On discontinuity waves and smooth waves in thermo-piezoelectric bodies

A. Montanaro
montanar@math.unipd.it
Department of Mathematics
University of Padua, Italy

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1 Abstract

The solid body $\mathcal{B}$ under consideration is composed of a linear thermo-piezoelectric medium, i.e., a non-magnetizable linearly elastic dielectric medium that is heat conducting and not electric conducting; $\mathcal{B}$ has a natural configuration, say a placement $\kappa[\mathcal{B}]$ of the three-dimensional Euclidean space that $\mathcal{B}$ can occupy with zero stress, uniform temperature and uniform electric field. Such natural configuration and state will be used as reference.

We consider processes of $\mathcal{B}$ constituted by small displacements, thermal deviations and small electric fields

$$(\mathbf{u},\, T,\, \mathbf{E}\,)$$

superposed to $\kappa[\mathcal{B}]$.

A smooth singular surface (or discontinuity surface) of order r in the triple of fields $(\mathbf{u},\, T,\, \mathbf{E}\,)$ is referred to as a weak (thermo-piezoelectric) wave if $r \geq 2$. Here it is shown that any singular surface of order $r \geq 2$ is characteristic (for the linear thermo-piezoelectric partial differential equations). This result generalizes to piezoelectric heat-conducting bodies the results of [1] that hold for piezoelectric bodies that are not heat-conducting.

Then smooth waves are considered.

(i) It is shown that the wavefront of a plane progressive wave is characteristic if and only if the wave is isothermal.

(ii) The differential equations are characterized for standing waves of a general type and for the standing waves which are sinusoidal. The latter are isothermal, isentropic, have wavefronts which are characteristic, and their directions of polarization and of propagation satisfy certain constitutive conditions.

(iii) The differential equations for plane progressive waves which are reversible in time are characterized.

2 Linear Thermo-Piezoelectricity

2.1 Introduction

We adopt the linearized theory for thermo-piezoelectricity that is developed in [2], [3]; such a general framing contains more particular theories as, for example, the theory in [4] A unique system of coordinates (x_1, x_2, x_3) for both the reference configuration and the ambient space will be used, and the notations in [2], [3] are in force:

$\mathbf{t}$ mechanical Cauchy stress tensor
$\mathbf{E}$ electric vector
ϕ electrostatic potential
T incremental absolute temperature
$\mathbf{D}$ electric displacement vector,

The linear constitutive equations are specified in terms of the constitutive quantities listed below.

σ_{klij} = elastic moduli,
e_{ikl} = piezoelectric moduli,
β_{kl} = thermal stress moduli,
κ^E_{kl} = dielectric susceptibility,
$\tilde{\omega}_k$ = pyroelectric polarizability,
ϵ_{kl} = permittivity moduli,
κ_{kl} = Fourier coefficients,
γ = heat capacity,
η_o = entropy at the natural state,
T_o = absolute temperature at the natural state,
ρ_o = mass-density at the natural state,

2.2 Constitutive Equations and Balance Laws

The following constitutive equations are assumed respectively for the Cauchy stress, electric displacement vector, heat flux vector and specific entropy:

$$t_{kl} = \sigma_{klij}\, u_{i,\,j} - e_{ikl}\, E_i - \beta_{kl}\, T\,, \tag{1}$$

$$D_k = e_{kij}\, u_{i,\,j} + \epsilon_{ki}\, E_i + \tilde{\omega}_k\, T\,, \tag{2}$$

$$q_k = \kappa_{kl}\, T_{,\,l} + \kappa^E_{kl}\, E_l\,, \tag{3}$$

$$\eta = \eta_0 + \frac{\gamma}{T_0}\, T + \frac{1}{\rho_o}\,(\beta_{ij}\, u_{i,\,j} + \tilde{\omega}_i\, E_i)\,, \tag{4}$$

where

$$E_i = -\phi_{,\,i} \tag{5}$$

and the following symmetries hold:

$$\sigma_{klij} = \sigma_{ijkl} = \sigma_{lkij} = \sigma_{klji}\,, \tag{6}$$

$$e_{kij} = e_{kji}\,, \qquad \beta_{ij} = \beta_{ji}\,, \tag{7}$$

$$\kappa_{kl} = \kappa_{lk}\,, \qquad \kappa^E_{kl} = \kappa^E_{lk}\,. \tag{8}$$

In the absence of external fields the field equations corresponding to the (i) balance law of linear momentum, (ii) Maxwell's equation, and (iii) balance law of conservation of energy, write as

$$t_{kl,\,k} - \rho_0 \ddot{u}_l = 0\,, \tag{9}$$

$$D_{k,\,k} = 0\,, \qquad \rho_o\,\theta\,\dot{\eta} - q_{k,\,k} = 0\,. \tag{10}$$

