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NEWTON'S METHOD USING  
SPREADSHEET EXCEL**

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# SOLVING NON-LINEAR SYSTEMS BY NEWTON'S METHOD USING SPREADSHEET EXCEL

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### Abstract

Tay (2006) had proposed solving numerical methods using the Casio fx-570MS calculator to overcome the tediousness of doing recursive calculations. Here, we present another alternative, that is solving a non-linear system using Newton's method in Microsoft Office Excel. For this, we just make use of the MULT function to do matrix multiplication and MINVERSE function to do the matrix inverse operation. The concept presented here can be developed into a solver where the user just needs to input the initial vector  $X$ , the corresponding formula of Jacobian matrix  $J(x, y)$  and non-linear system vector  $F(x, y)$ . The full solutions will be displayed automatically. The solver is easy and user friendly for students and educator who needs its full solutions quickly.

### Introduction

Many real-life problems are non-linear. A couple of non-linear equations form a system of non-linear equations. Systems of non-linear equations can be solved numerically by Newton's method, fixed-point iteration method. Solving a system of non-linear equations by Newton's method is tedious. Tay (2006) proposed a step by step instruction on how to use Casio fx-570MS in Numerical methods, while Guerreri-Garcia and Santos-Polomo (2008) presented a collection of keystroke sequences helping in solving numerical methods. Tay and Kek (2008) developed a solver to solve the most dominant eigenvalue by the power method through Microsoft Office Excel. On the other hand, Kek and Tay (2008) developed a solver to solve system of linear equations using Excel too. None of the literature works dealt with a system of non-linear equations. Hence, in this paper, we have developed a solver for solving system of non-linear equations by using Newton's method through Excel 2003.

### Newton's Method

Given the continuous functions

$$\begin{aligned} f(x, y) &= 0, \\ g(x, y) &= 0. \end{aligned} \tag{1}$$

we would like to find the values  $x = x^*$  and  $y = y^*$  such that  $f(x^*, y^*) = 0$  and  $g(x^*, y^*) = 0$ .

Assume that an initial approximate solution to equation (1) is known as  $(x_i, y_i)$ . Express  $f(x, y)$  and  $g(x, y)$  in two-variables Taylor series about  $(x_i, y_i)$ , we get

$$\begin{aligned} f(x_{i+1}, y_{i+1}) &= f(x_i, y_i) + \frac{\partial}{\partial x} f(x_i, y_i)(x_{i+1} - x_i) + \frac{\partial}{\partial y} f(x_i, y_i)(y_{i+1} - y_i) + \dots = 0, \\ g(x_{i+1}, y_{i+1}) &= g(x_i, y_i) + \frac{\partial}{\partial x} g(x_i, y_i)(x_{i+1} - x_i) + \frac{\partial}{\partial y} g(x_i, y_i)(y_{i+1} - y_i) + \dots = 0 \end{aligned} \tag{2}$$

$f(x_{i+1}, y_{i+1}) = 0$  and  $g(x_{i+1}, y_{i+1}) = 0$  because will be the roots. Truncating Eq. (2) after the first derivatives terms and rearranging yields

$$\begin{aligned} -f(x_i, y_i) &= \frac{\partial}{\partial x} f(x_i, y_i) \Delta x_i + \frac{\partial}{\partial y} f(x_i, y_i) \Delta y_i, \\ -g(x_i, y_i) &= \frac{\partial}{\partial x} g(x_i, y_i) \Delta x_i + \frac{\partial}{\partial y} g(x_i, y_i) \Delta y_i, \end{aligned} \quad (3)$$

where  $\Delta x_i = x_{i+1} - x_i$ ,  $\Delta y_i = y_{i+1} - y_i$ ,

Writing Eq. (3) in matrix notation yields the small changes below:

$$\begin{pmatrix} \frac{\partial}{\partial x} f(x_i, y_i) & \frac{\partial}{\partial y} f(x_i, y_i) \\ \frac{\partial}{\partial x} g(x_i, y_i) & \frac{\partial}{\partial y} g(x_i, y_i) \end{pmatrix} \begin{pmatrix} \Delta x_i \\ \Delta y_i \end{pmatrix} = \begin{pmatrix} -f(x_i, y_i) \\ -g(x_i, y_i) \end{pmatrix} \quad (4)$$

$$J(x_i, y_i) \Delta \mathbf{X}_i = -\mathbf{F}(x_i, y_i),$$

$$\Delta \mathbf{X}_i = -J(x_i, y_i)^{-1} \mathbf{F}(x_i, y_i),$$

where  $J(x, y)$  is the Jacobian matrix.

