

**A STUDY ON t -BEST APPROXIMATION
PROBLEMS IN FUZZY n -NORMED LINEAR
SPACES**

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A STUDY ON t -BEST APPROXIMATION PROBLEMS IN FUZZY n -NORMED LINEAR SPACES

ABSTRACT

A problem in fundamental mathematics is to find an approximation to the functions which are given explicitly or implicitly by operator equations, from a set of functions of simple structure. It is well known that the Approximation Theory of functions can be divided into Best, Good and Possible Approximation. The theory of best approximation is mainly concerned with the questions of existence, uniqueness, characterizations, qualitative properties of the minimizing functions and the approximations errors minimize when the complexity order of the system is increased. The best approximation is playing a vital role in engineering problems. It considers the problem of approximation defined on a set taking values in a normed and n -normed space. To find an approximation of the exact value, it must know something about the problem element. Usually the information is imprecise; thus there exist much information corresponding to same problem element. Under such an interpretation it seems natural to consider the information as fuzzy, rather than crisp. This is the motivation of the thesis to develop the concept of best approximation, best coapproximation in the framework of fuzzy normed and fuzzy n -normed spaces. The objective of this thesis is a systematize study of t -best approximation, t -best simultaneous approximation and t -best (simultaneous) coapproximation problems in fuzzy normed and fuzzy n -normed spaces X with respect to continuous t -norm. This study is useful for the development of fuzzy functional analysis and fuzzy approximation theory in order to get the precise solutions.

CHAPTER 1

INTRODUCTION

1.1 Background and Rationale of the Study

Functions are one of the basic mathematical tools for describing objects and processes from the real world. But these functions are known explicitly only in some environment. Very often, it is very necessary to construct approximations to those functions based on limited information about the underlying processes. A problem in fundamental mathematics is to find an approximation to the functions which are given explicitly or implicitly by operator equations, from a set of functions of simple structure. It is well known that the Approximation Theory of functions can be divided into three main parts namely Best, Good and Possible Approximation. The central one in this theory is the notion of the best approximation introduced by Chebyshev (13) in 1853.

The theory of best approximation is an important topic in functional analysis. It is a very extensive field which has various applications (22; 56). The theory of best approximation is mainly concerned with the questions of existence, uniqueness, characterizations, qualitative properties of the minimizing functions and the approximation errors minimize when the complexity order of the system is increased. While at the beginning researcher's attention was attracted to the investigation of the best approximation to a single element in normed space, then shifted to the problem of approximation to the whole class. The best approximation is playing a vital

role in engineering problems. A theory of n -normed space has been introduced and developed by Gunawan and Mashadi in (54).

The theory of fuzzy sets was introduced by L. Zadeh (111) in 1965. Fuzzy techniques (112) are powerful tools for knowledge representation and processing and also can manage the vagueness and ambiguity or imprecision. Later many researchers have applied this theory to the well known results in the classical set theory. So it has become an area of active research for nearly forty years. The fuzzy logic has been used not only in many engineering applications, such as, in bifurcation of non-linear dynamical systems (57), in the control of chaos (35), in the computer programming (48), in the population dynamics (7), in the quantum physics (28), but also in various branches of mathematics, such as, in metric and topological spaces (1; 33; 46; 62), in the theory of functions (10; 60), in the study of matrix and linear systems (9), in the approximation theory (3). Especially, there are a lot of studies in the field of fuzzy topology. Actually, the fuzzy topology has very important applications in the quantum particle physics (27; 29). One of the most important problems on fuzzy topology is to obtain an appropriate concept of fuzzy metric space. The problem has been investigated by many authors (19; 33; 46; 62; 72) from different points of view. In particular, George and Veeramani (46) have introduced and studied a notion of fuzzy metric space with the help of continuous t -norms, which constitutes a slight but appealing modification of the one due to Kramosil and Michalak (72). Veeramani (108) in 2001 introduced the concept of t -best approximation in fuzzy metric spaces and also, Vaezpour and Karimi (107) introduced the concept of t -best approximation in fuzzy normed spaces. Recently, the notion of intuitionistic fuzzy metric space has been introduced by Park (89).

Furthermore, Saadati and Park (95) gave the concept of intuitionistic fuzzy normed space. Recently, Kavikumar et. al., (65) have introduced the concept of Reisz theorem in fuzzy n -normed linear spaces. It considers the problem of approximation defined on a set taking values in an n -normed space. To find an approximation of the exact value, something about the problem element must be known. Usually the information is imprecise; thus there exist much information corresponding to same problem element. Under such an interpretation it seems natural to consider the information as fuzzy, rather than crisp. This is the motivation of this present study to develop the concept of best approximation problems in the framework of fuzzy normed spaces, fuzzy n -normed spaces. and intuitionistic fuzzy normed spaces.

