Exact solutions in Einstein's theory and beyond

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Zusammenfassung

Diese Arbeit befasst sich mit exakten Lösungen in „geometrischen" Theorien der Gravitation von der Art der Allgemeinen Relativitätstheorie. Am Beginn des ersten Kapitels geben wir eine kurze Einführung in den Kalkül der äußeren Differentialformen. Daran anschließend untersuchen wir die *Cotton 2-Form*, die zu konformen Eigenschaften der Raumzeit in Beziehung steht und von besonderer Bedeutung für Modelle der Gravitation in $1+2$ Dimensionen ist. Wir definieren die Cotton 2-Form mittels der zweiten Bianchi-Identität, führen eine irreduzible Zerlegung durch und bestimmen so die Anzahl der *unabhängigen Komponenten*. In drei Dimensionen leiten wir die Cotton 2-Form aus einem Variationsprinzip ab. Dann klassifizieren wir die Cotton 2-Form im 3-dimensionalen Riemann'schen Raum und geben entsprechende Beispiele an. Danach konstruieren wir mit Hilfe der Cotton 2-Form kovariant erhaltene geometrische Größen und untersuchen die entsprechenden Materie-Ströme. In diesem Rahmen führen wir die *Bach 3-Form* ein und setzen sie in Beziehung zur *Chevreton 3-Form*, einem Superenergietensor, der gegenwärtig in der Literatur diskutiert wird. Abschließend untersuchen wir die Eigenschaften der Cotton 2-Form im metrisch-affinen Raum.

Das zweite Kapitel ist der Gravitation in $1 + 2$ Dimensionen gewidmet. Wir beschäftigen uns mit der *Einstein-Cartan-Chern-Simons* (ECCS)-Theorie von Mielke und Baekler und finden eine "BTZ-artige" Lösung mit konstanter axialer Torsion. Wir bestimmen die Autoparallelen und Extremalen, die Killingvektoren und die globalen Ladungen. Daran anschließend leiten wir den Teleparallelismus, die Einstein-Cartan-Theorie und das Modell der topologisch-massiven Graviation aus der ECCS-Theorie ab. Wir zeigen, wie sich die BTZ-Lösung mit Torsion zu Lösungen der vorgenannten Spezialfälle reduzieren lässt. Abschließend leiten wir eine neue, konformflache Lösung der $1 + 2$-dimensionalen Einstein'schen Gleichung für perfekte Flüssigkeiten mit Hilfe in Kapitel 1 gefundener Techniken ab.

Im letzten Kapitel wenden wir uns der $1 + 3$-dimensionalen Metrisch-Affinen-Graviationstheorie (MAG) zu. Das Hauptziel besteht in der Entwicklung eines einfachen Modelles, in dem die Lorentz-Invarianz durch ein Vektorfeld verletzt wird. Dazu verwenden wir einen bestimmten Anteil der Nichtmetrizität. Wir stellen einen Lagrangian auf und wählen die Kopplungsparameter derart, dass sich die Feldgleichungen auf einen quasi-einsteinschen Anteil und eine Wellengleichung für das die Lorentz-Invarianz verletzende Vektorfeld reduzieren. Abschließend diskutieren wir eine einfache, von Baekler vorgeschlagene Lösung dieses Modells.

Abstract

In this thesis we present exact solutions of geometrical theories of gravity, i. e. those of a general relativistic type. In the first chapter, we give a short introduction into the calculus of exterior differential forms. Subsequently we investigate the *Cotton 2-form* which is related to the conformal properties of spacetime and plays an important role in three dimensional models of gravity. We derive the Cotton 2-form for arbitrary dimension by means of the second Bianchi identity. We perform an *irreducible decomposition* and determine the number of *independent components*. In three dimensions we derive it from a variational principle. We review its conformal properties in Riemannian spacetime. Then we perform a *classification* of the Cotton 2-form in three dimensional Riemannian spacetime and give examples for all classes. After that we construct conserved geometrical quantities from the Cotton 2-form and investigate the corresponding material currents. In this course we derive the *Bach 3-form* and relate it to the *Chevreton 3-form*, a superenergy tensor for the electromagnetic field recently discussed in the literature. We conclude with discussing the properties of the Cotton 2-form in metric-affine space.

The second chapter is devoted to gravity in three dimensions. We investigate the *Einstein-Cartan-Chern-Simons* (ECCS) theory of Mielke and Baekler and find a *"BTZ-like"* solution *with* constant axial *torsion*. We determine autoparallels and extremals, Killing vectors and global charges of this solution. Subsequently, we derive teleparallelism, Einstein-Cartan theory and topologically massive gravity from the more general framework of the ECCS theory. We show how the BTZ-solution with torsion reduces to solutions of the aforementioned subcases. In conclusion, we construct a new conformally flat perfect fluid solution of Einstein's field equation in three dimensions by using technics developed in chapter 1.

In the last chapter we turn to *four-dimensional metric-affine gravity*. The main goal is to devise a simple model which allows for the breaking of Lorentz invariance by means of a vector-like quantity. Therefore we take a vector-piece of the nonmetricity. Then we set up a Lagrangian and derive the field equations. By means of analyzing the field equations we find constraints on the coupling parameters which simplify the field equations considerably. Thereby we arrive at the desired simple model of Einstein's gravitational theory extended by a vector-like, *Lorentz violating* field. Finally, we discuss a simple solution of this model by Baekler.

Introduction: Einstein's theory of gravity and beyond

Why go beyond Einstein's theory of gravity? Today, Einstein's theory of gravity, nearly hundred years after it was proposed, is an experimentally well established framework for describing gravitational phenomena. Even in every day life it plays an important role; without general-relativistic corrections, GPS[1]-based navigation would mislead us by several kilometres per day from the desired destination. Indeed, in the vicinity of our earth general relativity predicts gravitational physics with astonishing accuracy. The gravitational redshift, time delay in the gravitational field, relativistic corrections of satellite orbits—all have been confirmed. Also in our solar system the relativistic corrections to the planetary orbits, time delays of radar pulses traveling between the earth and its neighboring planets and the bending of the light from distant stars in the gravitational field of the sun fit well into general relativity. Gravitational lensing of distant galaxies gives evidence for the validity of general relativity far away from us. The slowing down of the rotation period of the Hulse-Taylor pulsar is in good agreement with calculations assuming the genuinely general-relativistic effect of gravitational radiation. As important as these direct applications of general relativity are, the tests of its fundamental assumptions is encoded in the local validity of special relativity. Numerous experiments justify directly the axioms and confirm the predictions of local Lorentz invariance. Moreover, also quantum field theory is hardly imaginable without the rigid Poincaré group, and therewith special relativity.

However, going beyond the solar system, first doubts arise. An anomalous acceleration of the Pioneer spacecrafts 10 and 11 has been detected. Anomalous means not predictable by means of general relativity based celestial perturbation theory. Is there just a technical problem or new physics involved? Looking deeper into the cosmos the problems also become deeper ... In order to explain the rotation curves of galaxies and the expansion rate of the universe, we are forced to work with the hypothesis that most of the gravitating mass-energy is comprised of invisible and unknown forms of energy and matter. Up to the present, no direct experimental evidence for such forms of mass-energy could be found.

On the theoretical side, there is a deep conceptual conflict in unifying quantum

[1] *Global positioning system*: Sattelite-based positioning system

theory and in particular the standard model of elementary particle physics, and general relativity. Quantum field theory presupposes spacetime as background on which the fields propagate. In general relativity the gravitational field carries energy. According to quantum theory, this energy should be quantized, i. e. spacetime itself is suggested to be a quantum field. Does our world only consist of quantum fields whereas space and time, like macroscopical substances, appear only effectively due to quantum interaction? Or is spacetime an irreducible classical quality of nature? Despite many important advances the various quantum gravity theories until now have not provided a consistent and definite answer. This motivates to look beyond Einstein's theory of gravity.

It is important to understand the fundamental interaction between matter fields and spacetime (gravity). This is the reason for looking into a gauge theory of gravity. We start from matter currents in flat spacetime and then ask which degrees of freedom of spacetime should be coupled to it. By Noether's theorem we know that energy-momentum is related to translations in spacetime, angular momentum (and spin) is related to rotations. The matter Lagrangian is invariant under rigid spacetime translations and rotations, i. e., the rigid Poincaré group. The next step is, in the spirit of Einstein's equivalence principle, to soften the rigid invariance to a local one and thereby introducing the potential of translations, the coframe ϑ^α and the potential of rotations, the antisymmetric connection $\Gamma_{\alpha\beta} = -\Gamma_{\beta\alpha}$. In order to make these degrees of freedom propagating, we have to introduce the corresponding field strengths, the torsion $T^\alpha = D\vartheta^\alpha$, and the Riemann-Cartan curvature $R_{\alpha\beta} =$ "$D\Gamma_{\alpha\beta}$" and devise a dynamical Lagrangian. In this way we arrive at the Poincaré gauge theory. Experimental evidence suggests even a step beyond this structure. The Regge-trajectories in elementary particle physics give rise to the hypothesis of the existence of shear-currents. The corresponding shear potential turns out to be the nonmetricity $Q_{\alpha\beta} = -Dg_{\alpha\beta}$. We arrive at metric-affine gravity with independent variables coframe ϑ^α, connection $\Gamma_\alpha{}^\beta$ and metric $g_{\alpha\beta}$.

The interplay between the various geometrical quantities coframe, connection, metric, torsion, curvature and nonmetricity is mathematically very complex. This thesis is devoted to the analysis of the geometrical structures in order to construct exact solutions for metric affine gravity and various of its subcases, such as Poincaré gauge theory, teleparallelism, Einstein- and Einstein-Cartan theory, and topologically massive gravity. We thereby hope to contribute to the question of which generalization of Einstein's theory of gravity is most promising.

Chapter 1

Geometry of n-dimensional (post-)Riemannian spacetime

1.1 Introduction

We first give a short overview of differential geometry with exterior differential forms. Subsequently, we turn to the *Cotton 2-form* which is of special importance in three dimensional models of gravity.

The non–linear coupling of gravity to matter in general relativity presents difficult technical problems in attempts to understand the gravitational interaction of elementary particles and strings or to investigate details of the gravitational collapse. Progress in the former area has come mainly from treating quantum fields as propagating on fixed background geometries [98], whereas much of the progress in the latter has come from detailed numerical work [31, 1, 51]. Exact solutions of the relevant matter–gravity equations can play an important role by shedding light on questions of interest in both general relativity and string theory. One is often interested in certain classes of solutions with specified asymptotic properties, the most common of them are the asymptotically flat spacetimes. Recent work in string theory has, via the AdS/CFT conjecture, highlighted the importance of the asymptotically anti–de Sitter spacetimes [80]. The AdS/CFT correspondence relates a quantum field theory in d dimensions to a theory in $d+1$ dimensions that includes gravity [50, 117]. This is the motivation for looking at the *conformally flat* spaces and at the spaces of *constant curvature*. For this reason we decided to review the subject and to collect some old and new results that are nowadays important in the context of anti–de Sitter spacetimes and to present them in a modern language. These results seem presently not to be too well known in the community.

In the theory of conformal spaces the main geometrical objects to be analyzed are the Weyl [114] and the Cotton [33] tensors. It is well known that for conformally flat spaces the Weyl tensor has to vanish. Then the Cotton tensor vanishes, too. However, the Cotton tensor is only conformally invariant in three dimensions.

Recently, the study of three-dimensional spaces is becoming of great interest; for these spaces the Weyl tensor is always zero and the vanishing of the Cotton tensor depends on the type of relation between the Ricci tensor and the energy–momentum tensor of matter. Any three-dimensional space is conformally flat if the Cotton tensor vanishes. In Einstein's theory of gravity, the Ricci tensor is related to the energy–momentum tensor of matter by means of the Einstein equation. Then the vanishing of the Cotton tensor imposes severe restrictions on the energy–momentum tensor. The Cotton tensor also plays a role in the context of the Hamiltonian formulation of general relativity, see [4].

First we derive the Cotton 2-form in the context of the Bianchi identities. Subsequently we describe its characteristic properties and perform an irreducible decomposition with respect to the (pseudo-)orthogonal group. This allows us to determine the number of irreducible components in any dimension. Moreover, in four dimensions, we relate the Cotton to the Bach tensor. After that we show how to derive the Cotton 2-form in 3 dimensions by means of a variational procedure. We classify the Cotton 2-form in 3 dimensions by means of its eigenvalues and give examples for all classes. Eventually we discuss the role of the Cotton 2-form in the context of building covariantly conserved geometric quantities which may serve as gravitational counterparts to conserved matter currents. We will encounter the Bach 3-form and a certain super energy tensor of the electromagnetic field, the so-called Chevreton tensor.

1.2 Exterior calculus: Notation and Conventions

1.2.1 Coframe and frame, Levi-Civita density

We start from a given n-dimensional differentiable manifold M_n. At each point P of M_n there is the n-dimensional tangent vector space $T_P(M)$. In such a tangent space we can introduce a vector basis, the so-called *frame* $\{e_\alpha\}_{\alpha=0,\dots,n-1}$. Since we have a differentiable manifold, we are always supplied with a local coordinate system. The frame can then be expanded in terms of the local coordinate basis ∂_i according to

$$e_\alpha = e^i{}_\alpha \, \partial_i \, . \tag{1.1}$$

If there exists a coordinate system such that the (non-degenerated) coefficients $e^i{}_\alpha$ obey $e^i{}_\alpha = \delta^i_\alpha$ we call the frame *natural* or *holonomic*. We will give another criterion a few lines below.

In the cotangent space $T^*_P(M)$ there exists a local one-form basis or *coframe* which can be also expanded in terms of the local coordinate cobasis according to

$$\vartheta^\beta = e_j{}^\beta \, dx^j \, . \tag{1.2}$$

For every coframe there is a frame which is dual to that particular coframe,

$$e_\alpha \rfloor \vartheta^\beta = e^i{}_\alpha \, \partial_i \rfloor (e_j{}^\beta \, dx^j) = e^i{}_\alpha \, e_i{}^\beta = \delta^\beta_\alpha \, . \tag{1.3}$$

Again, the coframe is called *holonomic* or *natural* if there is a coordinate system such that $e_i{}^\alpha = \delta^\alpha_i$. This can be achieved if the coframe is integrable, i.e.

$$0 = \Omega^\gamma := d\vartheta^\gamma = \frac{1}{2} \, \Omega_{\alpha\beta}{}^\gamma \, \vartheta^\alpha \wedge \vartheta^\beta \, , \tag{1.4}$$

where we introduced the *object of anholonomity* Ω^γ. Its components read

$$\Omega_{\alpha\beta}{}^\gamma = e_\beta \rfloor e_\alpha \rfloor \Omega^\gamma = 2 \, e^i{}_\alpha \, e^j{}_\beta \, \partial_{[i} \, e_{j]}{}^\gamma \, . \tag{1.5}$$

In the case of a non-vanishing object of anholonomity we call coframe and frame *anholonomic*. From now on we obey the following conventions:

$$\begin{aligned} \alpha,\, \beta,\, \dots &= 0,\, 1,\, 2,\, \dots,\, (n-1) \quad \text{are} \quad \text{anholonomic or frame indices,} \\ i,\, j,\, \dots &= 0,\, 1,\, 2 \dots (n-1) \quad \text{are} \quad \text{holonomic or coordinate indices.} \end{aligned}$$

Under a coordinate transformation $x^i \to x^{i'}$ the coordinate basis and cobasis, respectively, transform according to

$$\partial_{i'} = \frac{\partial x^i}{\partial x^{i'}} \, \partial_i \, , \quad dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} \, dx^i \, . \tag{1.6}$$

Thus, for a vector $v = v^i\,\partial_i$ or a 1-form $\omega = \omega_i\,dx^i$ to remain invariant under a coordinate transformation, their respective components have to transform like

$$v^{i'} = \frac{\partial x^{i'}}{\partial x^i}\,v^i\,, \quad \omega_{i'} = \frac{\partial x^i}{\partial x^{i'}}\,\omega_i\,. \tag{1.7}$$

The generalization to arbitrary p-forms and tensors is straightforward. The anholonomic indices are not affected by coordinate transformations but by local frame transformations. In general, the anholonomic indices are referring to a representation of a Lie-algebra. Taken as an example, a p-form with one upper index, ω^α represents $\boldsymbol{\omega}= \omega^\alpha\,e_\alpha$. Thus, we call ω^α a vector-valued p-form. Analogously we term ω_α as covector-valued p-form, and a p-form without index a scalar valued form.

A *tensor density of weight* w is defined as

$$T_{i'_1\dots i'_p}{}^{j'_1\dots j'_q} = (\mathrm{sgn}\ \mathrm{J})^P\ |J|^w\ \frac{\partial x^{i_1}}{x^{i'_1}}\cdots\frac{\partial x^{i_p}}{x^{i'_p}}\,\frac{\partial x^{j'_1}}{x^{j_1}}\cdots\frac{\partial x^{j'_q}}{x^{j_q}}\,T_{i_1\dots i_p}{}^{j_1\dots j_q}\,, \tag{1.8}$$

where $J := \det\left(\frac{\partial x^i}{\partial x^{i'}}\right)$ is the Jacobi determinant and P takes the values 0 or 1. In the latter case we speak of a *pseudo*-density, which switches sign if the orientation of the coordinate system is reversed.

We introduce the *contravariant Levi-Civita density* independent of all other structures of the manifold, apart from its dimension, by putting (here $n = 4$),

$$\epsilon^{ijkl} = \begin{cases} 1 & \text{if } \{ijkl\} \text{ even permutation of } \{0,1,2,3\} \\ 0 & \text{if } \{ijkl\} \text{ no permutation of } \{0,1,2,3\} \\ -1 & \text{if } \{ijkl\} \text{ odd permutation } \{0,1,2,3\} \end{cases}\ . \tag{1.9}$$

The weight turns out to be $+1$. Via the relation

$$\hat{\epsilon}_{ijkl}\,\epsilon^{abcd} := \delta^{abcd}_{ijkl} := \begin{cases} 1 & \text{if } \{ijkl\} \text{ even permutation of} \{a,b,c,d\} \\ 0 & \text{if } \{ijkl\} \text{ no permutation of } \{a,b,c,d\} \\ -1 & \text{if } \{ijkl\} \text{ odd permutation of } \{a,b,c,d\} \end{cases}\ , \tag{1.10}$$

where δ^{abcd}_{ijkl} denotes the *generalized Kronecker symbol*, we define the *covariant Levi civita tensor density*, which is of weight -1. Thus,

$$\hat{\epsilon}_{ijkl} = \begin{cases} 1 & \text{if } \{ijkl\} \text{ even permutation of } \{0,1,2,3\} \\ 0 & \text{if } \{ijkl\} \text{ no permutation of } \{0,1,2,3\} \\ -1 & \text{if } \{ijkl\} \text{ odd permutation of } \{0,1,2,3\} \end{cases}\ . \tag{1.11}$$

For anholonomic indices we find completely analogous relations, apart from the fact that the determinant of the jacobian is replaced by the determinant of the local frame transformation. We then speak of a pseudo density of *anholonomic weight*, see A.1.1. in [58].

1.2.2 Connection, covariant derivative, and structure equations

In order to allow for a *covariant derivative* we introduce a *connection* 1-form by

$$\nabla_u e_\alpha = \Gamma_\alpha{}^\beta(u)\, e_\beta\,, \quad \Gamma_\alpha{}^\beta(\partial_i) = \Gamma_{i\alpha}{}^\beta\,, \quad \Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta\, dx^i\,. \tag{1.12}$$

It is comprised of n^3 independent components. The meaning of such a general connection and its role in defining parallel transport is explained in Schrödinger's classical text *Space-Time Structure* [104], e.g. Taken as an example, the exterior covariant derivative of a p-form $\omega_\alpha{}^\beta$ reads

$$D\omega_\alpha{}^\beta = d\omega_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta + \Gamma_\gamma{}^\beta \wedge \omega_\alpha{}^\gamma\,. \tag{1.13}$$

According to the expansion of the connection (1.12) we find for the components of the covariant derivative of a tensor of type (1,1), that is a 0-form $\Psi_\alpha{}^\beta$,

$$\nabla_\mu\, \Psi_\alpha{}^\beta = \partial_\mu\, \Psi_\alpha{}^\beta - \Gamma_{\mu\alpha}{}^\gamma\, \Psi_\gamma{}^\beta + \Gamma_{\mu\gamma}{}^\beta\, \Psi_\alpha{}^\gamma\,. \tag{1.14}$$

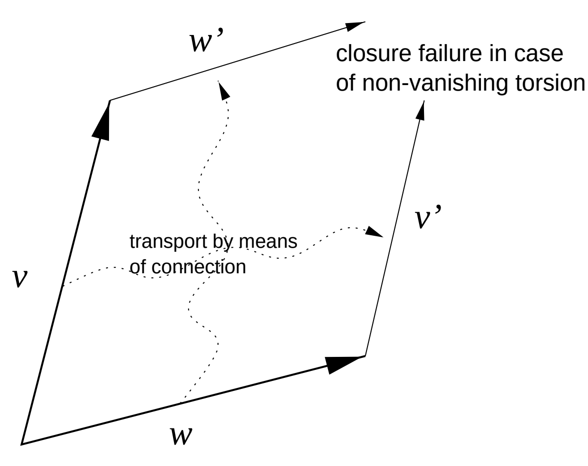
Subsequently, we define the *torsion*, a vector-valued two-form T^α by

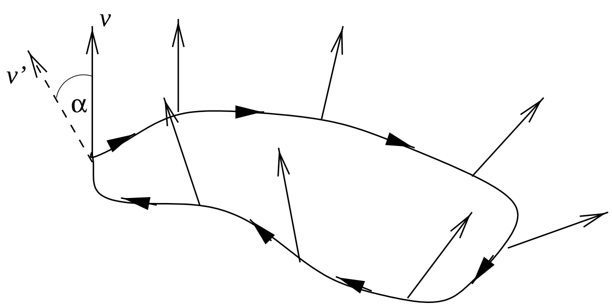
$$T^\alpha = \frac{1}{2}\, T_{\mu\nu}{}^\alpha\, \vartheta^\mu \wedge \vartheta^\nu := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta\,, \qquad 1^{\text{st}} \text{ structure eq.,} \tag{1.15}$$

and the *curvature*, an antisymmetric tensor-valued 2-form $R_\alpha{}^\beta$

$$R_\alpha{}^\beta = \frac{1}{2} R_{\mu\nu\alpha}{}^\beta\, \vartheta^\mu \wedge \vartheta^\nu := d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta\,, \qquad 2^{\text{nd}} \text{ structure eq.} \tag{1.16}$$

Both quantities have a geometrical interpretation. The torsion represents a closure failure of infinitesimal displacements: Let at a point P be given two vectors v and w. By means of the connection we perform a parallel displacement of the vector v along w, yielding v', and of w along v, yielding w', respectively. If there is torsion present a closure failure of the infinitesimal parallelogram will occur (see figure). To be more accurate, torsion measures the non commutativity of displacements of points, see [48].





In the presence of curvature a vector which is parallely transported around a closed loop in general will not return to its initial state but will be rotated by an angle α. Its length, however, remains the same. This rotation is represented by the antisymmetric curvature 2-form. In a spacetime with nonmetricity the curvature possesses also a symmetric piece. Then a vector is subjected to a general linear transformation and changes its length, moreover, the angle between two vectors will not be conserved under parallel transport. We will come back to this later.

The definitions of torsion, curvature and covariant exterior derivative are already sufficient to derive the *Bianchi identities*,

$$DT^\alpha = R_\beta{}^\alpha \wedge \vartheta^\beta \,, \qquad 1^{\text{st}} \text{ Bianchi}\,, \tag{1.17}$$
$$DR_\alpha{}^\beta = 0\,, \qquad 2^{\text{nd}} \text{ Bianchi}\,. \tag{1.18}$$

The first Bianchi identity follows from to the *Ricci identity*,

$$DD\omega_\alpha{}^\beta = -R_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta + R_\gamma{}^\beta \wedge \omega_\alpha{}^\gamma \,. \tag{1.19}$$

In order to prove the 2nd Bianchi identity we write the second structure equation in the form

$$d\Gamma_\alpha{}^\beta = \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta + R_\alpha{}^\beta \,. \tag{1.20}$$

Taking the exterior derivative of (1.20) yields

$$0 = dd\Gamma_\alpha{}^\beta = d\Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta - \Gamma_\alpha{}^\gamma \wedge d\Gamma_\gamma{}^\beta + dR_\alpha{}^\beta \,. \tag{1.21}$$

We can replace the derivatives of $\Gamma_\alpha{}^\beta$ by means of eq.(1.16), yielding

$$\begin{aligned} 0 &= d\Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta - \Gamma_\alpha{}^\gamma \wedge d\Gamma_\gamma{}^\beta + dR_\alpha{}^\beta \\ &= \left(\Gamma_\alpha{}^\delta \wedge \Gamma_\delta{}^\gamma + R_\alpha{}^\gamma\right) \wedge \Gamma_\gamma{}^\beta - \Gamma_\alpha{}^\gamma \wedge \left(\Gamma_\gamma{}^\delta \wedge \Gamma_\delta{}^\beta + R_\gamma{}^\beta\right) + dR_\alpha{}^\beta \,. \\ &= R_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta - \Gamma_\alpha{}^\gamma \wedge R_\gamma{}^\beta + dR_\alpha{}^\beta \,, \qquad (1.22) \\ &= DR_\alpha{}^\beta \,. \qquad (1.23) \end{aligned}$$

The Lie-derivative is a kind of derivative which is independent from any kind of connection or metric structure. We refer to [60] or [110] for the mathematical details.

The Lie-derivative for a scalar-valued p-form reads (Cartan's formula)

$$\mathcal{L}_u \omega = d(u \rfloor \omega) + u \rfloor d\omega \,. \tag{1.24}$$

The proof can be found in [8], e.g.

For a vector-valued p-form we have

$$\mathcal{L}_u\omega^\alpha = d(u\rfloor\omega^\alpha) + u\rfloor d\omega^\alpha - \omega^\beta\,(\partial_\beta u^\alpha + u^\gamma\,\Omega_{\gamma\beta}{}^\alpha)\,, \tag{1.25}$$

and for a covector-valued p-form

$$\mathcal{L}_u\omega_\alpha = d(u\rfloor\omega_\alpha) + u\rfloor d\omega_\alpha + \omega_\beta\,(\partial_\alpha u^\beta + u^\gamma\,\Omega_{\gamma\alpha}{}^\beta)\,. \tag{1.26}$$

The generalization to an arbitrary number of (mixed) indices is straightforward.

Sometimes it is desirable to replace the ordinary derivatives by covariant ones. Then, the object of anholonomity is replaced by the torsion,

$$\mathcal{L}_u\omega^\alpha = D(u\rfloor\omega^\alpha) + u\rfloor D\omega^\alpha - \omega^\beta\,(\nabla_\beta u^\alpha + u^\gamma\,T_{\gamma\beta}{}^\alpha)\ . \tag{1.27}$$

1.2.3 Metric, Hodge-, and Lie-duals

Now we introduce a geometric structure which allows us to represent length and angles, namely a Riemannian metric

$$g = g_{\alpha\beta}\,\vartheta^\alpha\otimes\vartheta^\beta\,,\quad g_{\alpha\beta} = g(e_\alpha, e_\beta)\,, \tag{1.28}$$

where $g_{\alpha\beta}$ is a symmetric, non-degenerated tensor field,

$$g_{\alpha\beta} = g_{\beta\alpha}\,,\quad \det g_{\alpha\beta} \neq 0\,. \tag{1.29}$$

Since $g_{\alpha\beta}$ is a symmetric, regular square matrix, there is a frame where it takes a diagonal form. The elements on the diagonal are the eigenvalues of g. According to *Sylvester's theorem of inertia*, the number $\mathrm{ind}(g)$ of negative eigenvalues is invariant under similarity transformations. By normalization of the basis vectors we arrive at the diagonal elements -1 or $+1$. A frame in which the components of the metric have this form is called *(pseudo) orthonormal* frame. The number of negative eigenvalues is called *index* such that

$$g_{\alpha\beta} = \mathrm{diag}(\underbrace{-1,-1,\ldots}_{\mathrm{ind}},\underbrace{1,1,\ldots}_{n-\mathrm{ind}})\,. \tag{1.30}$$

By means of the metric we can raise and lower indices in the usual way. It is quite easy to calculate that the determinant of the metric $g := \det(g_{ij})$ transforms like a tensor density of weight $+2$. This can be used to construct the *totally antisymmetric unit tensor* from the Levi-Civita tensor density:

$$\eta^{i_1\ldots i_n} := \frac{1}{\sqrt{|\det(g_{ij})|}}\,\epsilon^{i_1\ldots i_n} \tag{1.31}$$

Being a tensor we lower its indices by means of the metric

$$\eta_{i_1\ldots i_n} := g_{i_1j_1}\,\ldots\,g_{i_nj_n}\,\eta^{j_1\ldots j_n}\,. \tag{1.32}$$

Table 1.1: The η-basis

$$
\begin{array}{llllll}
\eta & := & {}^\star 1\,, & & & \text{basis of n-forms,} \\
\eta_{\alpha_1} & := & {}^\star\vartheta_{\alpha_1} & = & e_{\alpha_1} \rfloor \eta\,, & \text{basis of } (n-1)\text{-forms,} \\
\eta_{\alpha_1\alpha_2} & := & {}^\star(\vartheta_{\alpha_1}\wedge\vartheta_{\alpha_2}) & = & e_{\alpha_2} \rfloor \eta_{\alpha_1}\,, & \text{basis of } (n-2)\text{-forms,} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\eta_{\alpha_1\alpha_2\ldots\alpha_n} & := & {}^\star(\vartheta_{\alpha_1}\wedge\vartheta_{\alpha_2}\wedge\cdots\wedge\vartheta_{\alpha_n}) & = & e_{\alpha_n} \rfloor \eta_{\alpha_1\alpha_2\ldots\alpha_{(n-1)}}\,. &
\end{array}
\tag{1.36}
$$

By means of the Levi-Civita tensor density, we can define a duality operation which maps a contravariant tensor of degree p, $\Psi_{\alpha_1\ldots\alpha_p}$, into a covariant tensor density of degree $n-p$, namely $\Psi_{\alpha_1\ldots\alpha_p}\,\epsilon^{\alpha_1\ldots\alpha_p\,\beta_1\ldots\beta_{n-p}}$. In order to establish a proper duality operation between tensors/forms of degree p and degree $n-p$ we have 1) to raise and lower indices and 2) to get rid of the determinant of the Jacobian which arises from coordinate transformations. Both can obviously achieved by means of the η-tensor.

We define the *Hodge-duality operation* which maps a p-form Ψ into a $(n-p)$-form ${}^\star\Psi$

$$
{}^\star\Psi := \frac{1}{(n-p)!\,p!}\,\Psi_{\alpha_1\ldots\alpha_p}\,\eta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_{n-p}}\,\vartheta^{\beta_1}\wedge\,\ldots\,\wedge\vartheta^{\beta_{n-p}}\,. \tag{1.33}
$$

We give some important relations for the Hodge-dual in appendix A.1. As components we have

$$
{}^\star\Psi_{\beta_1\ldots\beta_{n-p}} = \frac{1}{p!}\,\Psi_{\alpha_1\ldots\alpha_p}\,\eta^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_{n-p}}\,. \tag{1.34}
$$

This suggests the following definition of the *Lie-dual* of a Lie-algebra valued p-form with q-indices

$$
\omega^{\star\,\alpha_1\ldots\alpha_{n-q}} := \frac{1}{q!}\,\omega^{\beta_1\ldots\beta_q}\,\eta_{\beta_1\ldots\beta_q}{}^{\alpha_1\ldots\alpha_{n-q}}\,. \tag{1.35}
$$

Supplied with a *metric* g and the corresponding Hodge-star duality operator *, we can define the *η-basis*[1], see table 1.1.

In general we assume independence of the metric and parallel transport, i. e. the connection. Then arises the *nonmetricity*

$$
Q_{\alpha\beta} := -Dg_{\alpha\beta} = -dg_{\alpha\beta} + 2\Gamma_{(\alpha\beta)}\,. \tag{1.37}
$$

Together with the definition of the torsion, $D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha\wedge\vartheta^\beta$, we then can give

[1] The *η-basis* seemingly was introduced by Trautman, see [110].

a decomposition of the connection according to

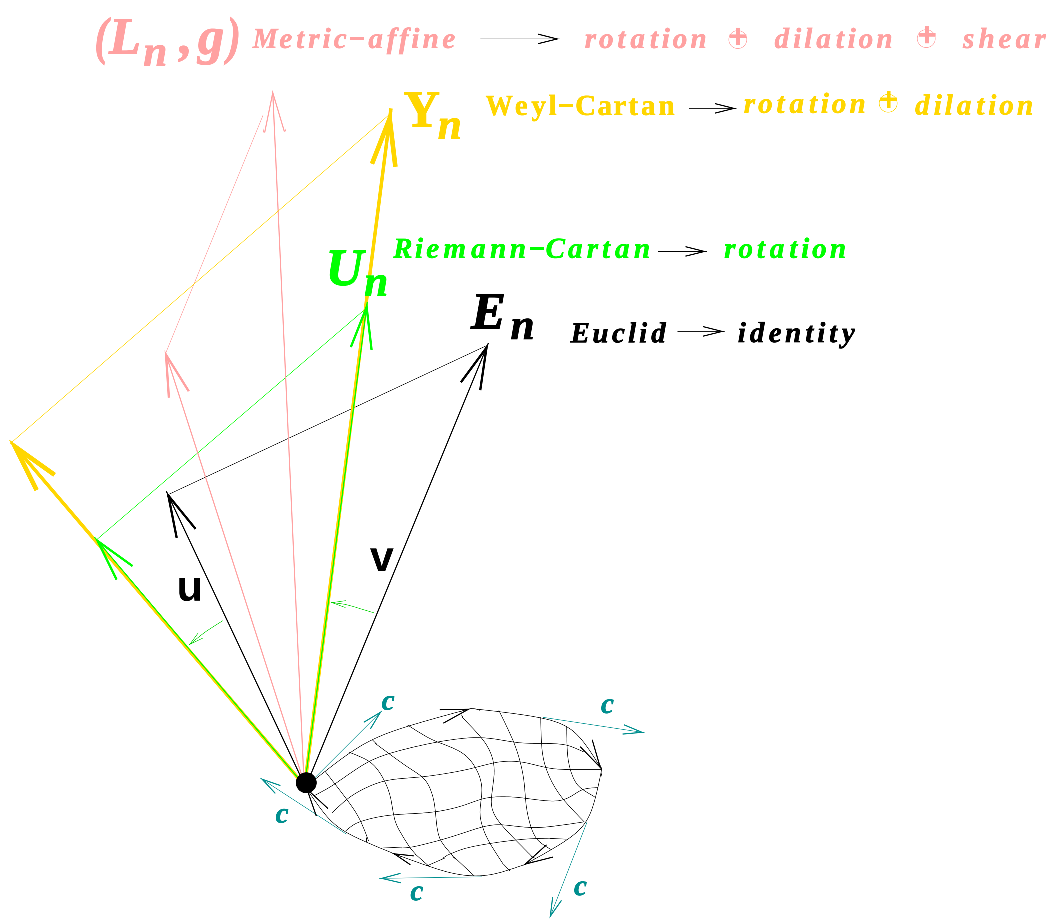
$$\begin{aligned}
\Gamma_{\alpha\beta} \;&=\; \frac{1}{2}\, dg_{\alpha\beta} + (e_{[\alpha}\rfloor dg_{\beta]\gamma})\, \vartheta^\gamma + e_{[\alpha}\rfloor \Omega_{\beta]} - \frac{1}{2}\,(e_\alpha\rfloor e_\beta\rfloor \Omega_\gamma)\,\vartheta^\gamma \\
&\quad - e_{[\alpha}\rfloor T_{\beta]} + \frac{1}{2}(e_\alpha\rfloor e_\beta\rfloor T_\gamma)\,\vartheta^\gamma \\
&\quad + \frac{1}{2}\, Q_{\alpha\beta} + (e_{[\alpha}\rfloor Q_{\beta]\gamma})\vartheta^\gamma\,. \qquad (1.38)
\end{aligned}$$

If nonmetricity is present, the curvature 2-form will no longer be antisymmetric,

$$DQ_{\alpha\beta} = -DDg_{\alpha\beta} = R_\alpha{}^\gamma\, g_{\gamma\beta} + R_\beta{}^\gamma\, g_{\gamma_\beta} = 2\, R_{(\alpha\beta)}\,. \qquad (1.39)$$

As a consequence, vectors, if parallely transported around a closed loop will not only be rotated but undergo a general linear transformation.

Figure 1.1: Take two vectors u and v and transport them in Euclidean, Riemann-Cartan, and Metric-affine space around a closed loop.



1.3 The Cotton 2-form in n-dimensional Riemannian space

1.3.1 Bianchi identities and the irreducible decomposition of the curvature

In a Riemannian space the torsion $T^\alpha = D\vartheta^\alpha$ vanishes. Thus, the first Bianchi identity reads

$$0 = DT^\alpha = DD\vartheta^\alpha = R_\beta{}^\alpha \wedge \vartheta^\beta\,, \tag{1.40}$$

or, in components,

$$R_{[\alpha\beta\gamma]}{}^\delta = 0\,. \tag{1.41}$$

The first Bianchi identity is a (co-)vector valued 3-form with

$$n \begin{pmatrix} n \\ 3 \end{pmatrix} = \frac{n^2(n-1)(n-2)}{3!} \tag{1.42}$$

independent components that imposes the same number of constraint equations on the components of the curvature. Accordingly, in n-dimensions, the curvature 2-form has

$$\begin{pmatrix} n \\ 2 \end{pmatrix} \begin{pmatrix} n \\ 2 \end{pmatrix} - n \begin{pmatrix} n \\ 3 \end{pmatrix} = \frac{n^2(n-1)(n+1)}{12} \tag{1.43}$$

independent components. For $n = 3$, we have 6 independent components and for $n = 4$ (the case of GR) 20 independent components.

The second Bianchi identity is

$$DR_\alpha{}^\beta = 0\,, \qquad \nabla_{[\lambda} R_{\mu\nu]\alpha}{}^\beta = 0\,. \tag{1.44}$$

We now perform the irreducible decomposition of the curvature with respect to the pseudo-orthogonal group [58]:

$$\begin{array}{llcl}
n = 1 & R_{\alpha\beta} & = & 0\,, \\
n = 2 & R_{\alpha\beta} & = & \mathrm{Scalar}_{\alpha\beta}\,, \\
n = 3 & R_{\alpha\beta} & = & \mathrm{Scalar}_{\alpha\beta} + \mathrm{Ri}\!\not{\mathrm{c}}\mathrm{i}_{\alpha\beta}\,, \\
n \geq 4 & R_{\alpha\beta} & = & \mathrm{Scalar}_{\alpha\beta} + \mathrm{Ri}\!\not{\mathrm{c}}\mathrm{i}_{\alpha\beta} + \mathrm{Weyl}_{\alpha\beta}\,.
\end{array} \tag{1.45}$$

- The $\mathrm{Scalar}_{\alpha\beta}$-piece is given by

$$\mathrm{Scalar}_{\alpha\beta} := -\frac{1}{n(n-1)}\, R\,\vartheta_\alpha \wedge \vartheta_\beta\,, \quad R := e_\alpha \rfloor \mathrm{Ric}^\alpha\,, \quad \mathrm{Ric}_\alpha := e_\beta \rfloor R_\alpha{}^\beta\,, \tag{1.46}$$

where R is the curvature scalar and Ric_α the Ricci 1-form. This piece has 1 independent component and is present in any dimension $n > 1$. In components we have

$$\mathrm{Ric}_\alpha = \mathrm{Ric}_{\mu\alpha}\,\vartheta^\mu\,, \quad \mathrm{Ric}_{\mu\alpha} = R_{\lambda\mu\alpha}{}^{\lambda}\,, \quad R = R_{\lambda\mu}{}^{\mu\lambda}\,, \tag{1.47}$$

and

$$\mathrm{Scalar}_{\mu\nu\alpha\beta} = -\frac{2}{n(n-1)}\,R\,g_{\mu[\alpha}g_{\beta]\nu}\,. \tag{1.48}$$

The Scalar piece enjoys the obvious symmetry

$$\mathrm{Scalar}_{\alpha\beta}\wedge\vartheta^\beta = 0\,, \qquad \mathrm{Scalar}_{[\mu\nu\alpha]\beta} = 0\,. \tag{1.49}$$

- From dimension three onwards the tracefree Ricci piece comes into play,

$$\mathrm{Ri}\not{\mathrm{c}}\mathrm{ci}_{\alpha\beta} := -\frac{2}{n-2}\,\vartheta_{[\alpha}\wedge\mathrm{Ri}\not{\mathrm{c}}_{\beta]}\,, \quad \mathrm{Ri}\not{\mathrm{c}}_\beta := \mathrm{Ric}_\beta - \frac{1}{n}\,R\,\vartheta_\beta\,. \tag{1.50}$$

It has $\frac{1}{2}(n+2)(n-1)$ independent components. In index notation this corresponds to

$$\mathrm{Ri}\not{\mathrm{c}}\mathrm{ci}_{\mu\nu\alpha\beta} = -\frac{4}{n-2}\,g_{[\mu|\,[\alpha}\,\mathrm{Ri}\not{\mathrm{c}}_{\beta]\,|\,\nu]}\,, \quad \mathrm{Ri}\not{\mathrm{c}}_{\alpha\beta} = \mathrm{Ric}_{\alpha\beta} - \frac{1}{n}\,R\,g_{\alpha\beta}\,. \tag{1.51}$$

If we contract the first Bianchi identity (1.40), we find

$$0 = e_\beta\rfloor(R_\alpha{}^\beta\wedge\vartheta^\alpha) = \mathrm{Ric}_\alpha\wedge\vartheta^\alpha\,, \tag{1.52}$$

since $R_\alpha{}^\alpha = 0$ in a Riemannian space. Thus, $\mathrm{Ric}_{\mu\alpha}\,\vartheta^\mu\wedge\vartheta^\alpha = 0$ or

$$\mathrm{Ric}_{\alpha\beta} = \mathrm{Ric}_{\beta\alpha}\,, \tag{1.53}$$

that is, the Ricci tensor is symmetric. This also implies

$$\mathrm{Ri}\not{\mathrm{c}}\mathrm{ci}_{\alpha\beta}\wedge\vartheta^\alpha = 0\,. \tag{1.54}$$

- Finally, in dimension greater than three, the Weyl 2-form emerges according to

$$\mathrm{Weyl}_{\alpha\beta} := R_{\alpha\beta} - \mathrm{Scalar}_{\alpha\beta} - \mathrm{Ri}\not{\mathrm{c}}\mathrm{ci}_{\alpha\beta}\,. \tag{1.55}$$

From the construction it is clear that the Weyl 2-form is totally traceless, i. e.,

$$e_\alpha\rfloor\mathrm{Weyl}^{\alpha\beta} = -e_\beta\rfloor\mathrm{Weyl}^{\alpha\beta} = 0\,, \quad e_\alpha\rfloor e_\beta\rfloor\mathrm{Weyl}^{\alpha\beta} = 0\,. \tag{1.56}$$

This property also explains the vanishing of the Weyl 2-form in 3 dimensions. An arbitrary antisymmetric tensor–valued 2-form $A_{\alpha\beta} = -A_{\beta\alpha} = A_{\mu\nu\alpha\beta}\,\vartheta^\mu\wedge\vartheta^\nu/2$ in 3 dimensions has 9 independent components. The condition $e_\alpha\rfloor A^{\alpha\beta} = 0$ results in 3 one-forms, i. e., 9 constraint equations that eventually yield the vanishing of all components.

According to [62], we can combine $\mathrm{Scalar}_{\alpha\beta}$ and $\mathrm{Ri}\cancel{\mathrm{cc}}\mathrm{i}_{\alpha\beta}$,

$$\mathrm{Scalar}_{\alpha\beta} + \mathrm{Ri}\cancel{\mathrm{cc}}\mathrm{i}_{\alpha\beta} = -\frac{2}{n-2}\,\vartheta_{[\alpha} \wedge L_{\beta]}\,, \tag{1.57}$$

with

$$L_\alpha := e_\beta \rfloor R_\alpha{}^\beta - \frac{1}{2(n-1)}\,R\,\vartheta_\alpha\,, \tag{1.58}$$

i.e., this sum can be expressed in a coherent way in terms of the 1-form L_α. From $\mathrm{Scalar}_{\alpha\beta}$ and $\mathrm{Ri}\cancel{\mathrm{cc}}\mathrm{i}_{\alpha\beta}$ it inherits the property

$$L_\alpha \wedge \vartheta^\alpha = 0\,. \tag{1.59}$$

We may expand L_α in components as

$$L_{\alpha\beta} = L_{\beta\alpha} = \mathrm{Ric}_{\alpha\beta} - \frac{1}{2(n-1)}\,R\,g_{\alpha\beta}\,. \tag{1.60}$$

This tensor is sometimes called *Schouten tensor*. Also the names *rho tensor* or $P_{\alpha\beta}$ can be found in the literature. Then the curvature 2-form can be expressed as

$$R_{\alpha\beta} = \mathrm{Weyl}_{\alpha\beta} - \frac{2}{n-2}\,\vartheta_{[\alpha} \wedge L_{\beta]} \tag{1.61}$$

or, in components,

$$R_{\alpha\beta\gamma\delta} = \mathrm{Weyl}_{\alpha\beta\gamma\delta} - \frac{4}{n-2}\,g_{[\alpha|[\gamma}L_{\delta]|\beta]}\,. \tag{1.62}$$

1.3.2 Cotton 2-form

By applying the exterior covariant derivative to (1.61), we obtain the following representation of the second Bianchi identity,

$$0 = DR_{\alpha\beta} = D\mathrm{Weyl}_{\alpha\beta} + \frac{2}{n-2}\vartheta_{[\alpha} \wedge C_{\beta]}\,, \tag{1.63}$$

where we encounter the *Cotton 2-form*

$$C_\alpha := DL_\alpha = \frac{1}{2}\,C_{\mu\nu\alpha}\,\vartheta^\mu \wedge \vartheta^\nu \tag{1.64}$$

or, in components,

$$C_{\alpha\beta\gamma} = 2\left(\nabla_{[\alpha}\mathrm{Ric}_{\beta]\gamma} - \frac{1}{2(n-1)}\,\nabla_{[\alpha}R g_{\beta]\gamma}\right)\,. \tag{1.65}$$

We perform an irreducible decomposition of the Cotton 2-form with respect to the Lorentz group. We can use the decomposition for the torsion, as given in [58], since this is also a vector-valued 2-form. Then we have

$$\begin{array}{rcccccc} C^\alpha & = & {}^{(1)}C^\alpha & + & {}^{(2)}C^\alpha & + & {}^{(3)}C^\alpha\,, \\ & = & \text{TENCOT} & + & \text{TRACOT} & + & \text{AXICOT}\,, \\ \tfrac{1}{2}\,n^2(n-1) & = & \tfrac{1}{3}\,n(n^2-4) & + & n & + & \tfrac{1}{6}\,n(n-1)(n-2)\,, \end{array} \tag{1.66}$$

where

$$ {}^{(2)}C^\alpha := \frac{1}{n-1}\,\vartheta^\alpha \wedge (e_\beta \rfloor C^\beta)\,, \tag{1.67}$$

$$ {}^{(3)}C^\alpha := \frac{1}{3}\,e^\alpha \rfloor (C_\beta \wedge \vartheta^\beta)\,, \tag{1.68}$$

$$ {}^{(1)}C^\alpha := C^\alpha - {}^{(2)}C^\alpha - {}^{(3)}C^\alpha\,, \tag{1.69}$$

or, in components,

$$ {}^{(2)}C_{\mu\nu}{}^\alpha = -\frac{2}{n-1}\,\delta^\alpha_{[\mu}\,C_{\nu]\beta}{}^\beta\,, \tag{1.70}$$

$$ {}^{(3)}C_{\mu\nu}{}^\alpha = \frac{1}{3!}\,C_{[\mu\nu\beta]}\,g^{\alpha\beta}\,, \tag{1.71}$$

$$ {}^{(1)}C_{\mu\nu}{}^\alpha = C_{\mu\nu}{}^\alpha - {}^{(2)}C_{\mu\nu}{}^\alpha - {}^{(3)}C_{\mu\nu}{}^\alpha\,. \tag{1.72}$$

TENCOT, TRACOT, and AXICOT are the computer algebra names of the pieces of the Cotton 2-form, denoting the tensor, the trace, and the axial pieces, respectively. The number of independent components of these pieces is given in the third line of (1.66). They arise as follows: TRACOT corresponds to a scalar-valued 1-form $C := e_\alpha \rfloor C^\alpha$ with n independent components. In general, a (co-)vector-valued 2-form has

$$ n\begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{n^2(n-1)}{2} \tag{1.73}$$

independent components. AXICOT corresponds to a scalar valued 3-form $(C^\alpha \wedge \vartheta_\alpha)$ and thus has

$$ \begin{pmatrix} n \\ 3 \end{pmatrix} = \frac{n(n-1)(n-2)}{6} \tag{1.74}$$

independent components. Thus, TENCOT is left with

$$ \frac{n^2(n-1)}{2} - \frac{n(n-1)(n-2)}{6} - n = \frac{n}{3}(n-2)(n+2) \tag{1.75}$$

independent components.

