Galleries and q-analogs in combinatorial representation theory

Inaugural – Dissertation

zur

Erlangung des Doktorgrades

der Mathematisch-Naturwissenschaftlichen Fakultät

der Universität zu Köln

vorgelegt von

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Hundt Druck GmbH, Köln
2006

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Tag der mündlichen Prüfung:	30. Juni 2006

Abstract

Schur functions and their q-analogs constitute an interesting branch of combinatorial representation theory. For Schur functions one knows several combinatorial formulas regarding their expansion in terms of monomial symmetric functions, their structure constants and their branching coefficients. In this thesis we prove q-analogs of these formulas for Hall–Littlewood polynomials. We give combinatorial formulas for the expansion of Hall–Littlewood polynomials in terms of monomial symmetric functions, for their structure constants and their branching coefficients. Specializing these formulas we get new proofs for the formulas involving Schur functions. As a combinatorial tool we use the gallery model introduced by Gaussent and Littelmann and show its relation to the affine Hecke algebra. All assertions are then proven in the more general context of the Macdonald basis of the spherical Hecke algebra.

We show a commutation formula in the affine Hecke algebra with which we obtain a Demazure character formula involving galleries. We give a geometric interpretation of Kostka numbers and Demazure multiplicities of a complex reductive algebraic group using the affine Grassmanian of its Langlands dual group. As a further application we prove some first results regarding the positivity of Kostka–Foulkes coefficients.

Kurzzusammenfassung

Schur Polynome und ihre q-Analoga sind ein interessantes Gebiet der kombinatorischen Darstellungstheorie. Kombinatorische Formeln für die Koeffizienten der Schur Polynome bezüglich der monomialen symmetrischen Funktionen, für ihre Strukturkonstanten und für die Verzweigungskoeffizienten sind wohlbekannt. In dieser Dissertation werden q-Analoga dieser Formeln für die Hall–Littlewood Polynome bewiesen. Es werden kombinatorische Formeln für die Koeffizienten der Hall–Littlewood Polynome bezüglich der monomialen symmetrischen Polynome, für ihre Strukturkonstanten und ihre Verzweigungskoeffizienten gezeigt. Spezialisiert man diese Formeln, erhält man neue Beweise für die klassischen Formeln bezüglich der Schur Polynome. Als kombinatorisches Hilfsmittel wird das Galerienmodell von Gaussent und Littelmann benutzt und mit der affinen Hecke Algebra in Verbindung gebracht. Die Aussagen über die Hall–Littlewood Polynome werden allgemeiner für die Macdonald Basis der sphärischen Hecke Algebra bewiesen.

Es wird eine Vertauschungsformel in der affinen Hecke Algebra gezeigt, die sich zu einer Demazure Charakterfomel spezialisieren lässt. Kostka Zahlen und Demazure Multiplizitäten von komplexen reduktiven algebraischen Gruppen werden mit Hilfe der affinen Grassmannschen der Langlands dualen Gruppe geometrisch interpretiert. Auch werden erste Resultate hinsichtlich der Positivität der Kostka–Foulkes Koeffizienten erzielt.

Introduction

The symmetries of many systems in mathematics and physics are governed by the representation theory of groups and algebras. The aim of combinatorial representation theory is to give combinatorial models for such representations and to obtain combinatorial formulas for interesting invariants. This both gives a way for calculating these invariants and leads to a better understanding of the representations. A detailed exposition of this area of mathematics can be found in the survey article [BR99] of Barcelo and Ram.

One interesting class of representations which is quite well understood and where there are good combinatorial models is the category of finite dimensional complex representations of a complex reductive algebraic group $G^\vee$. Here one is interested in weight multiplicities, tensor product decompositions and branching rules for the restriction to a Levi subgroup. Applying the character with respect to a maximal torus yields an isomorphism from the representation ring to the algebra of symmetric functions of the associated root datum. This isomorphism sends irreducible representations to Schur polynomials. So the above mentioned problems transform in giving formulas for the expansion of Schur polynomials with respect to monomial symmetric functions, calculating the structure constants with respect to Schur polynomials and describing their coefficients with respect to Schur polynomials of smaller rank.

For the special case of the general linear group the combinatorics of Young tableaus solves all these problems. All the mentioned entities are expressed as number of tableaus with certain additional properties. The path model of Littelmann [Lit94] is a model for general $G^\vee$. It replaces tableaus with piecewise linear paths in the dual of a Cartan subalgebra of the Lie algebra of $G^\vee$.

Many modern developments in combinatorics, representation theory and in the theory of reductive groups over local fields yield q-analogs of symmetric functions which specialize for certain values of q to the Schur polynomials. So it is a natural question to ask for generalizations of the combinatorial models to these q-analogs.

In [GL05] Gaussent and Littelmann introduce the gallery model as a tool for the geometric-combinatorial analysis of the affine Grassmanian associated to the Langlands dual group G of $G^\vee$. They show that it is a combinatorial model of the representations of $G^\vee$ equivalent to the path model. Moreover, they associate to these galleries explicitly given subsets of Mirković–Vilonen cycles, which by the work of Mirković and Vilonen [MV00, MV04] on the geometric Langlands duality are geometric models for the representations of $G^\vee$. As a byproduct of their work one gets a combinatorially defined polynomial for each gallery which reflects the geometric structure of the associated subset.

In this thesis we show that the gallery model together with these polynomials yields a combinatorial model for certain q-analogs of Schur polynomials, the Hall–Littlewood polynomials. To be more precise, we describe the expansion of Hall–Littlewood polynomials in terms of monomial symmetric functions, we calculate their structure constants and describe their coefficients with respect to Hall–Littlewood polynomials of smaller rank. The moral of this should be that the gallery model is a good model to calculate

q-analogs, not only the ones considered here in this thesis.

Specializing our results we get new proofs for the formulas involving Schur functions in [GL05]. In contrast to their approach (and the approach in [Lit94] based on paths) we do not use the combinatorics of root operators. But we show that their root operators are in some sense compatible with our approach. All of our arguments are based on calculations in affine Hecke algebras and certain specialization arguments and do not rely on results of [GL05]. It should be mentioned that our approach, despite of giving formulas for the Schur polynomials, does not work without introducing q-analogs. So in some sense working with q-analogs is easier than the classical case.

We use the Satake isomorphism to identify q-analogs of symmetric functions with the spherical Hecke algebra with equal parameters. Under this isomorphism, Hall–Littlewood polynomials correspond (up to some factor) to the Macdonald basis and the monomial symmetric functions correspond to the monomial basis of the spherical Hecke algebra. All assertions are then proven in the more general setting of spherical Hecke algebras with arbitrary parameters. We calculate the expansion of the Macdonald basis in terms of the monomial basis, the structure constants of the spherical Hecke algebra with respect to the Macdonald basis and their restriction coefficients. For doing this we introduce the alcove basis of the affine Hecke algebra and show its intimate relation to galleries.

In type A the expansion of (modified) two parameter Macdonald polynomials in terms of monomial symmetric functions for equal parameters was described by Haglund, Haiman and Loehr in [HHL05] using Young diagrams. Specializing their formula yields the expansion of Hall–Littlewood polynomials in this case.

It is well known that the Satake coefficients form a triangular matrix. With our combinatorial description of the Satake coefficients we can show that all remaining entries are in fact nonzero. This yields a new proof of a positivity result of Rapoport [Rap00] in the case of a spherical Hecke algebra of a reductive group over a local field since these geometrically defined spherical Hecke algebras arise as specializations of the combinatorially defined ones.

There are various other attempts to calculate the structure constants with respect to Hall–Littlewood polynomials (respectively to the Macdonald basis) when all parameters are specialized to a power of some prime number p. In type A, where up to normalization the Hall–Littlewood polynomials are the Hall polynomials known from the theory of p-groups, there exists an algorithm due to Macdonald [Mac95] calculating them using certain sequences of Young diagrams. But this algorithm is not very explicit. An improved version is given by Malley in [Mal96]. In [KM04] Kapovic and Millson proved the saturation conjecture. As a byproduct of their investigation [KM04, corollary 6.16] they prove a formula similar to ours (for equal parameters) for general type using folded geodesics in an affine building of G. As part of his thesis Parkinson [Par06] showed that for arbitrary parameters these structure constants can be interpreted as the number of certain intersections in a regular building. Using a geometric interpretation as the number of points in certain intersections in the affine Grassmanian of G Haines calculated the degree and the leading coefficients of the structure constants in [Hai03]. Using our results they can be expressed by a statistic very similar to the one used in [GL05]. This suggests that these intersections can be parameterized by galleries in the same way as in [GL05].

Calculating Demazure characters for $G^\vee$ was another challenge for combinatorial representation theory which was solved by the path model. Using our approach we can explicitly determine subsets of the galleries describing a Schur character which describe the corresponding Demazure characters and thus arrive at an explicit Demazure character formula involving galleries.

In [BD94] Billig and Dyer describe intersections of Iwahori and Iwasawa orbits in the affine flag variety of G. Their results can be formulated using galleries as in [GL05]. Using these results we show that the entries of the transition matrix from the alcove basis to the standard basis of the affine Hecke algebra (specialized at some prime power) can be described as the number of points of these intersections over a finite field. Using specialization arguments we give explicit formulas for the dimension of certain intersections in the complex affine Grassmanian of G and we show that Demazure multiplicities of $G^\vee$ are given by the number of top dimensional irreducible components of these intersections. This is a slight extension of the results in [GL05]. We get an indexing of these irreducible components by galleries counting Demazure multiplicities as in [GL05] for Kostka numbers. In contrast to [MV04] we have only a numerical coincidence and we do not prove (or conjecture) any deeper result explaining this. We compare our result with a similar result of Ion [Ion04, Ion05] obtained by specializing nonsymmetric two parameter Macdonald polynomials.

As a further result we get a commutation rule for the Bernstein representation of the affine Hecke algebra. This is a q-analog of a commutation formula of Pittie and Ram [PR99] in terms of galleries. As an application one recovers by specialization their Pieri–Chevalley rule in the equivariant K-theory of the generalized flag variety of $G^\vee$. This specialized formula is the same as the one obtained by Lenart and Postnikov [LP04] by different methods.

One of the most interesting q-analogs of weight multiplicities occurring in combinatorial representation theory are the Kostka–Foulkes polynomials. In our context they can be defined as entries of the transition matrix from Hall–Littlewood symmetric functions to Schur polynomials. But they have various other interpretations: For instance, they are special Kazhdan–Lusztig polynomials for the extended affine Weyl group and they encode the local intersection cohomology of the affine Grassmanian (see Lusztig's article [Lus83]). In particular, they have nonnegative coefficients. A combinatorial proof for type A of this nonnegativity was obtained by Lascoux and Schützenberger [LS78] using the *charge* function on tableaus. It is conjectured, that such a function exists for all types. We do not get such a function for the gallery model. But we calculate the expansion of certain sums of Schur polynomials with respect to Hall–Littlewood polynomials and show how it supports this conjecture.

Parts of the results of this thesis are available in the preprint [Sch05]. A more conceptual treatment of galleries and their relation to the affine Hecke algebra and q-analogs is given by Ram [Ram06]. The formulas for the structure constants and the restriction coefficients are there proven by introducing q-crystals.

This thesis is organized as follows: In section 1 we give a brief overview on symmetric functions, their q-analogs and the relation to representation theory. We give a precise definition of the coefficients we want to calculate and a first statement of some of our results. In section 2 we introduce the concept of generalized alcoves and show its relation to the extended affine Weyl group. In the following section several versions of affine Hecke algebras and their relation to Hall–Littlewood polynomials are discussed. Then we introduce galleries and various polynomials associated to them. This enables us to state the above mentioned formulas in the theorems 4.5, 4.10 and 4.13. We prove these theorems in the sections 5, 7 and 8. The commutation formula for the affine Hecke algebra and its specialization are proven in 6.1 being followed by the proof of the Demazure character formula. In section 9 we show that the root operators of [GL05] are compatible with our approach, at least after specialization. In the following two sections we show what happens when one regards the affine Hecke algebra as Hecke algebra of a reductive group over a local field. First we show that we do get these geometrically defined Hecke algebras and thus prove the above mentioned result of Rapoport in 10.2. Then we restrict to the case of split groups and give the geometric interpretations promised above. In the last section we give some first results relating the gallery model to the positivity of Kostka–Foulkes coefficients.

Acknowlegdements. I want to thank everyone who made this thesis possible. I am profoundly grateful to my advisor Prof. Dr. Peter Littelmann for his interest in my research and encouraging support. I want to express my gratitude to the Cusanuswerk for their (not only financial) support.

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1 Symmetric functions

In this section we introduce the algebra of symmetric functions associated to a root datum. We describe its relations to the representation theory of complex algebraic reductive groups and give a more precise meaning to our results.

Let $\Phi = (X, \phi, X^\vee, \phi^\vee)$ be a reduced root datum, i.e.

- X and $X^\vee$ are finitely generated free abelian groups with given subsets $\phi \subset X$ and $\phi^\vee \subset X^\vee$.
- We have a perfect pairing $\langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbb{Z}$.
- There is a bijection $\phi \to \phi^\vee, \alpha \mapsto \alpha^\vee$ such that for each $\alpha \in \phi$ we have $\langle \alpha, \alpha^\vee \rangle = 2$.
- For any $\alpha \in \phi$ the reflection $s_\alpha : X^\vee \to X^\vee, x \mapsto x - \langle \alpha, x \rangle \alpha^\vee$ leaves $\phi^\vee$ invariant.
- For any $\alpha^\vee \in \phi^\vee$ the reflection $s_{\alpha^\vee} : X \to X, x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ leaves ϕ invariant.
- If $\alpha \in \phi$ then the only other multiple of α in ϕ is $-\alpha$.

Let $V = X \otimes \mathbb{R}$ and $V^* \cong X^\vee \otimes \mathbb{R}$ such that the natural pairing $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{R}$ is induced by the pairing between X and $X^\vee$. Let $Q \subset X$ (respectively $Q^\vee \subset X^\vee$) be the subgroup generated by ϕ (respectively $\phi^\vee$). Then $(Q \otimes \mathbb{R}, \phi)$ and $(Q^\vee \otimes \mathbb{R}, \phi^\vee)$ are dual root systems in the sense of [Bou81]. For details on the combinatorics of root systems and Coxeter groups see also Humphreys' book [Hum90].

The Weyl group W of Φ is the subgroup of $GL(V^*)$ generated by the reflections s_α for $\alpha \in \phi$. Choose a set of simple roots Δ and denote by $\phi^+ \subset \phi$ the positive roots with respect to Δ. Denote by $S = \{s_\alpha \,|\, \alpha \in \Delta\} \subset W$ the set of simple reflections. Then (W, S) is a Coxeter system. Denote the corresponding length function by $l : W \to \mathbb{N}$.

A fundamental domain for the W-action on $X^\vee$ is given by the dominant cone

$$X^\vee_+ = \{x \in X^\vee \,|\, \langle \alpha, x \rangle \geq 0 \ \text{ for all } \alpha \in \phi^+\}.$$

The W-action on $X^\vee$ induces a natural action on the group algebra $\mathbb{Z}[X^\vee]$. For $\mu \in X^\vee$ denote by $x^\mu \in \mathbb{Z}[X^\vee]$ the corresponding basis element. The algebra of symmetric polynomials $\Lambda = \mathbb{Z}[X^\vee]^W$ is the algebra of invariants under this action. If the underlying root system of Φ is of type A_n, then Λ consists of symmetric Laurent polynomials in the usual sense.

Now let $\Delta_J \subset \Delta$ and denote by $\phi_J \subset \phi$ its span. Then $\Phi_J = (X, \phi_J, X^\vee, \phi^\vee_J)$ is a sub root datum of Φ. All entities with index J are the induced ones for Φ_J of the ones with the same name for Φ. We get an inclusion of algebras $\Lambda \hookrightarrow \Lambda_J$ and $W_J \subset W$ is a parabolic subgroup.

1.1 Classical situation

There are many interesting bases of Λ (as $\mathbb{Z}$-module) indexed by $X^\vee_+$. The simplest one is given by the monomial symmetric functions $\{m_\lambda\}$, where $m_\lambda = \sum_{\mu \in W\lambda} x^\mu$ is just the orbit sum. Another one is given by the Schur polynomials $\{s_\lambda\}$. They are defined as follows: Define $\mathcal{J} : \mathbb{Z}[X^\vee] \to \mathbb{Z}[X^\vee]$ by $\mathcal{J}(x^\mu) = \sum_{w\in W}(-1)^{l(w)}x^{w\mu}$. Then one has

$$s_\lambda = \mathcal{J}(x^{\lambda+\rho^\vee})/\mathcal{J}(x^{\rho^\vee})$$

where $\rho^\vee = \frac{1}{2}\sum_{\alpha\in\phi}\alpha^\vee$. Replacing W by W_J in the definitions yields monomial symmetric functions m^J_λ and Schur polynomials s^J_λ for all $\lambda \in {}^J\!X^\vee_+$ where

$${}^J\!X^\vee_+ = \{x \in X^\vee \mid \langle\alpha, x\rangle \geq 0 \text{ for all } \alpha \in \Delta_J\}.$$

Now define integers $k_{\lambda\mu}$ for $\lambda, \mu \in X^\vee_+$ (the Kostka numbers), $c^\nu_{\lambda\mu}$ for $\lambda, \mu, \nu \in X^\vee_+$ (the Littlewood–Richardson coefficients) and $b^J_{\lambda\mu}$ for $\lambda \in X^\vee_+$ and $\mu \in {}^J\!X^\vee_+$ (the branching coefficients) by

- $s_\lambda = \sum_{\mu\in X^\vee_+} k_{\lambda\mu}m_\mu$,
- $s_\lambda s_\mu = \sum_{\nu\in X^\vee_+} c^\nu_{\lambda\mu}s_\nu$ and
- $s_\lambda = \sum_{\mu\in {}^J\!X^\vee_+} b^J_{\lambda\mu}s^J_\mu$.

So the Kostka numbers are the entries of the transition matrix from monomial symmetric functions to Schur polynomials, the Littlewood–Richardson coefficients are the structure constants of Λ with respect to the Schur polynomials and the branching coefficients give the expansion of the s_λ with respect to Schur polynomials of the sub root datum Φ_J.

The relation to representation theory is as follows (see [Hum75]). Let $G^\vee$ be the unique complex reductive linear algebraic group with Borel subgroup $B^\vee$ and maximal torus $T^\vee$ such that the associated root datum together with the choice of simple roots is the dual of Φ. One is interested in the category of finite dimensional complex representations of $G^\vee$. It is well known that this category is semisimple and that the irreducible objects are given by highest weight modules $V(\lambda)$ with highest weight $\lambda \in X^\vee_+$. Assigning to such a representation its $T^\vee$-character yields an isomorphism from the Grothendieck ring of finite dimensional representations of $G^\vee$ to Λ. In the same way the whole algebra $\mathbb{Z}[X^\vee]$ is the representation ring of $T^\vee$. By Weyl's character formula the Schur polynomial s_λ for $\lambda \in X^\vee_+$ is the character of $V(\lambda)$.

The Kostka number $k_{\lambda\mu}$ for $\lambda, \mu \in X^\vee_+$ is the weight multiplicity of μ in $V(\lambda)$, i.e. the dimension of the μ-weight space $V(\lambda)_\mu$. The Littlewood–Richardson coefficient $c^\nu_{\lambda\mu}$ for $\lambda, \mu, \nu \in X^\vee_+$ is the multiplicity of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$, i.e. the dimension of $\mathrm{Hom}_{G^\vee}(V(\nu), V(\mu) \otimes V(\lambda))$.

Now let $P^\vee_J \subset G^\vee$ be the standard parabolic subgroup of type Δ_J and denote by $L^\vee_J$ its Levi part. Then $B^\vee \cap L^\vee_J$ is a Borel subgroup of $L^\vee_J$ and $T^\vee$ a maximal torus of $L^\vee_J$. The root datum associated to this choice is Φ_J with simple roots Δ_J. Now any representation of $G^\vee$ restricts to a representation of $L^\vee_J$. The branching coefficient $b^J_{\lambda\mu}$ for $\lambda \in X^\vee_+$ and $\mu \in {}^J\!X^\vee_+$ is the multiplicity (as a $L^\vee_J$ representation) of $V^J(\mu)$ in $V(\lambda)$, i.e. the dimension of $\mathrm{Hom}_{L^\vee_J}(V^J(\mu), V(\lambda))$.

Other numbers of representation theoretical interest are weight multiplicities of Demazure modules. Let $w \in W$ and denote by $V(\lambda)_{w\lambda} \subset V(\lambda)$ the one dimensional extremal weight space with weight $w\lambda$. Let $V_w(\lambda) \subset V(\lambda)$ be the $B^\vee$-module generated by $V(\lambda)_{w\lambda}$. It is called the Demazure module associated to λ and w. Again one is interested in its $T^\vee$-character which is in general only an element of $\mathbb{Z}[X^\vee]$ and not of Λ. Denote by $d^w_{\lambda\mu}$ for $\mu \in X^\vee$ the weight multiplicity of μ in $V_w(\lambda)$. Of course we have $V_{w_0}(\lambda) = V(\lambda)$ and thus $d^{w_0}_{\lambda\mu} = k_{\lambda\mu}$ for the longest element $w_0 \in W$. There is a formula (the Demazure character formula [Dem74]) describing this character. But it it is more convenient to introduce it in the context of the nil affine Hecke algebra so we postpone it to section 3.

As already mentioned in the introduction it was one of the main tasks of combinatorial representation theory to give combinatorial formulas for all these coefficients and it was solved in [Lit94].

1.2 q-analogs

Extending the base ring to $\mathcal{L}^- := \mathbb{Z}[q^{-1}]$ one gets new interesting bases. The Hall–Littlewood polynomials $\{P_\lambda(q^{-1})\}$ are a basis for $\Lambda_q := \mathcal{L}^-[X^\vee]^W$ (as $\mathcal{L}^-$-module). For $\lambda \in X^\vee_+$ they are defined by

$$P_\lambda(q^{-1}) = \frac{1}{W_\lambda(q^{-1})}\,\mathcal{J}\big(x^{\lambda+\rho^\vee}\prod_{\alpha\in\phi^+}(1-q^{-1}x^{-\alpha^\vee})\big)/\mathcal{J}(x^{\rho^\vee})$$

where $W_\lambda \subset W$ is the stabilizer of λ and $W_\lambda(q^{-1}) = \sum_{w\in W_\lambda} q^{-l(w)}$ is its Poincaré polynomial. From the definition it is not clear that $P_\lambda(q^{-1})$ is indeed an element of Λ_q. For this and other properties see the survey article [NR03] of Nelsen and Ram. But it is clear that the Hall–Littlewood polynomials are q-analogs of the Schur polynomials in the sense that $P_\lambda(0) = s_\lambda$. Moreover, one has $P_\lambda(1) = m_\lambda$. As above, we get the Hall–Littlewood polynomials $P^J_\lambda(q^{-1})$ of Φ_J by replacing W by W_J in all the definitions.

Having these q-analogs one asks the same questions as in the classical case, i.e. one looks for a combinatorial description of the transition matrix from the monomial symmetric functions to the Hall–Littlewood functions, for the structure constants of Λ_q with respect to them and for their branching coefficients. Specialization at $q^{-1} = 0$ then yields new proofs for the classical formula regarding the $k_{\lambda\mu}$, the $c^\nu_{\lambda\mu}$ and the $b^J_{\lambda\mu}$.

Define Laurent polynomials $L_{\lambda\mu}$ for $\lambda, \mu \in X^\vee_+$ by

$$P_\lambda(q^{-1}) = \sum_{\mu\in X^\vee_+} q^{-\langle\rho,\lambda+\mu\rangle} L_{\lambda\mu} m_\mu,$$

where $\rho := \frac{1}{2}\sum_{\alpha\in\phi^+}\alpha$. Since $P_\lambda(0) = s_\lambda$ we have $q^{-\langle\rho,\lambda+\mu\rangle}L_{\lambda\mu} \in \mathcal{L}^-$ and the constant term of $q^{-\langle\rho,\lambda+\mu\rangle}L_{\lambda\mu}$ is $k_{\lambda\mu}$. For non-dominant $\mu \in X^\vee$ we define $L_{\lambda\mu} = q^{\langle\rho,\mu-\mu^+\rangle}L_{\lambda\mu^+}$, where $\mu^+ \in X^\vee_+$ is the unique dominant element in the W-orbit of μ. In section 4 we introduce galleries and a monic polynomial L_σ for each positively folded gallery σ. We prove in section 5:

Theorem 1.1. *For $\lambda \in X^\vee_+$ and $\mu \in X^\vee$ we have $L_{\lambda\mu} = q^{-l(w_\lambda)}\sum_\sigma q^{l(w_0\iota(\sigma))}L_\sigma$, where the sum is over all positively folded galleries σ of type t^λ and weight μ with $\iota(\sigma) \in W^\lambda$.*

Remark 1.2. *For $\lambda, \mu \in X_+^\vee$ the exponent $\langle \rho, \lambda + \mu \rangle$ is not always in $\mathbb{Z}$. But it follows from the definition of galleries, that if $L_{\lambda\mu} \neq 0$ then $\lambda - \mu \in Q^\vee$. And under this hypothesis one has $\langle \rho, \lambda + \mu \rangle \in \mathbb{Z}$ since $2\rho \in Q$.*

From this we get a description of the $k_{\lambda\mu}$ in terms of galleries by evaluation at $q^{-1} = 0$. We introduce LS-galleries (roughly speaking these are the galleries which survive the specialization $q^{-1} = 0$) and get as in [GL05]

Corollary 1.3. *For $\lambda, \mu \in X_+^\vee$ the Kostka number $k_{\lambda\mu}$ is the number of LS-galleries of type t^λ and weight μ.*

In section 4 we also introduce a second monic polynomial C_σ for each gallery σ which is closely related to L_σ. We prove that with this statistic one can calculate the structure constants of Λ_q with respect to the Hall–Littlewood polynomials. More precisely, define $C_{\lambda\mu}^\nu$ for $\lambda, \mu, \nu \in X_+^\vee$ by

$$P_\lambda(q^{-1})P_\mu(q^{-1}) = \sum_{\nu \in X_+^\vee} q^{-\langle \rho, \mu - \lambda + \nu \rangle} C_{\lambda\mu}^\nu P_\nu(q^{-1}).$$

Theorem 1.4. *Let $\lambda, \mu, \nu \in X_+^\vee$. Then $C_{\lambda\mu}^\nu = q^{-l(w_\mu)} \sum_\sigma q^{l(w_0 \iota(\sigma))} C_\sigma W_{\mu\nu}^{\varepsilon(\sigma)}$. Here the sum is over all positively folded galleries of type t^μ and weight ν starting in λ such that they are contained in the dominant chamber and $\varepsilon(\sigma) \in W_\nu W^{w_0\mu}$. The correction factor $W_{\mu\nu}^{\varepsilon(\sigma)}$ is contained in $\mathcal{L}^-$.*

For $q^{-1} = 0$ this yields a Littlewood-Richardson rule in terms of galleries.

Corollary 1.5. *For λ, μ, ν in $X_+^\vee$ the Littlewood–Richardson coefficient $c_{\lambda\mu}^\nu$ is the number of LS-galleries σ of type t^μ and weight $\nu - \lambda$ such that the translated gallery $\lambda + \sigma$ is contained in the dominant chamber $\mathcal{C}$.*

Remark 1.6. *From [GL05] one would expect that the sum is over all galleries which are contained in the interior of the translated dominant chamber $-\rho^\vee + \mathcal{C}$. But the galleries leaving $\mathcal{C}$ are not LS. This is explained in remark 4.12.*

We also introduce monic polynomials C_σ^J for each gallery σ which specialize to the above C_σ for $\Delta_J = \Delta$. This statistic describes the branching coefficients of the Hall–Littlewood polynomials. Define $B_{\lambda\mu}^J$ for $\lambda \in X_J^\vee$ and $\mu \in {}^J X_+^\vee$ by

$$P_\lambda(q^{-1}) = \sum_{\mu \in {}^J X_+^\vee} q^{-\langle \rho, \lambda + \mu \rangle} B_{\lambda\mu}^J P_\mu^J(q^{-1}).$$

Theorem 1.7. *For $\lambda \in X_+^\vee$ and $\mu \in {}^J X_+^\vee$ we have $B_{\lambda\mu}^J = q^{-l(w_\lambda)} \sum_\sigma q^{w_0 \iota(\sigma)} C_\sigma^J \, {}^J W_{\lambda\mu}^{\varepsilon(\sigma)}$, where the sum is over all positively folded galleries of type t^λ and weight ν contained in the dominant chamber $\mathcal{C}^J$ with respect to Φ_J such that $\varepsilon(\sigma) \in W_\mu^J W^{w_0\lambda}$. Again the correction factor ${}^J W_{\lambda\mu}^{\varepsilon(\sigma)}$ is in $\mathcal{L}^-$.*

As above we can specialize at $q^{-1} = 0$ and get the following formula for the classical branching coefficients (where a remark similar to the one above applies).