1.1.1 Objective of the Study

1. To investigate the problems of t -best coapproximation and t -best simultaneous coapproximation sets in fuzzy normed space.
2. To study the t -best coapproximation problems in fuzzy anti- n -normed spaces.
3. To study the existence of t -best approximation and t -best simultaneous approximation problems in intuitionistic fuzzy normed spaces.

1.2 Preliminaries

1.2.1 Fuzzy Normed Space

Definition 1.2.1. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be continuous t-norm if $([0, 1], *)$ is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ ($a, b, c, d \in [0, 1]$). We call $*_1 \leq *_2$ if $a *_1 b \geq a *_2 b$ for all $a, b \in [0, 1]$.

Definition 1.2.2. The 3-tuple $(X, N, *)$ is said to be a fuzzy normed space if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$

- (i) $N(x, t) > 0$
- (ii) $N(x, t) = 1 \Leftrightarrow x = 0$
- (iii) $N(\alpha x, t) = N(x, t/|\alpha|)$ for all $\alpha \neq 0$
- (iv) $N(x, t) * N(y, s) \leq N(x + y, t + s)$
- (v) $N(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous
- (vi) $\lim_{t \rightarrow \infty} N(x, t) = 1$.

Lemma 1.2.3. *Let N be a fuzzy norm. Then:*

- (i) $N(x, t)$ is non decreasing with respect to t for each $x \in X$.
- (ii) $N(y - x, t) = N(x - y, t)$.

Example 1.2.4. Let $(X, \|\cdot\|)$ be a normed space. We define $a * b = ab$ or $a * b = \min\{a, b\}$ and

$$N(x, t) = \frac{kt^n}{kt^n + m\|x\|}, k, m, n \in \mathbb{R}^+.$$

Then $(X, N, *)$ is a fuzzy normed space. In particular if $k = m = n = 1$ we have

$$N(x, t) = \frac{t}{t + \|x\|},$$

which is called the standard fuzzy norm induced by the norm $\|\cdot\|$.

Remark 1.2.5. In (96), it was shown that every fuzzy norm induces a fuzzy metric and so every fuzzy normed space is a topological space.

Definition 1.2.6. Let $(X, N, *)$ be a fuzzy normed space. The open and closed ball $B(x, r, t)$ and $B[x, r, t]$ with the center $x \in X$, radius $0 < r < 1$ and $t > 0$ are defined as follows:

$$B(x, r, t) = \{y \in X : N(x - y, t) > 1 - r\}.$$

$$B[x, r, t] = \{y \in X : N(y - x, t) \geq 1 - r\}.$$

Proposition 1.2.7. Let $(X, N, *)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, \infty)$.

Corollary 1.2.8. Let $(X, N, *)$ be a fuzzy normed space. Then $N : X \times (0, \infty) \rightarrow [0, 1]$ is continuous.

Remark 1.2.9. For any $r_1 > r_2$, we can find r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 we can find r_5 such that $r_5 * r_5 \geq r_4$, $(r_1, r_2, r_3, r_4, r_5 \in (0, 1))$.

1.2.2 Fuzzy n -normed and Anti n -normed Space

Definition 1.2.10. Let X be a real vector space of dimension greater than 1 and let

$\|\bullet, \bullet\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

- (1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
- (2) $\|x, y\| = \|y, x\|$,
- (3) $\|\alpha x, y\| = |\alpha| \|x, y\|$, where α is real.
- (4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

$\|\bullet, \bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet, \bullet\|)$ is called a linear 2-normed space.

Definition 1.2.11. (109). Let $n \in \mathbb{N}$ (natural numbers) and X be a real linear space of dimension $d \geq n$. (Here we allow d to be infinite). A real valued function $\|\bullet, \bullet, \dots, \bullet\|$ on $X \times X \times \dots \times X$ (n times) $= X^n$ satisfying the following four properties:

- (N1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent.
- (N2) $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation of x_1, x_2, \dots, x_n .
- (N3) $\|x_1, x_2, \dots, cx_n\| = |c| \|x_1, x_2, \dots, x_n\|$, for any real c .
- (N4) $\|x_1, x_2, \dots, x_{n-1}, y + z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$

is called an n -norm on X and the pair $(X, \|\bullet, \dots, \bullet\|)$ is called an n -normed linear space.

Definition 1.2.12. (109). Let X be a linear space over a real field \mathbb{F} . A fuzzy subset N of $X^n \times [0, \infty)$ is called a fuzzy n -norm on X if and only if:

(FN1) $N(x_1, x_2, \dots, x_n, t) > 0$.

(FN2) $N(x_1, x_2, \dots, x_n, t) = 1 \Leftrightarrow x_1, x_2, \dots, x_n$ are linearly dependent.

(FN3) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of x_1, x_2, \dots, x_n .

