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Solution of the Forced Korteweg-de Vries-Burgers Nonlinear Evolution Equation

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Abstract This paper reports several findings on forced solitons solution generated by the forced Korteweg-de Vries-Burgers equation (fKdVB),

$$U_t + \varepsilon U U_x - \nu U_{xx} + \mu U_{xxx} = f(x), \qquad a \leq x \leq b.$$

The fKdVB equation is a nonlinear evolution equation that combines several effects such as forcing; $f(x)$, nonlinearity; $\varepsilon U U_x$, dissipation; νU_{xx} and dispersion; μU_{xxx}. The forcing term breaks those symmetries associated with the unforced systems. Thus, the traditional analytical method such as inverse scattering method and Bäcklund transformation do not work on forcing system anymore. Approximate and numerical solution seem to be the ways to solve the fKdVB equation. The semi-implicit pseudo-spectral method is used to develop a numerical scheme to solve the fKdVB equation with arbitrary forcing. A software package,(BURSO) that has user friendly graphical interface is developed using Matlab 7.0 to implement the above numerical scheme. Numerical simulation proves that it is very flexible since it can solve free and force system such as the KdV, Burgers, KdVB and fKdV equations efficiently. Thus it is able to solve the fKdVB equation faithfully. Our future research would sought the approximate solution of the fKdVB equation.

Keywords Korteweg-de Vries, Burgers, semi-implicit pseudo-spectral method, soliton.

Abstrak *Kertas kerja ini melaporkan beberapa dapatan tentang penyelesaian soliton paksaan yang dijanakan oleh persamaan paksaan Korteweg-de Vries-Burgers (fKdVB),*

$$U_t + \varepsilon U U_x - \nu U_{xx} + \mu U_{xxx} = f(x), \qquad a \leq x \leq b.$$

Persamaan fKdVB adalah satu persamaan evolusi tak linear yang menggabungkan beberapa kesan seperti paksaan; $f(x)$, tak linear; $\varepsilon U U_x$, disipasi; νU_{xx} dan penyerakan; μU_{xxx}. Kesan paksaan ini memecahkan sifat simetri yang berkaitan dengan sistem bebas paksaan. Justeru itu, secara tradisi kaedah analitik seperti kaedah penyerakan songsang dan Kaedah Trasformasi Bäcklund tidak boleh digunakan lagi ke atas suatu sistem paksaan. Kaedah berangka dan penyelesaian hampir merupakan cara yang mampu digunakan untuk menyelesaikan persamaan fKdVB. Kaedah semi-implicit pseudo-spectral digunakan untuk membangunkan skim berangka bagi menyelesaikan persamaan fKdVB dengan paksaan abitrari. Satu perisian (BURSO) yang mempunyai antaramuka grafik yang mesra pengguna telah dibangunkan dengan Matlab 7.0 untuk melaksanakan skim berangka tersebut. Simulasi berangka ini menunjukkan skim berangka ini fleksibel kerana ia boleh menyelesaikan sistem bebas dan paksaan seperti persamaan KdV, Burgers, KdVB dan fKdV dengan cekap. Justeru itu skim berangka ini boleh menyelesaikan persamaan fKdVB dengan jayanya. Penyelidikan masa depan adalah untuk mencari penyelesaian hampir bagi persamaan fKdVB.

Katakunci *Korteweg-de Vries, Burgers, kaedah semi-implicit pseudo-spectral, dan soliton.*

1   Introduction

The fKdVB equation is a nonlinear evolution equation that combines several effects such as forcing; $f(x)$, nonlinearity; $\varepsilon U U_x$, dissipation; νU_{xx} and dispersion; μU_{xxx}. terms. The unforced system has been studied intensively during the past 30 years. The forcing term breaks those symmetries associated with the unforced systems. Thus, the traditional analytical method such as inverse scattering method and Bäcklund transformation do not work on forced system anymore. Approximate and numerical solution seem to be the method to solve the fKdVB equation. In this paper, we solve the fKdVB equation numerically using semi-implicit pseudo-spectral method.

2   The Governing Equation

In this paper, the governing equations for nonlinear evolution equations is given by fKdVB equation,

$$U_t + \varepsilon U U_x - \nu U_{xx} + \mu U_{xxx} \quad = \quad f(x). \tag{1}$$

where ε, ν, μ are positive parameters. The parameter ε controls the nonlinearity effect, ν gives the effect of dissipation, the dispersion effect is controlled by μ whereas $f(x)$ gives the effect of forcing. By manipulating the values of these parameters, we will see a few cases as below.

Case 1

When $f(x)$ and ν both equal to zeros, we have the Korteweg-de Vries equation (KdV),

$$U_t + \varepsilon U U_x + \mu U_{xxx} = 0. \tag{2}$$

Its analytical solution is given by [6] as

$$U(\xi) = \frac{3c}{\varepsilon} \operatorname{sech}^2 \sqrt{\frac{c}{4\mu}} \xi. \tag{3}$$

Case 2

When $f(x)$ and μ both equal to zeros, we have the Burgers equation given by

$$U_t + \varepsilon U U_x - \nu U_{xx} = 0. \tag{4}$$

Its solution is given by [3] as

$$U(\xi) = \frac{u_{-\infty} + u_{\infty}}{2} - \frac{u_{-\infty} - u_{\infty}}{2} \tanh\left(\frac{\varepsilon(u_{-\infty} - u_{\infty})\xi}{4\nu}\right). \tag{5}$$