Corollary 1.8. *For $\lambda \in X_+^\vee$ and $\mu \in {}^J X_+^\vee$ the branching coefficient $b_{\lambda\mu}^J$ is given as the number of all LS-galleries of type t^λ contained in $\mathcal{C}^J$.*

Remark 1.9. *Now one might ask why the $L_{\lambda\mu}$ are not called $K_{\lambda\mu}$. But the last symbol is in general reserved for the Kostka–Foulkes polynomials which describe the transition matrix from Hall–Littlewood polynomials to Schur polynomials. See section 12.*

The combinatorial descriptions in the corollaries 1.3, 1.5 and 1.8 are more or less the same as the above mentioned descriptions in [GL05]. Although the results on the Littlewood–Richardson coefficients and the branching coefficients are not stated explicitly there, they follow quite immediately from the description of the crystal operators.

By our approach one also gets q-analogs of the Demazure characters $d_{\lambda\mu}^w$ in 6.2 which one can describe using galleries. Specialization yields

Corollary 1.10. *Let $\lambda \in X_+^\vee$, $\mu \in X^\vee$ and $w \in W$. Then $d_{\lambda\mu}^w$ is the number of LS-galleries of type t^λ and weight μ such that $\iota(\sigma) \leq w$.*

The condition on the initial alcove seems not to reflect the fact that $V_v(\lambda) = V_{vw}(\lambda)$ for any $w \in W_\lambda$. But from the precise definition of LS-galleries and remark 5.10 it follows that the condition indeed depends only on wW_λ.

2 Affine Weyl group and alcoves

In this section we recall some facts on the (extended) affine Weyl group and on alcoves as in [Bou81, Hum90]. Furthermore, we introduce the notion of generalized alcoves.

The group $Q^\vee$ acts on V^* by translations. The affine Weyl group is defined as the semidirect product $W^\mathfrak{a} = W \ltimes Q^\vee$. It acts on V^* by affine transformations. For $\lambda \in Q^\vee$ denote by $\tau_\lambda \in W^\mathfrak{a}$ the associated translation. The affine Weyl group is generated by its affine reflections. Let $H^\mathfrak{a}$ be the union of all reflection hyperplanes of reflections in $W^\mathfrak{a}$. Then $H^\mathfrak{a} = \bigcup_{\alpha \in \phi^+, m \in \mathbb{Z}} H_{\alpha,m}$, where $H_{\alpha,m} = \{x \in V^* \mid \langle \alpha, x \rangle = m\}$. Let $H_{\alpha,m}^\pm = \{x \in V^* \mid \langle \alpha, x \rangle \gtrless m\}$ be the associated affine half spaces.

The connected components of $V^* \setminus H^\mathfrak{a}$ are called open alcoves. Their closures are the alcoves in V^*. Denote by $\mathcal{A}$ the set of all alcoves. The action of $W^\mathfrak{a}$ on $\mathcal{A}$ is free and transitive. The fundamental alcove $A_f = \{x \in V^* | 0 \leq \langle \alpha, x \rangle \leq 1 \text{ for all } \alpha \in \phi^+\} \in \mathcal{A}$ is a fundamental domain for the $W^\mathfrak{a}$-action on V^*. We get a bijection $W^\mathfrak{a} \to \mathcal{A}, w \mapsto A_w := wA_f$. One also writes $\lambda + A$ for the alcove $\tau_\lambda A$ where $\lambda \in Q^\vee$ and $A \in \mathcal{A}$.

A face F of an alcove A is an intersection $F = A \cap H$ such that $H \subset H^\mathfrak{a}$ is a reflection hyperplane and $\langle F \rangle_{\mathrm{aff}} = H$. Here $\langle F \rangle_{\mathrm{aff}}$ is the affine subspace spanned by F. A wall of A is some hyperplane $H \subset H^\mathfrak{a}$ such that $H \cap A$ is a face of A.

The walls of A_f are of particular interest. They are given as follows: For each $\alpha \in \Delta$ one has the wall $H_{\alpha,0} \cap A_f$. Let $\Theta \subset \Phi^+$ be the set of maximal elements with respect to the usual dominance ordering on $X^\vee$. By this we mean $\mu \leq \lambda$ iff $\lambda - \mu = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee$ for nonnegative integers n_α. So the number of elements in Θ equals the number of irreducible components of the Dynkin diagram of Φ. Then each intersection $H_{\theta,1} \cap A_f$

with $\theta \in \Theta$ is a wall of A_f. For $\theta \in \Theta$ let $s_{\theta,1}$ be the affine reflection at $H_{\theta,1}$. Then we have the following

Lemma 2.1. *The group $W^{\mathfrak{a}}$ is generated by the reflections $S^{\mathfrak{a}}$ at the walls of A_f. One has $S^{\mathfrak{a}} = S \cup \{s_{\theta,1} \,|\, \theta \in \Theta\}$. Moreover, $(W^{\mathfrak{a}}, S^{\mathfrak{a}})$ is a Coxeter system.*

Let F be a face of A_f. The type of F is the reflection at $\langle F\rangle_{\text{aff}}$. Extend this definition to all faces by demanding that the $W^{\mathfrak{a}}$-action preserves types.

Remark 2.2. *More generally, the hyperplanes $H_{\alpha,m}$ define the structure of a labelled chamber complex on $Q^\vee \otimes \mathbb{R}$ which is a realization of the Coxeter complex of $(W^{\mathfrak{a}}, S^{\mathfrak{a}})$.*

Right multiplication of $W^{\mathfrak{a}}$ induces an action of $W^{\mathfrak{a}}$ on $\mathcal{A}$ from the right. For $A \in \mathcal{A}$ and $s \in S^{\mathfrak{a}}$ the alcove As is the unique alcove not equal to A having a common face of type s with A. Let $F_s \subset A$ be the face of type s and $\langle F_s\rangle_{\text{aff}} = H_{\alpha,m}$ for some $\alpha \in \phi^+$ and $m \in \mathbb{Z}$. The hyperplane $H_{\alpha,m}$ is called the separating hyperplane between A and As. Call A negative with respect to s if A is contained in $H^-_{\alpha,m}$ and denote this by $A \prec As$. Of course A is called positive with respect to s if As is negative with respect to s. We have $A \prec As$ iff $\lambda + A \prec \lambda + As$ for all $\lambda \in Q^\vee$.

Example 2.3.
- *For A_w and A_{ws} in the dominant chamber $A_w \prec A_{ws}$ iff $w < ws$, where '$\leq$' is the usual Bruhat order on $W^{\mathfrak{a}}$.*
- *Let $w \in W$ and $s \in S$. Then $A_w \prec A_{ws}$ iff $w > ws$.*
- *Let $w \in W$ and $s = s_{\theta,1}$ with $\theta \in \Theta$. Then $A_w \prec A_w s$ iff $w\theta \in \phi^+$. The wall H in A_w belonging to the face of type s is $H_{w\theta,1}$ and $A_w \in H^-_{w\theta,1}$. This is the negative half space of H iff $w\theta \in \phi^+$ and this is equivalent to $A_w \prec A_w s$.*

There is also a natural action of $X^\vee$ on V^* by translations. So we can extend the above definition and get the extended affine Weyl group $\tilde{W}^{\mathfrak{a}} := W \ltimes X^\vee$. Extending the above notation write τ_μ for the translation by $\mu \in X^\vee$. The action of $\tilde{W}^{\mathfrak{a}}$ on $\mathcal{A}$ is no longer free and type preserving. The stabilizer Ω of A_f is isomorphic to $X^\vee/Q^\vee$. The isomorphism is given by sending $g \in \Omega$ to the class of $g(0)$. So a set of representatives is given by $X^\vee \cap A_f$. We have $\tilde{W}^{\mathfrak{a}} \cong \Omega \ltimes W^{\mathfrak{a}}$ and every element $v \in \tilde{W}^{\mathfrak{a}}$ can be written as $v = wg$ for unique $w \in W^{\mathfrak{a}}$ and $g \in \Omega$. Although $\tilde{W}^{\mathfrak{a}}$ is no longer a Coxeter group, we can extend the definition of the length function by setting $l(v) = l(w)$. So multiplication by elements of Ω does not change the length. This length function has many of the important properties of the length function of a Coxeter group. We have $l(vw) \leq l(v) + l(w)$ for any $v, w \in \tilde{W}^{\mathfrak{a}}$ and $l(ws) = l(w) \pm 1$ for any $w \in \tilde{W}^{\mathfrak{a}}$ and $s \in S^{\mathfrak{a}}$. We also can extend the Bruhat order on $\tilde{W}^{\mathfrak{a}}$ as follows: Let $v = wg$ and $v' = w'g' \in \tilde{W}^{\mathfrak{a}}$ such that $w, w' \in W^{\mathfrak{a}}$ and $g, g' \in \Omega$. Then define $v \leq v'$ iff $g = g'$ and $w \leq w'$ (in the usual Bruhat order on $W^{\mathfrak{a}}$).

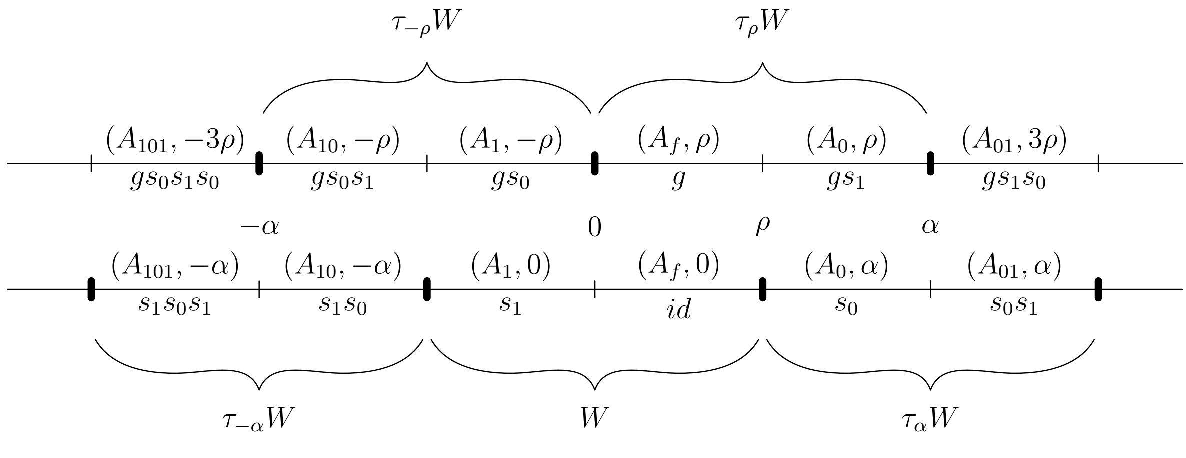
As mentioned above, the action of $\tilde{W}^{\mathfrak{a}}$ on $\mathcal{A}$ is no longer free. So we have to introduce the new notion of a generalized alcove in order to work with the extended affine Weyl group. This can be done as follows: Take an alcove $A \in \mathcal{A}$. Then some conjugate of Ω acts transitively on $A \cap X^\vee$ and this intersection is in natural bijection to $X^\vee/Q^\vee$. So if we define $\tilde{\mathcal{A}} = \{(A, \mu) \in \mathcal{A} \times X^\vee \,|\, \mu \in A\} \cong A \times \Omega$ then there is a natural free

$\tilde{W}^{\mathfrak{a}}$-action on $\tilde{\mathcal{A}}$. In the same way as above we also get a right $\tilde{W}^{\mathfrak{a}}$-action on $\tilde{\mathcal{A}}$ where Ω acts only on the second factor. The action of $W^{\mathfrak{a}}$ on $\tilde{\mathcal{A}}$ changes just the first factor but depends also on the second. The definitions of face and type of a face carry over to this situation by demanding that $\tilde{W}^{\mathfrak{a}}$ acts type preserving. The elements of $\tilde{\mathcal{A}}$ are called generalized alcoves. Every generalized alcove is of the form $\mu + A_w$ for unique $\mu \in X^\vee$ and $w \in W$. Then μ is called the weight of A and w its direction. Denote this by $wt(A) := \mu$ and $\delta(A) := w$. The alcoves $\mathcal{A}$ can then be identified with the generalized alcoves with weight in $Q^\vee$.

Here one has to be a aware of the fact that the type of a wall of a generalized alcove (A, μ) depends not only on the wall itself (as a subset of V^*), but also on the chosen μ. But the right multiplication of $S^{\mathfrak{a}}$ on $\tilde{\mathcal{A}}$ has the same geometrical interpretation in V^* as before and one has in general $wt(As) = wt(A)$ iff $s \in S$.

Example 2.4. *In order to give an idea how generalized alcoves look like we include a description of the rank one case. So let $V = \mathbb{R}^2$ with standard basis $\{e_1, e_2\}$ and dual basis $\{\epsilon_1, \epsilon_2\}$. Define $\alpha = e_1 - e_2 \in V$ and $\alpha^\vee = \epsilon_1 - \epsilon_2 \in V^*$. Define $d = \epsilon_1 + \epsilon_2 \in V^*$. Let $X = \mathbb{Z}\alpha \subset V$ and $X^\vee = \mathbb{Z}\alpha^\vee/2 \subset V^*/\mathbb{R}d$. Then $\Phi = (X, \{\pm\alpha\}, X^\vee, \{\pm\alpha^\vee\})$ is a root datum with pairing induced by the standard pairing between V and V^*. The corresponding algebraic group $G^\vee$ is $SL_2(\mathbb{C})$. As usual we identify V and V^* via the standard scalar product on V. So we can and will identify roots and coroots. The simple reflections are $S^{\mathfrak{a}} = \{s_1, s_0\}$ where $s_1 = s_\alpha \in S$ and $s_0 = s_{\alpha,1}$ is the additional affine reflection. Moreover, $\Omega = \{id, g\} \cong \mathbb{Z}/2$ where $g = s_{\alpha,1/2}$ is the affine reflection at the affine hyperplane $\alpha/4$ and $\tilde{W}^{\mathfrak{a}} = W^{\mathfrak{a}} \sqcup W^{\mathfrak{a}}g$. One has $gs_1g = s_0$. The extended affine Weyl group, the generalized alcoves and the types of their walls can be visualized as in the following picture.*

Any open unit interval in this picture is a generalized alcove. Above each generalized alcove we have its representation as usual alcove together with an element in $X^\vee$. We abbreviated the elements of $W^{\mathfrak{a}}$ in the indexing of the alcoves by a sequence of 0 and 1 corresponding to a reduced expression. Under each generalized alcove there is the corresponding element in $\tilde{W}^{\mathfrak{a}}$. The thick endpoints are the ones of type s_0. The bottom row consists of the alcoves respectively $W^{\mathfrak{a}}$. The top row consists of the coset $W^{\mathfrak{a}}g$. It represents the alcoves (A, μ) with $\mu \in \rho + Q^\vee$. We also indicated the right cosets $\tau_\lambda W$ for $\lambda \in X^\vee$.



One may replace Φ by the root datum $\tilde{\Phi} = (\frac{1}{2}X, \{\pm\alpha\}, 2X^\vee, \{\pm\alpha^\vee\})$. In this case the *extended affine Weyl group coincides with the affine Weyl group. The corresponding group is* $PGL_2(\mathbb{C})$. *There is one great difference which will be important later on: In* $W^{\mathfrak{a}}$ *the simple affine reflections* s_0 *and* s_1 *are not conjugate, whereas they are in the extended affine Weyl group of* Φ *as noted above.*

Replacing $SL_2(\mathbb{C})$ *by* $GL_2(\mathbb{C})$ *we get the root datum* $\Phi' = (\mathbb{Z}^2, \{\pm\alpha\}, \mathbb{Z}^2, \{\pm\alpha^\vee\})$ *where* α *and* $\alpha^\vee$ *are as above. The inclusion of* $SL_2(\mathbb{C})$ *in* $GL_2(\mathbb{C})$ *induces a morphism of root data* $\Phi \to \Phi'$ *which is the inclusion* $X \subset \mathbb{Z}^2$ *and the projection* $\mathbb{Z}^2 \to X^\vee$ *induced by* $V^* \to V^*/\mathbb{R}d$. *The picture now is as follows: Alcoves are of the form* $A \times \mathbb{R}d$ *for an alcove* A *of* Φ *and* Ω *is the free group generated by* $g = s_{\alpha,1}\tau_{e_1}$. *Observe that* $g^2 = \tau_d$ *and* $l(\tau_d) = 0$.

This is the general picture: Alcoves for the root datum of a reductive group $G^\vee$ *are always products of alcoves of its derived group by* $\mathbb{R}^k$ *where* k *is the rank difference between* $G^\vee$ *and its derived group.*

In various circumstances we will deal with stabilizer subgroups of W. We use the following notation for some notions related to them.

Definition 2.5. *Let* $\mu \in X^\vee$ *and* $W_\mu \subset W$ *its stabilizer. The maximal element of* W_μ *is denoted by* w_μ, *the minimal representatives of* W/W_μ *by* W^μ *and the minimal element in the coset* $\tau_\mu W$ *by* n^μ.

In particular, $W = W_0$ and w_0 is the longest element in W.

The dominant Weyl chamber is defined as $\mathcal{C} = \{x \in V^* \mid \langle \alpha, x\rangle \geq 0 \text{ for all } \alpha \in \phi^+\}$, the Φ_J-dominant chamber is given by $\mathcal{C}^J = \{x \in V^* \mid \langle \alpha, x\rangle \geq 0 \text{ for all } \alpha \in \phi_J^+\}$.

We will frequently use some facts about the length function on $\tilde{W}^{\mathfrak{a}}$ summarized in

Lemma 2.6. *Let* $\lambda \in X_+^\vee$.

(i) *We have* $l(\tau_\lambda) = 2\langle \rho, \lambda\rangle$. *In particular,* l *is additive on* $X_+^\vee$.

(ii) *One has* $\tau_\lambda w_\lambda = n^\lambda w_0$ *and* $l(\tau_\lambda) + l(w_\lambda) = l(n^\lambda) + l(w_0)$. *Moreover,* $n^\lambda \in W\tau_\lambda W$ *is minimal.*

Remark 2.7. *The inclusion of Coxeter systems* $(W_J, S_J) \subset (W, S)$ *induces an inclusion* $\tilde{W}_J^{\mathfrak{a}} \subset \tilde{W}^{\mathfrak{a}}$. *But the last inclusion is not an inclusion of Coxeter systems. The affine simple reflections* $S_J^{\mathfrak{a}}$ *of* Φ_J *are not necessarily contained in* $S^{\mathfrak{a}}$. *In particular, the length function on* $\tilde{W}^{\mathfrak{a}}$ *is not induced by the one of* $\tilde{W}^{\mathfrak{a}}$.

Remark 2.8. *At the beginning we started with a* reduced *root datum. Of course the construction makes sense also for any nonreduced root datum* $\Phi = (X, \phi, X^\vee, \phi^\vee)$, *i.e. there exists* $\alpha \in \phi$ *such that* $\frac{1}{2}\alpha \in \phi$. *As above one constructs the affine Weyl group and the extended affine Weyl group with explicitly given generators.*

But there exists a reduced *root datum leading to the same extended affine Weyl group as follows: Let* $\tilde{\phi}^\vee = \{\alpha^\vee \in \phi^\vee \mid \frac{1}{2}\alpha \notin \phi^\vee\}$ *be the set of indivisible coroots of* Φ. *Define* $\tilde{\phi} = \{\alpha \in \phi \mid \alpha^\vee \in \tilde{\phi}^\vee\} = \{\alpha \in \phi \mid 2\alpha \notin \phi\}$. *In particular, all the maximal roots of* ϕ *are contained in* $\tilde{\phi}$. *Then* $\tilde{\Phi} = (X, \tilde{\phi}, X^\vee, \tilde{\phi}^\vee)$ *is a reduced root datum (see [Bou81]) having* *the same Weyl group as Φ. Moreover, the Coxeter generators for Φ are also Coxeter generators for $\tilde{\Phi}$ and $Q^\vee(\Phi) = Q^\vee(\tilde{\Phi})$. One gets an isomorphism of Coxeter systems $(W^\mathfrak{a}(\Phi), S^\mathfrak{a}(\Phi)) \cong (W^\mathfrak{a}(\tilde{\Phi}), S^\mathfrak{a}(\tilde{\Phi}))$ and an isomorphism of the corresponding extended affine Weyl groups. So in order to study affine Hecke algebras of arbitrary root datums it is enough to consider reduced root datums.*

Be aware that this works only since we allowed root data and not only root systems in which case $X = Q$ and $X^\vee$ is the set of coweights. Here one gets more extended affine Weyl groups when one allows non reduced root systems. The choice of $X^\vee$ essentially gives more freedom in choosing the parameters of the affine Hecke algebra.

3 Hecke algebras

In this section we introduce the various Hecke algebras (extended affine Hecke algebra, spherical Hecke algebra and nil affine Hecke algebra) we want to work with. Details on affine Hecke algebras with unequal parameters can be found in Lusztig's article [Lus89]. For the spherical Hecke algebra (with equal parameters) and relations to Kazhdan–Lusztig polynomials see the survey article [NR03] of Nelsen and Ram where a slightly different notation is used. For the relation between the nil affine Hecke algebra and Demazure operators see [GR04] of Griffeth and Ram.

3.1 Affine Hecke algebra

We first have to fix parameters. Let $d : S^\mathfrak{a} \to \mathbb{N}$ be invariant under conjugation by elements of $\tilde{W}^\mathfrak{a}$. Let $\mathcal{L} := \mathbb{Z}\big[q^{\pm\frac{1}{2}}\big]$ and define $q_s = q^{d(s)}$ for $s \in S^\mathfrak{a}$. For $v \in W^\mathfrak{a}$ we set $q_v = \prod_{j=1}^k q_{s_{i_j}}$ where $v = s_{i_1} \cdot \ldots \cdot s_{i_k}$ is a reduced decomposition of v. For arbitrary $v \in \tilde{W}^\mathfrak{a}$ let $q_v = q_{v'}$ where $v = v'g$ with $v' \in W^\mathfrak{a}$ and $g \in \Omega$.

Often we will need some normalization factors. So define $H(q) = \sum_{w\in H} q_w$ and $H(q^{-1}) = \sum_{w\in H} q_w^{-1}$ for a subset $H \subset W$.

The standard representation of the extended affine Hecke algebra $\tilde{\mathcal{H}}^\mathfrak{a}$ associated to the root datum Φ and the above choice of d is as follows: As a $\mathcal{L}$-module it is free with basis $\{T_w\}_{w\in\tilde{W}^\mathfrak{a}}$ and multiplication is given by

- $T_s^2 = q_s T_{id} + (q_s - 1)T_s$ for all $s \in S^\mathfrak{a}$ and
- $T_v T_w = T_{vw}$ for all $v, w \in \tilde{W}^\mathfrak{a}$ such that $l(vw) = l(v) + l(w)$.

From this it follows immediately that for any $w \in \tilde{W}^\mathfrak{a}$ and $s \in S^\mathfrak{a}$ we have

$$T_w T_s = \begin{cases} T_{ws} & \text{if } l(ws) = l(w) + 1 \\ q_s T_{ws} + (q_s - 1)T_w & \text{if } l(ws) = l(w) - 1. \end{cases}$$

So $\tilde{\mathcal{H}}^\mathfrak{a}$ is a q-deformation of the group algebra of $\tilde{W}^\mathfrak{a}$ which we get by specializing at $q = 1$, i.e. taking the quotient by the ideal generated by $q - 1$.

Remark 3.1. *Usually Hecke algebras are only defined for Coxeter systems. But it is enough to have a set of generators together with a compatible length function as in the case of* $\tilde{W}^{\mathfrak{a}}$.

On $\tilde{\mathcal{H}}^{\mathfrak{a}}$ there is a natural $\mathbb{Z}$-algebra involution $\bar{\cdot} : \tilde{\mathcal{H}}^{\mathfrak{a}} \to \tilde{\mathcal{H}}^{\mathfrak{a}}$. It is given by $\overline{T}_w = T_{w^{-1}}^{-1}$ for $w \in \tilde{W}^{\mathfrak{a}}$ and $\overline{q^j} = q^{-j}$. Later on we will use the following formulas:

- For $s \in S^{\mathfrak{a}}$ we have $\overline{T}_s = q_s^{-1}(T_s + (1 - q_s)T_{id})$.
- For $s \in S^{\mathfrak{a}}$ and $w \in \tilde{W}^{\mathfrak{a}}$ we have

$$\overline{T}_w T_s = \begin{cases} q_s \overline{T}_{ws} + (q_s - 1)\overline{T}_w & \text{if } l(ws) = l(w) + 1 \\ \overline{T}_{ws} & \text{if } l(ws) = l(w) - 1. \end{cases}$$

For $\lambda \in X_+^\vee$ define $q_\lambda = q^{\frac{1}{2}\sum_{j=1}^k d(s_{i_j})}$ where $\tau_\lambda = s_{i_1} \cdot \ldots \cdot s_{i_k} g$ is a reduced decomposition with $g \in \Omega$. So we have $q_\lambda^2 = q_{\tau_\lambda}$. For arbitrary $\mu \in X^\vee$ define $q_\mu := q_\lambda q_{\lambda'}^{-1}$ where $\lambda, \lambda' \in X_+^\vee$ such that $\mu = \lambda - \lambda'$. Clearly q_μ is independent of the particular choice of λ, λ' because of the additivity of the length function on $X_+^\vee$ (see lemma 2.6).

There is a second presentation of $\tilde{\mathcal{H}}^{\mathfrak{a}}$ due to Bernstein which is closer to the definition of $\tilde{W}^{\mathfrak{a}}$ as a semi-direct product and also yields a large commutative subalgebra. For each $\mu \in X^\vee$ define an element $X_\mu \in \tilde{\mathcal{H}}^{\mathfrak{a}}$ by $X_\mu := q_\mu^{-1} T_{\tau_\lambda} T_{\tau_{\lambda'}}^{-1}$ where as above $\mu = \lambda - \lambda'$ with $\lambda, \lambda' \in X_+^\vee$. So for dominant λ we have $X_\lambda = q_\lambda^{-1} T_{\tau_\lambda}$. By the same reason as above X_μ does not depend on the choice of λ and λ' and we have $X_\lambda X_\mu = X_{\lambda+\mu}$ for λ, μ in $X^\vee$. Using this, $\tilde{\mathcal{H}}^{\mathfrak{a}}$ is generated (as a $\mathcal{L}$-algebra) by $\{X_\mu\}$ for $\mu \in X^\vee$ together with $\{T_w\}$ for $w \in W$. There are formulas relating the two presentations (see [Lus89, proposition 3.6] for details). But they are quite technical to state in the case of unequal parameters, for the case of equal parameters see below. However, they can be proven by our approach with galleries, see 6.1.

Using the commutativity of the X_μ one gets an inclusion of $\mathcal{L}$-algebras

$$\begin{aligned} \mathcal{L}[X^\vee] &\hookrightarrow \tilde{\mathcal{H}}^{\mathfrak{a}}. \\ x^\nu &\mapsto X_\nu \end{aligned}$$

We identify $\mathcal{L}[X^\vee]$ with its image. The image of $\mathcal{L}[X^\vee]^W$ under this inclusion is the center of $\tilde{\mathcal{H}}^{\mathfrak{a}}$ (see [Lus89, proposition 3.1]).

3.2 Spherical Hecke algebra

In $\tilde{\mathcal{H}}^{\mathfrak{a}}$ one has the symmetrizer $\mathbf{1}_0 = \sum_{w \in W} T_w$. It has the following properties:

- For $w \in W$ we have $T_w \mathbf{1}_0 = q_w \mathbf{1}_0$ and $\mathbf{1}_0^2 = W(q)\mathbf{1}_0$.
- $\overline{\mathbf{1}_0} = q_{w_0}^{-1} \mathbf{1}_0$.

The spherical Hecke algebra $\mathcal{H}^{sph}$ is defined by

$$\mathcal{H}^{sph} = \{h \in \frac{1}{W(q)}\tilde{\mathcal{H}}^{\mathfrak{a}} \mid T_w h = h T_w = q_w h \text{ for all } w \in W\}.$$

The Macdonald basis of $\mathcal{H}^{sph}$ is given by $\{M_\lambda\}_{\lambda \in X_+^\vee}$ where

$$\begin{aligned} M_\lambda &:= \frac{1}{W(q)} \sum_{w \in W\tau_\lambda W} T_w = \frac{1}{W(q)W_\lambda(q)} \mathbf{1}_0 T_{n^\lambda} \mathbf{1}_0 \\ &= \frac{q_\lambda q_{w_0}^{-1}}{W(q)W_\lambda(q^{-1})} \mathbf{1}_0 X_\lambda \mathbf{1}_0. \end{aligned}$$

For the second equality observe that by lemma 2.6 we have $X_\lambda = q_{-\lambda} T_{n^\lambda} T_{w_0} \overline{T}_{w_\lambda}$ and $W_\lambda(q^{-1}) = q_{w_\lambda}^{-1} W_\lambda(q)$. One obtains an isomorphism

$$\begin{aligned} \mathcal{L}[X^\vee] &\xrightarrow{\cong} \frac{1}{W(q)} \mathcal{H}\mathbf{1}_0 \\ x &\mapsto \frac{1}{W(q)} x \mathbf{1}_0. \end{aligned}$$

The restriction of this morphism to $\mathcal{L}[X^\vee]^W$ yields an isomorphism to $\mathcal{H}^{sph}$, the combinatorial Satake isomorphism. In particular, $\mathcal{H}^{sph}$ is commutative. For $\lambda \in X_+^\vee$ define Y_λ to be the image of m_λ under this isomorphism.

