)$. Finally, if u and v are both in V^x, we distinguish the cases where $x_u + x_v \leq b(u, v) - 1$ and $x_u + x_v = b(u, v)$:

In the first case, $x^*_u + x^*_v \leq b(u, v)$ follows immediately. In the latter case, $(u, v) \in R^x$ implies that at least one of the two vertices is in the blue stable set. Hence, $x^*_u + x^*_v \leq b(u, v)$, i.e., x^* is a solution of the WEIGHTED R/B-SPLIT PROBLEM of (G, b).

"Necessity": In case G^x is not an R/B-split graph, we know that there exists a forbidden cycle $C = C_1(p) + C_2(p)$ with odd cut c_p in G^x.

As any edge in G^x satisfies $x(e) = b(e)$, we know by Lemma 10.1 that C is a tight cycle. By the same Lemma, we know that for each feasible potential x' of (G, b) and each edge $e \in C$, $x'(e) = b(e)$ holds.

W.l.o.g., we assume that the odd cycle $C_1 = C_1(p)$ is red (otherwise we consider $C_2(p)$). Then

$$2x'_p = x'(C_1) = b(C_1) = x(C_1) = 2x_p.$$

Since $2x_p$ is odd, x'_p is not integral. Hence, there cannot exist an integral feasible potential of (G, b). □

10.3 Weighted excluded subgraphs

So far we have seen how to solve the WEIGHTED R/B-SPLIT PROBLEM: We first solve the SHORTEST-PATH PROBLEM of the corresponding weighted bipartite digraph $(D(G), b)$ to achieve a half-integral potential x. Thereafter, we solve the R/B-SPLIT PROBLEM of the auxiliary graph G^x.

In order to characterize weighted red-blue graphs (G, b) that allow an integral feasible potential by forbidden subgraphs, independently from a feasible potential x, we need a few more observations:

Let C be a tight cycle in (G, b) and c_p be a cut dividing C into two subcycles $C = C_1(p) + C_2(p)$. We define the *parity* of c_p to be

$$\text{Parity}(c_p) = \begin{cases} \text{odd} & : \quad b(C_1(p)) = -b(C_2(p)) \text{ odd }, \\ \text{even} & : \quad b(C_1(p)) = -b(C_2(p)) \text{ even }. \end{cases}$$

Lemma 10.4. *Let C be a tight normalized forbidden cycle in (G, b). Then any two odd cuts c_u and c_p of C have the same parity.*

Proof. In case c_u is parallel to c_p, we may assume $C_2(u)$ to be a subcycle of $C_2(p)$. Let us divide C into the subwalks $C = C_1(p) + P + C_2(u) + P^-$, where P is a path from p to u. The equality $b(C) = 0$ implies $-b(C_2(u)) = b(C_1(p)) + 2b(P)$. Therefore, c_u and c_p must have the same parity.

In case c_u crosses c_p, we can divide C into 4 subwalk $C = P_1 + P_2 + P_3 + P_4$, such that $C_1(p) = P_1 + P_2, C_2(p) = P_3 + P_4$, $C_1(u) = P_2 + P_3$ and $C_2(u) = P_4 + P_1$ (see Figure 10.2). Since $P_1 + P_3^-$ as well as $P_2 + P_4^-$ are tight cycles, we obtain $b(P_1) = -b(P_3)$ and $b(P_2) = -b(P_4)$. Hence, $b(C_1(p)) = b(P_1) + b(P_2)$ and $b(C_1(u)) = b(P_2) + b(P_3)$ are of the same parity. □

Given a tight normalized forbidden cycle C we may thus choose any odd cut c_p and define

$$\text{Parity}(C) = \text{Parity}(c_p).$$

Hence, the following is well-defined:

Definition 10.1 (Tight odd flower). *A* tight odd flower *in (G, b) is a flower in G, whose corresponding normalized forbidden cycle is tight and of odd parity.*

Observe that, given a tight normalized forbidden cycle C with odd cut $C = C_1(p) + C_2(p)$ and an arbitrary feasible potential $x : V \to \mathbb{R}$, we have

$$\text{Parity}(C) = \text{Parity}(c_p) = \text{parity of } b(C_1(p)) = \text{parity of } 2x_p.$$

I.e., if the parity of C is odd, the component x_p cannot be integral. Therefore, tight odd flowers cannot occur in solvable instances (G, b). We conclude:

Corollary 10.2. *The* WEIGHTED R/B-SPLIT PROBLEM *of $G = (V, R \cup B, b)$ is solvable if and only if G contains neither a negative even circuit or a negative simple handcuff, nor a tight odd flower. (See Figure 10.4.)*