We now show that in a Riemannian space the trace piece (TRACOT) and the axial piece (AXICOT) vanish. Hence, only the tensor piece (TENCOT) with its

$n(n^2-4)/3$ independent components survives. This insight seems to be new. For $n=3$, we have 5 and for $n=4$ (the case of GR) 16 independent components.

In order to see the vanishing of AXICOT, we contract the Cotton 2-form with the coframe and use (1.59):

$$\vartheta^\alpha \wedge C_\alpha = \vartheta^\alpha \wedge DL_\alpha = -D(\vartheta^\alpha \wedge L_\alpha) = 0\,, \tag{1.76}$$

or

$$C_{[\mu\nu\alpha]} = \frac{2}{3!}\,\nabla_{[\mu}L_{\nu\alpha]} = 0\,. \tag{1.77}$$

The second Bianchi identity leads to a vanishing trace of the Cotton 2-form (TRACOT), $C = e_\alpha \rfloor C^\alpha = 0$. In order to see this, we contract (1.63) twice:

$$0 = e_\beta \rfloor DR^{\alpha\beta} = e_\beta \rfloor D\mathrm{Weyl}^{\alpha\beta} - \frac{n-3}{n-2}\,C^\alpha - \frac{1}{n-2}\vartheta^\alpha \wedge C\,, \tag{1.78}$$

$$0 = e_\alpha \rfloor e_\beta \rfloor DR^{\alpha\beta} = e_\alpha \rfloor e_\beta \rfloor D\mathrm{Weyl}^{\alpha\beta} - 2C = -2C\,, \tag{1.79}$$

or

$$C_{\alpha\beta}{}^\alpha = \nabla_\alpha \left(\mathrm{Ric}_\beta{}^\alpha - \frac{1}{2}\,R\,\delta^\alpha_\beta\right) = 0. \tag{1.80}$$

As we see, the second Bianchi identity relates the derivative of the Weyl 2-form to the Cotton 2-form,

$$e_\beta \rfloor D\mathrm{Weyl}_\alpha{}^\beta = \frac{n-3}{n-2}\,C_\alpha\,. \tag{1.81}$$

This formula allows us to rewrite the Einstein equation as a Maxwell-like equation for the Weyl tensor, see [14], e. g. The Ricci identity intertwines the derivative of the Cotton 2-form with the Weyl 2-form,

$$\begin{aligned} DC_\alpha &= DDL_\alpha = -R_\alpha{}^\beta \wedge L_\beta = -\mathrm{Weyl}_\alpha{}^\beta \wedge L_\beta + \frac{2}{n-2}\,\vartheta_{[\alpha} \wedge L_{\beta]} \wedge L^\beta \\ &= -\mathrm{Weyl}_\alpha{}^\beta \wedge L_\beta\,. \end{aligned} \tag{1.82}$$

Consequently, in three dimensions, C_α is a covariantly conserved 2-form, with $DC_\alpha = 0$. Thus it is a candidate for a conserved current that can be derived by means of a variational procedure.

The Weyl 2-form is antisymmetric and tracefree, i. e.

$$\mathrm{Weyl}_{\alpha\beta} = -\mathrm{Weyl}_{\beta\alpha}\,, \qquad e_\beta \rfloor \mathrm{Weyl}_\alpha{}^\beta = 0\,,$$

and the L_α 1-form fulfills

$$e_\alpha \rfloor L_\beta = e_\beta \rfloor L_\alpha\,.$$

Table 1.2: Properties of the Cotton 2-form C_α in arbitrary dimensions

$C_\alpha := DL_\alpha\,,\quad L_\alpha := e_\beta\rfloor R_\alpha{}^\beta - \frac{1}{2(n-1)}\,R\,\vartheta_\alpha$	Cotton 2-form
$\vartheta^\alpha\wedge C_\alpha = 0 \qquad$ (axialfree)	1st Bianchi identity
$e_\alpha\rfloor C^\alpha = 0 \qquad$ (tracefree)	contr. 2nd Bianchi ident.
$D\mathrm{Weyl}_{\alpha\beta} = -\frac{2}{n-2}\,\vartheta_{[\alpha}\wedge C_{\beta]}$	2nd Bianchi identity
$DC_\alpha = -\mathrm{Weyl}_\alpha{}^\beta\wedge L_\beta$	Ricci identity
$\widehat{C}_\alpha = C_\alpha + (n-2)\,\sigma_{,\beta}\,\mathrm{Weyl}_\alpha{}^\beta$	conformal transformation

Therefore the contraction of DC_α with the frame vanishes,

$$\begin{aligned} e^\alpha\rfloor DC_\alpha &= -e^\alpha\rfloor\left(\mathrm{Weyl}_\alpha{}^\beta\wedge L_\beta\right) = -(e^\alpha\rfloor\mathrm{Weyl}_\alpha{}^\beta)\wedge L_\beta - \mathrm{Weyl}_\alpha{}^\beta\wedge e^\alpha\rfloor L_\beta \\ &= 0\,. \end{aligned} \tag{1.83}$$

This means, in components, that the divergence with respect to the last index vanishes,

$$\nabla^\alpha\,C_{\mu\nu\alpha} = 0\,. \tag{1.84}$$

The properties of the Cotton tensor are summarized in Table 1.2.

1.3.3 The Cotton 2-form as a variational derivative

It is well known [40, 7] that C_α can be obtained by means of varying the 3-dimensional Chern-Simons action

$$C_{\mathrm{RR}} = -\frac{1}{2}\left(\Gamma_\alpha{}^\beta\wedge d\Gamma_\beta{}^\alpha - \frac{2}{3}\,\Gamma_\alpha{}^\beta\wedge\Gamma_\beta{}^\gamma\wedge\Gamma_\gamma{}^\alpha\right) \tag{1.85}$$

with respect to the metric keeping the connection fixed. In order to enforce vanishing torsion

$$T^\alpha = D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha\wedge\vartheta^\beta \tag{1.86}$$

and vanishing nonmetricity

$$Q_{\alpha\beta} = -Dg_{\alpha\beta} = -dg_{\alpha\beta} + \Gamma_\alpha{}^\gamma\,g_{\gamma\beta} + \Gamma_\beta{}^\gamma\,g_{\alpha\gamma}\,, \tag{1.87}$$

we have to apply Lagrange multipliers. Then the total Lagrangian reads

$$L = C_{\mathrm{RR}} + \lambda_\alpha\wedge T^\alpha + \lambda^{\alpha\beta}\wedge Q_{\alpha\beta}\,, \tag{1.88}$$

where λ_α is a 1-form and $\lambda^{\alpha\beta} = \lambda^{\beta\alpha}$ a symmetric 2-form. The corresponding field equations read (for the explicit calculation see Appendix A.4.3)

$$\frac{\delta L}{\delta \lambda_\alpha} = T^\alpha = 0\,, \tag{1.89}$$

$$\frac{\delta L}{\delta \lambda^{\alpha\beta}} = Q_{\alpha\beta} = 0\,, \tag{1.90}$$

$$\frac{\delta L}{\delta \Gamma_\alpha{}^\beta} = -R_\beta{}^\alpha - \lambda_\beta \wedge \vartheta^\alpha + 2\lambda_\beta{}^\alpha = 0\,, \tag{1.91}$$

$$\frac{\delta L}{\delta \vartheta^\alpha} = D\lambda_\alpha = 0\,, \tag{1.92}$$

$$\frac{\delta L}{\delta g_{\alpha\beta}} = D\lambda^{\alpha\beta} = 0\,. \tag{1.93}$$

We can solve (1.91) for its symmetric and its antisymmetric parts,

$$R_{[\alpha\beta]} + \vartheta_{[\alpha} \wedge \lambda_{\beta]} = 0\,, \tag{1.94}$$

$$-R_{(\alpha\beta)} + \vartheta_{(\alpha} \wedge \lambda_{\beta)} + 2\lambda_{\alpha\beta} = 0\,. \tag{1.95}$$

Because of (1.90), the symmetric part of the curvature vanishes,

$$0 = DQ_{\alpha\beta} = -DDg_{\alpha\beta} = R_\alpha{}^\gamma\, g_{\gamma\beta} + R_\beta{}^\gamma g_{\alpha\gamma} = 2R_{(\alpha\beta)}\,. \tag{1.96}$$

Thus, by means of (1.95)

$$\lambda_{\alpha\beta} = -\frac{1}{2}\,\vartheta_{(\alpha} \wedge \lambda_{\beta)}\,. \tag{1.97}$$

According to (1.61), in three dimensions, $R_{\alpha\beta} = -2\,\vartheta_{[\alpha} \wedge L_{\beta]}$. We substitute this into (1.94) and find

$$\lambda_\beta = 2\,L_\beta\,, \qquad \lambda_{\alpha\beta} = -\vartheta_{(\alpha} \wedge L_{\beta)}\,. \tag{1.98}$$

Eventually,

$$\frac{1}{2}\,\frac{\delta L}{\delta \vartheta^\alpha} = C_\alpha\,, \qquad \frac{\delta L}{\delta g_{\alpha\beta}} = -\vartheta^{(\alpha} \wedge C^{\beta)}\,, \qquad -\frac{2}{n-1}\,e_\beta \rfloor \frac{\delta L}{\delta g_{\alpha\beta}} = C^\alpha\,. \tag{1.99}$$

In three dimensions the Cotton tensor arises from the variation of the topological Chern-Simons term. Recently, Jackiw et al. [67, 52, 66] defined a modified Cotton tensor in $n = 4$ which also can be defined by means of a variational principle. They start from the Pontryagin density

$$-\frac{1}{2}\,R_\alpha{}^\beta \wedge R_\beta{}^\alpha = dC_{\rm RR}\,,$$

where the Chern-Simons action appears as a boundary term. In order to get a contribution of $C_{\rm RR}$ to the field equations Jackiw et al. introduce an external embedding field θ,

$$V_{\rm Jackiw} = \frac{1}{2}\,\theta\, R_\alpha{}^\beta \wedge R_\beta{}^\alpha = -\theta\, dC_{\rm RR} = d\theta\wedge C_{\rm RR} - d\left(\theta\, C_{\rm RR}\right)\,. \tag{1.100}$$

The field θ is kept fixed, i. e. will not be varied. A similar procedure was also carried out for electromagnetism, where the embedding variable is related to symmetry breaking. By varying (1.100) with respect to the metric one finds

$$\tilde{C}^{ij} := \theta_{,m}\,\eta^{klm(i}C_{kl}{}^{j)} + \theta_{,k;l}\,{}^\star{\rm Weyl}^{k(ij)l}\,.$$

This modified Cotton tensor is manifestly symmetric and traceless. The vector field $\theta_{,i}$ induces polarization dependence of the intensity of gravitational waves.

The Cotton 2-form defined by us only in $n=3$ can be obtained from a variational principle. However, in $n=4$, the Bach 3-form can be obtained from the conformally invariant action

$$V_{\rm Bach} = -\frac{1}{2}\,{\rm Weyl}_{\alpha\beta}\wedge {}^\star{\rm Weyl}^{\alpha\beta}\,.$$

For vanishing nonmetricity and torsion we obtain, see appendix A.4.2,

$$B_\alpha = -\frac{\delta V_{\rm Bach}}{\delta\vartheta^\alpha} = D^\star C_\alpha + {}^\star{\rm Weyl}_\alpha{}^\beta\wedge L_\beta\,.$$

Like the Cotton 2-form in 3 dimensions, the Bach 3-form in 4 dimensions is tracefree, symmetric, covariantly conserved, conformally invariant, and vanishes in a conformally flat space. We will discuss the Bach 3-form in more detail in section 1.6.

1.4 Conformal correspondence

The conformal correspondence between two n–dimensional manifolds V_n and $\widehat{V}_n$ is achieved by means of a conformal transformation of the form [42, 102]

$$\hat{g}_{\alpha\beta} = \exp(2\sigma) g_{\alpha\beta}\,, \qquad \hat{g}^{\alpha\beta} = \exp(-2\sigma) g^{\alpha\beta}\,, \tag{1.101}$$

where σ is an arbitrary function. In general, a conformal transformation (1.101) is *not* associated with a transformation of coordinates, i.e., with a diffeomorphism of V_n; both metrics in (1.101) are given in the same coordinate system and frame. Since these transformations preserve angles between corresponding directions, the causal structure of the manifold is preserved. As a rule, indices of quantities *with* hat are raised and lowered by means of $\hat{g}^{\alpha\beta}$ or $\hat{g}_{\alpha\beta}$, respectively, those of untransformed quantities by $g^{\alpha\beta}$ or $g_{\alpha\beta}$. The transformed connection reads

$$\widehat{\Gamma}_\alpha{}^\beta = \Gamma_\alpha{}^\beta + \left(\delta_\alpha^\beta\, d\sigma - \vartheta_\alpha\,\sigma^{,\beta} + \sigma_{,\alpha}\,\vartheta^\beta\right) =: \Gamma_\alpha{}^\beta + S_\alpha{}^\beta\,, \tag{1.102}$$

a comma denotes partial and a semicolon covariant differentiation. If $\hat{D} = d + \Gamma_\alpha{}^\beta + S_\alpha{}^\beta$ is the exterior covariant derivative with respect to $\widehat{\Gamma}_\alpha{}^\beta$, the transformed curvature is

$$\widehat{R}_\alpha{}^\beta = d\widehat{\Gamma}_\alpha{}^\beta - \widehat{\Gamma}_\alpha{}^\gamma \wedge \widehat{\Gamma}_\gamma{}^\beta = R_\alpha{}^\beta + 2\,\vartheta_{[\alpha} \wedge S_{\gamma]}\, g^{\gamma\beta}\,, \tag{1.103}$$

with

$$S_\gamma := D\sigma_{,\gamma} - \sigma_{,\gamma}\, d\sigma + \frac{1}{2}\,\sigma^{,\alpha}\sigma_{,\alpha}\,\vartheta_\gamma\,. \tag{1.104}$$

By contracting (1.103) with the frame $e_\beta\rfloor$, we infer

$$\widehat{L}_\alpha = L_\alpha - (n-2)\,S_\alpha\,, \tag{1.105}$$

$$\widehat{\text{Weyl}}_\alpha{}^\beta = \text{Weyl}_\alpha{}^\beta\,, \tag{1.106}$$

$$\widehat{R} = \exp(-2\sigma)\,[R - 2(n-1)\,\sigma^{,\alpha}{}_{;\alpha} - (n-1)(n-2)\sigma_{,\alpha}\sigma^{,\alpha}]\,. \tag{1.107}$$

The Weyl 2-form is conformally invariant since a conformal transformation does not act on the trace-free part of the curvature. Application of $\widehat{D}$ onto (1.105) yields the transformation behavior of the Cotton 2-form,

$$\widehat{C}_\alpha = C_\alpha + (n-2)\,\sigma_{,\beta}\,\text{Weyl}_\alpha{}^\beta\,. \tag{1.108}$$

Thus, in $n = 3$, where the Weyl 2-form vanishes, the Cotton 2-form becomes conformally invariant.

Criteria for conformal flatness

In the following paragraphs we investigate the criteria for conformal flatness, i. e., the possibilities to transform the curvature to zero by means of a conformal transformation. We basically follow [102]. Since we have seen that the curvature 2-form in 2, 3, and more than 3 dimensions is built up rather differently, we have to investigate these cases separately.

$\mathbf{n = 2}$

In $n = 2$ the only non-vanishing curvature piece is the curvature scalar R. Its behavior under conformal transformation is given by

$$\widehat{R} = \exp(-2\sigma)\;(R - 2\,\sigma^{,\alpha}{}_{;\alpha}) = 0\,. \tag{1.109}$$

Thus,

$$\widehat{R} = 0 \quad \Longleftrightarrow \quad \sigma^{,\alpha}{}_{;\alpha} = \frac{R}{2}\,. \tag{1.110}$$

This is a scalar wave equation for the conformal factor σ with R as source. Since the wave equation always has a solution, we conclude that all 2-dimensional spaces are conformally flat.

$\mathbf{n \geq 3}$

For more than 2 dimensions we start from (1.61), namely

$$R_{\alpha\beta} = \mathrm{Weyl}_{\alpha\beta} - \frac{2}{n-2}\,\vartheta_{[\alpha} \wedge L_{\beta]}\,. \tag{1.111}$$

Since the Weyl 2-form is conformally invariant it cannot be transformed to zero by means of a conformal transformation. Consequently, the vanishing of the Weyl 2-form is a necessary condition for conformal flatness.

The L_α 1-form transforms according to

$$\widehat{L}_\alpha = L_\alpha - (n-2)\,S_\alpha\,. \tag{1.112}$$

We can transform L_α to zero if there is a function σ such that

$$L_\alpha = (n-2)\,S_\alpha\,. \tag{1.113}$$

This will impose a differential restriction on $L_{\alpha\beta}$. By means of (1.104), we rewrite the latter equation as a differential equation for $\sigma_{,\alpha}$,

$$D\sigma_{,\alpha} = \sigma_{,\alpha}\,\sigma_{,\beta}\,\vartheta^\beta - \frac{1}{2}\,\sigma^{,\beta}\,\sigma_{,\beta}\,\vartheta_\alpha + \frac{1}{n-2}\,L_\alpha\,. \tag{1.114}$$

If we apply the covariant derivative to both sides of (1.114), we obtain a necessary condition for the integrability,

$$-R_\alpha{}^\beta\,\sigma_{,\beta} = DD\sigma_{,\alpha} = \sigma_{,\beta}\,D\sigma_{,\alpha} \wedge \vartheta^\beta - \sigma^{,\beta}\,D\sigma_{,\beta} \wedge \vartheta_\alpha + \frac{1}{n-2}\,C_\alpha\,. \tag{1.115}$$

This becomes a necessary and sufficient condition of integrability if the dependence on $\sigma_{,\alpha}$ can be eliminated, see [102, 103]. Thus we substitute $D\sigma_{,\alpha}$ from (1.114) into (1.115):

$$-R_\alpha{}^\beta\,\sigma_{,\beta} = -\frac{2}{n-2}\,L_{[\alpha} \wedge \vartheta_{\beta]}\,\sigma^{,\beta} + \frac{1}{n-2}\,C_\alpha\,. \tag{1.116}$$

Using the decomposition (1.61) of the curvature, we finally arrive at

$$-(n-2)\,\mathrm{Weyl}_\alpha{}^\beta\,\sigma_{,\beta} = C_\alpha\,. \tag{1.117}$$

For $n = 3$, the Weyl 2-form is zero and $C_\alpha = 0$ is the integrability condition for the conformal factor. Thus, if the Cotton 2-form is zero, the space is conformally flat. Conversely, if the space is conformally flat, there is a conformal transformation such that $\widehat{R}_\alpha{}^\beta = 0 \Leftrightarrow \widehat{L}_\alpha = 0 \Rightarrow \widehat{C}_\alpha = 0$. Since the Cotton 2-form is conformally invariant in 3 dimensions, we find $C_\alpha = 0$. Hence, the vanishing of the Cotton 2-form is the necessary and sufficient condition for a V_3 to be conformally flat.

In more than 3 dimensions the vanishing of the Weyl 2-form is a necessary condition for conformal flatness. Thus, also in dimensions greater than 3, $C_\alpha = 0$ is the integrability condition for the conformal factor. However, for $n > 3$, the contracted second Bianchi identity (1.78) implies the vanishing of the Cotton 2-form when the Weyl 2-form is zero. Hence, the vanishing of the Weyl 2-form is also the sufficient condition for conformal flatness.

1.5 Classification of the Cotton 2-form in three dimensions

A vector-valued 2-form in 3 dimensions has 9 independent components, the same as the number of components of a 3×3 matrix. A mapping between these two can be achieved by means of the Hodge dual. The Hodge dual of a vector-valued 2-form in 3 dimensions is a vector-valued 1-form with the same number of independent components. Its components form a 2nd rank tensor ("matrix"),

$$C_{\alpha\beta} := e_\alpha \rfloor {}^\star C_\beta = {}^\star(C_\beta \wedge \vartheta_\alpha) \tag{1.118}$$

or, in components,

$$C_\alpha{}^\beta = \nabla_\mu \left(\mathrm{Ric}_{\nu\alpha} - \frac{1}{4} R g_{\nu\alpha} \right) \eta^{\mu\nu\beta} \,. \tag{1.119}$$

This alternative representation of the Cotton 2-form, often called *Cotton-York tensor* [120] (even though it was already discussed explicitly by ADM [4]), can only be defined in three dimensions. Sometimes it appears under the name Bach tensor in the literature, see [32], e.g. This seems to be a misnomer.

The Cotton tensor is tracefree

$$C_\alpha{}^\alpha = e_\alpha \rfloor {}^\star C^\alpha = {}^\star(C^\alpha \wedge \vartheta_\alpha) = 0 \,. \tag{1.120}$$

In three dimensions, the 2nd Bianchi identity (1.63) amounts to $\vartheta_{[\alpha} \wedge C_{\beta]} = 0$. In view of the definition (1.118), we infer that the Cotton tensor is symmetric $C_{\alpha\beta} = C_{\beta\alpha}$. Introducing this symmetry explicitly into (1.119), we obtain the alternative representation

$$C^{\alpha\beta} = C^{\beta\alpha} = \eta^{\mu\nu(\alpha} \nabla_\mu \mathrm{Ric}_\nu{}^{\beta)} \,. \tag{1.121}$$

We now perform a classification of the Cotton tensor with respect to its eigenvalues. The corresponding generalized eigenvalue problem reads:

$$\left(C^{\alpha\beta} - \lambda\, g^{\alpha\beta} \right) V_\beta = 0 \,, \quad C^{[\alpha\beta]} = 0 \,, \quad C^{\alpha\beta} g_{\alpha\beta} = 0 \,. \tag{1.122}$$

By lowering one index, we can reformulate this as ordinary eigenvalue problem for the matrix $C_\alpha{}^\beta$. However, in that case, the symmetry $C^{\alpha\beta} = C^{\beta\alpha}$ is no longer manifest:

$$\left(C_\alpha{}^\beta - \lambda\, \delta_\alpha^\beta \right) V_\beta = 0 \,, \quad C_\alpha{}^\alpha = 0 \,. \tag{1.123}$$

Euclidean signature

The case of Euclidean signature is simple: the generalized eigenvalue problem reduces to an ordinary one. As a real symmetric matrix $C^{\alpha\beta}$ possesses 3 real eigenvalues and the eigenvectors form a basis. With respect to this basis, $C^{\alpha\beta}$ takes a diagonal form. Since $C^{\alpha\beta}$ is tracefree, the sum of the eigenvalues is zero. Consequently, we can distinguish 3 classes:

- Class A
  Three distinct eigenvalues: $\lambda_1 \neq \lambda_2$ and $\lambda_3 = -(\lambda_1 + \lambda_2)$.
- Class B
  Two distinct eigenvalues: $\lambda_1 = \lambda_2 \neq 0$, $\lambda_3 = -2\lambda_1$.
- Class C
  One distinct eigenvalue: $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
  In the present context of Euclidean signature, this implies $C_{\alpha\beta} = 0$.

Lorentzian signature

In the case of an indefinite metric, the roots of the characteristic polynomial

$$\det\left(C^{\alpha\beta} - \lambda\, g^{\alpha\beta}\right) = 0 \tag{1.124}$$

may be complex. Accordingly, the matrix $C_\alpha{}^\beta$ is no longer symmetric and in the equivalent ordinary eigenvalue problem

$$\det\left(C_\alpha{}^\beta - \lambda\, \delta_\alpha^\beta\right) = 0 \tag{1.125}$$

complex eigenvalues occur, too. This point seems to have been overlooked by the authors of [12]. Consequently, the classification will not be as simple as it was the case for the Euclidean metric.

In the following, we will present a classification of $C_\alpha{}^\beta$. The tracefree condition $(1.123)_2$, in orthonormal coordinates, reads explicitly

$$C_1{}^1 + C_2{}^2 + C_3{}^3 = 0\,. \tag{1.126}$$

Accordingly, we can eliminate $C_3{}^3$, e.g., from $(1.123)_1$. Then the secular determinant reads

$$\det\begin{vmatrix} C_1{}^1 - \lambda & C_1{}^2 & C_1{}^3 \\ -C_1{}^2 & C_2{}^2 - \lambda & C_2{}^3 \\ -C_1{}^3 & C_2{}^3 & -C_1{}^2 - C_2{}^2 - \lambda \end{vmatrix} = 0\,, \tag{1.127}$$

with the 5 matrix elements $C_1{}^1, C_1{}^2, C_1{}^3, C_2{}^2, C_2{}^3$. We compute the determinant and order according to powers of λ,

$$\lambda^3 + b\,\lambda + c = 0\,, \tag{1.128}$$

where

$$
\begin{aligned}
b &:= -({C_1}^1)^2 - {C_1}^1{C_2}^2 - ({C_2}^2)^2 + ({C_1}^2)^2 + ({C_1}^3)^2 - ({C_2}^3)^2\,, && (1.129)\\
c &:= \big[({C_1}^1)^2{C_2}^2 + {C_1}^1({C_2}^2)^2 + {C_1}^1({C_1}^2)^2 + {C_1}^1({C_2}^3)^2 && (1.130)\\
&\qquad +({C_1}^2)^2{C_2}^2 + 2\,{C_1}^2{C_1}^3{C_2}^3 - ({C_1}^3)^2{C_2}^2\big]\,.
\end{aligned}
$$

The roots of (1.128) are given by

$$
\lambda_1 = A\,, \quad \lambda_2 = -\frac{A}{2} + i\frac{\sqrt{3}}{2}B\,, \quad \lambda_3 = -\frac{A}{2} - i\frac{\sqrt{3}}{2}B\,, \tag{1.131}
$$

with

$$
A := \frac{D^2 - 12b}{6D}\,, \quad B := \frac{D^2 + 12b}{6D}\,, \quad D := \left(-108c + 12\sqrt{12b^3 + 81c^2}\right)^{1/3}\,. \tag{1.132}
$$

A cubic polynomial with real coefficients has at least one real root and the complex roots have to be complex conjugates. The Jordan normal forms of the Cotton tensor read:

"Petrov"-type	Jordan form	Segre notation	eigenvalues
I	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix}$	[111]	$\lambda_1 \neq \lambda_2,\, \lambda_3 = -\lambda_1 - \lambda_2$
D	$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$	[(11)1]	$\lambda_1 = \lambda_2 \neq 0, \lambda_3 = -2\lambda_1$
II	$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & -2\lambda_1 \end{pmatrix}$	[21]	$\lambda_1 = \lambda_2 \neq 0, \lambda_3 = -2\lambda_1$
N	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	[(21)]	$\lambda_1 = \lambda_2 = \lambda_3 = 0$
III	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	[3]	$\lambda_1 = \lambda_2 = \lambda_3 = 0$
O	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$		

This parallels exactly the Petrov classification of the Weyl tensor in 4 dimensions [108]. This comes about since the Weyl tensor in 4D is equivalent to a (complex) 3×3 tracefree matrix, as ${C_\alpha}^\beta$ in 3D; for a similar classification of $C_{\alpha\beta}$, see [54].

Since one eigenvalue is real, types D and II with only one independent eigenvalue $\lambda_1 = \lambda_2 = -2\lambda_3$ are always real. For class I, besides the real eigenvalue, two complex eigenvalues may occur. In that case, they are complex conjugated. Therefore, class I can be subdivided into class I with 3 real eigenvalues, [111], and class I' with one real and two complex conjugated eigenvalues, $[1z\bar{z}]$. By performing a kind of null rotation, we can also give a real form for class I':

$$\text{I'} \qquad \begin{pmatrix} \operatorname{Re} z & \operatorname{Im} z & 0 \\ -\operatorname{Im} z & \operatorname{Re} z & 0 \\ 0 & 0 & -2\operatorname{Re} z \end{pmatrix} \qquad [1z\bar{z}] \quad \lambda_1 = -2\operatorname{Re} z,\ \lambda_2 = z,\ \lambda_3 = \bar{z}\ .$$

We can now specify simple criteria for deciding to which of these classes the Cotton tensor $C_\alpha{}^\beta$ belongs. First determine the eigenvalues.

1. Three different eigenvalues (2 independent)
   (a) all real $\Rightarrow$ Class I
   (b) one real, two complex $\Rightarrow$ Class I'
2. Two different eigenvalues (1 independent $\lambda_1 = \lambda_2 = -2\lambda_3$)
   (a) $(C_\alpha{}^\beta - \lambda_1\,\delta_\alpha^\beta)(C_\beta{}^\gamma + \frac{1}{2}\lambda_1\,\delta_\beta^\gamma) = 0 \Rightarrow$ Class D
   (b) else $\Rightarrow$ Class II
3. All eigenvalues zero
   (a) $C_\alpha{}^\beta = 0 \Rightarrow 0$
   (b) $C_\alpha{}^\beta\, C_\beta{}^\gamma = 0 \Rightarrow$ Class N
   (c) else $\Rightarrow$ Class III

Examples

We now give examples in order to demonstrate explicitly that all classes presented are non-empty indeed. All results have been checked by means of computer algebra, see Appendix A.5.1 for an explicit sample program.

- Class I'
  The generic example is the (1+2)D static and spherically symmetric spacetime, given in an orthonormal coframe with signature $(+--)$ by

$$\vartheta^{\hat{0}} = \sqrt{\psi}\,dt, \quad \vartheta^{\hat{1}} = \frac{dr}{\sqrt{\psi}}, \quad \vartheta^{\hat{2}} = r\,d\varphi\,, \quad \psi = \psi(r)\,. \tag{1.133}$$

  The Cotton tensor and its eigenvalues read, here $()' = d/dr$:

$$C_\alpha{}^\beta = \frac{\sqrt{\psi}\,\psi'''}{4}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,, \quad \lambda_1 = 0\,, \lambda_2 = -\lambda_3 = i\frac{\sqrt{\psi}\,\psi'''}{4}\,. \tag{1.134}$$

A well-known example is the 3D analog to the Reissner-Nordström solution, a solution of the 3D Einstein-Maxwell equation [11]:

$$\psi = \Lambda\, r^2 - q^2 \,\ln r - M\,. \tag{1.135}$$

- Class I
  In [92], eq.(4.1), the following solution for the vacuum DJT field equation is given: The orthonormal coframe with signature $(-++)$ reads

$$\begin{aligned}
\vartheta^{\hat{0}} &= a_0\,(d\psi + \sinh\theta\, d\phi)\,, && (1.136)\\
\vartheta^{\hat{1}} &= a_1\,(-\sin\psi\, d\theta + \cos\psi\,\cosh\theta\, d\phi)\,, && (1.137)\\
\vartheta^{\hat{2}} &= a_2\,(\cos\psi\, d\theta + \sin\psi \cosh\theta\, d\phi)\,, && (1.138)
\end{aligned}$$

  where the DJT field equations are fulfilled provided

$$a_0 + a_1 + a_2 = 0\,, \qquad \mu = -\frac{a_0^2 + a_1^1 + a_2^2}{a_0 a_1 a_2}\,. \tag{1.139}$$

  Then the Cotton tensor reads

$$C_\alpha{}^\beta = -4\frac{a_1^2 + a_1 a_2 + a_2^2}{(a_1+a_2)a_1^2 a_2^2}\begin{pmatrix} 1 & 0 & 0\\ 0 & \frac{-a_1}{a_1+a_2} & 0\\ 0 & 0 & \frac{-a_2}{a_1+a_2}\end{pmatrix}. \tag{1.140}$$

  The eigenvalues can be read off from the diagonal. For $a_1 = a_2$, the Cotton tensor degenerates to class D. The solution eq.(4.6) in [92] is analogous to the present case.

- Class D
  An example is the 3D Gödel solution (signature $(+--)$), see [97] eq.(4.1):

$$\begin{aligned}
\vartheta^{\hat{0}} &= \left(\frac{3}{\mu}\right)\left[dt - 2(\sqrt{r^2+1} - 1)\, d\phi\right], && (1.141)\\
\vartheta^{\hat{1}} &= \left(\frac{3}{\mu}\right)\frac{dr}{\sqrt{r^2+1}}\,, && (1.142)\\
\vartheta^{\hat{2}} &= \left(\frac{3}{\mu}\right) r\, d\phi\,, && (1.143)
\end{aligned}$$

  with

$$C_\alpha{}^\beta = \left(\frac{\mu}{3}\right)^3 \begin{pmatrix} -2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \lambda_1 = \lambda_2 = -\frac{1}{2}\lambda_3 = \left(\frac{\mu}{3}\right)^3. \tag{1.144}$$

  This is a vacuum solution of the DJT model as well as a solution of the 3D Einstein equation with matter.

Many of the other solutions known for the DJT field equation are also of Class D:

- The squashed 3-sphere solutions by Nutku and Baekler [92], eq.(4.10) and eq.(4.1), eq.(4.6) for a special choice of parameters (see above).
- The topologically massive planar universe with constant twist of Percacci et al. [97], eq.(3.20).
- The perfect fluid solution of Gürses [53], eq.(6).
- The DJT-black hole solution of Nutku [91], eq.(24).
- The recent black hole solution by Moussa et al. [87], eq.(4).

• Class N
In section 2.6, the following solution of the DJT-field equations is derived:

$$\vartheta^{\hat{0}} = e^{\mu y/2}\left[(1+\frac{1}{2}\,e^{-\mu y})\,dt + (1-\frac{1}{2}\,e^{-\mu y}\,dx)\right], \tag{1.145}$$

$$\vartheta^{\hat{1}} = \frac{1}{2}\,e^{-\mu y/2}\,(dt - dx)\,, \tag{1.146}$$

$$\vartheta^{\hat{2}} = dy\,. \tag{1.147}$$

The Cotton tensor, with all eigenvalues being zero, reads

$$C_\alpha{}^\beta = \frac{\mu^3}{2}\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_1 = \lambda_2 = \lambda_3 = 0\,. \tag{1.148}$$

Another class N solution is given in [97], eq.(4.9).

We have found no (sensible) solutions to the Einstein or DJT field equations which are of Class II or III. However, it is easy to find metrics for which the Cotton tensor is in general of class I but may degenerate to classes II or III. Just in order to show that these classes are nonempty, we will sketch corresponding examples:

• Class II
The following coframe (signature $(-++)$),

$$\vartheta^{\hat{0}} = e^{-2y}\,dt + dx\,, \quad \vartheta^{\hat{1}} = e^{y}\,dx\,, \quad \vartheta^{\hat{2}} = dy\,, \tag{1.149}$$

yields the Cotton matrix

$$C_\alpha{}^\beta = \begin{pmatrix} 8e^{-3y} & 6(3e^{-2y}-1) & 0 \\ -6(3e^{-2y}-1) & -4e^{3y}\,(3e^{2y}+1) & 0 \\ 0 & 0 & 4e^{-3y}\,(e^{2y}+1)) \end{pmatrix}. \tag{1.150}$$

In general we find three different eigenvalues

$$\lambda_1 = 2e^{-3y}\left(3\sqrt{1-e^{6y}+7e^{4y}-7e^{2y}}-3e^{2y}+1\right), \tag{1.151}$$
$$\lambda_2 = 2e^{-3y}\left(-3\sqrt{1-e^{6y}+7e^{4y}-7e^{2y}}-3e^{2y}+1\right), \tag{1.152}$$
$$\lambda_3 = 2e^{-3y}(3e^{2y}+1)\,. \tag{1.153}$$

Hence, in general, this yields class I (or I'). However, for $y=0$ this reduces to

$$C_\alpha{}^\beta \overset{y=0}{=} \begin{pmatrix} 8 & 12 & 0 \\ -12 & -16 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \qquad \lambda_1=\lambda_2=-4, \lambda_3=8\,. \tag{1.154}$$

Since

$$\left(C_\alpha{}^\beta - 8\delta_\alpha^\beta\right)\left(C_\alpha{}^\beta + 4\delta_\alpha^\beta\right) = 144 \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0\,, \tag{1.155}$$

this matrix belongs not to class D. Indeed we find

$$\left(C_\alpha{}^\beta - 8\delta_\alpha^\beta\right)\left(C_\alpha{}^\beta + 4\delta_\alpha^\beta\right)^2 = 0\,. \tag{1.156}$$

Thus, it belongs to class II.

- Class III
  The Cotton tensor for the following coframe (signature $(-++)$) is also of class I in general:

$$\vartheta^{\hat 0} = (x-t)\,dt\,, \quad \vartheta^{\hat 1} = (x+t)\,dx\,, \quad \vartheta^{\hat 2} = dy\,, \tag{1.157}$$

$$C_\alpha{}^\beta = \frac{4(2t^2+x^2)x}{(t+x)^5(t-x)^4}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{(t^2+2x^2)(t+x)t}{(2t^2+x^2)(t-x)x} \\ -1 & \frac{(t^2+2x^2)(t+x)t}{(2t^2+x^2)(t-x)x} & 0 \end{pmatrix}, \tag{1.158}$$

where the three different eigenvalues read

$$\lambda_1 = \frac{4\sqrt{t^8+2(t^7x+tx^7)+t^6x^2+16(t^5x^3+t^3x^5)-t^2x^6-x^8}}{(t+x)^5(t-x)^5}, \tag{1.159}$$
$$\lambda_2 = -\lambda_1\,, \tag{1.160}$$
$$\lambda_3 = 0\,. \tag{1.161}$$

Again, this leads to class I (or I'). On the hypersurface given by $x=t(\sqrt{13}+3)/2$, all eigenvalues collapse to zero. However, one can easily compute that also in that case

$$C_\alpha{}^\beta \neq 0\,, \quad C_\alpha{}^\beta\, C_\beta{}^\gamma \neq 0\,, \quad \text{but } C_\alpha{}^\beta\, C_\beta{}^\gamma\, C_\gamma{}^\delta = 0\,. \tag{1.162}$$

Therefore, the Cotton tensor degenerates to class III.

- Class 0: All conformally flat solutions.

1.6 The Cotton 2-form and automatically conserved quantities

In this section we want to shed some light on the subtle interplay between conserved matter currents and geometrical identities. Experience has told us that all forms of energy gravitate — the source of gravity is the energy density of matter. From special relativity we know that the energy density is accommodated in the time component of the energy-momentum vector. In this way we find the vector-valued energy-momentum $(n-1)$-form

$$\mathbf{\Sigma} = \Sigma^\alpha \, e_\alpha = \frac{1}{(n-1)!} \, \Sigma_{\mu_1 \dots \mu_{n-1}}{}^\alpha \, \vartheta^{\mu_1} \wedge \dots \wedge \vartheta^{\mu_{n-1}} \; e_\alpha =: T_\mu{}^\alpha \, \eta^\mu \, e_\alpha \, . \tag{1.163}$$

By means of the Noether identities the energy-momentum is conserved. In flat spacetime we have

$$d\Sigma_i = 0 \quad \Leftrightarrow \quad \nabla_j \, T^{ij} = 0 \, . \tag{1.164}$$

We have identified the source of gravity. What about the gravitational field? Einstein recognized that the connection of flat spacetime acts as the "inertial field strength". Switching on gravity means supplying the connection with own degrees of freedom and thereby arriving at a curved Riemannian spacetime. In electromagnetism, e. g., the divergence of the field strength is proportional to the source current. The (covariant) derivative of the connection is given by the curvature. Hence, we suspect the curvature to be proportial to the energy-momentum. Since the energy-momentum is a vector-valued, conserved $(n-1)$-form we have to look for a similar piece of the curvature. The basic relation in order to construct such a quantity is the second Bianchi identity,

$$DR_\alpha{}^\beta = 0 \, . \tag{1.165}$$

The conservation equation $D\Sigma_\alpha = 0$, as a vector valued n-form, has n independent components. The Bianchi identity is a tensor valued 3-form and thus has $n^2(n-1)(n-2)/6$ independent components. However we can extract a piece with n independent components in a straightforward manner by contracting it with the frame,

$$e_\alpha \rfloor e_\beta \rfloor e_\gamma \rfloor DR^{\beta\gamma} = 0 \, . \tag{1.166}$$

Then we have a covector-valued 0-form with n components. We can rewrite this equation by taking the Hodge dual,

$$\begin{aligned} {}^\star \left(e_\alpha \rfloor e_\beta \rfloor e_\gamma \rfloor DR^{\beta\gamma} \right) &= -\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma \wedge {}^\star DR^{\beta\gamma} \\ &= -DR^{\beta\gamma} \wedge {}^\star \left(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma \right) \\ &= -DR^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} = -D \left(R^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma} \right) \, , \end{aligned} \tag{1.167}$$

where we used some elementary rules of exterior algebra and the relation $D\eta^{\alpha\beta\gamma}=0$, which is valid in Riemannian spacetime. This suggests the definition of the *Einstein* $(n-1)$*-form*

$$G_\alpha := \frac{1}{2}\,\eta_{\alpha\beta\gamma}\wedge R^{\beta\gamma}\,, \qquad \text{with} \quad DG_\alpha = 0\,. \tag{1.168}$$

In this way we motivate the Einstein equation (ℓ is the gravitational constant)

$$G_\alpha = \ell\,\Sigma_\alpha\,, \tag{1.169}$$

which is valid in all dimensions n. We may generalize this field equation without departing too far from general relativity by adding further conserved geometrical $(n-1)$-forms at the left hand side.

Here, we draw our attention to *dimension dependent* equations. Also in this case our starting point is the second Bianchi identity. By differentiation of the second Bianchi identity we found, see table 1.2,

$$DC_\alpha = -\mathrm{Weyl}_\alpha{}^\beta\wedge L_\beta\,. \tag{1.170}$$

Since in $n=3$ the Weyl tensor vanishes, the Cotton 2-form, in three dimensions, is a conserved $(3-1)=2$-form. Thus, what about the gravitational field equation

$$G_\alpha + \mu C_\alpha = \ell\Sigma_\alpha\,? \tag{1.171}$$

This is nothing else than the well-known DJT-model of gravity!

Something similar can be obtained in 4 dimensions by using the Bianchi identity for the dual of the curvature. In Appendix A.4.2 it is shown that

$$DD^\star C_\alpha = -D\left({}^\star\mathrm{Weyl}_\alpha{}^\beta\wedge L_\beta\right)\,. \tag{1.172}$$

Thus,

$$B_\alpha := D^\star C_\alpha + {}^\star\mathrm{Weyl}_\alpha{}^\beta\wedge L_\beta =: B_\alpha{}^\beta\,\eta_\beta \tag{1.173}$$

or, in components,

$$B_{\alpha\beta} = \nabla^\mu C_{\alpha\mu\beta} + L^{\mu\nu}\,\mathrm{Weyl}_{\alpha\mu\beta\nu}\,, \tag{1.174}$$

is a covariantly conserved 3-form:

$$DB_\alpha = 0 \qquad (\nabla_\beta B_\alpha{}^\beta = 0)\,. \tag{1.175}$$

We recognize the *Bach tensor* $B_{\alpha\beta}$ [5, 102, 96, 112]. From the symmetry properties of C_α, L_α, and $\mathrm{Weyl}_{\alpha\beta}$ it follows that

$$B_\alpha\wedge\vartheta^\alpha = 0 \quad (B_\alpha{}^\alpha = 0)\,, \qquad e_\alpha\rfloor B^\alpha = 0 \quad (B_{[\alpha\beta]} = 0)\,. \tag{1.176}$$

Moreover, it transforms as a conformal density and can be derived from a variational principle, see appendix A.4.4. Since in a conformally flat space the Weyl and the Cotton tensors vanish, the vanishing of the Bach tensor is also a necessary (but not sufficient) condition for a four dimensional space to be conformally flat. Applying again our heuristical scheme we may put

$$G_\alpha + \alpha B_\alpha = \ell\,\Sigma_\alpha \tag{1.177}$$

and just have "found" Bach gravity!