Case 3

When $f(x)$ is zero, we will get the Korteweg-de Vries-Burgers equations (KdVB) given by

$$U_t + \varepsilon U U_x - \nu U_{xx} + \mu U_{xxx} = 0. \tag{6}$$

Its dispersion-dominant solution is given by [8] as

$$U(\xi) = \begin{cases} u_{-\infty} + \exp\left(\dfrac{\nu\xi}{2\mu}\right)\left[A\cos\left(\sqrt{\dfrac{\varepsilon(u_{-\infty} - u_{\infty})}{2\mu} - \left(\dfrac{\nu}{2\mu}\right)^2}\xi\right)\right] & \xi < 0 \\[4ex] \dfrac{3(u_{-\infty} - u_{\infty})}{2}\operatorname{sech}^2\sqrt{\dfrac{\varepsilon(u_{-\infty} - u_{\infty})}{8\mu}}\xi & \xi > 0 \end{cases}$$

where

$$\xi = x - ct. \tag{7}$$

Case 4

When ν is zero, we will have the forced Korteweg-de Vries equation (fKdV) given by

$$U_t + \varepsilon U U_x + \mu U_{xxx} = f(x). \tag{8}$$

Since this is a forced system, we will only see approximate and numerical solution as given by [10].

3   Semi-implicit Pseudo-Spectral Method

Nouri and Sloan [9] studied six Fourier pseudo-spectral methods that solve the KdV equation numerically namely leap-frog scheme of Fornberg and Witham, semi-implicit scheme of Chan and Kerhoven, modified basis function scheme of Chan and Kerhoven, split-step scheme based on Taylor expansion, split-step scheme based on characteristics and quasi-Newton implicit method. They found that the semi-implicit scheme of Chan and Kerhoven [4] to be the most efficient of the methods tested. Chan and Kerhoven integrated the KdV equation in time in Fourier space using two Fast Fourier Transform (FFT) per time step. They also used Crank-Nicolson method for the linear term and a leap-frog method for the nonlinear term.

Here, we extend the Chan and Kerkhoven [4] scheme for Equation (1) which is integrated in time by the leapfrog finite difference scheme in the spectral space. The infinite interval is replaced by $-L < x < L$ with L sufficiently large such that the periodicity assumptions holds

$$U(-L,t) = U(L,t) = 0.$$

When we apply the Chan-Kerkhoven scheme, the "noise" propagation problem due to the numerical scheme does not appear to be serious when L is large enough and a proper time step $\triangle t$ is chosen.

By introducing $\xi = sx + \pi$ where $s = \dfrac{\pi}{L}$ we will transform $U(x,t)$ into $V(\xi,t)$. By taking $f(x) = \frac{\gamma}{2}\delta_x(x)$, thus Equation (1) will be transformed into

$$V_t + \varepsilon s V V_\xi - \nu s^2 V_{\xi\xi} + \mu s^3 V_{\xi\xi\xi} \;\; = \;\; \frac{\gamma}{2}s\frac{d}{d\xi}\delta(\frac{\xi}{s} - L) \qquad (9)$$

By letting $W(\xi,t) = \frac{1}{2}sV^2$, then the nonlinear term $\varepsilon s V V_\xi$ can be written as εW_ξ, so Equation (9) will becomes

$$V_t + \varepsilon W_\xi - \nu s^2 V_{\xi\xi} + \mu s^3 V_{\xi\xi\xi} \;\; = \;\; \frac{\gamma}{2}s\frac{d}{d\xi}\delta(\frac{\xi}{s} - L) \qquad (10)$$

For the numerical solution of Equation (10), we discretize the interval $[0, 2\pi]$ by $N + 1$ equidistant points. We let $\xi_0 = 0, \xi_1, \xi_2, ..., \xi_N = 2\pi$, so that $\triangle\xi = \dfrac{2\pi}{N}$. In this case, N will always be even and is to be a power of two. So we let $m = \dfrac{N}{2}$. The Discrete Fourier Transform (DFT) of $V(\xi_j, t)$ for $j = 0, 1, 2, ..., N - 1$ is denoted by $\widehat{V}(p,t)$ is given by:-

$$\widehat{V}(p,t) \;\; = \;\; \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} V(\xi_j,t)e^{-(\frac{2\pi jp}{N})i}$$

where $\quad p = -m, -m+1, -m+2, ..., m-1$

whereas the inverse Fourier Transform of $\widehat{V}(p,t)$ for $p = -m, -m+1, -m+2, ..., m-1$ is denoted by $V(\xi_j, t)$ written as

$$V(\xi_j, t) = \frac{1}{\sqrt{N}} \sum_{p=-m}^{m-1} \widehat{V}(p,t) e^{(\frac{2\pi jp}{N})i}$$

where $\quad j = 0, 1, 2, ..., N-1$

and $i = \sqrt{-1}$ is the imaginary number. The DFT of Equation (10) with respect to ξ gives

$$\widehat{V}_t(p,t) + i\varepsilon p \widehat{W}(p,t) + \nu p^2 s^2 \widehat{V}(p,t) - i\mu p^3 s^3 \widehat{V}(p,t) \quad = i\frac{\gamma}{2} sp \frac{\sqrt{N}}{2L} e^{-i\pi p} \tag{11}$$