So we have two bases for $\mathcal{H}^{sph}$: The Macdonald basis and the monomial basis $\{Y_\lambda\}$ given by the images of the monomial symmetric functions under the Satake isomorphism. We are interested in the transition matrix from the monomial basis to the Macdonald basis. (Re)define $L_{\lambda\mu}$ for $\lambda, \mu \in X_+^\vee$ as modified entries of this transition matrix. More precisely, we have

$$M_\lambda = \sum_{\mu \in X_+^\vee} q_{-\mu} L_{\lambda\mu} Y_\mu.$$

For arbitrary $\mu \in X^\vee$ and dominant $\lambda \in X_+^\vee$ we set $L_{\lambda\mu} = q_{\mu-\mu^+} L_{\lambda\mu^+}$ where as before μ^+ is the unique dominant element in the W-orbit of μ.

Of course one can also ask for the structure constants of the spherical Hecke algebra with respect to the Macdonald basis. For this, (re)define $C_{\lambda\mu}^\nu$ for $\lambda, \mu, \nu \in X_+^\vee$ as modified structure constants by

$$M_\lambda M_\mu = \sum_{\nu \in X_+^\vee} q_{\lambda-\nu}^2 C_{\lambda\mu}^\nu M_\nu.$$

As for symmetric functions one can also ask for the branching coefficients. For this one first has to define restricted versions M_λ^J for $\lambda \in {}^J X_+^\vee$ of the Macdonald basis. Define them by

$$M_\lambda^J := \frac{q_\lambda^J q_{w_J}^{-1}}{W(q)W_\lambda^J(q^{-1})} \mathbf{1}_J X_\lambda \mathbf{1}_0.$$

Here q_λ^J is the J-analog of q_λ defined starting with a reduced decomposition of τ_λ in $\tilde{W}_J^{\mathfrak{a}}$, $\mathbf{1}_J = \sum_{w \in W_J} T_w$ and $w_J \in W_J$ is the element of maximal length. By definition we have $M^J_\lambda \in \frac{1}{W(q)}\tilde{\mathcal{H}}^{\mathfrak{a}}\mathbf{1}_0$. Now (re)define $B^J_{\lambda\mu}$ for $\lambda \in X^\vee_+$ and $\mu \in {}^JX^\vee_+$ by

$$M_\lambda = \sum_{\mu \in {}^JX^\vee_+} q_{-\mu} q^J_{-\mu} B^J_{\lambda\mu} M^J_\mu.$$

In the next section we give a description of the coefficients $L_{\lambda\mu}$, $C^\nu_{\lambda\mu}$ and $B^J_{\lambda\mu}$ using galleries.

Remark 3.2. *Now we want to clarify the relations of this section to symmetric polynomials and their q-analogs. In particular, we describe the relation between the coefficients defined above and the ones with the same names in section 1.*

For this regard the case of equal parameters, i.e. $d(s) = 1$ *for all* $s \in S^{\mathfrak{a}}$*. In this case we have* $q_v = q^{l(v)}$ *for* $v \in \tilde{W}^{\mathfrak{a}}$ *and* $q_\mu = q^{\langle \rho, \mu \rangle}$ *for* $\mu \in X^\vee$*. It is known (see for example [NR03, theorem 2.9]) that the image of* $P_\lambda(q^{-1})$ *under the Satake isomorphism is* $q_{-\lambda}M_\lambda$*. This is Macdonald's formula. So comparing the definitions of the* $L_{\lambda\mu}$ *and* $C^\nu_{\lambda\mu}$ *in section 1 with the ones given here shows that the first ones are special cases of the latter ones. So the theorems stated there will follow from theorems 4.5 and 4.10 given in the next section.*

For the branching coefficients one has to be more careful. We can identify the Hecke algebra $\tilde{\mathcal{H}}^{\mathfrak{a}}_J$ *of* Φ_J *with the subalgebra of* $\tilde{\mathcal{H}}^{\mathfrak{a}}$ *generated by* T_w *for* $w \in W_J$ *and* X_λ *for* $\lambda \in X^\vee$*. The Satake morphisms of* $\tilde{\mathcal{H}}^{\mathfrak{a}}$ *and* $\tilde{\mathcal{H}}^{\mathfrak{a}}_J$ *are compatible in the sense that the diagram*

$$\begin{tikzcd}
\mathcal{L}[X^\vee]^{W_J} \arrow[r, hook] & \mathcal{L}[X^\vee] \arrow[r, "\cong"] & \frac{1}{W_J(q)}\tilde{\mathcal{H}}^{\mathfrak{a}}_J\mathbf{1}_J \arrow[d] \\
\mathcal{L}[X^\vee]^{W} \arrow[r, hook] \arrow[u, hook] & \mathcal{L}[X^\vee] \arrow[r, "\cong"] \arrow[u, equal] & \frac{1}{W(q)}\tilde{\mathcal{H}}^{\mathfrak{a}}\mathbf{1}_0
\end{tikzcd}$$

commutes. Here the rightmost arrow from top to bottom is given by sending $\frac{1}{W_J(q)}X_\mu\mathbf{1}_J$ *to* $\frac{1}{W(q)}X_\mu\mathbf{1}_0$*, i.e by right multiplication with* $\frac{W_J(q)}{W(q)}\sum_{w\in W^J} T_{w^{-1}}$ *where* W^J *is the set of minimal representatives of* W/W_J*. So the image of* P^J_λ *in* $\frac{1}{W(q)}\tilde{\mathcal{H}}^{\mathfrak{a}}\mathbf{1}_0$ *is given by* $q^J_{-\lambda}M^J_\lambda$ *and the* M^J_λ *are the images of the Macdonald basis of* Φ_J*. In particular, the branching coefficients* $B^J_{\lambda\mu}$ *of this section coincide with the ones of section 1.*

Remark 3.3. *This is not the most general choice of parameters the affine Hecke algebra is defined for and where theorems 4.5 and 4.10 are true. One important example is the following: Replace* $\mathcal{L}$ *by the image of the morphism* $\mathcal{L} \to \mathbb{C}$ *evaluating the variable* q *at some fixed prime power. Hecke algebras of reductive groups over local fields are of this form (see sections 10 and 11 for more details on this).*

3.3 Nil affine Hecke algebra

In this section we introduce the nil affine Hecke algebra and its relations to the representation theory of $G^\vee$ and to the extended affine Hecke algebra.

For doing this we start with the affine Hecke algebra with equal parameters. Let $\alpha \in \Delta$ and $s = s_\alpha$. Then the commutation formula between the T_s and the X_λ is well known. We have

$$T_s X_\lambda = X_{s\lambda} T_s + (q-1)\frac{X_\lambda - X_{s\lambda}}{1 - X_{-\alpha^\vee}}. \tag{3.1}$$

Remark 3.4. *The quotient* $\frac{X_\lambda - X_{s\lambda}}{1-X_{-\alpha^\vee}}$ *is in* $\mathbb{Z}[X^\vee]$. *This can be seen easily by looking at the geometric series* $\frac{1}{1-X_{-\alpha^\vee}} = \sum_{k=0}^{\infty} X_{-k\alpha^\vee}$. *Setting* $k = \langle \alpha, \lambda \rangle$ *one gets the explicit relations*

- *If* $k \geq 0$ *then* $T_s X_\lambda = X_{s\lambda} T_s + (q-1)\sum_{j=0}^{k-1} X_{\lambda - j\alpha^\vee}$.
- *If* $k < 0$ *then* $T_s X_\lambda = X_{s\lambda} T_s - (q-1)\sum_{j=0}^{k-1} X_{s\lambda - j\alpha^\vee}$.

Now we change the generators to obtain the nil affine Hecke algebra. For doing this let $\tilde{T}_w = q_w^{-1} T_w$ for $w \in W$. Then we get the relations

$$\tilde{T}_s^2 = q^{-1}\tilde{T}_{\mathrm{id}} + (1-q^{-1})\tilde{T}_s$$

for $s \in S$ and

$$\tilde{T}_s X_\lambda = X_{s\lambda}\tilde{T}_s + (1-q^{-1})\frac{X_\lambda - X_{s\lambda}}{1 - X_{-\alpha^\vee}}$$

for $s \in S$ and $\lambda \in X^\vee$. These relations involve only negative powers of q.

So we can define $\tilde{\mathcal{H}}^{\mathfrak{a}}_{-} \subset \tilde{\mathcal{H}}^{\mathfrak{a}}$ to be the $\mathcal{L}^- = \mathbb{Z}[q^{-1}]$-module with basis $X_\mu \tilde{T}_w$ for $\mu \in X^\vee$ and $w \in W$. By the above relations it is a $\mathcal{L}^-$-algebra and $\tilde{T}_w X_\mu$ for $\mu \in X^\vee$ and $w \in W$ is also a $\mathcal{L}^-$-basis. The nil affine Hecke algebra $\mathcal{H}^{\mathrm{nil}}$ is the specialization of $\tilde{\mathcal{H}}^{\mathfrak{a}}_{-}$ at $q^{-1} = 0$, i.e. the quotient of $\tilde{\mathcal{H}}^{\mathfrak{a}}_{-}$ by the principal ideal generated by q^{-1}. Thus in $\mathcal{H}^{\mathrm{nil}}$ we get the relations

- $\tilde{T}_s^2 = \tilde{T}_s$ for $s \in S$ and
- $\tilde{T}_s X_\lambda = X_{s\lambda}\tilde{T}_s + \frac{X_\lambda - X_{s\lambda}}{1-X_{-\alpha^\vee}}$.

As in the case of the affine Hecke algebra we get an inclusion of algebras $\mathbb{Z}[X^\vee] \hookrightarrow \mathcal{H}^{\mathrm{nil}}$, $x^\mu \mapsto X_\mu$. This inclusion commutes with the various projections, i.e the diagram

$$\begin{tikzcd}
\mathcal{L}^-[X^\vee] \arrow[r, hook] \arrow[d, "\pi^-"] \arrow[dr] & \mathcal{L}[X^\vee] \arrow[dr] & \\
\mathbb{Z}[X^\vee] \arrow[dr] & \tilde{\mathcal{H}}^{\mathfrak{a}}_{-} \arrow[r, hook] \arrow[d, "\pi^-"] & \tilde{\mathcal{H}}^{\mathfrak{a}} \\
& \mathcal{H}^{\mathrm{nil}} &
\end{tikzcd}$$

is commutative. Here π^- denotes both the projection $\mathcal{L}^- \to \mathbb{Z}$ and $\tilde{\mathcal{H}}^{\mathfrak{a}}_{-} \to \mathcal{H}^{\mathrm{nil}}$.

Similar to the morphism above we get an isomorphism

$$\begin{aligned} \mathbb{Z}[X^\vee] &\to \mathcal{H}^{\mathrm{nil}}\tilde{T}_{w_0} \\ x &\mapsto x\tilde{T}_{w_0}. \end{aligned}$$

whose restriction to Λ yields an isomorphism to $\tilde{T}_{w_0}\mathcal{H}^{\mathrm{nil}}\tilde{T}_{w_0}$.

So we get two diagrams where the vertical arrows are the respective Satake isomorphisms.

$$\begin{tikzcd} \Delta \arrow[r, hook] \arrow[d, "\cong"] & \mathbb{Z}[X^\vee] \arrow[d, "\cong"] \\ \tilde{T}_{w_0}\mathcal{H}^{\mathrm{nil}}\tilde{T}_{w_0} \arrow[r, hook] & \mathcal{H}^{\mathrm{nil}}\tilde{T}_{w_0} \end{tikzcd} \qquad \begin{tikzcd} \Delta_q \arrow[r, hook] \arrow[d, "\cong"] & \mathcal{L}[X^\vee] \arrow[d, "\cong"] \\ \mathcal{H}^{sph} \arrow[r, hook] & \tilde{\mathcal{H}}^{\mathfrak{a}}\mathbf{1}_0 \end{tikzcd}$$

These two diagrams commute in the sense of the above diagram thanks to the following observation which also explains the normalizing factor $\frac{1}{W(q)}$: Since $W(q) = q_{w_0}W(q^{-1})$ we get

$$\frac{1}{W(q)}\mathbf{1}_0 = \frac{q_{w_0}^{-1}}{W(q^{-1})}\sum_{w\in W} q_w\tilde{T}_w = \frac{1}{W(q^{-1})}\sum_{w\in W} q_{w_0w}^{-1}\tilde{T}_w.$$

Thus we have $\frac{1}{W(q)}\mathbf{1}_0 \in \tilde{\mathcal{H}}^{\mathfrak{a}}_-$ and its image in $\mathcal{H}^{\mathrm{nil}}$ is $\tilde{T}_{w_0}$.

Let $w \in W$ and $\lambda \in X_+^\vee$. As promised in section 1 we also give combinatorial formulas for the Demazure weight multiplicities $d^w_{\lambda\mu}$. For this we first identify these numbers with coefficients appearing in $\mathcal{H}^{\mathrm{nil}}$. We claim that

$$\tilde{T}_w X_\lambda \tilde{T}_{w_0} = \sum_{\mu\in X^\vee} d^w_{\lambda\mu} X_\mu \tilde{T}_{w_0}.$$

This can be seen as follows: Via the isomorphism $\mathbb{Z}[X^\vee] \to \mathcal{H}^{\mathrm{nil}}\tilde{T}_{w_0}$ left multiplication by $\tilde{T}_s$ for $s \in S$ induces operators on $\mathbb{Z}[X^\vee]$. By the explicit description above one sees that they are nothing else than the usual Demazure operators on $\mathbb{Z}[X^\vee]$ and the statement made is nothing else than the Demazure character formula. For calculating the $d^w_{\lambda\mu}$ in 6.2 we calculate the coefficient of $X_\mu\mathbf{1}_0$ in the expansion of $T_wX_\lambda\mathbf{1}_0$ and specialize at $q^{-1} = 0$.

Remark 3.5. *For $s = s_{\theta,1} \in S^{\mathfrak{a}}$ we have $T_s = T_{\tau_{\theta^\vee}}\overline{T}_{s_\theta} = q^{\langle\rho,\theta^\vee\rangle}X_{\theta^\vee}\overline{T}_{s_\theta}$. So $q^{-1}T_s \in \tilde{\mathcal{H}}^{\mathfrak{a}}_-$ iff $\langle\rho,\theta^\vee\rangle = 1$ i.e. $\theta \in \Delta$. But this is only the case if the irreducible component of Φ containing θ is of rank one.*

There is a second way to specialize at $q^{-1} = 0$. Therefore define $\tilde{T}_v = q^{-l(v)}T_v$ for any $v \in \tilde{W}^{\mathfrak{a}}$ (and not only for elements of W). Then one can define the $\mathcal{L}^-$-subalgebra of $\tilde{\mathcal{H}}^{\mathfrak{a}}$ generated by the $\tilde{T}_v$. This is in general not $\tilde{\mathcal{H}}^{\mathfrak{a}}_-$ since $\tilde{T}_s$ for $s \in S^{\mathfrak{a}}$ would always be in this subalgebra.

Bringing this together we can conclude: For $\mathcal{H}^{\mathrm{nil}}$ we do not have a good standard basis labelled by $\tilde{W}^{\mathfrak{a}}$ which behaves nicely with respect to multiplication by the basis elements labelled by $S^{\mathfrak{a}}$. The lack of such a basis is why our approach using galleries does not work directly in $\mathcal{H}^{\mathrm{nil}}$.

4 Galleries

In this section we introduce galleries and some polynomials associated to them. We then give a precise meaning to the theorems stated in the introduction in the general setting of the last section. The galleries used here are a slight generalization of the usual galleries in a Coxeter complex since we regard generalized alcoves instead of alcoves.

Definition 4.1. *Let $t = (t_1, \dots, t_k)$ with $t_i \in S^{\mathfrak{a}} \cup \Omega$. Let $s \in S^{\mathfrak{a}}$.*

- *A gallery σ of type t connecting generalized alcoves A and B is a sequence $(A = A_0, \dots, B = A_k)$ of generalized alcoves such that $A_{i+1} = A_i t_{i+1}$ if $t_{i+1} \in \Omega$ and $A_{i+1} \in \{A_i, A_i t_{i+1}\}$ if $t_{i+1} \in S^{\mathfrak{a}}$. In the case of $t_{i+1} \in S^{\mathfrak{a}}$ this means that A_i and A_{i+1} have a common face of type t_{i+1}.*
- *The initial direction $\iota(\sigma)$ is defined to be the direction $\delta(A_0)$ of the first generalized alcove. The weight $wt(\sigma)$ of σ is $wt(A_k)$, the ending $e(\sigma)$ is A_k and the final direction $\varepsilon(\sigma)$ is $\delta(A_k)$.*
- *The gallery σ has a positive s-direction at i if $t_{i+1} = s$, $A_{i+1} = A_i s$ and A_i is negative with respect to s, i.e. $A_i \prec A_{i+1}$. The separating hyperplane is the wall of A_i corresponding to the face of type s.*
- *The gallery σ is s-folded at i if $t_{i+1} = s$ and $A_{i+1} = A_i$. The folding hyperplane is the wall of A_i corresponding to the face of type s. The folding is positive if $A_i \succ A_i s$.*

We call σ positively folded, if all foldings occurring are positive. A gallery is said to be minimal if it is of minimal length among all galleries connecting the same generalized alcoves.

For the precise statement on the $L_{\lambda\mu}$, the $C^{\nu}_{\lambda\mu}$ and the $B^{J}_{\lambda\mu}$ we need some statistics on galleries.

Definition 4.2. *Let σ be a positively folded gallery of type t. For $s \in S^{\mathfrak{a}}$ define*

- *$m_s(\sigma)$ the number of positive s-directions.*
- *$n_s(\sigma)$ the number of positive s-folds.*
- *$r_s(\sigma)$ the number of positive s-folds such that the folding hyperplane is not a wall of the dominant chamber $\mathcal{C}$.*
- *$p_s(\sigma)$ the number of positive s-folds such that the folding hyperplane is a wall of $\mathcal{C}$.*
- *$r^J_s(\sigma)$ the number of positive s-folds such that the folding hyperplane is not a wall of the J-dominant chamber $\mathcal{C}^J$.*
- *$p^J_s(\sigma)$ the number of positive s-folds such that the folding hyperplane is a wall of $\mathcal{C}^J$.*

In particular, $r_s(\sigma) + p_s(\sigma) = n_s(\sigma)$*. Now we can define*

- $L_\sigma = \prod_{s\in S^{\mathfrak{a}}} q_s^{m_s(\sigma)}(q_s-1)^{n_s(\sigma)}$,
- $C_\sigma = \prod_{s\in S^{\mathfrak{a}}} q_s^{m_s(\sigma)+p_s(\sigma)}(q_s-1)^{r_s(\sigma)}$ *and*
- $C_\sigma^J = \prod_{s\in S^{\mathfrak{a}}} q_s^{m_s(\sigma)+p_s^J(\sigma)}(q_s-1)^{r_s^J(\sigma)}$.

The polynomials C_σ^J somehow interpolate between the L_σ and the C_σ: For $\Delta_J = \emptyset$ we have $C_\sigma^J = L_\sigma$ and for $\Delta_J = \Delta$ we have $C_\sigma^J = C_\sigma$. By definition we have (in the case of equal parameters) $\deg L_\sigma = \deg C_\sigma = \deg C_\sigma^J$.

Fix some type $t = (t_1, \dots, t_k)$. For $A \in \tilde{\mathcal{A}}$ and $\mu \in X^\vee$ let $\Gamma_t^+(A,\mu)$ be the set of all positively folded galleries of type t starting in A with weight μ. Further let $\Gamma_t^+(\mu) = \coprod_{w\in W} \Gamma_t^+(A_w,\mu)$ be the set of all positively folded galleries of weight μ starting in the origin and let Γ_t^+ be the set of all positively folded galleries starting in the origin. Define

$$L_t(\mu) := \sum_{\sigma\in\Gamma_t^+(\mu)} q_{w_0\iota(\sigma)} L_\sigma.$$

So there is an additional contribution measuring the distance from $-A_f$ to the initial alcove.

Remark 4.3. *There is an alternative way of defining* $L_t(\mu)$*: For any* $w \in W$ *choose a minimal gallery* σ_w *of type* t_w *which connects* $-A_f$ *and* A_w*. Then* σ_w *is a nonfolded gallery of length* $l(w_0 w) = l(w_0) - l(w)$ *and it has only positive directions. The positively folded galleries of type* $t'_w = (t_w, t)$ *beginning in* $-A_f$ *correspond to the positively folded galleries of type* t *starting in* A_w*. We get*

$$L_t(\mu) = \sum_{w\in W}\Big(\sum_{\sigma\in\Gamma^+_{t'_w}(-A_f,\mu)} L_\sigma\Big).$$

Remark 4.4. *Let* $w \in W^{\mathfrak{a}}$*. The choice of a minimal gallery* σ *connecting* A_f *and* A_w *is equivalent to the choice of a reduced expression for* w*. Let* $t = (t_1, \dots, t_k)$ *be the type of* σ*. Then we have the reduced expression* $w = t_1 \cdot \ldots \cdot t_k$*. In particular, the length of a minimal gallery connecting* A_f *and* A_w *is* $l(w)$*. This is no longer true if we allow* $w \in \tilde{W}^{\mathfrak{a}}$*: Take for example* $w = g \in \Omega$*. Then a minimal gallery from* A_f *to* A_g *has type* (g) *and thus is of length length 1 but* $l(w) = 0$*. But this example reflects the general behavior: The length of a minimal gallery connecting* A_f *and* A_w *is either* $l(w)$ *or* $l(w)+1$*. Moreover, since any* $w \in \tilde{W}^{\mathfrak{a}}$ *has a unique expression* $w = vg$ *with* $v \in W^{\mathfrak{a}}$ *and* $g \in \Omega$*, one can always arrange that at most the last entry of its type is in* Ω*.*

Now we can give the formula for the $L_{\lambda\mu}$. Let $\lambda \in X_+^\vee$ and n^λ be the element of minimal length in $\tau_\lambda W$. Denote by W^λ the minimal representatives of W/W_λ as introduced in 2.5. Let σ^λ be a minimal gallery connecting A_f and A_{n^λ} and denote its type by t^λ. Using the last definition we get polynomials $L_{t^\lambda}(\mu)$ for all $\mu \in X^\vee$. Up to some factor these are the $L_{\lambda\mu}$. More precisely we prove in section 5:

Theorem 4.5. *For $\mu \in X^\vee$ we have*

$$L_{\lambda\mu} = \frac{1}{W_\lambda(q)} L_{t^\lambda}(\mu).$$

Furthermore,

$$L_{\lambda\mu} = q_{w_\lambda}^{-1} \sum_{\substack{\sigma \in \Gamma_{t^\lambda}^+(\mu) \\ \iota(\sigma) \in W^\lambda}} q_{w_0 \iota(\sigma)} L_\sigma.$$

In particular the $L_{t^\lambda}(\mu)$ do not depend on the choice of the minimal gallery σ^λ and $L_{t^\lambda}(\mu) = q_{\mu - w\mu} L_{t^\lambda}(w\mu)$ for all $w \in W$.

Remark 4.6. *One of the surprising implications of the last theorem is the W-invariance of the $L_{t^\lambda}(\mu)$ up to some power of q. This is surprising because even the cardinality of the sets $\Gamma_{t^\lambda}^+(w\mu)$ depends on w.*

Now it is quite natural to ask when $\Gamma_{t^\lambda}^+(\mu) \neq \emptyset$. Although the definition of galleries is a combinatorial one, it seems hard to give a combinatorial proof for the existence (or non existence) of a gallery of given type and weight. Let σ be any gallery of type t^λ starting in 0, ending in A_v of weight μ. Since the folding hyperplanes are root hyperplanes we always have $\lambda - wt(A_v) \in Q^\vee$. Moreover, $v \leq \iota(\sigma) n^\lambda$ by definition of the Bruhat order on $\tilde{W}^{\mathfrak{a}}$. This implies $\mu^+ \leq \lambda$. This also follows from the well known fact that the transition matrix from the monomial basis to the Macdonald basis is triangular with respect to the dominance ordering on $X_+^\vee$.

The question of the existence of a gallery in $\Gamma_{t^\lambda}^+(\mu)$ does not depend on the choice of parameters d. So we can take $d = 1$ as in remark 3.2. Since P_λ and m_μ are contained in Λ_q we have $q^{-\langle \rho, \lambda + \mu \rangle} L_{\lambda\mu} \in \mathcal{L}^-$. Moreover, $q^{-l(w_\lambda)} W_\lambda(q) = W_\lambda(q^{-1}) \in \mathcal{L}^-$ and thus $q^{-\langle \rho, \lambda+\mu\rangle - l(w_\lambda)} L_{t^\lambda}(\mu) \in \mathcal{L}^-$. So we get the upper bound

$$\deg(L_\sigma) + l(w_0 \iota(\sigma)) \leq \langle \rho, \mu + \lambda \rangle + l(w_\lambda) \tag{4.1}$$

for all $\sigma \in \Gamma_{t^\lambda}^+(\mu)$. The galleries with maximal degree are of special interest. So define

Definition 4.7. *A gallery $\sigma \in \Gamma_{t^\lambda}^+$ is a LS-gallery if we have equality in the above equation, i.e.* $\deg(L_\sigma) + l(w_0 \iota(\sigma)) = \langle \rho, wt(\sigma) + \lambda \rangle + l(w_\lambda)$.

Since all L_σ are monic we get the following corollary by evaluating theorem 4.5 at $q^{-1} = 0$ which answers the above question, proves corollary 1.3 and sharpens the triangularity.

Corollary 4.8. *The number of LS-galleries in $\Gamma_{t^\lambda}^+(\mu)$ is $k_{\lambda\mu^+}$. In particular we have $\Gamma_{t^\lambda}^+(\mu) \neq \emptyset$ iff μ occurs as a weight in $V(\lambda)$, i.e. $\mu^+ \leq \lambda$. Moreover, we have (for arbitrary parameters) $L_{\lambda\mu} \neq 0$ iff $\mu \leq \lambda$.*

The assertion on the triangularity (for the case of spherical Hecke algebras of a reductive group over a local field) was shown by Rapoport [Rap00].

Remark 4.9. *For regular λ the definition of galleries coincides with the one given in [GL05]. Instead of using generalized alcoves they regard galleries of alcoves together with an initial and final weight in $X^\vee$ contained in the first respectively last alcove. This is equivalent to our definition since we can always arrange such that at most the last component of t^λ is in Ω (compare remark 4.4). For nonregular λ they regard degenerate alcoves. This is more or less the same as our choice of the initial direction. See also remark 5.10 and section 9 for a discussion of this choice.*

We now give the formula for the $C_{\lambda\mu}^\nu$ replacing L_σ with C_σ. So let $\lambda \in X_+^\vee$ and t be any type. Define $\Gamma_{t,\lambda}^d$ as the set of all positively folded galleries of type t starting in λ which are contained in the dominant chamber. Here we allow that folding hyperplanes are contained in the walls of $\mathcal{C}$. For $\nu \in X_+^\vee$ let $\Gamma_{t,\lambda}^d(\nu) \subset \Gamma_{t,\lambda}^d$ be the subset of galleries ending in ν. Define

$$C_{\lambda t}(\nu) = \sum_{\Gamma_{t,\lambda}^d(\nu)} q_{w_0\iota(\sigma)} C_\sigma.$$

Now let $\lambda, \mu \in X_+^\vee$ and let t^μ be the type of a minimal gallery connecting A_f and A_{n^μ} where $n^\mu \in \tau_\mu W$ is the minimal representative in $\tau_\mu W$. The above definition yields $C_{\lambda t^\mu}(\nu)$ for any $\nu \in X_+^\vee$. Define $W_{\mu\nu}^w := q_w \sum_{v \in W^{w_0\mu} \cap W_\nu w} q_v^{-1}$ for $\mu, \nu \in X_+^\vee$ and $w \in W$. In section 7 we prove:

Theorem 4.10. *For $\lambda, \mu, \nu \in X_+^\vee$ we have*

$$C_{\lambda\mu}^\nu = \frac{W_\nu(q^{-1})}{W_\mu(q)} C_{\lambda t^\mu}(\nu).$$

Furthermore,

$$C_{\lambda\mu}^\nu = q_{w_\mu}^{-1} \sum_{\sigma \in \Gamma_{t^\mu,\lambda}^d(\nu)} q_{w_0\iota(\sigma)} C_\sigma W_{\mu\nu}^{\varepsilon(\sigma)}.$$

In particular, the $C_{\lambda t^\mu}(\nu)$ do not depend on the choice of the minimal gallery.

So in contrast to theorem 4.5 we have a condition on the final direction since $W_{\mu\nu}^w = 0$ iff $w \notin W_\nu W^{w_0\mu}$.

