A COMPARISON OF NUMERICAL METHODS FOR SOLVING THE BRATU AND BRATU-TYPE PROBLEMS

RUHAILA MD. KASMANI

UNIVERSITI TEKNOLOGI MALAYSIA

UNIVERSITI TEKNOLOGI MALAYSIA

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A COMPARISON OF NUMERICAL METHODS FOR SOLVING THE BRATU AND BRATU-TYPE PROBLEMS

RUHAILA MD.KASMANI

A dissertation submitted in partial fulfilment of the

requirements for the award of the degree of

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Faculty of Science

Universiti Teknologi Malaysia

MARCH 2005

I declare that this thesis entitled " *A Comparison of Numerical Methods for Solving the Bratu and Bratu-Type Problems* " is the result of my own research except as cited in the references. The thesis has not been accepted for any degree and is not concurrently submitted in candidature of any degree.

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Name : Ruhaila Md.Kasmani

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To my beloved family and friends.

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ABSTRACT

The Bratu problem $u''(x) + \lambda e^{u(x)} = 0$ with $u(0) = u(1) = 0$ has two exact solutions for values of $0 < \lambda < \lambda_c$, no solutions if $\lambda > \lambda_c$ while unique solution is obtained when $\lambda = \lambda_c$ where $\lambda_c = 3.513830719$ is the critical value. The First Bratu-Type problem corresponds $\lambda = -\pi^2$ while the Second Bratu-Type problem is $u''(x) + \pi^2 e^{-u(x)} = 0$. The exact solution of the First Bratu-Type problem blows up at $x = 0.5$ while the Second Bratu-Type problem is continuous. The present work seeks to compare various numerical methods for solving the Bratu and Bratu-Type problems. The numerical methods are the standard Adomian decomposition method, the modified Adomian decomposition method, the shooting method and the finite difference method. These methods are implemented using Maple. Convergence is achieved by applying the four methods when $0 < \lambda \leq 2$, however the shooting method is the most effective method as it gives the smallest maximum absolute error. When $\lambda = \lambda_c$, none of these methods give the convergence solutions. Due to the nature of the solution of the First Bratu-Type problem, only the shooting method and the modified Adomian decomposition method can give the convergence values to the exact solution. The finite difference method is proved to be the most effective method for the Second Bratu-Type problem compared to other methods.

Keywords: Bratu problem, Bratu-Type problems, standard Adomian decomposition method, modified Adomian decomposition method, shooting method, finite difference method.

ABSTRAK

Masalah Bratu $u''(x)+\lambda e^{u(x)} = 0$ dengan syarat sempadan $u(0) = u(1) = 0$ mempunyai dua penyelesaian bagi $0 < \lambda < \lambda_c$, tiada penyelesaian jika $\lambda > \lambda_c$ dan penyelesaian unik jika $\lambda = \lambda_c$ di mana $\lambda_c = 3.513830719$ adalah merupakan nilai genting. Masalah Jenis-Bratu Pertama mengambil nilai $\lambda = -\pi^2$ manakala masalah Jenis-Bratu Kedua adalah $u''(x) + \pi^2 e^{-u(x)} = 0$. Penyelesaian sebenar bagi masalah Jenis-Bratu Pertama meningkat secara mendadak pada titik $x = 0.5$ manakala penyelesaian sebenar masalah Jenis-Bratu Kedua adalah selanjar. Disertasi ini melaporkan mengenai perbandingan di antara beberapa kaedah berangka bagi menyelesaikan masalah Bratu dan Jenis-Bratu. Perbandingan ini melibatkan penggunaan empat kaedah iaitu kaedah penghuraian Adomian asal, kaedah penghuraian Adomian terubahsuai, kaedah penembakan dan kaedah pembeza terhingga. Setiap penyelesaian berangka dilaksanakan dengan menggunakan Maple. Bagi kes $0 < \lambda \leq 2$, keempat-empat kaedah tersebut telah memberikan penyelesaian berangka yang menumpu. Didapati kaedah penembakan merupakan kaedah yang paling efektif. Hanya kaedah penembakan dan kaedah penghuraian Adomian terubahsuai telah menunjukkan penyelesaian menumpu bagi masalah Jenis-Bratu Pertama. Kaedah pembeza terhingga merupakan kaedah yang paling efektif bagi mendapatkan penyelesaian berangka untuk masalah Jenis-Bratu Kedua berbanding kaedah yang lain.

Kata kunci: Masalah Bratu, masalah Jenis-Bratu, kaedah penghuraian Adomian asal, kaedah penghuraian Adomian terubahsuai, kaedah penembakan, kaedah pembeza terhingga.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

When mathematical modeling is used to describe physical, biological or chemical phenomena, one of the most common results of the modeling process is a system of partial differential equations.

The Bratu problem is a partial differential equation which appears in a number of applications such as the steady state model of the solid fuel ignition in thermal combustion theory and the Chandrasekhar model of the expansion of the universe. The former model stimulates a thermal reaction process in a rigid material, where the process depends on a balance between chemically generated heat addition and heat transfer by conduction (Averick *et al.*, 1992).