(FN4) $N(x_1, x_2, \dots, cx_n, t) = N(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in \mathbb{F}$ (field)

(FN5) $N(x_1, x_2, \dots, x_n + x'_n, s+t) \geq N(x_1, x_2, \dots, x_n, t) * N(x_1, x_2, \dots, x'_n, s)$ for all $s, t \in$

\mathbb{R}

(FN6) $N(x_1, x_2, \dots, x_n, \cdot)$ is left continuous and non-decreasing function of \mathbb{R} such

that $\lim_{t \rightarrow \infty} N(x_1, x_2, \dots, x_n, t) = 1$.

Then (X, N) is called a fuzzy n -normed linear space.

Definition 1.2.13. (65). Let X be a linear space over a real field \mathbb{F} . A fuzzy subset N of $X^n \times [0, \infty)$ is called a fuzzy anti n -norm on X if and only if:

(FN*1) for all $t \in \mathbb{R}$ with $t \leq 0, N(x_1, x_2, \dots, x_n, t) = 1$.

(FN*2) for all $t \in \mathbb{R}$ with $t > 0, N(x_1, x_2, \dots, x_n, t) = 0 \Leftrightarrow x_1, x_2, \dots, x_n$ are linearly dependent.

(FN*3) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of x_1, x_2, \dots, x_n .

(FN*4) for all $t \in \mathbb{R}$ with $t > 0, N(x_1, x_2, \dots, cx_n, t) = N(x_1, x_2, \dots, x_n, \frac{t}{|c|})$ if $c \neq 0, c \in$

\mathbb{F} (field)

(FN*5) for all $s, t \in \mathbb{R}$,

$$N(x_1, x_2, \dots, x_n + x'_n, s + t) \leq \max\{N(x_1, x_2, \dots, x_n, s), N^*(x_1, x_2, \dots, x'_n, t)\}$$

(FN*6) $N(x_1, x_2, \dots, x_n, \cdot)$ is right continuous and non-increasing function of \mathbb{R} such that

$$\lim_{t \rightarrow \infty} N(x_1, x_2, \dots, x_n) = 0$$

Then (X, N) is called a fuzzy anti n -normed linear space.

To strengthen the above definition, we present the following example.

Example 1.2.14. (65). Let $(X, \|\bullet, \bullet, \dots, \bullet\|)$ be a n -normed linear space

Define,

$$N(x_1, x_2, \dots, x_n, t) = \begin{cases} 1 - \frac{t}{t + \|x_1, x_2, \dots, x_n\|} & \text{when } t(> 0) \in \mathbb{R}, \forall x \in X \\ 1 & \text{when } t(\leq 0) \in \mathbb{R}, \forall x \in X \end{cases}$$

Then (X, N) is a fuzzy anti n -normed linear space.

Definition 1.2.15. (103). A sequence $\{x_k\}$ in a fuzzy anti- n -normed linear space (X, N) is said to be convergent to $x \in X$ if given $t > 0, 0 < r < 1$, there exists an integer $n_0 \in \mathbb{N}$ such that

$$N(x_1, x_2, \dots, x_{n_1}, x_k - x, t) < r, \forall k \geq n_0.$$

Theorem 1.2.16. (103). In a fuzzy anti- n -normed linear space (X, N) , a sequence

$\{x_k\}$ converges to $x \in X$ if and only if

$$\lim_{k \rightarrow \infty} N(x_1, x_2, \dots, x_{n_1}, x_k - x, t) = 0, \forall t > 0.$$

Definition 1.2.17. (103). Let (X, N) be a fuzzy anti- n -normed linear space. Let $\{x_k\}$ be a sequence in X then $\{x_k\}$ is said to be a Cauchy sequence if

$$\lim_{k \rightarrow \infty} N(x_1, x_2, \dots, x_{n_1}, x_{k+p} - x_k, t) = 0, \forall t > 0$$

and $p = 1, 2, 3, \dots$. A fuzzy anti- n -normed linear space (X, N) is said to be complete if every Cauchy sequence in X is convergent. A complete fuzzy anti- n -normed space (X, N) is called a fuzzy anti- n -Banach space. The open ball $B(x, r, t)$ and the closed ball $B[x, r, t]$ with the center $x \in X$ and radius $0 < r < 1, t > 0$ are defined as follows:

$$B(x, r, t) = \{y \in X : N(x_1, x_2, \dots, x_{n_1}, x - y, t) < r\},$$

$$B[x, r, t] = \{y \in X : N(x_1, x_2, \dots, x_{n_1}, x - y, t) \leq r\}.$$

A subset A of X is said to be open if there exists $r \in (0, 1)$ such that $B(x, r, t) \subset A$ for all $x \in A$ and $t > 0$. A subset A of X is said to be closed if for any sequence $\{x_k\}$ in A converges to $x \in A$, i.e., $\lim_{k \rightarrow \infty} N(x_1, x_2, \dots, x_{n_1}, x_k - x, t) = 0$, for all $t > 0$ implies that $x \in A$.

