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Steps For Constructing Magic Cube Using Two Orthogonal Latin Squares and A Magic Square

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Abstract - A magic cube of order n is an example of fine permutation for natural number 1,2,…,n^3 where the sum of rows, columns and diagonals are the same. It is a natural generalization of a magic square. Magic cube is constructed using two Orthogonal Latin Squares and a Magic Square. Unfortunately, steps for creating magic square from two orthogonal Latin square was not explicitly mentioned. This paper introduces steps for creating odd and even magic square from two orthogonal Latin squares. Subsequently, magic cubes are created based on orthogonal Latin Squares. Examples show that the proposed steps work well on explaining the construction process of magic cubes.

Keywords: Magic Cube, Magic Square, Latin Square

I. INTRODUCTION

A magic cube of order n $(n \neq 2 \text{ and } n \neq 4)$ [1,4] is an example of fine permutation for natural number 1,2,…,n^3 where the sum of rows, columns and diagonals are the same. It is a natural generalization of a magic square. Magic cube is constructed using two Orthogonal Latin Squares and a Magic Square. Magic Cube can be described using subset notation as follow: Natural Numbers ⊂ Latin Square ⊂ Orthogonal Latin Square ⊂ Magic Square ⊂ Magic Cube. The technique to construct a magic cube is explained in [1,2,3,4]. Unfortunately, steps for creating magic square from two orthogonal Latin square was not explicitly mentioned. This paper introduces steps for creating odd and even magic square from two orthogonal Latin square. The rest of the paper is organized as follows. Section 2 describes components for creating Magic Cube. Section 3 discusses two different magic cubes using two different pairs of orthogonal Latin squares. Section 4 concludes this paper with future work of this research.

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II. COMPONENTS FOR CREATING MAGIC CUBE

This section explain components and steps for creating magic cube: Latin Square, Orthogonal Latin Square, magic square and the magic cube.

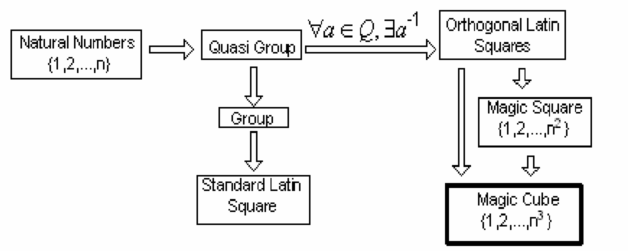

Figure 1. Process of creating Magic Cube

Figure 1 shows the process of creating magic cube from combination or permutation) of natural numbers. Many Latin squares can be generated from quasi group as compared to if we are using a group.

Latin Square

A Latin Square of order n, denoted as

$$R_n = [r(i,j) : 1 \leq i, j \leq n] \qquad (1)$$

is a two dimensional $(n \times n)$ matrix such that every row and every column is a permutation of the set of natural number {1, 2, …, n}. Operations on Latin Squares can be defined as:

a. Isotopism [6] of a Latin Square R is a
   (1) permutation of its rows,
   (2) permutation of its columns,
   (3) permutation of its elements
   (These permutations do not have to be the same.)
b. $\mathbb{R}_n$ is *reduced* if and only if its first row is $[1 \;\; 2 \;\; \cdots \;\; n]$ and its first column is $[1 \;\; 2 \;\; \cdots \;\; n]^T$.
c. $\mathbb{R}_n$ is *normal* if and only if its first row is $[1 \;\; 2 \;\; \cdots \;\; n]$.

Latin Square can be represented either as a group (Z_n^*) where * represent binary operators or as a quasi group. In a group, there exists only one unique Latin Square for every n. Multiple Latin squares can be created in quasi group which form Orthogonal Latin Square as shown in Figure 2. Each orthogonal Latin square element's has a unique inverse.

$$\begin{bmatrix}1&2&3&4\\2&1&4&3\\3&4&1&2\\4&3&2&1\end{bmatrix} \quad \begin{bmatrix}1&2&3&4\\3&4&1&2\\4&3&2&1\\2&1&4&3\end{bmatrix}$$

R_4 S_4

Figure 2: Orthogonal Latin squares (R_4 and S_4) of order 4

1 = r(1,1) = r(2,2) = r(3,3) = r(4,4)
2 = r(1,2) = r(2,1) = r(3,4) = r(4,3)
3 = r(1,3) = r(2,4) = r(3,1) = r(4,2)
4 = r(1,4) = r(2,3) = r(3,2) = r(4,1)

Figure 3.Representation using elements and coordinates

Alternatively, Latin Square can be viewed using the element and the coordinates r(i,j), where i represents the row and j represents the column as shown in figure 3. This representation is used to ease the understanding process for the construction of magic square and magic cube Each element of Latin Square appears in different rows and columns. A combination of two Latin Square can be used to generate another unique matrix called Orthogonal Latin Square.