It does not follow immediately from the theorem that the structure constants of $\mathcal{H}^{sph}$ are indeed polynomials in q. This is shown in theorem 6.4.

As above we can give an estimate for the degree of the $C_{\lambda t^\nu}(\nu)$ and prove corollary 1.5. From the last theorem we get $q^{-\langle\rho,\mu-\lambda+\nu\rangle - l(w_\mu)} C_{\lambda t^\mu}(\nu) \in \mathcal{L}^-$ and thus for any $\sigma \in \Gamma_{t^\mu,\lambda}^d(\nu)$ we have

$$\deg C_\sigma + l(w_0\iota(\sigma)) \leq \langle\rho, \mu - \lambda + \nu\rangle + l(w_\mu).$$

Since $\deg L_\sigma = \deg C_\sigma$ and translating a gallery by an element of $X^\vee$ does not change L_σ and the initial direction, corollary 1.5 is proven and we get

Corollary 4.11. *For $\lambda, \mu, \nu \in X_+^\vee$ we have $C_{\lambda\mu}^\nu \neq 0$ if $c_{\lambda\mu}^\nu \neq 0$.*

For equal parameters the last corollary was proven by Haines in [Hai03] by geometric arguments using the affine Grassmanian of the Langlands dual G of $G^\vee$ to calculate the degree and the leading coefficients of $C_{\lambda\mu}^\nu$.

Remark 4.12. *Looking at the results in [GL05] one would expect the following: $c^{\nu}_{\lambda\mu}$ is the number of LS-galleries σ in $\Gamma^+_{t^\mu}(\nu-\lambda)$ such that the translated gallery is contained in the interior of $-\rho^\vee+\mathcal{C}$ which means that the folding hyperplanes are not of the form $H_{\alpha,-1}$ for some $\alpha\in\Delta$. But a gallery σ leaving $\mathcal{C}$ with this property is not LS. In fact, one can apply one of the operators $\tilde{e}_\alpha$ from [GL05] to σ (compare section 9).*

In the same way we now proceed for the branching coefficients replacing C_σ with C^J_σ. So again let t be any type and let $\lambda\in X^\vee_+$. Let $\Gamma^J_t\subset\Gamma^+_t$ by the subset of galleries contained in $\mathcal{C}^J$ and for $\mu\in {}^JX^\vee_+$ define $\Gamma^J_t(\mu)\subset\Gamma^J_t$ to be the subset of galleries with weight μ. Define $B_t(\mu)=\sum_{\sigma\in\Gamma^J_t(\mu)}C^J_\sigma$. Define ${}^JW^w_{\lambda\mu}:=q_w\sum_{v\in W^{w_0\lambda}\cap W^J_\mu w}q_v^{-1}$ where $W^J_\mu\subset W_J$ is the stabilizer of μ in W_J. We prove in section 8 (where t^λ is as above)

Theorem 4.13. *For $\lambda\in X^\vee_+$ and $\mu\in {}^JX^\vee_+$ we have*

$$B^J_{\lambda\mu}=\frac{W^J_\mu(q^{-1})}{W_\lambda(q)}B_{t^\lambda}(\mu).$$

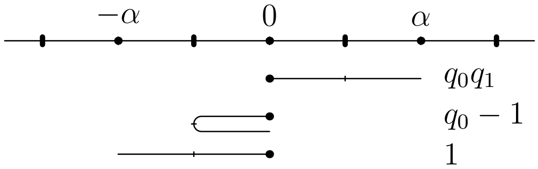
Furthermore,

$$B^J_{\lambda\mu}=q^{-1}_{w_\lambda}\sum_{\sigma\in\Gamma^J_{t^\lambda}(\mu)}q_{w_0\iota(\sigma)}C^J_\sigma\,{}^JW^{\varepsilon(\sigma)}_{\lambda\mu}$$

and the $B_{t^\lambda}(\mu)$ do not depend on the choice of t^λ.

Since $W_\lambda(q)=q_{w_\lambda}W_\lambda(q^{-1})$ one gets corollary 1.8 by specialization at $q^{-1}=0$.

Example 4.14. *In this example we want to illustrate theorem 4.5 in the case of not necessary equal parameters. So let $\tilde{\Phi}$ be of rank one as in 2.4 and thus $\tilde{W}^{\mathfrak{a}}=W^{\mathfrak{a}}$. As observed there, s_0 and s_1 are not conjugate in $W^{\mathfrak{a}}$. So we have two parameters $q_0:=q_{s_0}$ and $q_1:=q_{s_1}$. Let $\sigma=(A_f,A_{s_0})$ be the minimal gallery of type (s_0) from 0 to $\alpha^\vee$. Then $\Gamma^+_{(s_0)}$ and the corresponding $q_{w_0\iota(\sigma)}L_\sigma$ can be seen in the following picture.*



Since $q_{\alpha^\vee}=q_0^{\frac12}q_1^{\frac12}$ we get

$$q_{-\alpha^\vee}M_{\alpha^\vee}=X_{\alpha^\vee}\mathbf{1}_0+q_1^{-\frac12}(q_0^{\frac12}-q_0^{-\frac12})X_0\mathbf{1}_0+X_{-\alpha^\vee}\mathbf{1}_0$$

This shows that if one defines P_λ for arbitrary parameters as in the case of equal parameters, then the resulting symmetric polynomials are in general not in Λ_q. See Knop's article [Kno05, section 6] for a discussion of the relation between the choice of d and the existence and uniqueness of Kahzdan–Lusztig elements.

Example 4.15. *In the following we are in the case of equal parameters. Let $\lambda\in X^\vee_+$. Then $\Gamma^+_{t^\lambda}$ is more or less as in the last example. There are two nonfolded galleries of* *weight λ and $-\lambda$ and for any $-\lambda < \mu < \lambda$ we have a gallery $\sigma(\mu)$ starting in $-A_f$ of weight μ with one positive fold and $L_{\sigma(\mu)} = (q-1)q^{\langle \rho, \lambda+\mu\rangle -1}$. This yields*

$$q^{-\langle \rho,\lambda\rangle} M_\lambda = P_\lambda = m_\lambda + \sum_{\mu \in X_+^\vee, \mu < \lambda} (1-q^{-1}) m_\mu.$$

In this case all galleries are LS-galleries and we get the well known $s_\lambda = \sum_{\mu\in X_+^\vee, \mu \leq \lambda} m_\mu$.

5 Satake coefficients

In this section we introduce the alcove basis of the extended affine Hecke algebra and show that right multiplication of this alcove basis by elements of the standard basis can be calculated using positively folded galleries. From this theorem 4.5 follows. We also show that one can replace positively folded galleries by negatively folded galleries.

Definition 5.1. *Let $A \in \tilde{\mathcal{A}}$. Define $X_A = q_{-wt(A)} q_{\delta(A)} X_{wt(A)} \overline{T}_{\delta(A)}$.*

The set $\{X_A\}_{A\in\tilde{\mathcal{A}}}$ is a basis of $\tilde{\mathcal{H}}^{\mathfrak{a}}$. Before we proceed, we need some properties of this basis. First let $\lambda \in X^\vee$ and $A \in \tilde{\mathcal{A}}$. One calculates

$$X_\lambda X_A = q_\lambda X_{\lambda + A}. \tag{5.1}$$

Now assume $A = A_v$ to be dominant such that $\lambda := wt(A)$ is regular. Then $v = \tau_\lambda \delta(A)$. Moreover, τ_λ is of maximal length in $\tau_\lambda W$ by lemma 2.6 and $l(v) = l(\tau_\lambda) - l(\delta(A))$. So we get $T_{\tau_\lambda} \overline{T}_{\delta(A)} = T_{\tau_\lambda \delta(A)} = T_v$ and thus

$$X_A = q_{-\lambda} q_{\delta(A)} X_\lambda \overline{T}_{\delta(A)} = q_{\tau_\lambda}^{-1} q_{\delta(A)} T_{\tau_\lambda} \overline{T}_{\sigma(A)} = q_v^{-1} T_v. \tag{5.2}$$

Multiplying the elements of the alcove basis with T_s from the right can be expressed in terms of the alcove order. It is a q-analog of the $\tilde{W}^{\mathfrak{a}}$-action on $\tilde{\mathcal{A}}$.

Lemma 5.2. *Let $A \in \tilde{\mathcal{A}}$. In $\tilde{\mathcal{H}}^{\mathfrak{a}}$ we have*

$$X_A T_s = \begin{cases} q_s X_{As} & \text{if } A \prec As \\ X_{As} + (q_s - 1) X_A & \text{if } A \succ As. \end{cases}$$

Proof. By (5.1) the assertion is invariant under translation, i.e. under left multiplication with some X_μ. So it is enough to show the assertion for generalized alcoves $A = A_v$ such that $wt(A) - \alpha^\vee$ is dominant and regular for all $\alpha \in \phi$. By (5.2) we have $X_A = q_v^{-1} T_v$ and the multiplication law in $\tilde{\mathcal{H}}^{\mathfrak{a}}$ yields

$$T_v T_s = \begin{cases} T_{vs} & \text{if } l(v) < l(vs) \\ q_s T_{vs} + (q_s - 1) T_v & \text{if } l(v) > l(vs). \end{cases}$$

But for generalized alcoves in the dominant chamber increasing in the alcove order is equivalent to increasing the length of the corresponding elements of $\tilde{W}^{\mathfrak{a}}$ (see example 2.3). Moreover, by the choice of A we get $X_{As} = q_{vs}^{-1} T_{vs}$ as elements in $\tilde{\mathcal{H}}^{\mathfrak{a}}$ again by (5.2) and the assertion follows. □

Remark 5.3. *The last lemma can also be restated as follows. In [Lus80] Lusztig introduced the periodic Hecke module with equal parameters. It has a basis indexed by alcoves and the multiplication is given (up to some power of the parameters) by the relations of the last lemma. So what we really did was the following: We extended the periodic Hecke module to generalized alcoves and to arbitrary parameters and gave an explicit realization of an isomorphism (of right $\tilde{\mathcal{H}}^{\mathfrak{a}}$-modules) from the periodic Hecke module to $\tilde{\mathcal{H}}^{\mathfrak{a}}$.*

Using the same arguments and the fact that multiplying by T_g for $g \in \Omega$ does not change the length we get

Lemma 5.4. *For $A \in \tilde{\mathcal{A}}$ we have $X_A T_g = X_{Ag}$ as elements in $\tilde{\mathcal{H}}^{\mathfrak{a}}$.*

For later use we need the following: For $w \in \tilde{W}^{\mathfrak{a}}$ define $q_w^{\frac{1}{2}}$ by replacing q_s with $q_s^{\frac{1}{2}}$ in the definition of q_w. For $A \in \tilde{\mathcal{A}}$ define $q_A = q_{-wt(A)} q_{\delta(A)}^{\frac{1}{2}}$. Since $q_{\lambda+A} = q_{-\lambda} q_A$ for $\lambda \in X^\vee$ and $A \in \tilde{\mathcal{A}}$ and $q_{A_v}^2 = q_v^{-1}$ for A_v dominant with $wt(A_v)$ regular we get with the same arguments as in the proof of lemma 5.2

Lemma 5.5. *For $A \in \tilde{\mathcal{A}}$ and $s \in S^{\mathfrak{a}}$ one has*

$$q_{As} = \begin{cases} q_A q_s^{-\frac{1}{2}} & \text{if } A \prec As \\ q_A q_s^{\frac{1}{2}} & \text{if } A \succ As. \end{cases}$$

Now we can connect the multiplication in $\tilde{\mathcal{H}}^{\mathfrak{a}}$ to the L-polynomials. For generalized alcoves A and B and any type t define $\Gamma_t^+(A,B)$ to be the set of all positively folded galleries of type t connecting A and B and set $L_t(A,B) = \sum_{\sigma \in \Gamma_t^+(A,B)} L_\sigma$.

Lemma 5.6. *Let $t = (t_1, \ldots, t_k)$, $s \in S^{\mathfrak{a}}$, $t' = (t_1, \ldots, t_k, s)$, and fix generalized alcoves A and B. We have*

$$L_{t'}(A, Bs) = \begin{cases} L_t(A,B) & \text{if } B \succ Bs \\ q_s L_t(A,B) + (q_s - 1) L_t(A, Bs) & \text{if } B \prec Bs. \end{cases}$$

Proof. Let $\sigma' = (A, \ldots, C, Bs) \in \Gamma_{t'}^+(A, Bs)$. Then $C \in \{B, Bs\}$. We have $C = Bs$ iff σ' is s-folded at $k+1$. Let $\sigma = (A, \ldots, C)$ and distinguish two cases:
$B \succ Bs$: We then have $C = B$ and σ' is negative at $k+1$. So $\sigma \in \Gamma_t^+(A,B)$ and $L_{\sigma'} = L_\sigma$. Moreover, all galleries in $\Gamma_t^+(A,B)$ are obtained this way.
$B \prec Bs$: If $C = B$ we have $\sigma \in \Gamma_t^+(A,B)$ and σ' is positive at $k+1$. So $L_{\sigma'} = q_s L_\sigma$ and one gets all galleries in $\Gamma_t^+(A,B)$ this way. If $C = Bs$ we have $\sigma \in \Gamma_t^+(A,Bs)$, $L_{\sigma'} = (q_s - 1) L_\sigma$ and one obtains all galleries in $\Gamma_t^+(A,Bs)$ this way.
The lemma follows. □

Let $v \in \tilde{W}^{\mathfrak{a}}$ and σ be a minimal gallery of type t connecting A_f and A_v.

Theorem 5.7. *Given $A \in \tilde{\mathcal{A}}$ one has $X_A T_v = \sum_{B \in \tilde{\mathcal{A}}} L_t(A,B) X_B$.*

Proof. Because of lemma 5.4 and since the L-polynomials are not affected by elements of Ω in the type it is enough to show the theorem for $v \in W^{\mathfrak{a}}$. The proof is done by induction on $l(v)$.
Let first $v = s \in S^{\mathfrak{a}}$: Distinguish two cases.
$A \prec As$: In this case $L_{(s)}(A, A) = 0$, $L_{(s)}(A, As) = q_s$ and $L_{(s)}(A, B) = 0$ for all other B and $X_A T_s = q_s X_{As}$.
$A \succ As$: In this case $L_{(s)}(A, A) = q_s - 1$, $L_{(s)}(A, As) = 1$ and $L_{(s)}(A, B) = 0$ for all other B and $X_A T_s = X_{As} + (q_s - 1)X_A$.
Now let $v \in W^{\mathfrak{a}}$, $s \in S^{\mathfrak{a}}$ such that $l(v) < l(vs)$ and $\sigma' = (A_0, \ldots, A_v, A_{vs})$ is a minimal gallery of type t'. Using the last lemma we get

$$
\begin{aligned}
X_A T_{vs} = X_A T_v T_s &= \Big(\sum_{B \in W^{\mathfrak{a}}} L_t(A, B) X_B \Big) T_s \\
&= \sum_{B \prec Bs} q_s L_t(A, B) X_{Bs} + \sum_{B \succ Bs} L_t(A, B) X_{Bs} + \sum_{B \succ Bs} (q_s - 1) L_t(A, B) X_B \\
&= \sum_{B \prec Bs} \big(q_s L_t(A, B) + (q_s - 1) L_t(A, Bs) \big) X_{Bs} + \sum_{B \succ Bs} L_t(A, B) X_{Bs} \\
&= \sum_{B \in \tilde{\mathcal{A}}} L_{t'}(A, Bs) X_{Bs} = \sum_{B \in \tilde{\mathcal{A}}} L_{t'}(A, B) X_B
\end{aligned}
$$

□

In particular we get that $L_t(A, B)$ does not depend on σ and t but only on v. So we define $L_v(A, B) := L_t(A, B)$. For later use we also define Γ_v^+ to be the set of all positively folded galleries starting in the origin of the type of some minimal gallery joining A_f and A_v.

From this we get as a corollary (by setting $A = A_f$) the expansion of the standard basis in terms of the alcove basis.

Corollary 5.8. *Let $v \in \tilde{W}^{\mathfrak{a}}$ and fix some minimal gallery of type t connecting A_f and A_v. Then*

$$
T_v = \sum_{\sigma \in \Gamma_t^+, \iota(\sigma) = id} L_\sigma X_{e(\sigma)} = \sum_{\sigma \in \Gamma_t^+, \iota(\sigma) = id} q_{wt(\sigma)}^{-1} q_{\varepsilon(\sigma)} L_\sigma X_{wt(\sigma)} \overline{T}_{\varepsilon(\sigma)}.
$$

With these results we now can prove proposition 4.5.

Lemma 5.9. *For $\lambda \in X_+^\vee$ we have*

$$
\mathbf{1}_0 T_{n^\lambda} \mathbf{1}_0 = \sum_{\mu \in X^\vee} q_{-\mu} L_{t^\lambda}(\mu) X_\mu \mathbf{1}_0.
$$

Proof. We use the last theorem and the facts that $\overline{\mathbf{1}_0} = q_{w_0}^{-1} \mathbf{1}_0$ and $\overline{T}_w \mathbf{1}_0 = q_w^{-1} \mathbf{1}_0$ for all $w \in W$. So one calculates

$$\begin{aligned}
\mathbf{1}_0 T_{n^\lambda} \mathbf{1}_0 &= q_{w_0} \sum_{w \in W} \overline{T}_w T_{n^\lambda} \mathbf{1}_0 = q_{w_0} \sum_{w \in W} q_w^{-1} X_{A_w} T_{n^\lambda} \mathbf{1}_0 \\
&= q_{w_0} \sum_{w \in W} q_w^{-1} \sum_{\sigma \in \Gamma^+_{t^\lambda}, \iota(\sigma) = w} q_{-wt(\sigma)} q_{\varepsilon(\sigma)} L_\sigma X_{wt(\sigma)} \overline{T}_{\varepsilon(\sigma)} \mathbf{1}_0 \\
&= \sum_{w \in W} q_{w_0 w} \sum_{\sigma \in \Gamma^+_{t^\lambda}, \iota(\sigma) = w} q_{-wt(\sigma)} L_\sigma X_{wt(\sigma)} \mathbf{1}_0 \\
&= \sum_{\sigma \in \Gamma^+_{t^\lambda}} q_{w_0 \iota(\sigma)} q_{-wt(\sigma)} L_\sigma X_{wt(\sigma)} \mathbf{1}_0 = \sum_{\mu \in X^\vee} q_{-\mu} L_{t^\lambda}(\mu) X_\mu \mathbf{1}_0
\end{aligned}$$

where the last equality holds by the definition of $L_{t^\lambda}(\mu)$ in section 4. $\square$

From this we get

$$M_\lambda = \frac{1}{W(q) W_\lambda(q)} \sum_{\mu \in X^\vee} q_{-\mu} L_{t^\lambda}(\mu) X_\mu \mathbf{1}_0.$$

But on the other hand $q_{-\mu} L_{\lambda\mu}$ for dominant μ is the coefficient of M_λ with respect to Y_μ. Moreover, for arbitrary $\nu \in X^\vee$ we defined $L_{\lambda\nu} = q_{\nu - \nu^+} L_{\lambda\nu^+}$. So we get

$$\begin{aligned}
M_\lambda &= \sum_{\mu \in X^\vee_+} q_{-\mu} L_{\lambda\mu} Y_\mu = \frac{1}{W(q)} \sum_{\mu \in X^\vee_+} \Big(\sum_{\nu \in W\mu} q_{-\nu} L_{\lambda\nu} X_\nu \mathbf{1}_0 \Big) \\
&= \frac{1}{W(q)} \sum_{\mu \in X^\vee} q_{-\mu} L_{\lambda\mu} X_\mu \mathbf{1}_0.
\end{aligned}$$

Comparing coefficients of these two expansions we get

$$L_{\lambda\mu} = \frac{1}{W_\lambda(q)} L_{t^\lambda}(\mu)$$

which proves the first statement in 4.5. The second statement can be obtained as follows: Every $w \in W$ can be written as $w = w_1 w_2$ for unique $w_1 \in W^\lambda$ and $w_2 \in W_\lambda$ such that $l(w) = l(w_1) + l(w_2)$ (using the notation introduced in definition 2.5). Define $\mathbf{1}_\lambda = \sum_{w \in W_\lambda} T_w$. Since $\overline{T}_v \overline{T}_w = \overline{T}_{vw}$ for $v, w \in W$ with $l(v) + l(w) = l(vw)$ and $\overline{\mathbf{1}_\lambda} = q_{w_\lambda}^{-1} \mathbf{1}_\lambda$ we get

$$\mathbf{1}_0 = q_{w_0} \sum_{w \in W^\lambda} \overline{T}_w \overline{\mathbf{1}_\lambda} = q_{w_0 w_\lambda} \sum_{w \in W^\lambda} \overline{T}_w \mathbf{1}_\lambda.$$

If $v \in W_\lambda$ we have $l(v) + l(n^\lambda) = l(vn^\lambda)$. Moreover, $vn^\lambda = v\tau_\lambda w_\lambda w_0 = \tau_\lambda v w_\lambda w_0 = n^\lambda v'$ with $v' = w_0 w_\lambda v w_\lambda w_0$ by lemma 2.6. Then $l(v') = l(v)$ and $q_v = q_{v'}$. Thus $T_v T_{n^\lambda} \mathbf{1}_0 = T_{n^\lambda} T_{v'} \mathbf{1}_0 = q_v T_{n^\lambda} \mathbf{1}_0$ and we get

$$\mathbf{1}_0 T_{n^\lambda} \mathbf{1}_0 = q_{w_0 w_\lambda} W_\lambda(q) \sum_{w \in W^\lambda} \overline{T}_w \, T_{n^\lambda} \mathbf{1}_0.$$

Now the second statement follows the same way as in the proof of lemma 5.9 using $q_{w_0 w_\lambda} \sum_{w \in W^\lambda} \overline{T}_w = q_{w_\lambda}^{-1} \sum_{w \in W^\lambda} q_{w_0 w} X_{A_w}$.

Remark 5.10. *In the above considerations there are various other choices for the condition on the initial alcove. Let $v \in W_\lambda$. We have $\mathbf{1}_0 = q_{w_0 w_\lambda v} \sum_{w \in W^\lambda} \overline{T}_{wv} \mathbf{1}_\lambda$ since $\overline{T}_v \mathbf{1}_\lambda = q_v^{-1} \mathbf{1}_\lambda$ and the last equation becomes $\mathbf{1}_0 T_{n^\lambda} \mathbf{1}_0 = q_{w_0 w_\lambda v} W_\lambda(q) \sum_{w \in W^\lambda v} \overline{T}_w \, T_{n^\lambda} \mathbf{1}_0$. Thus one gets*

$$L_{\lambda\mu} = q_{w_\lambda v}^{-1} \sum_{\substack{\sigma \in \Gamma^+_{t^\lambda}(\mu) \\ \iota(\sigma) \in W^\lambda v}} q_{w_0 \iota(\sigma)} L_\sigma .$$

The case considered above was $v = id$. In the case of equal parameters we get for any gallery $\sigma \in \Gamma^+_{t^\lambda}$ such that $\iota(\sigma) \in W^\lambda v$ the upper bound

$$\deg L_\sigma + l(w_0 \iota(\sigma)) \leq \langle \rho, \lambda + wt(\sigma) \rangle + l(w_\lambda v).$$

One could define LS-galleries to be the ones such that $\iota(\sigma) \in W^\lambda v$ and where there is equality in the last equation. But only with the choice $v = id$ it is enough to impose this equality. The condition on the initial direction follows from this. In particular, for a LS-gallery σ we have $\iota(\sigma) \in W^\lambda$. See also section 9 for a connection of this choice to the crystal operators defined in [GL05].

For the definition of the $L_{t^\lambda}(\mu)$ we started with the minimal representative n^λ and we showed that $L_{t^\lambda}(\mu)$ is independent of the initially chosen minimal gallery. One can allow even more freedom in this initial choice. Let $v \in W \tau_\lambda W$ and let $w, w' \in W_\lambda$ such that $v = w n^\lambda w'$ and $l(w) + l(n^\lambda) + l(w') = l(v)$. If instead of t^λ we use the type t of a minimal gallery from A_f to A_v we get from the proof of 5.9 that $L_t(\mu) = q_w q_{w'} L_{t^\lambda}(\mu)$ for any $\mu \in X^\vee$. It is clear that the number of LS-galleries in $\Gamma^+_t(\mu)$ (with the appropriate changes of the degree condition in the definition) is the same as in $\Gamma^+_{t^\lambda}(\mu)$ since they always encode s_λ. One also has a canonical bijection between these different sets of LS-galleries. But the total number of galleries in $\Gamma^+_t(\mu)$ really depends on the choice of v and this number is minimal if we choose n^λ. There is another fact that singles out n^λ: All the nonfolded galleries are LS-galleries, which is false if we replace n^λ by v as can be seen in the following example.

Example 5.11. *As an example for this we regard the case of a rank one root datum Φ as in example 2.4 and continue the example 4.15. So let again $\lambda = \alpha$ and take $t = (s_0, s_1)$. Instead of the three galleries in the picture there we now have four galleries in Γ^+_t, two of them of weight $-\alpha$: The nonfolded gallery σ_0 starting in $-A_f$ and the gallery σ_1 starting in $-A_f$ and having a s_1-fold. Then $L_{\sigma_0} = 0$ and $L_{\sigma_1} = q - 1$, so σ_1 is LS and σ_0 is not despite of being nonfolded.*

This is also the smallest example where one can see that $|\Gamma^+_t(\mu)|$ is not W-invariant in contrast to $|LS_t(\mu)|$ and $|\Gamma_t(\mu)|$.

Remark 5.12. *In definition 4.1 one can replace positive (respectively positively folded) by negative (respectively negatively folded), i.e. one gets $m_s^-(\sigma)$ and $n_s^-(\sigma)$ for each negatively folded gallery σ. With the obvious changes this yields polynomials L_σ^- nonzero only for negatively folded galleries. Going further, one gets $\Gamma_t^-(A, B)$, $L_t^-(A, B)$ and recursions (using the same notations as in 5.6)*

$$L_{t'}^-(A, Bs) = \begin{cases} L_t^-(A, B) & \text{if } B \prec Bs \\ q_s L_t^-(A, B) + (q_s - 1) L_t^-(A, Bs) & \text{if } B \succ Bs. \end{cases}$$

Since $\overline{T}_s = q_s^{-1}(T_s + (1-q_s)T_{id})$ for $s \in S^{\mathfrak{a}}$ we get from lemma 5.2 that

$$X_A\overline{T}_s = \begin{cases} X_{As} + (q_s^{-1} - 1) & \text{if } A \prec As \\ q_s^{-1}X_{As} & \text{if } A \succ As \end{cases}$$

for any $A \in \tilde{\mathcal{A}}$ and $s \in S^{\mathfrak{a}}$. Under the hypotheses of theorem 5.7 we get

$$X_A\overline{T}_v = \sum_{B \in \mathcal{A}} \overline{L_t^-(A,B)} X_B.$$

If one defines

$$L_t^-(\mu) = \sum_{\sigma \in \Gamma_t^-(\mu)} q_{\iota(\sigma)} L_\sigma^-$$

we also can express the $L_{\lambda\mu}$ with negatively folded galleries. For this note that left multiplication by w_0 on $\tilde{\mathcal{A}}$ induces a type preserving bijection $\phi : \Gamma_t^+ \to \Gamma_t^-$ for any type t. Obviously we have $L_{\phi(\sigma)}^- = L_\sigma$ and $\iota(\phi(\sigma)) = w_0\iota(\sigma)$. In particular, we get the equality $L_t(\mu) = L_t^-(w_0\mu)$. Combining this with the semi-invariance of the $L_{\lambda\mu}$ with respect to μ we get

$$L_{\lambda\mu} = q_{\mu - w_0\mu} L_{\lambda, w_0\mu} = \frac{q_\mu^2}{W_\lambda(q)} L_{t^\lambda}(w_0\mu) = \frac{q_\mu^2}{W_\lambda(q)} L_{t^\lambda}^-(\mu)$$

which gives an expression of $L_{\lambda\mu}$ in terms of negatively folded galleries by the definition of $L_{t^\lambda}^-(\mu)$.

6 Commutation and Demazure character formula

In this section we prove a commutation rule for the affine Hecke algebra, i.e. a formula in terms of galleries for the coefficients appearing in $T_w X_\lambda = \sum R_{\lambda w}^{\mu v} X_\mu T_v$. Specializing this formula extends the Pieri–Chevalley formula of Pittie and Ram [PR99] using the path model to the non-dominant case. This specialization is equivalent to the formula of Lenart and Postnikov [LP04]. See the end of this section for the geometric significance of these coefficients. In the same way we calculate q-analogs of Demazure multiplicities and thus prove the Demazure character formula 1.10.

6.1 Commutation formula

Using corollary 5.8 we can express any element T_w of the standard basis in terms of the alcove basis. So we get a formula for $T_w X_\lambda = q_{-\lambda} T_w T_{\tau_\lambda}$ for dominant λ in terms of the alcove basis. But this method does not work for non-dominant λ since only for the dominant λ we have a good description of X_λ in terms of the standard generators T_s (which we need to apply 5.8). Using remark 5.12 we could derive a similar formula for antidominant λ using negatively folded galleries since there we have a description as a product of $\overline{T}_s$. In the general case one has to mix these two notions.