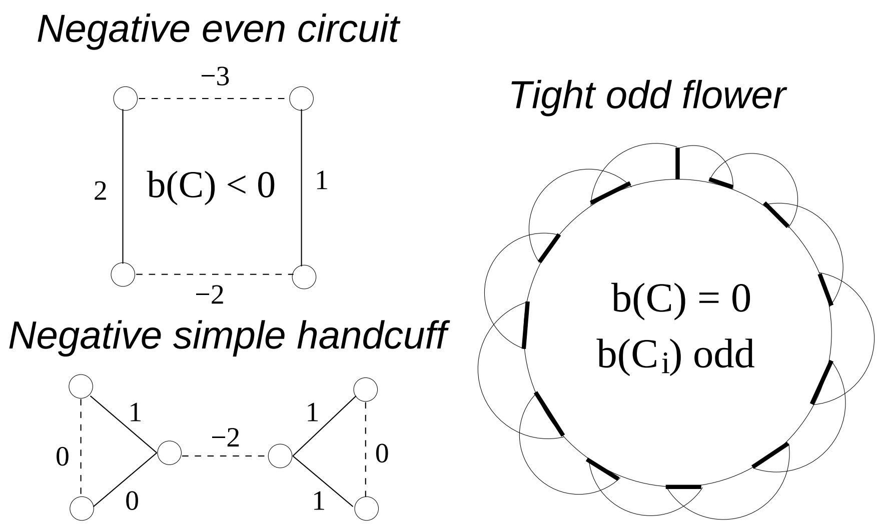


Figure 10.4: Excluded subgraphs in weighted R/B-split graphs.

Chapter 11

Maximal union of a red and a blue stable set

We now return to red-blue graphs $G = (V, R \cup B)$ without weights on the edge set. In case G is not an R/B-split graph itself, one might still be interested in the largest subset of V whose induced subgraph is an R/B-split graph.

Definition 11.1 (Max R/B-split problem). *Given a red-blue graph $G = (V, R \cup B)$, the* Max R/B-Split Problem *asks for a subsets $S \subseteq V$ of maximal cardinality such that S can be split into a red and a blue stable set.*

By taking complement graphs, we observe that it is an equivalent task to determine a maximal subset that can be split into a red and a blue clique, respectively, into a red clique and a blue stable set.

The general Max R/B-Split Problem is easily seen to be $\mathcal{NP}$-hard as it includes the Max Stable Set Problem (suppose G_R is a clique). Although the Max Stable Set Problem is well-known to be $\mathcal{NP}$-hard in general, it is polynomially solvable for a large class of graphs, e.g., perfect graphs. It would therefore be interesting to identify polynomially solvable cases of the Max R/B-Split Problem.

11.1 Reduction to a stable set problem

A general construction reduces the problem in G to the Stable Set Problem in an associated graph $H(G)$:

Let V' be a copy of V and consider the graph $H(G) = (V \cup V', E)$ with

$$(i,j) \in E \iff \begin{cases} i,j \in V \text{ and } (i,j) \in R, \\ i,j \in V' \text{ and } (i,j) \in B, \\ i \in V, j \in V' \text{ and j is the copy of i.} \end{cases}$$

This (uncolored) graph H is constructed by joining the red and the blue graph by a (special) perfect matching. Clearly, a maximal stable set S of $H(G)$ corresponds to a maximal union of a red and a blue stable set. Therefore, the MAX R/B-SPLIT PROBLEM is polynomially solvable if and only if a maximal stable set in $H(G)$ can be determined in polynomial time.

11.2 Chordal and comparability graphs

The reduction above shows that the MAX R/B-SPLIT PROBLEM can always be solved by solving a MAX STABLE SET PROBLEM. Would it be also possible to identify pairs of a red and a blue graph where the MAX R/B-SPLIT PROBLEM is always solvable independent of the way the red and the blue edges interact? One example of such a class of tractable pairs is formed by a comparability graph and the complement of a chordal graph (cf. Lemma 11.1).