Orthogonal Latin Square

Two Latin Squares, R_n = [r(i,j)] and S_n = [s(i,j)] are said to be orthogonal if i, i', j, j' $\in \{1,2,\ldots,n\}$are such that [r(i,j)] = [r(i',j')] and [s(i,j)] = [s(i',j')]. Then we must have i = i' and j = j'. Thus two *nxn* Latin Squares R_n = [r(i,j)] and S_n = [s(i,j)] are said to be orthogonal if and only if the n^2 pair (r(i,j) and s(i,j) are all different. Using similar approach, Orthogonal Latin Square Order 16 in Figure 3 can be represented using elements and coordinates as in figure 4.

$$\begin{bmatrix}11&22&33&44\\23&14&41&32\\34&43&12&21\\42&31&24&13\end{bmatrix}$$

T_{16}

Figure 4: Orthogonal Latin Square T_n order 16

11 = rs(1,1), 22 = rs(1,2), 33 = rs(1,3), 44 = rs(1,4)
23 = rs(2,1), 14 = rs(2,2), 41 = rs(2,3), 32 = rs(2,4)
34 = rs(3,1), 43 = rs(3,2), 12 = rs(3,3), 21 = rs(3,4)
42 = rs(4,1), 31 = rs(4,2), 24 = rs(4,3), 13 = rs(4,4)

Figure 5. Representation using elements and coordinates

Magic Square

Magic Square is generated using two Orthogonal Latin squares. A magic square of order n, denoted as

Mn = [m(i,j) : $1 \le i,j \le n$]

It is a two dimension *nxn* matrix containing $\{1,2, \ldots,n^2\}$ in some order such that the sum of the number along every row, column and main diagonal is a fixed constant of $\frac{n(n^2+1)}{2}$.

Alternatively, magic square can also be depicted using coordinates i and j as shown in figure 6.

m(1,1)	m(1,2)	m(1,3)	…	m(1,n-1)	m(1,n)
m(2,1)	m(2,2)	m(2,3)	…	m(2,n-1)	m(2,n)
m(3,1)	m(3,2)	m(3,3)	…	m(3,n-1)	m(3,n)
⋮	⋮	⋮	⋱	⋮	⋮
m(n-1,1)	m(n-1,2)	m(n-1,3)	…	m(n-1,n-1)	m(n-1,n)
m(n,1)	m(n,2)	m(n,3)	…	m(n,n-1)	m(n,n)

Figure 6. Coordinates for magic square of order n

The summation of every rows, columns and diagonals can be defined by:

Rows:

$$\sum_{j=1,1,\ldots,n;i=1} m(i,j) = \cdots = \sum_{j=1,\ldots,n;i=n} m(i,j) \quad (2)$$

Columns:

$$\sum_{i=1,1,\ldots,n;j=1} m(i,j) = \cdots = \sum_{i=1,1,\ldots,n;j=n} m(i,j) \quad (3)$$

Diagonals:

$$\sum_{i=j=1,\ldots,n} m(i,j) = \cdots = \sum_{i=1,\ldots,n;j=n,\ldots,1} m(i,j) \quad (4)$$

Using equation (2), (3) and (4), the associate Magic Square from two Orthogonal Latin Square R_n and S_n is shown in figure 6 with total summation of rows, columns and diagonals equal to 34 (the summation of numbers (elements) divided by number of rows or columns).

Steps For Creating Magic Squares

Forming the magic square can be done using the following steps:

a. Calculate the sum of numbers $(1+2+\ldots n^2) = n(n^2+1)/2$.
b. Find the average (C_{value}) by dividing the sum by the number of rows or columns.
c. The odd and even magic square can be constructed using the following formula:

a. Odd order magic square

$$M_k = 2k+1, k \in \mathrm{N},$$

$$Center\_value = \frac{n^2+1}{2},$$

$$m(i,j) = m(k+1, k+1)$$

b. Even order magic square

$$M_k = 2k, k \in \{2,4,6,\ldots\},$$

Center_diagonal coordinates,

$$m(i,j) = \begin{cases} a = m\left(\frac{n}{2}, \frac{n}{2}\right) \\ b = m\left(\frac{n}{2}, \frac{n}{2}+1\right) \\ c = m\left(\frac{n}{2}+1, \frac{n}{2}\right) \\ d = m\left(\frac{n}{2}+1, \frac{n}{2}+1\right) \end{cases}$$