What is now the physical content of this game? In Einstein gravity, the Bianchi identity $DG_\alpha = 0$ corresponds to $D\Sigma_\alpha = 0$. In turn, $D\Sigma_\alpha = 0$ is a conserved current due to the field equations of matter. In flat space, without gravity, we then find the corresponding conservation law $d\Sigma_\alpha = 0$ for the free matter fields.

Do also the Cotton or the Bach forms supply us with such independent conservation laws? Let us *assume Einstein gravity*, $G_\alpha = \ell\Sigma_\alpha$. The Einstein $(n-1)$-form is equivalent to the 1-form L_α,

$$G_\alpha = L^\beta \wedge \eta_{\beta\alpha} = \ell\,\Sigma_\alpha\,. \tag{1.178}$$

This equation can be inverted, yielding a 1-form representation of Einstein's field equation,

$$L_\alpha = \ell\,\widetilde{\Sigma}_\alpha\,, \tag{1.179}$$

where the modified energy-momentum 1-form is defined according to

$$\widetilde{\Sigma}_\alpha = (-1)^{n-1+\text{ind}}\left[{}^\star\Sigma_\alpha - \frac{1}{n-1}\,{}^\star\left(\Sigma_\gamma\wedge\vartheta^\gamma\right)\wedge\vartheta_\alpha\right]. \tag{1.180}$$

Similar to (1.178) we find[2]

$$\Sigma_\alpha = \widetilde{\Sigma}^\beta \wedge \eta_{\beta\alpha}\,. \tag{1.181}$$

Consequently, we can express the Cotton 2-form in terms of energy-momentum,

$$C_\alpha = DL_\alpha = \ell D\widetilde{\Sigma}_\alpha = (-1)^{n-1+\text{ind}}\,\ell D\left[{}^\star\Sigma_\alpha - \frac{1}{n-1}\,{}^\star\left(\Sigma_\gamma\wedge\vartheta^\gamma\right)\wedge\vartheta_\alpha\right]. \tag{1.182}$$

Since the vanishing of the Cotton 2-form is, in all dimensions, a necessary criterion for conformal flatness, matter sources which allow conformally flat solutions of Einstein's field equations must fulfill (1.182). In $n = 3$ this is also a sufficient condition

---

[2]Incidentally, this can be understood from a kind of dual variation. In [88] the formula of the variation of the η-basis is given (here adapted for an orthonormal coframe),

$$\delta\eta_\alpha = \delta\vartheta^\beta\wedge\eta_{\alpha\beta} \quad\Rightarrow\quad \delta L = \delta\vartheta^\alpha\wedge\frac{\delta L}{\delta\vartheta^\alpha} = \delta^\star\vartheta_\beta\wedge\frac{\delta L}{\delta^\star\vartheta_\beta} = \delta\vartheta^\alpha\wedge\eta_{\beta\alpha}\wedge\frac{\delta L}{\delta^\star\vartheta_\beta}\,,$$

or $\frac{\delta L}{\delta\vartheta^\alpha} = (-1)^n\,\frac{\delta L}{\delta^\star\vartheta_\beta}\wedge\eta_{\beta\alpha}$. This formula directly yields the relations between G_α, L_α and Σ_α, $\widetilde{\Sigma}_\alpha$.

for conformal flatness. We will make use of this fact when deriving the conformally flat perfect fluid solution in section 2.7.

In $n=3$, the Cotton 2-form is identically conserved,

$$DC_\alpha = -\mathrm{Weyl}_\alpha{}^\beta \wedge C_\beta \overset{n=3}{=} 0\,. \tag{1.183}$$

Hence, in $n=3$, the Einstein equation implies ($\ell = 1$)

$$DC_\alpha = DD\widetilde{\Sigma}_\alpha = -R_\alpha{}^\beta \wedge \widetilde{\Sigma}_\beta = 0\,. \tag{1.184}$$

Using the irreducible decomposition of the curvature and eq.(1.179), eq.(1.184) amounts to an algebraic constraint on the energy-momentum:

$$\widetilde{\Sigma}^\alpha \wedge \vartheta^\beta \wedge {}^\star\Sigma_\beta = 0\,. \tag{1.185}$$

However,

$$\vartheta^\beta \wedge {}^\star\Sigma_\beta = {}^\star(e^\beta \rfloor \Sigma_\beta) = {}^\star(\mathcal{T}_{\alpha\beta}\,\eta^{\alpha\beta}) = 0\,, \tag{1.186}$$

since the energy-momentum is symmetric which corresponds to angular momentum conservation, see [57, 83].

What about the Bach 3-form? We can proceed along the same line as before. Replacing L_α by $\widetilde{\Sigma}_\alpha$ we find the matter-counterpart of the Bach 3-form,

$$X_\alpha := D^\star D\left[{}^\star\Sigma_\alpha - \frac{1}{3}{}^\star\left(\Sigma_\gamma \wedge \vartheta^\gamma\right) \wedge \vartheta_\alpha\right] + {}^\star\mathrm{Weyl}_\alpha{}^\beta \wedge {}^\star\Sigma_\beta\,. \tag{1.187}$$

This is a conserved quantity provided the Einstein equation holds. Can we find a corresponding conserved quantity in flat spacetime, i. e. if we switch of gravity? In the course of studying literature on electromagnetic conservation laws, [13], we came across the *Chevreton tensor*

$$H_{ij} := (\nabla^k F_{il})\,(\nabla_k F_j{}^l) - \frac{1}{4}\,g_{ij}\,(\nabla_k F_{lm})\,(\nabla^k F^{lm})\,. \tag{1.188}$$

It is tracefree and symmetric and thus has 9 independent components. Moreover, it is conserved in flat spacetime. In [41], Edgar derives an alternative representation of the Chevreton tensor eq.(1.188) just by using the source free Maxwell equations, the definition of the electromagnetic energy-momentum and some geometrical identities in $n=4$,

$$2H_{ij} = \nabla^k\nabla_k T_{ij} - 2T_{kl}\,\mathrm{Weyl}_i{}^k{}_j{}^l + \frac{2R}{3}\,T_{ij}\,. \tag{1.189}$$

This occured to us rather similar to the Bachtensor, (1.174),

$$B_{ij} = \nabla^k\,C_{ikj} + L^{kl}\,\mathrm{Weyl}_{ikjl} = 2\nabla^k\nabla_{[i}L_{k]j} + L^{kl}\,\mathrm{Weyl}_{ikjl}\,, \tag{1.190}$$

if one replaces L_{ij} by T_{ij} and assuming $R = 0$. Indeed, a short calculation, see appendix A.4.8, showed

$$H_{ij} = -\ell \, B_{ij} \,, \tag{1.191}$$

provided the Einstein-Maxwell equation holds. This becomes even more apparent by using an alternative representation of the Bach tensor, using the trace-free Ricci-tensor (compare [112], but mind the signs due to different conventions for the curvature!),

$$\begin{aligned} B_{ij} \;\; = \;\; & -\nabla^k \nabla_k \not{\mathrm{Ric}}_{ij} + 2\mathrm{Weyl}_{ikjl} \not{\mathrm{Ric}}^{kl} \\ & +2 \left(\not{\mathrm{Ric}}_{ik} \not{\mathrm{Ric}}^k{}_j - \frac{1}{4} g_{ij} \not{\mathrm{Ric}}_{kl} \not{\mathrm{Ric}}^{kl} \right) \\ & + \left(\frac{1}{3} \not{\mathrm{Ric}}_{ij} - \frac{1}{12} g_{ij} \nabla^k \nabla_k + \frac{1}{3} \nabla_i \partial_j \right) R \,. \end{aligned} \tag{1.192}$$

Comparing this to the Chevreton tensor (1.189) we see that provided

$$G_{ab} = \ell \, T_{ab} \quad \text{(Einstein equation)}, \tag{1.193}$$

$$T^{ab} \, g_{ab} = 0 \qquad \text{(trace free)}, \tag{1.194}$$

$$T_{ai} T^i{}_b - \frac{1}{4} T^{ij} \, T_{ij} \, g_{ab} = 0 \quad \text{(Rainich condition)}, \tag{1.195}$$

we indeed find (1.191). In exterior calculus, we can define the Chevreton 3-form

$$\mathcal{H}_\alpha := H_\alpha{}^b \, \eta_b \,. \tag{1.196}$$

Incidentally, by making again use of the source-free Maxwell equations, we can give the following representation,

$$\mathcal{H}_\alpha \;\; = \;\; \frac{1}{2} e_\alpha \rfloor (\mathcal{F}_\beta \wedge^* \mathcal{F}^\beta) - \mathcal{F}_\beta \wedge (e_\alpha \rfloor^* \mathcal{F}^\beta) \tag{1.197}$$

$$\;\; = \;\; \frac{1}{2} \left[(e_\alpha \rfloor \mathcal{F}_\beta) \wedge^* \mathcal{F}^\beta) - \mathcal{F}_\beta \wedge (e_\alpha \rfloor^* \mathcal{F}^\beta) \right] \,, \tag{1.198}$$

where

$$\mathcal{F}_\alpha := D(e_\alpha \rfloor^\star F) \,. \tag{1.199}$$

This 3-form is manifestly traceless and symmetric by construction,

$$\vartheta_a \wedge \mathcal{H}^a = 0 \,, \qquad e^\alpha \rfloor \mathcal{H}_\alpha = 0 \,. \tag{1.200}$$

Additionally, it is conserved in *flat* space, see appendix A.4.9.

$$D\mathcal{H}_\alpha = 0 \qquad \text{flat space} \,. \tag{1.201}$$

Table 1.3: The mapping between conserved matter currents and geometrical identities

contracted 2nd Bianchi identity	$DG_\alpha = 0$	Einstein equation $G_\alpha = \ell\Sigma_\alpha$ $\longleftrightarrow$	$D\Sigma_\alpha = 0$	conservation of energy-momentum current
divergence free Bach 3-form	$DB_\alpha = 0$	$G_\alpha = \ell\,\Sigma_\alpha\,,$ $\Sigma_\alpha \wedge \vartheta^\alpha = 0\,,$ $\Sigma_\alpha \wedge {}^\star\Sigma_\beta - \frac{1}{4}\,g_{\alpha\beta}\,\Sigma_\gamma \wedge {}^\star\Sigma^\gamma = 0\,.$ $\longleftrightarrow$	$DH_\alpha = 0$	conservation of Chevreton current

In general, we do not have $D\mathcal{H}_\alpha = 0$ in an arbitrary spacetime. It is known from the literature [41] that the Chevreton 3-form is conserved in *Einstein spaces.*

These structures suggest that the Chevreton 3-form is connected to some Noether identity. Eq. (1.197) looks like the canonical energy-momentum for the Lagrangian $\mathcal{F}_\alpha \wedge {}^\star\mathcal{F}^\alpha = D(e_\alpha\rfloor{}^\star F)\wedge{}^\star D(e^\alpha\rfloor{}^\star F)$. Then $\mathcal{H}_\alpha$ would be related to translations. However, the Langrangian contains derivatives of the electromagnetic field F. Probably one should try to transform it to a first-order Langrangian by means of an appropriate change of variables in order to find a viable physical interpretation. The concrete example of a point charge suggests that the Chevreton 3-form is somehow related to quadropole interaction. In cartesian coordinates (electromagnetic potential $A = q/r\,dt$) the components of $\mathcal{H}_\alpha$ read $(A, B = 1, 2, 3)$

$$[H_\alpha] = \frac{q^2}{r^6}\left(\begin{array}{c|c} 3 & 0 \\ \hline 0 & -\dfrac{3x^A x^B - 2\delta^{AB}\,r^2}{r^2} \end{array}\right)\,. \tag{1.202}$$

The spatial part clearly resembles the usual electromagentic quadropole tensor. Further investigations are necessary.

1.7 Generalization of the Cotton 2-form to post-Riemannian spacetimes

Riemann-Cartan spacetime

In Riemannian space(time) the Cotton 2-form arises as the exterior covariant derivative of the 1-form part of the curvature in the context of the 2nd Bianchi identity,

$$C_\alpha = DL_\alpha\,, \qquad L_\alpha = e_\beta \rfloor R_\alpha{}^\beta - \frac{1}{2(n-1)}\, R\vartheta_\alpha\,.$$

This definition can naturally be generalized to Riemann-Cartan space(time). In this case, three additional irreducible parts of the curvature occur. We again identify a 1-form piece, see appendix, page 88,

$$^{(4)}R_{\alpha\beta} + {}^{(5)}R_{\alpha\beta} + {}^{(6)}R_{\alpha\beta} = -\frac{2}{n-2}\,\vartheta_{[\alpha} \wedge L_{\beta]}\,, \qquad L_\beta := e_\beta \rfloor R_\alpha{}^\beta - \frac{1}{2(n-1)}\, R\,\vartheta_\alpha\,, \tag{1.203}$$

and a 2-form piece

$$M^{\alpha\beta} := R^{\alpha\beta} + \frac{2}{n-2}\,\vartheta^{[\alpha} \wedge L^{\beta]}\,, \tag{1.204}$$

where

$$M^{\alpha\beta} = {}^{(1)}R^{\alpha\beta} + {}^{(2)}R^{\alpha\beta} + {}^{(3)}R^{\alpha\beta}\,. \tag{1.205}$$

Then we apply the exterior covariant derivative to eq.(1.204),

$$DR^{\alpha\beta} = DM^{\alpha\beta} - \frac{2}{n-2}\, T^{[\alpha} \wedge L^{\beta]} + \frac{2}{n-2}\,\vartheta^{[\alpha} \wedge C^{\beta]} = 0\,, \tag{1.206}$$

where C^α is the *Cotton 2-form* $C^\alpha = DL^\alpha$ as introduced in the previous sections. We expand it in a coframe

$$\begin{aligned} C^\alpha &= DL^\alpha = D\left(L_\mu{}^\alpha\, \vartheta^\mu\right) = \left(DL_\mu{}^\alpha\right) \wedge \vartheta^\mu + L_\mu{}^\alpha\, D\vartheta^\mu \\ &= \left(\nabla_\beta\, L_\mu{}^\alpha\right) \vartheta^\beta \wedge \vartheta^\mu + L_\mu{}^\alpha\, T^\mu\,, \end{aligned} \tag{1.207}$$

and read off the components

$$C_{\mu\nu}{}^\alpha = 2\,\nabla_{[\mu} L_{\nu]}{}^\alpha + L_\beta{}^\alpha\, T_{\mu\nu}{}^\beta\,. \tag{1.208}$$

We also may rearrange eq.(1.206) in order to isolate the 1- and 2-form pieces.

$$DM^{\alpha\beta} - \frac{2}{n-2}\, \not{T}^{[\alpha} \wedge \not{\mathrm{Ric}}^{\beta]} + \frac{2}{n-2}\,\vartheta^{[\alpha} \wedge C^{\beta]} - \frac{2}{n-2}\,\vartheta^{[\alpha} \wedge E^{\beta]} = 0\,, \tag{1.209}$$

where

$$E^{\beta} := \frac{1}{n-1}\, T\wedge \not{\mathrm{Ric}}^{\,\beta} - \frac{n-2}{2n(n-1)}\, R\not{T}^{\beta} + \frac{n-2}{2n(n-1)^2}\, R\,T\wedge\vartheta^{\beta}\,, \tag{1.210}$$

$$\not{T}^{\alpha} := T^{\alpha} - \frac{1}{n-1}\,\vartheta^{\alpha}\wedge T = {}^{(1)}T^{\alpha} + {}^{(3)}T^{\alpha}\,. \tag{1.211}$$

All irreducible pieces of $C_\alpha \neq 0$, the trace is

$$C = e_\alpha\rfloor C^{\alpha} = \frac{1}{2}\,(e_\alpha\rfloor e_\beta\rfloor T^{\gamma})\, e_\gamma\rfloor M^{\alpha\beta} + \frac{1}{n-2}\,\left[(L^{\beta} e_\beta\rfloor T) + 2\,(e_\beta\rfloor T^{[\alpha})\,(e_\alpha\rfloor L^{\beta]})\right]\,, \tag{1.212}$$

and the axial piece reads

$$C_\alpha\wedge\vartheta^{\alpha} = \frac{n-2}{n-3}\,\vartheta_\alpha\wedge(e_\beta\rfloor DM^{\alpha\beta}) - \frac{2}{n-3}\,\vartheta_\alpha\wedge\left[e_\beta\rfloor\left(T^{[\alpha}\wedge L^{\beta]}\right)\right]\,. \tag{1.213}$$

The Divergence is now given by

$$DC_\alpha = -M_\alpha{}^{\beta}\wedge L_\beta + \frac{1}{n-2}\,\left[(\vartheta_\beta\wedge L^{\beta})\wedge L_\alpha\right]\,. \tag{1.214}$$

Even in $n=3$, where $M^{\alpha\beta}=0$, DC_α is non-vanishing. There is no specific relation to the conformal transformation behaviour of the Riemann-Cartan curvature which actually is conformally invariant, see below.

Metric-affine space

In metric-affine space, the curvature is asymmetric. It can be split into a symmetric and an anti-symmetric piece,

$$R_{\alpha\beta} = R_{[\alpha\beta]} + R_{(\alpha\beta)} =: W_{\alpha\beta} + Z_{\alpha\beta}\,. \tag{1.215}$$

Hence, there exist two contractions ($Z_\alpha := e^{\beta}\rfloor Z_{\alpha\beta}$)

$$e^{\alpha}\rfloor R_{\alpha\beta} = Z_\beta - (L_\beta + \frac{1}{n-2}\,L\,\vartheta_\beta)\,, \tag{1.216}$$

$$e^{\beta}\rfloor R_{\alpha\beta} = Z_\alpha + (L_\beta + \frac{1}{n-2}\,L\,\vartheta_\beta)\,. \tag{1.217}$$

The curvature does therefore not contain a unique 1-form piece but two independent 1-form pieces. In order to cover the 1-form content of the curvature we have to consider both (or two linear combinations, respectively). Consequently, one would have to define two "Cotton 2-forms",

$${}^{(W)}C_\alpha := DL_\alpha\,, \tag{1.218}$$

$${}^{(Z)}C_\alpha := DZ_\alpha\,. \tag{1.219}$$

Conformal Transformation in metric-affine space

According to [58], conformal transformations in metric-affine space are give by[3]

$$\tilde{g}: \quad \begin{cases} \tilde{g}_{\alpha\beta} = \Omega^{L-2F}\, g_{\alpha\beta}\,, \qquad \tilde{\vartheta}^{\alpha} = \Omega^{F}\,\vartheta^{\alpha}\,, \qquad \tilde{e}_{\alpha} = \Omega^{-F}\, e_{\alpha}\,, \\ \tilde{\Gamma}_{\alpha}{}^{\beta} = \Gamma_{\alpha}{}^{\beta} - C\,\delta_{\alpha}^{\beta}\, d\ln\Omega \qquad \leftrightarrow \qquad \tilde{R}_{\alpha}{}^{\beta} = R_{\alpha}{}^{\beta}\,. \end{cases} \tag{1.220}$$

Since the curvature is conformally invariant, the Cotton 2-form will play no special role. Moreover, a metric-affine space can not be conformally flat. Torsion and nonmetricity transform according to

$$\tilde{Q}_{\alpha\beta} = \Omega^{L-2F}\left[Q_{\alpha\beta} - (L-2F+2C)\, g_{\alpha\beta}\, d\ln\Omega\right], \tag{1.221}$$

$$\tilde{T}^{\alpha} = \Omega^{F}\left[T^{\alpha} + (F-C)\, d\ln\Omega\wedge\vartheta^{\alpha}\right]. \tag{1.222}$$

The conformal transformation acts only on the trace pieces,

$$\tilde{T} = T + (C-F)(n-1)d\ln\Omega\,, \tag{1.223}$$

$$\tilde{Q} = Q - (L-2F+2C)\, d\ln\Omega\,. \tag{1.224}$$

Hence, torsion and nonmetricity can be transformed to zero if

$${}^{(1)}T^{\alpha} = {}^{(2)}T^{\alpha} = 0\,, \qquad dT = 0\,, \tag{1.225}$$

$${}^{(1)}Q_{\alpha\beta} = {}^{(2)}Q_{\alpha\beta} = {}^{(3)}Q_{\alpha\beta} = 0\,, \qquad dQ = 0\,, \tag{1.226}$$

where, if both, T and Q, are present we additionally need

$$T = -\frac{(C-F)(n-1)}{L-2F+C}\, Q\,. \tag{1.227}$$

The curvature is conformally invariant. In its definition enters only the connection. In our approach, the connection is transformed only by a piece proportional to a total differential, see eq.(1.220). This additional piece obviously does not contribute to the curvature $R_{\alpha}{}^{\beta} = d\Gamma_{\alpha}{}^{\beta} - \Gamma_{\alpha}{}^{\gamma}\wedge\Gamma_{\gamma}{}^{\beta}$. The Riemannian piece of the curvature does, of course, transform, see appendix A.4.7.

In conclusion it can be stated that we may keep the definition $C_{\alpha} = DL_{\alpha}$ also in Riemann-Cartan and metric-affine spaces. However, already in Riemann-Cartan space(time) the Cotton 2-form loses most of its distinctive properties which made it of specific interest in Riemannian spacetimes.

[3] *In order to compare our results to [58], in this section, we use C_{α} for the object of anholonomity and Ω for the conformal factor!*

Chapter 2

Gravity in three dimensions: Models and solutions

2.1 Introduction

On first sight, $(1+2)$–dimensional gravity seems to be rather boring. In 3 dimensions (3D), the Weyl tensor vanishes and the curvature is fully determined by the Ricci tensor and thus, via the Einstein equation, by the energy-momentum alone. Outside the sources the curvature is zero and there are no propagating degrees of freedom, i.e., no gravitational waves. Moreover, there is no Newtonian limit. But even if spacetime is flat, it is not trivial globally. A point particle, e.g., generates the spacetime geometry of a cone. In such a geometry we have light bending, double images, etc. The spacetime for N particles can be constructed similarly by gluing together patches of $(1+2)$D Minkowski space. This was occasionally investigated since the late 1950s, see Deser et al. [38] and the review of Carlip [23].

Some problems in $(1+3)$D gravity reduce to an effective $(1+2)$D theory, like the cosmic string, e.g.; the high–temperature behavior of $(1+3)$D theories also motivates the study of $(1+2)$D theories. In this context, Deser, Jackiw, and Tempelton (DJT) proposed a $(1+2)$D gravitational gauge model with topologically generated mass [40]. However, the real push for $(1+2)$D gravitational models came when Witten formulated the $(1+2)$D Einstein theory as a Chern–Simons theory, in a similar way as proposed by Achúcarro and Townsend [3], and showed its exact solvability in terms of a finite number of degrees of freedom [115, 116]. Also de Sitter gravity, conformal gravity, and supergravity, in $(1+2)$D, turn out to be equivalent to Chern–Simons theories [64, 73, 34, 35], see also the recent monograph of Blagojević [16].

Mielke and Baekler (MB) proposed a $(1+2)$D topological gauge model with torsion and curvature [84, 7, 85] from which the DJT–model can be derived by imposing the constraint of vanishing torsion by means of a Lagrange multiplier term. Gravitational theories in $(1+2)$D with torsion, see also Tresguerres [111] and Kawai [71], are analogous to the continuum theory of lattice defects in crystal physics, in particular, the corresponding theory of dislocations relates to a torsion of the underlying continuum, see Kröner [76], Kleinert [72], Dereli and Verçin [36, 37], Katanaev and Volovich [70], Kohler [74], and Puntigam and Soleng [101]. The fresh approach of Lazar [77, 78, 79] promises additional insight.

The next important impact on $(1+2)$D gravity was the discovery of a black hole solution by Bañados, Teitelboim, and Zanelli (BTZ) [11]. The BTZ black hole is locally isometric to anti-de Sitter (AdS) spacetime. It can be obtained, see Brill [20], from the AdS spacetime as a quotient of the latter with the group of finite isometries. It is asymptotically anti–de Sitter and has no curvature singularity at the origin. Nevertheless, it is clearly a black hole: it has an event horizon and, in the rotating case, an inner horizon. Also electrically and magnetically charged generalizations are known. For extensive discussions see the reviews [10, 24, 25, 23, 9, 15]. The relevance to $(1+3)$D gravity can also be seen from the fact that the BTZ solution can be derived from the $(1+3)$D Plebański–Carter metric by means of a dimensional reduction procedure, see Cataldo et al. [27]. By means of the BTZ solution, many interesting questions can be addressed in the context of quantum gravity. For example, Strominger computed the entropy of the BTZ black hole microscopically [109]. There is also a relationship between the BTZ black hole and string theory, see Hemming and Keski–Vakkuri [63]. Some recent work on solutions in 3 dimensional gravity can be found in [93], [39], e. g.

Thus, although $(1+2)$D gravity lacks many important features of real, $(1+3)$D gravity, it keeps enough characteristic structure to be of interest, especially in view of the fact that in the $(1+2)$D case many calculations can be done which are far too involved in $(1+3)$D for the time being.

In this chapter, we will first construct a general Lagrangian for quadratic Poincaré gauge theory enriched by a rotational and a translational Chern-Simons term and derive the corresponding field equations. Then we specify to a model proposed by Mielke and Baekler which could be called "Einstein-Cartan-Chern-Simons theory" (ECCS). We derive the general vacuum solution which carries constant axial curvature and constant axial torsion. From this we derive a suitable coframe and connection 1-form. We arrive at a generalized BTZ-solution with torsion, see table 2.1. Subsequently we determine the Killing vectors, the autoparallels and global charges. The ECCS theory contains various gravitational models as subcases:

- Teleparallel gravity in case of vanishing curvature and rotational Chern-Simons term,
- Topologically massive gravity (Deser-Jackiw-Tempelton model) in case of vanishing torsion,
- Einstein-Cartan theory in case of vanishing of both Chern-Simons terms,
- Einstein gravity in case of vanishing of both Chern-Simons terms and vanishing torsion.

We show that the global charges in all this subcases reduce to the results known from the literature. Subsequently we derive the subcases in a rigorous manner by means of imposing Lagrange multipliers. We show that our BTZ-solution with torsion is also a solution of proper teleparallelism and of 3D Einstein-Cartan theory with matter. We also derive a new solution of the DJT-model and a conformally flat perfect fluid solution of Einstein's theory, that is, an analog of the interior Schwarzschild solution.

2.2 Topological Poincaré gauge theory

Now we construct a Lagrangian for $n=3$ which encompasses the variables ϑ^α and ${\Gamma_\alpha}^\beta$ and the field-strengths T^α and ${R_\alpha}^\beta$. Additionally, we use the Hodge dual. We consider a first order field theory. Moreover we demand the Lagrangian to be at most quadratic in the field strengths and cubic in the gauge potentials.

First, we consider 3-forms linear in the field strengths. In the case of the curvature we need a 1-form with two indices. The natural choice for this is $\eta_{\alpha\beta}$, yielding

$$V_{\rm EC} := -\frac{1}{2\ell}\, R^{\alpha\beta} \wedge \eta_{\alpha\beta}\,. \tag{2.1}$$

This is the usual Einstein-Cartan (Hilbert-Einstein in the case of vanishing torsion) Lagrangian. We have introduced a *fundamental length* ℓ in order to guarantee the same physical dimension of all terms in the Lagrangian. To build a 3-form linear in torsion, we are in need of a 1-form with one index. Therefore, we may use the coframe ϑ^α,

$$\mathcal{C}_{\rm T} := \frac{1}{2\ell^2}\, \vartheta^\alpha \wedge T_\alpha\,. \tag{2.2}$$

Since ϑ_α may be interpreted as gauge potential of local translations, we call $\mathcal{C}_{\rm T}$ the *translational Chern-Simons* term. The Chern-Simons 3-form for the curvature reads [55]

$$\mathcal{C}_{\rm L} := -\frac{1}{2}\left({\Gamma_\alpha}^\beta \wedge d{\Gamma_\beta}^\alpha - \frac{2}{3}\,{\Gamma_\alpha}^\beta \wedge {\Gamma_\beta}^\gamma \wedge {\Gamma_\gamma}^\alpha\right)\,. \tag{2.3}$$

This term is cubic in the gauge potential Γ. In 4D, $\mathcal{C}_{\rm L}$ is a boundary term and its exterior derivative $d\mathcal{C}_{\rm L} = -R^{\alpha\beta} \wedge R_{\alpha\beta}/2$ is quadratic in the curvature.

Second, we turn to 3-forms quadratic in the field strengths. They are built up from the field strength and linear combinations of its contractions multiplied by its dual. It was shown that these products can be most appropriately given in terms of the irreducible pieces of the field strengths. The torsion has 3 irreducible pieces, a tensor piece ${}^{(1)}T^\alpha$, a trace piece ${}^{(2)}T^\alpha$, and an axial piece ${}^{(3)}T^\alpha$ (for their definition see appendix A.3 and [58]). The curvature in general has 6 irreducible components. However, in 3 dimensions only three survive, namely the scalar piece ${}^{(6)}R^{\alpha\beta}$, the tracefree symmetric Ricci piece ${}^{(4)}R^{\alpha\beta}$, and the (post-Riemannian) antisymmetric Ricci piece ${}^{(5)}R^{\alpha\beta}$. For the precise definition of the irreducible pieces, we refer to appendix A.3. Then we can write the most general quadratic pieces of the Lagrangian as

$$V_{\mathrm{T}^2} = \frac{1}{2\ell}T^\alpha \wedge {}^\star\left(a_1\,{}^{(1)}T_\alpha + a_2\,{}^{(2)}T_\alpha + a_3\,{}^{(3)}T_\alpha\right)\,, \tag{2.4}$$

$$V_{\mathrm{R}^2} = \frac{1}{2\ell^2}\,R^{\alpha\beta} \wedge {}^\star\left(b_4\,{}^{(4)}R_{\alpha\beta} + b_5\,{}^{(5)}R_{\alpha\beta} + b_6\,{}^{(6)}R_{\alpha\beta}\right)\,. \tag{2.5}$$

Finally, we include a cosmological term

$$V_\Lambda = -\frac{\Lambda}{\ell}\,\eta\,. \tag{2.6}$$

Thus, our Lagrangian reads

$$V_\infty = \chi\,V_{\mathrm{EC}} + V_\Lambda + V_{\mathrm{T}^2} + V_{\mathrm{R}^2} + \theta_{\mathrm{T}}\,\mathcal{C}_{\mathrm{T}} + \theta_{\mathrm{L}}\,\mathcal{C}_{\mathrm{L}}\,. \tag{2.7}$$

We multiply the Einstein-Cartan piece with a dimensionless constant χ and the Chern-Simons parts with "vacuum angles" θ_{T} and θ_{L}. This Lagrangian includes many known models of $(1+2)$D gravity. For $\chi = 1$, $a_1 = a_2 = a_3 = 0$, $b_4 = b_5 = b_6 = 0$, $\theta_{\mathrm{T}} = \theta_{\mathrm{L}} = 0$, we arrive at the standard Einstein theory with cosmological constant in $(1+2)$D. If we include the curvature Chern-Simons term ($\theta_{\mathrm{L}} \neq 0$) and the Riemannian quadratic curvature terms ($a_1 = a_2 = a_3 = 0$, $b_5 = 0$), we find the "quadratic gravity in $(2+1)$D with topological Chern-Simons term" advocated by Accioly et al.[2]. By setting the quadratic terms to zero and keeping Einstein-Cartan, cosmological, and both Chern-Simons terms, we recover the 3D topological gravity model of Mielke and Baekler [84, 7].

Our following investigation will mainly circle around solutions with constant axial torsion. The most general case we consider in this context is the Mielke-Baekler model which also may be called *Einstein-Cartan-Chern-Simons* theory (ECCS). We find the general vacuum solution which is a prolongation of the well-known BTZ solution of 3D Einstein gravity. Subsequently we show that this solution also solves the field equations of Einstein teleparallel gravity and Einstein-Cartan theory. The Deser-Jakiw-Tempelton (DJT) [40] model of topological massive gravity arises by means of imposing a Lagrange multiplier for vanishing torsion. Thus, we consider the following Lagrangian

$$\begin{aligned} V = {} & \frac{1}{2\ell}\left[-\chi\,R^{\alpha\beta}\wedge\eta_{\alpha\beta} - 2\,\Lambda\,\eta + T^\alpha\wedge{}^\star\left(\sum_{I=1}^{3} a_I\,{}^{(I)}T_\alpha\right)\right] \\ & + \frac{\Theta_T}{2\ell^2}\,\vartheta^\alpha\wedge T_\alpha - \frac{\Theta_L}{2}\left(\Gamma_\alpha{}^\beta\wedge d\Gamma_\beta{}^\alpha - \frac{2}{3}\,\Gamma_\alpha{}^\beta\wedge\Gamma_\beta{}^\gamma\wedge\Gamma_\gamma{}^\alpha\right) \\ & + Q_{\alpha\beta}\wedge\mu^{\alpha\beta} + T^\alpha\wedge\lambda_\alpha + R_\alpha{}^\beta\wedge\lambda^\alpha{}_\beta\,. \end{aligned} \tag{2.8}$$

The field equations follow from the variation with respect to $g_{\alpha\beta}$ (zeroth field equation), ϑ^α (first field equation), $\Gamma_\alpha{}^\beta$ (second field equation), and the Lagrange multipliers (constraint equations). However, in [58] it is shown that the zeroth field equation can be eliminated by means of the Noether identities. We are then left with the first field equation and the antisymmetric part of the second field equation. These read

$$\begin{aligned} DH_\alpha + \frac{\chi}{2\ell}\,\eta_{\alpha\beta\gamma}\,R^{\beta\gamma} + \frac{\Lambda}{\ell}\,\eta_\alpha - \frac{\theta_{\mathrm{T}}}{\ell^2}\,T_\alpha - E_\alpha - D\lambda_\alpha &= \Sigma_\alpha\,, && (2.9)\\ \frac{\chi}{2\ell}\,\eta_{\alpha\beta\gamma}\,T^\gamma - \frac{\theta_{\mathrm{T}}}{2\ell^2}\,\vartheta_\alpha\wedge\vartheta_\beta - \theta_{\mathrm{L}}\,R_{\alpha\beta} + \vartheta_{[\alpha}\wedge H_{\beta]} - D\lambda_{[\alpha\beta]} - \vartheta_{[\alpha}\wedge\lambda_{\beta]} &= \tau_{\alpha\beta}\,, && (2.10)\end{aligned}$$

where we also included the respective matter currents and

$$\begin{aligned} H_\alpha &= -\frac{1}{\ell}\,{}^\star\left(\sum_{I=1}^{3} a_I\,{}^{(I)}T_\alpha\right), && (2.11)\\ E_\alpha &= \frac{1}{2}\,\left[(e_\alpha\rfloor T^\beta)\wedge H_\beta - T^\beta\wedge(e_\alpha\rfloor H_\beta)\right]. && (2.12)\end{aligned}$$

2.3 Einstein-Cartan-Chern-Simons theory

In this section we assume

$$\lambda_\alpha = \lambda^\alpha{}_\beta = a_1 = a_2 = a_3 = 0\,, \qquad (2.13)$$

and obtain the Lagrangian

$$V_{\mathrm{MB}} = \chi V_{\mathrm{EC}} + V_\Lambda + \theta_{\mathrm{T}}\,\mathcal{C}_{\mathrm{T}} + \theta_{\mathrm{L}}\,\mathcal{C}_{\mathrm{L}} + L_{\mathrm{mat}}\,, \qquad (2.14)$$

where we included a matter Lagrangian. We find the field equations by variation with respect to coframe and connection, respectively:

$$\begin{aligned} \frac{\chi}{2}\,\eta_{\alpha\beta\gamma}\,R^{\beta\gamma} + \Lambda\,\eta_\alpha - \frac{\theta_{\mathrm{T}}}{\ell}\,T_\alpha &= \ell\,\Sigma_\alpha\,, && (2.15)\\ \frac{\chi}{2}\,\eta_{\alpha\beta\gamma}\,T^\gamma - \frac{\theta_{\mathrm{T}}}{2\ell}\,\vartheta_\alpha\wedge\vartheta_\beta - \theta_{\mathrm{L}}\,\ell\,R_{\alpha\beta} &= \ell\,\tau_{\alpha\beta}\,. && (2.16)\end{aligned}$$

The 3-forms of the material energy-momentum and spin currents are defined by $\Sigma_\alpha := \delta L_{\mathrm{mat}}/\delta\vartheta^\alpha$ and $\tau_{\alpha\beta} := \delta L_{\mathrm{mat}}/\delta\Gamma^{\alpha\beta}$, respectively. For vacuum, $\Sigma_\alpha = 0$, $\tau_{\alpha\beta} = 0$, we can algebraically solve the two equations with respect to curvature $R^{\alpha\beta}$ and torsion T^α. It turns out that both, $R^{\alpha\beta}$ and T^α, have only an axial piece with one independent component:

$$R_{\alpha\beta} \sim \vartheta_\alpha\wedge\vartheta_\beta\,, \qquad T_\alpha \sim \eta_\alpha\,. \qquad (2.17)$$

The coefficients can be expressed in terms of the coupling constants. We put the right-hand side of (2.16) to zero and multiply it by $\chi/(2\theta_{\rm L}\ell)\,\eta^{\mu\alpha\beta}$. We arrive at ($\theta_{\rm L}\neq 0$)

$$\frac{\chi^2}{4\theta_{\rm L}\ell}\underbrace{\eta^{\mu\alpha\beta}\,\eta_{\alpha\beta\gamma}}_{=-2\,\delta^\mu_\gamma}T^\gamma-\frac{\theta_{\rm T}\chi}{4\ell^2\theta_{\rm L}}\underbrace{\eta^{\mu\alpha\beta}\,\vartheta_\alpha\wedge\vartheta_\beta}_{=2\,\eta^\mu}-\frac{\chi}{2}\,\eta^{\mu\alpha\beta}\,R_{\alpha\beta}=0\,. \tag{2.18}$$

We substitute this into the first field equation (2.15). Together with (2.16), we find

$$T_\alpha = \frac{2\Lambda\ell^2\theta_{\rm L}-\theta_{\rm T}\chi}{\chi^2+2\theta_{\rm T}\theta_{\rm L}}\;\frac{\eta_\alpha}{\ell} =: 2\,\frac{\mathcal{T}}{\ell}\,\eta_\alpha\,, \tag{2.19}$$

$$R_{\alpha\beta} = -\frac{\chi\Lambda\ell^2+\theta_T^2}{\chi^2+2\theta_{\rm T}\theta_{\rm L}}\;\frac{\vartheta_\alpha\wedge\vartheta_\beta}{\ell^2} =: \frac{\mathcal{R}}{\ell^2}\,\vartheta_\alpha\wedge\vartheta_\beta\,, \tag{2.20}$$

where

$$\chi^2+2\theta_{\rm T}\theta_{\rm L}\neq 0 \tag{2.21}$$

has to be required. By redoing this derivation for the case $\theta_{\rm L}=0$, we see that (2.19) and (2.20) remain valid. Thus (2.21) is the only constraint of the coupling constants.

Eqs.(2.19) and (2.20) represent the *general exact vacuum solution* of the Einstein-Cartan-Chern-Simons model with the MB-Lagrangian (2.14).

2.3.1 Vacuum solution of the ECCS theory

The (1+2)–dimensional model of Mielke and Baekler contains the DJT model in the limit of vanishing torsion. Therefore it is expected that the general solution of the field equations (2.15) and (2.16) contains a kind of BTZ–like solution with torsion which, in the same limit, reduces to the standard BTZ solution.

According to (2.19) and (2.20), the vacuum solution has constant curvature and torsion. Thus, we make a static and spherically symmetric ansatz for the (orthonormal) coframe,

$$\vartheta^{\hat t} = N(r)\,dt\,, \tag{2.22}$$

$$\vartheta^{\hat r} = \frac{dr}{N(r)}\,, \tag{2.23}$$

$$\vartheta^{\hat\phi} = G(r)\,\left[-W(r)\,dt+d\phi\right]\,, \tag{2.24}$$

where $N(r)$, $G(r)$ and $W(r)$ are free functions. Our ansatz for $\vartheta^{\hat r}$ does not restrict the generality. We could introduce another free function $\vartheta^{\hat r}=dr/F(r)$. However, rescaling of the radial coordinate according to $dr=(dR/N)\,F$ leads back to our ansatz.

The metric reads

$$g=-\vartheta^{\hat t}\otimes\vartheta^{\hat t}+\vartheta^{\hat r}\otimes\vartheta^{\hat r}+\vartheta^{\hat\phi}\otimes\vartheta^{\hat\phi}\,. \tag{2.25}$$

Together with the torsion, see (2.19),

$$T^\alpha = 2\,\frac{\mathcal{T}}{\ell}\,\eta^\alpha\,, \quad \mathcal{T} = \frac{2\Lambda\ell^2\theta_{\rm L} - \theta_{\,\rm T}\chi}{2\,(\chi^2 + 2\theta_{\rm T}\theta)}\,, \tag{2.26}$$

we calculate the connection according to, c. f. eq.(1.38), ($\Omega^\alpha := d\vartheta^\alpha$)

$$\Gamma_{\alpha\beta} = e_{[\alpha}\rfloor\Omega_{\beta]} - \frac{1}{2}\,(e_\alpha\rfloor e_\beta\rfloor\Omega_\gamma)\,\vartheta^\gamma - e_{[\alpha}\rfloor T_{\beta]} + \frac{1}{2}\,(e_\alpha\rfloor e_\beta\rfloor T_\gamma)\,\vartheta^\gamma\,,$$

see also [58] Eq.(3.10.6), for $dg_{\alpha\beta} = 0$ and $Q_{\alpha\beta} = 0$. Subsequently, we compute the curvature and the components of the field equations. Linear combination of the components of the 1st field equation (2.15) yields

$$\frac{d^2\,G}{dr^2}\,\frac{\chi n^2}{G} = 0\,. \tag{2.27}$$

Consequently, we put

$$G = A + Br\,, \tag{2.28}$$

where A and B are integration constants. After substitution of (2.28) into (2.15), one component turns out to be

$$\chi\,\frac{N}{2}\,\left[\frac{d^2\,W}{dr^2}\,(A+Br) + 3B\,\frac{d\,W}{dr}\right] = 0\,. \tag{2.29}$$

This requires

$$W = \frac{\alpha}{(A+Br)^2} + \beta\,. \tag{2.30}$$

Again we have introduced two integration constants, α and β. Together with (2.30), the 1st field equation (2.15) yields a first order ordinary differential equation for N^2, which can be integrated (m is another integration constant),

$$N^2(r) = m + \frac{\alpha^2}{(rB+A)^2} - \frac{\Lambda_{\rm eff}}{B^2}\,\left(A^2 - 2ABr - B^2r^2\right)\,, \tag{2.31}$$

where we introduced an effective cosmological constant

$$\Lambda_{\rm eff} = \frac{\mathcal{T}^2 + \mathcal{R}}{\ell^2}\,. \tag{2.32}$$

This agrees with [7] Eq.(9.3). By means of the coordinate transformation $r \to Ar+B$ and $\phi \to \phi + \beta\,t$ and some change in notation, we arrive at our new BTZ-like solution with torsion, see Table 2.1 for its explicit form. The topological terms in the Lagrangian induce an effective cosmological constant even if the 'bare' cosmological constant Λ vanishes. If we put $\theta_{\rm L} = \theta_{\rm T} = 0$, then $\Lambda_{\rm eff} = -\Lambda$ and $T^\alpha = 0$, and we fall back to the standard BTZ solution [11].

The $M = 1$ and $J = 0$ metric is then recognized as the three–dimensional anti–de Sitter space AdS_3. It is straightforward to see that the general solution (2.19) and (2.20) represents a space of constant negative curvature. Thus, the BTZ black hole with torsion is also locally isometric to AdS_3.

Table 2.1: **Exact vacuum solution of the 3D Einstein-Cartan-Chern-Simons theory: BTZ-like solution with torsion**

vacuum field eqs.	$\frac{\chi}{2}\,\eta_{\alpha\beta\gamma}\,R^{\beta\gamma} + \Lambda\,\eta_\alpha - \frac{\theta_{\rm T}}{\ell}\,T_\alpha \;=\; 0$	(2.33)
	$\frac{\chi}{2}\,\eta_{\alpha\beta\gamma}\,T^{\gamma} - \frac{\theta_{\rm T}}{2\ell}\,\vartheta_\alpha\wedge\vartheta_\beta - \theta_{\rm L}\,\ell\,R_{\alpha\beta} \;=\; 0\,.$	(2.34)
coframe	$\vartheta^{\hat t} = \psi(r)\,dt$	(2.35)
	$\vartheta^{\hat r} = \frac{dr}{\psi(r)} \qquad \psi(r) := \sqrt{\left(\frac{J}{2r}\right)^2 - M + \Lambda_{\rm eff}\,r^2}$	(2.36)
	$\vartheta^{\hat\phi} = r\left(-\frac{J}{2r^2}\,dt + d\phi\right)$	(2.37)
metric	$g = -\vartheta^{\hat t}\otimes\vartheta^{\hat t} + \vartheta^{\hat r}\otimes\vartheta^{\hat r} + \vartheta^{\hat\phi}\otimes\vartheta^{\hat\phi}$	(2.38)
connection	$\Gamma^{\hat t\hat r} = -\Gamma^{\hat r\hat t} = \left(\frac{\mathcal{T}}{\ell}\,\frac{J}{2r} - \Lambda_{\rm eff}\,r\right)dt + \left(\frac{J}{2r} - \frac{\mathcal{T}}{\ell}\,r\right)\,d\phi$	(2.39)
	$\Gamma^{\hat r\hat\phi} = -\Gamma^{\hat\phi\hat r} = \psi(r)\,\left(\frac{\mathcal{T}}{\ell}\,dt + d\phi\right)$	(2.40)
	$\Gamma^{\hat\phi\hat t} = -\Gamma^{\hat t\hat\phi} = -\left(\frac{J}{2r^2} + \frac{\mathcal{T}}{\ell}\right)\,\frac{dr}{\psi(r)}$	(2.41)
torsion	$T^\alpha = 2\,\frac{\mathcal{T}}{\ell}\,\eta^\alpha$	(2.42)
curvature	Riemann-Cartan $\quad R^{\alpha\beta} = \frac{\mathcal{R}}{\ell^2}\,\vartheta^\alpha\wedge\vartheta^\beta$	(2.43)
	Riemann $\quad \widetilde{R}^{\alpha\beta} = \Lambda_{\rm eff}\,\vartheta^\alpha\wedge\vartheta^\beta$	(2.44)
Cotton	Riemann-Cartan $\quad C^\alpha = -\frac{\mathcal{T}\,\mathcal{R}}{\ell^3}\,\eta^\alpha$	(2.45)
	Riemann $\quad \widetilde{C}^\alpha = 0$	(2.46)
constants	$\mathcal{T} := \frac{-\frac{\theta_{\rm T}}{2}\,\chi + \Lambda\ell^2\theta_{\rm L}}{\chi^2 + 2\theta_{\rm T}\theta_{\rm L}} \qquad \mathcal{R} := -\frac{\theta_{\rm T}^2 + \chi\Lambda\ell^2}{\chi^2 + 2\theta_{\rm T}\theta_{\rm L}} \qquad \Lambda_{\rm eff} := \frac{\mathcal{T}^2 + \mathcal{R}}{\ell^2}$	(2.47)

2.3.2 General conformally flat vacuum solution with torsion

As we have seen, the vacuum field equations (2.15), (2.16) imply constant *Riemann-Cartan* curvature. We can decompose the Riemann-Cartan curvature into the *Riemannian* curvature $\widetilde{R}_{\alpha\beta}$ and a torsion dependent part,

$$R_{\alpha\beta} = \widetilde{R}_{\alpha\beta} - D\,K_{\alpha\beta} + K_{\alpha}{}^{\gamma} \wedge K_{\gamma\beta}\,, \tag{2.48}$$

where the contortion $K_{\alpha}{}^{\beta}$ is defined according to

$$T^{\alpha} =: \vartheta^{\beta} \wedge K_{\beta}{}^{\alpha}\,. \tag{2.49}$$

Then also the Riemannian curvature turns out to be constant, see appendix A.4.10. Consequently, the Riemannian Cotton 2-form is zero and the metric has to be conformally flat. Hence we make the ansatz

$$\vartheta^{\hat{0}} = \frac{dx}{\Psi}\,, \qquad \vartheta^{\hat{1}} = \frac{dy}{\Psi}\,, \qquad \vartheta^{\hat{2}} = \frac{dz}{\Psi}\,, \tag{2.50}$$

where $\Psi = \Psi(x,y,z)$. The 1st field equation (2.15) yields that the mixed second derivatives vanish. Therefore Ψ has the form

$$\Psi = \Psi^{(x)}(x) + \Psi^{(y)}(y) + \Psi^{(z)}(z)\,. \tag{2.51}$$

Then, the 1st field equation requires

$$-\partial_{yy}\,\Psi^{(y)} = \partial_{xx}\,\Psi^{(x)} = -\partial_{zz}\Psi^{(z)}\,. \tag{2.52}$$

This leads to the general solution with 5 Parameters A, B, C, D, E

$$\Psi = A\,\left(-x^2 + y^2 + z^2\right) + Bx + Cy + Dz + E\,. \tag{2.53}$$

The field equations impose only one constraint equation on the parameters,

$$0 = B^2 - C^2 - D^2 + 4AE + \Lambda_{\text{eff}}\,. \tag{2.54}$$

For $B = C = D = 0$, $E = 1$ we recover the usual form of the (anti-) de Sitter metric, for $A = B = D = E = 0$ the Poincaré metric. Coordinate transformations which yield the BTZ-metric are given in [23].