Thus the new approximation solution will be

$$x_{i+1} = x_i + \Delta x_i,$$

$$y_{i+1} = y_i + \Delta y_i,$$

$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \begin{pmatrix} \Delta x_i \\ \Delta y_i \end{pmatrix}$$

or equivalently

$$\mathbf{X}_{i+1} = \mathbf{X}_i + \Delta \mathbf{X}_i.$$

#### Procedure

for solving nonlinear system

$$f(x, y) = 0,$$

$$g(x, y) = 0.$$

by Newton's method.

**Step 1:** Obtain Jacobian matrix,  $J(x, y) = \begin{pmatrix} \frac{\partial}{\partial x} f(x_i, y_i) & \frac{\partial}{\partial y} f(x_i, y_i) \\ \frac{\partial}{\partial x} g(x_i, y_i) & \frac{\partial}{\partial y} g(x_i, y_i) \end{pmatrix}$

**Step 2:** Obtain  $\mathbf{F}(x_i, y_i) = \begin{pmatrix} f(x_i, y_i) \\ g(x_i, y_i) \end{pmatrix}$

**Step 3:** Solve  $J(x_i, y_i) \Delta \mathbf{X}_i = -\mathbf{F}(x_i, y_i)$  for  $\Delta \mathbf{X}_i$ .

**Step 4:** Calculate  $\mathbf{X}_{i+1} = \mathbf{X}_i + \Delta \mathbf{X}_i$ .

**Step 5:** Repeat Steps 3 and 4 until  $|\mathbf{X}_{i+1} - \mathbf{X}_i| < \varepsilon$ .

**Step 6:** Roots is  $\mathbf{X}_{i+1}$ .

### Numerical Example

Given the system of non-linear equation below:

$$f(x, y) = x^2 - y - 1 = 0,$$

$$g(x, y) = y^2 - x = 0.$$

Their graphs are shown in the Figure 1 below:

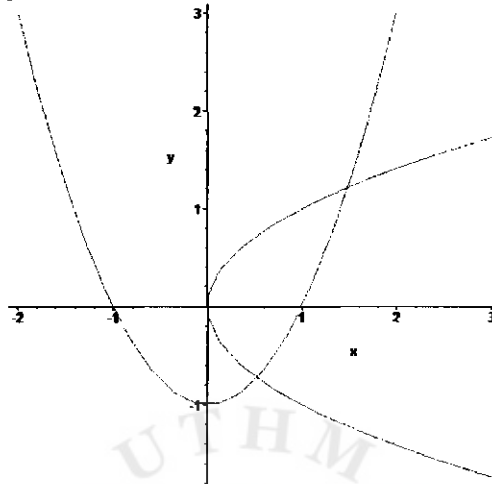


Figure 1: Graph of  $f(x, y)$  and  $g(x, y)$

Solve the system of non-linear equation above by using Newton's method with initial guess  $(0, -1)$  and  $(1, 1)$ .

*Solution*

**Step 1**

$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & -1 \\ -1 & 2y \end{bmatrix}$$

**Step 2**

$$F(x, y) = \begin{bmatrix} x^2 - y - 1 \\ y^2 - x \end{bmatrix}$$

**Step 3**

By using  $\mathbf{X}_{i+1} = \mathbf{X}_i + \Delta\mathbf{X}_i$  where  $\Delta\mathbf{X}_i = -J(x_i, y_i)^{-1}\mathbf{F}(x_i, y_i)$ , we obtain the numerical solution with initial guesses  $(0, -1)$  and  $(1, 1)$  as in Tables 1 and 2, respectively.