We first consider a slightly more general situation. Let $t = (t_1, \ldots, t_k)$ be a type and let $N \subset \{j \,|\, t_j \in S^{\mathfrak{a}}\}$ be any subset. Define $\varepsilon(j) = -1$ if $j \in N$ and $\varepsilon(j) = 1$ otherwise. We then define $T_{t,N} = T_{t_1}^{\varepsilon(1)} \cdot \ldots \cdot T_{t_k}^{\varepsilon(k)}$ and $q_{t,N} = q_{t_1}^{\varepsilon(1)} \cdot \ldots \cdot q_{t_k}^{\varepsilon(k)}$.

Let σ be a gallery of type t. Then σ is called N-folded if the following holds: If σ is positively (respectively negatively) folded at j then $j \notin N$ (respectively $j \in N$). So for $N = \emptyset$ we get positively folded galleries and for $N = \{j \,|\, j \in S^{\mathfrak{a}}\}$ we get negatively folded galleries. Define Γ_t^N to be the set of N-folded galleries starting in the origin. For a N-folded gallery σ and $s \in S^{\mathfrak{a}}$ we define

- $m_s^+(\sigma)$ the number of $j \notin N$ such that σ is s-positive at j.
- $m_s^-(\sigma)$ the number of $j \in N$ such that σ is s-negative at j.
- $n_s^+(\sigma)$ the number of $j \notin N$ such that σ has a positive s-fold at j.
- $n_s^-(\sigma)$ the number of $j \in N$ such that σ has a negative s-fold at j.

Of course all these entities depend on N, but in order to simplify notation we suppress this dependency. The corresponding N should be clear from the context. Now define $L_\sigma^N := \prod_{s \in S^{\mathfrak{a}}} q_s^{m_s^+(\sigma) - m_s^-(\sigma)} (q-1)^{n_s^+(\sigma)} (q^{-1}-1)^{n_s^-(\sigma)}$. In particular we have $L_\sigma^\emptyset = L_\sigma$.

Combining lemma 5.2, its negative counterpart in remark 5.12 and lemma 5.4 we get by induction on k the following

Lemma 6.1. *For $A \in \tilde{\mathcal{A}}$ we have*

$$X_A T_{t,N} = \sum_\sigma L_\sigma^N X_{e(\sigma)},$$

where the sum is over all N-folded galleries starting in A.

In particular we get the following: Let $A, B \in \tilde{\mathcal{A}}$ and σ a nonfolded gallery connecting A and B. Denote its type by t and define $N = \{j \,|\, \sigma \text{ is negative at } j\}$ to be its set of negative directions. Then the last lemma reads

$$X_A T_{t,N} = L_\sigma^N X_B = q_{t,N} X_B \tag{6.1}$$

since in this case no foldings are allowed and $L_\sigma^N = q_{t,N}$ by definition of L_σ^N. Moreover, by lemma 5.5 we know that

$$q_A q_{t,N}^{-\frac{1}{2}} = q_B \tag{6.2}$$

where $q_{t,N}^{\frac{1}{2}}$ is defined as $q_{t,N}$ replacing all q_s by $q_s^{\frac{1}{2}}$.

Now we want to apply this to compute $T_w X_\lambda$ for $\lambda \in X^\vee$ and $w \in W$. For doing this choose a gallery γ connecting A_{w_0} with $\lambda + A_{w_0}$. Denote by t its type and define N as above to be the set of negative directons of γ. By (6.1) we have $X_{A_{w_0}} T_{t,N} = q_{t,N} X_{\lambda + A_{w_0}}$ and $q_{t,N} = q_{A_{w_0}}^2 q_{\lambda + A_{w_0}}^{-2} = q_\lambda^2$ by (6.2). Again using the last lemma we calculate

$$T_w X_{\lambda + A_{w_0}} = q_{t,N}^{-1} T_w X_{A_{w_0}} T_{t,N} = q_{t,N}^{-1} q_w X_{A_{ww_0}} T_{t,N} = q_{t,N}^{-1} q_w \sum_{\sigma \in \Gamma_t^N, \iota(\sigma) = ww_0} L_\sigma^N X_{e(\sigma)}.$$

Moreover, $X_{\lambda+A_{w_0}}T_{w_0} = q_{w_0}X_{\lambda+A_f} = q_{w_0}q_{-\lambda}X_\lambda$. So multiplying from the right by T_{w_0} yields

$$q_{w_0}T_wX_\lambda = q_{-\lambda}q_w\sum_\sigma L_\sigma^N X_{e(\sigma)}T_{w_0} = q_{-\lambda}q_w\sum_\sigma q_{-wt(\sigma)}q_{\varepsilon(\sigma)}L_\sigma^N X_{wt(\sigma)}T_{\varepsilon(\sigma)w_0}$$

where the sum is over $\sigma \in \Gamma_t^N$ starting in A_{ww_0}.

Summarizing we get

Theorem 6.2. *Let $w \in W$ and $\lambda \in X^\vee$. Let t be the type of a non-folded gallery γ connecting A_{w_0} and $\lambda + A_{w_0}$ and denote by N the set of negative directions of γ. Then*

$$T_wX_\lambda = q_{w_0w}^{-1}\sum_{\mu\in X^\vee} q_{-\lambda-\mu}\sum_{wt(\sigma)=\mu} q_{\varepsilon(\sigma)}L_\sigma^N X_\mu T_{\varepsilon(\sigma)w_0}$$

where the sum is over all galleries $\sigma \in \Gamma_t^N$ starting in ww_0.

Example 6.3. *We include some examples which we will use subsequently. These results are well known (see [Lus89, proposition 3.6]). The usual proofs (at least for unequal parameters) use that $\tilde{\mathcal{H}}^{\mathfrak{a}}$ is a quotient of the group algebra of the affine braid group.*

Let $\alpha \in \Delta$ and $\lambda \in X^\vee$ such that $\langle\alpha, \lambda\rangle = 0$. Let $s = s_\alpha$. Then $s \in W_\lambda$. Let σ be a minimal gallery from A_{w_0} to $\lambda + A_{w_0}$ of type t and let N be as above. Then σ is completely contained in $H_{\alpha,0}^- \cap H_{\alpha,-1}^+$. In particular, no separating hyperplane of σ is of the form $H_{\alpha,j}$. Let γ be the nonfolded gallery of type t starting in sw_0, i.e. $\gamma = s\sigma$. So $wt(\gamma) = \lambda$ and $\varepsilon(\gamma) = sw_0$. Since s only changes directions where the separating hyperplane is of the form $H_{\alpha,j}$, σ is positive at j iff γ is positive at j. In particular, in γ no foldings are allowed and thus γ is the only gallery in Γ_t^N starting in sw_0. Moreover, we have $L_\gamma^N = L_\sigma^N = q_{t,N} = q_\lambda^2$ and so the last theorem yields $T_sX_\lambda = X_\lambda T_s$.

Assume now $\langle\alpha, \lambda\rangle = 1$. Then $s\lambda = \lambda - \alpha^\vee$. Define $s' = w_0sw_0 \in S$. So we have $q_s = q_{s'}$. Let σ' be a minimal gallery from $A_{w_0s'}$ to $\lambda + A_{w_0}$ and denote by t' its type. Then σ' is contained in $H_{\alpha,0}^+ \cap H_{\alpha,1}^-$. So again no separating hyperplane is of the form $H_{\alpha,j}$. Extend σ' to a minimal gallery from A_{w_0} to $\lambda + A_{w_0}$ by adding A_{w_0} at the beginning. Then the type t of σ is the concatenation of s' and t'. Denote by N the set of negative directions of σ. This time there are two galleries in Γ_t^N starting in sw_0: As above there is $\gamma = s\sigma$ with $wt(\gamma) = s\lambda$ and $\varepsilon(\sigma) = sw_0$. In contrast to the situation there γ has a negative s'-direction at the beginning whereas σ is positive there. All other directions remain as in σ and thus $L_\gamma^N = q_s^{-1}L_\sigma^N = q_s^{-1}q_\lambda^2$. Moreover, one can fold between the first two alcoves A_{sw_0} and A_{w_0} of γ and obtains γ' with $wt(\gamma') = \lambda$ and $\varepsilon(\gamma) = w_0$. All other directions in γ' are as in γ and so $L_{\gamma'}^N = (q_s-1)L_\gamma^N = (1-q_s^{-1})q_\lambda^2$. Using the last theorem we have $T_sX_\lambda = q_{\alpha^\vee}q_s^{-1}X_{s\lambda}T_s + (q_s-1)X_\lambda$. In this case we have $q_{\alpha^\vee} = q_s$ which can be seen as follows: $s\sigma'$ is a nonfolded gallery of type t' from A_{w_0} to $s\lambda + A_{sw_0}$. Denote by N' its set of negative directions. Then $q_{t',N'} = q_s^{-1}q_{t,N} = q_s^{-1}q_\lambda^2$ by (6.2) and $q_{A_{w_0}}q_{t',N'}^{-\frac{1}{2}} = q_{s\lambda+A_{sw_0}} = q_{\alpha^\vee}q_s^{-\frac{1}{2}}q_{-\lambda}q_{A_{w_0}}$. We obtain $q_{\alpha^\vee} = q_s$ and thus $T_sX_\lambda = X_{s\lambda}T_s + (q_s-1)X_\lambda$.

In general it is not true that $q_{\alpha^\vee} = q_s$. See example 4.14.

A general formula for T_sX_λ can be obtained with the same methods. But in the case of unequal parameters it is quite hard to get the correct coefficients. In the case of equal *parameters it is merely a calculation as in the rank one case and one arrives easily at the formula* (3.1)*.*

Before we proceed and specialize the last theorem to obtain a commutation rule in $\mathcal{H}^{\text{nil}}$ we provide some results on the relation between positive and negative directions needed for a more precise analysis of the structure constants of $\mathcal{H}^{sph}$.

Let t be any type and $\sigma \in \Gamma_t^+$. Define $q_+ := \prod_{s \in S^{\mathfrak{a}}} q_s^{m_s^+(\sigma)}$ and $q_- := \prod_{s \in S^{\mathfrak{a}}} q_s^{-m_s^-(\sigma)}$. So q_+ is the contribution of the positive directions to L_σ and q_- its negative counterpart. Deleting all entries in t corresponding to foldings of σ yields a new type t'. Let σ' be the nonfolded gallery of type t' starting in the same alcove as σ and denote by N' its negative directions. Then $e(\sigma) = e(\sigma')$ and $q_{t',N'} = q_+ q_-$. Using (6.2) we get $q^2_{A_{\iota(\sigma)}} q^{-1}_{t',N'} = q^2_{e(\sigma)} = q^2_{-wt(\sigma)} q_{\varepsilon(\sigma)}$ and thus

$$q^2_{-wt(\sigma)} q_{w_0 \iota(\sigma)} q_+ = q_{w_0 \varepsilon(\sigma)} q_-^{-1} \in \mathbb{Z}[q]. \tag{6.3}$$

Now let $\lambda, \mu, \nu \in X_+^\vee$ and t^μ a minimal gallery from A_f to A_{n^μ} as in the situation before theorem 4.10. Recall that the coefficient of M_ν in $M_\lambda M_\mu$ is given by $q_{w_\mu}^{-1} \sum_{\sigma \in \Gamma^d_{t^\mu,\lambda}(\nu)} q^2_{\lambda-\nu} q_{w_0 \iota(\sigma)} C_\sigma W_{\mu\nu}^{\varepsilon(\sigma)}$ where $W^w_{\mu\nu} = q_w \sum_{v \in W^{w_0 \mu} \cap W_\nu w} q_v^{-1}$. Let $\sigma \in \Gamma^d_{t^\mu,\lambda}(\nu)$. Then the translated gallery $-\lambda + \sigma$ is in $\Gamma^+_{t^\mu}(-\lambda+\nu)$. Since translation does not change q_+ and q_- we get from (6.3) that

$$q_{w_\mu}^{-1} q^2_{\lambda-\nu} q_{w_0 \iota(\sigma)} q_+ W^{\varepsilon(\sigma)}_{\mu\nu} \in \mathbb{Z}[q]$$

since $q_{w_\mu}^{-1} q_{w_0 \varepsilon(\sigma)} W^{\varepsilon(\sigma)}_{\mu\nu} = q_{w_0 w_\mu} \sum_{v \in W^{w_0 \mu} \cap W_\nu \varepsilon(\sigma)} q_v^{-1} \in \mathbb{Z}[q]$. So we get

Theorem 6.4. *The structure constants of* $\mathcal{H}^{sph}$ *with respect to the Macdonald basis are polynomials in* q*. Moreover, regarded as polynomials in the* $q_s - 1$ *they have nonnegative coefficients.*

This theorem is one of the results in [Par06]. He shows it by interpreting the structure constants as intersections in a regular affine building but does not give a combinatorial formula.

Now we want to specialize the commutation formula of $\tilde{\mathcal{H}}^{\mathfrak{a}}$ to obtain a commutation formula for the nil affine Hecke algebra. So we assume from now on that all parameters are equal. Rewrite the assertion of theorem 6.2 as

$$\tilde{T}_w X_\lambda = \sum_{\mu \in X^\vee} q^{-\langle \rho, \lambda+\mu \rangle} \sum_{\substack{\sigma \in \Gamma_t^N \\ \iota(\sigma) = w w_0}} L^N_\sigma X_\mu \tilde{T}_{\varepsilon(\sigma) w_0}$$

The coefficients of the right hand side in the last equation are Laurent-polynomials in q, i.e. we get the estimate

$$\deg L^N_\sigma \leq \langle \rho, \lambda + wt(\sigma) \rangle.$$

In contrast to the case of positively folded galleries the leading term of L^N_σ can be -1. So before specializing one has to think about the sign. Since L^N_σ is by definition of the form $q^*(q-1)^\#(q^{-1}-1)^{n^-(\sigma)}$ where $n^-(\sigma)$ is the total number of negative foldings, the leading term is $(-1)^{n^-(\sigma)}$ if it is nonzero. So we get

Theorem 6.5. *Under the hypotheses of the last theorem*

$$\tilde{T}_w X_\lambda = \sum_\sigma (-1)^{n^-(\sigma)} X_{wt(\sigma)} \tilde{T}_{\varepsilon(\sigma)w_0}$$

where the sum is over all $\sigma \in \Gamma_t^N$ *starting in* ww_0 *such that* $\deg L_\sigma = \langle \rho, \lambda + wt(\sigma) \rangle$.

Remark 6.6. *This formula is positive in the case where* $N = \emptyset$. *This can be achieved if* λ *is dominant. In this case any* minimal *gallery from* A_{w_0} *to* $\lambda + A_{w_0}$ *has only positive directions. There is one special choice for this minimal gallery: First take a minimal gallery from* A_f *to* A_{w_λ} *and then one (with type* t^λ*) to* $\lambda + A_{w_0}$.

The assertion of the last theorem is exactly the commutation formula [LP04, theorem 6.1]. The translation between these two formulations is given by introducing the companion of a gallery defined in [GL05, definition 22]. For a discussion of the equivalence between these formulas see the appendix of [LP04] and [GL05, remark 12]. But their approach is quite different from ours. Instead of labeling the walls crossed by the initial gallery with the simple affine reflections corresponding to right multiplication they label them by the root of the separating hyperplane corresponding to left multiplication. Then they define operators R_β for any root β, show that they satisfy certain compatibility conditions and use this to calculate the coefficients of the last theorem. They work entirely in the nil affine Hecke algebra and so do not get any q-analogs.

There is a close connection of this formula to the $T^\vee$-equivariant K-theory of the flag variety $G^\vee/B^\vee$. It is an algebra over the representation ring $R(T^\vee) = \mathbb{Z}[X^\vee]$ and has a natural basis over $R(T^\vee)$ given by the classes of the structure sheaves $\mathcal{O}_w$ for $w \in W$ of the Schubert varieties. One is interested in the following question: Given the class of a structure sheaf $\mathcal{O}_w$ with $w \in W$ and a line bundle $\mathcal{L}_\lambda$ of weight λ. Then one asks for the expansion of the class of the tensor product $\mathcal{L}_\lambda \otimes \mathcal{O}_w$ in terms of the structure sheaves. By [PR99] the coefficients appearing are exactly the ones from the last corollary. So these coefficients do have an interesting geometric interpretation.

Now one may ask for a geometric interpretation of the q-analogs of the last two theorems. It is known by the work of Lusztig that there is an isomorphism from the affine Hecke algebra to the equivariant K-theory of the Steinberg variety associated to $G^\vee$ and $B^\vee$. But it is not clear, if there is any nice geometric interpretation for the coefficients in this context.

6.2 Demazure character formula

In this section we calculate Demazure characters and thus give a proof for corollary 1.10. For this we again regard $\tilde{\mathcal{H}}^{\mathfrak{a}}$ with equal parameters. Of course we could multiply the formula of theorem 6.5 from the right by $\tilde{T}_{w_0}$. But in order to get the connection with LS-galleries it is more convenient to restart.

Let $\lambda \in X_+^\vee$ and $w \in W$. By lemma 2.6 we have $T_{\tau_\lambda} = T_{n^\lambda} T_{w_0 w_\lambda}$. Using corollary 5.8 we get

$$T_w X_\lambda \mathbf{1}_0 = q^{-\langle \rho, \lambda \rangle} T_w T_{n^\lambda} T_{w_0 w_\lambda} \mathbf{1}_0 = q^{l(w_0 w_\lambda)} \sum_{\substack{\sigma \in \Gamma_t^+ \\ \iota(\sigma) = id}} q^{-\langle \rho, \lambda + wt(\sigma) \rangle} L_\sigma X_{wt(\sigma)} \mathbf{1}_0.$$

where $t = (t_w, t^\lambda)$ is the concatenation of a type t_w of a minimal gallery from A_f to A_w and t^λ. Rewriting this in terms of the basis $\tilde{T}_v$ one gets

$$\tilde{T}_w X_\lambda \tilde{\mathbf{1}_0} = q^{l(w_0 w_\lambda)-l(w)} \sum_{\substack{\sigma\in\Gamma_t^+ \\ \iota(\sigma)=id}} q^{-\langle \rho, \lambda + wt(\sigma)\rangle} L_\sigma X_{wt(\sigma)} \tilde{\mathbf{1}_0}.$$

In particular we get the estimate

$$\deg L_\sigma \leq \langle \rho, \lambda + wt(\sigma)\rangle - l(w_0 w_\lambda) + l(w) \tag{6.4}$$

for any $\sigma \in \Gamma_t^+$ with $\iota(\sigma) = id$. We can regard the image of the equation above in $\mathcal{H}^{\text{nil}}$ and get

$$\tilde{T}_w X_\lambda \tilde{T}_{w_0} = \sum_\sigma X_{wt(\sigma)} \tilde{T}_{w_0}, \tag{6.5}$$

where the sum is over all galleries $\sigma \in \Gamma_t^+$ with $\iota(\sigma) = id$ such that the degree of L_σ is maximal.

But we would like to get rid of the initial part of type t_w to get formulas with galleries of a type depending only on λ. This can be achieved as follows: Within the t_w-part σ stays at the origin, i.e. the alcoves A have $wt(A) = 0$. Let A_v be the ending alcove of the t_w-part, i.e. $v \in W$. The t^λ-part of σ thus starts at A_v. The polynomial L_σ can be split the same way to get $L_\sigma = L_\sigma^{t_w} L_\sigma^{t^\lambda}$. For the t_w-part we have

Lemma 6.7. *Let $v \in W$. Then $q^{l(v)-l(w)} L_{t_w}(A_f, A_v) \in 1 + q^{-1}\mathcal{L}^-$ if $v \leq w$ and 0 otherwise.*

The proof is done by induction on $l(w)$ using lemma 5.6. For $v, w \in W$ the polynomials $L_{t_w}(A_f, A_v)$ coincide with the R-polynomials $R_{w,v}$ of Deodhar [Deo85] by example 2.3.

From the last lemma we get that for $v \leq w$ there is exactly one gallery σ_v of type t_w from A_f to A_v with $\deg L_{\sigma_v} = l(w) - l(v)$. But on the other hand we know from (4.1) that $\deg L_\sigma^{t^\lambda} \leq \langle \rho, \lambda + wt(\sigma)\rangle - l(w_0 w_\lambda) + l(v)$ and that we have equality iff σ is a LS-gallery. So in order to have equality in (6.4) the t_w-part has to be σ_v and the t^λ-part has to be a LS-gallery. Bringing this together we get that in (6.5) it is enough to sum over all LS-galleries σ in $\Gamma_{t^\lambda}^+$ such that $\iota(\sigma) \leq w$. This proves corollary 1.10.

Now we can refine the discussion on the existence of LS-galleries with a given weight. Let $\sigma \in \Gamma_{t^\lambda}^+(\mu)$ with $\iota(\sigma) \leq w$. Then $w\lambda \leq \iota(\sigma)\lambda \leq \mu$. But we also have the general condition $\mu^+ \leq \lambda$. These two conditions together are equivalent to $d_{\lambda\mu}^w > 0$. So with the last corollary we know that there is a gallery $\sigma \in \Gamma_{t^\lambda}^+(\mu)$ with $\iota(\sigma) \leq w$ iff $d_{\lambda\mu}^w > 0$.

See also section 11 for a geometric interpretation of these multiplicities.

7 Structure constants

In this section we calculate the structure constants of the spherical Hecke algebra with respect to the Macdonald basis and prove theorem 4.10 and thus theorem 1.4 and its corollary.

Therefore we need some preparation. For a generalized alcove A and a type t define $\Gamma^+_{t,A}$ to be the set of all positively folded galleries of type t with initial alcove A.

Lemma 7.1. *Let $A = \mu + A_w$ be a dominant generalized alcove such that As is no longer dominant. Let $H_{\alpha,0}$ be the hyperplane separating A and As with $\alpha \in \Delta$. Then we have $X_A T_s = T_{s_\alpha} X_A$.*

Proof. We have $s_\alpha A = As$ and $A \succ As$. So s_α and s are conjugate in $\tilde{W}^{\mathfrak{a}}$ and thus $q_{s_\alpha} = q_s$. Distinguish two cases:
If $s = s_{\theta,1}$ with $\theta \in \Theta$ we have $\langle \alpha_i, \mu \rangle = 1$ and thus $s_\alpha(\mu) = \mu - \alpha_i^\vee$ and $s_\alpha A = s_\alpha(\mu) + A_{s_\alpha w}$. But on the other hand we have $As = (\mu + w\theta_k^\vee) + A_{ws_\theta}$ and so $w\theta^\vee = -\alpha^\vee$. In particular, $s_\alpha w < w$. From example 6.3 we know that $q_{\alpha^\vee} = q_s$ in this case and

$$T_{s_\alpha} X_\mu = X_{\mu - \alpha_i^\vee} T_{s_\alpha} + (q_{s_\alpha} - 1) X_\mu.$$

Together with $s_\alpha w < w$ this yields

$$T_{s_\alpha} X_\mu \overline{T}_w = X_{\mu - \alpha_i^\vee} \overline{T}_{s_\alpha w} + (q_{s_\alpha} - 1) X_\mu \overline{T}_w$$

and thus

$$T_{s_\alpha} X_A = X_{A_s} + (q_{s_\alpha} - 1) X_A = X_A T_s$$

where the last equality follows from $A \succ As$.
If $s = s_\beta \in S$ we have $s_\alpha(\mu) = \mu$ and $w^{-1}(\alpha) = \beta$. So here $s_\alpha w > w$. Using $T_{s_\alpha} X_\mu = X_\mu T_{s_\alpha}$ one obtains the desired equality as above. $\square$

We keep the notation of the last lemma and get $\mathbf{1}_0 X_A T_s = \mathbf{1}_0 T_{s_\alpha} X_A = q_s \mathbf{1}_0 X_A$ (recall that $q_{s_\alpha} = q_s$).

Let $t = (t_1, \ldots, t_k)$ be a type and define $T_t = T_{t_1} \cdot \ldots \cdot T_{t_k}$. From theorem 5.7 we get

$$X_A T_t = \sum_{\sigma \in \Gamma^+_{t,A}} L_\sigma X_{e(\sigma)}.$$

This yields

$$\mathbf{1}_0 X_A T_t = \sum_{\sigma \in \Gamma^+_{t,A}} L_\sigma \mathbf{1}_0 X_{e(\sigma)}.$$

Setting $t' = (s, t)$ we obtain by the same arguments

$$\mathbf{1}_0 X_A T_s T_t = \sum_{\sigma \in \Gamma^+_{t',A}} L_\sigma \mathbf{1}_0 X_{e(\sigma)}.$$

Since $\mathbf{1}_0 X_A T_s T_t = q_s \mathbf{1}_0 X_A T_t$ we get the following

Lemma 7.2. *Let t be any type and let A be a dominant generalized alcove such that As is no longer dominant. Setting $t' = (s, t)$ we have*

$$q_s \sum_{\sigma \in \Gamma^+_{t,A}} L_\sigma \mathbf{1}_0 X_{e(\sigma)} = \sum_{\sigma \in \Gamma^+_{t',A}} L_\sigma \mathbf{1}_0 X_{e(\sigma)}.$$

Now let $\lambda \in X_+^\vee$. Then the generalized alcove $A := \lambda + A_w$ is dominant iff $w^{-1} \in W^\lambda$. Let $w^{-1} \in W^\lambda$ and $v \in W_\lambda$. Then we get

$$\overline{T}_v X_A = q_v^{-1} X_{\lambda + A_{vw}} = q_v^{-1} X_{vA}.$$

Since $v \in W_\lambda$ we get the equality (using the notation introduced before the last lemma)

$$\mathbf{1}_0 X_{vA} T_t = q_v \mathbf{1}_0 \overline{T}_v X_A T_t = \mathbf{1}_0 X_A T_t.$$

For later use observe that $vA = \lambda + A_{vw}$ and thus vA is no longer dominant. We get

Lemma 7.3. *Let $\lambda \in X_+^\vee$, $w^{-1} \in W^\lambda$ and $v \in W_\lambda$. Let $A = \lambda + A_w$. For any type t we have*

$$\sum_{\sigma \in \Gamma^+_{t,A}} L_\sigma \mathbf{1}_0 X_{e(\sigma)} = \sum_{\sigma \in \Gamma^+_{t,vA}} L_\sigma \mathbf{1}_0 X_{e(\sigma)}.$$

Now let $\lambda, \mu \in X_+^\vee$. Let $w_\mu \in W_\mu$ and $n^\mu \in \tau_\mu W$ as in definition 2.5. Let t^μ denote the type of a minimal gallery from A_f to A_{n^μ}. As in the proof of lemma 5.9 we get by lemma 5.8

$$\mathbf{1}_0 X_\lambda \, \mathbf{1}_0 T_{n^\mu} = \mathbf{1}_0 X_\lambda \sum_{\sigma \in \Gamma^+_{t^\mu}} q_{w_0 \iota(\sigma)} L_\sigma X_{e(\sigma)} \tag{7.1}$$

$$= q_\lambda \sum_{\sigma \in \Gamma^+_{t^\mu,\lambda}} q_{w_0 \iota(\sigma)} L_\sigma \mathbf{1}_0 X_{e(\sigma)}. \tag{7.2}$$

Here $\Gamma^+_{t^\mu,\lambda}$ is the set of all galleries of type t^μ starting in λ and the last equality holds since translating a gallery σ by λ does not change L_σ. So we have an expansion for the product in terms of X_A for $A \in \tilde{\mathcal{A}}$. But we need the expansion in terms of X_A for dominant A to compute the structure constants.

Theorem 7.4. *For $\lambda, \mu \in X_+^\vee$ we have*

$$\mathbf{1}_0 X_\lambda \mathbf{1}_0 T_{n^\mu} = q_\lambda W_\lambda(q^{-1}) \sum_{\sigma \in \Gamma^d_{t^\mu,\lambda}} q_{w_0 \iota(\sigma)} C_\sigma \mathbf{1}_0 X_{e(\sigma)}.$$

Proof. For the proof of this theorem we use lemmas 7.2 and 7.3 to show that the contribution of the galleries with non-dominant weights in the formula (7.1) is exactly the contribution of the p_s.