In the anti-de Sitter case we can display the solution, coframe and connection, very compactly as

$$\vartheta^{\alpha} = \frac{dx^{\alpha}}{\psi}\,, \qquad \psi = 1 - \frac{\Lambda_{\text{eff}}}{4}(-x^2 + y^2 + z^2)\,, \tag{2.55}$$

$$\Gamma^{\alpha\beta} = \frac{\mathcal{T}}{\ell}\,\eta^{\alpha\beta} + x^{[\alpha}\,\vartheta^{\beta]}\,\frac{\Lambda_{\text{eff}}}{3}\,. \tag{2.56}$$

For $\theta_T = 0$, this corresponds to the solution of Dereli and Verçin [37].

In the "teleparallel" case, where the Riemann-Cartan curvature vanishes, we have $\mathcal{T} = \ell\sqrt{\Lambda_{\text{eff}}}$. Then we recover the solution of Fjelstad and Hwang [45].

For $\Lambda_{\text{eff}} = 0$, i.e. vanishing Riemannian curvature, we recover Cartan's *spiral staircase solution* discussed in [26] as an example for a 3D space with torsion. However, the requirement $\Lambda_{\text{eff}} = 0 = \mathcal{R} + \mathcal{T}^2$ amounts to a constraint on the coupling parameters. We will discuss this case in section 2.5.

2.3.3 Properties of our solution

Autoparallels and extremals

In a Riemann–Cartan space, the autoparallels (straightest lines) and the extremals or geodesics (longest/shortest lines) do not coincide in general. An autoparallel curve $x^i(s)$ obeys, in terms of a suitable affine parameter s, the equation

$$\frac{d^2\, x^k(s)}{ds^2} + \Gamma_{ij}{}^k \frac{d\, x^i(s)}{ds} \frac{d\, x^j(s)}{ds} = 0\,. \tag{2.57}$$

The (holonomic) components of the connection $\Gamma_{ij}{}^k$ depend on metric and torsion according to

$$\Gamma_{ij}{}^k = \widetilde{\Gamma}_{ij}{}^k - K_{ij}{}^k\,, \qquad K_{ij}{}^k := \frac{1}{2}\left(-T_{ij}{}^k + T_j{}^k{}_i - T^k{}_{ij}\right)\,, \tag{2.58}$$

where $\widetilde{\Gamma}_{ij}{}^k$ is the Christoffel symbol and $K_{ij}{}^k$ the contortion. In (2.57), only the symmetric part of the connection enters. By means of (2.58), it can be expressed as

$$\Gamma_{(ij)}{}^k = \widetilde{\Gamma}_{(ij)}{}^k + T^k{}_{(ij)}\,. \tag{2.59}$$

The extremals are determined by the metrical properties of spacetime alone and follow from the variation of the world length $\int \sqrt{-g_{ij}\,\dot{x}^i\,\dot{x}^j}$ in the standard way:

$$\frac{d^2\, x^k(s)}{ds^2} + \widetilde{\Gamma}_{ij}{}^k \frac{d\, x^i(s)}{ds} \frac{d\, x^j(s)}{ds} = 0\,. \tag{2.60}$$

For our solution, see Table 1,

$$T_{ijk} = 2\,\frac{\mathcal{T}}{\ell}\,\eta_{ijk} \qquad \Longrightarrow \qquad T_{i(jk)} = 0\,. \tag{2.61}$$

Thus, the torsion dependent piece drops out in (2.59) and (2.57). Autoparallels and extremals coincide and we get the same geodesics as in the case of the standard BTZ–solution in Riemannian spacetime.

Killing vectors

In a Riemann-Cartan space we call $\xi = \xi^\alpha\, e_\alpha$ a Killing vector if the latter is the generator of a symmetry transformation of the metric and of the connection according to

$$\pounds_\xi\, g = 0\,, \quad \pounds_\xi\, \Gamma_\alpha{}^\beta = 0\,, \tag{2.62}$$

see [58, p.83]. These two relations can be recast into a more convenient form,

$$e_{(\alpha} \rfloor \widetilde{D}\xi_{\beta)} \;=\; 0\,, \tag{2.63}$$

$$D\left(e_\alpha \rfloor\, \widehat{D}\,\xi^\beta\right) + \xi \rfloor R_\alpha{}^\beta \;=\; 0\,, \tag{2.64}$$

where $\widetilde{D}$ refers to the Riemannian part of the connection (Levi–Civita connection) and $\widehat{D}$ to the transposed connection: $\widehat{D} := d + \widehat{\Gamma}_\alpha{}^\beta := d + \Gamma_\alpha{}^\beta + e_\alpha \rfloor T^\beta$. For our solution we find two Killing vectors, namely

$$\overset{(t)}{\xi} := \partial_t \quad \text{and} \quad \overset{(\phi)}{\xi} := \partial_\phi\,, \tag{2.65}$$

that is, the same Killing vectors as in the case of the standard BTZ solution.

Quasilocal conserved quantities

Now we consider the conserved quantities of our solution. Nester, Chen, and Wu [90], see also the literature quoted there, proposed a quasi–local boundary expression within metric–affine gravity, a theory the spacetime of which goes beyond the Riemann–Cartan structure in that it carries additionally a nonmetricity. We adapt the formulas of [90] for the case of vanishing nonmetricity. The derivation starts from a first–order Lagrange n–form V that is at most quadratic in its field strengths T^α and $R^{\alpha\beta}$. The corresponding momenta read $H_\alpha := -\partial V/\partial T^\alpha$ and $H_{\alpha\beta} := -\partial V/\partial R^{\alpha\beta}$. The Lagrangian can be decomposed with respect to a vector field N, with $N\rfloor d\nu = 1$:

$$\begin{aligned} V &= d\nu \wedge N\rfloor V \\ &=: d\nu \wedge \Big[-\left(\pounds_N \vartheta^\alpha\right) \wedge H_\alpha - \left(\pounds_N \Gamma_\alpha{}^\beta\right) \wedge H^\alpha{}_\beta - N^\alpha \mathfrak{H}_\alpha - d\,\mathfrak{B}\Big]\,. \end{aligned} \tag{2.66}$$

The Hamilton 2–form $\mathfrak{H}$ is defined by $\mathfrak{H} := N^\alpha\,\mathfrak{H}_\alpha + d\,\mathfrak{B}$. Since $\mathfrak{H}_\alpha$ turns out to be proportional to the field equations, only the spatial boundary 1–form $\mathfrak{B}$ contributes to the boundary integral of $\mathfrak{H}$. In order to obtain finite values for the quasi–local "charges", the boundary term has to be compared to a reference or background solution which will be denoted by a bar over the corresponding symbol. As background, we choose our solution with $M = 0$, $J = 0$. Moreover, the difference of a quantity α between a solution and the background is $\Delta\alpha := \alpha - \overline{\alpha}$. Then, the quasi–local charges are given by [90]

$$\begin{aligned} \mathfrak{B}(N) \;:=\; & -\left\{ \begin{array}{lcl} (N\rfloor\vartheta^\alpha)\,\Delta H_\alpha & + & \Delta\vartheta^\alpha\left(N\rfloor\overline{H}_\alpha\right) \\ (N\rfloor\overline{\vartheta}^\alpha)\,\Delta H_\alpha & + & \Delta\vartheta^\alpha\left(N\rfloor H_\alpha\right) \end{array} \right\} \\ & -\left\{ \begin{array}{lcl} (\widehat{D}\,{}^\alpha N^\beta)\,\Delta H_{\alpha\beta} & + & \Delta\Gamma^{\alpha\beta}\left(N\rfloor\overline{H}_{\alpha\beta}\right) \\ \overline{(\widehat{D}\,{}^\alpha N^\beta)}\,\Delta H_{\alpha\beta} & + & \Delta\Gamma^{\alpha\beta}\left(N\rfloor H_{\alpha\beta}\right) \end{array} \right\}. \end{aligned} \tag{2.67}$$

The upper (lower) line in the braces is chosen if the field strengths (momenta) are prescribed on the boundary. The momenta of our solution read $H_\alpha = -(\theta_{\rm T}/2\ell^2)\,\vartheta_\alpha$ and $H_{\alpha\beta} = (\chi/2\ell)\,\eta_{\alpha\beta} - (\theta_{\rm L}/2)\,\Gamma_{\alpha\beta}$.

We derive the quasi–local energy and angular momentum by taking for the vector field N the Killing vectors ∂_t or ∂_ϕ, respectively:

$$\begin{aligned}
\ell\,\mathfrak{B}(\partial_t) &= \left[\theta_{\rm L}\left(\Lambda_{\rm eff}\ell\,J-\mathcal{T}\,M\right)+\chi\left(\Lambda_{\rm eff}r^2-\sqrt{\Lambda_{\rm eff}}\,r\psi\right)\right]d\phi\\
&\quad -\frac{1}{2\ell}\left[\left(2\theta_{\rm L}\,\mathcal{T}^2-\theta_{\rm T}\right)M-2\ell\theta_{\rm L}\,\Lambda_{\rm eff}J\mathcal{T}+\chi\left(\ell\Lambda_{\rm eff}J-2M\mathcal{T}\right)\right]\,dt, \qquad (2.68)\\
\ell\,\mathfrak{B}(\partial_\phi) &= -\left[\frac{\chi}{2}\,J+\theta_{\rm L}\left(\ell\,M-\mathcal{T}\,J\right)\right]d\phi\\
&\quad -\left[\chi\left(\Lambda_{\rm eff}r^2-\sqrt{\Lambda_{\rm eff}}\,r\psi\right)+\frac{1}{\ell}\left(\theta_{\rm L}\mathcal{T}-\chi\right)\left(\ell\,M-\mathcal{T}\,J\right)+\frac{\theta_{\rm T}}{2\ell}\,J\right]dt\,. \qquad (2.69)
\end{aligned}$$

In order to obtain total energy and angular momentum, we have to integrate, for $t=\text{const}$, the $\mathfrak{B}$'s over a full circle and to perform the limit $r\rightarrow\infty$.

$$E_\infty = \frac{1}{\ell}\lim_{r\to\infty}\int_0^{2\pi}\left[\theta_{\rm L}\left(\Lambda_{\rm eff}\ell\,J-\mathcal{T}\,M\right)+\chi\left(\Lambda_{\rm eff}r^2-\sqrt{\Lambda_{\rm eff}}\,r\psi\right)\right]d\phi\,, \qquad (2.70)$$

$$L_\infty = \frac{1}{\ell}\lim_{r\to\infty}\int_0^{2\pi}-\left[\frac{\chi}{2}\,J+\theta_{\rm L}\left(\ell\,M-\mathcal{T}\,J\right)\right]d\phi\,. \qquad (2.71)$$

By carrying out the integration and taking the limit we find[1]

$$E_\infty = \frac{2\pi}{\ell}\left[\theta_L\left(\Lambda_{\rm eff}\,\ell J-\mathcal{T}\,M\right)+\chi\,\frac{M}{2}\right], \qquad (2.72)$$

$$L_\infty = -\frac{2\pi}{\ell}\left[\frac{\chi}{2}\,J+\theta_L\left(\ell M-\mathcal{T}\,J\right)\right]. \qquad (2.73)$$

In each case occur admixtures from the other charge, respectively. This is not too surprising, since torsion and curvature emerge in both field equations. We can recover some specific cases well known from the literature and compare our results.

In the first place, of course, we consider 3D Einstein theory, where only the Riemann-Cartan and cosmological constant terms survive,

$$\text{Einstein theory} \qquad \Theta_T=\Theta_L=0\,,\quad \chi=1\,,\quad \Lambda\neq 0\,. \qquad (2.74)$$

Indeed we arrive at the usual interpretation of the parameters M and J as mass and angular momentum,

$$E_\infty=\frac{\pi}{\ell}\,M\,,\qquad L_\infty=-\frac{\pi}{\ell}\,J\,, \qquad (2.75)$$

provided we identify the 3D gravitational constant as $\ell=\pi$ and introduce a factor -1 into the angular momentum. The last point seems to be somewhat awkward. However, one has to keep in mind that a corresponding factor $-1/2$ is also introduced in classical general relativity, see Wald [113]. Moreover we have to keep in

[1] In our publication, [47], we adapted the choice $\ell=\pi$ and introduced a factor -1 into the angular momentum. We will discuss this choice on the next page.

mind that there is no clear and unambiguous prescription of which timelike and spacelike Killing vectors are to be used in order to get the "right" energy and angular momentum. The parameters J and M can be expressed in terms of the proper angular momentum L_∞ and the proper mass-energy E_∞:

$$J = -\frac{\ell}{\pi}\,\frac{(\chi - 2\theta_L\,\mathcal{T})\,L_\infty + 2\ell\theta_L\,E_\infty}{\chi^2 - 4\chi\theta_L\,\mathcal{T} - 4\theta_L^2\,\mathcal{R}} =: -\frac{\ell}{\pi}\,(A\,L_\infty + B\,E_\infty)\,, \tag{2.76}$$

$$M = \frac{\ell}{\pi}\,\frac{(\chi - 2\theta_L\,\mathcal{T})\,E_\infty + 2\Lambda_{\text{eff}}\,\ell\theta_L\,L_\infty}{\chi^2 - 4\chi\theta_L\,\mathcal{T} - 4\theta_L^2\,\mathcal{R}} =: \frac{\ell}{\pi}\,(A\,E_\infty + C\,L_\infty)\,. \tag{2.77}$$

The Killing equations and the quasi-local charges are *linear* in the Killing vectors. Hence, we could define the new Killing vectors

$$\xi^{(t)} := A\,\partial_t + C\,\partial_\phi\,, \qquad \xi^{(\phi)} := -A\,\partial_t - B\,\partial_\phi\,. \tag{2.78}$$

These "automatically" yield the canonical M and J as mass and angular momentum, respectively.

Also the "teleparallel limit" gives the expected result. Here we assume $\theta_L = 0$ and $\mathcal{R} = 0$. This implies $\theta_T^2 = -\chi\Lambda\ell^2$, i. e. we have to keep the Einstein-Cartan piece. We will understand the reason in the next section. This also explains that θ_T does not independently contribute to the charges: Without either Riemann-Cartan or rotational Chern-Simons piece the corresponding Lagrangian allows only vanishing torsion and curvature! Thus, we fall back to the Einsteinian case, as it should be since it turns out that the MB-Lagrangian constrained to yield vanishing curvature is indeed equivalent to the teleparallel Einstein Lagrangian.

Finally, we consider the subcase of vanishing torsion. As we are going to show in section 2.6 this corresponds to the DJT model of topologically massive gravity. We find it especially satisfactory that we obtain a result in agreement with the literature. In [87] the authors assume the following identification of parameters

$$2\ell\theta_L = -\frac{1}{\mu}\,, \quad \Lambda_{\text{eff}} = -\Lambda = l^{-2}\,, \quad \ell = \kappa = \pi\,, \quad \chi = 1\,, \tag{2.79}$$

where μ is the DJT coupling constant. We stress that this is *exactly* the identification which is enforced by the proper procedure to deduce the DJT field equations form the Mielke-Baekler model by means of a Lagrange multiplier for vanishing torsion! Then

$$E_\infty = M - \frac{J}{\mu l^2}\,, \quad L_\infty = J - \frac{M}{\mu}\,. \tag{2.80}$$

2.4 Teleparallelism

In teleparallel gravity we demand vanishing curvature. The basic variable is the coframe, acting as a translational gauge potential. The Lagrangian than is taken to be quadratic in the field strength, i. e. torsion,

$$V := V_{\mathrm{T}^2} - \frac{\Lambda}{\ell}\,\eta + \lambda^{\alpha\beta} \wedge R_{\alpha\beta} + \mu^{\alpha\beta} \wedge Q_{\alpha\beta}\,. \tag{2.81}$$

In [58] it is shown that due to the Noether identities the field equation for $g_{\alpha\beta}$ drop out completely and we are left with the antisymmetric part of the field equation for $\Gamma_{\alpha\beta}$,

$$\begin{aligned} R_{\alpha\beta} &= 0\,, & (2.82)\\ DH_\alpha + \frac{\Lambda}{\ell}\,\eta_\alpha - E_\alpha &= 0\,, & (2.83)\\ D\lambda_{[\alpha\beta]} - \vartheta_{[\alpha} \wedge H_{\beta]} &= 0\,, & (2.84) \end{aligned}$$

where, as a reminder,

$$\begin{aligned} H_\alpha &= -\frac{1}{\ell}\,{}^\star\left(\sum_{I=1}^{3} a_I\,{}^{(I)}T_\alpha\right)\,, & (2.85)\\ E_\alpha &= \frac{1}{2}\,\left[(e_\alpha\rfloor T^\beta) \wedge H_\beta - T^\beta \wedge (e_\alpha\rfloor H_\beta)\right]\,. & (2.86) \end{aligned}$$

In view of the identity, see appendix A.4.6 ,

$$D\eta_{\alpha\beta} = \eta_{\alpha\beta\gamma} \wedge T^\gamma = -2\,\vartheta_{[\alpha} \wedge {}^\star\left({}^{(1)}T_{\beta]} - (n-2)\,{}^{(2)}T_{\beta]} - \frac{1}{2}\,{}^{(3)}T_{\beta]}\right)\,, \tag{2.87}$$

we can give a general solution for the Lagrange multiplier provided we choose

$$a_1 = -1\,, \qquad a_2 = n-2\,, \qquad a_3 = \frac{1}{2}\,. \tag{2.88}$$

Consequently, we find for the Lagrange multiplier

$$\lambda_{\alpha\beta} = -\frac{1}{2\ell}\,\eta_{\alpha\beta}\,. \tag{2.89}$$

In this way we recovered nothing else than Einstein's teleparallelism which is equivalent to general relativity because of, see appendix A.4.5,

$$\begin{aligned} R^{\alpha\beta} \wedge \eta_{\alpha\beta} &= R^{\{\}\alpha\beta} \wedge \eta_{\alpha\beta} - 2\,d(\vartheta_\alpha \wedge {}^\star T^\alpha)\\ &\quad + T^\alpha \wedge {}^\star\left(-{}^{(1)}T_\alpha + (n-2)\,{}^{(2)}T_\alpha + \frac{1}{2}\,{}^{(3)}T_\alpha\right)\,. \end{aligned} \tag{2.90}$$

Hence the Lagrangian (2.81) with the choice of parameters (2.88) is effectively equivalent to the conventional Hilbert-Einstein Lagrangian. Moreover, in [48] it is shown that (2.88) makes the Lagrangian (2.81) invariant under local Lorentz transformations.

In the following we show that our "BTZ-solution with torsion" is also a solution of the proper, Einstein teleparallelism in $n = 3$ where

$$H_\alpha = -\frac{1}{\ell}{}^\star\left(-{}^{(1)}T_\alpha + {}^{(2)}T_\alpha + \frac{1}{2}\,{}^{(3)}T_\alpha\right). \tag{2.91}$$

The field equations to fulfill are given by

$$DH_\alpha - E_\alpha + \frac{\Lambda}{\ell}\,\eta_\alpha = 0\,, \qquad R_{\alpha\beta} = 0\,. \tag{2.92}$$

Our aim was to embed the solution of constant axial torsion into teleparallel gravity. Hence we make the ansatz

$$T^\alpha = 2\frac{\mathcal{T}}{\ell}\,\eta^\alpha \qquad \Rightarrow \qquad H_\alpha = \frac{\mathcal{T}}{\ell^2}\,\vartheta^\alpha\,. \tag{2.93}$$

A simple calculation (using $D\vartheta^\alpha = T^\alpha$ and some relations for the eta basis) yields

$$DH_\alpha = 2\,\frac{\mathcal{T}^2}{\ell^3}\,\eta_\alpha\,, \quad E_\alpha = \frac{\mathcal{T}^2}{\ell^3}\,\eta_\alpha\,. \tag{2.94}$$

Hence the field equation (2.92) reduce to

$$\frac{\mathcal{T}^2}{\ell^3}\,\eta_\alpha + \frac{\Lambda}{\ell}\,\eta_\alpha = 0 \qquad \Leftrightarrow \qquad \mathcal{T} = \pm\sqrt{-\Lambda}\,\ell\,. \quad R_{\alpha\beta} = 0\,. \tag{2.95}$$

It remains the task to find a suitable metric/coframe such that the Riemann-Cartan curvature vanishes. Therefore we split the connection 1-form into its Riemannian piece and the contortion

$$\Gamma_\alpha{}^\beta = \Gamma^{\{\}}{}_\alpha{}^\beta - K_\alpha{}^\beta\,, \tag{2.96}$$

which is determined by the torsion according to $K^\alpha{}_\beta \wedge \vartheta^\beta = T^\alpha$, and find

$$R_\alpha{}^\beta = R^{\{\}}{}_\alpha{}^\beta - DK_\alpha{}^\beta + K_\alpha{}^\gamma \wedge K_\gamma{}^\beta\,. \tag{2.97}$$

For our ansatz of constant axial torsion we compute by means of some simple algebra

$$R_{\alpha\beta} = R^{\{\}}_{\alpha\beta} - \frac{\mathcal{T}^2}{\ell^2}\,\vartheta_\alpha \wedge \vartheta_\beta\,. \tag{2.98}$$

Again we arrive at constant Riemannian curvature. A suitable coframe and metric have already been derived in section 2.3.1. This solution is a subcase of the vacuum solution of the Mielke-Baekler model. The teleparallel constraint $R_{\alpha\beta} = 0$ here results in a constraint on the coupling parameters,

$$\theta_T^2 + \chi\Lambda\ell^2 = 0\,, \qquad \theta_L = 0\,. \tag{2.99}$$

This case was extensively studied by Blagojevic [19, 18, 17].

2.5 É. Cartan's spiral staircase

In 1922, É. Cartan gave the following prescription of parallel transport in a space with curvature and torsion:

> "...imagine a space F which corresponds point by point with a Euclidean space E, the correspondence preserving distances. The difference between the two spaces is following: two orthogonal triads issuing from two points A and A' infinitesimally nearby in F will be parallel when the corresponding triads in E may be deduced one from the other by a given helicoidal displacement (of right–handed sense, for example), having as its axis the line joining the origins. The straight lines in F thus correspond to the straight lines in E: They are geodesics. The space F thus defined admits a six parameter group of transformations; it would be our ordinary space as viewed by observers whose perceptions have been twisted. Mechanically, it corresponds to a medium having constant pressure and constant internal torque."

Obviously, Cartan's prescriptions are reflected in the solution (2.55, 2.56),

$$\vartheta^\alpha = \frac{dx^\alpha}{\psi}\,, \quad \psi = 1 - \frac{\Lambda_{\rm eff}}{4}(-x^2+y^2+z^2)\,, \quad \Gamma^{\alpha\beta} = \frac{\mathcal{T}}{\ell}\,\eta^{\alpha\beta} + x^{[\alpha}\,\vartheta^{\beta]}\,\frac{\Lambda_{\rm eff}}{3}\,, \tag{2.100}$$

where the effective cosmological constant has to be set to zero, $\Lambda_{\rm eff} = 0$, then autoparallels and extremals coincide, see 2.3.3. Thus, in the spiral staircase, extremals are *Euclidean* straight lines. This is apparent in Cartan's construction. We can view this as a subcase of the BTZ-solution with torsion of the MB-model provided we impose the following constraint on the coupling parameters, see table 2.1,

$$0 = \Lambda_{\rm eff} = \frac{\mathcal{T}^2 + \mathcal{R}}{\ell^2}\,, \tag{2.101}$$

or

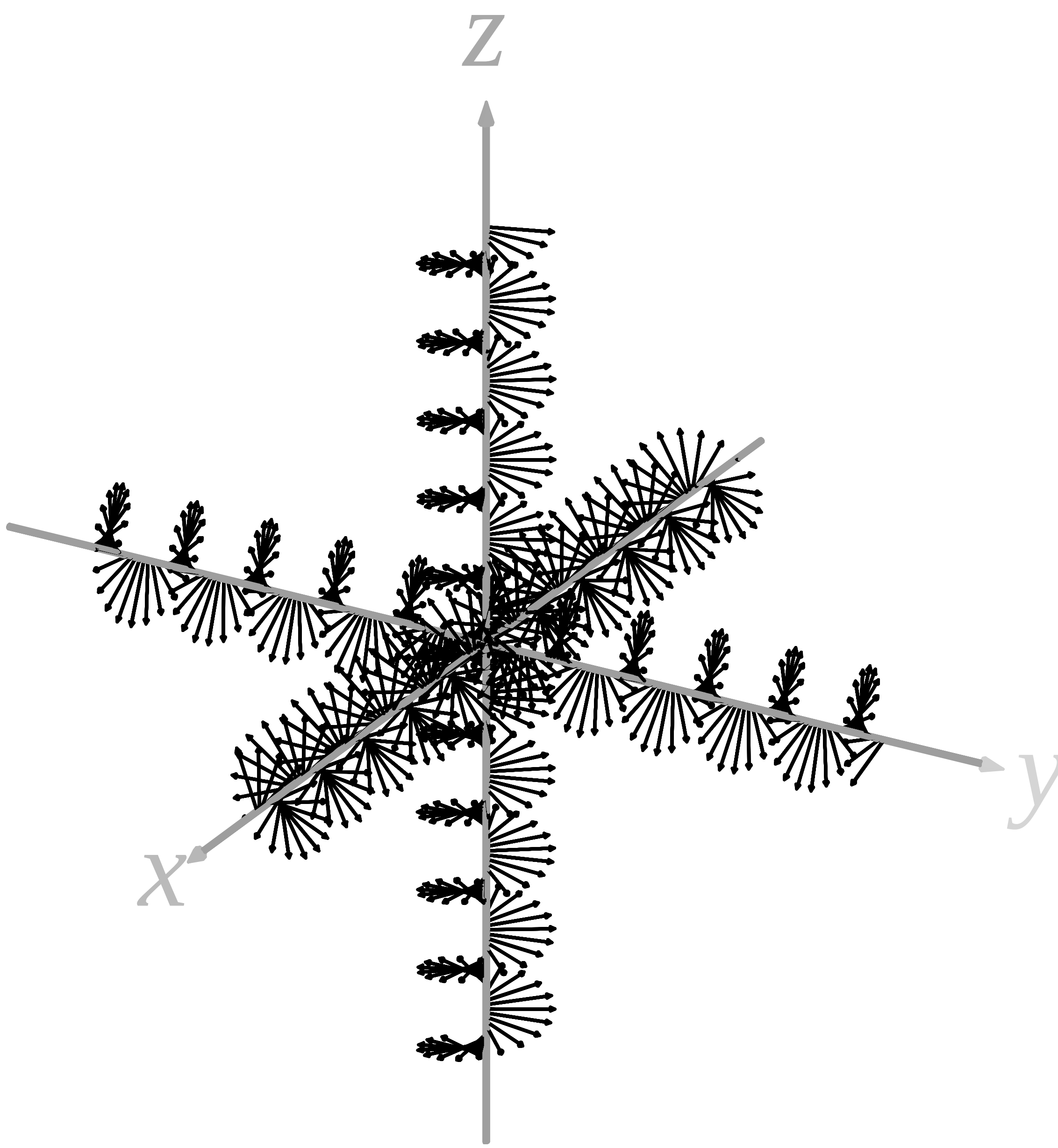
$$4\Lambda^2\ell^4\theta_L^2 - 3\theta_T^2\chi^2 - 12\chi\Lambda\ell^2\theta_T\theta_L - 8\theta_T^3\theta_L - 4\kappa^3\Lambda\ell^2 = 0\,. \tag{2.102}$$

Cartan apparently had in mind a 3D space with Euclidean signature. For an alternative interpretation of Cartan's spiral staircase we consider the 3D Einstein–Cartan field equations without cosmological constant. This is a subcase of our "Master Lagrangian" (2.8) with $\Lambda = a_1 = a_2 = a_3 = \theta_T = \theta_L = \lambda_\alpha = 0$. The field equations are then

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\,R^{\beta\gamma} \;=\; \ell\,\Sigma_\alpha\,, \tag{2.103}$$

$$\frac{1}{2}\,\eta_{\alpha\beta\gamma}\,T^{\gamma} \;=\; \ell\,\tau_{\alpha\beta}\,. \tag{2.104}$$

Figure 2.1: *Cartan's spiral staircase.* Cartan's rules [26] for the introduction of a non-Euclidean connection in a 3D Euclidean space are as follows: (i) A vector which is parallelly transported along itself does not change (cf. a vector directed and transported in x-direction). (ii) A vector that is orthogonal to the direction of transport rotates with a prescribed constant "velocity" (cf. a vector in y–direction transported in x–direction). The winding sense around the three coordinate axes is always positive.



If we put $\Lambda_{\text{eff}} = 0$, then, see (2.55,2.56), we arrive at

$$\vartheta^\alpha = \delta_i^\alpha \, dx^i \,, \qquad \Gamma^{\alpha\beta} = \frac{\mathcal{T}}{\ell}\, \eta^{\alpha\beta} \,. \tag{2.105}$$

The components of the connection are totally antisymmetric: $\Gamma_{\gamma\alpha\beta} = e_\gamma \rfloor \Gamma_{\alpha\beta} = (\mathcal{T}/\ell)\, \eta_{\gamma\alpha\beta}$. The Riemannian curvature vanishes. By simple algebra we find,

$$T^\alpha = 2\, \frac{\mathcal{T}}{\ell}\, \eta^\alpha \,, \qquad \widetilde{R}^{\alpha\beta} = 0 \,, \qquad R^{\alpha\beta} = -\frac{\mathcal{T}^2}{\ell^2}\, \vartheta^{\alpha\beta} \,. \tag{2.106}$$

The coframe and the connection of (2.105), Euclidean signature assumed, form a solution of the Einstein–Cartan field equations *with matter* provided the energy–momentum current (for Euclidean signature the force stress tensor $\mathfrak{t}_\alpha{}^\beta$) and the spin current (here the torque or moment stress tensor $\mathfrak{s}_{\alpha\beta}{}^\gamma$) are constant,

$$\Sigma_\alpha =: \mathfrak{t}_\alpha{}^\beta\, \eta_\beta = -\frac{\mathcal{T}^2}{\ell^3}\, \eta_\alpha \quad \text{and} \quad \tau_{\alpha\beta} =: \mathfrak{s}_{\alpha\beta}{}^\gamma\, \eta_\gamma = -\frac{\mathcal{T}}{\ell^2}\, \vartheta_{\alpha\beta} \,. \tag{2.107}$$

Inversion yields

$$\mathfrak{t}_\alpha{}^\beta = -\frac{\mathcal{T}^2}{\ell^3}\, \delta_\alpha^\beta \,, \qquad \mathfrak{s}_{\alpha\beta\gamma} = -\frac{\mathcal{T}}{\ell^2}\, \eta_{\alpha\beta\gamma} \,. \tag{2.108}$$

We find a constant hydrostatic *pressure* $-\mathcal{T}^2/\ell^3$ and a constant *torque* $-\mathcal{T}/\ell^2$, exactly as foreseen by Cartan. In solid state physics, this corresponds to a superposition of three “forests” of *screw dislocations* that are parallel to the coordinate axes with constant and equal densities. However, in a real crystal, the Riemann–Cartan curvature $R^{\alpha\beta}$ has to vanish (instead of the Riemannian curvature $\widetilde{R}^{\alpha\beta}$, as in our exact solution) and no pressure would emerge macroscopically.

Thus we can either view the spiral staircase as a vacuum solution and special case of our (constrained) solution of Table 1 or as a material solution of 3D Einstein–Cartan theory (with Euclidean signature) carrying constant pressure and constant torque.

2.6 The Deser-Jackiw-Templeton model

In our geometrical section we already mentioned the fact that the Cotton 2-form in 3 dimensions is covariantly conserved and can be derived from the rotational Chern-Simons Lagrangian by imposing vanishing nonmetricity and torsion. Thus, it can be consistently coupled to matter. In our Master-Lagrangian we have to put $\theta_T = a_1 = a_2 = a_3 = 0$, yielding

$$\begin{aligned} L_{\text{DJT}} &= \theta_L\, C_{\text{RR}} + V_{\text{HE}} + V_\Lambda + \lambda_\alpha \wedge T^\alpha + \lambda^{\alpha\beta}\, Q_{\alpha\beta} + L_{\text{mat}} \\ &= \theta_L\, C_{\text{RR}} - \frac{1}{2\ell}\, R^{\alpha\beta} \wedge \eta_{\alpha\beta} - \frac{\Lambda}{\ell}\, \eta + \lambda_\alpha \wedge T^\alpha + \lambda^{\alpha\beta} \wedge Q_{\alpha\beta} + L_{\text{mat}} \,. \end{aligned} \tag{2.109}$$

Using (1.99) and (2.9, 2.10) we find

$$G_\alpha + \Lambda\eta_\alpha + \frac{1}{\mu}\,C_\alpha = \ell\Sigma_\alpha\,, \tag{2.110}$$

where $G_\alpha = \frac{1}{2}\,\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma}$ is the Einstein 2-form and the DJT coupling constant $1/\mu = -2\theta_{\rm L}\ell$, see [40]. The Einstein $(n-1)$-form G_α is equivalent to the 1-form L_α according to

$$G_\alpha = L^\beta \wedge \eta_{\beta\alpha}\,, \tag{2.111}$$

see [62]. Hence, we may rewrite the DJT-field equation as a differential equation for L_α,

$$DL_\alpha + \mu\, L^\beta \wedge \eta_{\beta\alpha} = \ell\mu\,\Sigma_\alpha - \mu\Lambda\eta_\alpha\,. \tag{2.112}$$

The Bianchi identities imply full integrability of this system.

In the following we derive a new solution of the DJT-model. Our motivation was to find a Class N spacetime c. f. section 1.5. Inspired by the corresponding $(1+3)$D metrics, we start with the ansatz

$$\vartheta^{\hat 0} = dt + dx\,, \quad \vartheta^{\hat 1} = dt - dx\,, \quad \vartheta^{\hat 2} = dy\,, \tag{2.113}$$

with the *non-orthonormal* metric

$$g = \vartheta^{\hat 0} \otimes \vartheta^{\hat 1} + \psi\,\vartheta^{\hat 1} \otimes \vartheta^{\hat 1} - \vartheta^{\hat 2} \otimes \vartheta^{\hat 2}\,, \quad \psi = \psi(y)\,. \tag{2.114}$$

The Cotton tensor, in this frame, reads ($()' = d/dy$):

$$C_\alpha{}^\beta = \psi''' \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad \lambda_1 = \lambda_2 = \lambda_3 = 0\,. \tag{2.115}$$

The vacuum DJT field equation reduces to

$$\frac{1}{\mu}\,\psi''' - \psi'' = 0\,, \tag{2.116}$$

with the general solution

$$\psi = Ay + B\,e^{\mu y} + C\,. \tag{2.117}$$

In an orthonormal coframe with signature $(-++)$ and $A = C = 0$ and $B = 1$, coframe and Cotton tensor can be brought into the more familiar form

$$\vartheta^{\hat 0} = e^{\mu y/2}\left[(1 + \frac{1}{2}\,e^{-\mu y})\,dt + (1 - \frac{1}{2}\,e^{-\mu y}\,dx)\right]\,, \tag{2.118}$$

$$\vartheta^{\hat 1} = \frac{1}{2}\,e^{-\mu y/2}\,(dt - dx)\,, \tag{2.119}$$

$$\vartheta^{\hat 2} = dy\,. \tag{2.120}$$

The Cotton tensor, with all eigenvalues being zero, in this frame, reads

$$C_\alpha{}^\beta = \frac{\mu^3}{2}\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_1 = \lambda_2 = \lambda_3 = 0\,. \tag{2.121}$$

2.7 Einstein: Conformally flat perfect fluid solution

As an application of the relation between energy-momentum 2-form and Cotton 2-form, eq. (1.182) and as an example for a class 0 solution, we will derive the spherically symmetric, conformally flat, perfect fluid solution to Einstein's field equation. We use the ansatz

$$\vartheta^{\hat 0} = N(r)\,dt\,, \quad \vartheta^{\hat 1} = dr/F(r)\,, \quad \vartheta^{\hat 2} = r\,d\phi\,, \tag{2.122}$$

with signature $(-++)$. The energy–momentum of the perfect fluid is given by

$$\Sigma_\alpha = [\rho(r) + p(r)]\,u_\alpha\,u^\beta\eta_\beta + p\,\eta_\alpha\,, \tag{2.123}$$

where u^α is the 4-velocity of the fluid elements which, in an orthonormal frame, is given by $u^\alpha = (1,0,0)$. By using (1.182), we find

$$C_{\hat 0} = -\left\{\frac{F}{2N}\left(2\,\partial_r\left[N\left(p+\rho\right)\right] - N\,\partial_r\rho\right)\right\}\vartheta^{\hat 0}\wedge\vartheta^{\hat 1}\,, \tag{2.124}$$

$$C_{\hat 1} = 0\,, \tag{2.125}$$

$$C_{\hat 2} = -\frac{F}{2}\,\partial_r\rho\,\vartheta^{\hat 1}\wedge\vartheta^{\hat 2}\,. \tag{2.126}$$

Consequently, we have to demand constant energy density $\rho = const$ for a conformally flat solution with $C_\alpha = 0$. By using $\rho = const$, we infer from (2.124)

$$N(r) = \frac{c_1}{\rho + p(r)}\,, \tag{2.127}$$

where c_1 is an integration constant. The 0-component of the Einstein field equation (1.8) yields

$$F^2(r) = c_2 - (\ell\rho + \Lambda)\,r^2\,. \tag{2.128}$$

The remaining components of the field equation are fulfilled provided

$$\frac{dp}{dr} = \frac{(\ell\rho - \Lambda)\,(p+\rho)\,r}{F^2}\,. \tag{2.129}$$

This ordinary differential equation can be integrated yielding (c_3 is another integration constant)

$$p = \frac{c_3\, F\,(\ell\rho + \Lambda) + (c_3)^2 \ell\Lambda + \rho F^2}{(c_3)^2\, \ell^2 - F^2}\,. \tag{2.130}$$

Finally, the solution is given by (2.122,2.127,2.128,2.130), compare the solutions in [46].

This is the analog to the interior Schwarzschild solution which is also conformally flat. The constants can be chosen in such a way that the pressure is positive and finite. A smooth joining to the BTZ-metric also is possible.

Chapter 3

Einstein-aether theory

3.1 Introduction

In the previous sections we considered gravitational theories living in Riemann-Cartan or Riemannian spacetimes. The independent variables have been the coframe and the connection with the field strength torsion and curvature, respectively. The metric hasn't been an independent variable; the corresponding field strength, the nonmetricity, which characterizes the non-compatibility of metric and parallel transport, has been forced to be zero by means of a Lagrange multiplier. As a consequence, the symmetric part of the connection can be eliminated from the field equations and only the antisymmetric part remains as an independent variable. In this last chapter we are going to liberate the action principle from the last constraint, the vanishing of the nonmetricity, thereby promoting the connection to a truly independent variable. Allowing for nonmetricity corresponds to abolishing *local Lorentz invariance*. In Riemann-Cartan spacetime, the length and the angles between vectors remain absolute structures in the sense that these are preserved under parallel transport. In the presence of nonmetricity

$$Q_{\alpha\beta} = -Dg_{\alpha\beta}\,, \tag{3.1}$$

the scalar product between two vectors changes under parallel displacement along a curve with tangent vector u as (Ł is the gauge covariant Lie derivative)

$$\text{Ł}_u g(V,W) = -u\rfloor\left[\not{Q}_{\alpha\beta}V^\alpha W^\beta + Q\, g(V,W)\right]\,, \tag{3.2}$$

where

$$Q := \frac{1}{n}\,Q_\alpha{}^\alpha\,, \qquad \not{Q}_{\alpha\beta} := Q_{\alpha\beta} - Q\, g_{\alpha\beta}\,, \tag{3.3}$$

are the trace (Weyl covector) and the traceless (shear) part of the nonmetricity. In the case of vanishing shear, the light-cone is left intact—we have a rigid (conformally) light-cone structure which may be viewed as the epitome of local Lorentz invariance.

Hence, by allowing for nonmetricity, we break down the local Lorentz invariance of Riemann-Cartan spacetime.

Violation of Lorentz invariance is a generic feature of many models of quantum gravity, see [68, 75], e. g. In classical general relativity, the flat Minkowski space is, in a way, the ground-state of spacetime in the absence of matter fields, i. e. vacuum. If now spacetime becomes a quantum field itself, there is no apriori reason why the ground state of this quantum field should be Lorentz invariant. Indeed, even in classical electrodynamics it is conceivable that there are vacuum spacetimes which are characterized by a non-Lorentz invariant constitutive relation which obstructs the familiar light-cone structure of conventional Maxwell-Lorentz electrodynamics, see [61]. Thus one should be open-minded for dissolving the light-cone ... The deformation of the light cone leads to observable consequences and there is an intensive search going on to see these effects by means of astrophysical observations [68, 99, 100].

The most common reaction in the literature to the challenge of Lorentz symmetry violation is the introduction of a preferred frame and thereby reintroducing a kind of "aether field". Jacobson and Mattingly [69] introduce an extra timelike 4-vector field u which, in addition to the metric, describes the properties of spacetime. In order to make the theory general covariant, they take this field to be dynamical, determined by a Lagrangian carrying a kinetic term $(\nabla u)^2$ and a massive term u^2. Consequently, the aether field equations are Yang-Mills like,

$$\nabla H + \text{lower order terms} \sim \ell u \,, \tag{3.4}$$

where $H \sim \nabla u$ is the field momentum (excitation) and ℓ some constant.

We will propose a different scheme. It doesn't seem very natural to associate u with the properties of spacetime. What tools do we have to characterize spacetime itself? We measure length and angles ($\longrightarrow$ metric $g_{\alpha\beta}$) in a local frame of reference ($\longrightarrow$ ϑ^α) and compare these measurements carried out at different points ($\longrightarrow$ parallel transport $\Gamma_\alpha{}^\beta$). Thus, the intrinsic properties of spacetime are encoded in ($g_{\alpha\beta}$, ϑ^α, $\Gamma_\alpha{}^\beta$) and quantities derived therefrom. An additional, external vector field is foreign to this structure! Moreover, the geometry of metric-affine spacetime, by carrying nonmetricity, is rich enough to account for a violation of Lorentz invariance. As additional benefit we can give a consistent framework of how to couple the "aether field" to matter, a problem which seems to be unsolved in the theory of J&M. We assume, in the spirit of the equivalence principle, minimal coupling, i. e., partial derivatives are replaced by covariant ones.

In the theory of J&M the "aether" is represented by the vector field u. In order to compare our approach to theirs, we first look for vector-like, geometrical quantities which characterize Lorentz violation in metric-affine geometry. In the next step we propose a Lagrangian for our "geometrical-aether" theory. Eventually we analyze the field equations and construct some simple solutions.