By using the following approximation,

$$\widehat{V}_t(p,t) \quad \approx \quad \frac{\widehat{V}(p, t+\triangle t) - \widehat{V}(p, t-\triangle t)}{2\triangle t} \tag{12}$$

$$\widehat{V}(p,t) \quad \approx \quad \left[\frac{\widehat{V}(p, t+\triangle t) + \widehat{V}(p, t-\triangle t)}{2} \right] \tag{13}$$

$$\tag{14}$$

and denote $\widehat{V}(p, t+\triangle t)$ by $\widehat{V}_{pt}$, $\widehat{V}(p, t-\triangle t)$, by $\widehat{V}_{mt}$ and $\widehat{V}(p,t)$ by $\widehat{V}_t$, so Equation (11) becomes,

$$\frac{\widehat{V}_{pt} - \widehat{V}_{mt}}{2\triangle t} + i\varepsilon p \widehat{W}(p,t) + \nu s^2 p^2 \left[\frac{\widehat{V}_{pt} + \widehat{V}_{mt}}{2} \right]$$

$$-i\mu s^3 p^3 \left[\frac{\widehat{V}_{pt} + \widehat{V}_{mt}}{2} \right] = i\frac{\gamma}{2} sp \frac{\sqrt{N}}{2L} e^{-i\pi p} \tag{15}$$

By multiplying Equation (15) with $2\triangle t$, we get

$$\widehat{V}_{pt} - \widehat{V}_{mt} + 2i\varepsilon p \triangle t \widehat{W}(p,t) + \nu s^2 p^2 \triangle t (\widehat{V}_{pt} + \widehat{V}_{mt})$$

$$-i\mu s^3 p^3 \triangle t (\widehat{V}_{pt} + \widehat{V}_{mt}) = i\gamma sp \triangle t \frac{\sqrt{N}}{2L} e^{-i\pi p} \tag{16}$$

Collecting the terms in Equation (16) will gives us

$$(1 + \nu s^2 p^2 \triangle t - i\mu s^3 p^3 \triangle t) \widehat{V}_{pt} \quad = \quad (1 - \nu s^2 p^2 \triangle t + i\mu s^3 p^3 \triangle t) \widehat{V}_{mt}$$

$$-2i\varepsilon p \triangle t \widehat{W}(p,t) + i\gamma sp \triangle t \frac{\sqrt{N}}{2L} e^{-i\pi p} \tag{17}$$

Then, Equation (18) will be our forward scheme given by,

$$\widehat{V}_{pt} = \frac{1}{1 + \nu s^2 p^2 \triangle t - i\mu s^3 p^3 \triangle t} \left[\widehat{V}_{mt} (1 - \nu s^2 p^2 \triangle t + i\mu s^3 p^3 \triangle t) \right.$$

$$\left. -2i\varepsilon p \triangle t \widehat{W}(p,t) + i\gamma sp \triangle t \frac{\sqrt{N}}{2L} e^{-i\pi p} \right] \tag{18}$$

4   Numerical Simulation

Figure (1) shows the graphical user interface, BURSO built using the Matlab 7.0 software. With BURSO, we just need to input the relevant data, select appropriate initial condition and lastly click the plot button to see the desired numerical simulation of the fKdVB equation. This is indeed user friendly since we can just change those parameters and redo the process again and again so fast that this numerical simulation tends to be our virtual laboratories in solving the fKdVB equation. With BURSO, we are able to generate graphical outputs for the KdV, Burgers, KdVB, fKdV and fKdVB and it is done so efficiently.
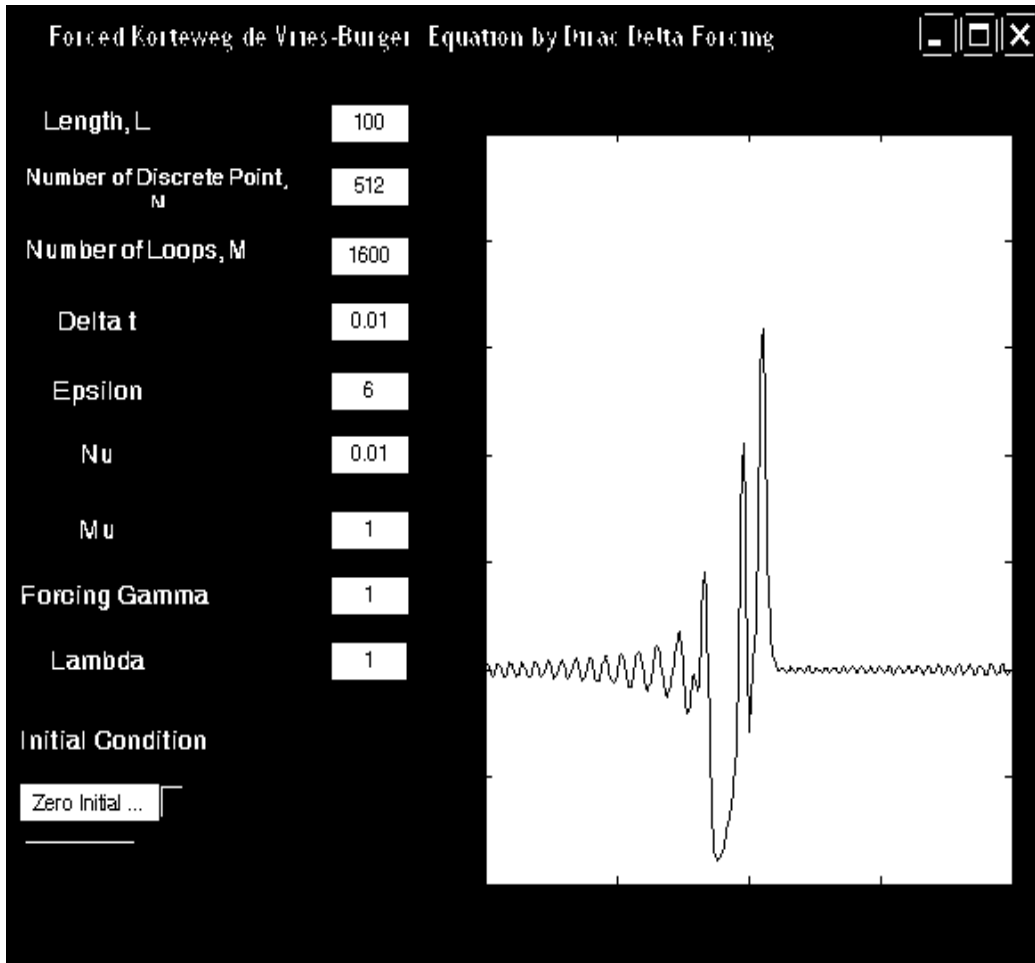
Figure 1: The Graphical User Interface