Corollary 1.2.18. (103). Let (X, N) be a fuzzy anti- n -normed linear space. Then N

is a continuous function on

$$\underbrace{X \times X \times \dots \times X}_n \times \mathbb{R}.$$

Let G be a non-empty subset of a linear n -normed space X . An element $g_0 \in G$ is called a best approximation to $x \in X$ from G if for every $g \in G$ and for every $x_2, x_3, \dots, x_n \in X \setminus G$ which is independent of x and G ,

$$\|x - g_0, x_2, x_3, \dots, x_n\| \leq \|x - g, x_2, x_3, \dots, x_n\|,$$

The set of all elements of best approximation to $x \in X$ from G is denoted by $P_{G, x_2, x_3, \dots, x_n}(x)$

Let $(X, \|\bullet, \dots, \bullet\|)$ be a linear n -normed space and let G be an arbitrary nonempty subset of X and $x_0 \in X$. Then, for every $x \in X$ and for every $x_2, x_3, \dots, x_n \in X \setminus G$ which is independent of x and x_0 ,

$$d_{x_2, x_3, \dots, x_n}(x, G) \leq \|x - x_0, x_2, x_3, \dots, x_n\| + d_{x_2, x_3, \dots, x_n}(x_0, G),$$

where

$$d_{x_2, x_3, \dots, x_n}(x, G) = \inf_{g \in G} \|x - g, x_2, x_3, \dots, x_n\|$$

. For each $G \subset X$ and $x_0 \in X$, we define

$$\begin{aligned} D_{x_2, x_3, \dots, x_n}(x_0, G) \\ = \{x \in X : d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\| \\ + d_{x_2, x_3, \dots, x_n}(x_0, G)\} \end{aligned}$$

for any $x_2, x_3, \dots, x_n \in X \setminus G$ which is independent of x and x_0 . We denote

$$\begin{aligned} P_{G, x_2, x_3, \dots, x_n}(x) \\ = \{g_0 \in G : \|x - g_0, x_2, x_3, \dots, x_n\| = d_{x_2, x_3, \dots, x_n}(x, G)\}, \end{aligned}$$

$$\begin{aligned} P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0) \\ = \{x \in X : \|x - x_0, x_2, x_3, \dots, x_n\| = d_{x_2, x_3, \dots, x_n}(x, G)\}, \end{aligned}$$

where $x_0 \in G$.

The following important property of n -norm was established by Hahng-Yun Chu et.al.,(55).

Theorem 1.2.19. *Let x_1, x_2, \dots, x_n be elements of a linear n -normed space X and γ a real number. Then*

$$\|x_1, \dots, x_i, \dots, x_j, \dots, x_n\| = \|x_1, \dots, x_i, \dots, x_j + \gamma x_i, \dots, x_n\|$$

for all $1 \leq i \neq j \leq n$.

Riesz (94) obtained the following theorem in a normed space

Theorem 1.2.20. *Let Y and Z be subspaces of a normed space X and Y a closed proper subset of Z . For each $\theta \in (0, 1)$, there exists an element $z \in Z$ such that*

$$\|z\| = 1, \quad \|z - y\| \geq \theta$$

for all $y \in Y$.

Definition 1.2.21. (54). A sequence $\{x_n\}$ in a linear n -normed space $(X, \|\bullet, \dots, \bullet\|)$ is said to n -convergent to $x \in X$ and denoted by $x_k \rightarrow x$ as $k \rightarrow \infty$ if

$$\lim_{k \rightarrow \infty} \|x_1, x_2, \dots, x_{n-1}, x_n - x\| = 0$$

From the above definitions, Park and Chu (88) obtained the following theorem in a n -normed spaces.

Theorem 1.2.22. *Let Y and Z be subspaces of a linear n -normed space X and Y an n -compact proper subset of Z with codimension greater than $n - 1$. For each $\theta \in (0, 1)$, there exists an element $(z_1, z_2, \dots, z_n) \in Z^n$ such that*

$$\|z_1, z_2, \dots, z_n\| = 1, \quad \|z_1 - y, z_2 - y, \dots, z_n - y\| \geq \theta$$

for all $y \in Y$.

Definition 1.2.23. (109). A sequence $\{x_n\}$ in a fuzzy n -normed space (X, N) is said to converge to x if given $r > 0, t > 0, 0 < r < 1$, there exists an integer $n_0 \in \mathbb{N}$ such that $N(x_1, x_2, \dots, x_{n-1}, x_n - x, t) > 1 - r$ for all $n \geq n_0$.