3.2 Vector-like Lorentz-violating quantities in MAG

We argued in section 3.1 that the genuine quantity describing Lorentz violation is the nonmetricity. In this section we look for vector-like pieces of the nonmetricity. In n dimensions, as a symmetric tensor-valued 1-form, the nonmetricity has $n^2(n+1)/2$ independent components. We immediately recognize a vector-like piece with n components by splitting off the trace,

$$Q := \frac{1}{n}\, Q_\alpha{}^\alpha\,, \qquad \not{Q}_{\alpha\beta} := Q_{\alpha\beta} - Q\, g_{\alpha\beta}\,. \tag{3.5}$$

The *Weyl covector* Q is a scalar-valued 1-form and therefore has n independent components. It is related to scale transformations, see (3.3), and thus extends the Lorentz to the conformal group. The presence of Q does not touch the (conformal) light-cone structure. The nonmetricity $Q_{\alpha\beta} = Q_{i\alpha\beta}\, dx^i$, from a geometrical point of view, can be understood as a strain measure for the different directions specified by the 1-forms dx^i. Accordingly, $\not{Q}_{\alpha\beta}$ defines a *shear* measure since the *dilation* measure Q is subtracted out. In order to find a vector-like degree of freedom of $\not{Q}_{\alpha\beta}$, we contract it with the frame and find

$$\Lambda^\alpha := e^\beta \rfloor\, \not{Q}_{\alpha\beta}\,, \qquad \Lambda := \Lambda_\alpha\, \vartheta^\alpha\,, \qquad\qquad n \text{ components}\,. \tag{3.6}$$

Apparently, the 1-form Λ is our desired vector-like shear measure. It remains the task to resolve (3.6) with respect to $\not{Q}_{\alpha\beta}$. Of course, Λ_α with n components does not contain all information of $\not{Q}_{\alpha\beta}$ with $n(n+1)(n-2)/2$ independent components. We will denote the piece of $\not{Q}_{\alpha\beta}$ which corresponds to Λ by ${}^{(3)}Q_{\alpha\beta}$. The superscript 3 is conventional and will be explained later. The excess of $\not{Q}_{\alpha\beta}$ over ${}^{(3)}Q_{\alpha\beta}$ is

$$\tilde{\not{Q}}_{\alpha\beta} := \not{Q}_{\alpha\beta} - {}^{(3)}Q_{\alpha\beta}\,, \tag{3.7}$$

with

$$\tilde{\not{Q}}_\alpha{}^\alpha = {}^{(3)}Q_\alpha{}^\alpha = \not{Q}_\alpha{}^\alpha = 0\,, \tag{3.8}$$

$$e^\alpha \rfloor {}^{(3)}Q_{\alpha\beta} = \Lambda_\beta\,, \qquad e^\alpha \rfloor\, \tilde{\not{Q}}_{\alpha\beta} = 0\,. \tag{3.9}$$

We now make an ansatz for ${}^{(3)}Q_{\alpha\beta}$ in order to obtain its explicit form. The only possibility to construct a symmetric tensor-valued 1-form out of scalar-valued 1-forms $\Lambda^{(\mathrm{I})}$ and $\Lambda^{(\mathrm{II})}$, is

$${}^{(3)}Q_{\alpha\beta} = \vartheta_{(\alpha}\, e_{\beta)} \rfloor \Lambda^{(\mathrm{I})} + g_{\alpha\beta}\, \Lambda^{(\mathrm{II})}\,, \tag{3.10}$$

The 1-forms $\Lambda^{(\mathrm{I})}$, $\Lambda^{(\mathrm{II})}$ can be determined using the constraints (3.8,3.9),

$${}^{(3)}Q_\alpha{}^\alpha = \Lambda^{(\mathrm{I})} + n\, \Lambda^{(\mathrm{II})} = 0\,, \tag{3.11}$$

$$e^\alpha \rfloor {}^{(3)}Q_{\alpha\beta} = \frac{n+1}{2}\, e_\beta \rfloor \Lambda^{(\mathrm{I})} + e_\beta \rfloor \Lambda^{(\mathrm{II})} = e_\beta \rfloor \Lambda\,. \tag{3.12}$$

Thus we find

$$\Lambda^{(\mathrm{II})} = -\frac{1}{n}\,\Lambda^{(\mathrm{I})}\,, \qquad \Lambda^{(\mathrm{I})}_{\beta} = \frac{2n}{(n-1)(n+2)}\,\Lambda_{\beta}\,, \tag{3.13}$$

and consequently

$$^{(3)}Q_{\alpha\beta} = \frac{2n}{(n-1)(n+2)}\left(\Lambda_{(\alpha}\,\vartheta_{\beta)} - \frac{1}{n}\,g_{\alpha\beta}\,\Lambda\right)\,. \tag{3.14}$$

It turns out that $^{(3)}Q_{\alpha\beta}$ already is an irreducible piece of the nonmetricity which can be decomposed according to

$$\begin{aligned} Q_{\alpha\beta} &= {}^{(1)}Q_{\alpha\beta} + {}^{(2)}Q_{\alpha\beta} + {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}\,, \\ &= \mathrm{TRINOM} + \mathrm{BINOM} + \mathrm{VECNOM} + \mathrm{CONOM}\,, \\ \frac{1}{2}\,n^2(n+1) &= \frac{1}{6}\,n(n-1)(n+4) + \frac{1}{3}n(n^2-4) + n + n\,, \end{aligned}$$

where the remaining pieces are defined by

$$^{(2)}Q_{\alpha\beta} = -\frac{2}{3}\,e_{(\alpha}\rfloor P_{\beta)}\,, \qquad P_\alpha = \not{Q}_{\alpha\beta}\wedge\vartheta^\beta - \frac{1}{n-1}\vartheta_\alpha\wedge\Lambda\,, \tag{3.15}$$

$$^{(4)}Q_{\alpha\beta} = g_{\alpha\beta}\,Q\,, \tag{3.16}$$

$$^{(1)}Q_{\alpha\beta} = Q_{\alpha\beta} - {}^{(2)}Q_{\alpha\beta} - {}^{(3)}Q_{\alpha\beta} - {}^{(4)}Q_{\alpha\beta}\,. \tag{3.17}$$

In conclusion, we have found a vector-like piece of the nonmetricity which is related to shear and therewith genuine Lorentz violating. This piece, $^{(3)}Q_{\alpha\beta}$, will be in the center of interest in the following.

In order to set up a “J&M-like Lagrangian” we require a massive term proportional to $\Lambda^\alpha\Lambda_\alpha$. Indeed such a term can be naturally constructed from the respective irreducible piece $^{(3)}Q_{\alpha\beta}$,

$$^{(3)}Q_{\alpha\beta}\wedge{}^{\star(3)}Q^{\alpha\beta} = \frac{2n}{(n+2)(n-1)}\,\Lambda\wedge{}^\star\Lambda = \frac{2n}{(n+2)(n-1)}\,\Lambda^\alpha\Lambda_\alpha\,\eta\,. \tag{3.18}$$

It remains the task to find an appropriate kinetic term. This is the subject of the next section.

3.3 The quest for the kinetic aether term: The strain curvature

In the last section we motivated the use of the 1-form piece Λ of the nonmetricity as “aether 1-form”. The corresponding aether field strength should be of the type $F \sim d\Lambda$. Then, the canonical, Yang-Mills like kinetic term for a 1-form field Λ would look like

$$F\wedge{}^\star F \sim d\Lambda\wedge{}^\star d\Lambda\,. \tag{3.19}$$

How could we construct such a piece for our aether 1-form? First we note that the nonmetricity acts on the level of a connection, i. e., a gauge potential. We can split off the post-Riemannian part of the connection, the distortion

$$N_\alpha{}^\beta := \Gamma_\alpha{}^\beta - \Gamma^{\{\}\beta}_{\alpha} \,. \tag{3.20}$$

We can read off the components of $N_{\alpha\beta}$ from the decomposition of the connection (1.38) and find

$$Q_{\alpha\beta} = 2N_{(\alpha\beta)} \,. \tag{3.21}$$

If we opt for interpreting the nonmetricity, and therewith its irreducible pieces, as connections, the corresponding field strength, i. e. the exterior covariant derivatives, act as curvature 2-forms. Indeed, by employing the zeroth Bianchi identity we find the symmetric strain curvature 2-form

$$2Z_{\alpha\beta} := DQ_{\alpha\beta} = 2R_{(\alpha\beta)} \,. \tag{3.22}$$

Obviously, it has one distinctive piece, namely its trace $Z := g^{\alpha\beta}Z_{\alpha\beta} = Z_\gamma{}^\gamma$. It should be noted that Z is related to a *premetric* quantity. In a space with only a linear connection, the curvature $R_\alpha{}^\beta$ can be contracted, $R_\alpha{}^\alpha$, even if a metric is not present. Thus $R_\gamma{}^\gamma$ and, as a consequence, also Z is rightfully called `dilcurv`, the part of the curvature related to dil(at)ations. This is an irreducible piece of $Z_{\alpha\beta}$ and we call it

$$^{(4)}Z_{\alpha\beta} := \frac{1}{4}\, g_{\alpha\beta} Z \,. \tag{3.23}$$

Since on the level of the nonmetricity dilations are related to $^{(4)}Q_{\alpha\beta}$, we denoted the related curvature piece by the same number. In fact, the zeroth Bianchi identity, if contracted, yields $g^{\alpha\beta}DQ_{\alpha\beta} = 2Z_\gamma{}^\gamma = Z$. By partial integration, we find

$$Z = 2dQ \qquad \text{or} \qquad {}^{(4)}Z_{\alpha\beta} = \frac{1}{2}\, g_{\alpha\beta}\, dQ = \frac{1}{2}\left(D\,{}^{(4)}Q_{\alpha\beta} + \not{Q}_{\alpha\beta} \wedge Q\right) . \tag{3.24}$$

The tracefree piece can be further decomposed into various "traces", see appendix A.3:

$$\not{Z}_\alpha := e^\beta\rfloor\, \not{Z}_{\alpha\beta}, \qquad \hat{\Delta} := \frac{1}{n-2}\,\vartheta^\alpha \wedge \not{Z}_\alpha, \qquad Y_\alpha := {}^*(\not{Z}_{\alpha\beta} \wedge \vartheta^\beta) \,. \tag{3.25}$$

Subsequently we can subtract out the traces:

$$\Xi_\alpha := \not{Z}_\alpha - \frac{1}{2} e_\alpha \rfloor (\vartheta^\gamma \wedge \not{Z}_\gamma), \qquad \Upsilon_\alpha := Y_\alpha - \frac{1}{n-2}\, e_\alpha \rfloor (\vartheta^\gamma \wedge Y_\gamma) \,. \tag{3.26}$$

The irreducible pieces may then be written as (see [58])

$$
\begin{aligned}
{}^{(2)}Z_{\alpha\beta} &:= -\frac{1}{2}\,{}^*(\vartheta_{(\alpha}\wedge\Upsilon_{\beta)})\,, && (3.27)\\
{}^{(3)}Z_{\alpha\beta} &:= \frac{1}{n+2}\,\left(n\,\vartheta_{(\alpha}\wedge e_{\beta)}\rfloor - 2\,g_{\alpha\beta}\right)\hat{\Delta}\,, && (3.28)\\
{}^{(4)}Z_{\alpha\beta} &:= \frac{1}{n}\,g_{\alpha\beta}\,Z\,, && (3.29)\\
{}^{(5)}Z_{\alpha\beta} &:= \frac{2}{n}\,\vartheta_{(\alpha}\wedge\Xi_{\beta)}\,, && (3.30)\\
{}^{(1)}Z_{\alpha\beta} &:= Z_{\alpha\beta} - {}^{(2)}Z_{\alpha\beta} - {}^{(3)}Z_{\alpha\beta} - {}^{(4)}Z_{\alpha\beta} - {}^{(5)}Z_{\alpha\beta}\,. && (3.31)
\end{aligned}
$$

Our main interest focuses on ${}^{(3)}Q_{\alpha\beta}$. Is it possible to relate its exterior covariant derivative to one of the irreducible pieces of $Z_{\alpha\beta}$? In order to answer this question we calculate

$$
D\,{}^{(3)}Q_{\alpha\beta} = \frac{2n}{(n-1)(n+2)}\,\left(T_{(\alpha}\,\Lambda_{\beta)} - \vartheta_{(\alpha}\wedge D\Lambda_{\beta)} + \frac{1}{n}\,Q_{\alpha\beta}\wedge\Lambda - \frac{1}{n}\,g_{\alpha\beta}\,d\Lambda\right)\,. \qquad (3.32)
$$

We use the following representation of the exterior covariant derivative of the zero-form Λ_α:

$$
D\Lambda_\beta = \vartheta^\alpha\,e_\alpha\rfloor D\Lambda_\beta = \vartheta^\alpha\,D_\alpha\,\Lambda_\beta = \vartheta^\alpha\,D_\alpha\Lambda_\beta = \vartheta^\alpha\,D_{(\alpha}\Lambda_{\beta)} + \vartheta^\alpha\,D_{[\alpha}\Lambda_{\beta]} \qquad (3.33)
$$

Let us denote the "symmetric exterior derivative" as

$$
\mathcal{D}\Lambda_\beta = \vartheta^\alpha\,D_{(\alpha}\Lambda_{\beta)}\,. \qquad (3.34)
$$

The antisymmetric part can be retrieved from the exterior derivative,

$$
\vartheta^\alpha\,D_{[\alpha}\Lambda_{\beta]} = -\frac{1}{2}\,e_\beta\rfloor d\Lambda + \frac{1}{2}\,e_\beta\rfloor\left(T^\alpha\,\Lambda_\alpha\right)\,. \qquad (3.35)
$$

Thus,

$$
D\Lambda_\beta = \mathcal{D}\Lambda_\beta - \frac{1}{2}\,e_\beta\rfloor\left(d\Lambda - T^\alpha\Lambda_\alpha\right)\,. \qquad (3.36)
$$

Substituting this into (3.32) yields

$$
\begin{aligned}
D\,{}^{(3)}Q_{\alpha\beta} = \frac{2n}{(n-1)(n+2)}\,\Big[&T_{(\alpha}\,\Lambda_{\beta)} - \vartheta_{(\alpha}\wedge\mathcal{D}\Lambda_{\beta)} + \tfrac{1}{2}\,\vartheta_{(\alpha}\wedge e_{\beta)}\rfloor\left(d\Lambda - T^\gamma\Lambda_\gamma\right)\\
&+\tfrac{1}{n}\,Q_{\alpha\beta}\wedge\Lambda - \tfrac{1}{n}\,g_{\alpha\beta}\,d\Lambda\Big]\,. \qquad (3.37)
\end{aligned}
$$

Apparently, the right-hand side of (3.37) contains all irreducible pieces of the symmetric curvature $Z_{\alpha\beta}$. Contraction with the metric, for instance, yields

$$
g^{\alpha\beta}\,D\,{}^{(3)}Q_{\alpha\beta} = Q\wedge\Lambda\,. \qquad (3.38)
$$

Thus, $D^{(3)}Q_{\alpha\beta}$ contributes to ${}^{(4)}Z_{\alpha\beta}$. For a better understanding of this structure we make the simplifying assumptions

$$T^\alpha = 0\,, \qquad Q_{\alpha\beta} = {}^{(3)}Q_{\alpha\beta}\,. \tag{3.39}$$

Then we find

$$2\,{}^{(3)}Z_{\alpha\beta} \stackrel{*}{=} {}^{(3)}D^{(3)}Q_{\alpha\beta} = \frac{1}{(n+2)(n-1)}\,\left(n\,\vartheta_{(\alpha}\wedge e_{\beta)}\rfloor d\Lambda - 2g_{\alpha\beta}\,d\Lambda\right)\,, \tag{3.40}$$

$$2\,{}^{(5)}Z_{\alpha\beta} \stackrel{*}{=} {}^{(5)}D^{(3)}Q_{\alpha\beta} = \frac{4}{n}\,\vartheta_{(\alpha}\wedge \Xi_{\beta)}\,, \tag{3.41}$$

$$\begin{aligned}\Xi_\beta = -\frac{n^2}{(n-1)(n+2)}\,\Big[\mathcal{D}\Lambda_\beta - \frac{1}{n}\,\vartheta_\beta\, e^\alpha\rfloor\mathcal{D}\Lambda_\alpha \\ - \frac{2}{(n-1)(n+2)}\,\left(\Lambda_\beta\,\Lambda - \vartheta_\beta\;\Lambda^\alpha\Lambda_\alpha/n\right)\Big]\,.\end{aligned} \tag{3.42}$$

In general, derivatives of Λ will occur in different irreducible pieces of the symmetric curvature $Z_{\alpha\beta}$. However, in view of (3.40), the piece ${}^{(3)}Z_{\alpha\beta}$ seems to be the simplest generalization of $d\Lambda$, especially in view of the identity, see appendix A.4.12,

$${}^{(3)}Z^{\alpha\beta}\wedge {}^\star{}^{(3)}Z_{\alpha\beta} = \frac{n(n-2)}{n+2}\,\hat{\Delta}\wedge{}^\star\hat{\Delta}\,, \tag{3.43}$$

which obviously should parallel $d\Lambda\wedge{}^\star d\Lambda$. We can give the general relation between $\hat{\Delta}$ and Λ, see appendix A.4.11,

$$\begin{aligned}\hat{\Delta} \;=\; & \frac{1}{2(n-1)}\,d\Lambda - \frac{1}{2(n-2)}\,e^\alpha\rfloor DP_\alpha - \frac{1}{2(n-2)}\left\{\frac{1}{n-1}\,P_\alpha\,e^\alpha\rfloor\left[(n-1)\,Q - \Lambda + T\right]\right. \\ & \left. + \left(\frac{n+1}{n+2}\,{}^{(1)}T_\alpha + \frac{n}{n-1}\,{}^{(3)}T_\alpha\right)\,\Lambda^\alpha - \left({}^{(1)}Q_{\alpha\beta} + {}^{(2)}Q_{\alpha\beta}\right)\wedge e^\alpha\rfloor{}^{(1)}T^\beta\right\}.\end{aligned} \tag{3.44}$$

In the special case

$${}^{(1)}T^\alpha = {}^{(3)}T^\alpha = {}^{(2)}Q_{\alpha\beta} = 0\,, \tag{3.45}$$

we indeed find the desired simple relation

$$\hat{\Delta} \sim d\Lambda\,, \tag{3.46}$$

and arrive at

$${}^{(3)}Z_{\alpha\beta}\wedge{}^\star{}^{(3)}Z^{\alpha\beta} \sim d\Lambda\wedge{}^\star d\Lambda\,. \tag{3.47}$$

We conclude that

$${}^{(3)}Q_{\alpha\beta}\wedge{}^\star{}^{(3)}Q^{\alpha\beta} + {}^{(3)}Z_{\alpha\beta}\wedge{}^\star{}^{(3)}Z^{\alpha\beta} \tag{3.48}$$

presents itself as the centerpiece for embedding the J&M-aether field into metric-affine gravity. However, life will not be so simple ... The first Bianchi identity

$$0 = B^\alpha = DT^\alpha - R_\beta{}^\alpha \wedge \vartheta^\beta \,, \tag{3.49}$$

interweaves various pieces of nonmetricity, torsion, and curvature in a complicated manner. By contraction we find

$$\begin{aligned} e_\beta \rfloor B^\beta &= \mathcal{RIC} + (n-2)\,\Delta - \frac{n-2}{2}\, dQ + e_\alpha \rfloor \left[D({}^{(1)}T^\alpha + {}^{(3)}T^\alpha) \right] \\ &\quad + \frac{1}{n-1}\,({}^{(1)}T^\alpha + {}^{(3)}T^\alpha)\, e_\alpha \rfloor T - \frac{n-2}{n-1}\, dT = 0\,, \end{aligned} \tag{3.50}$$

$$\vartheta^\beta \wedge B_\beta = \tilde{X} - D(\vartheta^\alpha \wedge {}^{(3)}T_\alpha) + T^\alpha \wedge T_\alpha = 0\,. \tag{3.51}$$

Even if we assume that the torsion only possesses its trace piece, i. e.

$$T^\alpha = {}^{(2)}T = \frac{1}{n-1}\, \vartheta^\alpha \wedge T\,, \tag{3.52}$$

and therewith $\hat{\Delta} = d\Lambda/(2(n-1))$, we have

$$\mathcal{RIC} - \frac{n-2}{2(n-1)}\, d\,(2T + (n-1)\,Q - \Lambda) = 0\,, \tag{3.53}$$

i. e. a dynamical aether field requires the presence of torsion.

When setting up a Lagrangian, it seems we are forced to include also torsion. This leads us straight to the next section.

3.4 Lagrangian for metric-affine gravity

We now set up a first-order Lagrangian for gravity and aether. The independent variables are $(g_{\alpha\beta},\, \vartheta^\alpha,\, \Gamma_\alpha{}^\beta)$. These are coupled minimally to matter fields Ψ,

$$L_{\text{tot}} = V(g_{\alpha\beta}, \vartheta^\alpha, Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta) + L(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi)\,. \tag{3.54}$$

The variation of the matter Lagrangian

$$\delta L = \frac{1}{2}\delta g_{\alpha\beta}\, \sigma^{\alpha\beta} + \delta\vartheta^\alpha \wedge \Sigma_\alpha + \delta\Gamma_\alpha{}^\beta \wedge \Delta^\alpha{}_\beta + \delta\Psi \wedge \frac{\delta L}{\delta\Psi} \tag{3.55}$$

allows us to identify the material currents coupled to the potentials as metric energy-momentum and canonical energy-momentum, and as hypermomentum $(\sigma_{\alpha\beta}, \Sigma_\alpha, \Delta^\alpha{}_\beta)$. The energy-momenta $\sigma_{\alpha\beta}$ and Σ_α are related to each other by a Belinfante-Rosenfeld type of relation. The hypermomentum splits in spin current $\oplus$ dilation current $\oplus$ shear current:

$$\Delta_{\alpha\beta} = \tau_{\alpha\beta} + \frac{1}{4}\, g_{\alpha\beta}\, \Delta^\gamma{}_\gamma + \widehat{\not{\Delta}}_{\alpha\beta}\,, \qquad \tau_{\alpha\beta} = -\tau_{\beta\alpha}\,, \quad \widehat{\not{\Delta}}_{\alpha\beta} = \widehat{\not{\Delta}}_{\beta\alpha}\,. \tag{3.56}$$

The hypothetical shear current $\widehat{\not{\Delta}}_{\alpha\beta}$ is discussed in [89, 59], see also the literature given there.

The gauge Lagrangian is built in "Yang-Mills fashion", quadratic in the field strengths,

$$V_{\rm MAG} \sim \frac{1}{\kappa}(R_{\rm EC} + \lambda_0 + T^2 + TQ + Q^2) + \frac{1}{\rho}(W^2 + Z^2)\,. \tag{3.57}$$

The part in the first parentheses describes "weak gravity", governed by the conventional gravitational constant κ. It contains the usual Einstein-Cartan type term, a cosmological term and terms quadratic in torsion and nonmetricity. Since we have seen that torsion and nonmetricity enter into the connection and the weak-gravity part contains no derivatives of the connection (namely the curvature), only new contact interactions in addition to usual Newton-Einstein gravity will arise. In order to make the connection propagating, we have to allow for the "strong gravity" part in the last parentheses, characterized by the strong gravitational constant ρ. It turns out that the most general (parity conserving) quadratic Lagrangian is most appropriately displayed in terms of the $4+3+6+5$ irreducible pieces of $Q_{\alpha\beta}$, T^α, $W_\alpha{}^\beta$, and $Z_\alpha{}^\beta$, respectively (see [43], [95], [56], and references given):

$$\begin{aligned}
V_{\rm MAG} \;=\;& \frac{1}{2\kappa}\Bigg[-a_0\, R^{\alpha\beta}\wedge\eta_{\alpha\beta} - 2\lambda_0\,\eta + T^\alpha\wedge{}^*\left(\sum_{I=1}^{3} a_I\,{}^{(I)}T_\alpha\right)\\
&+\; 2\left(\sum_{I=2}^{4} c_I\,{}^{(I)}Q_{\alpha\beta}\right)\wedge\vartheta^\alpha\wedge{}^*T^\beta + Q_{\alpha\beta}\wedge{}^*\left(\sum_{I=1}^{4} b_I\,{}^{(I)}Q^{\alpha\beta}\right)\\
&+\; b_5\left({}^{(3)}Q_{\alpha\gamma}\wedge\vartheta^\alpha\right)\wedge{}^*\left({}^{(4)}Q^{\beta\gamma}\wedge\vartheta_\beta\right)\Bigg]\\
&-\; \frac{1}{2\rho}\,R^{\alpha\beta}\wedge{}^*\Bigg(\sum_{I=1}^{6} w_I\,{}^{(I)}W_{\alpha\beta} + w_7\,\vartheta_\alpha\wedge\left(e_\gamma\rfloor{}^{(5)}W^\gamma{}_\beta\right)\\
&+\; \sum_{I=1}^{5} z_I\,{}^{(I)}Z_{\alpha\beta} + z_6\,\vartheta_\gamma\wedge\left(e_\alpha\rfloor{}^{(2)}Z^\gamma{}_\beta\right) + \sum_{I=7}^{9} z_I\,\vartheta_\alpha\wedge\left(e_\gamma\rfloor{}^{(I-4)}Z^\gamma{}_\beta\right)\Bigg)\,.
\end{aligned} \tag{3.58}$$

Here κ is the dimensionful (weak) gravitational constant, λ_0 the "bare" cosmological constant, and the dimensionless ρ is the strong gravity coupling constant. The constants $a_0,\ldots a_3$, $b_1,\ldots b_5$, c_2, c_3, c_4, $w_1,\ldots w_7$, $z_1,\ldots z_9$ are dimensionless and should be of order unity. Note the nontrivial formula

$$R_{\alpha\beta}\wedge\eta^{\alpha\beta} = {}^{(6)}W_{\alpha\beta}\wedge\eta^{\alpha\beta}\,. \tag{3.59}$$

The field equations then read

$$
\begin{aligned}
DM^{\alpha\beta} - m^{\alpha\beta} &= \sigma^{\alpha\beta}\,, && \text{(zeroth)} && (3.60)\\
DH_\alpha - E_\alpha &= \Sigma_\alpha\,, && \text{(first)} && (3.61)\\
DH^\alpha{}_\beta - E^\alpha{}_\beta &= \Delta^\alpha{}_\beta\,, && \text{(second)} && (3.62)\\
\frac{\delta L}{\delta\Psi} &= 0\,. && \text{(matter)} && (3.63)
\end{aligned}
$$

The excitations or gauge momenta are given by

$$M^{\alpha\beta} = -2\,\frac{\partial V}{\partial Q_{\alpha\beta}}\,,\qquad H_\alpha = -\frac{\partial V}{\partial T^\alpha}\,,\qquad H^\alpha{}_\beta = -\frac{\partial V}{\partial R_\alpha{}^\beta}\,. \tag{3.64}$$

The metric energy-momentum of the gauge fields is

$$m^{\alpha\beta} := 2\,\frac{\partial V}{\partial g_{\alpha\beta}} = \vartheta^\alpha\wedge E^\beta + Q^\beta{}_\gamma\wedge M^{\alpha\gamma} - T^\alpha\wedge H^\beta - R_\gamma{}^\alpha\wedge H^{\gamma\beta} + R^{\beta\gamma}\wedge H^\alpha{}_\gamma\,, \tag{3.65}$$

the canonical energy-momentum of the gauge fields[1]

$$E_\alpha := \frac{\partial V}{\partial\vartheta^\alpha} = e_\alpha\rfloor V + (e_\alpha\rfloor T^\beta)\wedge H_\beta + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge H^\beta{}_\gamma + \frac{1}{2}(e_\alpha\rfloor Q_{\beta\gamma})\,M^{\beta\gamma}\,, \tag{3.66}$$

and the hypermomentum of the gauge fields

$$E^\alpha{}_\beta := \frac{\partial V}{\partial\Gamma_\alpha{}^\beta} = -\vartheta^\alpha\wedge H_\beta - g_{\beta\gamma}\,M^{\alpha\gamma}\,. \tag{3.67}$$

One can show that if the second field equation is fulfilled, either the zeroth or the first field equation is redundant due to the Noether identities. The excitations can be read off from the Lagrangian since we can use the Euler theorem for homogeneous functionals. Take for instance

$$V(R^{\alpha\beta}) = R^{\alpha\beta}\wedge{}^*\left(\sum_{I=1}^{6} w_I\,{}^{(I)}W_{\alpha\beta}\right)\,, \tag{3.68}$$

and substitute $R^{\alpha\beta}\longrightarrow\lambda R^{\alpha\beta}$. Then we have

$$V(\lambda\,R^{\alpha\beta}) = \lambda^2\,V(R^{\alpha\beta})\,, \tag{3.69}$$

and find by differentiation with respect to λ

$$\frac{\partial\,V}{\partial(\lambda\,R^{\alpha\beta})}\wedge\frac{d\,(\lambda\,R^{\alpha\beta})}{d\lambda} = \frac{\partial\,V}{\partial(\lambda\,R^{\alpha\beta})}\wedge R^{\alpha\beta} = 2\lambda\,V\,, \tag{3.70}$$

[1] For the relations between different energy-momentum currents in gravitational theory one should also compare Itin [65].

or, for $\lambda = 1$, and employing the definition of the momenta,

$$V = -\frac{1}{2}\, H_{\alpha\beta} \wedge R^{\alpha\beta}\,. \tag{3.71}$$

Then we can read off $H_{\alpha\beta}$ by comparison with the explicit form of V. Since all terms in (3.58) are homogeneous in this sense, we may generally apply this scheme. This method does not work in case of the rotational Chern-Simons Lagrangian considered in section 2.6. Then the variation has to be carried out by hand, see appendix A.4.3.

In the following we will not have to consider the full Lagrangian in (3.58)—remember that our main interest is focused on ${}^{(3)}Q_{\alpha\beta}$ and ${}^{(3)}Z_{\alpha\beta}$. Additionally we will of course keep the Einstein-Cartan part. Moreover we assume some more pieces which will allow us to cancel some parts of the field equations by imposing algebraic constraints on the coupling parameters.

3.5 Simple gravity-aether model and search for exact solutions

Now we are going to devise explicitly a simple gravity-aether model. Our idea is to arrive at a wave-like equation for the aether field

$$D^{\star\,(3)}Z_{\alpha}{}^{\beta} \sim D^{\star}D^{(3)}Q_{\alpha}{}^{\beta} = 0\,, \tag{3.72}$$

and the usual Einstein-like equations for the gravitational part. Therefore we surely keep Einstein-Cartan, cosmological, ${}^{(3)}Q^2$ and ${}^{(3)}Z^2$ terms. Additionally, we allow for the torsion-square term ${}^{(2)}T^2$, the cross term ${}^{(2)}T\,{}^{(3)}Q$, and the piece ${}^{(6)}W^2$. These will allow us to set the excitations H_α and $M^{\alpha\beta}$ to zero by means of appropriate constraints on the coupling parameters. We thus assume the Lagrangian

$$\begin{aligned} V_{\rm EA} \;=\; & \frac{1}{2\kappa}\,\Big[-a_0\,\big({}^{(6)}W^{\alpha\beta}\wedge\eta_{\alpha\beta}+2\lambda\,\eta\big)+a_2\,T^{\alpha}\wedge{}^{\star\,(2)}T_{\alpha}+2c_3\,{}^{(3)}Q_{\alpha\beta}\wedge\vartheta^{\alpha}\wedge{}^{\star}T^{\beta} \\ & +b_3 Q_{\alpha\beta}\wedge{}^{\star(3)}Q^{\alpha\beta}\Big]-\frac{1}{2}\,R^{\alpha\beta}\wedge{}^{\star}\big(w_6\,{}^{(6)}W_{\alpha\beta}+z_3\,{}^{(3)}Z_{\alpha\beta}\big)\;. \end{aligned}$$

By means of prolongation methods developed in [6], Baekler suggested a solution to the field equations. The coframe/metric will be assumed to be of the Schwarzschild-de Sitter form,

$$\vartheta^0 = e^{\mu(r)}dt\,,\quad \vartheta^1 = e^{-\mu(r)}dr\,,\quad \vartheta^2 = r\,d\theta\,,\quad \vartheta^3 = r\sin\theta\,d\phi\,, \tag{3.73}$$

where

$$g = -\vartheta^{\hat 0}\otimes\vartheta^{\hat 0}+\vartheta^{\hat 1}\otimes\vartheta^{\hat 1}+\vartheta^{\hat 2}\otimes\vartheta^{\hat 2}+\vartheta^{\hat 3}\otimes\vartheta^{\hat 3}\,, \tag{3.74}$$

together with the Schwarzschild-de Sitter function

$$e^{2\mu(r)} = 1-2\frac{M}{r}-\frac{\lambda}{3}r^2\,. \tag{3.75}$$

The torsion 2-form is

$$ {}^{(2)}T^{\alpha} = \frac{\mathcal{L}_0 e^{-\mu(r)}}{4r^2}\,, \begin{pmatrix} \vartheta^{01} \\ \vartheta^{01} \\ \vartheta^{02} - \vartheta^{12} \\ \vartheta^{03} - \vartheta^{13} \end{pmatrix}\,, \tag{3.76} $$

whereas the tensor part ${}^{(1)}T^{\alpha}$ and the axially symmetric piece ${}^{(3)}T^{\alpha}$ vanish identically. The nonmetricity $Q^{\alpha\beta}$ is given by

$$ Q^{\alpha\beta} = {}^{(1)}Q^{\alpha\beta} + {}^{(3)}Q^{\alpha\beta} = \frac{\mathcal{L}_0 e^{-\mu(r)}}{2r^2} \begin{pmatrix} \vartheta^0 + \vartheta^1 & 0 & \vartheta^2 & \vartheta^3 \\ 0 & \vartheta^0 + \vartheta^1 & \vartheta^2 & \vartheta^3 \\ \vartheta^2 & \vartheta^2 & 0 & 0 \\ \vartheta^3 & \vartheta^3 & 0 & 0 \end{pmatrix}\,. \tag{3.77} $$

The curvature pieces for this solution are explicitly displayed in appendix A.4.13. In order to form a solution of the field equation we have to require certain constraints on the coupling parameters,

$$ 8b_3 + 3c_3 = 0\,, \quad a2 + 2c_3 = 0\,, \quad 3a_0 = 2\kappa\lambda w_6\,, \quad z_3 = 0\,. \tag{3.78} $$

When we checked this solution by means of computer algebra, we realized that under these conditions the aether 1-form is, in fact, completely arbitrary apart from the fact that it has to be light-like $\Lambda \wedge {}^{\star}\Lambda = 0$. This is a pity but not very astonishing since the ${}^{(3)}Z$ part, which governs the aether dynamics, has been canceled. In the following we try to understand this structure better, especially the role of the constraints.

We first analyze the excitations H_α and $M^{\alpha\beta}$. For the Lagrangian (3.73) we find

$$ \begin{aligned} H_\alpha &= -\frac{1}{\kappa}{}^{\star}\left(a_2\,{}^{(2)}T_\alpha + c_3\,{}^{(3)}Q_{\alpha\beta} \wedge \vartheta^\beta\right) \\ &= -\frac{1}{3\kappa}{}^{\star}\left[\vartheta_\alpha \wedge (c_3\,\Lambda + a_2\,T)\right]\,. \end{aligned} \tag{3.79} $$

Hence, any two of the equations

$$ (i) \qquad H_\alpha = 0\,, \tag{3.80} $$

$$ (ii) \qquad a_2 + 2c_3 = 0\,, \tag{3.81} $$

$$ (iii) \quad T = \tfrac{\Lambda}{2} \quad \Leftrightarrow {}^{(3)}Q_{\alpha\beta} \wedge \vartheta^\beta = 2\,{}^{(2)}T_\alpha\,, \tag{3.82} $$

imply the remaining equation. We also have

$$ \begin{aligned} M^{\alpha\beta} &= -\frac{2}{\kappa}\left[b_3\,{}^{(3)}Q^{\alpha\beta} + c_3\left(\vartheta^{(\alpha} \wedge {}^{\star(2)}T^{\beta)} + \frac{1}{4}\,g^{\alpha\beta}\,{}^{\star}T\right)\right] \\ &= -\frac{2}{3\kappa}\left[\vartheta^{(\alpha}\left(\frac{4}{3}\,b_3\,\Lambda^{\beta)} + c_3\,e^{\beta)}\rfloor T\right) - g^{\alpha\beta}\left(\frac{b_3}{3}\,\Lambda + \frac{c_3}{4}\,T\right)\right]\,. \end{aligned} \tag{3.83} $$

Again, any two of

$$(i) \qquad M^{\alpha\beta} = 0\,, \tag{3.84}$$
$$(ii) \quad 8b_3 + 3c_3 = 0\,, \tag{3.85}$$
$$(iii) \qquad T = \tfrac{1}{2}\,\Lambda\,, \tag{3.86}$$

imply the remaining equation to hold. Thus, by assuming

$$a_2 + 2c_3 = 0\,, \quad 8b_3 + 3c_3 = 0\,, \quad T = \frac{1}{2}\,\Lambda\,, \tag{3.87}$$

the field equations reduce to

$$\text{FIRST} \quad e_\alpha\rfloor V + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge H^\beta{}_\gamma = 0\,, \tag{3.88}$$
$$\text{SECOND} \qquad DH^\alpha{}_\beta = 0\,, \tag{3.89}$$

where

$$H^\alpha{}_\beta = \frac{a_0}{2\kappa}\,\eta^\alpha{}_\beta + w_6\,{}^{\star(6)}W^\alpha{}_\beta + z_3\,{}^{\star(3)}Z^\alpha{}_\beta\,. \tag{3.90}$$

When assuming $\Lambda = 2T$ and $T^\alpha = {}^{(2)}T^\alpha$, the distortion simply evaluates to (half of) the nonmetricity,

$$N_{\alpha\beta} = \frac{1}{2}\,Q_{\alpha\beta}\,. \tag{3.91}$$

Since we were mainly interested in the nonmetricity, this seems to be rather satisfactory. As a consequence we also get a particularly simple splitting of the curvature into Riemannian and post-Riemannian parts. In general, we find the formula

$$\begin{aligned} R_\alpha{}^\beta &= R^{\{\}\beta}_\alpha + D^{\{\}}N_\alpha{}^\beta - N_\alpha{}^\gamma\wedge N_\gamma{}^\beta \\ &= R^{\{\}\beta}_\alpha + DN_\alpha{}^\beta + N_\alpha{}^\gamma\wedge N_\gamma{}^\beta\,. \end{aligned} \tag{3.92}$$

Hence, if (3.91) holds,

$$\begin{aligned} R_{\alpha\beta} &= R^{\{\}\beta}_\alpha + \frac{1}{2}\,D\left(Q_{\alpha\gamma}g^{\gamma\beta}\right) + \frac{1}{4}\,Q_\alpha{}^\gamma\wedge Q_\gamma{}^\beta \\ &= R^{\{\}\beta}_\alpha + \frac{1}{2}\,g^{\gamma\beta}\,DQ_{\alpha\gamma} - \frac{1}{2}\,Q_{\alpha\gamma}\wedge Q^{\gamma\beta} + \frac{1}{4}\,Q_\alpha{}^\gamma\wedge Q_\gamma{}^\beta \\ &= R^{\{\}\beta}_\alpha + Z_\alpha{}^\beta - \frac{1}{4}\,Q_\alpha{}^\gamma\wedge Q_\gamma{}^\beta\,. \end{aligned} \tag{3.93}$$

From this equation we directly derive the contribution of the aether field to the irreducible piece ${}^{(6)}W_{\alpha\beta}$, i.e., to the curvature scalar $W = e_\alpha\rfloor e_\beta\rfloor R^{\alpha\beta}$. Because of $e_\alpha\rfloor e_\beta\rfloor = -e_\beta\rfloor e_\alpha\rfloor$, the symmetric $Z^{\alpha\beta}$ drops out,

$$W = R^{\{\}} - \frac{1}{4}\,e^\alpha\rfloor e^\beta\rfloor\left(Q_\alpha{}^\gamma\wedge Q_{\gamma\beta}\right)\,. \tag{3.94}$$

Under the assumption $Q_{\alpha\beta} = {}^{(3)}Q_{\alpha\beta}$, the last term simply evaluates to

$$e^\alpha \rfloor e^\beta \rfloor \left({}^{(3)}Q_\alpha{}^\gamma \wedge {}^{(3)}Q_{\gamma\beta} \right) = -\frac{8}{9}\, \Lambda^\alpha \, \Lambda_\alpha \,. \tag{3.95}$$

If we assume a light-like aether,

$$0 = \Lambda^\alpha \, \Lambda_\alpha = {}^\star(\Lambda \wedge {}^\star\Lambda)\,, \tag{3.96}$$

the curvature scalar collapses to its Riemannian part. This allows us to cancel the ${}^{(6)}W$ and Einstein-Cartan like pieces in the field equation SECOND by assuming a metric/coframe which yields a constant Riemannian curvature, the usual Schwarzschild-de Sitter form, for instance,

$$\vartheta^{\hat{0}} = e^{\mu(r)}\, dt\,, \quad \vartheta^{\hat{1}} = e^{-\mu(r)}\, dr\,, \quad \vartheta^{\hat{2}} = r d\theta\,, \quad \vartheta^{\hat{3}} = r \sin\theta \, d\phi\,, \tag{3.97}$$

$$e^{2\mu(r)} = 1 - 2\,\frac{M}{r} - \frac{\lambda}{3}\, r^2\,. \tag{3.98}$$

The well-known Riemannian curvature scalar for this metric reads

$${}^{(6)}R^{\alpha\beta} = -\frac{\lambda}{3}\, \vartheta^\alpha \wedge \vartheta^\beta\,. \tag{3.99}$$

By assuming

$$a_0 = 2\kappa w_6 \frac{\lambda}{3}\,, \tag{3.100}$$

we arrive at

$$H^\alpha{}_\beta = z_3\, {}^{\star(3)}Z^\alpha{}_\beta\,. \tag{3.101}$$

Moreover, because of the light-likeness of Λ, we have

$$T^\alpha \wedge {}^{\star(2)}T_\alpha \sim T \wedge {}^\star T \sim \Lambda \wedge {}^\star\Lambda \;=\; 0\,, \tag{3.102}$$

$$Q_{\alpha\beta} \wedge {}^{\star(3)}Q^{\alpha\beta} \sim \Lambda \wedge {}^\star\Lambda \;=\; 0\,, \tag{3.103}$$

$${}^{(3)}Q_{\alpha\beta} \wedge \vartheta^\alpha \wedge {}^{\star(2)}T^\beta = 2T^\alpha \wedge {}^{\star(2)}T_\alpha \;=\; 0\,. \tag{3.104}$$

Consequently, the field equations read

$$\frac{z_3}{2}\,\left[\left(e_\alpha \rfloor Z^{\beta\gamma}\right) \wedge^{\star}\, {}^{(3)}Z_{\beta\gamma} - Z^{\beta\gamma} \wedge \left(e_\alpha \rfloor^{\star(3)} Z_{\beta\gamma}\right)\right] \;=\; 0\,, \tag{3.105}$$

$$z_3\, D^{\star(3)} Z_\alpha{}^\beta \;=\; 0\,. \tag{3.106}$$

These equations form an algebraic constraint on the components of ${}^{(3)}Z_{\alpha\beta}$ and a wave-like equation for ${}^{(3)}Q_{\alpha\beta}$ and are thus exactly what we were looking for. Unfortunately, we were not able to find an exact solution for the aether field Λ. Thus it seems that we have either to allow for different pieces governing the aether dynamics in the Lagrangian or alter the structure of the underlying Riemannian spacetime.

Both possibilities are not too far-fetched. Since the aether 1-form as a vector-like quantity introduces a preferred direction in spacetime, it seems plausible to assume a cylindrically symmetric structure of spacetime and not a spherically symmetric one, as we did in our simple ansatz (3.73). On the other hand, dynamical aether terms also do occur in other pieces of the curvature than ${}^{(3)}Z$, as we have seen in (3.37). Thus, it may also be viable to postulate a $({}^{(5)}Z)^2$-term instead or additional to the $({}^{(3)}Z)^2$-term in the Lagrangian. We made extensive computer algebra experiments in both directions but did not succeed in finding a sensible solution. Further investigations are necessary.

Appendix A

Appendix

A.1 Rules for exterior calculus

We would like to remind the reader of the following relations which hold for p-forms ϕ and ψ, and a q-form ω. Furthermore we denote the exterior product by $\wedge$, the interior product by $\rfloor$, the Hodge-star operator by * and the exterior derivative by d. The symbols $\mathbf{u}$, $\mathbf{v}$ stand for vectors, a and b are numbers, ind is the index of the metric.

$$
\begin{aligned}
(\phi+\psi)\wedge\omega &= \phi\wedge\omega+\psi\wedge\omega && \text{(A.1)}\\
(a\,\phi)\wedge\omega &= \phi\wedge(a\omega)=a\,(\phi\wedge\omega) && \text{(A.2)}\\
\phi\wedge\omega &= (-1)^{pq}\,\omega\wedge\phi && \text{(A.3)}\\
\mathbf{u}\rfloor(a\psi+b\phi) &= a\,\mathbf{u}\rfloor\psi+b\,\mathbf{u}\rfloor\phi && \text{(A.4)}\\
(\mathbf{u}+\mathbf{v})\rfloor\omega &= \mathbf{u}\rfloor\omega+\mathbf{v}\rfloor\omega && \text{(A.5)}\\
\mathbf{u}\rfloor(\phi\wedge\omega) &= (\mathbf{u}\rfloor\phi)\wedge\psi+(-1)^{p}\,\phi\wedge(\mathbf{u}\rfloor\omega) && \text{(A.6)}\\
\mathbf{u}\rfloor\mathbf{v}\rfloor\phi &= -\mathbf{v}\rfloor\mathbf{u}\rfloor\phi && \text{(A.7)}\\
\vartheta^{\alpha}\wedge e_{\alpha}\rfloor\phi &= p\,\phi && \text{(A.8)}\\
{}^\star(a\psi+b\phi) &= a^\star\psi+b^\star\phi && \text{(A.9)}\\
{}^{\star\star}\psi &= (-1)^{p(n-p)+\mathrm{ind}}\,\psi && \text{(A.10)}\\
{}^\star(e_{\alpha}\rfloor\psi) &= (-1)^{(p-1)}\,\vartheta_{\alpha}\wedge{}^\star\psi && \text{(A.11)}\\
e_{\alpha}\rfloor{}^\star\psi &= {}^\star(\psi\wedge\vartheta_{\alpha}) && \text{(A.12)}\\
{}^\star\psi\wedge\phi &= {}^\star\phi\wedge\psi && \text{(A.13)}\\
d(a\psi+b\phi) &= a\,d\psi+b\,d\phi && \text{(A.14)}\\
d(\omega\wedge\phi) &= d\omega\wedge\phi+(-1)^{q}\,\omega\wedge d\phi && \text{(A.15)}\\
dd\phi &= 0 && \text{(A.16)}
\end{aligned}
$$

A.2 Conventions of index notation

Our index notation is based on the conventions of Schouten [102] and for exterior calculus we refer to [58]. For quick and easy reference, we display our conventions for index positions and signs of the Christoffel symbol, the Riemann tensor, and the Ricci tensor (holonomic indices $i, j, \dots = 0, 1, 2, 3$). The sign of the Ricci tensor is the same as those of the L_{ij} tensor and the Cotton tensor. In particular, the Ric_{ij} sign introduces a relative sign between the L_{ij} tensor and the Weyl tensor in the decomposition of the curvature:

$$\begin{aligned}
\nabla_i T_j{}^k &= \partial_i T_j{}^k - \Gamma_{ij}{}^\ell T_\ell{}^k + \Gamma_{i\ell}{}^k T_j{}^\ell \,, && \text{(A.17)}\\
+R_{ijk}{}^\ell &= \partial_i \Gamma_{jk}{}^\ell - \partial_j \Gamma_{ik}{}^\ell + \Gamma_{im}{}^\ell \Gamma_{jk}{}^m - \Gamma_{jm}{}^\ell \Gamma_{ik}{}^m \,, && \text{(A.18)}\\
+\mathrm{Ric}_{jk} &= R_{ijk}{}^i \,, && \text{(A.19)}\\
+R &= R_{ij}{}^{ji} \,, && \text{(A.20)}\\
\mathrm{Weyl}_{ijk\ell} &= R_{ijk\ell} + \frac{4}{n-2}\, g_{[i|[k} L_{\ell]|j]} \,. && \text{(A.21)}
\end{aligned}$$

An extensive comparison between the various conventions can be found in [86].