In our numerical simulation, we will show all the solutions of five nonlinear evolution equations using BURSO. By choosing the appropriate values for the parameters, we will solve each of these equations given as follow:

A The Korteweg-de Vries (KdV) Equation

By considering Equation (2) and we input the following set of data into BURSO to get the 3-soliton solution for the KdV equation.

Set	L	N	M	$\triangle t$	ε	ν	μ	Initial Condition
K1	50	1024	1000	0.001	6	0	1	$12\operatorname{sech}^2 x$

The BURSO can reduce Equation (1) to the KdV equation and accurately yields the 3-soliton solution. The result is shown in Figure (2).
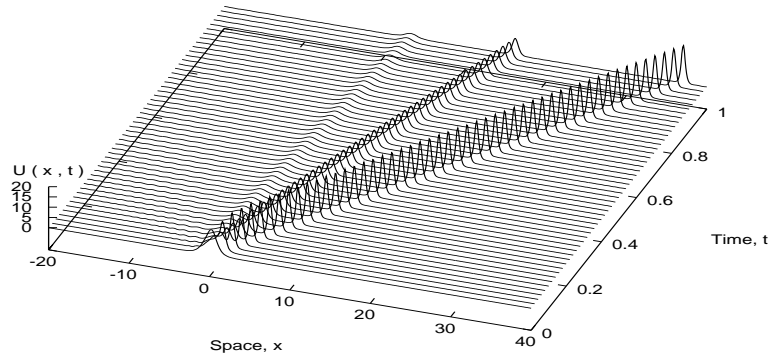
Figure 2: The 3-soliton solution

B The Burgers Equation

To get the Burgers type solution, we will then input the following set of data into BURSO. We actual reduce Equation (1) to the Burgers equation.

Set	L	N	M	$\triangle t$	ε	ν	μ	Initial Condition
B1	50	256	1000	0.01	1	1	0	$\operatorname{sech}^2 x$
B2	50	256	1000	0.01	1	2	0	$\operatorname{sech}^2 x$
B3	50	256	1000	0.01	1	0.1	0	$\operatorname{sech}^2 x$
B4	50	1024	1000	0.01	1	0.01	0	$\operatorname{sech}^2 x$

Figure (3-6) show the 2D-plot of the Burgers solution for time $t = 0 - 10$ with $\varepsilon = 1$ and $\nu = 1, 2, 0.1, 0.01$ respectively. The top curve is at time $t = 0$ and the bottom curve at time $t = 10$ with an increment of one. All four figures show that the amplitude of the initial wave is slowly diminishing with time due to the effect of the viscosity which damps the amplitude of the initial wave. Comparing Figure (3) and (4), we notice that when we increased the viscosity effect from 1 to 2, the amplitude of the wave damps much faster. On the other hand, when we reduced the viscosity effect tends to zero (from 0.1 to 0.01) in Figure (5) and (6), we observe