2.3 Field Equations

The linearized field equations, that are obtained by replacing the constitutive equations in the balance laws and neglecting the higher order terms, in the homogeneous case write as

$$\sigma_{klij}\,u_{i,\,jk} + e_{ijl}\,\phi_{,\,ij} - \beta_{kl}\,T_{,\,k} = \rho_o\,\ddot{u}_l\,, \qquad (l = 1,\,2,\,3)\,, \tag{11}$$

$$e_{kji}\,u_{j,\,ik} - \epsilon_{kj}\,\phi_{,\,jk} + \tilde{\omega}_k\,T_{,\,k} = 0\,, \tag{12}$$

$$-\kappa_{kj}\,T_{,\,jk} + \kappa^E_{jk}\,\phi_{,\,jk} + T_0\,\beta_{kj}\,\dot{u}_{k,\,j} + \rho_o\,\gamma\,\dot{T} - T_0\,\tilde{\omega}_k\,\dot{\phi}_{,k} = 0\,. \tag{13}$$

2.4 Characteristic Hypersurfaces of the Linear Thermo-Piezoelectric Equations

A *characteristic manifold* of a system of partial differential equations is a surface in $\mathbb{R}^4$ that is exceptional for the assignment of data in the appropriate Cauchy initial value problem.

The *characteristic equation* of the linear differential equations (11-13) is the equation obtained by equating to zero the determinant of the coefficients of the system of five equations

$$\begin{array}{rrrrl} & (\sigma_{klij}n_j n_k - \rho_o V^2 \delta_{li})\lambda_i & + & e_{ilj}n_i n_j\,\varphi & = 0 \\ & e_{kji}n_i n_k\,\lambda_j & - & \epsilon_{kj}n_j n_k\,\varphi & = 0 \\ -\kappa_{kj}n_j n_k\,\tau \;+ & T_0 V \beta_{ij} n_j\,\lambda_i & + & (\kappa^E_{jk}n_j n_k - T_0 n_k\,\tilde{\omega}_k V)\,\varphi & = 0 \end{array} \tag{14}$$

in the five scalar unknowns τ, λ_i, φ (see [5], where this result is shown to hold also for a non homogeneous body).

3 Discontinuity Waves

3.1 Compatibility Conditions for Jumps of Partial Derivatives

Let E^3 denote the three-dimensional Euclidean ambient space, $I = [t_o,\,t_1]$ a time interval and $\mathcal{E} = I \times E^3$. We consider a smooth hypersurface $\mathcal{S}$ in $\mathcal{E}$ that admits a suitably regular representation

$$x_i = \psi_i(t,\,\xi_1,\,\xi_2), \quad i = 1,\,2,\,3\,, \tag{15}$$

with the parameter pair belonging to an open subset of $I\!R^2$. For any value of t equation (15) defines a surface $\mathcal{S}_t$ in E^3, referred to the curvilinear coordinates ξ_1, ξ_2. The totality of surfaces $\mathcal{S}_t$ for $t \in I$ is a moving surface in E^3. Thus $\mathcal{S}$ can be interpreted as both the hypersurface of $\mathcal{E}$ of equations (15) and the associated moving surface in E^3.

The *speed* $\mathbf{V}$ of the surface $\mathcal{S}$ at time t has x components

$$V_i = \frac{\partial \psi_i}{\partial t} \tag{16}$$

and the *speed* of $\mathcal{S}_t$ in direction of $\mathbf{n}$ is

$$V = V_i \, n_i \, . \tag{17}$$

Now let $f : \mathcal{N} \to I\!R$ be a real scalar-valued function, where $\mathcal{N} = I \times N$ with N open subset of E^3 having, for all $t \in I$, non-empty intersection with $\mathcal{S}_t$. Since the results below refer only to the part of $\mathcal{S}$ contained in $\mathcal{N}$, we replace $\mathcal{S} \cap \mathcal{N}$ by $\mathcal{S}$ and $\mathcal{S}_t \cap \mathcal{N}$ by $\mathcal{S}_t$. Let $\partial f / \partial n$ denote the derivative of f in the direction of $\mathbf{n}$ on $\mathcal{S}_t$, where n is distance measured from $\mathcal{S}_t$. Hence $\partial / \partial n \equiv n_i \, \partial / \partial x_i$.