Recall that a graph is said to be *chordal* if each cycle with more than four edges posesses a chord (i.e., an edge dividing the cycle into two subcycles). Since a stable set in G_B corresponds to a clique in the complement graph $\bar{G}_B$, and since the maximal cliques of a chordal graph can be listed in polynomial time, we get:

Lemma 11.1. *If G_R is the complement graph of a chordal graph $\bar{G}_R$, and G_B is a comparability graph, the* MAX R/B-SPLIT PROBLEM *is polynomial.*

Proof. We have to split the vertex set V into a clique in the chordal graph $\bar{G}_R$ and an antichain in the partial order P corresponding to G_B. Observe that, in a disjoint union of a red clique and a blue stable set, we may assume the red clique to be inclusionwise maximal. Fulkerson and Gross [FG65] showed that all maximal cliques of a chordal graph $\bar{G}_R = (V, \bar{R})$ can be calculated in time $\mathcal{O}(|V| + |\bar{R}|)$. Therefore, for each maximal clique C in $\bar{G}_R$, we simply have to determine a maximal antichain in the reduced poset $(P \setminus C, \leq)$. But this is a polynomial task. $\square$

Note that the same approach of the preceding proof works for any pair of graphs where the maximal stable sets of the red graph can be listed in polynomial time and a maximal stable set of the blue graph can be calculated efficiently even if the blue graph is reduced by a subset of vertices.

If the red edge set and the blue edge set are identical, the MAX R/B-SPLIT PROBLEM becomes the problem to determine a maximal union of two stable sets. For graphs in general, this problem is still $\mathcal{NP}$-hard as it is a special instance of the known $\mathcal{NP}$-hard MAXIMUM INDUCED SUBGRAPH WITH PROPERTY Π PROBLEM (see [ACG+98], p. 381). However, if we restrict to partial orders, the problem becomes polynomial:

11.3 Maximal unions of chains and antichains

If $G_R = G_B$ is the comparability graph of a partial order $P = (P, \leq)$, the Max R/B-Split Problem is the problem to determine the maximal union of two antichains in P. Due to Greene and Kleitman [GK67], this problem is efficiently solvable even for the maximal union of k antichains. Dually, Greene [Gre67] showed that the maximal union of k chains can be found in polynomial time. The question arises whether the maximal union of a chain and an antichain of the same poset can be calculated efficiently.

We discuss the three special instances where $G_R = G_B$, $\bar{G}_R = \bar{G}_B$ or $G_R = \bar{G}_B$ are comparability graphs, of our Max R/B-Split Problem: We show that the maximal union of a chain and an antichain is easy to determine. Moreover, we characterize those posets that contain a disjoint pair of a maximum chain and a maximum antichain, a disjoint pair of two maximum chains, or a disjoint pair of two maximum antichains.

Given a poset $P = (P, \leq)$, we denote the value

$$w(P) = \max\{|A| \mid A \text{ antichain in } P\}$$

the *width* of P, and the value

$$l(P) = \max\{|C| \mid C \text{ chain in } P\}$$

the *length* of P. We first consider the Max R/B-Split Problem where $G_R = \bar{G}_B$ is the comparability graph of some poset $(P, \leq)$. I.e., we ask for the maximal union of a chain and an antichain in $(P, \leq)$:

11.3.1 Maximal union of a chain and an antichain

As a chain and an antichain of the same poset $P = (P, \leq)$ can intersect in at most one element, the following result is immediate:

$$\max\{|A \cup C| \mid A \text{ antichain}, C \text{ chain in } P\} \in \{w(P) + l(P), w(P) + l(P) - 1\}.$$

Therefore, in the special case where $G_R = \bar{G}_B$ is a comparability graph of poset P, the Max R/B-Split Problem reduces to the question:

"Does there exist a disjoint pair of a maximum chain and a maximum antichain in P?"

The following Theorem 11.1 characterizes posets allowing a positive answer to this question. Additionally, the proof provides an algorithm to find a chain and an antichain of maximal union.

Given an element $a \in P$, we define the sets

$$a^{\downarrow} = \{p \in P \mid p \leq a\} \quad \text{and}$$

$$a^{\uparrow} = \{p \in P \mid a \leq p\}.$$

For simplicity, any subset $A \subseteq P$ corresponds to the partial order $(A, \leq)$ induced by P.