that the wave front becomes steeper. In fact, all four figures of Burgers solution that we get is same as [1], [2], [5] and [11] . On top of that, when ν tends to zero , we have nonlinear equation $U_t + \varepsilon U U_x = 0$. The solution of the nonlinear equation is $U(x,t) = f(x - \varepsilon U t)$. Comparing with the linear equation $U_t + cU_x = 0$ which has solution of $U(x,t) = f(x - ct)$. We now see εU itself is the velocity. Thus, the nonlinearity effect ($\varepsilon U U_x$) will make higher value of U, that is the top wave (crest) move faster than bottom wave (trough). As a result, the wave front will get steeper and tends to turn over and then break. Consequently, we can conclude that the numerical scheme indeed can reduce Equation (1) to the Burgers equation and later solve it accurately.
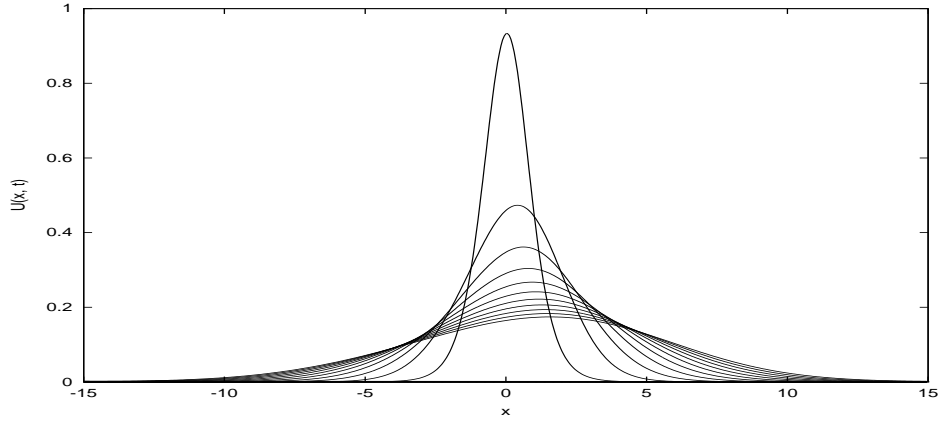
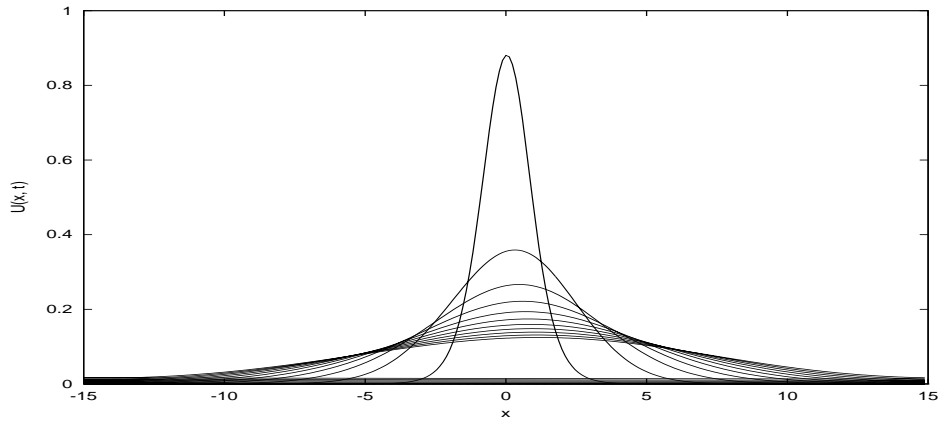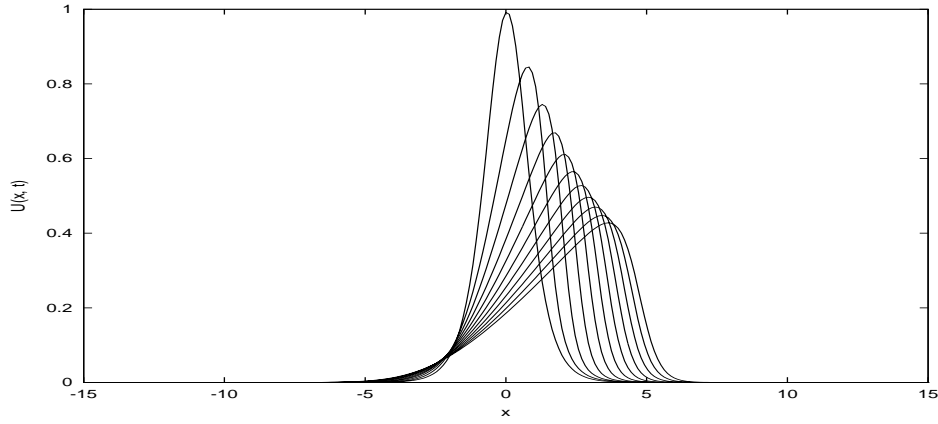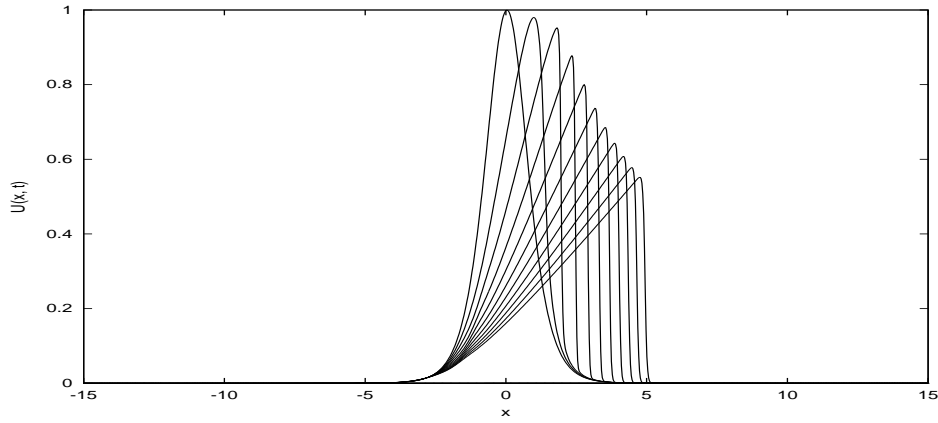Figure 3: The Burgers type solution with $\varepsilon = 1; \nu = 1$ for time $t = 0 - 10$

Figure 4: The Burgers type solution with $\varepsilon = 1; \nu = 2$ for time $t = 0 - 10$

Figure 5: The Burgers type solution with $\varepsilon = 1$; $\nu = 0.1$ for time $t = 0 - 10$

Figure 6: The Burgers type solution with $\varepsilon = 1$; $\nu = 0.01$ for time $t = 0 - 10$

C The Korteweg-de Vries-Burgers (KdVB) Equation

To obtain the KdVB type solution, we consider Equation (6). Then, we input the following set of data into BURSO.