1.2.3 Intuitionistic Fuzzy Normed Space

Definition 1.2.24. A binary operation $*$: $[0, 1] \rightarrow [0, 1]$ is said to be a continuous t-norm if it satisfies the following conditions:

- (a) $*$ is associative and commutative,
- (b) $*$ is continuous,
- (c) $a * 1 = a$ for all $a \in [0, 1]$,
- (d) $a * d \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 1.2.25. A binary operation \diamond : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

- (a) \diamond is associative and commutative,
- (b) \diamond is continuous,
- (c) $a \diamond 0 = a$ for all $a \in [0, 1]$,
- (d) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Using the continuous t-norm and t-conorm, Saadati and Park (95) has recently introduced the concept of intuitionistic fuzzy normed spaces as follows:

Definition 1.2.26. The 5-tuple $(V, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (IFNS) if V is a vector space, $*$ is a continuous t-norm, \diamond is a continuous t-conorm, and μ, ν fuzzy sets on $V \times (0, \infty)$ satisfy the following conditions for every $x, y \in V$ and $s, t > 0$:

- (a) $\mu(x, t) + v(x, t) \leq 1$,
- (b) $\mu(x, t) > 0$,
- (c) $\mu(x, t) = 1$ if and only if $x = 0$,
- (d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
- (e) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$,
- (f) $\mu(x, \cdot) : (0, \infty) \rightarrow [0, 1]$, is continuous,
- (g) $\lim_{t \rightarrow \infty} \mu(x, t) = 1$ and $\lim_{t \rightarrow 0} \mu(x, t) = 0$,
- (h) $v(x, t) < 1$,
- (i) $v(x, t) = 0$ if and only if $x = 0$,
- (j) $v(\alpha x, t) = v(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$
- (k) $v(x, t) \diamond v(y, s) \geq v(x + y, t + s)$,
- (l) $v(x, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous,
- (m) $\lim_{t \rightarrow \infty} v(x, t) = 0$ and $\lim_{t \rightarrow 0} v(x, t) = 1$.

In this case (μ, v) is called an *intuitionistic fuzzy norm*.

As a standard example, we can give the following: Let $(V, \|\cdot\|)$ be a normed space, and let $a * b = ab$ and $a \diamond b = \min\{a + b, 1\}$ for all $a, b \in [0, 1]$. For all $x \in V$ and every $t > 0$, consider

$$\mu_0(x, t) := \frac{t}{t + \|x\|}$$

and

$$v_0(x, t) := \frac{\|x\|}{t + \|x\|}$$

Then observe that $(V, \mu, v, *, \diamond)$ is an intuitionistic fuzzy normed space.

Definition 1.2.27. A sequence $\{x_n\}$ in an intuitionistic fuzzy normed space $(V, \mu, v, *, \diamond)$ is called a Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mu(x_n - x_m, t) > 1 - \varepsilon$$

and

$$v(x_n - x_m, t) < \varepsilon$$

for each $n, m \geq n_0$. The sequence $\{x_n\}$ is said to be convergent to $x \in V$ in intuitionistic fuzzy normed space $(V, \mu, v, *, \diamond)$ and denote by $x_n \xrightarrow{(\mu, v)} x$ if $\mu(x_n - x, t) \rightarrow 1$ and $v(x_n - x, t) \rightarrow 0$ whenever $n \rightarrow \infty$ for every $t > 0$. An intuitionistic fuzzy normed space is said to be complete if and only if every Cauchy sequence is convergent.

Lemma 1.2.28. Let (μ, v) be an intuitionistic fuzzy norm, then

- (i) $\mu(x, t)$ and $v(x, t)$ are nondecreasing and non-increasing with respect to t , respectively.
- (ii) $\mu(x - y, t) = \mu(y - x, t)$ and $v(x - y, t) = v(y - x, t)$ for every $t > 0$.

Definition 1.2.29. Let $(V, \mu, v, *, \diamond)$ be an intuitionistic fuzzy normed space. We define an open ball $B(x, r, t)$ with the center $x \in V$ and the radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in V : \mu(x - y, t) > 1 - r, v(x - y, t) < r\},$$

for every $t > 0$. Also a subset $A \subseteq V$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subseteq A$. Let $\tau_{(\mu, \nu)}$ denote the family of all open subset of V . $\tau_{(\mu, \nu)}$ is called the topology induced by intuitionistic fuzzy norm.

Note that this topology is the same as the topology induced by intuitionistic fuzzy metric sense Park (see (89) Remark 3.3).

Definition 1.2.30. Let A be an intuitionistic fuzzy normed linear space, for all $t > 0$, $\mu(x, t) > 0$ implies $x = 0$. Define $\|x\|_\alpha = \inf\{t > 0 : \mu(x, t) \geq \alpha \text{ and } \nu(x, t) \leq (1 - \alpha)\}$, $\alpha \in (0, 1)$. Then $\{\|x\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on X . We call these norms as α -norms on X corresponding to the intuitionistic fuzzy normed linear space A .

1.2.4 Modified Intuitionistic Fuzzy Normed Space

Lemma 1.2.31. (21). Consider the set L^* and operation \leq_{L^*} defined by:

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1 \text{ and } x_2 \geq y_2$, for every $(x_1, x_2), (y_1, y_2) \in L^*$. Then (L^*, \leq_{L^*}) is a complete lattice.