Theorem 11.1. *A poset $P = (P, \leq)$ contains a disjoint pair of a maximum chain and a maximum antichain if and only if there exists an upper neighbour b of some element a in P such that*

$$l(P) = l(b^{\uparrow}) + l(a^{\downarrow}), \text{ and} \tag{11.1}$$
$$w(P) = w(P \setminus (b^{\uparrow} \cup a^{\downarrow})). \tag{11.2}$$

Proof. Suppose $|A \cup C| = w(P) + l(P)$ for some antichain A and some chain C in P. Let b be the minimal member of C such that b is greater than at least one member of A. Let a be the predecessor of b in C (Figure 11.1). Since C is a longest chain, b must

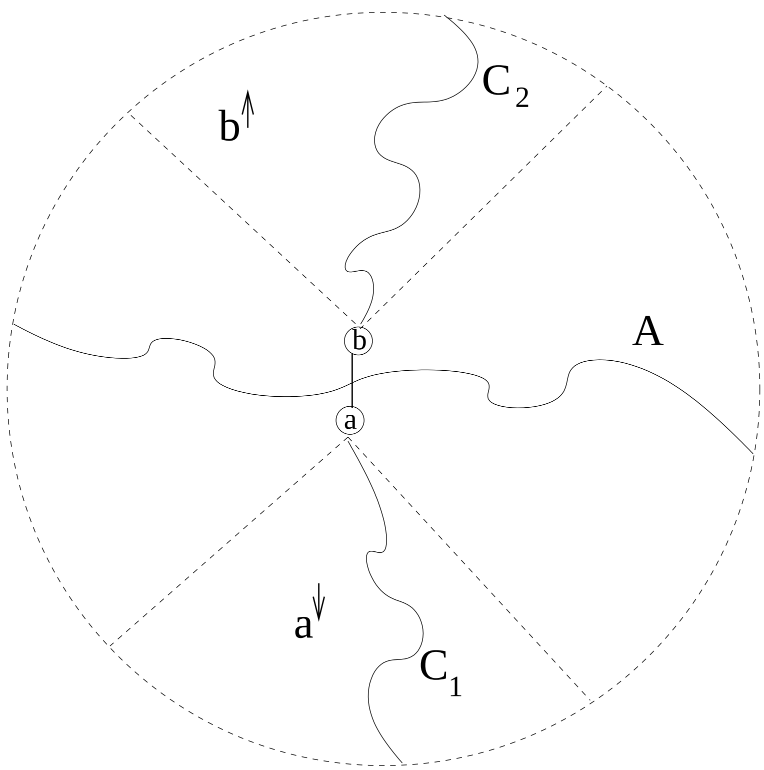


Figure 11.1: b is upper neighbour of a.

be an upper neighbor of a in P. Chain C divides into the subchains $C_1 = C \cap a^{\downarrow}$ and $C_2 = C \cap b^{\uparrow}$, where C_1 is a longest chain in ideal $a^{\downarrow}$, and C_2 is a longest chain in filter $b^{\uparrow}$.

Hence, $l(P) = l(b^{\uparrow}) + l(a^{\downarrow})$. Moreover, A can neither contain an element of $a^{\downarrow}$ nor of $b^{\uparrow}$, implying $w(P) = w(P \setminus (b^{\uparrow} \cup a^{\downarrow}))$.

Suppose there exists an upper neighbor b of some element a in P such that $l(P) = l(b^{\uparrow}) + l(a^{\downarrow})$ and $w(P) = w(P \setminus (b^{\uparrow} \cup a^{\downarrow}))$. Choose longest chains C_1 in $a^{\downarrow}$ and C_2 in $b^{\uparrow}$, and a widest antichain A in poset $P \setminus (b^{\uparrow} \cup a^{\downarrow})$. Then $C = C_1 \cup C_2$ is a longest chain in P, A is a widest antichain in P, and A and C are disjoint. □

Since we could characterize posets containing a disjoint pair of a maximum chain and a maximum antichain, the question arises how posets containing a disjoint pair of two maximum antichains, respectively, chains look like. We use results of Greene and Kleitman about the maximal union of two antichains, respectively,chains to characterize posets such that the size of this union equals $2w(P)$, respectively $2l(P)$:

11.3.2 Maximal union of two antichains

As already observed, if $G_R = G_B$ is the comparability graph of some poset $P = (P, \leq)$, the Max R/B-Split Problem is the problem to determine the maximal union of two antichains in P. This problem is well-known to be efficiently solvable even for k antichains:

Theorem 11.2 (Greene and Kleitman [GK67]). *Let $(P, \leq)$ be a partially ordered set and $k \in \mathbb{Z}_+$. Then the maximum size of the union of k antichains is equal to the minimum value of*

$$\sum_{C \in \mathcal{C}} \min\{k, |C|\}, \tag{11.3}$$

where $\mathcal{C}$ ranges over partitions of P into chains.

The proof methods of Fomin [Fom78] and Frank [Fra80], based on minimum-cost circulations, provide a polynomial-time algorithm to find a maximum union of k antichains.