First assume λ is regular. Then the first generalized alcove of every gallery starting in λ is dominant. Let $\eta \in \Gamma^+_{t^\mu,\lambda}$ be a gallery leaving the dominant chamber. Let γ be the maximal initial subgallery of η contained in $\mathcal{C}$ and let A be $e(\gamma)$. Then η is not folded after A and the next generalized alcove in η is of the form As for some $s \in S^{\mathfrak{a}}$. Denote by $\Gamma^+_\gamma \subset \Gamma^+_{t^\mu,\lambda}$ the set of galleries starting with γ. By lemma 7.2 we have that

$$\frac{q_s}{q_s - 1} \sum_{\sigma \in \Gamma^+_\gamma, \sigma \text{ folded at } A} L_\sigma \mathbf{1}_0 X_{e(\sigma)} = \sum_{\sigma \in \Gamma^+_\gamma} L_\sigma \mathbf{1}_0 X_{e(\sigma)}.$$

So the contribution of all galleries starting with γ is the same as the contribution of the galleries starting with γ and staying in $\mathcal{C}$ at A, if the contribution of the folding at A is q_s instead of $q_s - 1$. Iteration of this procedure eventually yields

$$\sum_{\sigma\in\Gamma_\gamma^+} L_\sigma \mathbf{1}_0 X_{e(\sigma)} = \sum_{\sigma\in\Gamma_\gamma^+,\sigma\subset\mathcal{C}} C_\sigma \mathbf{1}_0 X_{e(\sigma)}$$

which proves the theorem for regular λ.
If λ is non-regular we have to apply lemma 7.3 to obtain the theorem because in this case the first alcove of a gallery starting in λ can be non-dominant. In this case its contribution has a part coming from the initial direction, which we did not need to consider in the regular case. But lemma 7.3 tells us that the contribution arising from these alcoves is the same as the contribution from the dominant ones. More precisely we have for $w^{-1} \in W^\lambda$ and $v \in W_\lambda$

$$\sum_{\sigma\in\Gamma^+_{t^\mu,\lambda},\iota(\sigma)=w} q_{w_0 w} L_\sigma \mathbf{1}_0 X_{e(\sigma)} = q_v \sum_{\sigma\in\Gamma^+_{t^\mu,\lambda},\iota(\sigma)=vw} q_{w_0 v w} L_\sigma \mathbf{1}_0 X_{e(\sigma)}$$

and thus

$$W_\lambda(q^{-1}) \sum_{\sigma\in\Gamma^+_{t^\mu,\lambda},\iota(\sigma)=w} q_{w_0 w} L_\sigma \mathbf{1}_0 X_{e(\sigma)} = \sum_{\sigma\in\Gamma^+_{t^\mu,\lambda},\iota(\sigma)\in W_\lambda w} q_{w_0\iota(\sigma)} \mathbf{1}_0 X_{e(\sigma)}.$$

Since the sum over all $w^{-1} \in W^\lambda$ of the left hand side of the last equation is exactly the contribution of the galleries starting in $\mathcal{C}$, the theorem follows. $\square$

Remark 7.5. *The proofs for multiplying Schur polynomials using paths are of a similar type as above (see for example [Lit94, section 6]). First one gets a formula involving also Schur polynomials associated to paths leaving the dominant chamber. Then one shows that the contributions of the leaving paths cancel each other. This is done by combinatorial arguments, i.e. one can see which paths cancel each other. In contrast to this we do not have any concrete information about this cancellation process.*

Now we can prove the first part of theorem 4.10 respectively theorem 1.4. We multiply the equation of the last theorem from the right by $\mathbf{1}_0$ and get by the definition of the Macdonald basis

$$\begin{aligned}
M_\lambda M_\mu &= \frac{q_\lambda q_{w_0}^{-1}}{W(q)W_\lambda(q^{-1})} \frac{1}{W(q)W_\mu(q)} \mathbf{1}_0 X_\lambda \mathbf{1}_0\, \mathbf{1}_0 T_{n^\mu} \mathbf{1}_0 \\
&= \frac{q_\lambda^2 q_{w_0}^{-1}}{W(q)W_\mu(q)} \sum_{\sigma\in\Gamma^d_{t^\mu,\lambda}} q_{w_0\iota(\sigma)} C_\sigma \mathbf{1}_0 X_{e(\sigma)} \mathbf{1}_0 \\
&= \frac{q_\lambda^2 q_{w_0}^{-1}}{W(q)W_\mu(q)} \sum_{\sigma\in\Gamma^d_{t^\mu,\lambda}} q_{-wt(\sigma)} q_{w_0\iota(\sigma)} C_\sigma \mathbf{1}_0 X_{wt(\sigma)} \mathbf{1}_0 \\
&= \frac{q_\lambda^2}{W_\mu(q)} \sum_{\sigma\in\Gamma^d_{t^\mu,\lambda}} q^2_{-wt(\sigma)} q_{w_0\iota(\sigma)} C_\sigma W_{wt(\sigma)}(q^{-1}) M_{wt(\sigma)} \\
&= \frac{q_\lambda^2}{W_\mu(q)} \sum_{\nu\in X_+^\vee} q^2_{-\nu} W_\nu(q^{-1}) C_{t^\mu,\lambda}(\nu) M_\nu.
\end{aligned}$$

To prove the second part of theorem 4.10 and thus theorem 1.4 we need one more step. It is not possible to impose conditions on the initial direction as in theorem 4.5. Instead we impose conditions on the final direction to get rid of the fraction $\frac{1}{W_\mu(q)}$. For doing this we need some preparation. The situation is more difficult than the case of Satake coefficients since now two stabilizers instead of one are involved. So we first need some information on the interplay between them.

We use the notation for stabilizer subgroups introduced in definition 2.5. Moreover, for any $\nu \in X^\vee$ let $\mathbf{1}_\nu = \sum_{w\in W_\nu} T_w$ be the corresponding symmetrizer. Note that $W_{w_0\mu} = w_0 W_\mu w_0$ and thus $q_{w_\mu} = q_{w_{w_0\mu}}$ and $W_\mu(q) = W_{w_0\mu}(q)$.

Let $Y = \sum_{w\in W} R_w \overline{T}_w \in \tilde{\mathcal{H}}^{\mathfrak{a}}$ with $R_w \in \mathcal{L}$. Assume $Y \in \tilde{\mathcal{H}}^{\mathfrak{a}}\mathbf{1}_{w_0\mu}$. Then $R_w = R_{wv}$ for any $w \in W$ and $v \in W_{w_0\mu}$ and thus $Y = q_{w_\mu}^{-1} \sum_{w\in W^{w_0\mu}} R_w \overline{T}_w \mathbf{1}_{w_0\mu}$ since for $w \in W^{w_0\mu}$ we have $\overline{T}_w \overline{\mathbf{1}_{w_0\mu}} = \sum_{v\in W_{w_0\mu}} \overline{T}_{wv}$ and $\overline{\mathbf{1}_{w_0\mu}} = q_{w_\mu}^{-1}\mathbf{1}_{w_0\mu}$.

Now let $\nu \in X^\vee$ and take Y of a special form, namely let $Y = \sum_{w^{-1}\in W^\nu} R_w \mathbf{1}_\nu \overline{T}_w$. For $w \in W$ denote by w^ν the minimal element of the coset $W_\nu w$. In particular $(w^\nu)^{-1} \in W^\nu$. Expanding Y in terms of the $\overline{T}_w$ yields

$$Y = q_{w_\nu} \sum_{w\in W} R_{w^\nu} \overline{T}_w.$$

So if in addition $Y \in \tilde{\mathcal{H}}^{\mathfrak{a}}\mathbf{1}_{w_0\mu}$ we get $Y = q_{w_\nu} q_{w_\mu}^{-1} \sum_{w\in W^{w_0\mu}} R_{w^\nu} \overline{T}_w \mathbf{1}_{w_0\mu}$ by the considerations above.

We want to calculate $Y\mathbf{1}_0$. We get $Y\mathbf{1}_0 = q_{w_\nu} q_{w_\mu}^{-1} W_\mu(q) \sum_{w\in W^{w_0\mu}} q_w^{-1} R_{w^\nu} \mathbf{1}_0$ and thus

$$Y\mathbf{1}_0 = q_{w_\nu} W_\mu(q^{-1}) \sum_{w^{-1}\in W^\nu} q_w^{-1} W_{\mu\nu}^w R_w \mathbf{1}_0 \tag{7.3}$$

where $W_{\mu\nu}^w := q_w \sum_{v\in W^{w_0\mu}\cap W_\nu w} q_v^{-1}$. Observe that $W^{w_0\mu} \cap W_\nu w \neq \emptyset$ iff $w \in W_\nu W^{w_0\mu}$. In particular, we get for regular ν that $W_{\mu\nu}^w = 1$ if $w \in W^{w_0\mu}$ and 0 else.

Now we relate this to our problem. We have $W_\nu(q)\mathbf{1}_0 X_\nu = \mathbf{1}_0 X_\nu \mathbf{1}_\nu$ since $T_w X_\nu = X_\nu T_w$ for any $w \in W_\mu$. Moreover,

$$W_\mu(q)\mathbf{1}_0 T_{n^\mu} = \mathbf{1}_0 \mathbf{1}_\mu T_{\tau_\mu} T_{w_\mu} \overline{T}_{w_0} = \mathbf{1}_0 T_{\tau_\mu} T_{w_\mu} \mathbf{1}_\mu \overline{T}_{w_0} = \mathbf{1}_0 T_{n^\mu} T_{w_0} \mathbf{1}_\mu \overline{T}_{w_0}.$$

But $T_{w_0} T_w \overline{T}_{w_0} = T_{w_0 w w_0}$ for all $w \in W$ and thus $T_{w_0}\mathbf{1}_\mu \overline{T}_{w_0} = \mathbf{1}_{w_0\mu}$. So $\mathbf{1}_0 T_{n^\mu} \in \tilde{\mathcal{H}}^{\mathfrak{a}}\mathbf{1}_{w_0\mu}$.

Consider the contribution of X_ν in theorem 7.4 given by

$$\sum_{\sigma\in\Gamma^d_{t^\mu,\lambda}(\nu)} q_{w_0\iota(\sigma)} C_\sigma \mathbf{1}_0 X_{e(\sigma)} = \frac{q_{-\nu}}{W_\nu(q)} \mathbf{1}_0 X_\nu \sum_{\sigma\in\Gamma^d_{t^\mu,\lambda}(\nu)} q_{w_0\iota(\sigma)} q_{\varepsilon(\sigma)} C_\sigma \mathbf{1}_\nu \overline{T}_{\varepsilon(\sigma)}.$$

As already observed before, $\nu + A_v \subset \mathcal{C}$ with $v \in W$ iff $v^{-1} \in W^\nu$. So the final directions of the galleries σ occurring in the last equation satisfy $(\varepsilon(\sigma))^{-1} \in W^\nu$. If we define $Y := \sum_{\sigma\in\Gamma^d_{t^\mu,\lambda}(\nu)} q_{w_0\iota(\sigma)} q_{\varepsilon(\sigma)} C_\sigma \mathbf{1}_\nu \overline{T}_{\varepsilon(\sigma)}$ then Y is of the kind considered above. So we can apply (7.3) and get

$$\sum_{\sigma\in\Gamma^d_{t^\mu,\lambda}(\nu)} q_{w_0\iota(\sigma)} q_{\varepsilon(\sigma)} C_\sigma \mathbf{1}_\nu \overline{T}_{\varepsilon(\sigma)} \mathbf{1}_0 = q_{w_\nu} W_\mu(q^{-1}) \sum_{\sigma\in\Gamma^d_{t^\mu,\lambda}(\nu)} q_{w_0\iota(\sigma)} C_\sigma W_{\mu\nu}^{\varepsilon(\sigma)} \mathbf{1}_0.$$

Bringing all this together we can multiply the assertion of theorem 7.4 from the right by $\mathbf{1}_0$ and get

$$\mathbf{1}_0 X_\lambda \mathbf{1}_0 T_{n^\mu} \mathbf{1}_0 = q_\lambda W_\lambda(q^{-1}) W_\mu(q^{-1}) \sum_{\nu \in X_+^\vee} \frac{q_{-\nu}}{W_\nu(q^{-1})} \sum_{\sigma \in \Gamma^d_{t^\mu,\lambda}(\nu)} q_{w_0 \iota(\sigma)} C_\sigma W_{\mu\nu}^{\varepsilon(\sigma)} \mathbf{1}_0 X_\nu \mathbf{1}_0.$$

Now we can calculate the coefficient of M_ν in the product $M_\lambda M_\mu$ as above. It is equal to

$$q^2_{\lambda-\nu} q^{-1}_{w_\mu} \sum_{\sigma \in \Gamma^d_{t^\mu,\lambda}(\nu)} q_{w_0 \iota(\sigma)} C_\sigma W_{\mu\nu}^{\varepsilon(\sigma)}$$

which proves the second part of theorem 4.10 and thus 1.4.

Remark 7.6. *Consider the case of equal parameters and let $w^{-1} \in W^\nu$. Then we have $W^w_{\mu\nu} = q^{l(w)} \sum_{v \in W^{w_0\mu} \cap W_\nu w} q^{-l(v)}$. By definition of W^ν we have $l(v) \geq l(w)$ for all $v \in W_\nu w$ and thus $W^w_{\mu\nu} \in \mathcal{L}^-$. Moreover, the constant term of $W^w_{\mu\nu}$ is 1 iff $w \in W^{w_0\mu}$.*

Remark 7.7. *One can proceed the same way to obtain a formula for the Satake coefficients as in the second part of theorem 4.5 with a condition on the final direction. For stating the results we consider again the situation of section 5. So $\lambda \in X_+^\vee$ and t^λ is the type of a minimal gallery from A_f to A_{n^λ}. Applying the above considerations (for λ instead of μ) yields $\mathbf{1}_0 T_{n^\lambda} \in \tilde{\mathcal{H}}^{\mathfrak{a}} \mathbf{1}_{w_0\lambda}$. A formula for $\mathbf{1}_0 T_{n^\lambda}$ is given by (see the proof of lemma 5.9) $\sum_{\sigma \in \Gamma^+_{t^\lambda}} q_{-wt(\sigma)} q_{\varepsilon(\sigma)} q_{w_0\iota(\sigma)} L_\sigma X_{wt(\sigma)} \overline{T}_{\varepsilon(\sigma)}$. So we get*

$$\mathbf{1}_0 T_{n^\lambda} = q^{-1}_{w_\lambda} \sum_{\substack{\sigma \in \Gamma^+_{t^\lambda} \\ \varepsilon(\sigma) \in W^{w_0\lambda}}} q_{-wt(\sigma)} q_{\varepsilon(\sigma)} q_{w_0\iota(\sigma)} L_\sigma X_{wt(\sigma)} \overline{T}_{\varepsilon(\sigma)} \mathbf{1}_{w_0\lambda}.$$

Multiplying by $\mathbf{1}_0$ from the right then yields

$$M_\lambda = \frac{q^{-1}_{w_\lambda}}{W(q)} \sum_{\mu \in X^\vee} q_{-\mu} \sum_{\substack{\sigma \in \Gamma^+_{t^\lambda}(\mu) \\ \varepsilon(\sigma) \in W^{w_0\lambda}}} q_{w_0\iota(\sigma)} L_\sigma X_\mu \mathbf{1}_0$$

and thus $L_{\lambda\mu} = q^{-1}_{w_\lambda} \sum_{\substack{\sigma \in \Gamma^+_{t^\lambda}(\mu) \\ \varepsilon(\sigma) \in W^{w_0\lambda}}} q_{w_0\iota(\sigma)} L_\sigma$.

Moreover, we see that for a LS-gallery σ we have $\varepsilon(\sigma) \in W^{w_0\lambda}$.

8 Restriction coefficients

In this section we prove theorem 4.13 and thus the restriction formula 1.7. The proof is quite similar to the proof of 4.10 in the last section so we omit the details.

Let $\lambda \in X_+^\vee$ and denote by t^λ the type of a minimal gallery from A_f to A_{n^λ} as in section 5. We start with the observation $W_J(q)\mathbf{1}_0 = \mathbf{1}_J \mathbf{1}_0$. So we have

$$W_J(q) \mathbf{1}_0 T_{n^\lambda} = \mathbf{1}_J \mathbf{1}_0 T_{n^\lambda} = \sum_{\sigma \in \Gamma^+_{t^\lambda}} q_{w_0\iota(\sigma)} L_\sigma \mathbf{1}_J X_{e(\sigma)}.$$

As in the case of the $C^\nu_{\lambda\mu}$ we now have to reduce the galleries appearing in the formula to galleries having the final alcove in the J-dominant chamber by replacing L_σ with some other polynomial. For this we use the lemmas 7.2 and 7.3 and apply them as in the proof of theorem 7.4, but this time only for the walls of $\mathcal{C}^J$, i.e. for the walls $H_{\alpha,0}$ with $\alpha \in \Delta_J$. Recall that $\Gamma^J_{t^\lambda}$ is the set of positively folded galleries of type t^λ contained in $\mathcal{C}^J$. Repeating the steps in this proof we eventually get

Theorem 8.1. *For $\lambda \in X^\vee_+$ we have*

$$\mathbf{1}_0 T_{n^\lambda} = q_J^{-1} \sum_{\sigma \in \Gamma^J_{t^\lambda}} q_{w_0\iota(\sigma)} C^J_\sigma \mathbf{1}_J X_{e(\sigma)}.$$

From this the first part of theorem 4.13 follows since (where as before W^J_μ is the stabilizer of μ in W_J)

$$\begin{aligned} M_\lambda &= \frac{1}{W(q)W_\lambda(q)} \mathbf{1}_0 T_{n^\lambda} \mathbf{1}_0 = \frac{q_J^{-1}}{W(q)W_\lambda(q)} \sum_{\sigma\in\Gamma^J_{t^\lambda}} q_{-wt(\sigma)} q_{w_0\iota(\sigma)} C^J_\sigma \mathbf{1}_J X_{wt(\sigma)} \mathbf{1}_0 \\ &= \frac{1}{W_\lambda(q)} \sum_{\sigma\in\Gamma^J_{t^\lambda}} q_{-wt(\sigma)} q^J_{-wt(\sigma)} C^J_\sigma \, W^J_{wt(\sigma)}(q^{-1}) M^J_{wt(\sigma)}. \end{aligned}$$

Now observe that since $e(\sigma) \subset \mathcal{C}^J$ we have $(\varepsilon(\sigma))^{-1} \in W_J^{wt(\sigma)}$ where by $W_J^{wt(\sigma)}$ we denote the minimal representatives of $W/W^J_{wt(\sigma)}$. Applying the considerations after theorem 7.4 with W_λ instead of W_μ and W^J_μ instead of W_ν leads to

$$\mathbf{1}_0 T_{n^\lambda} = q_{w_\lambda}^{-1} q_J^{-1} \sum_{\mu\in {}^J X^\vee_+} \frac{q_{-\mu}}{W^J_\mu(q^{-1})} \sum_{\substack{\sigma\in\Gamma^J_{t^\lambda}(\mu) \\ \varepsilon(\sigma)\in W^{w_0\lambda}}} q_{w_0\iota(\sigma)} \, {}^J W^{\varepsilon(\sigma)}_{\lambda\mu} C^J_\sigma \mathbf{1}_J X_\mu \mathbf{1}_0.$$

where ${}^J W^w_{\lambda\mu} := q_w \sum_{v\in W^{w_0\lambda}\cap W^J_\mu w} q_v^{-1}$. So the second part of theorem 4.13 and theorem 1.7 follow as above.

Similar to the situation in the last section we have that ${}^J W^w_{\lambda\mu} \in \mathcal{L}^-$ and the constant term is 1 iff $w \in W^{w_0\lambda}$.

9 Crystals

In this section we want to show the relationship between our approach and the root operators defined in [GL05].

Let t be any type and denote by Γ_t the set of all galleries of type t starting in 0. In [GL05, section 6] they define root operators e_α and f_α on Γ_t for any $\alpha \in \Delta$ and show that this makes Γ_t into a crystal in the sense of Kashiwara [Kas95]. If $t = t^\lambda$ for regular λ then the set of LS-galleries is closed under the root operators and it is the highest weight crystal for λ. However, for non-regular λ they regard degenerate galleries, i.e. not galleries of alcoves but of simplices of smaller dimension. But using some of their general results on properties of the root operators and the additional operators $\tilde{e}_\alpha$ for $\alpha \in \Delta$ one can show that the LS-galleries for non-regular λ are closed under the root operators and define the highest weight crystal for λ.

For stating these results we recall the notion of dimension of a gallery. For $\sigma \in \Gamma_t$ it is defined by $\dim \sigma := l(w_0\iota(\sigma)) + \sum_{s \in S^{\mathfrak{a}}} m_s(\sigma) + n_s(\sigma)$, i.e. if we regard the case of equal parameters we have $\dim \sigma = l(w_0\iota(\sigma)) + \deg L_\sigma$.

Now let $\lambda \in X_+^\vee$. We know from section 4 that $\dim \sigma \leq \langle \rho, \lambda + wt(\sigma) \rangle + l(w_\lambda)$ for $\sigma \in \Gamma_{t^\lambda}^+$ and the galleries of maximal dimension are precisely the LS-galleries. The result we need regarding the operators $\tilde{e}_\alpha$ is the following.

Theorem 9.1 ([GL05, lemma 7]). *Let $\sigma \in \Gamma_t^+$ and $\alpha \in \Delta$.*

(i) *If $\tilde{e}_\alpha(\sigma)$ is defined, then $wt(\tilde{e}_\alpha(\sigma)) = wt(\sigma)$. Moreover, $\tilde{e}_\alpha(\sigma) \in \Gamma_t^+$ and we have $\dim \tilde{e}_\alpha(\sigma) = \dim \sigma + 1$.*

(ii) *If $e_\alpha(\sigma)$ is defined, then $wt(e_\alpha(\sigma)) = wt(\sigma) + \alpha^\vee$ and $\dim e_\alpha(\sigma) = \dim \sigma + 1$.*

(iii) *If $\tilde{e}_\alpha(\sigma)$ is not defined but $e_\alpha(\sigma)$ is, then $e_\alpha(\sigma)$ is again positively folded.*

Let $\sigma \in \Gamma_{t^\lambda}^+$ be a LS-gallery such that $e_\alpha(\sigma)$ is defined. By (i) and the maximality of $\dim \sigma$ one gets that $\tilde{e}_\alpha(\sigma)$ is not defined. So (iii) yields that $e_\alpha(\sigma)$ is again positively folded and from (ii) we know that $e_\alpha(\sigma)$ is again a LS-gallery. This shows that the set of LS-galleries is closed under the root operators e_α. Since the sum over the weights of all LS-galleries is s_λ by 1.3 they yield the highest weight crystal for λ.

Remark 9.2. *This assertion makes the definition of LS-galleries more plausible. Recall that we had many choices for the definition and we chose the ones having maximal dimension. For any other suitable definition of LS-galleries (in the sense of remark 5.10) the last assertion does not have to be true. There exist choices, where the corresponding LS-galleries are not closed under the root operators. In these cases the image of a LS-gallery under a root operator is not necessarily positively folded.*

We want to relate the root operators to $\mathcal{H}^{\mathrm{nil}}$. We recall the definition of the root operator f_α for $\alpha \in \Delta$. Let $\sigma = (A_0, \ldots, A_l)$ be a gallery of type t. For $1 \leq i \leq l$ with $t_i \in S^{\mathfrak{a}}$ denote by H_i the wall of A_i of type t_i and set $H_0 = wt(A_0)$ and $H_{l+1} = wt(A_l)$. Let $m \in \mathbb{Z}$ be minimal such that there exists $H_i \subset H_{\alpha,m}$. Let j be maximal with $H_j \subset H_{\alpha,m}$. If $j = l+1$ then $f_\alpha \sigma$ is not defined. Else let $k > j$ be minimal such that $H_k \subset H_{\alpha,m+1}$. Then

$$f_\alpha \sigma := (A_0, \ldots, A_{j-1}, s_{\alpha,m}A_j, \ldots, s_{\alpha,m}A_{k-1}, -\alpha^\vee + A_k, \ldots, -\alpha^\vee + A_l).$$

A careful case by case analysis (we omit the details) yields the following

Lemma 9.3. *Let $\sigma \in \Gamma_{t^\lambda}^+$ such that $f_\alpha(\sigma)$ is defined. Let $\varepsilon = \varepsilon(\sigma)$, $\varepsilon_\alpha = \varepsilon(f_\alpha \sigma)$ and $s = s_\alpha$.*

(i) *Assume $m = 0$. Let $k < l+1$. Then we have $\varepsilon_\alpha = \varepsilon$. If moreover $j = 0$, $h = 1$ and $\sigma \in LS_{t^\lambda}$ then $s\varepsilon > \varepsilon$ and $f_\alpha^2 \sigma$ is not defined.*
Let $k = l+1$. In this case one has $\varepsilon_\alpha = s\varepsilon < \varepsilon$ and $f_\alpha^2 \sigma$ is not defined.

(ii) *Assume $m < 0$. If $k < l+1$ then $\varepsilon_\alpha = \varepsilon$. Moreover, $h = m+1$ iff $f_\alpha^2\sigma$ is not defined and if $h = m+1$ then $s\varepsilon > \varepsilon$.*
If $k = l+1$ then $\varepsilon_\alpha = s\varepsilon < \varepsilon$ and $f_\alpha^2\sigma$ is not defined.

Now let σ be a LS-gallery such that $e_\alpha\sigma$ is not defined, i.e. $m = 0$. Define $\mu = wt(\sigma)$ and $w = \varepsilon(\sigma)$. Let $h = \langle\alpha, \mu\rangle$. Since $e_\alpha(\sigma)$ is not defined we have $h \geq 0$. The α-string of σ is the set $\{\sigma, f_\alpha\sigma, \dots, f_\alpha^h\sigma\}$ and f_α^{h+1} is not defined. From the last lemma we get the following: $\varepsilon(f_\alpha^i\sigma) = \varepsilon(\sigma)$ for all $i < h$ and $\varepsilon(f_\alpha^h\sigma) = \varepsilon(\sigma)$ if $s_\alpha\varepsilon(\sigma) > \varepsilon(\sigma)$ and $\varepsilon(f_\alpha^h\sigma) = s_\alpha\varepsilon(\sigma)$ if $s_\alpha\varepsilon(\sigma) < \varepsilon(\sigma)$. By the multiplication rules in $\mathcal{H}^{\mathrm{nil}}$ this yields

$$\sum_{j=0}^{h} X_{wt(f_\alpha^j\sigma)}\tilde{T}_{\varepsilon(f_\alpha^j\sigma)w_0} = \sum_{j=0}^{h-1} X_{\mu - j\alpha^\vee}\tilde{T}_{ww_0} + X_{s\mu}\tilde{T}_s\tilde{T}_{ww_0}.$$

which proves the following theorem using the commutation rule for $\tilde{T}_s X_\mu$.

Theorem 9.4. *Under the above hypotheses we have*

$$\tilde{T}_s X_{wt(\sigma)}\tilde{T}_{\varepsilon(\sigma)w_0} = \sum_{j=0}^{h} X_{wt(f_\alpha^j\sigma)}\tilde{T}_{\varepsilon(f_\alpha^j\sigma)w_0}.$$

Here one should be aware of the fact that the last theorem does not follow immediately from the existence of root operators since from this existence one gets only the expansions of the symmetrized versions $\tilde{T}_w X_\mu \tilde{T}_{w_0}$.

Now one may ask for q-analogs for the last theorem using the L_σ and replacing $\tilde{T}_s$ by $C_s = q^{-1}(T_s + 1)$ (so $C_s^2 = C_s$ and the image of C_s in $\mathcal{H}^{\mathrm{nil}}$ is $\tilde{T}_s$) and thus for some sort of q-analogs of the crystal operators. It does not follow from the commutation rules in $\tilde{\mathcal{H}}^{\mathfrak{a}}$ that such q-analogs exist. In general it is not true that (under the above hypotheses)

$$C_s q^{l(w_0\iota(\sigma))} L_\sigma X_{e(\sigma)} = \sum_{j=0}^{h} q^{l(w_0\iota(f_\alpha^j\sigma))} L_{f_\alpha^j\sigma} X_{e(f_\alpha^j\sigma)}.$$

Multiplying this formula from the right by T_{w_0} and specializing it would imply the last theorem. But it is almost true. If one examines the $L_{f_\alpha^j\sigma}$ as in the lemma above one can see, that if the formula does not hold, it can be corrected by adding summands at the beginning or at the end of the α-string. By case by case considerations it can be shown that these additional summands come from non LS-galleries. But it is far from being clear how general q-analogs of root operators could be defined so we omit the details and give just one example in the rank one case. Ram defines such q-analogs in [Ram06] and introduces q-crystals.

Example 9.5. *We continue the examples 4.15 and 5.11. Let again $\lambda \in X_+^\vee$ and start with $t = t^\lambda$. Denote $s = s_1$. Then the final direction of $\sigma(\mu)$ is s for $\mu \neq -\lambda$ and the final direction of $\sigma(-\lambda)$ is id. This yields*

$$qC_s X_\lambda \overline{T}_s = X_{-\lambda} + (q-1)\sum_{-\lambda<\mu<\lambda} X_\mu \overline{T}_s + qX_\lambda \overline{T}_s.$$

After multiplying by $q^{\langle\rho,\lambda\rangle}$ the left hand side gets $C_s q^{w_0\iota(\sigma(\lambda))} L_{\sigma(\lambda)} X_{e(\sigma(\lambda))}$ and the right hand side of this equation corresponds term by term to

$$\sum_{\sigma\in\Gamma_t^+} q^{l(w_0\iota(\sigma))} L_\sigma X_{e(\sigma)} = L_{\sigma(-\lambda)} X_{e(\sigma(-\lambda))} + \sum_{-\lambda<\mu<\lambda} L_{\sigma(\mu)} X_{e(\sigma(\mu))} + q L_{\sigma(\lambda)} X_{e(\sigma(\lambda)}.$$

So in this case the above equation really holds since all these galleries are LS-galleries and constitute a single α-string.