Definition 1.2.32. For every $z_\alpha = (x_\alpha, y_\alpha) \in L^*$ we define

$$\bigvee(z_\alpha) = (\sup(x_\alpha), \inf(y_\alpha)),$$

$$\bigwedge(z_\alpha) = (\inf(x_\alpha), \sup(y_\alpha))$$

Since $z_\alpha \in L^*$ hence $x_\alpha + y_\alpha \leq 1$ so $\sup(x_\alpha) + \inf(y_\alpha) \leq \sup(x_\alpha + y_\alpha) \leq 1$, i.e., $\vee(z_\alpha), \wedge(z_\alpha) \in L^*$. We denote its units by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$.

From now on, we will assume that if $x \in L^*$, then x_1 and x_2 denote respectively the first and the second component of x , i.e., $x = (x_1, x_2)$.

Using this lattice, we easily see that with every intuitionistic fuzzy set $\mathcal{A} = \{(u, \zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) \mid u \in U\}$ corresponds an L^* -fuzzy set. i.e., a mapping $\mathcal{A} : U \rightarrow L^* : u \mapsto (\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u))$. In the sequel we will use the same notation for an intuitionistic fuzzy set and its associated L^* -fuzzy set. So for the intuitionistic fuzzy set \mathcal{A} we will also use the notation $\mathcal{A}(u) = (\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u))$.

Definition 1.2.33. (4). An intuitionistic fuzzy set $\mathcal{A}_{\zeta, \eta}$ in a universe U is an object $\mathcal{A}_{\zeta, \eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) \mid u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0, 1]$ and $\eta_{\mathcal{A}}(u) \in [0, 1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta, \eta}$; we always have $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Remark 1.2.34. Interpreting intuitionistic fuzzy sets as L^* -fuzzy sets gives way to greater flexibility in calculating with membership and non-membership degrees, since the pair formed by the two degrees is an element of L^* .

Classically a triangular norm $* = T$ on $[0, 1]$ is defined as an increasing, commutative, associative mapping $T : [0, 1]^2 \rightarrow [0, 1]$ satisfying $T(1, x) = 1 * x = x$, for all $x \in [0, 1]$. A triangular conorm $\diamond = S$ is defined as an increasing, commutative, associative mapping $S : [0, 1]^2 \rightarrow [0, 1]$ satisfying $S(0, x) = 0 \diamond x = x$, for all $x \in [0, 1]$. Using the lattice (L^*, \leq_{L^*}) these definitions can be straightforwardly extended.

Definition 1.2.35. (16; 20). A triangular norm (t-norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \rightarrow L^*$ satisfying the following conditions:

$$(\forall x \in L^*) (\mathcal{T}(x, 1_{L^*}) = x), \text{ (boundary condition)}$$

$$(\forall (x, y) \in (L^*)^2) (\mathcal{T}(x, y) = \mathcal{T}(y, x)), \text{ (commutativity)}$$

$$(\forall (x, y, z) \in (L^*)^3) (\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)), \text{ (associativity)}$$

$$(\forall (x, x', y, y') \in (L^*)^4) [x \leq_{L^*} x' \text{ and } y \leq_{L^*} y' \implies \mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(x', y')].$$

(monotonicity)

Definition 1.2.36. (15). A continuous t-norm \mathcal{T} on L^* is called continuous t-representable if and only if there exist a continuous t-norm $*$ and a continuous t-conorm \diamond on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

Now define a sequence \mathcal{T}^n recursively by $\mathcal{T}^1 = \mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)}, \dots, x^{(n+1)}) = \mathcal{T}(\mathcal{T}^{n-1}(x^{(1)}, \dots, x^{(n)}), x^{(n+1)})$$

for $n \geq 2$ and $x^{(i)} \in L^*$.

We say the continuous t-representable norm is *natural* and write \mathcal{T}_n whenever $\mathcal{T}_n(a, b) = \mathcal{T}_n(c, d)$ and $a \leq_{L^*} c$ implies $b \geq_{L^*} d$.

Definition 1.2.37. (16; 20). A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \rightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$,

then \mathcal{N} is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $N: [0, 1] \rightarrow [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0, 1]$ defined as $N_s(x) = 1 - x$ for all $x \in [0, 1]$.

Remark 1.2.38. The mapping \mathcal{N}_s defined by $\mathcal{N}_s(x_1, x_2) = (x_2, x_1)$, for all $(x_1, x_2) \in L^*$, will be called the standard negator.