Greene and Kleitman [GK67] also showed that for each $h \in \mathbb{Z}_+$ there is a chain partition $\mathcal{C}$ of P attaining the minimum of formula (11.3) for both $k = h$ and $k = h + 1$. Using this result, we can characterize posets containing two disjoint maximum antichains. They turn out to be those posets such that no element lies in every maximum antichain:

Corollary 11.1. *A partially ordered set $(P, \leq)$ contains two disjoint antichains of maximal size if and only if*

$$w(P) = w(P \setminus v) \text{ for all } v \in P.$$

Proof. Let $\alpha_2(P)$ denote the maximal size of the union of 2 antichains in P. We have to show

$$\alpha_2(P) = 2w(P) \iff w(P) = w(P \setminus v) \quad \forall v \in P.$$

By Dilworth' Theorem, there exists a chain partition of P with $w(P)$ many chains. And by Greene and Kleitman's result, we may choose such a chain partition $\mathcal{C}$ with $w(P)$ many chains such that additionally $\alpha_2(P) = \sum_{C \in \mathcal{C}} \min\{2, |C|\}$ holds.

In case $\alpha_2(P) < 2w(P)$, there must exist a trivial chain $C = \{v\}$ in $\mathcal{C}$. Then $\mathcal{C} \setminus v$ is a chain partition of poset $(P \setminus v, \leq)$ with $w(P) - 1$ many chains. By Dilworth' Theorem, we therefore obtain $w(P \setminus v) = w(P) - 1$.

In case $w(P \setminus v) = w(P) - 1$, there exists a chain partition $\mathcal{C}'$ of poset $(P \setminus v, \leq)$ with $w(P) - 1$ many chains. Hence, $\mathcal{C} = \mathcal{C}' \cup \{v\}$ is a chain partition of P such that

$$\alpha_2(P) = \sum_{C \in \mathcal{C}} \min\{2, |C|\} \leq 2(w(P) - 1) + 1 < 2w(P).$$

□

11.3.3 Maximal union of two chains

Dual results for the maximal union of two resp. k chains can be formulated by interchanging the terms 'chain' and 'antichain':

Theorem 11.3 (Greene [Gre67], Edmonds and Giles [EG75]). *Let $(P, \leq)$ be a partially ordered set and $k \in \mathbb{Z}_+$. Then the maximum size of the union of k chains is equal to the minimum value of*

$$\sum_{A \in \mathcal{A}} \min\{k, |A|\}, \tag{11.4}$$

where $\mathcal{A}$ ranges over partitions of P into antichains.

Again, the proof methods of Fomin [Fom78] and Frank [Fra80] provide a polynomial-time algorithm to find a maximum union of k chains. Moreover, for each $h \in \mathbb{Z}_+$ there is an antichain partition $\mathcal{A}$ of P attaining the minimum of formula (11.4) for both $k = h$ and $k = h + 1$.

Dually to Corollary 11.1, we can characterize posets that contain two disjoint maximum antichains as exactly those posets, such that no element lies in every maximum chain:

Corollary 11.2. *A partially ordered set $(P, \leq)$ contains two disjoint chains of maximal size if and only if*

$$l(P) = l(P \setminus v) \text{ for all } v \in P.$$

Proof. The proof is analogue to the proof of Corollary 11.1: Let $c_2(P)$ denote the maximal size of the union of 2 chains in P. We have to show

$$c_2(P) = 2l(P) \iff l(P) = l(P \setminus v) \quad \forall v \in P.$$

By Dilworth' Theorem, there exists an antichain partition of P with $l(P)$ many chains. Again, we may choose such an antichain partition $\mathcal{A}$ with $l(P)$ many antichains such that additionally $c_2(P) = \sum_{A\in\mathcal{A}} \min\{2, |A|\}$ holds.

In case $c_2(P) < 2l(P)$, there must exist a trivial antichain $A = \{v\}$ in $\mathcal{A}$. Then $\mathcal{A} \setminus v$ is an antichain partition of poset $(P \setminus v, \leq)$ with $l(P) - 1$ many antichains. By Dilworth' Theorem, we therefore yield $l(P \setminus v) = l(P) - 1$.

In case $l(P \setminus v) = l(P) - 1$, there exists an antichain partition $\mathcal{A}'$ of poset $(P \setminus v, \leq)$ with $l(P) - 1$ many antichains. Hence, $\mathcal{A} = \mathcal{A}' \cup \{v\}$ is an antichain partition of P such that

$$c_2(P) = \sum_{A\in\mathcal{A}} \min\{2, |A|\} \leq 2(l(P) - 1) + 1 < 2l(P).$$

$\square$

As the maximal union of two antichains resp. chains of the same poset can be calculated efficiently the question raises whether the Max R/B-Split Problem is generally polynomial in case G_R and G_B are comparability graphs. In Section 11.4 we show that this problem to determine the maximal union of a red and a blue antichain is $\mathcal{NP}$-hard even for series-parallel orders.