Table 1

*Initial guess  $(0, -1)$*

$i$	$\mathbf{X}_i$	$J(x_i, y_i)$	$\mathbf{F}(x_i, y_i)$	$\Delta\mathbf{X}_i$
0	0	0	0	1
	-1	1	1	0

1	1	2	-1	1	-0.4
	-1	-1	-2	0	0.2
2	0.6			0.16	-0.074
	-0.8			0.04	0.0712
3	0.526	1.052055	-1	0.0055	-0.001
	-0.729	-1	-1.45753	0.0051	0.0043
4	0.5249			1E-06	6E-06
	-0.725			2E-05	8E-06
5	0.5249	1.049777	-1	4E-11	2E-12
	-0.724	-1	-1.44898	7E-11	4E-11
6	0.5249			0	0
	-0.724			0	0

Since  $|X_6 - X_5| = 0.0000 < \varepsilon = 0.0005$ , so one of the root is (0.5249, -0.7240).

Table 2  
Initial guess (1, 1)

i	$X_i$	$J(x_i, y_i)$	$F(x_i, y_i)$	$\Delta X_i$
0	1		-1	0.6667
	1		0	0.3333
1	1.6667	3.333333	-1	0.4444
	1.3333	-1	2.666667	0.1111
2	1.5023	3.004695	-1	0.027
	1.23		2.60094	0.0107
3	1.4903	2.980573	-1	0.0001
	1.2208	-1	2.441616	9E-05
4	1.4902	2.980437	-1	5E-09
	1.2207	-1	2.441488	4E-09

Since  $|X_4 - X_3| = 0.0001 < \varepsilon = 0.0005$ , so the other root is (1.4902, 1.2207).

### Steps In Using The Solver For Solving The System Of Non-Linear Equation

**Step 1:** User input the initial vector  $X_0$ .

**Step 2:** User input formula for initial Jacobian matrix,  $J(x_0, y_0)$ .

**Step 3:** User input formula for initial vector,  $F(x_0, y_0)$ .

**Step 3:** The results will be automatically displayed up to 10 iterations.

**Step 4:** User needs to copy and drag the iteration if the stopping criteria is not met yet.

**Step 5:** User needs to cut the appropriate number of iterations if the number of iterations is less than 10 iterations.

Figure 1 shows the solver in the Excel environment:

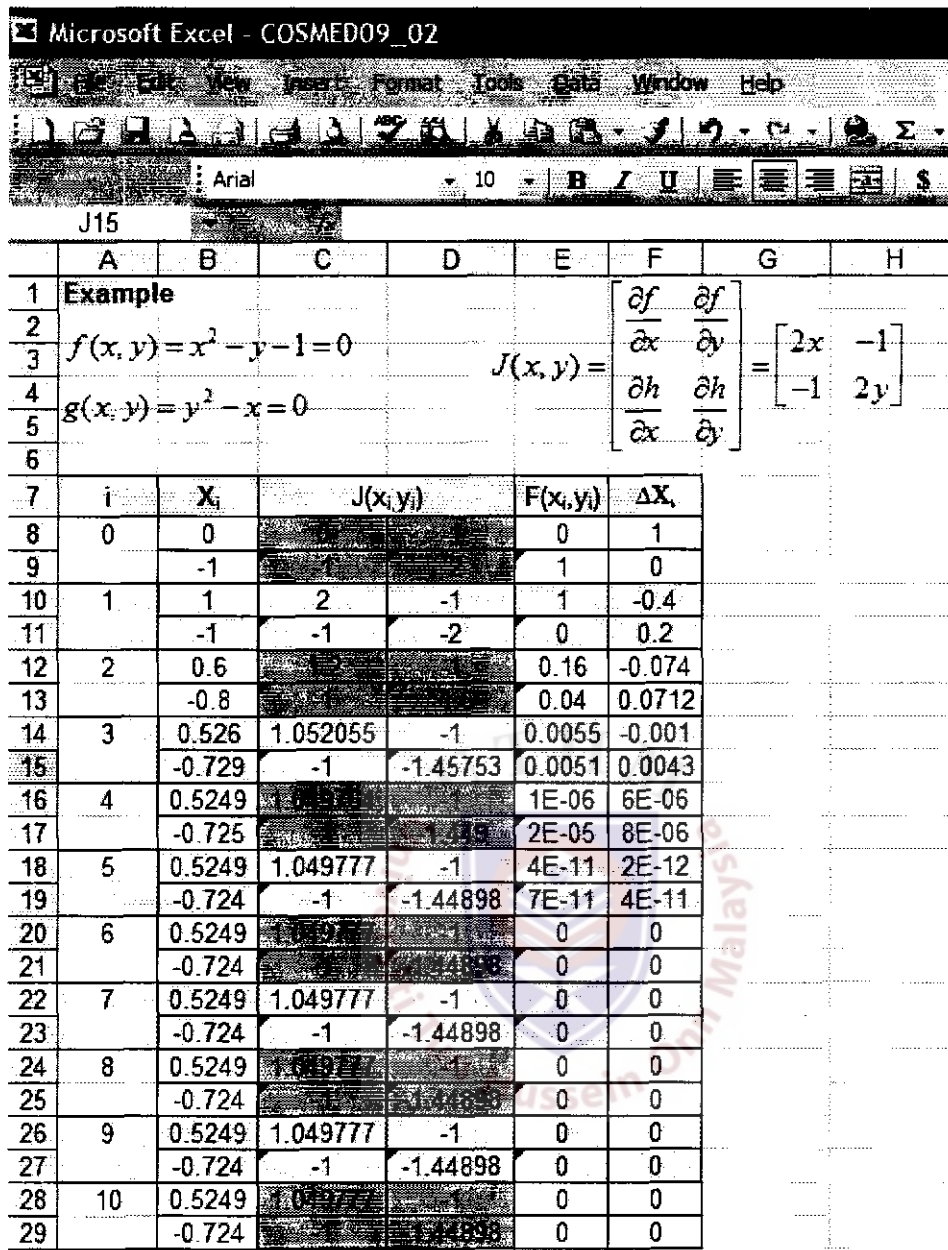


Figure 1. Excel Solver for Solving the System of Non-Linear Equations

### The Excel Command Involved In Building the Solver

- Step 1:** We select cells F8:F9 and type in = -MMULT(MINVERSE(C8:D9), E8:E9), then press SHIFT, CTRL and ENTER at the same time to calculate  $\Delta X_0$ .
- Step 2:** We select cells B10:B11 and type in = B8:B9+F8:F9 and then press SHIFT, CTRL and ENTER at the same time to calculate  $X_1$ .
- Step 3:** We then select cells C8:F9 and then copy it to F11.
- Step 4:** Finally we select A10:F11 and then copy them till F29 to get the full solution as shown in Figure 1.

## Advantages and Shortcomes of The Solver

This solver helps educators and students who need the full numerical solutions of a system of non-linear equations by inputting the initial vector  $X_0$ , Jacobian matrix  $J(x_0, y_0)$  and initial vector  $F(x_0, y_0)$ . Even though it is designed for a system of two non-linear equations, the EXCEL command involved is not that difficult if compared to the MATHCAD, MAPLE, MATLAB, or C Programming. Hence users can extend the concept presented here to a system of  $n$  non-linear equations. One of the limitations here is the solver is not yet fully automated so that the users need to control the number of iterations themselves manually by dragging to increase the number of iterations or select only the desired number of iterations from the given ten iterations. However, this manual option is suitable for the purpose of teaching and learning in the class room and evaluation in the examination hall, where students need to know where to stop the iterations.

## Conclusion

A solver is developed to get the full numerical solutions of a system of two non-linear equations by Newton's method through Excel. This solver helps educators and students who need full numerical solutions of a system of two non-linear equations by just inputting the initial vector  $X_0$ , Jacobian matrix  $J(x_0, y_0)$  and initial vector  $F(x_0, y_0)$ , but the solver is not yet fully automated where user needs to control the number of iterations themselves manually. So, we intend to study the Microsoft Excel VBA language to overcome this shortage in future.

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