Definition 1.2.39. (58). Let μ, ν be fuzzy sets from $V \times (0, +\infty)$ to $[0, 1]$ such that $\mu(x, t) + \nu(x, t) \leq 1$ for all $x \in V$ and $t > 0$. The 3-tuple $(V, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed space* if V is a vector space, \mathcal{T} is a continuous t -representable norm and $\mathcal{P}_{\mu, \nu}$ is a mapping $V \times (0, +\infty) \rightarrow L^*$ satisfying the following conditions for every $x, y \in V$ and $t, s > 0$:

- (a) $\mathcal{P}_{\mu, \nu}(x, t) >_{L^*} 0_{L^*}$;
- (b) $\mathcal{P}_{\mu, \nu}(x, t) = 1_{L^*}$ if and only if $x = 0$;
- (c) $\mathcal{P}_{\mu, \nu}(\alpha x, t) = \mathcal{P}_{\mu, \nu}(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;
- (d) $\mathcal{P}_{\mu, \nu}(x + y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{P}_{\mu, \nu}(x, t), \mathcal{P}_{\mu, \nu}(y, s))$;
- (e) $\mathcal{P}_{\mu, \nu}(x, \cdot) : (0, \infty) \rightarrow L^*$ is continuous;
- (f) $\lim_{t \rightarrow \infty} \mathcal{P}_{\mu, \nu}(x, t) = 1_{L^*}$ and $\lim_{t \rightarrow 0} \mathcal{P}_{\mu, \nu}(x, t) = 0_{L^*}$.

Then $\mathcal{P}_{\mu, \nu}$ is called an *intuitionistic fuzzy norm*. Here,

$$\mathcal{P}_{\mu, \nu}(x, t) = (\mu(x, t), \nu(x, t)).$$

Lemma 1.2.40. (95). Let $\mathcal{P}_{\mu,\nu}$ be an intuitionistic fuzzy norm. Then, for any $t > 0$, the following hold:

(i) $\mathcal{P}_{\mu,\nu}(x, t)$ is nondecreasing with respect to t , in (L^*, \leq_{L^*}) .

(ii) $\mathcal{P}_{\mu,\nu}(x - y, t) = \mathcal{P}_{\mu,\nu}(y - x, t)$.

Example 1.2.41. (58). Let $(V, \|\cdot\|)$ be a normed space and let $\mathcal{I}(a, b) = (a_1 b_1, \min(a_2 + 2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$. Now let μ, ν be fuzzy sets in $V \times (0, \infty)$ and define

$$\mathcal{P}_{\mu,\nu}(x, t) = (\mu(x, t), \nu(x, t)) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right),$$

for all $t \in \mathbb{R}^+$. Then $(V, \mathcal{P}_{\mu,\nu}, \mathcal{I})$ is an intuitionistic fuzzy normed space.

Definition 1.2.42. Let $(V, \mathcal{P}_{\mu,\nu}, \mathcal{I})$ be an intuitionistic fuzzy normed space. For $t > 0$, define the *open ball* and *closed ball* $B(x, r, t)$ and $B[x, r, t]$ with center $x \in V$ and radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in V : \mathcal{P}_{\mu,\nu}(x - y, t) >_{L^*} (N_s(r), r)\},$$

$$B[x, r, t] = \{y \in V : \mathcal{P}_{\mu,\nu}(x - y, t) \geq_{L^*} (N_s(r), r)\}.$$

Remark 1.2.43. In (58), it was shown that every intuitionistic fuzzy norm induces an intuitionistic fuzzy metric and so every intuitionistic fuzzy normed space is a topological space $(\tau_{\mu,\nu})$.

Lemma 1.2.44. (95). Let $(V, \mathcal{P}_{\mu,\nu}, \mathcal{I})$ be an intuitionistic fuzzy normed space. Then $\mathcal{P}_{\mu,\nu} : V \times (0, \infty) \rightarrow [0, 1]$ is continuous.

Lemma 1.2.45. (95). A subset A of \mathbb{R} is IF-bounded in $(\mathbb{R}, \mathcal{P}_{\mu_0, \nu_0}, \mathcal{I})$ if and only if it is bounded in \mathbb{R} .

1.3 Literature Review

1.3.1 Origin of Fuzziness

Fuzzy sets were introduced by Zadeh in 1965 to manipulate data and information possessing non-statistical uncertainties. It was specifically designed to represent uncertainty and vagueness mathematically and to provide formalized tools for dealing with the imprecision, intrinsic to many problems. The first publication in fuzzy set theory by Zadeh (111) and then by Goguen (51; 52) show the intention of the authors to generalize the classical set. In classical set theory, a subset A of a set X can be defined by its characteristic function χ_A as a mapping from the elements of the universal set X to the values of the set $\{0, 1\} : \chi_A \rightarrow \{0, 1\}$.