The classical Bratu problem can be described as follows:

$$\Delta u + \lambda e^{u} = 0 \quad \text{on} \quad \Omega : \{(x, y) \in 0 \leq x \leq 1 \ , \ 0 \leq y \leq 1\} \qquad (1.1)$$

with

$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where Δ is the Laplace operator and Ω is a bounded domain in $\Re^2$. According to Jacobsen and Schmitt (2001), equation (1.1) arises in the study of the quasilinear parabolic problem :

$$\begin{aligned} \nu_t &= \Delta\nu + \lambda(1-\varepsilon\nu)^m e^{\nu/(1+\varepsilon\nu)}, \quad x \in \Omega \qquad (1.2) \\ \nu &= 0, \quad x \in \partial\Omega, \end{aligned}$$

which is also known as the solid fuel ignition model and is derived as a model for the thermal reaction process in a combustible, nondeformable material of constant density during the ignition period. Here λ is known as the *Frank-Kamenetskii* parameter, ν is a dimensionless temperature and $\frac{1}{\varepsilon}$ is the activation energy.

The derivation of equation (1.2) from general principles is accounted in the comprehensive work by Frank and Kamenetskii in year 1955, who are interested in what happens when combustible medium is placed in a vessel whose walls are maintained at a fixed temperature. Intuitively, they expected that for a large value of λ, the reaction term will dominate and drive the temperature to infinity (explosion) whereas for smaller λ , the steady state might be possible.

Boyd (1986) developed a pseudo spectral method to generate approximate solutions to the classical two-dimensional planar Bratu problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda e^u = 0 \qquad (1.3)$$

on

$$\{(x,y) \in -1 \le x \le 1 \ , \ -1 \le y \le 1\},$$

with $u = 0$ on the boundary of the square. The basic idea is that the unknown solution $u(x,y)$ can be completely represented as an infinite series of spectral basis functions

$$u(x,y) \;=\; \sum_{k=1}^{N} a_k \phi_k(x,y). \tag{1.4}$$

The basis functions $\phi_k(x,y)$ are chosen so that they obey the boundary conditions and have the property that the more terms of the series are kept, the more accurate the representation of the solution $u(x,y)$ is. In other words, as $N \to \infty$ the error diminishes to zero. For finite N, the series expansion in (1.4) is substituted into (1.3) to produce the residual R. The residual function will depend on the spatial variables x, y, the unknown coefficients a_k and the parameter λ. The goal of Boyd's pseudo spectral method is to find a_k so that the residual function R is zero at N collocation points. The collocation points are usually chosen to be the roots of orthogonal polynomials that fall into the same family as the basis functions $\phi_k(x,y)$. Boyd (1986) uses the Gegenbauer polynomials to define the collocation points. The Gegenbauer polynomials are orthogonal on the interval $[-1,1]$ with respect to the weight function $w(x) = (1-x^2)^b$ where $b = -\frac{1}{2}$ corresponds to the Chebyshev polynomials and $b = 1$ is the choice Boyd (1986) uses. The second order Gegenbauer polynomial is

$$G_2(x) \;=\; \frac{3}{2}(5x^2-1), \qquad -1 \le x \le 1.$$

Using a 1-point collocation method at the point $x_1 = \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ and the choice of $\phi_1(x,y) = (1-x^2)(1-y^2)$, Boyd is able to obtain an approximation to the value of λ_c with a relative error of 8%. Note that this choice for $\phi_1(x,y)$ satisfies the boundary conditions since $\phi(1,y) = \phi(-1,y) = \phi(x,-1) = \phi(x,1) = 0$. The solution produced by Boyd's pseudo spectral does not have the deficiency of being unable to converge to both solutions of the Bratu problem for $\lambda < \lambda_c$.

The Bratu problem in one-dimensional planar coordinates is also often used as a benchmarking tool for numerical methods. The one-dimensional of this problem is

$$u''(x) + \lambda e^{u(x)} = 0, \quad 0 \leq x \leq 1, \tag{1.5}$$

with the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 0.$$

The nonlinear eigenvalue problem (1.5) has two known bifurcated exact solutions for values of $\lambda < \lambda_c$, no solutions for $\lambda > \lambda_c$ and a unique solution when $\lambda = \lambda_c$, where $\lambda_c = 3.513830719$ is denoted as the critical value (Buckmire, 2003).

In Aregbesola (1996), the method of weighted residuals was used to solve the Bratu problem (1.5). The idea is to approximate the solution with a polynomial involving a set of parameters. The polynomial is of the form

$$V(x) = \Phi_0(x) + \sum_{i=1}^{N} A_i \Phi_i(x),$$

where $\Phi_0(x)$ satisfies the given boundary conditions and each $\Phi_i(x)$ satisfies the homogenous form of the boundary conditions. The function $V(x)$ is then used as an approximation to the exact solution in the equation

$$L(U(x)) = Q(x)$$

to give

$$R(x) = L(U(x)) - Q(x).$$

where the function $R(x)$ is the residual. The aim is to make $R(x)$ as small as possible. One of the methods of minimizing $R(x)$ is the collocation method where $R(x)$ is set to zero at some points in the interval. The system of the resulting nonlinear equations is then solved to determine the parameters A_i. The polynomial $V(x)$ is then considered as the approximate solution. The weighted residual method provides accurate results and was found suitable for bifurcation problems.