Now start with a minimal gallery from A_f to A_{τ_λ} as in example 5.11. Then one has for each $-\lambda \leq \mu \leq \lambda$ a LS-gallery $\sigma(\mu)$ as above and the additional gallery $\tilde{\sigma}$ which is nonfolded of weight $-\lambda$. One has that the final direction of $\sigma(\mu)$ is id for all μ and $\varepsilon(\tilde{\sigma}) = s$. The equation

$$qC_s X_\lambda = X_{-\lambda}(q\overline{T}_s + (q-1)\overline{T}_{id}) + (q-1) \sum_{-\lambda<\mu<\lambda} X_\mu + qX_\lambda$$

then corresponds after multiplying by $q^{\langle\rho,\lambda\rangle+1}$ term by term to

$$\sum_{\sigma\in\Gamma_t^+} q^{l(w_0\iota(\sigma))} L_\sigma X_{e(\sigma)} = qL_{\tilde{\sigma}} X_{e(\tilde{\sigma})} + qL_{\sigma(-\lambda)} X_{e(\sigma(-\lambda))} + \sum_{-\lambda<\mu<\lambda} L_{\sigma(\mu)} X_{e(\sigma(\mu))} + qL_{\sigma(\lambda)} X_{e(\sigma(\lambda))}.$$

So in this case the above equation does not hold if we restrict to the α-string, i.e. to LS-galleries. The q-α-string should include also $\tilde{\sigma}$.

10 Geometric spherical Hecke algebras

In this section we consider spherical Hecke algebras of reductive groups over local fields. We show that these arise as spherical Hecke algebras of some root datum specialized at a prime power $\mathbf{q}$. So using theorem 4.5 we get a new proof of a positivity result of Rapoport obtained in [Rap00]. For details on reductive groups over local fields see Tits' survey article [Tit79].

Let $\mathcal{K}$ be a local field with finite residue field k. Denote by ω its valuation and let $\mathbf{q}$ be the cardinality of k. For any algebraic group H defined over $\mathcal{K}$ denote by $H(\mathcal{K})$ the group of $\mathcal{K}$-valued points. Let G be a quasi-split connected reductive group over $\mathcal{K}$, i.e. there exists a Borel subgroup B defined over $\mathcal{K}$. Let $S \subset B$ be a maximal $\mathcal{K}$-split torus of G and denote by N its normalizer. Then $T = Z_G(S) \subset B$ is a maximal torus and $B = T \ltimes U$ where $U \subset B$ is the unipotent radical. Let $\tilde{\Phi} = (X(S), \tilde{\phi}, X^\vee(S), \tilde{\phi}^\vee)$ be the restricted root datum of (G, S), i.e. $X(S) = \mathrm{Hom}_{\mathcal{K}}(S, \mathbb{G}_m)$ is the character group, $X^\vee(S) = \mathrm{Hom}_{\mathcal{K}}(\mathbb{G}_m, S)$ the cocharacter group of S and $\langle\cdot,\cdot\rangle : X(S) \times X^\vee(S) \to \mathbb{Z}$ the natural pairing. Denote by $\tilde{\phi}$ (respectively $\tilde{\phi}^\vee$) the roots (respectively coroots) of G with respect to S and let $V = X(S) \otimes \mathbb{R}$. In general, $\tilde{\phi}$ may be non reduced.

Define $X(T)$ and $X^\vee(T)$ as above. Then $X(T)$ is a subgroup of finite index in $X(S)$. We get a map $\nu : T(\mathcal{K}) \to X^\vee(T)$ by demanding that $\langle\beta, \nu(t)\rangle = -\omega(\beta(t))$ for all $t \in T(\mathcal{K})$ and $\beta \in X(T)$. The image $X^\vee$ of ν is a free abelian group of rank $\dim S = \dim V$. One has inclusions $X^\vee(S) \subset X^\vee \subset X^\vee(T)$. If S is a maximal torus, i.e. G is split, then all inclusions are equalities. If G splits over some unramified extension of $\mathcal{K}$ then $X^\vee = X^\vee(S)$. Define $X = \mathrm{Hom}_{\mathbb{Z}}(X^\vee, \mathbb{Z}) \subset X(S)$ and denote the pairing between X and $X^\vee$ induced by $\langle \cdot, \cdot \rangle$ with the same symbol.

The valuation ω induces a filtration on all root subgroups of G. Using this filtration one gets a configuration $\mathcal{H}$ of hyperplanes in the affine space A underlying V^* which is locally finite. The group $\tilde{W}$ generated by the affine reflections at the hyperplanes in $\mathcal{H}$ is called the affine Weyl group of the pair (G, S). It can also be described as $\tilde{W} = N(\mathcal{K})/\ker(\nu)$ and one has $\tilde{W} = W \ltimes X^\vee$. Similar as in section 2 one gets a polysimplicial structure on A. It is the apartment of the Bruhat–Tits building of $G(\mathcal{K})$. Identify V^* with A such that the origin gets a special point for the induced polysimplicial structure on V^*. The Borel subgroup B defines a fundamental alcove containing 0. One gets a set of reflections $\tilde{S^{\mathfrak{a}}}$ which together with the stabilizer of the fundamental alcove generates $\tilde{W}$.

In general $\tilde{W}$ is neither the extended affine Weyl group of the root datum of (G, S) nor of the root datum $(X, \tilde{\phi}, X^\vee, \tilde{\phi}^\vee)$. But there exists a unique reduced set of roots $\phi \subset X$ such that the extended affine Weyl $\tilde{W}^{\mathfrak{a}}$ group of $(X, \phi, X^\vee, \phi^\vee)$ is isomorphic to $\tilde{W}$, the collection of hyperplanes $\mathcal{H}$ coincides with $\{H_{\alpha,m} \mid \alpha \in \phi^+, m \in \mathbb{Z}\}$ and the generators $S^{\mathfrak{a}}$ correspond to $\tilde{S^{\mathfrak{a}}}$ under this isomorphism.

Remark 10.1. *Any coroot in $\phi^\vee$ is a positive multiple of a coroot in $\tilde{\phi}^\vee$. But this multiple may depend on the coroot, even if $\tilde{\phi}$ is reduced and irreducible. But ϕ and $\tilde{\phi}$ give rise to an* échelonnage *in the sense of Bruhat and Tits [BT72] or an affine root system in the sense of Macdonald [Mac03].*

Let $K \subset G(\mathcal{K})$ be the stabilizer of the origin of V^* in the Bruhat–Tits building of $G(\mathcal{K})$. It is a special, good, maximal compact subgroup of G. We have the Cartan decomposition $G(\mathcal{K}) = \coprod_{\lambda \in X_+^\vee} K\lambda K$ and the Iwasawa decomposition $G(\mathcal{K}) = \coprod_{\mu \in X^\vee} U(\mathcal{K})\mu K$. The spherical Hecke algebra of $(G(\mathcal{K}), K)$ is the set of K-biinvariant functions on $G(\mathcal{K})$ with multiplication given by convolution using a Haar measure giving volume 1 to K. As a consequence of the Cartan decomposition the spherical Hecke algebra is isomorphic to the abstract spherical Hecke algebra of $\tilde{W}^{\mathfrak{a}}$ specialized at $\mathbf{q}$. In this setting M_λ is the characteristic function on the double coset $K\lambda K$. Up to some normalizing factor the Satake isomorphism is given by integration over $U(\mathcal{K})$ and so the coefficients $L_{\lambda\mu}$ are (up to normalization) the measure of the intersections $K\lambda K \cap U(\mathcal{K})\mu K$. In particular, $K\lambda K \cap U(\mathcal{K})\mu \neq \emptyset$ iff $L_{\lambda\mu} \neq 0$. So by our considerations we get a new proof of the following theorem [Rap00, theorem 1.1].

Theorem 10.2. *Let $\lambda, \mu \in X_+^\vee$ such that $\mu \leq \lambda$. Then $L_{\lambda\mu} > 0$. In particular, $K\lambda K \cap U(\mathcal{K})\mu \neq \emptyset$ in this case.*

Moreover, our approach yields an algorithm to calculate the measure of $K\lambda K \cap U(\mathcal{K})\mu K$ explicitly. In contrast to the case of equal parameters we do not get an explicit formula for the degree of this measure (as a polynomial in $\mathbf{q}$).

Remark 10.3. *In this geometric setting the parameters of the Hecke algebra have the following interpretation: Let $I \subset G$ be an Iwahori subgroup (i.e. I is the stabilizer of the fundamental chamber as subset of the affine building of G). Choose a Haar measure on G giving volume 1 to I. Then the double cosets IwI have measure q_w for any $w \in \tilde{W}^{\mathfrak{a}}$.*

The structure constants of $\mathcal{H}^{sph}$ have the following interpretation in this setting: Let $\lambda, \mu, \nu \in X_+^\vee$. Then the coefficient of M_ν in $M_\lambda M_\mu$ is the volume of the intersection $\lambda K \mu K \cap K \nu K$ with respect to a Haar measure giving volume 1 to K. By theorem 6.4 it is given by a polynomial in $\mathbf{q}$.

11 Geometric interpretations

In this section we give geometric interpretations of the L-polynomials with equal parameters and of Demazure multiplicities using the affine flag variety of the Langlands dual group G of $G^\vee$. For this we regard a special case of the setting of the last section, the split case, which in turn yields equal parameters. All Hecke algebras considered in this section are specialized at a prime power $\mathbf{q}$.

For any linear algebraic group H defined over some field F and any F-algebra A denote by $H(A)$ the group of its A-valued points. More explicitly let $F[H]$ be the coordinate algebra of H over F. Then $H(A) = \mathrm{Hom}_{F-\mathrm{algebras}}(F[H], A)$. Let $F \subset F'$ be a field extension. Then denote by $H_{F'}$ the linear algebraic group over F' obtained from H.

Details of the following constructions and their relation to affine Kac-Moody algebras can be found in Kumar's book [Kum02]. Let k be any field and K its algebraic closure. Let G be the connected reductive algebraic group over K with Borel subgroup B and maximal torus $T \subset B$ such that the associated root datum is Φ and the associated simple roots are given by Δ. Let U^- be the unipotent radical of the opposite Borel of B. Assume that all groups are defined and split over k.

Let $\mathcal{K} = k((t))$ be the field of Laurent series and denote by $\mathcal{O} = k[[t]] \subset \mathcal{K}$ the ring of formal power series. Then $\mathcal{K}$ is a local field with residue field k. The valuation $\omega : \mathcal{K} \to \mathbb{Z}$ is given by the order in 0. Moreover, $\mathcal{O}$ is the corresponding valuation ring and t is a uniformizing element. The map $\mathcal{O} \to k$ induces a morphism of groups $ev : G(\mathcal{O}) \to G(k)$. Define $\mathcal{B} = ev^{-1}(B(k))$. Further we set $\mathcal{G} = G(\mathcal{K})$ and let $\mathcal{N} \subset \mathcal{G}$ be the normalizer of $T(k)$ in $\mathcal{G}$. Then $(\mathcal{G}, \mathcal{B}, \mathcal{N}, T(k))$ is a Tits system with Weyl group $\tilde{W}^{\mathfrak{a}}$. For all $\alpha \in \phi$ one has a root subgroup U_α of $G_{\mathcal{K}}$ together with an isomorphism $\varphi_\alpha : \mathbb{G}_a \to U_\alpha$ defined over $\mathcal{K}$ from the additive group. For each $n \in \mathbb{Z}$ we denote by $U_{\alpha,n}$ the image of kt^n under $\varphi_\alpha(\mathcal{K})$.

The filtration on $U_\alpha(k)$ mentioned in the last section is given by the image of the standard filtration on $\mathbb{G}_a(\mathcal{K}) = \mathcal{K}$, i.e. $\mathcal{K}_n = \{f \mid f \in t^n\mathcal{O}\}$ which corresponds to $U^+_{\alpha,n} = \prod_{m \geq n} U_{\alpha,m}$ under ϕ_α. The hyperplane configuration of the last section in this special case is really $\{H_{\alpha,m} \mid \alpha \in \phi, m \in \mathbb{Z}\}$.

There are two decompositions of $\mathcal{G}$ into double cosets. The first one is given by the Iwahori decomposition $\mathcal{G} = \coprod_{w \in \tilde{W}^{\mathfrak{a}}} \mathcal{B} w \mathcal{B}$. In this case one has the additional property that for each $w \in W$ one gets a subgroup $U_w \subset \mathcal{B}$ as a product of certain $U_{\alpha,n}(K)$ isomorphic to $k^{l(w)}$ such that for any $x \in \mathcal{B}w\mathcal{B}$ there exist unique $u \in U_w$ and $b \in \mathcal{B}$ such that $x = uwb$. If $s \in S^{\mathfrak{a}}$ then $U_s = U_{\alpha,0}(K)$ if $s \in S$ and $U_s = U_{-\theta,1}$ if $s = s_{\theta,1}$ with $\theta \in \Theta$. We denote by $\varphi_s : k \to U_s$ the corresponding isomorphism.

On the other hand there is the Iwasawa decomposition $\mathcal{G} = \coprod_{w \in W^{\mathfrak{a}}} U^-(\mathcal{K}) w \mathcal{B}$. Again one can strengthen this decomposition to obtain uniqueness in the decomposition. But this time the resulting subgroups are affine spaces of infinite dimension. These two decompositions are compared in [BD94].

Theorem 11.1 ([BD94]). *For $w \in \tilde{W}^{\mathfrak{a}}$ and $s \in S^{\mathfrak{a}}$ one has*

$$U^-(\mathcal{K})wU_s s\mathcal{B} = \begin{cases} U^-(\mathcal{K})ws\mathcal{B} & \text{if } w \prec ws \\ U^-(\mathcal{K})w\mathcal{B} \sqcup U^-(\mathcal{K})ws\mathcal{B} & \text{if } w \succ ws. \end{cases}$$

More precisely for $w \succ ws$:

$$w\varphi_s(x)s\mathcal{B} \in \begin{cases} U^-(\mathcal{K})ws\mathcal{B} & \text{if } x = 0 \\ U^-(\mathcal{K})w\mathcal{B} & \text{if } x \neq 0. \end{cases}$$

Now we can connect these geometric results to the combinatorics. Let $w \in W^{\mathfrak{a}}$ and let σ be a minimal gallery of type $t = (t_1, \dots, t_k)$ which connects A_f and A_w. Define a map $\eta : U_w \to \Gamma_t^+$ as follows: For $u \in U_w$ define

$$\eta(u) = (A_f, A_{w_1}, \dots, A_{w_k}) \text{ iff } ut_1 \cdot \ldots \cdot t_j \in U^-(\mathcal{K})w_j\mathcal{B} \text{ for all } j \in \{1, \dots, k\}.$$

It follows from the last theorem that η is well defined. For a positively folded gallery σ let $m(\sigma)$ be the total number of positive directions and $n(\sigma)$ the total number of positive folds. The connection of our combinatorics with geometry is given by

Theorem 11.2 ([BD94]). (i) *If $\sigma \in \Gamma_t^+$ then $\eta^{-1}(\sigma) \cong k^{m(\sigma)} \times (k^*)^{n(\sigma)}$.*

(ii) *If $v \in W^{\mathfrak{a}}$ then $\mathcal{B}w \cdot \mathcal{B} \cap U^-(\mathcal{K})v \cdot \mathcal{B} = \bigsqcup_{\sigma \in \Gamma_t^+(A_f, A_v)} \eta^{-1}(\sigma)w \cdot \mathcal{B}$.*

Remark 11.3. *In [BD94] the cited results are shown for any Kac–Moody group and any generalized system of positive roots. Theorem 11.1 is then formulated with distinguished subexpressions instead of positively folded galleries. It should be mentioned that they prove the above results just in the case that G is semisimple and simply connected since in this case $G(\mathcal{K})$ fits in the general theory of affine Kac–Moody groups. But the results one needs for the proof are valid for any reductive group.*

All the results above are more or less contained in [GL05] since one can lift their methods from the affine Grassmanian to the affine flag variety. Compare also [GL05, section 3] for a discussion of the reduction to the simply connected case.

11.1 Geometric interpretation of the L-polynomials

Now let $\mathbf{q}$ be any prime power and let $k = \mathbb{F}_{\mathbf{q}}$ be the finite field with $\mathbf{q}$ elements. So we have a special case (the split case) of the situation of the last section (with G there being $G_{\mathcal{K}}$ here). In this case $K = G(\mathcal{O})$ and the resulting Hecke algebra is the specialization at $\mathbf{q}$ of the abstract Hecke algebra of its root datum Φ with equal parameters. For any subset $M \subset \mathcal{G}$ denote by $M \cdot \mathcal{B}$ its image in the quotient $\mathcal{G}/\mathcal{B}$ and by $|M \cdot \mathcal{B}|$ the number of elements. From the last theorem we get

Corollary 11.4. *If $v, w \in \tilde{W}^{\mathfrak{a}}$ then $|\mathcal{B}w \cdot \mathcal{B} \cap U^-(\mathcal{K})v \cdot \mathcal{B}| = L_w(A_f, A_v)$.*

Remark 11.5. *Looking at positively folded galleries starting in $-A_f$ one can calculate the number of elements in the intersections $\mathcal{B}^- w \cdot \mathcal{B} \cap U^-(\mathcal{K})v \cdot \mathcal{B}$ in the same way as in the last corollary. Here $\mathcal{B}^-$ is obtained from the opposite Borel B^- of B in the same way as $\mathcal{B}$ from B.*

Now we want to give interpretations of the coefficients appearing in the expansion of $T_w X_\lambda \mathbf{1}_0$ for $w \in W^\lambda$ and $\lambda \in X_+^\vee$. Recall that Γ_v^+ is the set of all positively folded galleries starting in 0 of the type of a minimal gallery joining A_f and A_v for $v \in \tilde{W}^{\mathfrak{a}}$. We showed in 6.2 that

$$T_w X_\lambda \mathbf{1}_0 = \mathbf{q}^{l(w_0 w_\lambda)} \sum_{\mu \in X^\vee} \mathbf{q}^{-\langle \rho, \lambda+\mu \rangle} \sum_{\substack{\sigma \in \Gamma^+_{wn^\lambda}(\mu) \\ \iota(\sigma)=id}} L_\sigma X_\mu \mathbf{1}_0. \tag{11.1}$$

By corollary 11.4 we have

$$\sum_{\substack{\sigma \in \Gamma^+_{wn^\lambda}(\mu) \\ \iota(\sigma)=id}} L_\sigma = |\coprod_{v \in W} \mathcal{B} wn^\lambda \cdot \mathcal{B} \cap U^-(\mathcal{K}) \tau_\mu v \cdot \mathcal{B}|.$$

The last equation can be stated better considering intersections in the affine Grassmanian $\mathcal{G}/G(\mathcal{O})$. For $\nu, \mu \in X^\vee$ define $X_{\nu\mu} = \mathcal{B}\tau_\nu \cdot G(\mathcal{O}) \cap U^-(\mathcal{K})\tau_\mu \cdot G(\mathcal{O})$. The group $G(\mathcal{O})$ is the parabolic subgroup of $\mathcal{G}$ associated to the classical Weyl group $W \subset W^{\mathfrak{a}}$, i.e. $G(\mathcal{O}) = \bigsqcup_{w \in W} \mathcal{B} w \mathcal{B}$. Let $\pi : \mathcal{G}/\mathcal{B} \to \mathcal{G}/G(\mathcal{O})$ be the canonical projection. From general theory of Tits systems we know that

$$\pi_{|\mathcal{B}v \cdot \mathcal{B}} : \mathcal{B}v \cdot \mathcal{B} \to \mathcal{B}v \cdot G(\mathcal{O})$$

is an isomorphism iff v is of minimal length in vW and that $\mathcal{B}v \cdot G(\mathcal{O}) = \mathcal{B}vx \cdot G(\mathcal{O})$ for all $x \in W$. Moreover, we have

$$\pi^{-1}(U^-(\mathcal{K})\tau_\mu \cdot G(\mathcal{O})) = \coprod_{x \in W} U^-(\mathcal{K})\tau_\mu x \cdot \mathcal{B}.$$

Since $w \in W^\lambda$ we have that $v = wn^\lambda$ is minimal in vW. Moreover, $\tau_{w\lambda} \in vW$ and thus we get an isomorphism

$$\pi_{|\mathcal{B}wn^\lambda \cdot \mathcal{B} \cap \coprod_{x \in W} U^-(\mathcal{K})\tau_\mu x \cdot \mathcal{B}} : \mathcal{B}wn^\lambda \cdot \mathcal{B} \cap \coprod_{x \in W} U^-(\mathcal{K})\tau_\mu x \cdot \mathcal{B} \to X_{w\lambda,\mu}$$

and thus

$$|\coprod_{v \in W} \mathcal{B}wn^\lambda \cdot \mathcal{B} \cap U^-(\mathcal{K})\tau_\mu v \cdot \mathcal{B}| = |X_{w\lambda,\mu}|.$$

Combining corollary 11.4 and (11.1) this yields

$$T_w X_\lambda \mathbf{1}_0 = \mathbf{q}^{l(w_0 w_\lambda)} \sum_{\mu \in X^\vee} \mathbf{q}^{-\langle \rho, \lambda+\mu \rangle} |X_{w\lambda,\mu}| X_\mu \mathbf{1}_0.$$

To give a geometric interpretation to the polynomials $L_{\lambda\mu}$ we proceed as above. For $\lambda \in X_+^\vee$ and $\mu \in X^\vee$ define $Z_{\lambda\mu} = G(\mathcal{O})\tau_\lambda \cdot G(\mathcal{O}) \cap U^-(\mathcal{K})\tau_\mu \cdot G(\mathcal{O})$. Using remark 11.5 and $G(\mathcal{O}) = \coprod_{w\in W} \mathcal{B}^- w\mathcal{B}$ we get

$$\begin{aligned}
|Z_{\lambda\mu}| &= \sum_{\substack{w\in W^\lambda \\ v\in W}} |\mathcal{B}^- w_0 wn^\lambda \cdot \mathcal{B} \cap U^-(\mathcal{K})\tau_\mu v \cdot \mathcal{B}| \\
&= \sum_{\substack{w\in W^\lambda \\ v\in W}} L_{wn^\lambda}(-A_f, \tau_\mu v) = \sum_{w\in W^\lambda} L_{wn^\lambda}(-A_f, \mu) \\
&= \sum_{w_0 w\in W^\lambda} q^{l(w_0 w)} L_{n^\lambda}(A_{w_0 w}, \mu) = \sum_{\substack{\sigma\in\Gamma^+_{t^\lambda}(\mu) \\ w_0\iota(\sigma)\in W^\lambda}} q^{l(w_0\iota(\sigma))} L_\sigma = L_{\lambda\mu}.
\end{aligned}$$

The last equalities follow from the remarks 4.3 and 5.10 with $v = w_\lambda$. So in this geometric setting it is more natural to regard galleries $\sigma \in \Gamma^+_{t^\lambda}$ with $\iota(\sigma) \in W^\lambda w_\lambda = w_0 W^\lambda$ instead of $\iota(\sigma) \in W^\lambda$.

Here we also see the meaning of the correction factor $\frac{1}{W_\lambda(\mathbf{q})}$: The restriction of the projection π induces a map $G(\mathcal{O})\tau_\lambda \cdot \mathcal{B} \to G(\mathcal{O})\tau_\lambda \cdot G(\mathcal{O})$ with fibers isomorphic to P_λ/B where P_λ is the parabolic subgroup of G associated to λ. By the Bruhat decomposition for G we know that $|P_\lambda/B| = W_\lambda(\mathbf{q})$.

So we obtain as in [GL05]

Corollary 11.6. *For all $\lambda \in X_+^\vee$ and $\mu \in X^\vee$ we have*

$$|Z_{\lambda\mu}| = \sum_{\substack{\sigma\in\Gamma^+_{t^\lambda}(\mu) \\ \iota(\sigma)\in W^\lambda w_\lambda}} \mathbf{q}^{l(w_0\iota(\sigma))} L_\sigma.$$

Of course we would not need the results of this section to prove this corollary. By the very definition of the *geometric* Satake isomorphism given by integration over $U^-(\mathcal{K})$ one knows that the coefficients $L_{\lambda\mu}$ are given by the number of points in $Z_{\lambda\mu}$.

11.2 Geometric interpretation of $d^w_{\lambda\mu}$ and $k_{\lambda\mu}$

Now take $k = K = \mathbb{C}$ and identify varieties with their closed points. The affine flag variety $\mathcal{G}/\mathcal{B}$ and the affine Grassmanian $X := \mathcal{G}/G(\mathcal{O})$ can be interpreted as the set of closed points of an ind-variety defined over $\mathbb{C}$. All isomorphisms mentioned above then become isomorphisms of complex algebraic varieties. A filtration of X by finite dimensional projective varieties is given by the generalized Schubert varieties $X_\lambda = \overline{G(\mathcal{O})\tau_\lambda \cdot G(\mathcal{O})}$ for $\lambda \in X_+^\vee$. One knows that $\dim X_\lambda = 2\langle\rho, \lambda\rangle$. Using the above results we now can give some information on the dimension and the number of irreducible components of the intersections $X_{\lambda\mu}$ and $Z_{\lambda\mu}$ and relate this to Demazure multiplicities and Kostka numbers.

Let $\lambda \in X_+^\vee$ and $w \in W^\lambda$. Recall the formula (11.1) for the expansion of $T_w X_\lambda$. We associate the locally closed irreducible subvariety $X_\sigma = \pi(\eta^{-1}(\sigma)wn^\lambda \cdot \mathcal{B})$ of $X_{w\lambda,\mu}$ to a gallery $\sigma \in \Gamma^+_{wn^\lambda}(\mu)$. By theorem 11.2 we know that $\dim X_\sigma = \deg L_\sigma$. In particular, the dimension of $\dim X_{w\lambda,\mu}$ is given by the maximum of $\{\deg L_\sigma \mid \sigma \in \Gamma^+_{wn^\lambda}(\mu)\}$ and the irreducible components of maximal dimension are the closures of the X_σ such that $\deg L_\sigma$ is maximal. But by (6.4) we know that for $\sigma \in \Gamma^+_{wn^\lambda}(\mu)$ we have

$$\deg L_\sigma \leq \langle \rho, \lambda + \mu \rangle - l(w_0 w_\lambda) + l(w)$$

and the number of galleries where we have equality is the Demazure multiplicity $d^w_{\lambda\mu}$. Moreover, if $d^w_{\lambda\mu} = 0$ then $\Gamma^+_{t^\lambda}(\mu) = \emptyset$. Bringing this together yields a proof of the

Theorem 11.7. *For $\lambda \in X^\vee_+$, $w \in W^\lambda$ and $\mu \in X^\vee$ the dimension of the intersection $X_{w\lambda,\mu}$ is $\langle \rho, \lambda + \mu \rangle - l(w_0 w_\lambda) + l(w)$ and the Demazure multiplicity $d^w_{\lambda\mu}$ is the number of top dimensional irreducible components of $X_{w\lambda,\mu}$. These components are given by the closures $\overline{X}_\sigma$ for $\sigma \in \Gamma^+_{wn^\lambda}(\mu)$ such that $\deg L_\sigma$ is maximal.*

This theorem is some refinement of the geometric results in [GL05]. They associate to each $\sigma \in \Gamma^+_{t^\lambda}(\mu)$ a locally closed subset Y_σ of $Z_{\lambda\mu}$ and they show that the closures of the Y_σ for σ a LS-gallery are the irreducible components of this intersection. Here one does not have to emphasize the maximality of the dimension since these intersections are of pure dimension by [MV04].

In 6.2 we have seen that there is a bijection between the galleries σ such that $\deg L_\sigma$ is maximal (as in the last theorem) with LS-galleries σ' such that $\iota(\sigma') \leq w$. So one may ask for the connection between X_σ and $Y_{\sigma'}$ where σ' is the corresponding LS-gallery. By the construction of the bijection σ and σ' are almost the same, they differ only at the beginning. So it is enough to compare the contribution of the beginning part of σ with the contribution of the initial direction y of σ and thus we work in $G(\mathcal{O})/\mathcal{B}$ which we can identify with the flag variety G/B. Under this identification the contribution of y is given by $B^- y \cdot B$, an affine space of dimension $l(w_0 y)$. If we take $w \in W$ in corollary 11.4 then $\mathcal{B}w \cdot \mathcal{B} \subset G(\mathcal{O})/\mathcal{B}$ and identifies to $Bw \cdot B$ and the intersection $\mathcal{B}w \cdot \mathcal{B} \cap U^-(\mathcal{K})v \cdot \mathcal{B}$ for $v \in W$ corresponds to $Bw \cdot B \cap B^- v \cdot B$. This again is the statement that the L-polynomials for W are nothing else than Deodhar's R-polynomials which calculate the last intersection [Deo85]. Applying this to σ we get, that the contribution of its initial part is contained in $Bw \cdot B \cap B^- y \cdot B$. Moreover, it is open and dense there. So we have $X_\sigma \subset Y_{\sigma'}$. In the case $w = w_0$ we even get that X_σ is dense in $Y_{\sigma'}$.