A.3 Irreducible decompositions

A.3.1 Nonmetricity

The nonmetricity splits into 4 irreducible parts:

$$\begin{aligned}
Q_{\alpha\beta} &= {}^{(1)}Q_{\alpha\beta} + {}^{(2)}Q_{\alpha\beta} + {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta} \,,\\
&= \mathrm{TRINOM} + \mathrm{BINOM} + \mathrm{VECNOM} + \mathrm{CONOM} \,,\\
\frac{1}{2} n^2 (n+1) &= \frac{1}{6} n(n-1)(n+4) + \frac{1}{3} n(n^2-4) + n + n \,,
\end{aligned}$$

$$\begin{aligned}
{}^{(2)}Q_{\alpha\beta} &= -\frac{2}{3}\, e_{(\alpha} \rfloor P_{\beta)} \,, && \text{(A.22)}\\
{}^{(3)}Q_{\alpha\beta} &= \frac{2n}{(n-1)(n+2)} \left(\Lambda_{(\alpha}\, \vartheta_{\beta)} - \frac{1}{n}\, g_{\alpha\beta}\, \Lambda \right) , && \text{(A.23)}\\
{}^{(4)}Q_{\alpha\beta} &= g_{\alpha\beta}\, Q \,, && \text{(A.24)}\\
{}^{(1)}Q_{\alpha\beta} &= Q_{\alpha\beta} - {}^{(2)}Q_{\alpha\beta} - {}^{(3)}Q_{\alpha\beta} - {}^{(4)}Q_{\alpha\beta} \,. && \text{(A.25)}
\end{aligned}$$

A.3.2 Torsion

The torsion is a one-indexed 2-form. There are again two possibilities two build a one-index form out of scalar valued forms, namely by taking the exterior product of the frame with a 1-form and by taking the interior product with the frame. What is left over is the irreducible 3rd rank piece.

$$\begin{aligned} T^\alpha &= {}^{(1)}T^\alpha + {}^{(2)}T^\alpha + {}^{(3)}T^\alpha\,, \\ &= \text{TENTOR} + \text{TRATOR} + \text{AXITOR}\,, \\ \frac{1}{2}n^2(n-1) &= \frac{1}{3}n(n^2-4) + n + \frac{1}{6}n(n-1)(n-2)\,, \end{aligned} \tag{A.26}$$

where

$$ {}^{(2)}T^\alpha = \frac{1}{n-1}\,\vartheta^\alpha \wedge (e_\beta \rfloor T^\beta)\,, \tag{A.27}$$

$$ {}^{(3)}T^\alpha = \frac{1}{3}\,e_\alpha \rfloor (T^\beta \wedge \vartheta_\beta)\,, \tag{A.28}$$

$$ {}^{(1)}T^\alpha = T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha\,. \tag{A.29}$$

A.3.3 Curvature

According to [58, 82] the curvature 2-form in a metric-affine spacetime decomposes into 11 irreducible pieces. First we split it into a symmetric and an antisymmetric part,

$$ R_{\alpha\beta} = W_{\alpha\beta} + Z_{\alpha\beta}\,, \quad W_{\alpha\beta} = R_{[\alpha\beta]}\,, \quad Z_{\alpha\beta} = R_{(\alpha\beta)}\,. \tag{A.30}$$

The antisymmetric piece $W_{\alpha\beta}$ can be further decomposed with respect to the pseudo-orthogonal group:

$$\begin{aligned} W^{\alpha\beta} &= {}^{(1)}W^{\alpha\beta} + {}^{(2)}W^{\alpha\beta} + {}^{(3)}W^{\alpha\beta} + {}^{(4)}W^{\alpha\beta} + {}^{(5)}W^{\alpha\beta} + {}^{(6)}W^{\alpha\beta} \\ &= \text{WEYL} + \text{PAIRCOM} + \text{PSCALAR} + \text{RICSYMF} + \text{RICANTI} + \text{SCALAR}\,, \end{aligned}$$

where

$$ {}^{(2)}W_{\alpha\beta} = (-1)^{\text{ind}}\,{}^\star(\vartheta_{[\alpha} \wedge \psi_{\beta]}) = e_{[\alpha} \rfloor \tilde{\psi}_{\beta]}\,, \tag{A.31}$$

$$ {}^{(3)}W_{\alpha\beta} = (-1)^{\text{ind}}\frac{1}{12}{}^\star(X \wedge \vartheta_\alpha \wedge \vartheta_\beta) = -\frac{1}{12}\,e_\alpha \rfloor e_\beta \rfloor \tilde{X}\,, \tag{A.32}$$

$$ {}^{(4)}W_{\alpha\beta} = -\frac{2}{n-2}\vartheta_{[\alpha} \wedge \phi_{\beta]}\,, \tag{A.33}$$

$$ {}^{(5)}W_{\alpha\beta} = -\frac{1}{n-2}\vartheta_{[\alpha} \wedge \xi_{\beta]}\,, \tag{A.34}$$

$$ {}^{(6)}W_{\alpha\beta} = -\frac{1}{n(n-1)}W\,\vartheta_\alpha \wedge \vartheta_\beta\,, \tag{A.35}$$

$$ {}^{(1)}W_{\alpha\beta} = W_{\alpha\beta} - {}^{(2)}W_{\alpha\beta} - {}^{(3)}W_{\alpha\beta} - {}^{(4)}W_{\alpha\beta} - {}^{(5)}W_{\alpha\beta} - {}^{(6)}W_{\alpha\beta}\,, \tag{A.36}$$

with the traces

$$W^{\alpha} := e_{\beta}\rfloor W^{\alpha\beta}\,, \quad W := e_{\alpha}\rfloor W^{\alpha}\,, \quad X^{\alpha} := {}^{\star}(W^{\beta\alpha}\wedge\vartheta_{\beta})\,, \quad X := e_{\alpha}\rfloor X^{\alpha}\,, \tag{A.37}$$

and the covector-valued forms

$$\psi_{\alpha} := X_{\alpha} - \frac{1}{4}\vartheta_{\alpha}\wedge X - \frac{1}{n-2}e_{\alpha}\rfloor(\vartheta^{\beta}\wedge X_{\beta})\,, \tag{A.38}$$

$$\phi_{\alpha} := W_{\alpha} - \frac{1}{n}W\,\vartheta_{\alpha} - \frac{1}{2}e_{\alpha}\rfloor(\vartheta^{\beta}\wedge W_{\beta})\,, \tag{A.39}$$

$$\xi_{\alpha} := e_{\alpha}\rfloor(\vartheta^{\beta}\wedge W_{\beta}) =: e_{\alpha}\rfloor\mathcal{RIC}\,. \tag{A.40}$$

We can get rid of the stars in the definition of the various traces,

$$\mathcal{RIC} = \vartheta^{\alpha}\wedge(e_{\beta}\rfloor W_{\alpha}{}^{\beta})\,, \tag{A.41}$$

$$\tilde{X} = W_{\alpha\beta}\wedge\vartheta^{\alpha}\wedge\vartheta^{\beta} = (-1)^{\mathrm{ind}}\,{}^{\star}X\,, \tag{A.42}$$

$$\tilde{\psi}_{\alpha} = W_{\beta\alpha}\wedge\vartheta^{\beta} + \frac{1}{4}\,e_{\alpha}\rfloor\tilde{X} + \frac{1}{n-2}\,\vartheta_{\alpha}\wedge\mathcal{RIC} = (-1)^{(n-1)+\mathrm{ind}}\,{}^{\star}\psi^{\alpha}\,. \tag{A.43}$$

We find characteristic contraction properties,

$$e^{\alpha}\rfloor\phi_{\alpha} = 0\,, \qquad \phi_{\alpha}\wedge\vartheta^{\alpha} = 0\,, \tag{A.44}$$

$$e^{\alpha}\rfloor\tilde{\psi}_{\alpha} = 0\,, \qquad \tilde{\psi}_{\alpha}\wedge\vartheta^{\alpha} = 0\,, \tag{A.45}$$

$$e^{\alpha}\rfloor\xi_{\alpha} = 0\,, \qquad \xi_{\alpha}\wedge\vartheta^{\alpha} = -2\,\mathcal{RIC}\,. \tag{A.46}$$

The 1-form pieces can be collected,

$$L_{\alpha} = \phi_{\alpha} + \frac{1}{2}\,\xi_{\alpha} + \frac{n-2}{2n(n-1)}\,W\,\vartheta_{\alpha} = e_{\beta}\rfloor W_{\alpha}{}^{\beta} - \frac{1}{2(n-1)}\,W\,\vartheta_{\alpha}\,. \tag{A.47}$$

We may give a very intuitive representation of the irreducible decomposition. In the first line of (A.48) we have the irreducible 4th-rank piece for which all contractions vanish. In the second line are pieces displayed which effectively correspond to a 1-form with one index multiplied with the coframe. In the last line of (A.48) we display the pieces which can be obtained by taking the interior product with a 3-form.

$$\begin{aligned} W_{\alpha\beta} = {} & {}^{(1)}W_{\alpha\beta} \\ & - \frac{2}{n-2}\,\vartheta_{[\alpha}\wedge\phi_{\beta]} - \frac{1}{n-2}\,\vartheta_{[\alpha}\wedge\xi_{\beta]} - \frac{1}{n(n-1)}\,R\vartheta_{\alpha}\wedge\vartheta_{\beta} \\ & + e_{[\alpha}\rfloor\tilde{\psi}_{\beta]} - \frac{1}{12}e_{\alpha}\rfloor e_{\beta}\rfloor\tilde{Z}\,. \end{aligned} \tag{A.48}$$

For the symmetric part we find 5 irreducible pieces,

$$Z_{\alpha\beta} = {}^{(1)}Z_{\alpha\beta} + {}^{(2)}Z_{\alpha\beta} + {}^{(3)}Z_{\alpha\beta} + {}^{(4)}Z_{\alpha\beta} + {}^{(5)}Z_{\alpha\beta}\,, \tag{A.49}$$

where the single pieces are defined according to

$$
\begin{aligned}
{}^{(2)}Z_{\alpha\beta} &= \frac{1}{2}\,(-1)^{\mathrm{ind}\,\star}\left\{\vartheta_{(\alpha}\wedge\Upsilon_{\beta)}\right\}\,, && \text{(A.50)}\\
{}^{(3)}Z_{\alpha\beta} &= \frac{1}{n+2}\left\{n\,\vartheta_{(\alpha}\wedge(e_{\beta)}\rfloor\hat{\Delta})-2g_{\alpha\beta}\,\hat{\Delta}\right\}\,, && \text{(A.51)}\\
{}^{(4)}Z_{\alpha\beta} &= \frac{1}{n}\,g_{\alpha\beta}\,Z\,, && \text{(A.52)}\\
{}^{(5)}Z_{\alpha\beta} &= \frac{2}{n}\,\vartheta_{(\alpha}\wedge\Xi_{\beta)}\,, && \text{(A.53)}\\
{}^{(1)}Z_{\alpha\beta} &= Z_{\alpha\beta}-{}^{(2)}Z_{\alpha\beta}-{}^{(3)}Z_{\alpha\beta}-{}^{(4)}Z_{\alpha\beta}-{}^{(5)}Z_{\alpha\beta}\,, && \text{(A.54)}
\end{aligned}
$$

where

$$Z_{\alpha\beta} =: \not{Z}_{\alpha\beta}+\tfrac{1}{n}\,g_{\alpha\beta}\,Z_{\gamma}{}^{\gamma}\,, \qquad \text{(A.55)}$$

$$\not{Z}_{\alpha}:=e^{\beta}\rfloor\not{Z}_{\alpha\beta}\,,\qquad \hat{\Delta}:=\tfrac{1}{n-2}\,\vartheta^{\alpha}\wedge\not{Z}_{\alpha}\,,\qquad Y_{\alpha}:={}^{\star}(\not{Z}_{\alpha\beta}\wedge\vartheta^{\beta})\,, \quad \text{(A.56)}$$

$$\Xi_{\alpha}:=\not{Z}_{\alpha}-\tfrac{1}{2}\,e_{\alpha}\rfloor(\vartheta^{\gamma}\wedge\not{Z}_{\gamma})\,,\Upsilon_{\alpha}:=Y_{\alpha}-\tfrac{1}{n-2}\,e_{\alpha}\rfloor(\vartheta^{\gamma}\wedge Y_{\gamma})\,. \qquad \text{(A.57)}$$

The contractions of Ξ and Υ vanish,

$$e^{\alpha}\rfloor\Xi_{\alpha}=0\,,\qquad \vartheta^{\alpha}\wedge\Xi_{\alpha}=0\,,\qquad e^{\alpha}\rfloor\Upsilon_{\alpha}=0\,,\qquad \vartheta^{\alpha}\wedge\Upsilon_{\alpha}=0\,. \quad \text{(A.58)}$$

Contractions of the irreducible pieces of the curvature

Contractions with the coframe

$$
\begin{aligned}
{}^{(1)}W_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= 0\,, && \text{(A.59)}\\
{}^{(2)}W_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= \tilde{\Psi}^{\beta}\,, && \text{(A.60)}\\
{}^{(3)}W_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= -\frac{1}{4}\,e^{\beta}\rfloor\tilde{X}\,, && \text{(A.61)}\\
{}^{(4)}W_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= 0\,, && \text{(A.62)}\\
{}^{(5)}W_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= -\frac{1}{n-2}\,\vartheta^{\beta}\wedge\mathcal{RIC}\,, && \text{(A.63)}\\
{}^{(6)}W_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= 0\,, && \text{(A.64)}\\
{}^{(1)}Z_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= 0\,, && \text{(A.65)}\\
{}^{(2)}Z_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= (-1)^{n-1+\mathrm{sgn}\,\star}\Upsilon^{\beta}\,, && \text{(A.66)}\\
{}^{(3)}Z_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= -\hat{\Delta}\wedge\vartheta^{\beta}\,, && \text{(A.67)}\\
{}^{(4)}Z_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= \frac{1}{2}\,dQ\wedge\vartheta^{\beta}\,, && \text{(A.68)}\\
{}^{(5)}Z_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} &= 0\,. && \text{(A.69)}
\end{aligned}
$$

Contractions with the frame

$$e_\beta\rfloor{}^{(1)}W_\alpha{}^\beta = 0\,, \tag{A.70}$$
$$e_\beta\rfloor{}^{(2)}W_\alpha{}^\beta = 0\,, \tag{A.71}$$
$$e_\beta\rfloor{}^{(3)}W_\alpha{}^\beta = 0\,, \tag{A.72}$$
$$e_\beta\rfloor{}^{(4)}W_\alpha{}^\beta = \Phi_\alpha\,, \tag{A.73}$$
$$e_\beta\rfloor{}^{(5)}W_\alpha{}^\beta = \frac{1}{2}\,e^\alpha\rfloor\mathcal{RIC}\,, \tag{A.74}$$
$$e_\beta\rfloor{}^{(6)}W_\alpha{}^\beta = \frac{1}{n}\,W\,\vartheta_\alpha\,, \tag{A.75}$$
$$e_\beta\rfloor{}^{(1)}Z_\alpha{}^\beta = 0\,, \tag{A.76}$$
$$e_\beta\rfloor{}^{(2)}Z_\alpha{}^\beta = 0\,, \tag{A.77}$$
$$e_\beta\rfloor{}^{(3)}Z_\alpha{}^\beta = \frac{n-2}{2}\,e_\alpha\rfloor\hat{\Delta}\,, \tag{A.78}$$
$$e_\beta\rfloor{}^{(4)}Z_\alpha{}^\beta = \frac{1}{n}\,e_\alpha\rfloor Z\,, \tag{A.79}$$
$$e_\beta\rfloor{}^{(5)}Z_\alpha{}^\beta = \Xi_\alpha\,. \tag{A.80}$$

Contractions with the frame and the coframe

$$e_\beta\rfloor\left({}^{(1)}W_\alpha{}^\beta\wedge\vartheta^\alpha\right) = 0\,, \tag{A.81}$$
$$e_\beta\rfloor\left({}^{(2)}W_\alpha{}^\beta\wedge\vartheta^\alpha\right) = 0\,, \tag{A.82}$$
$$e_\beta\rfloor\left({}^{(3)}W_\alpha{}^\beta\wedge\vartheta^\alpha\right) = 0\,, \tag{A.83}$$
$$e_\beta\rfloor\left({}^{(4)}W_\alpha{}^\beta\wedge\vartheta^\alpha\right) = 0\,, \tag{A.84}$$
$$e_\beta\rfloor\left({}^{(5)}W_\alpha{}^\beta\wedge\vartheta^\alpha\right) = -\mathcal{RIC}\,, \tag{A.85}$$
$$e_\beta\rfloor\left({}^{(6)}W_\alpha{}^\beta\wedge\vartheta^\alpha\right) = 0\,, \tag{A.86}$$
$$e_\beta\rfloor\left({}^{(1)}Z_\alpha{}^\beta\wedge\vartheta^\alpha\right) = 0\,, \tag{A.87}$$
$$e_\beta\rfloor\left({}^{(2)}Z_\alpha{}^\beta\wedge\vartheta^\alpha\right) = 0\,, \tag{A.88}$$
$$e_\beta\rfloor\left({}^{(3)}Z_\alpha{}^\beta\wedge\vartheta^\alpha\right) = -(n-2)\,\hat{\Delta}\,, \tag{A.89}$$
$$e_\beta\rfloor\left({}^{(4)}Z_\alpha{}^\beta\wedge\vartheta^\alpha\right) = \frac{n-2}{2}\,dQ\,, \tag{A.90}$$
$$e_\beta\rfloor\left({}^{(5)}Z_\alpha{}^\beta\wedge\vartheta^\alpha\right) = 0\,. \tag{A.91}$$

Double-duality relations in $n = 4$

$$
\begin{aligned}
{}^{(1)}W^{*\alpha\beta} &= \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(1)}W_{\mu\nu} = {}^{\star}\,{}^{(1)}W^{\alpha\beta}\,, && \text{(A.92)}\\
{}^{(2)}W^{*\alpha\beta} &= \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(2)}W_{\mu\nu} = -{}^{\star}\,{}^{(2)}W^{\alpha\beta}\,, && \text{(A.93)}\\
{}^{(3)}W^{*\alpha\beta} &= \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(3)}W_{\mu\nu} = {}^{\star}\,{}^{(3)}W^{\alpha\beta}\,, && \text{(A.94)}\\
{}^{(4)}W^{*\alpha\beta} &= \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(4)}W_{\mu\nu} = -{}^{\star}\,{}^{(4)}W^{\alpha\beta}\,, && \text{(A.95)}\\
{}^{(5)}W^{*\alpha\beta} &= \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(5)}W_{\mu\nu} = {}^{\star}\,{}^{(5)}W^{\alpha\beta}\,, && \text{(A.96)}\\
{}^{(6)}W^{*\alpha\beta} &= \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(6)}W_{\mu\nu} = {}^{\star}\,{}^{(6)}W^{\alpha\beta}\,. && \text{(A.97)}
\end{aligned}
$$

A.4 Calculations

A.4.1 Double-duality relations for the curvature

In this section we proof the double-duality relations for the irreducible pieces of the antisymmetric curvature. The proofs all work in a similar way by computing

$$e^{\nu}\rfloor e^{\mu}\rfloor(\eta^{\alpha\beta}\wedge{}^{(i)}W_{\mu\nu}) = e^{\nu}\rfloor e^{\mu}\rfloor({}^{\star}\,{}^{(i)}W_{\mu\nu}\wedge\vartheta^{\alpha}\wedge\vartheta^{\beta})\,,$$

and making use of the contraction properties of the respective curvature pieces.

1) $\mathbf{{}^{(1)}W^{*\alpha\beta} = \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(1)}W_{\mu\nu} = {}^{\star}\,{}^{(1)}W^{\alpha\beta}}$

Because of $e^{\alpha}\rfloor{}^{(1)}W_{\alpha\beta} = 0$ we directly find

$$e^{\nu}\rfloor e^{\mu}\rfloor(\eta^{\alpha\beta}\wedge{}^{(1)}W_{\mu\nu}) = \eta^{\alpha\beta\mu\nu}\,{}^{(1)}W_{\mu\nu}\,. \qquad \text{(A.98)}$$

On the other hand ($\vartheta^{\alpha}\wedge{}^{(1)}W_{\alpha\beta} = e^{\alpha}\rfloor{}^{\star(1)}W_{\alpha\beta} = 0$),

$$
\begin{aligned}
e^{\nu}\rfloor e^{\mu}\rfloor(\eta^{\alpha\beta}\wedge{}^{(1)}W_{\mu\nu}) &= e^{\nu}\rfloor e^{\mu}\rfloor\left({}^{\star}(\vartheta^{\alpha}\wedge\vartheta^{\beta})\wedge{}^{(1)}W_{\mu\nu}\right)\\
&= e^{\nu}\rfloor e^{\mu}\rfloor\left({}^{\star}\,{}^{(1)}W_{\mu\nu}\wedge\vartheta^{\alpha}\wedge\vartheta^{\beta}\right) = 2^{\star}\,{}^{(1)}W_{\alpha\beta}\,. \qquad \text{(A.99)}
\end{aligned}
$$

2) $\mathbf{{}^{(2)}W^{*\alpha\beta} = \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(2)}W_{\mu\nu} = -{}^{\star}\,{}^{(2)}W^{\alpha\beta}}$

Since $e^{\alpha}\rfloor{}^{(2)}W_{\alpha\beta} = 0$ we again have

$$e^{\nu}\rfloor e^{\mu}\rfloor(\eta^{\alpha\beta}\wedge{}^{(2)}W_{\mu\nu}) = \eta^{\alpha\beta\mu\nu}\,{}^{(2)}W_{\mu\nu}\,. \qquad \text{(A.100)}$$

Shifting the Hodg-star and using ${}^{(2)}W_{\alpha\beta} = e_{[\alpha}\rfloor\tilde{\psi}_{\beta]}$, we find

$$
\begin{aligned}
e^\nu\rfloor e^\mu\rfloor(\eta^{\alpha\beta}\wedge {}^{(2)}W_{\mu\nu}) &= e^\nu\rfloor e^\mu\rfloor \left({}^{\star\,(2)}W_{\mu\nu}\wedge\vartheta^\alpha\wedge\vartheta^\beta\right)\\
&= {}^\star({}^{(2)}W_{\mu\nu}\wedge\vartheta^\mu\wedge\vartheta^\nu)\wedge\vartheta^\alpha\wedge\vartheta^\beta\\
&\quad +2^{\star\,(2)}W^{\alpha\beta} - 4\,{}^\star({}^{(2)}W_\mu{}^{[\alpha|}\wedge\vartheta^\mu)\wedge\vartheta^{|\beta]}\\
&= 2^{\star\,(2)}W^{\alpha\beta} - 4\,{}^\star\tilde{\psi}^{[\alpha}\wedge\vartheta^{\beta]} = 2^{\star\,(2)}W^{\alpha\beta} - 4\,{}^\star(e^{[\alpha}\rfloor\tilde{\psi}^{\beta]})\\
&= -2^{\star\,(2)}W^{\alpha\beta} \qquad\qquad (A.101)
\end{aligned}
$$

3) $\mathbf{{}^{(3)}W^{*\alpha\beta} = \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(3)}W_{\mu\nu} = {}^{\star\,(3)}W^{\alpha\beta}}$

We have once more $e^\alpha\rfloor{}^{(3)}W_{\alpha\beta} = 0$,

$$
e^\nu\rfloor e^\mu\rfloor(\eta^{\alpha\beta}\wedge {}^{(3)}W_{\mu\nu}) = \eta^{\alpha\beta\mu\nu}\,{}^{(3)}W_{\mu\nu}\,. \qquad (A.102)
$$

On the other hand, remembering

$$
{}^{(3)}W_{\alpha\beta} = -\frac{1}{12}\,e_\alpha\rfloor e_\beta\rfloor\widetilde{X}\,,
$$

we obtain,

$$
\begin{aligned}
e^\nu\rfloor e^\mu\rfloor(\eta^{\alpha\beta}\wedge {}^{(3)}W_{\mu\nu}) &= e^\nu\rfloor e^\mu\rfloor \left({}^{\star\,(3)}W_{\mu\nu}\wedge\vartheta^\alpha\wedge\vartheta^\beta\right)\\
&= {}^\star({}^{(3)}W_{\mu\nu}\wedge\vartheta^\mu\wedge\vartheta^\nu)\wedge\vartheta^\alpha\wedge\vartheta^\beta\\
&\quad +2\,{}^{\star\,(3)}W^{\alpha\beta} - 4\,{}^\star({}^{(3)}W_\mu{}^{[\alpha|}\wedge\vartheta^\mu)\wedge\vartheta^{|\beta]}\\
&= {}^\star\widetilde{X}\,\vartheta^\alpha\wedge\vartheta^\beta + 2\,{}^{\star\,(3)}W^{\alpha\beta} + {}^\star(e^{[\alpha}\rfloor\widetilde{X})\wedge\vartheta^{\beta]}\\
&= 2\,{}^{\star\,(3)}W^{\alpha\beta} \qquad\qquad (A.103)
\end{aligned}
$$

4) $\mathbf{{}^{(4)}W^{*\alpha\beta} = \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(4)}W_{\mu\nu} = -{}^{\star\,(4)}W^{\alpha\beta}}$

In this case the contraction with the frame is non-vanishing,

$$
e^\nu\rfloor e^\mu\rfloor(\eta^{\alpha\beta}\wedge {}^{(4)}W_{\mu\nu}) = \eta^{\alpha\beta\mu\nu}\,{}^{(4)}W_{\mu\nu} + 2\,\phi_\nu\wedge\eta^{\alpha\beta\nu}\,. \qquad (A.104)
$$

In view of

$$
{}^{(4)}W^{\alpha\beta} = -\vartheta_{[\alpha}\wedge\phi_{\beta]}\,,
$$

the last term on the right hand side can be reformulated,

$$
\begin{aligned}
\phi_\nu\wedge\eta^{\alpha\beta\nu} &= \phi_\nu\wedge\vartheta_\mu\wedge e^\mu\rfloor\eta^{\alpha\beta\nu} = \phi_\nu\wedge\vartheta_\mu\,\eta^{\alpha\beta\nu\mu}\\
&= -\,{}^{(4)}W_{\mu\nu}\,\eta^{\alpha\beta\mu\nu}\,. \qquad\qquad (A.105)
\end{aligned}
$$

Hence,

$$
e^\nu\rfloor e^\mu\rfloor(\eta^{\alpha\beta}\wedge {}^{(4)}W_{\mu\nu}) = -\eta^{\alpha\beta\mu\nu}\,{}^{(4)}W_{\mu\nu}\,. \qquad (A.106)
$$

Now we use $\vartheta^\alpha\wedge{}^{(4)}W_{\alpha\beta} = e^\alpha\rfloor{}^{\star\,(4)}W_{\alpha\beta} = 0$,

$$
e^\nu\rfloor e^\mu\rfloor(\eta^{\alpha\beta}\wedge {}^{(4)}W_{\mu\nu}) = e^\nu\rfloor e^\mu\rfloor\left({}^{\star\,(4)}W_{\mu\nu}\wedge\vartheta^\alpha\wedge\vartheta^\beta\right) = 2\,{}^{\star\,(4)}W^{\alpha\beta}\,, \qquad (A.107)
$$

which completes the proof.

5) $\mathbf{{}^{(5)}W^{*\alpha\beta} = \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(5)}W_{\mu\nu} = {}^{\star\,(5)}W^{\alpha\beta}}$

Since only the double contractions ${}^{(5)}W_{\alpha\beta}\wedge\vartheta^\alpha\wedge\vartheta^\beta = 0$ and $e^\alpha\rfloor e^\beta\rfloor{}^{(5)}W_{\alpha\beta} = 0$ vanish the proof is a little bit more involved. We use

$$ {}^{(5)}W_{\alpha\beta} = -\frac{1}{2}\,\vartheta_{[\alpha}\wedge\xi_{\beta]}\,, \qquad \xi_\alpha = e_\alpha\rfloor\mathcal{RIC}\,, $$

and calculate

$$ \begin{aligned} e^\nu\rfloor e^\mu\rfloor(\eta^{\alpha\beta}\wedge{}^{(5)}W_{\mu\nu}) &= -\frac{1}{2}\,e^\nu\rfloor e^\mu\rfloor(\eta^{\alpha\beta}\wedge\vartheta_\mu\wedge\xi_\nu) \\ &= -\frac{1}{2}(\eta^{\alpha\beta\mu\nu}\,\vartheta_\mu\wedge\xi_\nu + 2\,\eta^{\alpha\beta\nu}\wedge\xi_\nu) \\ &= -\frac{1}{2}(\eta^{\alpha\beta\mu\nu}\,\vartheta_\mu\wedge\xi_\nu + 2\,\vartheta_\mu\wedge\eta^{\alpha\beta\nu\mu}\wedge\xi_\nu) \\ &= \frac{1}{2}(\eta^{\alpha\beta\mu\nu}\,\vartheta_\mu\wedge\xi_\nu) = -\eta^{\alpha\beta\mu\nu}\,{}^{(5)}W_{\mu\nu}\,. \end{aligned} \tag{A.108} $$

On the other hand

$$ \begin{aligned} e^\nu\rfloor e^\mu\rfloor(\eta^{\alpha\beta}\wedge{}^{(5)}W_{\mu\nu}) &= e^\nu\rfloor e^\mu\rfloor\left({}^{\star\,(5)}W_{\mu\nu}\wedge\vartheta^\alpha\wedge\vartheta^\beta\right) \\ &= {}^\star({}^{(5)}W_{\mu\nu}\wedge\vartheta^\mu\wedge\vartheta^\nu)\wedge\vartheta^\alpha\wedge\vartheta^\beta \\ &\quad +2\,{}^{\star\,(5)}W^{\alpha\beta} - 4\,{}^\star({}^{(5)}W_\mu{}^{[\alpha|}\wedge\vartheta^\mu)\wedge\vartheta^{|\beta]} \\ &= 2\,{}^{\star\,(5)}W^{\alpha\beta} + 2\,{}^\star(\vartheta^{[\alpha}\wedge\mathcal{RIC})\wedge\vartheta^{\beta]} \\ &= 2\,{}^{\star\,(5)}W^{\alpha\beta} + 2\,{}^\star(\vartheta^{[\alpha}\wedge e^{\beta]}\rfloor\mathcal{RIC}) \\ &= 2\,{}^{\star\,(5)}W^{\alpha\beta} + 2\,{}^\star(\vartheta^{[\alpha}\wedge\xi^{\beta]}) = -2\,{}^{\star\,(5)}W^{\alpha\beta}\,, \end{aligned} \tag{A.109} $$

which completes the proof.

6) $\mathbf{{}^{(6)}W^{*\alpha\beta} = \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(6)}W_{\mu\nu} = {}^{\star\,(6)}W^{\alpha\beta}}$

This is quite easy,

$$ \begin{aligned} \frac{1}{2}\,\eta^{\alpha\beta\mu\nu}\,{}^{(6)}W_{\mu\nu} &= \frac{1}{24}\,W\,\eta^{\alpha\beta\mu\nu}\,\vartheta_\mu\wedge\vartheta_\nu = \frac{1}{24}\,W\,\vartheta_\mu\wedge\vartheta_\nu\wedge(e^\nu\rfloor e^\mu\rfloor\eta^{\alpha\beta}) \\ &= \frac{1}{12}\,W\,\eta^{\alpha\beta} = \frac{1}{12}\,W\,{}^\star(\vartheta^\alpha\wedge\vartheta^\beta) = {}^{\star\,(6)}W^{\alpha\beta}\,. \end{aligned} \tag{A.110} $$

A.4.2 Bach 3-form and Bianchi identity

By means of the Ricci identity and the decomposition of the curvature we have

$$ \begin{aligned} DD^\star C_\alpha &= -R_\alpha{}^\beta\wedge{}^\star C_\beta \\ &= -\mathrm{Weyl}_\alpha{}^\beta\wedge{}^\star C_\beta - \mathrm{Ri}\!\not{\mathrm{c}}\mathrm{ci}_\alpha{}^\beta\wedge{}^\star C_\beta - \mathrm{Scalar}_\alpha{}^\beta\wedge{}^\star C_\beta\,. \end{aligned} \tag{A.111} $$

For p-forms ϕ, ψ of the same degree, there holds ${}^\star\phi \wedge \psi = {}^\star\psi \wedge \phi$. By means of $\vartheta^\alpha \wedge {}^\star\phi = (-1)^{p-1}\,{}^\star(e^\alpha \rfloor \phi)$, we can prove that $\mathrm{Scalar}_{\alpha\beta} \wedge {}^\star C^\alpha = 0$. Performing a "partial integration" we arrive at

$$DD^\star C_\alpha = -D\left({}^\star\mathrm{Weyl}_\alpha{}^\beta \wedge L_\beta\right) + \left(D^\star\mathrm{Weyl}_\alpha{}^\beta\right) \wedge L_\beta - {}^\star\mathrm{Ri\not{c}ci}\,{}_\alpha{}^\beta \wedge C_\beta\,. \quad \text{(A.112)}$$

Next, we use the "double duality relations" for the irreducible pieces of the curvature,

$$\begin{aligned} {}^\star\mathrm{Weyl}_{\alpha\beta} &= \mathrm{Weyl}_{\mu\nu}\,\frac{1}{2}\,\eta^{\mu\nu}{}_{\alpha\beta}\,, && \text{(A.113)}\\ {}^\star\mathrm{Ri\not{c}ci}\,{}_{\alpha\beta} &= -\mathrm{Ri\not{c}ci}\,{}_{\mu\nu}\,\frac{1}{2}\,\eta^{\mu\nu}{}_{\alpha\beta}\,, && \text{(A.114)}\\ {}^\star\mathrm{Scalar}_{\alpha\beta} &= \mathrm{Scalar}_{\mu\nu}\,\frac{1}{2}\,\eta^{\mu\nu}{}_{\alpha\beta}\,. && \text{(A.115)} \end{aligned}$$

Together with eqs.(1.63) and (1.57), we obtain

$$\begin{aligned} \left(D^\star\mathrm{Weyl}_\alpha{}^\beta\right) \wedge L_\beta &= \frac{1}{2}\,\eta^{\mu\nu}{}_\alpha{}^\beta\, D\mathrm{Weyl}_{\mu\nu} \wedge L_\beta = -\frac{1}{2}\,\eta^{\mu\nu}{}_\alpha{}^\beta\,\vartheta_{[\mu} \wedge C_{\nu]} \wedge L_\beta \\ &= -\frac{1}{2}\,\eta^{\mu\beta\nu}{}_\alpha\,\vartheta_\mu \wedge L_\beta \wedge C_\nu = -{}^\star\mathrm{Ri\not{c}ci}\,{}^\nu{}_\alpha \wedge C_\nu + {}^\star\mathrm{Scalar}^\nu{}_\alpha \wedge C_\nu \\ &= {}^\star\mathrm{Ri\not{c}ci}\,{}_\alpha{}^\nu \wedge C_\nu\,. \end{aligned} \quad \text{(A.116)}$$

Substituting this into (A.112) completes the proof.

A.4.3 Variation of the ECCS Lagrangian

We consider the Lagrangian

$$C_{\mathrm{RR}} = -\frac{1}{2}\left(\Gamma_\alpha{}^\beta \wedge d\Gamma_\beta{}^\alpha - \frac{2}{3}\,\Gamma_\alpha{}^\beta \wedge \Gamma_\beta{}^\gamma \wedge \Gamma_\gamma{}^\alpha\right)\,. \quad \text{(A.117)}$$

The variation of this Chern-Simons Lagrangian, which only depends on the connection, turns out to be

$$\delta C_{\mathrm{RR}} = -\delta\Gamma_\alpha{}^\beta \wedge R_\beta{}^\alpha + \frac{1}{2}\,d\left(\Gamma_\alpha{}^\beta \wedge \delta\Gamma_\beta{}^\alpha\right)\,. \quad \text{(A.118)}$$

In the next step, we enforce vanishing torsion and nonmetricity by means of respective Lagrange multiplier terms:

$$L = C_{\mathrm{RR}} + \lambda_\alpha \wedge T^\alpha + \lambda^{\alpha\beta} \wedge Q_{\alpha\beta}\,. \quad \text{(A.119)}$$

The variation then yields

$$
\begin{aligned}
\delta L &= \delta C_{\rm RR} + \delta\lambda_\alpha \wedge T^\alpha + \lambda_\alpha \wedge \delta T^\alpha + \delta\lambda^{\alpha\beta} \wedge Q_{\alpha\beta} + \lambda^{\alpha\beta} \wedge \delta Q_{\alpha\beta} \\
&= -\delta\Gamma_\alpha{}^\beta \wedge R_\beta{}^\alpha + \delta\lambda_\alpha \wedge T^\alpha + \lambda_\alpha \wedge \left(d\delta\vartheta^\alpha + \delta\Gamma_\beta{}^\alpha \wedge \vartheta^\beta + \Gamma_\beta{}^\alpha \wedge \delta\vartheta^\beta\right) \\
&\quad +\delta\lambda^{\alpha\beta} \wedge Q_{\alpha\beta} + \lambda^{\alpha\beta} \wedge (-d\delta g_{\alpha\beta} + \delta\Gamma_\alpha{}^\gamma\, g_{\gamma\beta} + \Gamma_\alpha{}^\gamma\, \delta g_{\gamma\beta} + \delta\Gamma_\beta{}^\gamma\, g_{\alpha\gamma} + \Gamma_\beta{}^\gamma\, \delta g_{\alpha\gamma}) \\
&\quad +\frac{1}{2}\, d\left(\Gamma_\alpha{}^\beta \wedge \delta\Gamma_\beta{}^\alpha\right) \\
&= -\delta\Gamma_\alpha{}^\beta \wedge R_\beta{}^\alpha + \delta\lambda_\alpha \wedge T^\alpha + \delta\lambda^{\alpha\beta} \wedge Q_{\alpha\beta} + \frac{1}{2}\, d\left(\Gamma_\alpha{}^\beta \wedge \delta\Gamma_\beta{}^\alpha\right) + \lambda_\alpha \wedge D\delta\vartheta^\alpha \\
&\quad -\delta\Gamma_\beta{}^\alpha \wedge \lambda_\alpha \wedge \vartheta^\beta - \lambda^{\alpha\beta} \wedge D\delta g_{\alpha\beta} + \delta\Gamma_\alpha{}^\beta \wedge (\lambda^\alpha{}_\beta + \lambda_\beta{}^\alpha) \\
&= \delta\lambda_\alpha \wedge T^\alpha + \delta\lambda^{\alpha\beta} \wedge Q_{\alpha\beta} + \delta\vartheta^\alpha \wedge D\lambda_\alpha + \delta g_{\alpha\beta}\, D\lambda^{\alpha\beta} \\
&\quad -\delta\Gamma_\alpha{}^\beta \wedge (R_\beta{}^\alpha + \lambda_\beta \wedge \vartheta^\alpha - \lambda^\alpha{}_\beta - \lambda_\beta{}^\alpha) \\
&\quad -d\left(-\lambda_\alpha \wedge \delta\vartheta^\alpha + \frac{1}{2}\Gamma_\alpha{}^\beta \wedge \delta\Gamma_\beta{}^\alpha - \lambda^{\alpha\beta}\, \delta g_{\alpha\beta}\right)\, .
\end{aligned} \tag{A.120}
$$

A.4.4 Variational principle for the Bach 3-form

Our aim is to carry out the variation of $\mathrm{Weyl}_{\alpha\beta} \wedge {}^\star\mathrm{Weyl}^{\alpha\beta}$ with respect to the coframe assuming vanishing torsion and nonmetricity. In order to use our canonical formalism we note that due to the orthogonality relations for the irreducible pieces,

$$
{}^{(i)}R_{\alpha\beta} \wedge {}^{\star\,(j)}R^{\alpha\beta} = 0 \qquad \text{for } i \neq j\,, \tag{A.121}
$$

we may start from the Lagrangian

$$
V_{\rm Bach} = -\frac{1}{2}\, R^{\alpha\beta} \wedge {}^\star\mathrm{Weyl}_{\alpha\beta} + \frac{1}{2}\, Q_{\alpha\beta} \wedge \mu^{\alpha\beta} + T^\alpha \wedge \lambda_\alpha\,. \tag{A.122}
$$

We again can eliminate the zeroth field equation by means of the Noether identity and arrive at

$$
\begin{aligned}
Q_{\alpha\beta} &= 0\,, && \text{(A.123)}\\
T_\alpha &= 0\,, && \text{(A.124)}\\
D^\star\mathrm{Weyl}_{\alpha\beta} - \vartheta_{[\alpha} \wedge \lambda_{\beta]} &= 0\,, && \text{(A.125)}\\
D\lambda_\alpha + E_\alpha &= 0\,, && \text{(A.126)}
\end{aligned}
$$

where

$$
E_\alpha = \frac{1}{2}\, \left[(e_\alpha \rfloor R^{\beta\gamma}) \wedge {}^\star\mathrm{Weyl}_{\beta\gamma} - R^{\beta\gamma} \wedge (e_\alpha \rfloor {}^\star\mathrm{Weyl}_{\beta\gamma})\right]\,. \tag{A.127}
$$

In $n = 4$ we can express $D^\star\mathrm{Weyl}_{\alpha\beta}$ in terms of the Cotton 2-form since, only in $n = 4$, we have the double-duality relation

$$
{}^\star\mathrm{Weyl}_{\alpha\beta} = \frac{1}{2}\, \eta^{\mu\nu}{}_{\alpha\beta}\, \mathrm{Weyl}_{\mu\nu}\,. \tag{A.128}
$$

Then we can express the covariant derivative of the Weyl 2-form by means of the Cotton 2-form, see table 1.2. Moreover we use some relations for the Hodge-dual and that the contractions of the Cotton 2-form with the frame and the coframe, respectively, are zero. Thus we find

$$\begin{aligned} D^\star \mathrm{Weyl}_{\alpha\beta} &= \frac{1}{2}\,\eta^{\mu\nu}{}_{\alpha\beta}\, D\mathrm{Weyl}_{\mu\nu} \\ &= -\frac{1}{2}\,\eta^{\mu\nu}{}_{\alpha\beta}\,\vartheta_\mu \wedge C_\nu = \frac{1}{2}\,\eta_{\alpha\beta}{}^{\nu} \wedge C_\nu = \frac{1}{2}\,(e^\nu \rfloor \eta_{\alpha\beta}) \wedge C_\nu \\ &= \frac{1}{2}\,e^\nu \rfloor (\eta_{\alpha\beta} \wedge C_\nu) = \frac{1}{2}\,e^\nu \rfloor ({}^\star C_\nu \wedge \vartheta_\alpha \wedge \vartheta_\beta) = -\vartheta_{[\alpha} \wedge {}^\star C_{\beta]}\,. \end{aligned} \tag{A.129}$$

Substituting this into (A.125) directly yields

$$\lambda_\alpha = -{}^\star C_\alpha\,. \tag{A.130}$$

In order to compute E_α, we also use the double-duality relation and additionally the irreducible decomposition of the curvature. First, we find

$$\begin{aligned} (e_\alpha \rfloor R^{\beta\gamma}) \wedge {}^\star\mathrm{Weyl}_{\beta\gamma} &= (e_\alpha \rfloor \mathrm{Weyl}^{\beta\gamma}) \wedge {}^\star\mathrm{Weyl}_{\beta\gamma} \\ &\quad - \big(e_\alpha \rfloor (\vartheta^\beta \wedge L^\gamma)\big) \wedge {}^\star\mathrm{Weyl}_{\beta\gamma}\,, \end{aligned} \tag{A.131}$$

$$\begin{aligned} (e_\alpha \rfloor {}^\star\mathrm{Weyl}_{\beta\gamma}) \wedge R^{\beta\gamma} &= (e_\alpha \rfloor {}^\star\mathrm{Weyl}_{\beta\gamma}) \wedge \mathrm{Weyl}^{\beta\gamma} \\ &\quad - (e_\alpha \rfloor {}^\star\mathrm{Weyl}_{\beta\gamma}) \wedge \vartheta^\beta \wedge L^\gamma\,. \end{aligned} \tag{A.132}$$

The double-duality relation amounts to the equality of the first terms on the respective right-hand sides,

$$\begin{aligned} (e_\alpha \rfloor {}^\star\mathrm{Weyl}^{\beta\gamma}) \wedge \mathrm{Weyl}_{\beta\gamma} &= \left[e_\alpha \rfloor \left(\frac{1}{2}\,\eta^{\beta\gamma\mu\nu}\,\mathrm{Weyl}_{\mu\nu}\right)\right] \wedge \mathrm{Weyl}_{\beta\gamma} \\ &= (e_\alpha \rfloor \mathrm{Weyl}_{\mu\nu}) \wedge \left(\frac{1}{2}\,\eta^{\beta\gamma\mu\nu}\,\mathrm{Weyl}_{\beta\gamma}\right) \\ &= (e_\alpha \rfloor \mathrm{Weyl}_{\mu\nu}) \wedge {}^\star\mathrm{Weyl}^{\mu\nu}\,. \end{aligned} \tag{A.133}$$

Now we treat the mixed terms. Here we use the property of the Weyl 2-form that the contractions with the frame and the coframe vanish. Consequently, also the respective contractions with the Hodge-dual of the Weyl 2-form vanish (since $e\rfloor{}^\star(\dots) = {}^\star(\dots \wedge \vartheta_\alpha)$),

$$\begin{aligned} \big(e_\alpha \rfloor (\vartheta^\beta \wedge L^\gamma)\big) \wedge {}^\star\mathrm{Weyl}_{\beta\gamma} &= L^\gamma \wedge {}^\star\mathrm{Weyl}_{\alpha\gamma} - \vartheta^\beta \wedge (e_\alpha \rfloor L^\gamma) \wedge {}^\star\mathrm{Weyl}_{\beta\gamma} \\ &= L^\gamma \wedge {}^\star\mathrm{Weyl}_{\alpha\gamma} \end{aligned} \tag{A.134}$$

$$\begin{aligned} (e_\alpha \rfloor {}^\star\mathrm{Weyl}_{\beta\gamma}) \wedge \vartheta^\beta \wedge L^\gamma &= L^\gamma \wedge \left[e_\alpha \rfloor \left(\vartheta^\beta \wedge {}^\star\mathrm{Weyl}_{\beta\gamma}\right) - {}^\star\mathrm{Weyl}_{\alpha\gamma}\right] \\ &= -L^\gamma \wedge {}^\star\mathrm{Weyl}_{\alpha\gamma}\,. \end{aligned} \tag{A.135}$$

Consequently,

$$E_\alpha = -L^\gamma \wedge {}^\star\mathrm{Weyl}_{\alpha\gamma}\,. \tag{A.136}$$

Eventually we arrive at

$$\frac{\delta V_{\mathrm{Bach}}}{\delta\vartheta^\alpha} = -(D^\star C_\alpha + L^\gamma \wedge {}^\star\mathrm{Weyl}_{\alpha\gamma}) = -B_\alpha\,, \tag{A.137}$$

where the expression for the Bach 3-form does exactly coincide with the one constructed from the Bianchi identity, see appendix A.4.2.