where

$$Center\_diagonal\_value = a_{value} + c_{value} = b_{value} + d_{value}$$

$$= \frac{n(n^2+1)}{4}.$$

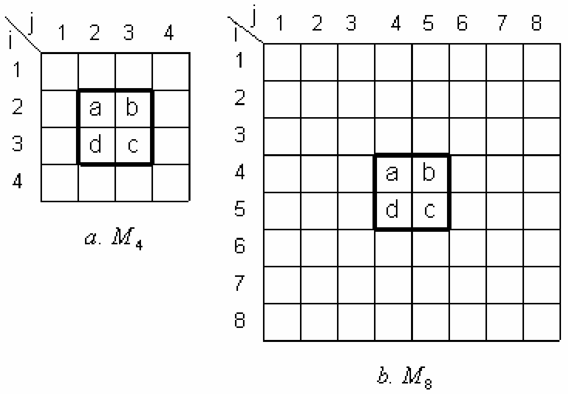

Figure 7. Coordinates representation for M_4 and M_8.

One example of magic square (n=4) is shown in figure 7 where a = m(2,2), b= m(2,3), c= m(3,2), d= m(3,3) form the center of the magic square with the diagonal value of 17. The total value for m(1,1) and m(4,4) is also 17. Using this technique, we can construct magic square where n = 4 as shown in figure 8.

$$M_4 = \begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

Figure 8. Magic Square with n = 4.

III. MAGIC CUBE

Magic cube is a generalization of magic square and two orthogonal Latin squares. A magic cube of order n is a cubical array containing natural numbers $1,2,3,\ldots n^3$. The three-dimensional coordinates for magic cube where n=4 is shown in figure 9.

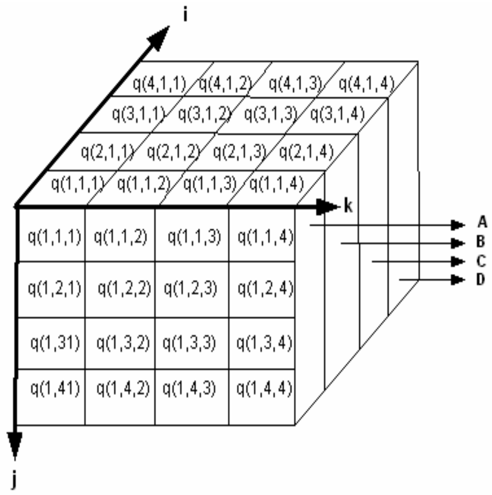

Figure 9. Coordinates for magic square with n=4.

From Figure 8, note that

a. The rows, i.e. n-tuple of elements having the same coordinates on two places are $3n^2$, they are

From block A (row)
$[\mathbf{q}(1,1,k)]_{k=1,2,3,4}$,
$[\mathbf{q}(1,2,k)]_{k=1,2,3,4}$, $[\mathbf{q}(1,3,k)]_{k=1,2,3,4}$.

From block A (column)
$[\mathbf{q}(1,j,1)]_{j=1,2,3,4}$, $[\mathbf{q}(1,j,2)]_{j=1,2,3,4}$,
$[\mathbf{q}(1,j,3)]_{j=1,2,3,4}$.

From block B (row)
$[\mathbf{q}(2,1,k)]_{k=1,2,3,4}$,
$[\mathbf{q}(2,2,k)]_{k=1,2,3,4}$, $[\mathbf{q}(2,3,k)]_{k=1,2,3,4}$.

From block B (column)
$[\mathbf{q}(2,j,1)]_{j=1,2,3,4}$, $[\mathbf{q}(2,j,2)]_{j=1,2,3,4}$,
$[\mathbf{q}(2,j,3)]_{j=1,2,3,4}$.

From block C (row)
$[\mathbf{q}(3,1,k)]_{k=1,2,3,4}$,
$[\mathbf{q}(3,2,k)]_{k=1,2,3,4}$, $[\mathbf{q}(3,3,k)]_{k=1,2,3,4}$.

From block C (column)
$[\mathbf{q}(3,j,1)]_{j=1,2,3,4}$, $[\mathbf{q}(3,j,2)]_{j=1,2,3,4}$,
$[\mathbf{q}(3,j,3)]_{j=1,2,3,4}$.