Set	L	N	M	$\triangle t$	ε	ν	μ	Initial Condition
KdVB1	150	512	800	0.05	2	0.0001	0.1	$1 - \tanh \dfrac{|x| - 25}{5}$
KdVB2	150	512	800	0.05	2	1	0.0001	$1 - \tanh \dfrac{|x| - 25}{5}$

The KdVB equation is a nonlinear evolution equation that involves of nonlinearity, dissipation and dispersion. If ν tends to zero, we should get the KdVB equation tends

to behave like the KdV equation. Whereas, if we let μ tends to zero, we should get the KdVB equation tends to behave like the Burgers equation. Figure (7) shows the graph of a tangent hyperbolic function as in [2] and [11]. In order to achieve the KdV type solution from the KdVB equation, we will let the viscosity so small ($\nu = 0.0001$) as in [2]. From Figure (8), we observe that a train of 10 solitons is generated. The number of solitons generated and its amplitude are exactly the same as in [2], [11] and [5]. In fact, the solution in Figure (8) is indistinguishable with the KdV type solution using identical parameters given by [7]. Next, we will obtain the Burgers type solution from the KdVB equation. So, we let the dispersion so small ($\mu = 0.0001$) . We notice that a triangular wave is generated in Figure (9). In fact Figure (9) is indistinguishable with the Burgers type solution obtained in previous section. Thus, we can conclude that BURSO can reduce Equation (1) to the KdVB equation and solve it faithfully.
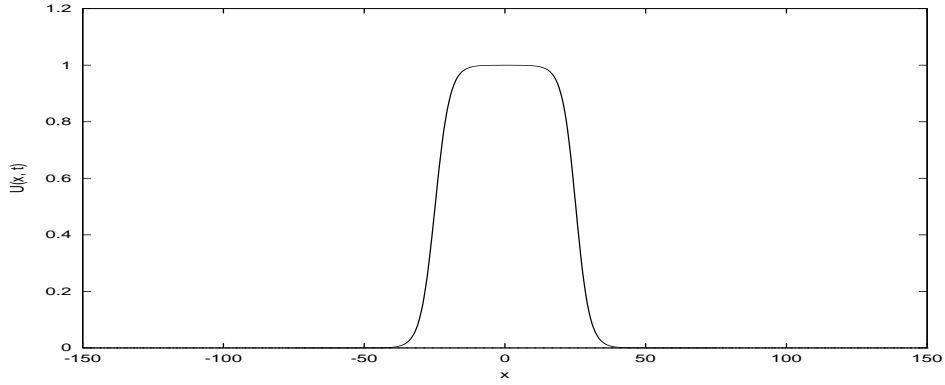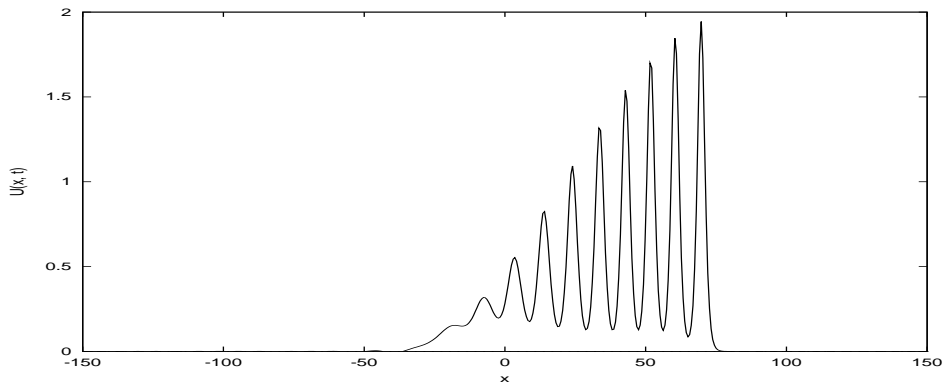

Figure 7: Initial tangent hyperbolic function

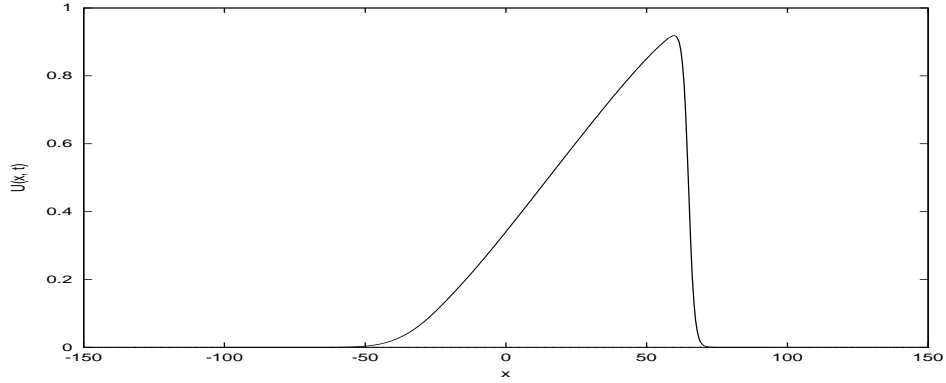Figure 8: The KdVB type solution with $\varepsilon = 2$; $\nu = 0.0001$; $\mu = 1$ at time $t = 40$

Figure 9: The KdVB type solution with $\varepsilon = 2$; $\nu = 1$; $\mu = 0.0001$ at time $t = 40$

D The forced Korteweg-de Vries (fKdV) Equation

By considering Equation (8), we will have the fKdV equation which describes the free surface profile of the water flows over bump on the bottom of a two dimensional channel [10]. Then, we input the following set of data into BURSO.