11.4 Maximal union of a red and a blue antichain

We have seen that it is easy to either decide whether we can cover all vertices of a graph with a red and a blue stable set, or to find two antichains covering a maximal number of elements relative to one partial order.

Suppose now that we are given a red and a blue partial order $P_R = (P, \leq_R)$ and $P_B = (P, \leq_B)$ on the same ground set P. Let $w_R = w(P_R)$ resp. $w_B = w(P_B)$ denote the size of a maximum red resp. blue antichain. In case $w_R + w_B < |P|$, it is obviously impossible to cover all elements with a red and a blue antichain. However, we might still wonder if we can find a red and a blue antichain that cover $w_R + w_B$ elements, i.e., if we can find a disjoint pair of a maximum red and a maximum blue antichain.

It turns out that the problem of deciding whether there exist two disjoint differently colored maximum antichains is $\mathcal{NP}$-complete already on the class of series-parallel orders. This fact directly implies $\mathcal{NP}$-hardness of the Max R/B-Split Problem.

Theorem 11.4. *Given two partial orders $P_R = (P, \leq_R)$ and $P_B = (P, \leq_B)$ on the same ground set P, it is $\mathcal{NP}$-hard to decide whether there exist maximum antichains A_R in P_R and A_B in P_B with $A_R \cap A_B = \varnothing$.*

Proof. We show $\mathcal{NP}$-hardness by a reduction from 3-SAT. Consider a 3-SAT instance with k clauses on n variables x_i,

$$\bigwedge_{j=1}^{k} (\ell_1^j \vee \ell_2^j \vee \ell_3^j),$$

i.e. $\ell_p^j \in \{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\}$ for $p \in \{1,2,3\}$. The ground set P contains all literals and their negations, where appearances of the same literal in different clauses are distinguished:

$$P = \{x_i^j, \neg x_i^j \mid \exists p \in \{1,2,3\} : x_i^j = \ell_p^j \text{ or } \neg x_i^j = \ell_p^j\}.$$

In the following, when referring to a literal ℓ_p^j, we mean its incarnation in clause j, *i.e.* $\ell_p^j = (\neg)x_i^j$ for the appropriate i. The red and blue orders are defined as follows:

$$\begin{aligned} \forall\ j: &\quad \ell_1^j < \ell_2^j < \ell_3^j \\ \forall\ i, j, j': &\quad x_i^j < \neg x_i^{j'} \end{aligned}$$

Fig. 11.2 shows the Hasse diagram of the reduction for an example formula (without the uncomparable items).

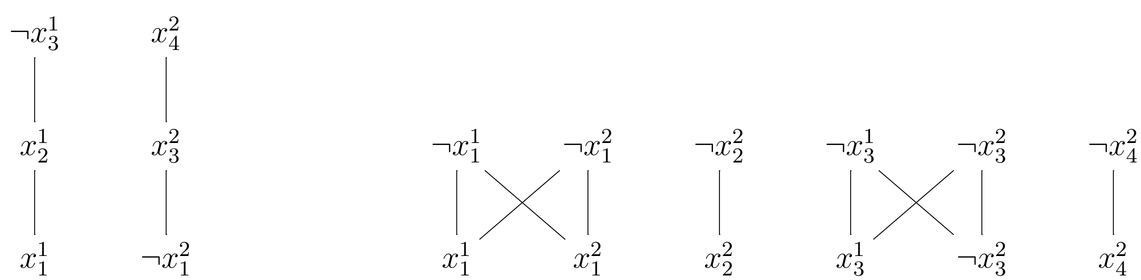


Figure 11.2: The red order (left) and the blue order (right) corresponding to 3–SAT instance $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_3 \vee \neg x_4)$

Obviously, a maximum red antichain covers exactly one literal per clause, whereas a maximum blue antichain corresponds to a consistent assignment of the variables. Note that a maximum red and a maximum blue antichain are disjoint if and only if the literals covered by the red antichain are *false* in the variable assignment corresponding to the blue antichain.