A.4.5 Decomposition of the Einstein-Hilbert Lagrangian in a Riemann-Cartan space

We use the basic definitions

RC-curvature	$R_\alpha{}^\beta$	$=$	$d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta$,
Torsion	T^α	$=$	$d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta$,
Contortion ([58] 3.10.10)	T^α	$=:$	$K^\alpha{}_\beta \wedge \vartheta^\beta$,
	$K_{\alpha\beta} = -K_{\beta\alpha}$	$=$	$e_{[\alpha}\rfloor T_{\beta]} - \frac{1}{2}\left(e_\alpha\rfloor e_\beta\rfloor T_\gamma\right)\vartheta^\gamma$,
Riemannian connection	$\Gamma^{\{\}}_{\alpha\beta}$	$=$	$\Gamma_{\alpha\beta} + K_{\alpha\beta}$,
Riemann curvature	$R^{\{\}}{}_\alpha{}^\beta$	$:=$	$d\Gamma^{\{\}}{}_\alpha{}^\beta - \Gamma^{\{\}}{}_\alpha{}^\gamma \wedge \Gamma^{\{\}}{}_\gamma{}^\beta$.

Then we have:

$$\begin{aligned} R_\alpha{}^\beta &= d\Gamma^{\{\}}{}_\alpha{}^\beta - dK_\alpha{}^\beta - \left(\Gamma^{\{\}}{}_\alpha{}^\gamma - K_\alpha{}^\gamma\right) \wedge \left(\Gamma^{\{\}}{}_\gamma{}^\beta - K_\gamma{}^\beta\right) \\ &= d\Gamma^{\{\}}{}_\alpha{}^\beta - dK_\alpha{}^\beta - \Gamma^{\{\}}{}_\alpha{}^\gamma \wedge \Gamma^{\{\}}{}_\gamma{}^\beta + K_\alpha{}^\gamma \wedge \Gamma^{\{\}}{}_\gamma{}^\beta + \Gamma_\alpha{}^\gamma \wedge K_\gamma{}^\beta \\ &\quad \boldsymbol{-\Gamma_\gamma{}^\beta \wedge K_\alpha{}^\gamma + \Gamma_\gamma{}^\beta \wedge K_\alpha{}^\gamma} \\ &= R^{\{\}}{}_\alpha{}^\beta - DK_\alpha{}^\beta + K_\alpha{}^\gamma \wedge K_\gamma{}^\beta\,. \end{aligned} \tag{A.138}$$

Next, we use

$$\begin{aligned} d\left(K^{\alpha\beta} \wedge \eta_{\alpha\beta}\right) &= D\left(K^{\alpha\beta} \wedge \eta_{\alpha\beta}\right) = \left(DK^{\alpha\beta}\right) \wedge \eta_{\alpha\beta} - K^{\alpha\beta} \wedge D\eta_{\alpha\beta} \\ &\overset{[58]\ 3.8.5}{=} \left(DK^{\alpha\beta}\right) \wedge \eta_{\alpha\beta} - K^{\alpha\beta} \wedge T^\gamma \wedge \eta_{\alpha\beta\gamma}\,. \end{aligned} \tag{A.139}$$

Hence,

$$R^{\alpha\beta} \wedge \eta_{\alpha\beta} = R^{\{\}\,\alpha\beta} \wedge \eta_{\alpha\beta} - d\left(K^{\alpha\beta} \wedge \eta_{\alpha\beta}\right) - K^{\alpha\beta} \wedge T^\gamma \wedge \eta_{\alpha\beta\gamma} + K^{\alpha\gamma} \wedge K_\gamma{}^\beta \wedge \eta_{\alpha\beta}\,. \tag{A.140}$$

The last two terms are proportional:

$$\begin{aligned} K^{\alpha\beta} \wedge T^\gamma \wedge \eta_{\alpha\beta\gamma} &= -K^{\alpha\beta} \wedge K_\delta{}^\gamma \wedge \vartheta^\delta \wedge \eta_{\alpha\beta\gamma} \\ &\overset{1}{=} -K^{\alpha\beta} \wedge K_\delta{}^\gamma \wedge {}^\star\left[e^\delta\rfloor\left(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma\right)\right] \\ &= -K^{\alpha\beta} \wedge K_\delta{}^\gamma \wedge {}^\star\left[\left(\delta^\delta_\alpha\,\vartheta_\beta \wedge \vartheta_\gamma - \delta^\delta_\beta\,\vartheta_\alpha \wedge \vartheta_\gamma + \delta^\delta_\gamma\,\vartheta_\alpha \wedge \vartheta_\beta\right)\right] \\ &= 2\,K^{\alpha\gamma} \wedge K_\gamma{}^\beta \wedge \eta_{\alpha\beta}\,. \end{aligned} \tag{A.141}$$

The last equality follows because $K_{\alpha\beta}$ is antisymmetric.

Up to know, we have

$$R^{\alpha\beta}\wedge\eta_{\alpha\beta}=R^{\{\}\,\alpha\beta}\wedge\eta_{\alpha\beta}-d\left(K^{\alpha\beta}\wedge\eta_{\alpha\beta}\right)-\frac{1}{2}\,K^{\alpha\beta}\wedge T^{\gamma}\wedge\eta_{\alpha\beta\gamma}\,. \tag{A.142}$$

Let us express

$$K^{\alpha\beta}\wedge\eta_{\alpha\beta\gamma}=e^{[\alpha}\rfloor T^{\beta]}\wedge\eta_{\alpha\beta\gamma}-\frac{1}{2}\,\left(e^{\alpha}\rfloor e^{\beta}\rfloor T_{\delta}\right)\,\vartheta^{\delta}\wedge\eta_{\alpha\beta\gamma}$$

in terms of irreducible pieces of the torsion.

We consider the two terms separately:

$$\begin{aligned}
e^{[\alpha}\rfloor T^{\beta]}\wedge\eta_{\alpha\beta\gamma} &= e^{\alpha}\rfloor T^{\beta}\wedge\eta_{\alpha\beta\gamma}=e^{\alpha}\rfloor\left(T^{\beta}\wedge\eta_{\alpha\beta\gamma}\right)-\underbrace{T^{\beta}\wedge e^{\alpha}\rfloor\eta_{\alpha\beta\gamma}}_{=0}\\
&= -e^{\alpha}\rfloor\left[T^{\beta}\wedge(e_{\beta}\rfloor\eta_{\alpha\gamma})\right]=-e^{\alpha}\rfloor\left[e_{\beta}\rfloor\left(T^{\beta}\wedge\eta_{\alpha\gamma}\right)-\left(e_{\beta}\rfloor T^{\beta}\right)\wedge\eta_{\alpha\gamma}\right]\\
&\overset{2}{=} -e^{\alpha}\rfloor\left[e_{\beta}\rfloor\left(\vartheta_{\alpha}\wedge\vartheta_{\gamma}\wedge{}^{\star}T^{\beta}\right)-(e_{\beta}\rfloor T^{\beta})\wedge\eta_{\alpha\gamma}\right]\\
&\overset{3}{=} -e^{\alpha}\rfloor\left[\vartheta_{\gamma}\wedge{}^{\star}T_{\alpha}-\vartheta_{\alpha}\wedge{}^{\star}T_{\gamma}+\vartheta_{\alpha}\wedge\vartheta_{\gamma}\wedge{}^{\star}(T^{\beta}\wedge\vartheta_{\beta})-(e_{\beta}\rfloor T^{\beta})\wedge\eta_{\alpha\gamma}\right]\\
&\overset{4}{=} -{}^{\star}T_{\gamma}+\vartheta_{\gamma}\wedge{}^{\star}(T_{\alpha}\wedge\vartheta^{\alpha})+n\,{}^{\star}T_{\gamma}-(n-2)\,{}^{\star}T_{\gamma}-n\,\vartheta_{\gamma}\wedge{}^{\star}(T^{\beta}\wedge\vartheta_{\beta})\\
&\quad +(n-2)\,\vartheta_{\gamma}\wedge{}^{\star}(T^{\beta}\wedge\vartheta_{\beta})+(e^{\alpha}\rfloor e_{\beta}\rfloor T^{\beta})\wedge\eta_{\alpha\gamma}\\
&= {}^{\star}T_{\gamma}-\vartheta_{\gamma}\wedge{}^{\star}(T^{\beta}\wedge\vartheta_{\beta})+(e^{\alpha}\rfloor e_{\beta}\rfloor T^{\beta})\wedge\eta_{\alpha\gamma}\,.
\end{aligned} \tag{A.143}$$

Moreover:

$$\begin{aligned}
(e^{\alpha}\rfloor e_{\beta}\rfloor T^{\beta})\,\eta_{\alpha\gamma} &= e_{\gamma}\rfloor\left[(e^{\alpha}\rfloor e_{\beta}\rfloor T^{\beta})\,\eta_{\alpha}\right]=e_{\gamma}\rfloor\left\{e^{\alpha}\rfloor\left[\vartheta_{\alpha}\wedge{}^{\star}(e_{\beta}\rfloor T^{\beta})\right]\right\}\\
&= e_{\gamma}\rfloor\left[n\,{}^{\star}(e_{\beta}\rfloor T^{\beta})-(n-1)\,{}^{\star}(e_{\beta}\rfloor T^{\beta})\right]\\
&= -{}^{\star}\left[\vartheta_{\gamma}\wedge(e_{\beta}\rfloor T^{\beta})\right]\,.
\end{aligned} \tag{A.144}$$

We turn to the second term:

$$\begin{aligned}
(e^{\alpha}\rfloor e^{\beta}\rfloor T_{\delta})\,\vartheta^{\delta}\wedge\eta_{\alpha\beta\gamma} &= (e^{\alpha}\rfloor e^{\beta}\rfloor T_{\delta})\,{}^{\star}\left[e^{\delta}\rfloor(\vartheta_{\alpha}\wedge\vartheta_{\beta}\wedge\vartheta_{\gamma})\right]\\
&= (e^{\alpha}\rfloor e^{\beta}\rfloor T_{\alpha})\,\eta_{\beta\gamma}-(e^{\alpha}\rfloor e^{\beta}\rfloor T_{\beta})\,\eta_{\alpha\gamma}+(e^{\alpha}\rfloor e^{\beta}\rfloor T_{\gamma})\,\eta_{\alpha\beta}\\
&= -2\,\left[e^{\alpha}\rfloor(e_{\beta}\rfloor T^{\beta})\right]\,\eta_{\alpha\gamma}-2\,{}^{\star}T_{\gamma}\,.
\end{aligned} \tag{A.145}$$

The last equality follows because

$${}^{\star}T_{\gamma}={}^{\star}\left(\frac{1}{2}\,T_{\mu\nu\gamma}\,\vartheta^{\mu}\wedge\vartheta^{\nu}\right)=\frac{1}{2}\,\left(e^{\nu}\rfloor e^{\mu}\rfloor T_{\gamma}\right)\,\eta_{\mu\nu}\,. \tag{A.146}$$

---

[1]For a p-form ψ we have $\vartheta_{\alpha}\wedge{}^{\star}\psi=(-1)^{(p-1)}\,{}^{\star}\,(e_{\alpha}\rfloor\psi)$

[2]$\phi\wedge{}^{\star}\psi=\psi\wedge{}^{\star}\phi$ for $\operatorname{rank}\phi=\operatorname{rank}\psi$

[3]$e_{\beta}\rfloor{}^{\star}\phi={}^{\star}(\phi\wedge\vartheta_{\alpha})$

[4]$\vartheta_{\alpha}\wedge(e^{\alpha}\rfloor\phi)=p\phi$ for $p=\operatorname{rank}\phi$

The irreducible decomposition of the torsion reads

$$
\begin{aligned}
{}^{(2)}T^{\alpha} &= \frac{1}{n-1}\,\vartheta^{\alpha}\wedge(e_{\beta}\rfloor T^{\beta})\,, && \text{(A.147)}\\
{}^{(3)}T^{\alpha} &= (-1)^{\mathrm{ind}(g)}\,\frac{1}{3}\,{}^{\star}\left[\vartheta^{\alpha}\wedge{}^{\star}(T^{\beta}\wedge\vartheta_{\beta})\right]\,, && \text{(A.148)}\\
{}^{(1)}T^{\alpha} &= T^{\alpha}-{}^{(2)}T^{\alpha}-{}^{(3)}T^{\alpha}\,. && \text{(A.149)}
\end{aligned}
$$

We also need

$$
{}^{\star(3)}T^{\gamma}=(-1)^{s}\,\frac{1}{3}\,\left[\vartheta^{\alpha}\wedge{}^{\star}(T^{\beta}\wedge\vartheta_{\beta})\right]\,(-1)^{(n-2)(n-(n-2))+\mathrm{ind}(g)}=\frac{1}{3}\,\vartheta^{\alpha}\wedge{}^{\star}(T^{\beta}\wedge\vartheta_{\beta})\,. \tag{A.150}
$$

(according to [82] (2.10) and [58] (3.7.5) $_{\mathrm{ind}}$ and s is used synonymous). Then,

$$
\begin{aligned}
K^{\alpha\beta}\wedge\eta_{\alpha\beta\gamma} &= e^{\alpha}\rfloor T^{\beta}\wedge\eta_{\alpha\beta\gamma}-\frac{1}{2}\,(e^{\alpha}\rfloor e^{\beta}\rfloor T_{\delta})\,\vartheta^{\delta}\wedge\eta_{\alpha\beta\gamma}\\
&= 2\,{}^{\star}T_{\gamma}-\vartheta_{\gamma}\wedge{}^{\star}(T^{\beta}\wedge\vartheta_{\beta})-2\,{}^{\star}\left[\vartheta_{\gamma}\wedge(e_{\beta}\rfloor T^{\beta})\right]\\
&= 2\,{}^{\star(1)}T_{\gamma}+2\,{}^{\star(2)}T_{\gamma}+2\,{}^{\star(3)}T_{\gamma}-3\,{}^{\star(3)}T_{\gamma}-2(n-1)\,{}^{\star(2)}T_{\gamma}\\
&= 2\,\left({}^{\star(1)}T_{\gamma}-(n-2)\,{}^{\star(2)}T_{\gamma}-\frac{1}{2}\,{}^{\star(3)}T_{\gamma}\right)\,. && \text{(A.151)}
\end{aligned}
$$

Last, we rewrite

$$
\begin{aligned}
K^{\alpha\beta}\wedge\eta_{\alpha\beta} &= e^{\alpha}\rfloor T^{\beta}\wedge\eta_{\alpha\beta}-\frac{1}{2}\,(e^{\alpha}\rfloor e^{\beta}\rfloor T_{\gamma})\,\vartheta^{\gamma}\wedge\eta_{\alpha\beta}\\
&= e^{\alpha}\rfloor(T^{\beta}\wedge\eta_{\alpha\beta})+(-1)^{n-2}\,{}^{\star}T^{\gamma}\wedge\vartheta_{\gamma}=e^{\alpha}\rfloor(\vartheta_{\alpha}\wedge\vartheta_{\beta}\wedge{}^{\star}T^{\beta})+\vartheta_{\gamma}\wedge{}^{\star}T^{\gamma}\\
&= n\,\vartheta_{\beta}\wedge{}^{\star}T^{\beta}-(n-1)\,\vartheta_{\beta}\wedge{}^{\star}T^{\beta}+\vartheta_{\gamma}\wedge{}^{\star}T^{\gamma}=2\,\vartheta_{\gamma}\wedge{}^{\star}T^{\gamma}\,. && \text{(A.152)}
\end{aligned}
$$

Substituting this into eq.(A.142) we find the final result:

$$
\begin{aligned}
R^{\alpha\beta}\wedge\eta_{\alpha\beta} &= R^{\{\}\alpha\beta}\wedge\eta_{\alpha\beta}-2\,d(\vartheta_{\alpha}\wedge{}^{\star}T^{\alpha})\\
&\quad +T^{\alpha}\wedge{}^{\star}\left(-{}^{(1)}T_{\alpha}+(n-2)\,{}^{(2)}T_{\alpha}+\frac{1}{2}\,{}^{(3)}T_{\alpha}\right)\,. && \text{(A.153)}
\end{aligned}
$$

A.4.6 On the Einstein choice in metric-affine space

In a n-dimensional metric-affine space we have the following geometric identity:

$$
\eta_{\alpha\beta\gamma}\wedge T^{\gamma}=-2\,\vartheta_{[\alpha}\wedge{}^{\star}\left({}^{(1)}T_{\beta]}-(n-2)\,{}^{(2)}T_{\beta]}-\frac{1}{2}\,{}^{(3)}T_{\beta]}\right)\,. \tag{A.154}
$$

We start from ($T:=e_{\alpha}\rfloor T^{\alpha}$):

$$
\eta_{\alpha\beta\gamma}\wedge T^{\gamma}=e_{\gamma}\rfloor\,(\eta_{\alpha\beta}\wedge T^{\gamma})-(-1)^{n-2}\,\eta_{\alpha\beta}\wedge T\,. \tag{A.155}
$$

The first term on the rhs can be rewritten as

$$
\begin{aligned}
e_\gamma \rfloor (\eta_{\alpha\beta} \wedge T^\gamma) &= e_\gamma \rfloor ({}^\star(\vartheta_\alpha \wedge \vartheta_\beta) \wedge T^\gamma) = e_\gamma \rfloor ({}^\star T^\gamma \wedge \vartheta_\alpha \wedge \vartheta_\beta) \\
&= (e_\gamma \rfloor {}^\star T^\gamma) \wedge \vartheta_\alpha \wedge \vartheta_\beta + (-1)^{n-2} \left[{}^\star T^\gamma \, g_{\alpha\gamma} \wedge \vartheta_\beta - {}^\star T^\gamma \wedge \vartheta_\alpha \, g_{\beta\gamma} \right] \\
&= {}^\star(T^\gamma \wedge \vartheta_\gamma) \wedge \vartheta_\alpha \wedge \vartheta_\beta + 2(-1)^{n-2}\, {}^\star T_{[\alpha} \wedge \vartheta_{\beta]} \\
&= {}^\star(T^\gamma \wedge \vartheta_\gamma) \wedge \vartheta_\alpha \wedge \vartheta_\beta - 2\vartheta_{[\alpha} \wedge {}^\star T_{\beta]} \, . \qquad \text{(A.156)}
\end{aligned}
$$

We can rewrite this formula in terms of the irreducible components of the torsion,

$$
{}^{\star(3)}T^\alpha = (-1)^s \, \frac{1}{3} \, {}^{\star\star} \left[\vartheta^\alpha \wedge {}^\star(T^\beta \wedge \vartheta_\beta) \right] = \frac{1}{3} \, \vartheta^\alpha \wedge {}^\star(T^\beta \wedge \vartheta_\beta) \, . \qquad \text{(A.157)}
$$

Hence,

$$
\vartheta_{[\alpha} \wedge {}^{\star(3)}T_{\beta]} = \frac{1}{3} \, {}^\star(T^\gamma \wedge \vartheta_\gamma) \wedge \vartheta_{[\alpha} \wedge \vartheta_{\beta]} = \frac{1}{3} \, {}^\star(T^\gamma \wedge \vartheta_\gamma) \wedge \vartheta_\alpha \wedge \vartheta_\beta \, . \qquad \text{(A.158)}
$$

Substituting $T^\alpha = {}^{(1)}T^\alpha + {}^{(2)}T^\alpha + {}^{(3)}T^\alpha$, and (A.156, A.158) into (A.155) we obtain

$$
\begin{aligned}
\eta_{\alpha\beta\gamma} \wedge T^\gamma &= 3\, \vartheta_{[\alpha} \wedge {}^{\star(3)}T_{\beta]} - 2\, \vartheta_{[\alpha} \wedge {}^\star \left({}^{(1)}T_{\beta]} + {}^{(2)}T_{\beta]} + {}^{(3)}T_{\beta]} \right) - T \wedge \eta_{\alpha\beta} \\
&= -2\vartheta_{[\alpha} \wedge {}^\star \left({}^{(1)}T_{\beta]} + {}^{(2)}T_{\beta]} - \frac{1}{2} \, {}^{(3)}T_{\beta]} \right) - T \wedge \eta_{\alpha\beta} \, . \qquad \text{(A.159)}
\end{aligned}
$$

The term $T \wedge \eta_{\alpha\beta}$ can be expressed by the piece ${}^{(2)}T_\alpha$ alone. Since $\eta \wedge T$ is a $n+1$-form we have

$$
\begin{aligned}
0 &= e_\beta \rfloor e_\alpha \rfloor (\eta \wedge T) = e_\beta \rfloor \left[\eta_\alpha \wedge T + (-1)^n \, \eta \, (e_\alpha \rfloor T) \right] \\
&= \eta_{\alpha\beta} \wedge T + 2(-1)^{n-1} \, \eta_{[\alpha} \, e_{\beta]} \rfloor T \, , \qquad \text{(A.160)}
\end{aligned}
$$

or,

$$
T \wedge \eta_{\alpha\beta} = 2 \, \eta_{[\alpha} \, e_{\beta]} \rfloor T \, . \qquad \text{(A.161)}
$$

For ${}^{\star\,(2)}T_\alpha$ we find

$$
\begin{aligned}
\vartheta_\alpha \wedge {}^{\star\,(2)}T_\beta &= \frac{1}{n-1} \, \vartheta_\alpha \wedge {}^\star(\vartheta_\beta \wedge T) = -\frac{1}{n-1} \, \vartheta_\alpha \wedge e_\beta \rfloor {}^\star T \\
&= \frac{1}{n-1} \left[e_\beta \rfloor (\vartheta_\alpha \wedge {}^\star T) - g_{\alpha\beta} \, {}^\star T \right] \\
&= \frac{1}{n-1} \left[(-1)^{n-1} \, e_\beta \rfloor ({}^\star T \wedge \vartheta_\alpha) - g_{\alpha\beta} {}^\star T \right] \\
&= \frac{1}{n-1} \left[(-1)^{n-1} \, e_\beta \rfloor (\eta_\alpha \wedge T) - g_{\alpha\beta} \, {}^\star T \right] \\
&= \frac{1}{n-1} \left[-T \wedge \eta_{\alpha\beta} + \eta_\alpha \, (e_\beta \rfloor T) - g_{\alpha\beta} \, {}^\star T \right] \, . \qquad \text{(A.162)}
\end{aligned}
$$

Antisymmetrizing and substituting (A.162) we arrive at

$$\begin{aligned}\vartheta_{[\alpha} \wedge {}^{\star\,(2)}T_{\beta]} &= \frac{1}{n-1}\left(-T \wedge \eta_{\alpha\beta} + \eta_{[\alpha}\, e_{\beta]}\rfloor T\right) \\ &= -\frac{1}{2(n-1)}\, T \wedge \eta_{\alpha\beta}\,. \end{aligned}\tag{A.163}$$

Substituting (A.163) into (A.159) yields

$$\begin{aligned}\eta_{\alpha\beta\gamma} \wedge T^{\gamma} &= -2\vartheta_{[\alpha} \wedge {}^{\star}\left({}^{(1)}T_{\beta]} + {}^{(2)}T_{\beta]} - \frac{1}{2}\,{}^{(3)}T_{\beta]}\right) + 2(n-1)\,\vartheta_{[\alpha} \wedge {}^{\star\,(2)}T_{\beta]} \\ &= -2\vartheta_{[\alpha} \wedge {}^{\star}\left({}^{(1)}T_{\beta]} - (n-2)\,{}^{(2)}T_{\beta]} - \frac{1}{2}\,{}^{(3)}T_{\beta]}\right)\,. \end{aligned}\tag{A.164}$$

A.4.7 Conformal transformation of the Riemannian connection

In order to compare our results to [58], in this section, we use C_α for the object of anholonomity and Ω for the conformal factor!

According to (1.38), the Riemannian connection reads

$$\Gamma^{\{\}}_{\alpha\beta} = \frac{1}{2}\, dg_{\alpha\beta} + (e_{[\alpha}\rfloor dg_{\beta]\gamma})\,\vartheta^{\gamma} + e_{[\alpha}\rfloor C_{\beta]} - \frac{1}{2}\,(e_{\alpha}\rfloor e_{\beta}\rfloor C_{\gamma})\,\vartheta^{\gamma}\,,\tag{A.165}$$

where the anholonomity is given by

$$C_{\alpha} = g_{\alpha\beta}\, C^{\beta} = g_{\alpha\beta}\, d\vartheta^{\beta}\,.\tag{A.166}$$

We consider the combined transformations

$$\tilde{g}_{\alpha\beta} = \Omega^{L-2F}\, g_{\alpha\beta}\,, \quad \tilde{\vartheta}^{\alpha} = \Omega^{F}\,\vartheta^{\alpha}\,, \quad \tilde{e}_{\alpha} = \Omega^{-F}\, e_{\alpha}\,.\tag{A.167}$$

Substitution into 3.10.9 yields:

$$d(\tilde{g}_{\alpha\beta}) = (L-2F)\,\Omega^{L-2F}\,(d\ln\Omega)\, g_{\alpha\beta} + \Omega^{L-2F}\, dg_{\alpha\beta}\,,\tag{A.168}$$

$$\begin{aligned}(\tilde{e}_{[\alpha}\rfloor d\tilde{g}_{\beta]\gamma})\,\tilde{\vartheta}^{\gamma} &= e_{[\alpha}\rfloor d(\Omega^{L-2F}\, g_{\beta]\gamma})\,\vartheta^{\gamma} \\ &= (L-2F)\,\Omega^{L-2F}\,(e_{[\alpha}\rfloor d\ln\Omega)\, g_{\beta]\gamma}\,\vartheta^{\gamma} \\ &\quad +\Omega^{L-2F}\,(e_{[\alpha}\rfloor dg_{\beta]\gamma})\,\vartheta^{\gamma}\,. \end{aligned}\tag{A.169}$$

With the definition (A.166) we obtain:

$$\begin{aligned}\tilde{e}_{[\alpha}\rfloor \tilde{C}_{\beta]} &= \Omega^{L-3F}\, e_{[\alpha}\rfloor g_{\beta]\gamma}\, d(\Omega^{F}\,\vartheta^{\gamma}) \\ &= \Omega^{L-2F}\, e_{[\alpha}\rfloor C_{\beta]} + F\,\Omega^{L-2F}\, e_{[\alpha}\rfloor (g_{\beta]\gamma}\, d\ln\Omega \wedge \vartheta^{\gamma}) \\ &= \Omega^{L-2F}\, e_{[\alpha}\rfloor C_{\beta]} + F\,\Omega^{L-2F}\, e_{[\alpha}\rfloor (d\ln\Omega \wedge \vartheta_{\beta]})\,, \end{aligned}\tag{A.170}$$

$$\begin{aligned}(\tilde{e}_{\alpha}\rfloor \tilde{e}_{\beta}\rfloor \tilde{C}_{\gamma})\,\tilde{\vartheta}^{\gamma} &= \Omega^{-F}\,(e_{\alpha}\rfloor e_{\beta}\rfloor \tilde{C}_{\gamma})\,\vartheta^{\gamma} = \Omega^{L-3F}\, e_{\alpha}\rfloor e_{\beta}\rfloor (g_{\gamma\delta}\, d(\Omega^{F}\,\vartheta^{\delta}))\,\vartheta^{\gamma} \\ &= \Omega^{L-2F}\,(e_{\alpha}\rfloor e_{\beta}\rfloor C_{\gamma})\,\vartheta^{\gamma} \\ &\quad +\Omega^{L-2F}\, F\,(e_{\alpha}\rfloor e_{\beta}\rfloor (d\ln\Omega \wedge \vartheta_{\gamma}))\,\vartheta^{\gamma}\,. \end{aligned}\tag{A.171}$$

Thus,

$$\begin{aligned}\tilde{\Gamma}^{\{\}}_{\alpha\beta} &= \Omega^{L-2F}\,\Gamma^{\{\}}_{\alpha\beta} + \frac{1}{2}\,(L-2F)\,\Omega^{L-2F}\,[g_{\alpha\beta}\,d\ln\Omega + 2(\,e_{[\alpha}\rfloor d\ln\Omega)\,g_{\beta]\gamma}\vartheta^{\gamma}] \\ &\quad -\frac{1}{2}\,F\,\Omega^{L-2F}\,[(e_{\alpha}\rfloor e_{\beta}\rfloor(d\ln\Omega\wedge\vartheta_{\gamma}))\vartheta^{\gamma} - 2\,e_{[\alpha}\rfloor(d\ln\Omega\wedge\vartheta_{\beta]})]\,. \qquad \text{(A.172)}\end{aligned}$$

It should be noted that this result differs from the corresponding formula eq.(3.14.12) in [58]. It seems there is used the incorrect equation $C_{\alpha} = d\vartheta_{\alpha}$.

Remark. Because of the simple structure of $d\ln\Omega = d\sigma = \sigma_{,\beta}\,\vartheta^{\beta}$ (with $\Omega = \exp\sigma$), the terms can be considerably simplified (the only possibilities to built 2-indexed 1-forms from $\sigma_{,\alpha}$, $g_{\alpha\beta}$ and ϑ_{α} are $d\sigma\, g_{\alpha\beta}$ and $\sigma_{,\alpha}\,\vartheta_{\beta}$).

$$\begin{aligned}(e_{[\alpha}\rfloor(d\ln\Omega)\,g_{\beta]\gamma})\,\vartheta^{\gamma} &= \frac{1}{2}\,(\sigma_{,\alpha}\,\vartheta_{\beta} - \sigma_{,\beta}\,\vartheta_{\alpha})\,, && \text{(A.173)}\\ g_{\alpha\beta}\,d\ln\Omega &= g_{\alpha\beta}\,d\sigma\,, && \text{(A.174)}\\ (e_{\alpha}\rfloor e_{\beta}\rfloor(d\ln\Omega\wedge\vartheta_{\gamma}))\,\vartheta^{\gamma} &= -2\,\sigma_{,[\alpha}\,\vartheta_{\beta]}\,, && \text{(A.175)}\\ e_{[\alpha}\rfloor(d\ln\Omega\wedge\vartheta_{\beta]}) &= \sigma_{,[\alpha}\,\vartheta_{\beta]}\,, && \text{(A.176)}\end{aligned}$$

such that

$$\begin{aligned}\tilde{\Gamma}^{\{\}}_{\alpha\beta} &= \Omega^{L-2F}\,\Gamma^{\{\}}_{\alpha\beta} + \frac{1}{2}\,(L-2F)\,\Omega^{L-2F}\,(g_{\alpha\beta}\,d\sigma + \sigma_{,\alpha}\,\vartheta_{\beta} - \sigma_{,\beta}\,\vartheta_{\alpha}) \\ &\quad + F\,\Omega^{L-2F}\,(\sigma_{,\alpha}\,\vartheta_{\beta} - \sigma_{,\beta}\,\vartheta_{\alpha})\,. \qquad \text{(A.177)}\end{aligned}$$

The pieces arising from the conformal transformation are tensorial,

$$\begin{aligned}S_{\alpha\beta} &= \frac{1}{2}\,(L-2F)\,\Omega^{L-2F}\,(g_{\alpha\beta}\,d\sigma + \sigma_{,\alpha}\,\vartheta_{\beta} - \sigma_{,\beta}\,\vartheta_{\alpha}) \\ &\quad + F\,\Omega^{L-2F}\,(\sigma_{,\alpha}\,\vartheta_{\beta} - \sigma_{,\beta}\,\vartheta_{\alpha})\,. \qquad \text{(A.178)}\end{aligned}$$

We have:

$$S_{(\alpha\beta)} = \frac{1}{2}\,(L-2F)\,\Omega^{L-2F}\,g_{\alpha\beta}\,d\sigma \qquad \text{(A.179)}$$

which corresponds to $Q_{\alpha\beta} = 0 \;\Rightarrow\; \tilde{Q}_{\alpha\beta} = 0$.

$$S_{\alpha}{}^{\beta}\wedge\vartheta^{\alpha} = F\,\Omega^{L-2F} d\sigma\wedge\vartheta^{\beta}\,, \qquad \text{(A.180)}$$

i.e. $T^{\alpha} = 0 \;\Rightarrow \tilde{T}^{\alpha} = 0$ if $F = 0$.

A.4.8 Bach- and Chevreton tensor

We proof that under the conditions

$$\begin{aligned}\nabla^{i}T_{ij} = 0 &\quad \text{(divergence free)}, && \text{(A.181)}\\ G_{ij} = \kappa\,T_{ij} &\quad \text{(Einstein equation)}, && \text{(A.182)}\\ T^{ij}\,g_{ij} = 0 &\quad \text{(trace free)}, && \text{(A.183)}\\ T_{ij}T^{j}{}_{j} - \frac{1}{4}\,T^{kl}\,T_{kl}\,g_{ij} = 0 &\quad \text{(Rainich condition)}, && \text{(A.184)}\end{aligned}$$

the Bach tensor

$$B_{ij} = \nabla^k C_{ikj} + L^{kl}\,\mathrm{Weyl}_{ikjl} = 2\nabla^k\nabla_{[i}L_{k]j} + L^{kl}\,\mathrm{Weyl}_{ikjl}\,, \tag{A.185}$$

can be rewritten as

$$B_{ij} = -\mathcal{H}_{ij} := -\Box_G\, T_{ij} := -\Box T_{ij} + 2C_{ikjl}\,T^{kl}\,. \tag{A.186}$$

We use holonomic coordinates a, b, c, d, i, j and assume $\kappa = 1$.

Because the energy momentum is trace free, the field equation yields

$$T_{ab} = G_{ab} = \mathrm{Ric}_{ab} = L_{ab}\,. \tag{A.187}$$

Substituting this into (A.185) we obtain

$$B_{ab} = \nabla^c\nabla_a T_{cb} - \nabla^c\nabla_c T_{ab} + T^{cd}\,\mathrm{Weyl}_{acbd}\,. \tag{A.188}$$

We employ the Ricci identity,

$$\nabla_c\nabla_a T^c{}_b - \nabla_a\nabla_c T^c{}_b = R_{cad}{}^c\, T^d{}_b - R_{cab}{}^d\, T^c{}_d = \mathrm{Ric}_{ad}\, T^d{}_b - R_{cab}{}^d\, T^c{}_d\,. \tag{A.189}$$

We use the irreducible decomposition of the curvature and arrive at

$$\nabla_c\nabla_a T^c{}_b - \nabla_a\nabla_c T^c{}_b = \mathrm{Ric}_{ad}\, T^d{}_b - \mathrm{Weyl}_{cab}{}^d\, T^c{}_d + 2(g_{[c|[b}\, L_{d]|a]})\, T^{cd}\,. \tag{A.190}$$

Replacing L_{ab} and Ric_{ab} via the field equation yields

$$\nabla_c\nabla_a T^c{}_b - \nabla_a\nabla_c T^c{}_b = -\mathrm{Weyl}_{cab}{}^d\, T^c{}_d + 2\left(T_{ac}T^c{}_b - \frac{1}{4}\, g_{ab}\, T_{cd}T^{cd}\right)\,. \tag{A.191}$$

The second term on the right hand side vanishes because of the Rainich identity. Using $\nabla_c T^c{}_b = 0$ and substituting the result into (A.188) we get

$$B_{ab} = -\nabla^c\nabla_c T_{ab} + 2\, T^{cd}\mathrm{Weyl}_{acbd} = -\mathcal{H}_{ab}\,. \tag{A.192}$$

A.4.9 Conservation of the Chevreton 3-form in flat spacetime

The Chevreton 3-form reads

$$\begin{aligned} \mathcal{H}_\alpha &= \frac{1}{2} e_\alpha \rfloor (\mathcal{F}_\beta \wedge^* \mathcal{F}^\beta) - \mathcal{F}_\beta \wedge (e_\alpha \rfloor^* \mathcal{F}^\beta) && \text{(A.193)} \\ &= \frac{1}{2}\left[(e_\alpha \rfloor \mathcal{F}_\beta) \wedge^* \mathcal{F}^\beta) - \mathcal{F}_\beta \wedge (e_\alpha \rfloor^* \mathcal{F}^\beta) \right]\,, && \text{(A.194)} \end{aligned}$$

where

$$\mathcal{F}_\alpha := D(e_\alpha \rfloor^* F)\,. \tag{A.195}$$

We now prove the conservation of $\mathcal{H}_\alpha$ in *flat space*. We use holonomic coordinates $a, b, i, j, k\ldots$ By construction we have, in flat space,

$$d\mathcal{F}_i = dd(e_i\rfloor{}^*F) = 0\,. \tag{A.196}$$

Moreover,

$$\begin{aligned} d^*\mathcal{F}_a &= d^*d(e_a\rfloor{}^*F) = d^*d(F_{ij}\eta^{ij}{}_a/2) = d^*(\partial_k F_{ij}\vartheta^k \wedge \eta^{ij}{}_a/2) \\ &= d^*(\partial_k F_{ij}(g^{ki}\eta^j{}_a - g^{kj}\eta^i{}_a + \delta^k_a\eta^{ij})/2) \\ &\overset{\partial_i F^i{}_j = 0}{=} d^*(\partial_a{}^*F) = -d\partial_a F = -\partial_a dF = 0\,. \end{aligned} \tag{A.197}$$

Both equations will not hold in a general spacetime! Using $d\mathcal{F}_a = 0 = d^*\mathcal{F}_a$, the exterior derivative of $\mathcal{H}_a$ amounts to

$$\begin{aligned} -2d\mathcal{H}_a &= \mathcal{F}_b \wedge d(e_a\rfloor{}^*\mathcal{F}^b) - d(e_a\rfloor\mathcal{F}_b) \wedge{}^*\mathcal{F}^b = \mathcal{F}_b \wedge \pounds_{e_a}{}^*\mathcal{F}^b - (\pounds_{e_a}\mathcal{F}_b) \wedge{}^*\mathcal{F}^b \\ &= \mathcal{F}_b \wedge{}^*\pounds_{e_a}\mathcal{F}^b - (\pounds_{e_a}\mathcal{F}_b) \wedge{}^*\mathcal{F}^b \\ &= (\pounds_{e_a}\mathcal{F}^b) \wedge{}^*\mathcal{F}_b - (\pounds_{e_a}\mathcal{F}_b) \wedge{}^*\mathcal{F}^b = 0\,. \end{aligned} \tag{A.198}$$

I have used here

$$\pounds^*_{e_a} = {}^*\pounds_{e_a}\,, \quad \pounds_{e_a}\mathcal{F}^b = e_a\rfloor d\mathcal{F}^b + d(e_a\rfloor\mathcal{F}^b)\,. \tag{A.199}$$

A.4.10 Contortion and curvature for constant axial torsion

We start from our ansatz for the torsion

$$T^\alpha = 2\,\frac{\mathcal{T}}{\ell}\,\eta^\alpha\,. \tag{A.200}$$

The contortion 1-form is defined according to

$$K_{\alpha\beta} = e_{[\alpha}\rfloor T_{\beta]} - \frac{1}{2}\,(e_\alpha\rfloor e_\beta\rfloor T_\gamma)\;\vartheta^\gamma\,. \tag{A.201}$$

The first term thus gives

$$e_{[\alpha}\rfloor T_{\beta]} = 2\,\frac{\mathcal{T}}{\ell}\,e_{[\alpha}\rfloor\eta_{\beta]} = -2\,\frac{\mathcal{T}}{\ell}\,\eta_{\alpha\beta}\,. \tag{A.202}$$

For the second term we find

$$(e_\alpha\rfloor e_\beta\rfloor T_\gamma)\;\vartheta^\gamma = 2\,\frac{\mathcal{T}}{\ell}\vartheta^\gamma\,e_\alpha\rfloor e_\beta\rfloor\eta_\gamma = -2\,\frac{\mathcal{T}}{\ell}\vartheta^\gamma\,e_\gamma\rfloor\eta_{\alpha\beta} = -2\,\frac{\mathcal{T}}{\ell}\eta_{\alpha\beta}\,. \tag{A.203}$$

Consequently,

$$K_{\alpha\beta} = -\frac{\mathcal{T}}{\ell}\,\eta_{\alpha\beta}\,. \tag{A.204}$$

In order to calculate the curvature we proceed from

$$R_{\alpha\beta} = R^{\{\}}_{\alpha\beta} - DK_{\alpha\beta} + K_\alpha{}^\gamma \wedge K_{\gamma\beta}\,. \tag{A.205}$$

We have the identity

$$D\eta_{\alpha_1\dots\alpha_p} = -\frac{1}{2}\,n\,Q \wedge \eta_{\alpha_1\dots\alpha_p} + T^\beta \wedge \eta_{\alpha_1\dots\alpha_p\beta}\,. \tag{A.206}$$

Hence,

$$DK_{\alpha\beta} = -\frac{\mathcal{T}}{\ell}\,D\eta_{\alpha\beta} = -\frac{\mathcal{T}}{\ell}\,T^\gamma \wedge \eta_{\alpha\beta\gamma} = -2\frac{\mathcal{T}^2}{\ell^2}\,\eta^\gamma \wedge \eta_{\alpha\beta\gamma}\,. \tag{A.207}$$

For the product of the η's, in $n = 3$, it holds

$$\begin{aligned} \eta^\gamma \wedge \eta_{\alpha\beta\gamma} &= e^\gamma\rfloor(\eta\,\eta_{\alpha\beta\gamma}) = e^\gamma\rfloor({}^\star 1\,{}^\star(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma)) = e^\gamma\rfloor({}^{\star\star}(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma)) \\ &= (-1)^{\rm ind} e^\gamma\rfloor(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma) = (-1)^{\rm ind}\,\vartheta_\alpha \wedge \vartheta_\beta\,. \end{aligned} \tag{A.208}$$

Consequently, for odd index of the metric,

$$DK_{\alpha\beta} = 2\frac{\mathcal{T}^2}{\ell^2}\,\vartheta_\alpha \wedge \vartheta_\beta\,. \tag{A.209}$$

Finally we treat

$$\begin{aligned} K_\alpha{}^\gamma \wedge K_{\gamma\beta} &= \frac{\mathcal{T}^2}{\ell^2}\,\eta_\alpha{}^\gamma \wedge \eta_{\gamma\beta} = \frac{\mathcal{T}^2}{\ell^2}\,e^\gamma\rfloor(\eta_\alpha \wedge \eta_{\gamma\beta}) = \frac{\mathcal{T}^2}{\ell^2}\,e^\gamma\rfloor({}^\star\vartheta_\alpha \wedge {}^\star(\vartheta_\gamma \wedge \vartheta_\beta)) \\ &= \frac{\mathcal{T}^2}{\ell^2}\,e^\gamma\rfloor({}^{\star\star}(\vartheta_\gamma \wedge \vartheta_\beta) \wedge \vartheta_\alpha) = (-1)^{\rm ind}\,\frac{\mathcal{T}^2}{\ell^2}\,e^\gamma\rfloor(\vartheta_\gamma \wedge \vartheta_\beta \wedge \vartheta_\alpha) \\ &= \frac{\mathcal{T}^2}{\ell^2}\,\vartheta_\alpha \wedge \vartheta_\beta\,. \end{aligned} \tag{A.210}$$

Eventually we arrive at

$$R_{\alpha\beta} = R^{\{\}}_{\alpha\beta} - \frac{\mathcal{T}^2}{\ell^2}\,\vartheta_\alpha \wedge \vartheta_\beta\,. \tag{A.211}$$

A.4.11 General relation between $\hat{\Delta}$ and $d\Lambda$

We start from the definition of $\hat{\Delta}$ in (3.25) and move the interior product to the left:

$$\hat{\Delta} = \frac{1}{n-2}\,\vartheta^\alpha \wedge \left(e^\beta\rfloor\; \not{Z}_{\alpha\beta}\right) = \frac{1}{n-2}\,\left[-e^\beta\rfloor(\vartheta^\alpha \wedge\; \not{Z}_{\alpha\beta}) +\; \not{Z}_\alpha{}^\alpha\right] = -\frac{1}{n-2} e^\alpha\rfloor(\not{Z}_{\alpha\beta} \wedge \vartheta^\beta)\,. \tag{A.212}$$

Obviously, we have to express $\not{Z}_{\alpha\beta} \wedge \vartheta^\beta$ in terms of nonmetricity and torsion. This is possible by means of the zeroth Bianchi identity

$$DQ_{\alpha\beta} = -DDg_{\alpha\beta} = R_\alpha{}^\gamma\,g_{\gamma\beta} + R_\beta{}^\gamma\,g_{\alpha\gamma} = 2R_{(\alpha\beta)} = 2Z_{\alpha\beta}\,. \tag{A.213}$$

We wedge with ϑ^β from the right and obtain

$$D\left(Q_{\alpha\beta}\wedge\vartheta^\beta\right) = 2Z_{\alpha\beta}\wedge\vartheta^\beta - Q_{\alpha\beta}\wedge T^\beta\,. \tag{A.214}$$