Ion showed in [Ion05] a very similar result. Define $Y_{\nu\mu} := G(\mathcal{O})\tau_{-\nu} \cdot \mathcal{B} \cap U^-(\mathcal{K})\tau_{-\mu} \cdot \mathcal{B}$ for $\nu, \mu \in X^\vee$. He shows

Theorem 11.8 ([Ion05]). *For $\lambda \in X^\vee_+$, $w \in W^\lambda$ and $\mu \in X$ the dimension of the intersection $Y_{w\lambda,\mu}$ is $\langle \rho, \lambda - \mu \rangle + l(w) + l(w_\lambda)$ and the Demazure multiplicity $d^w_{\lambda\mu}$ is the number of top dimensional irreducible components of $Y_{w\lambda,\mu}$.*

The approach in [Ion05] is quite different from ours. We are again in the case of a finite residue field k with $\mathbf{q}$ elements. There he shows, using his results obtained in [Ion04], that the numbers $|Y_{w\lambda,\mu}|$ occur as coefficients of specialized non-symmetric Macdonald polynomials. He calculates the asymptotic behavior of $|Y_{w\lambda,\mu}|$ considered as a function in $\mathbf{q}$. Using the Lefschetz fixed point formula and results from Deligne's proof of the Weil conjectures he calculates the dimension and the number of irreducible components of maximal dimension. In contrast to our approach he does not get a description of the irreducible components (or some open part of them).

How does this compare to the theorem before? In the same way as above one can calculate $|Y_{w\lambda,\mu}|$ using galleries. For this observe first that $G(\mathcal{O})v \cdot \mathcal{B} = G(\mathcal{O})wv \cdot \mathcal{B}$ for any $v \in \widetilde{W}^{\mathfrak{a}}$ and $w \in W$. Since $\tau_{-w\lambda} = ww_\lambda w_0(wn^\lambda)^{-1}$ by lemma 2.6 we get

$$|Y_{w\lambda,\mu}| = \sum_{\substack{\sigma\in\Gamma^+_{(wn^\lambda)^{-1}}(-\mu)\\ \varepsilon(\sigma)=id}} \mathbf{q}^{l(w_0\iota(\sigma))}L_\sigma.$$

Now let $\sigma \in \Gamma^+_{wn^\lambda}(\mu)$ with $\iota(\sigma) = id$. Recall that these are the galleries describing the coefficients of $X_\mu \mathbf{1}_0$ in $T_w X_\lambda$. Translate it by $-\mu$ and reverse its direction. This yields a positively folded gallery σ' of weight $-\mu$ with initial direction $\varepsilon(\sigma)$. Moreover, the type of σ' is $(wn^\lambda)^{-1}$. We also have $L_{\sigma'} = q^{-2\langle\rho,\mu\rangle+l(\varepsilon(\sigma))}L_\sigma$ which can be seen as follows: The number of foldings is the same for σ and σ', so it is enough to look at the positive directions. But the positive directions of σ' are the negative directions of σ. So if we denote by m^+ and m^- the number of positive respectively negative directions of σ we get $L_{\sigma'} = q^{m^- - m^+}L_\sigma$. Applying (6.3) to σ we get

$$q^{m^+-m^-} = q^{2\langle\rho,wt(\sigma)\rangle - l(\varepsilon(\sigma))} = q^{2\langle\rho,\mu\rangle - l(\varepsilon(\sigma))}.$$

since $\iota(\sigma) = id$.

We get a bijection $\{\sigma \in \Gamma^+_{wn^\lambda}(\mu) \mid \iota(\sigma) = id\} \to \{\gamma \in \Gamma^+_{(wn^\lambda)^{-1}}(-\mu) \mid \varepsilon(\gamma) = id\}, \sigma \mapsto \sigma'$ such that

$$\sum_{\substack{\sigma\in\Gamma^+_{wn^\lambda}(\mu)\\ \iota(\sigma)=id}} L_\sigma = q^{2\langle\rho,\mu\rangle - l(w_0)} \sum_{\substack{\sigma\in\Gamma^+_{(wn^\lambda)^{-1}}(-\mu)\\ \varepsilon(\sigma)=id}} q^{l(w_0\iota(\sigma))}L_\sigma.$$

This yields (again for $k = \mathbb{F}_\mathbf{q}$) that $|X_{w\lambda,\mu}| = \mathbf{q}^{2\langle\rho,\mu\rangle - l(w_0)}|Y_{w\lambda,\mu}|$ and thus it follows from theorem 11.7 and (6.4) that $\mathbf{q}^{-\langle\rho,\lambda-\mu\rangle - l(w) - l(w_\lambda)}|Y_{w\lambda,\mu}| = d^w_{\lambda\mu} + R(\mathbf{q})$ where $R \in q^{-1}\mathcal{L}^-$. This is the main ingredient in the proof of the last theorem in [Ion05].

Now one may ask for the dimension and the number of irreducible components of maximal dimension of general intersections $\mathcal{B}v \cdot \mathcal{B} \cap U^-(\mathcal{K})w \cdot \mathcal{B}$. Using corollary 11.4 one can calculate these for given v and w by calculating all the corresponding galleries. But one is interested in formulas not involving galleries as in 11.7. Such a formula was asked for by Görtz, Haines, Kottwitz and Reuman [GHKR05] in the context of affine Deligne–Lusztig varieties. Unfortunately, our approach cannot yield such a formula. The difference of this general case to the ones consideres in 11.7 is as follows: There we had an expansion in the affine Hecke algebra which we could specialize in the nil affine Hecke algebra. This yields upper bounds on the degrees of the involved galleries (and works in general). But then we used the fact (following from representation theory) that a coefficient in $\widetilde{\mathcal{H}}^{\mathfrak{a}}$ is non-zero iff the corresponding coefficient in $\mathcal{H}^{\text{nil}}$ is non-zero to show that there exist galleries attaining the maximal degree. And the last argument fails in general.

11.3 Geometric interpretation of the alcove basis

Now return to the situation $k = \mathbb{F}_{\mathbf{q}}$. In this case it is well known that the affine Hecke algebra for specialized $\mathbf{q}$ can be interpreted as the algebra of $\mathcal{B}$-invariant functions with finite support on the affine flag variety with the convolution product (see [IM65]). Using the second part of theorem 11.1 one can interpret the $U^-(\mathcal{K})$-invariant functions as a module over the Hecke algebra which is a known to be a free $\mathcal{H}^{\mathfrak{a}}$-module of rank one. This module can be identified explicitly with $\mathcal{H}^{\mathfrak{a}}$ using the alcove basis.

Let $F = \{f : X \to \mathbb{Z}\}$ and define $H \subset F$ as the subset of $\mathcal{B}$-(left)-invariant functions with finite support. There is a natural action of H on F by right convolution. More precisely $(f * h)(x \cdot \mathcal{B}) = \sum_{y \in \mathcal{G}/\mathcal{B}} f(y)h(y^{-1}x)$ for all $f \in F$ and $h \in H$. Restricting this action yields an algebra structure on H. Then it is known that H is isomorphic to the affine Hecke algebra specialized at $\mathbf{q}$. Under this isomorphism the generator T_w corresponds to the characteristic function on $\mathcal{B}w \cdot \mathcal{B}$. Thus F gets a right $\mathcal{H}^{\mathfrak{a}}$-module.

Let $t_w \in F$ be the characteristic function on $U^-(\mathcal{K})w \cdot \mathcal{B}$ (which in general does not have finite support) and let $M \subset F$ be the subspace spanned by all t_w. Let $s \in S^{\mathfrak{a}}$. In the same way as one uses the Bruhat decomposition for calculating the structure constants of H one now can use the second part of theorem 11.1 to show that M is closed under the right $\mathcal{H}^{\mathfrak{a}}$-operation and that

$$t_w * T_s = \begin{cases} t_{ws} & \text{if } w \prec ws \\ \mathbf{q}t_{ws} + (\mathbf{q}-1)t_w & \text{if } w \succ ws. \end{cases}$$

So by lemma 5.2 the map

$$\begin{aligned} &M \to \mathcal{H}^{\mathfrak{a}} \\ &t_v \mapsto \mathbf{q}^{\langle \rho, wt(A_v)\rangle} X_{wt(A_v)} \overline{T}_{\delta(A_v)} = \mathbf{q}_{A_v}^{-2} X_{A_v} \end{aligned}$$

is an isomorphism of right $\mathcal{H}^{\mathfrak{a}}$-modules.

The realization of $\tilde{\mathcal{H}}^{\mathfrak{a}}$ by functions was lifted (via the 'faisceaux-fonctions' correspondence of Grothendieck) by Springer [Spr82] to a algebra-geometric realization using sheaves on $\mathcal{G}/\mathcal{B}$ over $\mathbb{C}$. The characteristic function T_w corresponds to the constant sheaf on the Bruhat cell $\mathcal{B}w \cdot \mathcal{B}$ which is a finite dimensional subvariety of $\mathcal{G}/\mathcal{B}$. A similar construction for M is not known.

12 Kostka–Foulkes polynomials

In this section we want to describe a first result concerning the positivity of the coefficients of Kostka–Foulkes polynomials. It supports a conjecture concerning these polynomials. This approach can be seen as a symmetrized version of the approach of Deodhar in [Deo90]. Consider the case of equal parameters.

Before we proceed, let us shortly describe the conjecture. The Kostka–Foulkes polynomials $K_{\lambda\mu}$ for $\lambda, \mu \in X_+^\vee$ are defined as the entries of the transition matrix from Hall–Littlewood polynomials to Schur polynomials, i.e. $s_\lambda = \sum_{\mu \in X_+^\vee} K_{\lambda\mu} P_\mu(q^{-1})$. So we have $K_{\lambda\mu}(1) = k_{\lambda\mu}$ since $P_\mu(1) = m_\mu$. It is known that the coefficients of the $K_{\lambda\mu}$ are nonnegative [Lus83]. Except for type A where there is the formula of Lascoux and Schützenberger, there is no combinatorial proof showing this positivity. Since $K_{\lambda\mu}(1) = k_{\lambda\mu}$ one is led to the following conjecture. Of course this can be stated using any combinatorial model for the highest weight representations, but in our case it is convenient to use galleries. For $\lambda \in X_+^\vee$ define LS_{t^λ} to be the set of LS-galleries in $\Gamma_{t^\lambda}^+$ where t^λ is the type of a minimal gallery from A_f to A_{n^λ} as in section 4.

Conjecture 12.1. *Let $\lambda \in X_+^\vee$. There exists a function $c : LS_{t^\lambda} \to \mathbb{Z}$ such that $K_{\lambda\mu} = \sum_{\sigma \in LS_{t^\lambda}(\mu)} q^{c(\sigma)}$ for any $\mu \in X_+^\vee$.*

In type A this conjecture is solved by Lascoux and Schützenberger using the *charge* statistics on semistandard Young tableaus (for a detailed proof see [NR03]). However, for general type this remains open. We do not solve this conjecture here, but we show in theorem 12.4 that certain sums of the $K_{\lambda\mu}$ are given as in the conjecture. However, we do not have any idea how to split this up.

To state the theorem we need some preparation. Let t be any type. As already mentioned before there is a crystal structure on Γ_t introduced in [GL05]. Denote by $HW_t \subset \Gamma_t$ the set of highest weight galleries with respect to this crystal structure. So for $\sigma \in HW_t$ we have that $\sum_\gamma x^{wt(\gamma)} = s_{wt(\sigma)}$ where the sum is over the irreducible component of Γ_t containing σ. For the statement of the theorem we need one more statistic on galleries introduced by Deodhar in [Deo90].

Definition 12.2. *For a gallery $\sigma = (A_0, \ldots, A_k)$ of type t the defect $d(\sigma)$ is defined as $d(\sigma) = \{j \mid t_j \in S^{\mathfrak{a}} \text{ and } A_j > A_j t_{j+1}\}$.*

Remark 12.3. *If σ is completely contained in the interior of the dominant chamber, then $d(\sigma)$ is the number of positive foldings plus the number of negative directions of σ. In [Deo90] the statistic is more generally defined on subexpressions of a reduced expression in a Coxeter group. But in our special case subexpressions are nothing else than galleries of a fixed type.*

For a gallery σ define $p(\sigma) = l(e(\sigma)) - l(n^{wt(\sigma)})$. In particular, $p(\sigma) = l(w_0\varepsilon(\sigma))$ if $wt(\sigma)$ is dominant and regular. We also need the dimension of a gallery as introduced in section 4. Recall that $\dim(\sigma) = l(w_0\iota(\sigma)) + \deg L_\sigma$.

Now we can state the theorem. The proof will be given later.

Theorem 12.4. *For $\lambda, \nu \in X_+^\vee$ we have*

$$\sum_{\sigma \in HW_{t^\lambda}} q^{-\langle \rho, wt(\sigma)\rangle + \dim(\sigma)} K_{wt(\sigma),\nu} = q^{\langle \rho, \nu \rangle} \sum_{\sigma \in \Gamma_{t^\lambda}(\nu)} q^{p(\sigma)+d(\sigma)}.$$

This shows that the conjecture is true after replacing the $K_{\lambda\mu}$ by an appropriate sum and LS-galleries by all galleries. Now the problem remains to split up this equation, i.e. to identify subsets of Γ_{t^λ} such that the corresponding summands sum up to a Schur polynomial.

One might hope, that for a highest weight gallery σ one has

$$q^{-\langle \rho, wt(\sigma)+\nu\rangle + \dim(\sigma)} K_{wt(\sigma),\nu} = \sum_{\gamma} q^{p(\sigma)+d(\sigma)} \tag{12.1}$$

where the sum on the right hand side is over all galleries in $\Gamma_{t^\lambda}(\nu)$ which are in the irreducible component of σ in Γ_{t^λ}. But this already fails in type A_1.

But the theorem above gives us a reduction of the conjecture to the following

Conjecture 12.5. *There is a second crystal structure on Γ_t such that* (12.1) *holds for this new crystal structure.*

As one can see already in type A_1 this new structure is by no means unique. This rephrases the statement that the function c asked for in the first conjecture is not unique, too.

We now prove theorem 12.4. Define $C_s := T_s + 1$ for $s \in S^{\mathfrak{a}}$. This is up to a power of q the Kahzdan–Lusztig element associated to s. For $g \in \Omega$ define $C_g = T_g$. Now let $t = (t_1, \ldots, t_k)$ be any type. Define $C_t = C_{t_1} \cdot \ldots \cdot C_{t_k}$. We calculate the Schur and Macdonald expansions of $\mathbf{1}_0 C_t \mathbf{1}_0$ using galleries. Setting $t = t^\lambda$ will prove the theorem.

Remark 12.6. *Even if $w = t_1 \cdot \ldots \cdot t_k$ is a reduced expression, C_t does not depend only on w but on t. Consider for example $t = (s_1, s_2, s_1)$ and $t' = (s_2, s_1, s_2)$ in a root system of type A_2, i.e. $W \cong S_3$. Then t and t' are both reduced expressions for w_0, but $C_t = \mathbf{1}_0 + T_{s_1}^2$ and $C_{t'} = \mathbf{1}_0 + T_{s_2}^2$. However, it is not yet clear, if the symmetrized version does depend only on w. In the example, $\mathbf{1}_0 C_t \mathbf{1}_0 = (W(q)^2 + q^2)\mathbf{1}_0 = \mathbf{1}_0 C_{t'} \mathbf{1}_0$.*

We first calculate the Schur expansion of $\mathbf{1}_0 C_t \mathbf{1}_0$ using the crystal structure on Γ_t. For a generalized alcove $A \in \tilde{\mathcal{A}}$ and $s \in S^{\mathfrak{a}}$ we have (using lemma 5.2)

$$X_A C_s = \begin{cases} qX_{As} + X_A & \text{if } A \prec As \\ X_{As} + qX_A & \text{if } A \succ As. \end{cases}$$

By induction on the length of t we obtain

$$X_A C_t = \sum_{\sigma \in \Gamma_{t,A}} q^{m(\sigma)+n(\sigma)} X_{e(\sigma)} \tag{12.2}$$

where $m(\sigma) = \sum_{s \in S^{\mathfrak{a}}} m_s(\sigma)$ is the number of positive directions and $n(\sigma) = \sum_{s \in S^{\mathfrak{a}}} n_s(\sigma)$ is the number of positive foldings. Thus we get as in the proof of lemma 5.9

$$\begin{aligned}
\mathbf{1}_0 C_t \mathbf{1}_0 &= q^{l(w_0)} \overline{\mathbf{1}_0} C_t \mathbf{1}_0 = q^{l(w_0)} \sum_{w \in W} q^{-l(w)} X_{A_w} C_t \mathbf{1}_0 \\
&= \sum_{w \in W} q^{l(w_0 w)} \sum_{\sigma \in \Gamma_t, \iota(\sigma) = w} q^{m(\sigma)+n(\sigma)} X_{e(\sigma)} \mathbf{1}_0 \\
&= \sum_{\sigma \in \Gamma_t} q^{-\langle \rho, wt(\sigma)\rangle + \dim(\sigma)} X_{wt(\sigma)} \mathbf{1}_0.
\end{aligned}$$

From theorem 9.1 we know that for any gallery σ and any $\alpha \in \Delta$ such that $f_\alpha(\sigma)$ is defined one has $\dim(f_\alpha(\sigma)) = \dim \sigma - 1$. Since we also have $wt(f_\alpha(\sigma)) = wt(\sigma) - \alpha^\vee$ we conclude that $-\langle \rho, wt(\sigma)\rangle + \dim(\sigma)$ is constant on the irreducible components of the crystal Γ_t. We thus get

$$\mathbf{1}_0 C_t \mathbf{1}_0 = \sum_{\sigma\in\Gamma_t} q^{-\langle\rho,wt(\sigma)\rangle+\dim(\sigma)} X_{wt(\sigma)}\mathbf{1}_0 = W(q)\sum_{\sigma\in HW_t} q^{-\langle\rho,wt(\sigma)\rangle+\dim(\sigma)} s_{wt(\sigma)}\mathbf{1}_0. \quad (12.3)$$

Now we calculate the expansion of $\mathbf{1}_0 C_t \mathbf{1}_0$ with respect to the M_μ. First observe that for any $v \in \tilde{W}^{\mathfrak{a}}$ we have

$$T_v C_i = \begin{cases} T_{vs_i} + T_v & \text{if } vs_i > v \\ q(T_{vs_i} + T_v) & \text{if } vs_i < v. \end{cases}$$

By induction on the length of t we get for $v \in \tilde{W}^{\mathfrak{a}}$ that

$$T_v C_t = \sum_{\substack{\sigma\in\Gamma_t \\ \sigma \text{ starting in } A_v}} q^{d(\sigma)} T_{e(\sigma)} = \sum_{x\in\tilde{W}^{\mathfrak{a}}} \Big(\sum_{\sigma\in\Gamma_t(v,x)} q^{d(\sigma)} \Big) T_x.$$

where by $\Gamma_t(v,x)$ we mean the set of all galleries of type t connecting A_v and A_x.

Remark 12.7. *For $v = id$ we have that $\sum_{\sigma\in\Gamma_t(id,x)} q^{d(\sigma)}$ is the number of points of the fiber over x of the Bott–Samelson variety associated to t as stated in [Deo90, proposition 3.9]. A detailed proof can be found in [Gau01].*

We calculate

$$\mathbf{1}_0 C_t \mathbf{1}_0 = \sum_{\sigma\in\Gamma_t} q^{d(\sigma)} T_{e(\sigma)}\mathbf{1}_0 = \sum_{\sigma\in\Gamma_t} q^{p(\sigma)+d(\sigma)} T_{n^{wt(\sigma)}}\mathbf{1}_0 = \sum_{\mu\in X^\vee} \Big(\sum_{\sigma\in\Gamma_t(\mu)} q^{p(\sigma)+d(\sigma)} \Big) T_{n^\mu}\mathbf{1}_0.$$

But on the other hand $\mathbf{1}_0 C_t \mathbf{1}_0 \in \mathcal{H}^{sph}$. So the coefficients of $T_{n^\mu}\mathbf{1}_0$ have to be W-invariant and we can rewrite the last equation as

$$\begin{aligned} \mathbf{1}_0 C_t \mathbf{1}_0 &= \sum_{\mu\in X^\vee_+} \frac{1}{W_\mu(q)} \Big(\sum_{\sigma\in\Gamma_t(\mu)} q^{p(\sigma)+d(\sigma)} \Big) \mathbf{1}_0 T_{n^\mu}\mathbf{1}_0 \\ &= W(q) \sum_{\mu\in X^\vee_+} \Big(\sum_{\sigma\in\Gamma_t(\mu)} q^{p(\sigma)+d(\sigma)} \Big) M_\mu. \end{aligned}$$

Using the Schur expansion (12.3) we get the following equation in Λ_q for any type t:

$$\sum_{\sigma\in HW_t} q^{-\langle\rho,wt(\sigma)\rangle+\dim(\sigma)} s_{wt(\sigma)} = \sum_{\sigma\in\Gamma_t, wt(\sigma) \text{ dominant}} q^{\langle\rho,wt(\sigma)\rangle+p(\sigma)+d(\sigma)} P_{wt(\sigma)}(q^{-1}). \quad (12.4)$$

Setting $t = t^\lambda$ this yields theorem 12.4 by the definition of the $K_{\mu\nu}$.

In [Deo90] Deodhar regards the following general situation. He compares the expansion of C_t in terms of Kahzdan–Lusztig polynomials with the expansion in terms of the standard basis using subexpressions. In contrast to our situation he does not have the combinatorial positivity result regarding the KL-expansion of C_t and one has no formula for this expansion.

Example 12.8. *In type A_1 the condition for a gallery to be a highest weight gallery is quite simple. A gallery σ is a highest weight gallery iff σ is contained in the dominant half line. In particular, it has to start in A_f.*

Let $\lambda \in X_+^\vee$ and define $k = \langle \rho, \lambda \rangle$. Denote by t^λ the type of a minimal gallery joining 0 and n^λ. One calculates that for any $\sigma \in \Gamma_{t^\lambda}$ we have $-\langle \rho, wt(\sigma) \rangle + \dim(\sigma) = k$. So (12.4) *in this case is*

$$\sum_{\sigma \in HW_{t^\lambda}} s_{wt(\sigma)} = \sum_{\sigma \in \Gamma_{t^\lambda}, wt(\sigma) \text{ dominant}} q^{-\langle \rho, \lambda - wt(\sigma) \rangle + p(\sigma) + d(\sigma)} P_{wt(\sigma)}(q^{-1}).$$

But we know $s_\mu = \sum_{0 \le \nu \le \mu} q^{-\langle \rho, \mu - \nu \rangle} P_\nu(q^{-1})$. Now in the last equation there is only one summand s_λ on the left hand side. The corresponding summands on the right hand side then must satisfy $p(\sigma) + d(\sigma) = 0$. But the galleries σ with $p(\sigma) + d(\sigma) = 0$ are precisely the following: Either σ starts in A_f, then σ has to have some consecutive foldings at the beginning and no foldings afterwards. Or σ starts in $-A_f$. Then it is non-folded. For each weight in $V(\lambda)$ there is exactly one such gallery ending there. So the summands corresponding to them on the right hand side are exactly those summing up to s_λ and this is the only possible choice. Even in this case we do not get a good combinatorial description of a possible new crystal structure in the sense of the last conjecture.

In [Dye88] Dyer showed a similar result (in the general setting mentioned above) for universal Coxeter systems.

There is some geometric content hidden in (12.4). For this assume $t = t^\lambda$ for $\lambda \in X_+^\vee$. It encodes two different ways to calculate the number of points (over $k = \mathbb{F}_\mathbf{q}$) in the fiber over $U^-(\mathcal{K})\nu \cdot G(\mathcal{O})$ of the Bott–Samelson resolution associated to t^λ. For explaining this let $\pi_{t^\lambda} : BS_t \to \mathcal{G}/G(\mathcal{O})$ the Bott–Samelson resolution, i.e. BS_{t^λ} is smooth, π_{t^λ} is $G(\mathcal{O})$-equivariant and the image is X_λ. For more details see [GL05].

There are two ways to calculate this fiber: Since π_{t^λ} is $G(\mathcal{O})$-invariant, one can calculate the fiber over all $\mu \in X_+^\vee$ and multiply it with $|Z_{\mu\nu}| = L_{\mu\nu}$. Then the sum over all μ of these products gives the whole fiber. As mentioned above Gaussent showed in [Gau01] that the number of points of the fiber $\pi_{t^\lambda}^{-1}(\mu)$ is given by $\sum_{\sigma \in \Gamma_{t^\lambda}(\mu)} q^{d(\sigma)+p(\sigma)}$. This way of calculating the fiber is contained in the right hand side of (12.4) since the sum there may be written as

$$\begin{aligned}
&\sum_{\mu \in X_+^\vee} \Big(\sum_{\sigma \in \Gamma_t(\mu)} q^{d(\sigma)+p(\sigma)} \Big) M_\mu = \sum_{\mu \in X_+^\vee} \Big(\sum_{\sigma \in \Gamma_{t^\lambda}(\mu)} \mathbf{q}^{d(\sigma)+p(\sigma)} \Big) \Big(\sum_{\nu \le \mu} \mathbf{q}^{-\langle \rho, \nu \rangle} L_{\mu\nu} Y_\nu \Big) \\
&= \sum_{\nu \in X_+^\vee} \mathbf{q}^{-\langle \rho, \nu \rangle} \Big(\sum_{\mu \ge \nu} L_{\mu\nu} \sum_{\sigma \in \Gamma_{t^\lambda}(\mu)} \mathbf{q}^{d(\sigma)+p(\sigma)} \Big) Y_\nu = \sum_{\nu \in X_+^\vee} \mathbf{q}^{-\langle \rho, \nu \rangle} \Big(\sum_{\mu \ge \nu} |Z_{\mu\nu}| \; |\pi_{t^\lambda}^{-1}(\mu)| \Big) Y_\nu \\
&= \sum_{\nu \in X_+^\vee} \mathbf{q}^{-\langle \rho, \nu \rangle} |\pi_{t^\lambda}^{-1}(U^-(\mathcal{K})\nu \cdot G(\mathcal{O}))| Y_\nu.
\end{aligned}$$

The second way to calculate the fiber is as follows: To each gallery $\sigma \in \Gamma_{t^\lambda}(\nu)$ one can associate an affine cell B_σ of dimension $\dim \sigma$ in the fiber over $\pi_t^{-1}(U^-(\mathcal{K})\nu \cdot G(\mathcal{O}))$ and the whole fiber is the disjoint union all these cells [GL05]. So the left hand in (12.4) can be rewritten as

$$\begin{aligned}
&\sum_{\mu\in X_+^\vee} q^{-\langle\rho,\mu\rangle} \sum_{\sigma\in HW_t(\mu)} q^{\dim(\sigma)} s_\mu \mathbf{1}_0 = \sum_{\mu\in X_+^\vee} \mathbf{q}^{-\langle\rho,\mu\rangle}\Big(\sum_{\sigma\in HW_{t^\lambda}(\mu)} \mathbf{q}^{\dim(\sigma)}\Big)\Big(\sum_{\nu\leq\mu} k_{\mu\nu} Y_\nu\Big)\\
&= \sum_{\nu\in X_+^\vee} \mathbf{q}^{-\langle\rho,\nu\rangle}\Big(\sum_{\mu\geq\nu}\sum_{\sigma\in HW_{t^\lambda}(\mu)} q^{\dim(\sigma)+\langle\rho,\nu-\mu\rangle} k_{\mu\nu}\Big)Y_\nu = \sum_{\nu\in X_+^\vee} \mathbf{q}^{-\langle\rho,\nu\rangle}\Big(\sum_{\sigma\in\Gamma_{t^\lambda}(\nu)} q^{\dim(\sigma)}\Big)Y_\nu\\
&= \sum_{\nu\in X_+^\vee} \mathbf{q}^{-\langle\rho,\nu\rangle}\Big(\sum_{\sigma\in\Gamma_{t^\lambda}(\nu)} |B_\sigma|\Big)Y_\nu = \sum_{\nu\in X_+^\vee} \mathbf{q}^{-\langle\rho,\nu\rangle} |\pi_{t^\lambda}^{-1}(U^-(\mathcal{K})\nu\cdot G(\mathcal{O}))| Y_\nu.
\end{aligned}$$

This geometric interpretation seems quite interesting. It leads to he geometric counterpart of the problem of the correct distribution of the summands above: The left hand side in theorem 12.4 encodes the stalks of the push-forward of the constant sheaf on BS_{t^λ}. By the decomposition theorem it decomposes as a direct sum of intersection cohomology complexes $\mathcal{IC}_\mu$ on $X_\mu \subset X_\lambda$ for $\mu \leq \lambda$ whose stalks are given by Kostka–Foulkes polynomials. Instead of looking at the stalks one can look at the cohomology of the fiber. This has a natural basis indexed by galleries by [Gau01] coming from a Bialynicki–Birula decomposition. Now the question is, if this basis is compatible with the decomposition in intersection cohomology complexes and how it splits.

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Teilpublikation

[Sch05] C. Schwer. Galleries, Hall-Littlewood polynomials and structure constants of the spherical Hecke algebra. *arXiv:math.CO/0506287*, 2005.