Set	L	N	M	$\triangle t$	ε	ν	μ	γ
fKdV1	100	512	4000	0.01	-1.5	0	$-\frac{1}{6}$	1

A 3D-plot of the forced solitons genererated by Equation (8) is given in Figure (10). From Figure (11), we observe at time $t = 40s$, 7 matured solitons and 1 almost matured soliton are generated at upstream. Besides, a depression zone is generated immediately behind the disturbance followed by a train of cnoidal like waves gradually attenuating in the far field downstream. In fact, the figure produced is the same as [10]. Thus, we can say that BURSO can reduce Equation (1) to the fKdV equation and solve it effectively.
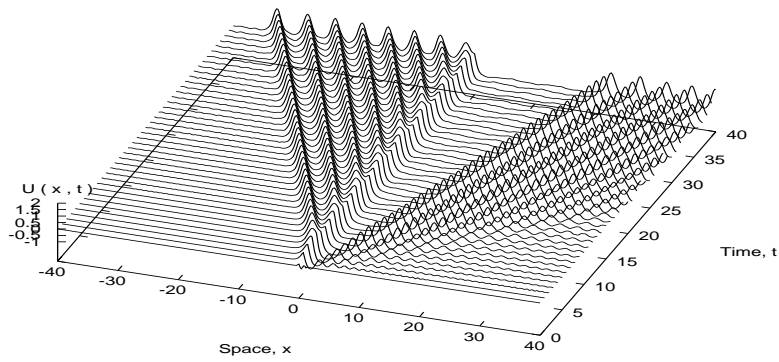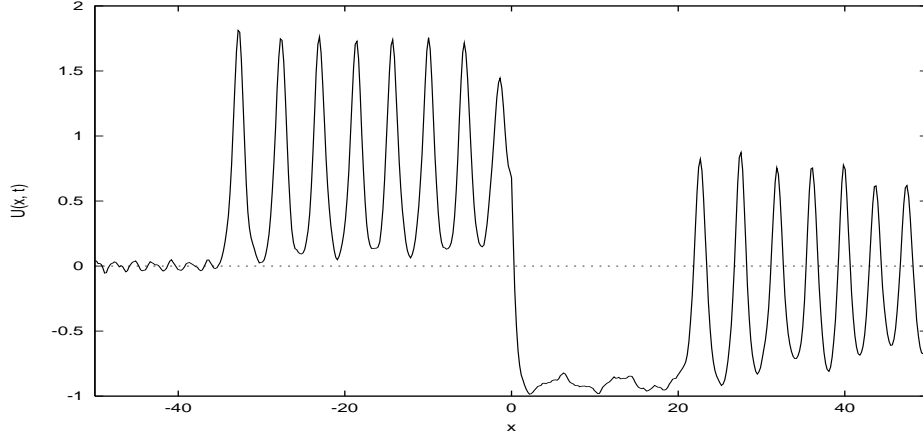

Figure 10: A 3D-plot of the solution of the fKdV equation with Dirac-Delta forcing

Figure 11: The solution of the FKdV equation with Dirac-Delta forcing at time $t = 40$

E The forced Korteweg-de Vries-Burgers (fKdVB) Equation

By considering Equation (1) and letting , $f(x) = \frac{\gamma}{2}\delta_x(x)$ which is a Dirac-Delta forcing. We then input the following set of data into BURSO.

Set	L	N	M	$\triangle t$	ε	ν	μ	γ
A1	150	512	6400	0.01	6	0.01	1	1

We do observe that the forced uniform solitons are generated at downstream, a depression zone is seen immediately to the left of the forcing site and some wakes moving upstream. A 3D-plot of forced uniform solitons generated by Dirac-Delta forcing of fKdVB is shown in Figure (12).
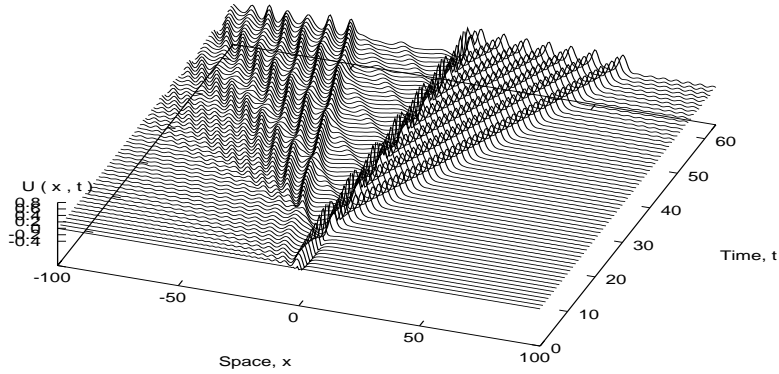