On the other hand, by making use of

$$Q_{\alpha\beta}\wedge\vartheta^\beta = P_\alpha + \frac{1}{n-1}\,\vartheta_\alpha\wedge\Lambda + Q\wedge\vartheta_\alpha \tag{A.215}$$

and of

$$D\vartheta_\alpha = D(g_{\alpha\beta}\,\vartheta^\beta) = (Dg_{\alpha\beta})\wedge\vartheta^\beta + g_{\alpha\beta}\,D\vartheta^\beta = -Q_{\alpha\beta}\wedge\vartheta^\beta + T_\alpha\,, \tag{A.216}$$

we can calculate

$$\begin{aligned}
D\left(Q_{\alpha\beta}\wedge\vartheta^\beta\right) \overset{(A.215)}{=}\;& D\left(Q\wedge\vartheta_\alpha + \frac{1}{n-1}\,\vartheta_\alpha\wedge\Lambda + P_\alpha\right)\\
=\;& dQ\wedge\vartheta_\alpha - Q\wedge D\vartheta_\alpha + \frac{1}{n-1}\,D\vartheta_\alpha\wedge\Lambda - \frac{1}{n-1}\vartheta_\alpha\wedge d\Lambda + DP_\alpha\\
\overset{(A.216)}{=}\;& dQ\wedge\vartheta_\alpha - Q\wedge\left(-Q_{\alpha\beta}\wedge\vartheta^\beta + T_\alpha\right)\\
&+\frac{1}{n-1}\,\left(-Q_{\alpha\beta}\wedge\vartheta^\beta + T_\alpha\right)\wedge\Lambda - \frac{1}{n-1}\,\vartheta_\alpha\wedge d\Lambda + DP_\alpha\\
\overset{(A.215)}{=}\;& dQ\wedge\vartheta_\alpha + Q\wedge\left(Q\wedge\vartheta_\alpha + \frac{1}{n-1}\,\vartheta_\alpha\wedge\Lambda + P_\alpha\right) - Q\wedge T_\alpha\\
&-\frac{1}{n-1}\,\left(Q\wedge\vartheta_\alpha + \frac{1}{n-1}\,\vartheta_\alpha\wedge\Lambda + P_\alpha\right)\wedge\Lambda + \frac{1}{n-1}\,T_\alpha\wedge\Lambda\\
&-\frac{1}{n-1}\,\vartheta_\alpha\wedge d\Lambda + DP_\alpha\\
=\;& dQ\wedge\vartheta_\alpha + Q\wedge Q\wedge\vartheta_\alpha + \frac{1}{n-1}\,Q\wedge\vartheta_\alpha\wedge\Lambda + Q\wedge P_\alpha - Q\wedge T_\alpha\\
&-\frac{1}{n-1}Q\wedge\vartheta_\alpha\wedge\Lambda - \frac{1}{(n-1)^2}\,\vartheta_\alpha\wedge\Lambda\wedge\Lambda - \frac{1}{n-1}\,P_\alpha\wedge\Lambda\\
&+\frac{1}{n-1}\,T_\alpha\wedge\Lambda - \frac{1}{n-1}\,\vartheta_\alpha\wedge d\Lambda + DP_\alpha\\
=\;& dQ\wedge\vartheta_\alpha + Q\wedge P_\alpha - Q\wedge T_\alpha - \frac{1}{n-1}\,P_\alpha\wedge\Lambda + \frac{1}{n-1}\,T_\alpha\wedge\Lambda\\
&-\frac{1}{n-1}\,\vartheta_\alpha\wedge d\Lambda + DP_\alpha\,.
\end{aligned} \tag{A.217}$$

Now we can compare (A.214) and (A.217). We find

$$\begin{aligned}
2\,\not{Z}_{\alpha\beta}\wedge\vartheta^\beta =\;& 2Z_{\alpha\beta}\wedge\vartheta^\beta - 2\,{}^{(4)}Z_{\alpha\beta}\wedge\vartheta^\beta\\
=\;& Q\wedge P_\alpha - Q\wedge T_\alpha - \frac{1}{n-1}\,P_\alpha\wedge\Lambda\\
&+\frac{1}{n-1}\,T_\alpha\wedge\Lambda - \frac{1}{n-1}\,\vartheta_\alpha\wedge d\Lambda + DP_\alpha + Q_{\alpha\beta}\wedge T^\beta\,.
\end{aligned} \tag{A.218}$$

We expand the last terms by means of the irreducible decomposition of torsion and nonmetricity:

$$\begin{aligned} Q_{\alpha\beta}\wedge T^{\beta} &= {\not}Q_{\alpha\beta}\wedge T^{\beta}+Q\wedge T_{\alpha} \\ &= {\not}Q_{\alpha\beta}\wedge\left({}^{(1)}T^{\beta}+{}^{(3)}T^{\beta}\right)+{\not}Q_{\alpha\beta}\wedge\left(\frac{1}{n-1}\,\vartheta^{\beta}\wedge T\right)+Q\wedge T_{\alpha} \\ &= {\not}Q_{\alpha\beta}\wedge\left({}^{(1)}T^{\beta}+{}^{(3)}T^{\beta}\right)+\frac{1}{(n-1)^2}\,\vartheta_{\alpha}\wedge\Lambda\wedge T \\ &\quad +\frac{1}{n-1}\,P_{\alpha}\wedge T+Q\wedge T_{\alpha}\,. \end{aligned} \tag{A.219}$$

Moreover,

$$\frac{1}{n-1}\,T_{\alpha}\wedge\Lambda=\frac{1}{n-1}\,\left({}^{(1)}T_{\alpha}+{}^{(3)}T_{\alpha}\right)\wedge\Lambda+\frac{1}{(n-1)^2}\,\vartheta_{\alpha}\wedge T\wedge\Lambda\,. \tag{A.220}$$

Substituting (A.219,A.220) into (A.218) yields

$$\begin{aligned} 2\,{\not}Z_{\alpha\beta}\wedge\vartheta^{\beta} &= DP_{\alpha}-\frac{1}{n-1}\,\vartheta_{\alpha}\wedge d\Lambda+{\not}Q_{\alpha\beta}\wedge\left({}^{(1)}T^{\beta}+{}^{(3)}T^{\beta}\right) \\ &\quad +\frac{1}{n-1}\,\left({}^{(1)}T_{\alpha}+{}^{(3)}T_{\alpha}\right)\wedge\Lambda \\ &\quad +P_{\alpha}\wedge\left[Q-\frac{1}{n-1}\,(\Lambda-T)\right]\,. \end{aligned} \tag{A.221}$$

We use the following properties of the irreducible pieces:

$$e^{\alpha}\rfloor P_{\alpha}=e^{\alpha}\rfloor{}^{(1)}T_{\alpha}=e^{\alpha}\rfloor{}^{(3)}T_{\alpha}=0\,,\qquad e^{\alpha}\rfloor\,{\not}Q_{\alpha\beta}=\Lambda_{\beta}\,. \tag{A.222}$$

Then we find

$$\begin{aligned} 2e^{\alpha}\rfloor({\not}Z_{\alpha\beta}\wedge\vartheta^{\beta}) &= P_{\alpha}\,e^{\alpha}\rfloor\left[Q-\frac{1}{n-1}\,(\Lambda-T)\right]+\frac{n}{n-1}\,\left({}^{(1)}T_{\alpha}+{}^{(3)}T_{\alpha}\right)\,\Lambda^{\alpha} \\ &\quad -\frac{n-2}{n-1}\,d\Lambda+e^{\alpha}\rfloor DP_{\alpha}-{\not}Q_{\alpha\beta}\wedge e^{\alpha}\rfloor\left({}^{(1)}T^{\beta}+{}^{(3)}T^{\beta}\right)\,. \end{aligned} \tag{A.223}$$

We can further simplify the last term. First we note that

$$\begin{aligned} e^{\alpha}\rfloor{}^{(3)}T^{\beta} &= (-1)^{s}\,e^{\alpha}\rfloor\frac{1}{3}{}^{\star}\left[\vartheta^{\beta}\wedge{}^{\star}(T^{\gamma}\wedge\vartheta_{\gamma})\right]=(-1)^{s}\,\frac{1}{3}{}^{\star}\left[\vartheta^{\beta}\wedge{}^{\star}(T^{\gamma}\wedge\vartheta_{\gamma})\wedge\vartheta^{\alpha}\right] \\ &= -e^{\beta}\rfloor{}^{(3)}T^{\alpha}\,. \end{aligned} \tag{A.224}$$

Hence,

$$\begin{aligned} {\not}Q_{\alpha\beta}\wedge e^{\alpha}\rfloor\left({}^{(1)}T^{\beta}+{}^{(3)}T^{\beta}\right) &= {\not}Q_{\alpha\beta}\wedge e^{\alpha}\rfloor{}^{(1)}T^{\beta} \\ &= {}^{(1)}Q_{\alpha\beta}\wedge e^{\alpha}\rfloor{}^{(1)}T^{\beta}+{}^{(2)}Q_{\alpha\beta}\wedge e^{\alpha}\rfloor{}^{(1)}T^{\beta} \\ &\quad +{}^{(3)}Q_{\alpha\beta}\wedge e^{\alpha}\rfloor{}^{(1)}T^{\beta}\,. \end{aligned} \tag{A.225}$$

The last term can be further rewritten as

$$\begin{aligned} {}^{(3)}Q_{\alpha\beta} \wedge e^\alpha \rfloor {}^{(1)}T^\beta &= \frac{2n}{(n-1)(n+2)} \left(\vartheta_{(\alpha}\, \Lambda_{\beta)} - \frac{1}{n}\, g_{\alpha\beta}\, \Lambda \right) \wedge e^\alpha \rfloor {}^{(1)}T^\beta \\ &= \frac{n}{(n-1)(n+2)} \left(\vartheta_\beta\, \Lambda_\alpha \wedge e^\alpha \rfloor {}^{(1)}T^\beta + \Lambda_\beta \vartheta_\alpha e^\alpha \rfloor {}^{(1)}T^\beta \right) \\ &= \frac{n}{(n-1)(n+2)} \left[\Lambda_\alpha \left(-e^\alpha \rfloor (\vartheta_\beta \wedge {}^{(1)}T^\beta) + {}^{(1)}T^\alpha \right) + 2\Lambda_\beta\, {}^{(1)}T^\beta \right] \\ &= \frac{3n}{(n-1)(n+2)}\, \Lambda_\alpha\, {}^{(1)}T^\alpha\,, \end{aligned} \tag{A.226}$$

where we used $\vartheta_\beta \wedge {}^{(1)}T^\beta = 0$. Finally we arrive at

$$\begin{aligned} \hat{\Delta} = {} & \frac{1}{2(n-1)}\, d\Lambda - \frac{1}{2(n-2)}\, e^\alpha \rfloor DP_\alpha \\ & - \frac{1}{2(n-2)} \left\{ \frac{1}{n-1}\, P_\alpha\, e^\alpha \rfloor \left[(n-1)\, Q - \Lambda + T \right] \right. \\ & \qquad + \left(\frac{n+1}{n+2}\, {}^{(1)}T_\alpha + \frac{n}{n-1}\, {}^{(3)}T_\alpha \right) \Lambda^\alpha \\ & \qquad \left. - \left({}^{(1)}Q_{\alpha\beta} + {}^{(2)}Q_{\alpha\beta} \right) \wedge e^\alpha \rfloor {}^{(1)}T^\beta \right\}. \end{aligned} \tag{A.227}$$

Note that in the last line we could substitute ${}^{(2)}Q_{\alpha\beta} = -2\, e_{(\alpha} \rfloor P_{\beta)}/3$.

A.4.12 On the square of ${}^{(3)}Z$

We proof that the following relation holds for *arbitrary* spacetimes:

$${}^{(3)}Z^{\alpha\beta} \wedge {}^{\star\,(3)}Z_{\alpha\beta} = \frac{n(n-2)}{n+2}\, \hat{\Delta} \wedge {}^\star\hat{\Delta}\,. \tag{A.228}$$

This comes about since ${}^{(3)}Z^{\alpha\beta}$ corresponds to a *scalar*-valued degree of freedom, namely to the two-form $\hat{\Delta}$, see (3.28). For a p-form ϕ, we have the rules for the Hodge dual ${}^{\star\star}\phi = (-1)^{p(n-p)-1}\phi$ in the case of Lorentz signature, furthermore, $\vartheta^\alpha \wedge (e_\alpha \rfloor \phi) = p\,\phi$ and ${}^\star(\phi \wedge \vartheta_\alpha) = e_\alpha \rfloor {}^\star\phi$. Thus, the terms quadratic in contractions of $\hat{\Delta}$ in the end evaluate to $\hat{\Delta} \wedge {}^\star\hat{\Delta}$,

$$(e^\alpha \rfloor \hat{\Delta}) \wedge {}^\star(e_\alpha \rfloor \hat{\Delta}) = -(e^\alpha \rfloor \hat{\Delta}) \wedge {}^\star(e_\alpha \rfloor {}^{\star\star}\hat{\Delta}) = -(e^\alpha \rfloor \hat{\Delta}) \wedge \vartheta_\alpha \wedge {}^\star\hat{\Delta} = 2\, \hat{\Delta} \wedge {}^\star\hat{\Delta}\,. \tag{A.229}$$

We recall the definition (3.28)

$${}^{(3)}Z_{\alpha\beta} = \frac{1}{n+2} \left[n\, \vartheta_{(\alpha} \wedge e_{\beta)} \rfloor \hat{\Delta} - 2\, g_{\alpha\beta}\, \hat{\Delta} \right]. \tag{A.230}$$

It is symmetric ${}^{(3)}Z^{[\alpha\beta]} = 0$ and tracefree ${}^{(3)}Z^{\gamma}{}_{\gamma} = 0$. Thus,

$$
\begin{aligned}
{}^{(3)}Z_{\alpha\beta} \wedge {}^{\star\,(3)}Z^{\alpha\beta} &= \frac{1}{n+2}\left[n\,\vartheta_\alpha \wedge (e_\beta \rfloor \hat{\Delta}) \wedge {}^{\star(3)}Z^{\alpha\beta}\right] \\
&= \frac{1}{(n+2)^2}\left[n^2\,\vartheta_\alpha \wedge (e_\beta \rfloor \hat{\Delta}) \wedge {}^\star\left(\vartheta^{(\alpha} \wedge (e^{\beta)} \rfloor \hat{\Delta})\right)\right. \\
&\qquad \left. -2n\,\vartheta_\alpha \wedge (e_\beta \rfloor \hat{\Delta}) \wedge g^{\alpha\beta\,\star}\hat{\Delta}\right]. \qquad \text{(A.231)}
\end{aligned}
$$

In order to calculate the first term, we apply the rules for commuting the Hodge star with the exterior/interior product.

$$
\begin{aligned}
{}^\star\left(\vartheta^{(\alpha} \wedge (e^{\beta)} \rfloor \hat{\Delta})\right) &= -{}^\star\left[(e^{(\beta} \rfloor \hat{\Delta}) \wedge \vartheta^{\alpha)}\right] = -\,e^{(\alpha} \rfloor {}^\star\left(e^{\beta)} \rfloor \hat{\Delta}\right) \\
&= -(-1)^{n-1}\,e^{(\alpha} \rfloor \left({}^\star\hat{\Delta} \wedge \vartheta^{\beta)}\right) = e^{(\alpha} \rfloor \left(\vartheta^{\beta)} \wedge {}^\star\hat{\Delta}\right) \\
&= g^{\alpha\beta\,\star}\hat{\Delta} - \vartheta^{(\alpha} \wedge (e^{\beta)} \rfloor {}^\star\hat{\Delta}). \qquad \text{(A.232)}
\end{aligned}
$$

The second term in the brackets of (A.231) simply evaluates to

$$
-2n\,\vartheta_\alpha \wedge (e_\beta \rfloor \hat{\Delta}) \wedge g^{\alpha\beta\,\star}\hat{\Delta} = -2n\,\vartheta^\alpha \wedge (e_\alpha \rfloor \hat{\Delta}) \wedge {}^\star\hat{\Delta} = -4n\,\hat{\Delta} \wedge {}^\star\hat{\Delta}. \qquad \text{(A.233)}
$$

Substituting (A.232) and (A.233) into (A.231), we find

$$
\begin{aligned}
{}^{(3)}Z_{\alpha\beta} \wedge {}^{\star\,(3)}Z^{\alpha\beta} &= \frac{1}{(n+2)^2}\left[2n^2\hat{\Delta} \wedge {}^\star\hat{\Delta} - n^2\,\vartheta_\alpha \wedge (e_\beta \rfloor \hat{\Delta}) \wedge \vartheta^{(\alpha} \wedge (e^{\beta)} \rfloor {}^\star\hat{\Delta})\right. \\
&\qquad \left. -4n\,\hat{\Delta} \wedge {}^\star\hat{\Delta}\right]. \qquad \text{(A.234)}
\end{aligned}
$$

The term in the middle yields ($\vartheta^\alpha \wedge \vartheta_\alpha = 0$)

$$
\begin{aligned}
(\vartheta_\alpha \wedge e_\beta \rfloor \hat{\Delta}) \wedge (\vartheta^{(\alpha} \wedge e^{\beta)} \rfloor {}^\star\hat{\Delta}) &= \frac{1}{2}\left[\vartheta_\alpha \wedge (e_\beta \rfloor \hat{\Delta}) \wedge \vartheta^\alpha \wedge (e^\beta \rfloor {}^\star\hat{\Delta})\right. \\
&\qquad \left. +\vartheta_\alpha \wedge (e_\beta \rfloor \hat{\Delta}) \wedge \vartheta^\beta \wedge (e^\alpha \rfloor {}^\star\hat{\Delta})\right] \\
&= \frac{1}{2}\left[-\vartheta^\beta \wedge (e_\beta \rfloor \hat{\Delta}) \wedge \vartheta_\alpha \wedge (e^\alpha \rfloor {}^\star\hat{\Delta})\right] \\
&= -(n-2)\,\hat{\Delta} \wedge {}^\star\hat{\Delta}. \qquad \text{(A.235)}
\end{aligned}
$$

Eventually,

$$
\begin{aligned}
{}^{(3)}Z_{\alpha\beta} \wedge {}^{\star\,(3)}Z^{\alpha\beta} &= \frac{1}{(n+2)^2}\left[2n^2 + n^2(n-2) - 4n\right]\,\hat{\Delta} \wedge {}^\star\hat{\Delta} \\
&= \frac{n(n-2)}{n+2}\,\hat{\Delta} \wedge {}^\star\hat{\Delta}. \qquad \text{(A.236)}
\end{aligned}
$$

A.4.13 Curvature of the spherically symmetric aether solution

The various non-vanishing pieces of the curvature of the solution on page 79 read:

$$
\begin{aligned}
{}^{(1)}W_{\alpha\beta} &= \frac{m}{r^3}\begin{pmatrix} 0 & 2\vartheta^{01} & -\vartheta^{02} & -\vartheta^{03} \\ \bullet & 0 & \vartheta^{12} & \vartheta^{13} \\ \bullet & \bullet & 0 & -2\vartheta^{23} \\ \bullet & \bullet & \bullet & 0 \end{pmatrix}, \\
{}^{(2)}W_{\alpha\beta} &= 0\,, \\
{}^{(3)}W_{\alpha\beta} &= 0\,, \\
{}^{(4)}W_{\alpha\beta} &= -\vartheta_{[\alpha} \wedge \phi_{\beta]}\,, \\
\phi_0 &= \frac{\mathcal{L}_0^2}{8r^4\, e^{2\mu}}\,(\vartheta^0 + \vartheta^1)\,, \\
\phi_1 &= \frac{\mathcal{L}_0^2}{8r^4\, e^{2\mu}}\,(\vartheta^0 + \vartheta^1)\,, \\
\phi_2 &= 0\,, \qquad \phi_3 = 0\,, \\
{}^{(5)}W_{\alpha\beta} &= 0\,, \\
{}^{(6)}W_{\alpha\beta} &= -\frac{1}{12}\,W\,\vartheta_{\alpha\beta}\,, \qquad W = 4\lambda\,, \\
{}^{(1)}Z_{\alpha\beta} &= -\frac{\mathcal{L}_0}{24r^3\,B}\begin{pmatrix} 2A\,\vartheta^{01} & -6B\,\vartheta^{01} & 3B\,\vartheta^{02} - A\,\vartheta^{12} & 3B\vartheta^{03} - A\,\vartheta^{13} \\ \bullet & -2C\,\vartheta^{01} & C\,\vartheta^{02} + 3B\,\vartheta^{12} & C\,\vartheta^{03} + 3B\,\vartheta^{13} \\ \bullet & \bullet & 4B\,\vartheta^{01} & 0 \\ \bullet & \bullet & \bullet & 4B\,\vartheta^{01} \end{pmatrix}, \\
& \quad A = 2\lambda r^3 - 15m + 3r\,, \qquad B = \lambda r^3 + 6m - 3r\,, \qquad C = 4B - A\,, \\
{}^{(2)}Z_{\alpha\beta} &= 0\,, \\
{}^{(3)}Z_{\alpha\beta} &= \frac{1}{6}\,\left[4\vartheta_{(\alpha} \wedge e_{\beta)}\rfloor\Delta - 2g_{\alpha\beta}\,\Delta\right]\,, \qquad \Delta = -\frac{\mathcal{L}_0}{2r^3}\vartheta^{01}\,, \\
{}^{(4)}Z_{\alpha\beta} &= 0\,, \\
{}^{(5)}Z_{\alpha\beta} &= \frac{1}{2}\,\vartheta_{(\alpha} \wedge \Xi_{\beta)}\,, \\
\Xi_0 &= \frac{\mathcal{L}_0}{2r^3\,B}\,\left[(2\lambda r^3 - 6m)\,\vartheta^0 - (2\lambda r^3 + 3m - 3r)\,\vartheta^1\right]\,, \\
\Xi_1 &= \frac{\mathcal{L}_0}{2r^3\,B}\,\left[-(2\lambda r^3 + 3m - 3r)\,\vartheta^0 + (4\lambda r^3 + 6m - 6r)\vartheta^1\right]\,, \\
\Xi_2 &= -\frac{\mathcal{L}_0}{2r^3}\,\vartheta^2\,, \qquad \Xi_3 = -\frac{\mathcal{L}_0}{2r^3}\,\vartheta^3\,.
\end{aligned}
$$

A.5 Computer algebra

A.5.1 Classification of the Cotton tensor

```
% Calculation of Cotton-York tensor form given coframe/metric

out "baekler1.exo";

load excalc ;

% Definition of coframe/metric
% Nutku, Baekler Ann. Phys (NY) 195 (1989) 16, eq 4.1

coframe o(0) =  a0 * (d psi + sinh(theta)* d phi)  ,
        o(1) =  a1 * (-sin(psi) * d theta
                        + cos(psi)*cosh(theta) * d phi),
        o(2) =  a2 * ( cos(psi) * d theta + sin(psi) *cosh(theta)* d phi)
with metric g = -o(0) * o(0) + o(1)*o(1) + o(2)*o(2) ;

frame e ;

displayframe ;

% calculation of curvature

pform riem2(a,b) = 2;
riemannconx chris1 ;
chris1(a,b) := chris1(b,a) ;
riem2(-a,b) := d chris1(-a,b) - chris1(-a,c) ^ chris1(-c,b) ;

% calculation of L_a and Cotton

pform ll1(a)=1,cotton2(a)=2;
ll1(a) :=    e(-b) _| riem2(a,b)
- 1/4 * (e(-c) _| ( e(-d) _| riem2(c,d) )) * o(a) ;

cotton2(a) := d ll1(a) + chris1(-b,a) ^ ll1(b) ;

% Definition of Cotton tensor
pform cotmat(a,b) = 0 ;
cotmat(a,b) :=  #(cotton2(a)^o(b)) ;

% Definitio of Cotton matrix
```

```
matrix cotm(3,3) $
for a:= 1:3 do {for b:= 1:3 do{
        cotm(a,b) := cotmat(-(a-1),b-1) }}$

% Definition Einstein 2-form
pform einstein2(a) =2;
einstein2(a) := (1/2) * #(o(a)^o(b)^o(c)) ^ riem2(-b,-c);

% Test of DJT field equations
pform null(a)=2;
null(a) := einstein2(a)+(1/mu)*cotton2(a);

let cos(~a)**2 + sin(~a)**2 => 1 ;
let cosh(~a)**2 - sinh(~a)**2 => 1 ;

a0+a1+a2:=0;
mu := -(a0**2+a1**2+a2**2)/(a0*a1*a2);

null(a) := null(a);

% test cotm
write "cotm/cotm(1,1);";
cotm/cotm(1,1);

% prefector cotm(1,1)
write " prefector cotm(1,1)";
off exp;
on gcd;
cotm(1,1);

write "cotm";
cotm;

end;
```

A.5.2 Test of the BTZ-solution with torsion

```
% chh 2003-01-02
% ansatz in curvature

load excalc ;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% general spherical symmetric  %
% coframe                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

pform {nn,ff,gg,ww}=0;
fdomain nn=nn(r),ff=ff(r),gg=gg(r),ww=ww(r);
coframe o(t) = nn * d t,
        o(r) =  (d r)/nn,
        o(phi) = gg * ( d phi - ww * d t )
with signature (-1,1,1) ;

frame e ;

pform {curv2(a,b),curv2_ansatz(a,b)}   = 2 $
pform torsion2(a)=2,conx1(a,b) = 1 $


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Ansatz for torsion: only axial piece %
% Nuo. Cim B 107, 91--110              %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

torsion2(a) := 2*tor/ell * #o(a) ;
curv2_ansatz(a,b) := curv/ell**2 * o(a) ^ o(b) ;



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Connection according to PR 258, 1-171, eq. 3.10.6 %
% with g=const                                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

conx1(a,b) :=   1/2 * ( e(a) _| d o(b) - e(b) _| d o(a) )
              - 1/2 * ( e(a) _| ( e(b) _| d o(-c) ) ) * o(c)
              - 1/2 * ( e(a) _| torsion2(b) - e(b) _| torsion2(a) )
              + 1/2 * ( e(a) _| ( e(b) _| torsion2(-c) )) * o(c) ;

%%%%%%%%%%%%%%%%
% RC-Curvature %
%%%%%%%%%%%%%%%%

curv2(-a,b)     := d conx1(-a,b) - conx1(-a,c) ^ conx1(-c,b) $
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% equation for torsion is identically fulfilled %
% now fulfill equation for curvature            %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

pform aux(a,b) = 2 ;
aux(a,b) := curv2(a,b) - curv2_ansatz(a,b) ;

pform curv_comp(a,b) = 0 ;
curv_comp(a,b) := eps(a,c,d)*eps(b,d,e) * e(-c) _| (e(-d) _| aux(-d,-e)) ;

% (t,t) + (r,r)
curv_comp(t,t) + curv_comp(r,r) ;

% yields
gg := aa + bb * r ;

% (t,phi)
curv_comp(t,phi) ;

% yields
ww := alpha/((aa+bb*r)**2) + beta ;

% finally, (t,t)
curv_comp(t,t) ;

% yields an 1st order diff. eq. for nn**2
solve(ws,@(nn,r)) ;

aux2 := 2*nn*
(aa**4*curv + aa**4*tor**2 + 4*aa**3*bb*curv*r + 4*aa**3*bb*r*tor**2 +
6*aa**2*bb**2*curv*r**2 + 6*aa**2*bb**2*r**2*tor**2 + 4*aa*bb**3*curv*r**3 +
4*aa*bb**3*r**3*tor**2 - alpha**2*bb**2*ell**2 + bb**4*curv*r**4 +
bb**4*r**4*tor**2)/(aa**3*bb*ell**2*nn + 3*aa**2*bb**2*ell**2*nn*r +
3*aa*bb**3*ell**2*nn*r**2 + bb**4*ell**2*nn*r**3) ;

nn := sqrt(m+int(aux2,r)) ;

% test
curv_comp(a,b) ;


% define effective cosmological constant
```

```
(tor**2+curv)/ell**2 := lam_eff ;

on exp, div, rat ;
off gcd, allfac ;

factor m,lam_eff,alpha ;

nn**2 ;
```

A.5.3 Test of 3D perfect fluid solution

```
% Test of interior BTZ solution

%out "interior.exo";

load excalc ;

pform {nn,ff,pp,rho} = 0;
fdomain nn = nn(r),ff=ff(r),pp=pp(r),rho=rho(r);
coframe o(0) =  nn*d t  ,
        o(1) = d r/ff ,
        o(2) = r*d phi
with metric g = -o(0) * o(0) + o(1)*o(1) + o(2)*o(2) ;

frame e ;

displayframe ;

pform riem2(a,b) = 2;
riemannconx chris1 ;
chris1(a,b) := chris1(b,a) ;
riem2(-a,b) := d chris1(-a,b) - chris1(-a,c) ^ chris1(-c,b) ;


pform ll1(a)=1,cotton2(a)=2;

ll1(a) :=    e(-b) _| riem2(a,b)
- 1/4 * (e(-c) _| ( e(-d) _| riem2(c,d) )) * o(a) ;

cotton2(a) := d ll1(a) + chris1(-b,a) ^ ll1(b) ;

pform cotmat(a,b) = 0 ;
cotmat(a,b) := (1/2) * #(o(a)^o(c)^o(d)) * e(-c) _| (e(-d) _| cotton2(b)) ;
```

```
matrix cotm(3,3) $
for a:= 1:3 do {for b:= 1:3 do{
        cotm(a,b) := cotmat(-(a-1),b-1) }}$


pform energy(a,b) = 0 ;
pform u(a) =0;
u(a) := 0;
u(0) := 1 ;

energy(a,b) := (rho+pp)*u(a)*u(b)+pp*g(a,b);
pform coten(a)=1 ;
pform energy(a) =2 ;
energy(a) := energy(a,b)*#o(-b) ;

coten(a) := #(energy(a)-(1/2)*e(a) _| (o(b)^energy(-b)));

d coten(a) + chris1(-b,a)^coten(b);


@(rho,r) := 0;
nn := c1/(rho+pp);

end;

ff := sqrt(c2-(kappa*rho+lam)*r**2) ;

pp:=(c4*ff*(kappa*rho+lam)+c4**2*kappa*lam+rho*ff**2)/
(c4**2*kappa**2-ff**2);


% Definition Einstein 2-form
pform einstein2(a) =2;
einstein2(a) := (1/2) * #(o(a)^o(b)^o(c)) ^ riem2(-b,-c);

einstein2(a) + lam* #o(a) - kappa * energy(a);

end;
```

A.5.4 Quasi local energy after Nester et al.

```
load excalc ;
```

```
load limits;

a0 := -kappa/(2*ell);
a1 := theta_t/(2*ell**2);
a2 := -theta_l / 2;

pform{conx1(a,b)}=1;
pform {o_ref(a),conx_ref1(a,b),hh1_ref(a),hhhh1_ref(a,b)} = 1;
pform {hh1(a),hh1_ref(a),hhhh1(a,b),hhhh1_ref(a,b),delo1(a),delconx1(a,b),
transconx1(a,b),delhh1(a),delhhhh1(a,b)} = 1;

psi := sqrt((j/(2*r))**2 -m + lam_eff * r**2 ) ;
coframe o(0) = psi * d t,
        o(1) = d r / psi,
        o(2) = -j/(2*r) * d t + r * d theta
with signature (-1,1,1) ;

frame e ;

pform torsion2(a) = 2;
torsion2(a) := 2*tor/ell * #o(a) ;

conx1(a,b) :=   1/2 * ( e(a) _| d o(b) - e(b) _| d o(a) )
              - 1/2 * ( e(a) _| ( e(b) _| d o(-c) ) ) * o(c)
              - 1/2 * ( e(a) _| torsion2(b) - e(b) _| torsion2(a) )
              + 1/2 * ( e(a) _| ( e(b) _| torsion2(-c) )) * o(c) ;

hh1(a) := a1 * o(a) ;
hhhh1(a,b) := a0 * #(o(a)^o(b))  + a2* conx1(a,b);


o_ref(0) := sqrt(lam_eff * r**2)* d t;
o_ref(1) := d r / sqrt(lam_eff * r **2);
o_ref(2) := r * d theta;

tor_ref := tor ;
conx_ref1(a,b) := 0;
index_symmetries conx_ref1(a,b): antisymmetric ;
conx_ref1(0,1) := - r *lam_eff * d t -tor_ref/ell* r * d theta;
conx_ref1(1,2) :=  sqrt(lam_eff*r**2) *(tor_ref/ell*d t+ d theta);
conx_ref1(2,0) := - tor_ref/ell * d r / sqrt(lam_eff * r **2);
```

```
hh1_ref(0) :=  a1 * o_ref(0) ;
hh1_ref(1) :=  a1 * o_ref(1) ;
hh1_ref(2) :=  a1 * o_ref(2) ;



index_symmetries hhhh1_ref(a,b): antisymmetric ;

hhhh1_ref(0,1) :=   - a0 * o_ref(2) + a2 * conx_ref1(0,1);
hhhh1_ref(1,2) := a0 * o_ref(0) + a2 * conx_ref1(1,2);
hhhh1_ref(2,0) :=   -a0 * o_ref(1) + a2 * conx_ref1(2,0);



delhh1(a)     := hh1(a) - hh1_ref(a) ;
delhhhh1(a,b) := hhhh1(a,b) - hhhh1_ref(a,b) ;
delo1(a)      := o(a) - o_ref(a) ;
delconx1(a,b) := conx1(a,b) - conx_ref1(a,b) ;

%transconx1(a,b) := e(a) _| conx1(-c,b)*o(c) - e(a) _| torsion2(b) ;
%transconx1(a,b) :=  conx1(a,b) + e(a) _| torsion2(b) ;

n := @(t);

transconx1(a,b) := (-e(b) _| (d(n _|o(a)) +conx1(a,-c) * (n _|o(c)) + n _|
torsion2(a) ));

pform bb1 = 1;
bb1 :=  (n _| o(a)) ^ delhh1(-a)
      + delo1(b) ^ (n _| hh1_ref(-b))
      - transconx1(c,d)  ^ delhhhh1(-c,-d)
      - delconx1(e,f) ^ (n _| hhhh1_ref(-e,-f)) ;

2*pi * limit(@(theta) _| bb1,r,infinity) ;
limit(@(t) _| bb1,r,infinity) ;


pform bb2 = 1;
bb2 :=   (n _| o_ref(a)) ^ delhh1(-a)
       + delo1(a) ^ (n _| hh1(-a))
       - (n _| conx_ref1(a,b)) ^ delhhhh1(-a,-b)
       - delconx1(a,b) ^ (n _| hhhh1(-a,-b) );
```

```
2*pi * limit(@(theta) _| bb2,r,infinity) ;
limit(@(t) _| bb2,r,infinity) ;



%killing eq

pform transconx(a,b) = 1;
transconx(a,b) :=  conx1(a,b) + e(a) _| torsion2(b) ;

pform aux(a,b) = 1 ;
aux(a,b) := e(a) _| (d (xi _| o(b)) + transconx(-c,b)^(xi _| o(c))) ;

d aux(a,b) + conx1(-c,a) ^ aux(c,b) + conx1(-c,b) ^ aux(a,c) + xi _|
curv2(a,
```

A.5.5 Spherically symmetric aether solution

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%     file baekler5.exi, chh, 2004-04-16, 2004-11-30           %
%     test for spherically-symmetric aether-solution             %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

load "excalc";

%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% input coframe and metric %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%

pform mu = 0;
fdomain mu = mu(r);

coframe   o(0) = exp(mu) * d t,
          o(1) = exp(-mu) * d r ,
          o(2) = r * d theta,
          o(3) = r *sin(theta) * d phi
with
    metric g =  -o(0)*o(0) + o(1)*o(1) + o(2)*o(2) + o(3)*o(3)$
frame e$

%%%%%%%%%%%%%%%%%%%%%%
% input nonmetricity %
```

```
%%%%%%%%%%%%%%%%%%%%%%

pform{nonmet1(a,b)}=1 ;

nonmet1(a,b) := 0;
index_symmetries nonmet1(a,b): symmetric ;

pform aether = 0;

% input aether function
%aether := 10*exp(-mu)/(2*r**2) ;
fdomain aether = aether(r) ;

nonmet1(0,0) := aether * (o(0) + o(1)) ;
nonmet1(0,2) := aether * o(2) ;
nonmet1(0,3) := aether * o(3) ;
nonmet1(1,1) := aether * (o(0) + o(1)) ;
nonmet1(1,2) := aether * o(2) ;
nonmet1(1,3) := aether * o(3) ;

% alternative non-metricity for arbitrary, light-like aether field
%pform lamz(a) = 0 ;
%fdomain lamz = lamz(r);
%lamz(0)**2-lamz(1)**2-lamz(2)**2-lamz(3)**2 :=0;
%nonmet1(-a,-b) := (4/9)*((o(-b)*lamz(-a)+o(-a)*lamz(-b))/2
%                          -g(-a,-b)*(lamz(-c)*o(c))/4);


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% calculation torsion according to prolongation %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

pform torsion2(a) = 2;

torsion2(a) := (1/2) * nonmet1(a,b) ^ o(-b) ;

%%%%%%%%%%%%%%
% connection %
%%%%%%%%%%%%%%

pform {conx1(a,b)}=1 ;

%conx1(-a,-b) :=   (1/2) * ( e(-a) _| d o(-b) - e(-b) _| d o(-a) )
```

```
%                 - (1/2) * ( e(-a) _| ( e(-b) _| d o(-c) ) ) ^ o(c)
%                 + (1/2) * nonmet1(-a,-b)
%                 + (1/2) * (e(-a) _| nonmet1(-b,-c)
%                          - e(-b) _| nonmet1(-a,-c)) *o(c)
%                 - 1/2 * ( e(-a) _| torsion2(-b) - e(-b) _| torsion2(-a) )
%                 + 1/2 * ( e(-a) _| ( e(-b) _| torsion2(-c) )) ^ o(c) ;

% alternative torsion
riemannconx chris1 ;
chris1(-a,b) := chris1(b,-a) ;

pform contor1(a,b) = 1;

contor1(-a,-b) := (1/2) * ( e(-a) _| torsion2(-b) - e(-b) _| torsion2(-a))
         - (1/2) * (e(-a) _| (e(-b) _| torsion2(-c)))^o(c) ;

conx1(-a,b) :=   chris1(-a,b) -contor1(-a,b)
               + (1/2) * nonmet1(-a,b)
               + (1/2) * (( e(-a) _| nonmet1(b,-c))
                        - ( e(b) _| nonmet1(-a,-c))) ^ o(c) ;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% irreducible decomp. torsion %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

pform {tentor2(a),trator2(a),axitor2(a)}=2$

trator2(a):=  (1/3)*o(a)  ^ (e(-b) _| torsion2(b));
axitor2(a):=  (1/3)*e(a) _| (o(-b)  ^ torsion2(b));
tentor2(a):=   torsion2(a)-trator2(a)-axitor2(a);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% irreducible decompostition nonmetr. %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

pform lamzero(a)=0, {weylcovector1,nomtracefree1(a,b),lamone1,
      binom1(a,b),vecnom1(a,b),trinom1(a,b),conom1(a,b)}=1,
      {thetatwo2(a),omega2(a)}=2, thetathree3=3$

weylcovector1        := nonmet1(-c,c)/4$
nomtracefree1(-a,-b) := nonmet1(-a,-b) - g(-a,-b)*weylcovector1$
lamzero(-a)          := e(b)_|nomtracefree1(-a,-b)$
lamone1              := lamzero(-a)*o(a)$
```

```
thetatwo2(-a)         := #(nomtracefree1(-a,-b)^o(b))$
thetathree3           := o(a)^thetatwo2(-a)$
omega2(-a)            := thetatwo2(-a)-(1/3)*e(-a)_|thetathree3$

binom1(-a,-b)  := (1/3)*#(o(-a)^omega2(-b)+o(-b)^omega2(-a));
vecnom1(-a,-b) := (4/9)*((o(-b)*lamzero(-a)+o(-a)*lamzero(-b))/2
                          -g(-a,-b)*lamone1/4);
conom1(-a,-b)  :=  g(-a,-b)*weylcovector1;
trinom1(-a,-b) :=  nonmet1(-a,-b) - binom1(-a,-b)
                  -vecnom1(-a,-b) - conom1(-a,-b);

%%%%%%%%%%%%%%%%%%%%%%%%%
% calculation curvature %
%%%%%%%%%%%%%%%%%%%%%%%%%

pform curv2(a,b) = 2 ;
curv2(-a,b)     := d conx1(-a,b) - conx1(-a,c) ^ conx1(-c,b) $

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% irreducible decomp. curvature %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

pform {ztracef1(a),yy1(a),xi1(a),upsilon1(a)}=1,
      {z2(a,b),ztracef2(a,b),delta2,
       zcurvone2(a,b), zcurvtwo2(a,b),zcurvthree2(a,b),
        dilcurv2(a,b),zcurvfive2(a,b)}=2$

z2(-a,-b)       := (1/2)*(curv2(-a,-b)+curv2(-b,-a))$
ztracef2(-a,-b) := z2(-a,-b)-(1/4)*g(-a,-b)*z2(-c,c)$
ztracef1(-a)    := e(b)_|ztracef2(-a,-b)$
delta2          := (1/2)*o(a)^ztracef1(-a)$
yy1(-a)         := #(ztracef2(-a,-b)^o(b))$
xi1(-a)        := ztracef1(-a)-(1/2)*e(-a)_|(o(c)^ztracef1(-c))$
upsilon1(-a)    := yy1(-a)-(1/2)*e(-a)_|(o(c)^yy1(-c))$

zcurvtwo2(-a,-b)  := -(1/4)*#(o(-a)^upsilon1(-b)
                             +o(-b)^upsilon1(-a));
zcurvthree2(-a,-b):= (1/6)*(2*(o(-a)^(e(-b)_|delta2)
                     +o(-b)^(e(-a)_|delta2))-2*g(-a,-b)*delta2);
dilcurv2(-a,-b)   := (1/4)*g(-a,-b)*z2(-c,c);
zcurvfive2(-a,-b) := (1/4)*(o(-a)^xi1(-b)+o(-b)^xi1(-a));
zcurvone2(-a,-b)  := z2(-a,-b)-zcurvtwo2(-a,-b)
         -zcurvthree2(-a,-b)-dilcurv2(-a,-b)-zcurvfive2(-a,-b);
```

```
pform {wzero,xzero}=0,
      {wone1(a),xone1(a),psi1(a),phi1(a)}=1,
      {w2(a,b),paircom2(a,b),pscalar2(a,b),ricsymf2(a,b),
               ricanti2(a,b), scalar2(a,b),   weyl2(a,b)}=2$

w2(-a,-b) := (1/2)*(curv2(-a,-b)-curv2(-b,-a))$
wone1(a)  := e(-b)_|w2(a,b)$
wzero     := e(-b)_|wone1(b)$
xone1(a)  := #(w2(b,a)^o(-b))$
xzero     := e(-a)_|xone1(a)$
psi1(-a)  := xone1(-a)-(1/4)*o(-a)^xzero
                      -(1/2)*e(-a)_|(o(b)^xone1(-b))$
phi1(-a)  := wone1(-a)-(1/4)*o(-a)*wzero
                      -(1/2)*e(-a)_|(o(b)^wone1(-b))$

paircom2(-a,-b) := -(1/2)*#(o(-a)^psi1(-b)-o(-b)^psi1(-a));
pscalar2(-a,-b) := -(1/12)*#(xzero^o(-a)^o(-b));
ricsymf2(-a,-b) := -(1/2)*(o(-a)^phi1(-b)-o(-b)^phi1(-a));
ricanti2(-a,-b) := -(1/4)*(o(-a)^(e(-b)_|(o(c)^wone1(-c)))
                          -o(-b)^(e(-a)_|(o(c)^wone1(-c))));
scalar2(-a,-b)  := -(1/12)*wzero*o(-a)^o(-b);
weyl2(-a,-b)    := w2(-a,-b)-paircom2(-a,-b)-pscalar2(-a,-b)-
                 ricsymf2(-a,-b)-ricanti2(-a,-b)-scalar2(-a,-b);


factor o(0),o(1),o(2),o(3) ;

%%%%%%%%%%%%%%%%%%%%
% input lagrangian %
%%%%%%%%%%%%%%%%%%%%

pform {lag41,lag42,lag43,lag44,lag45,lag4} = 4;

lag41 := (1/(2*kap)) * (-aa0*(curv2(a,b) ^ #(o(-a)^o(-b)) + 2 * lam * #1));
lag42 := (1/(2*kap)) *  aa2 * torsion2(a) ^ # trator2(-a);
lag43 := (1/kap)     * cc3 * vecnom1(-a,-b) ^ o(a) ^ # torsion2(b);
lag44 := (1/(2*kap)) * bb3 * nonmet1(a,b) ^ # vecnom1(-a,-b) ;
lag45 := - (1/2) * curv2(a,b) ^ #(  zz3 * zcurvthree2(-a,-b)
                                 + ww6 * scalar2(-a,-b) ) ;

lag4 := lag41 + lag42 + lag43 + lag44 + lag45 ;
```

```
%%%%%%%%%%%%%%%
% excitations %
%%%%%%%%%%%%%%%

pform {energy3(a),capm3(a,b)} = 3 ;
pform {htr2(a),hrot2(a,b)}= 2 ;

htr2(-a) := - (1/kap) *  # (aa2 * trator2(-a) + cc3 * vecnom1(-a,-b) ^ o(b)) ;

hrot2(a,-b) :=   (aa0/(2*kap)) * #(o(a)^o(-b))
                 + # ( ww6 * scalar2(a,-b) + zz3 * zcurvthree2(a,-b) ) ;

capm3(a,b) := -(2/kap) * (   bb3 * # vecnom1(a,b)
         + cc3 *  (1/2) * ( o(a) ^ #trator2(b) + o(b) ^ #trator2(a) )
         + (1/4)* cc3 * g(a,b) * #(e(-c) _| torsion2(c)) ) ;


energy3(a) :=   e(-a) _| lag4
              + (e(-a) _| torsion2(b)) ^ htr2(-b)
              + (e(-a) _| curv2(-b,c)) ^ hrot2(b,-c)
              + (1/2) * ( e(-a) _| nonmet1(-b,-c)) ^ capm3(b,c) ;

%%%%%%%%%%%%%%%%%%%%%%
%   field equations  %
%%%%%%%%%%%%%%%%%%%%%%


pform first3(a) = 3 ;
first3(-a) := d htr2(-a) - conx1(-a,b) ^ htr2(-b) - energy3(-a) ;

pform second3(a,b) = 3 ;
second3(a,-b) := d hrot2(a,-b) + conx1(-c,a)^hrot2(c,-b)
                               - conx1(-b,c)^hrot2(a,-c)
                + o(a) ^ htr2(-b) + capm3(a,-b) ;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% substitution of constraints %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

mu := log(1-2*m/r-lam/3*r**2)/2 ;

aa2 := - 2 * cc3 ;
bb3 := - 3* cc3 / 8 ;
```

```
aa0 := 2*kap*lam*ww6/3 ;
zz3 := 0 ;

on exp ;
on gcd ;
factor o(0),o(1),o(2),o(3) ;
factor ^ ;

first3(a);

second3(a,b);
```

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