Therefore, if we can find two maximum disjoint antichains, negating the variable assignment corresponding to the blue antichain produces a satisfying variable assignment for the original 3–SAT instance. On the other hand, if there are no such two antichains, there also is no variable assignment satisfying all clauses. As this reduction is obviously polynomial, we have shown $\mathcal{NP}$-completeness. $\square$

Remark 11.1. *As the orders produced in the reduction of Theorem 11.4 are series-parallel, the* MAX R/B-SPLIT PROBLEM *is already $\mathcal{NP}$-hard for series-parallel orders. Furthermore, finding two maximal disjoint chains or a maximal chain and a disjoint maximal antichain is also $\mathcal{NP}$-hard (already in series-parallel orders), as one can demonstrate via complementary constructions.*

Now consider the red and the blue order constructed in the proof of Theorem 11.4. Since the comparability graph of the red order consists of disjoint cliques, any maximum antichain of the red order is a basis in the corresponding partition matroid $\mathcal{M} = (P, \mathcal{F})$ whose ground set P is split into the single clauses of the 3–SAT instance $f = \bigwedge_{j=1}^{k} (\ell_1^j \vee \ell_2^j \vee \ell_3^j)$. Moreover, any basis of $\mathcal{M}$ which is stable in the comparability graph of the blue order, corresponds to a satisfying assignment of f. Thus, we conclude:

Corollary 11.3 (Theorem 7.7). *If $M = (P, \mathcal{F})$ is a partition matroid, and $G = (P, E)$ the comparability graph of a series-parallel order, then the* STABLE BASIS PROBLEM *is $\mathcal{NP}$-complete.*

Chapter 12

Summary

It is a very important task in discrete optimization to identify "greedy structures", i.e., families $\mathcal{F}$ of subsets of some finite set E such that greedy-type algorithms determine a member X in $\mathcal{F}$ of maximal weight $w(X) = \sum\{w(e) \mid e \in X\}$ for any weight function $w : E \to \mathbb{R}$. The more general such greedy structures are, the greater the chance that a discrete optimization problem occuring in theory or praxis can be shown to be solvable with a greedy-type algorithm. The probably most populare greedy structures are matroids, which are abstract generalizations of linear independent columns of a matrix. Less populare, but more general than matroids are greedy structures like Gauss greedoids, strong exchange structures, Δ-matroids and jump systems.

In the first part of this thesis, we generalize strong exchange structures, Gauss greedoids, Δ-matroids and jump systems by considering an arbitrary partial order on the ground set. A subset $X \subseteq E$ is an ideal of the partially ordered set ("poset") $(E, \leq)$ if X contains with each $x \in X$ all predecessors $y \leq x$ as well. Instead of (unordered) set systems we consider families of ideals of a given poset $(E, \leq)$. Certainly, any subset family can always be interpreted as a family of ideals of the trivial poset having no comparability constraints. Moreover, any family of n-dimensional integral vectors can be seen as an ideal system with respect to the poset consisting of n disjoint chains.

We identify certain exchange properties for ideal systems that guarantee the correctness of greedy-type algorithms. Since these algorithms are defined on the distributive lattice of all ideals, we call the structures characterized by these exchange properties "distributive strong exchange structures", "distributive Gauss greedoids" and "distributive Δ-matroids".

Hoffman and Dietrich established a dual greedy algorithm for certain "pseudolattices" on subsets of E. They claimed that their model covers and properly extends known and well-studied greedily solvable structures like, for example, polymatroids, distributive supermatroids and submodular systems. We prove that the pseudolattices in their model are in fact distributive lattices and reduce their model to submodular systems which can be solved with the generalized polymatroid greedy algorithm of Faigle and Kern.

In the second part of this thesis, we investigate the matching problem and its dual vertex cover problem: the bipartite matching algorithm, which goes back to König and Egerváry, works in some sense in a greedy way and determines a maximal matching and a minimal vertex cover of any bipartite graph. The algorithm has been extended in one direction by Edmonds and Frank to solve the matroid intersection problem, and in another direction by Edmonds to solve the matching problem in general graphs. In contrast to the matching problem, the problem to determine a vertex cover of minimal cardinality in general graphs is $\mathcal{NP}$-complete. However, if the graph is a König-Egerváry graph, i.e., if it contains a matching and a cover of identical size, then a maximal matching can be used to construct a minimal vertex cover.