Figure 12: 3D plot of the fKdVB solution with Dirac-Delta forcing

At specific times $t = 16$, $t = 32$ and $t = 48$ the forced uniform solitons generated by Equation (1) under Dirac-Delta forcing are given by Figure (13), Figure (14) and Figure (15) respectively. We notice that at time $t = 16$, one matured and one almost matured solitons are generated. When time is doubled, at time $t = 32$, the number of solitons generated is doubled. Now 3 matured and one almost matured solitons are generated. And lastly when time is tripled, at time $t = 32$, the number of solitons generated is tripled. Now 5 matured and one almost matured solitons are generated.
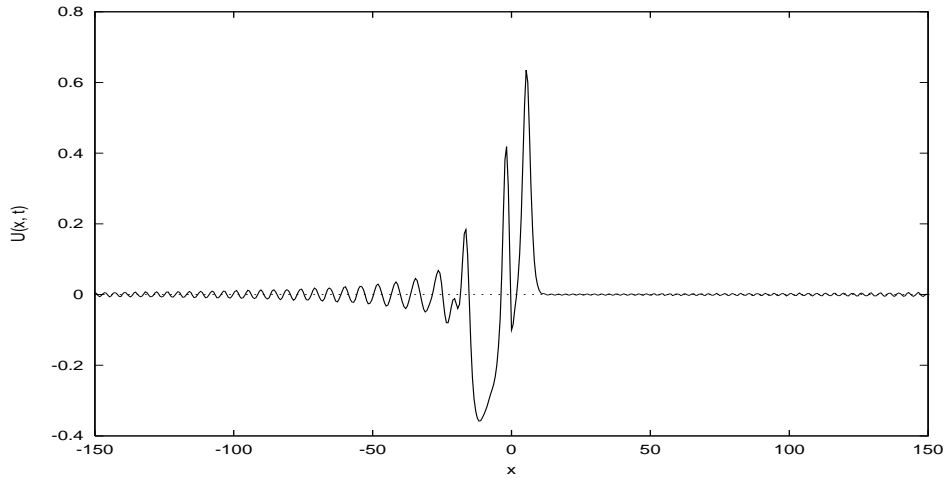
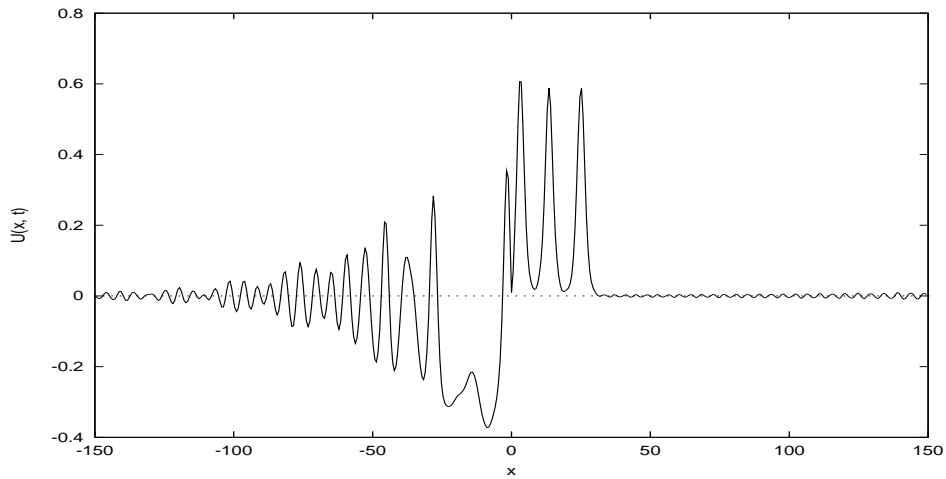Figure 13: The fKdVB solution with Dirac-Delta forcing at time $t = 16$

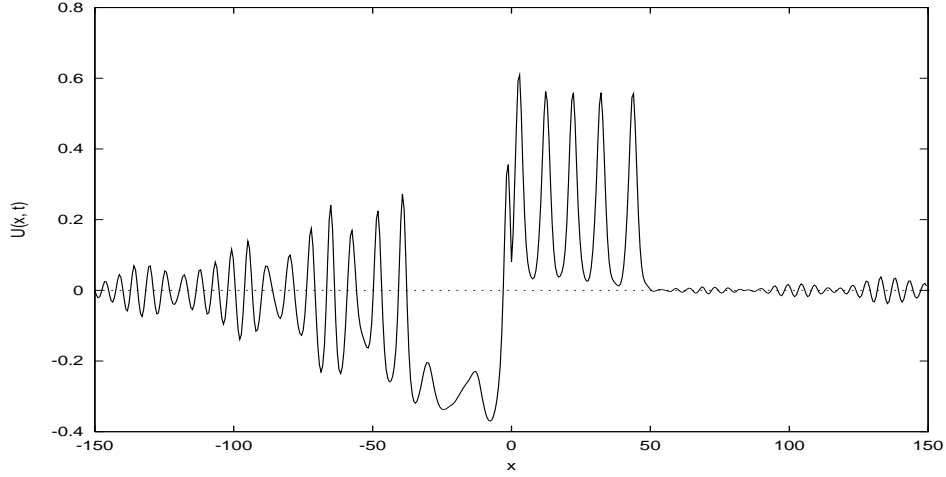Figure 14: The fKdVB solution with Dirac-Delta forcing at time $t = 32$

Figure 15: The fKdVB solution with Dirac-Delta forcing at time $t = 48$

5   Conclusion

The forcing term in the fKdVB equation causes the lost of group symmetries. Thus, traditional group-theoretical approach can no longer generate analytical solution. Consequently, the ways to solve for the fKdVB equation is through approximate and numerical method. In this paper, we have set up our numerical scheme using semi-implicit pseudo-spectral method. A user friendly graphical user interface (BURSO) has been develop to implement the numerical scheme using Matlab 7.0 software. Numerical simulation proved that BURSO is very flexible since it can solve free and force system such as the KdV, Burgers, KdVB, fKdV and fKDVB efficiently. In our future attempt, we will look for approximate solution to Equation (1) and later compare the results with those we have obtained numerically.

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