From block A, B and C
$[\mathbf{q}(i,1,1)]_{i=1,2,3,4}$, $[\mathbf{q}(i,1,2)]_{i=1,2,3,4}$, $[\mathbf{q}(i,1,3)]_{i=1,2,3,4}$
$[\mathbf{q}(i,2,1)]_{i=1,2,3,4}$,
$[\mathbf{q}(i,2,2)]_{i=1,2,3,4}$, $[\mathbf{q}(i,2,3)]_{i=1,2,3,4}$
$[\mathbf{q}(i,3,1)]_{i=1,2,3,4}$, $[\mathbf{q}(i,3,2)]_{i=1,2,3,4}$, $[\mathbf{q}(i,3,3)]_{i=1,2,3,4}$

The 4 diagonals are

$$[\mathbf{q}(1,1,1),\mathbf{q}(2,2,2),\mathbf{q}(3,3,3)],$$
$$[\mathbf{q}(1,1,3),\mathbf{q}(2,2,2),\mathbf{q}(3,3,1)],$$
$$[\mathbf{q}(1,3,1),\mathbf{q}(2,2,2),\mathbf{q}(3,1,3)],$$
$$[\mathbf{q}(1,3,3),\mathbf{q}(2,2,2),\mathbf{q}(3,1,1)].$$

b. The sum of the numbers along every row and diagonal

$$\sum_{i=1,\cdots,3}\mathbf{q}(i,j,k)=\sum_{j=1,\cdots,3}\mathbf{q}(i,j,k)=\sum_{k=1,\cdots,3}\mathbf{q}(i,j,k)=\frac{n(n^3+1)}{2}$$

.

Using Latin Square (R_4, S_4) and magic square M_4, we can obtain Magic Cube (Q_4) using formula given in [1].

Magic cube $Q_4 = [q(i,j,k) : 1 \le i, j, k \le 4]$ entries as,

$$q(i,j,k) = [s(i,r(j,k)) -1].4^2 + m(i,s(j,k))$$

for all i,j,k = 1,2,3,4.

Result using two different combinations of orthogonal Latin squares are shown in figure 10 and figure 11. In figure 10, we reverse the order of the orthogonal Latin square in [1] such that $R_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix}$ and $S_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$. The diagonal value of this magic cube is also equal to 130. In figure 11, we use two different orthogonal Latin squares, R_4 and S_4.

$R_4 = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 \\ 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ and $S_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix}$.

The magic cube generated is not standard in the sense that the diagonal values are not 130. From this exercise, it can be concluded that not all combinations of orthogonal Latin square will create a standard magic cube as described in [1].

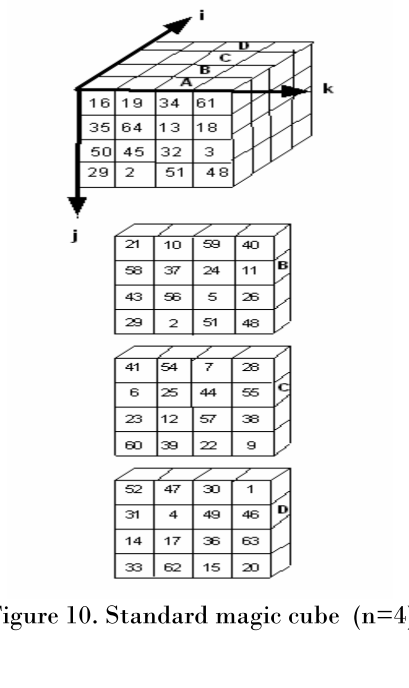

Figure 10. Standard magic cube (n=4)

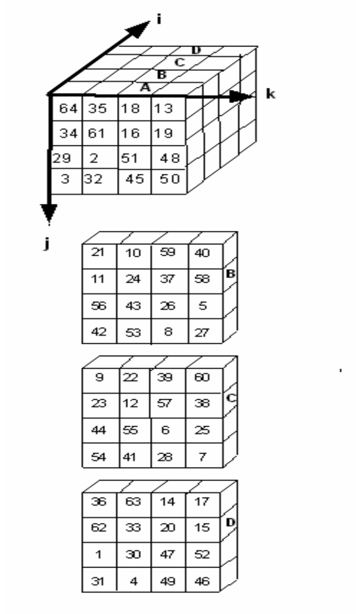

Figure 11. Non-standard magic cube (n=4)

IV. CONCLUSION

We have shown a process for creating magic cube with additional technique for constructing magic square for numbers $1,2,\ldots,n^2$. The center coordinate of the square matrix and the average values of elements can be used as the basis for constructing the magic square. Fixing the magic square and changing the orthogonal Latin square seem to give an interesting result where we can generate non-standard magic cube (different diagonal values). In the future, we would expand the technique for finding the generalize formula for generating magic square with n=3,4,5,7,8,…,n.

ACKNOWLEGMENT

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