The availability of the exact solution of the Bratu problem (1.5) together with universal applicability of the standard finite difference method, provides an important application of the nonstandard finite difference method. Solving a boundary value problem using the standard and nonstandard finite difference methods involve replacing each of the derivatives by an appropriate difference-quotient approximation. The interval $[a, b]$ is divided into N equal subintervals where

$$a = x_0 < x_1 < x_2 < \ldots < x_j < \cdots < x_N = b.$$

For a uniform subintervals, the step size h is constant and $h = \dfrac{1}{N}$ with $x_i = a+ih$ for $i = 0, 1, 2, \ldots, N$. The approximation of the second derivative by using the centered-difference formula is

$$u'' \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \tag{1.6}$$

However in the nonstandard finite difference method, the denominator of (1.6), h^2 is replaced by the denominator function $\phi(h)$. Therefore, the nonstandard finite difference method for the second derivative is

$$u'' \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\phi(h)}, \tag{1.7}$$

where the denominator function $\phi(h)$ has the property that $\phi(h) = h^2 + O(h^2)$. Buckmire (2003) has employed the standard finite difference method and the nonstandard finite difference method for solving the Bratu problem (1.5). Using the standard finite difference method, the discrete version of the Bratu problem (1.5) is

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \lambda e^{u_i} = 0, \quad i = 1, 2, \ldots, N-1. \tag{1.8}$$

The nonstandard finite difference method for solving the Bratu problem (1.5) is

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{2\ln[\cosh(h)]} + \lambda e^{u_i} = 0, \quad i = 1, 2, \ldots, N-1, \tag{1.9}$$

where the denominator function, $\phi(h) = 2\ln[\cosh(h)] = h^2 + O(h^2)$. Thus, in the limit as $h \to 0$, the standard finite difference method (1.8) and the nonstandard finite difference method (1.9) will be identical.

Buckmire extended his research in the application of nonstandard finite difference scheme to the cylindrical Bratu-Gelfand problem. The cylindrical Bratu-Gelfand is a particular boundary value problem related to the classical Bratu problem (1.1), with cylindrical radial operator. Jacobsen and Schmitt (2002) considered the nature of solutions to a version of the classical Bratu problem (1.1) generalized to more complicated operators in more dimensions that they called the Liouville-Bratu-Gelfand problem. The Liouville-Bratu-Gelfand problem for the class of quasilinear elliptic equations is defined by

$$\begin{aligned} r^{-\gamma}(r^{\alpha} \mid u' \mid^{\beta} u')' + \lambda e^{u} &= 0, \quad 0 < r < 1, \\ u &> 0, \\ u'(0) = u(1) &= 0, \end{aligned} \tag{1.10}$$

where the inequalities $\alpha \leq 0$, $\gamma + 1 > \alpha$ and $\beta + 1 > 0$ hold. The Bratu-Gelfand problem to be considered by Buckmire (2003) is the special case when $\alpha = 1$, $\beta = 0$, $\gamma = 1$:

$$\begin{aligned} \frac{1}{r}(ru')' + \lambda e^{u} &= 0, \quad 0 < r < 1, \\ u &> 0, \\ u'(0) = u(1) &= 0. \end{aligned} \tag{1.11}$$

The assumption has been made that $u = u(r)$ in order to the other derivatives in Laplacian can be ignored. In his previous works (Buckmire (1996) and (2003)), he has shown that the usefulness in applying a particular nonstandard finite difference scheme to boundary value problems in cylindrical coordinates that contain the expression of $r(\frac{du}{dr})$. The expression $r(\frac{du}{dr})$ is then approximated by the forward difference formula,

$$r\frac{du}{dr} \approx r_k \frac{u_{k+1} - u_k}{r_{k+1} - r_k}. \tag{1.12}$$

However, the following nonstandard finite difference scheme has been shown (Buckmire, 2003) to be a superior method, especially for singular problems where $r \to 0$:

$$r\frac{du}{dr} \approx \frac{u_{k+1} - u_k}{\log(r_{k+1}) - \log(r_k)}. \tag{1.13}$$

Using the approximation in (1.12), the Bratu-Gelfand problem (1.11) will be

$$\frac{1}{r_j}\left(r_{j+1/2}\frac{u_{j+1} - u_j}{r_{j+1} - r_j} - r_{j-1/2}\frac{u_j - u_{j-1}}{r_j - r_{j-1}}\right) + \lambda e^{u_j} = 0. \tag{1.14}$$

The nonstandard version finite difference scheme (1.13) for problem (1.11) will be

$$\frac{1}{r_j}\left(\frac{u_{j+1} - u_j}{\log(r_{j+1}/r_j)} - \frac{u_j - u_{j-1}}{\log(r_j/r_{j-1})}\right) + \lambda e^{u_j} = 0. \tag{1.15}$$