The mapping may be represented as a set of ordered pairs $\{x, \chi_A(x)\}$ with exactly one ordered pair present for each element of X . The first element of the ordered pair is an element of the set X and the second is its value in $\{0, 1\}$. The value "0" is used to represent non-membership and the value "1" is used to represent membership of the element A . The truth and falsity of the statement " x is in A " is determined by the ordered pair. The statement is true, if the second element of the ordered pair is "1", and the statement is false, if it is "0". Similarly, a fuzzy set A of a set X can be defined as a set of ordered pairs $\{(x, \mu_A(x)) : x \in X\}$, each with the first element from X and the second element from the unit interval $[0, 1]$ with exactly one ordered pair present for each element of X . This defines a mapping μ_A between elements of the set X and values in the interval $[0, 1] : \mu_A : X \rightarrow [0, 1]$. The value "0" is used to represent complete non-membership, the value "1" is used to represent complete membership and values in between are used to represent intermediate degree of

membership. The set X is referred to as universe of discourse for the fuzzy set A . Frequently, the mapping μ_A is described as a function, the membership function of A , the degree to which the statement " x is in A " is true, is determined by finding the ordered pair $(x, \mu_A(x))$. The degree of truth of the statement is the second element of the ordered pair.

1.3.2 Intuitionistic Fuzzy Sets

The theory of fuzzy sets proposed by Zadeh (111) has successfully applied in numerous fields such as engineering, finance, biology, and etc. In fuzzy set theory, the degree of belonging of element to the set is represented by a membership value in the real interval $[0,1]$ and there exists degree of non-membership which is complementary in nature. From latter point of view, it is true and acceptable that grade of membership and non-membership are complementary. Conversely in Atanassov (4) critical sense, some hesitation degree needs to be introduced in the concept of Intuitionistic Fuzzy Sets. While, Bustince and Burillo (11) found that this notion coincides with the notion of vague sets proposed by Gau and Buehrer (40), the Intuitionistic fuzzy sets make description of the objective world become more realistic, practical, and accurate, making it very promising. Instead of using fuzzy approach, past researchers have studied Intuitionistic fuzzy sets to be applied in variety area such as decision making problems (104), medical diagnostics (17) and pattern recognition (18) and seem to be more popular than fuzzy sets in recent years.

1.3.3 Approximation Theory in Normed and Fuzzy Normed Spaces

The concept of 2-norm and n -norm on a linear space has been introduced and developed by Gähler in (39; 38). Following Misiak (83), Kim and Cho (69) and Malčeski (76) developed the theory of n -normed space. In (54), Gunawan and Mashadi gave a simple way to derive an $(n - 1)$ norm from the n -norm and realized that any n -normed space is an $(n - 1)$ normed space. Proximality and Chebyshevity results for best approximation in linear 2-normed spaces have been obtained by S. Elumalai and R. Ravi (31), S. Elumalai et. al., (32) I. Franić (36), S. S. Kim and Y. J. Cho (70), S. A. Mariadoss (77), R. Ravi (92).

The problems of best approximation were initiated first by P. L. Chebyshev (13) in 1853. Singer (100) wrote about best approximation in normed spaces. One can see that in our days many mathematicians are interested in best approximation, such as, Geetha S. Rao and Saravana and Mazaheri (78; 79; 80; 44). Brosowski (8) and Meinardus (81) established some interesting results on invariant approximations in normed spaces using fixed point theory. Later, Jungck and Sessa (61) have obtained some results on approximation theory in the setting of normed spaces. The theory of best simultaneous approximation has been studied by many authors, e.g., (14; 23; 26; 45; 49; 50; 74; 75; 82; 90; 97; 98; 105; 106; 110). Diaz and McLaughlin (23; 24), Dunham (26) and Ling et al. (74) have considered the simultaneous approximation of two real valued functions defined on the closed interval $[a, b]$. The characterizations of best simultaneous approximations from a finite-dimensional subspace of a normed linear space can be found in (101). The concept of best coapproximation was introduced by Franchetti and Furi (37), in order to

study some characteristic properties of real Hilbert spaces among real reflexive Banach spaces, and such problems were considered further by Papini and Singer, (87) and Rao and Saravanan (91). This concept was named best coapproximation by Papini and Singer (87). Subsequently, Geetha S. Rao et. al have developed the theory of best coapproximation to a considerable extent (41; 42; 43). Further there are some results on coapproximation in (78; 85)

Geetha S. Rao and Saravanan (45) obtained some theorems on best coapproximation in quotient spaces. Modarres and Dehghani (84) introduced and discussed the concept of the best simultaneous coapproximation in normed linear spaces, which is the generalization of best coapproximation in normed spaces and also studied best simultaneous coapproximation in quotient spaces.

One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space and fuzzy normed spaces. This problem has been investigated by many authors (19; 33; 46; 62; 72) from different point of view. In particular, George and Veeramani (46) had introduced and studied the notion of fuzzy metric space with the help of continuous t-norms, which constitutes a slight but appealing modification of the one due to Kramosil and Michalak (72).

The concept of fuzzy norm was initiated by Katsaras in (63) and further, Narayanan and Vijayabalaji (86) introduced the concept of fuzzy n -normed linear space. Moreover, Vijayabalaji and Thillaigovindan (109) introduced the notion of convergent sequence and Cauchy sequence in fuzzy n -normed linear space. In (59) Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the

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