We characterize König-Egerváry graphs by the exclusion of certain subgraphs from which Lovász and Korach's characterizations of König-Egerváry graphs can be deduced. Our characterization of König-Egerváry graphs follows as an easy consequence of our characterization of "Red/Blue-split graphs", a common generalization of König-Egerváry graphs and classical split graphs: Red/Blue-split graphs consist of red and blue colored edges and allow a partition of the vertices into a red and a blue stable set. We present an algorithm that either determines a feasible partition of the vertices into a red and a blue stable set, or returns a handcuff characterizing non-Red/Blue-split graphs. Since handcuffs are not necessarly vertex disjoint, we normalize handcuffs such that the induced subgraphs are of a certain type which we call "flower".

We also investigate a weighted version of Red/Blue-split graphs which models integrally solvable inequality systems $Ax \leq b$ where the sum of absolute values in each row of the integral matrix A does not exceed the value two. Matrices with this property are called "simple". We solve the problem whether such an inequality system has an integral solution, or, equivalently, whether a graph with red and blue colored weighted edges is a "weighted Red/Blue-split graph", by a shortest-path- and our Red/Blue-split algorithm. These two algorithms lead to a characterization of weighted Red/Blue-split graphs by the exclusion of "negative even circuits", "negative simple handcuffs", and "tight odd flowers". This characterization can be viewed as a refinement of Schrijver's characterization of integrally solvable inequality systems with simple matrix.

Further on, we discuss some polynomially solvable instances of the problem to determine a maximal union of a red and blue stable set. In particular, we consider a red and a blue comparability graph. Interestingly, even though the problem on posets is polynomial if the red and the blue graph are identical, we show that is is $\mathcal{NP}$-complete for two differently colored posets.

Additionally, we investigate some related problems: for example, given a matroid $\mathcal{M}$ whose ground set is the vertex set of a graph G, we show that the problem to determine a basis of $\mathcal{M}$ which is stable in the graph G is $\mathcal{NP}$-complete in case $\mathcal{M}$ is a partition matroid, whereas it can be solved with our Red/Blue-split algorithm in case $\mathcal{M}$ is the dual of a partition matroid. Moreover, we characterize posets containing disjoint chains or antichains.

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Erklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbstständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschliesslich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegt hat; dass sie – abgesehen von unten angegebener Teilpublikationen – noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen dieser Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Professor Dr. Ulrich Faigle betreut worden.

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- E. Korach, T. Nguyen and B. Peis,
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  Covering graphs by colored stable sets,
  Zentrum für Angewandte Informatik Köln, Tech. report zaik2004-478.

Lebenslauf

Ich bin am 15.12.1975 als Tochter von Walter und Elisabeth Wienand in Winterberg geboren und gemeinsam mit meinen Geschwistern, Ingo und Kirsten, in Siedlinghausen, einem Vorort von Winterberg, aufgewachsen. Mittlerweile (d.h. seit dem 28. Juli 2005) bin ich mit Thorsten Peis verheiratet und seit dem 13. Juni 2006 Mutter unserer Tochter Emma.

Von 1982-86 besuchte ich die Grundschule in Winterberg-Siedlinghausen und daran anschliessend, von 1986-92, das Gymnasium in Winterberg. 1992 wechselte ich auf das Skigymnasium in Willingen und schloss meine Schulausbildung im Juni 1995 mit dem Abitur ab.

Da ich von 1994-98 der Skilanglauf Nationalmanschaft angehörte, studierte ich in meinem ersten Studienjahr zunächst nur die Fächer Sport- und Erziehungswissenschaften an der Sporthochschule Köln.

Im Oktober 1996 begann ich mein Mathematikstudium mit dem Studienziel Lehramt an der Fernuniversität Hagen, wechselte allerdings nach einem Semester an das mathematische Institut der Universität zu Köln.

Nach der Zwischenprüfung nahm ich ein Stipendium der University of Denver (Colorado, USA) an und studierte dort während der Wintersemester 1998/99 und 1999/2000 Mathematik und Informatik. Im Sommersemester 1999 wechselte ich an der Universität in Köln den Studiengang vom Lehramt zum Diplom in Mathematik mit Nebenfach Informatik.

Nach Erhalt meines Diploms im Januar 2002 bis September 2006 arbeitete ich als wissenschaftliche Mitarbeiterin am Zentrum für Angewandte Informatik Köln (ZAIK) in der Arbeitsgruppe von Professor Ulrich Faigle und Professor Rainer Schrader.

Während dieser Zeit setzte ich mein sport- und erziehungswissenschaftliches Studium an der Sporthochschule Köln fort und schloss es im Juni 2004 mit der Ersten Staatsprüfung für die Lehrämter an der Sekundarstufe I und II für die Fächer Mathematik, Informatik und Sport ab.

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