If $\mathcal{S}$ is a singular hypersurface in $\mathcal{E}$ of order $r \geq 2$ for the function $f = f(x_1, \, x_2, \, x_3, \, t)$, then the compatibility conditions (Hadamard [6] , pp.103-104)

$$\left[\frac{\partial^r f}{\partial x_i \partial x_j \ldots \partial x_l \partial t^{r-s}} \right] = (-V)^{r-s} \left[\frac{\partial^r f}{\partial n_r} \right] n_i \, n_j \ldots n_l \qquad (0 \leq s \leq r) \, , \tag{18}$$

hold on $\mathcal{S}$, where

$$\frac{\partial^r f}{\partial n^r} = \partial^r f \partial x_p \ldots \partial x_q \, n_p \ldots n_q \quad (r \text{ indexes}) \, , \tag{19}$$

and V is the local speed of propagation with respect to the medium, apply to the derivatives of f.

3.2 Weak Discontinuity Waves

The *l.p.d.e.*s (14) are of second order; thus the name *weak thermo-piezoelectric wave*, briefly *weak wave*, is applied to a singular hypersurface $\mathcal{S} \subset \mathcal{E} := I \times I\!R^3$ for the dependent variables $(u_i, \, \phi, \, T\,)$ of order $r \geq 2$.

Proposition 3.1 *Assume* (a). *Then weak waves are characteristic for the l.p.d.e.s (14).*

Proof. Let $\mathcal{S}$ be a weak wave; then, across $\mathcal{S}$ the jumps of the rth partial derivatives of $(u_i, \, \phi, \, T\,)$ are defined and the jumps of the partial derivatives of order lower than r identically vanish. For $r > 2$ the *l.p.d.e.*s (14) hold on $\mathcal{B}' := I \times \mathcal{B}$ and for $r = 2$ they hold on $\mathcal{B}' \setminus \mathcal{S}$. As a consequence, for all $r \geq 2$ the three equations below, that are obtained by applying to (14) the differential operator

$$\frac{\partial^{r-2}}{\partial x_a \ldots x_c} \qquad (r-2 \quad \text{summed indexes}) \, , \tag{20}$$

hold on $\mathcal{B}' \setminus \mathcal{S}$. That is, we have

$$\sigma_{klij}\,\frac{\partial^r u_i}{\partial x_a\ldots x_c\partial x_j\partial x_k}\;+\;e_{ijl}\,\frac{\partial^r \phi}{\partial x_a\ldots x_c\partial x_i\partial x_j}\;-\;\rho_o\,\frac{\partial^r u_l}{\partial x_a\ldots x_c\partial t^2}\;=\;0 \quad (21)$$

$$e_{kij}\,\frac{\partial^r u_j}{\partial x_a\ldots x_c\partial x_i\partial x_k}\;-\;\epsilon_{kj}\,\frac{\partial^r \phi}{\partial x_a\ldots x_c\partial x_j\partial x_k}\;=\;0 \quad (22)$$

$$-\kappa_{kj}\,\frac{\partial^r T}{\partial x_a\ldots x_c\partial x_j\partial x_k}\;+\;\kappa^E_{jk}\,\frac{\partial^r \phi}{\partial x_a\ldots x_c\partial x_j\partial x_k}\;+\;T_0\beta_{kj}\,\frac{\partial^r u_k}{\partial x_a\ldots x_c\partial x_j\partial t}$$
$$-\,T_0\,\tilde{\omega}_k\,\frac{\partial^r \phi}{\partial x_a\ldots x_c\partial x_k\partial t}\;=\;0 \quad (23)$$

Now, by (c) it follows that the right-hand sides in equations (21), (22) and (23) are terms involving derivatives of order lower than r. Thus their jumps across $\mathcal{S}$ identically vanish. As a consequence, forming the jumps across $\mathcal{S}$ of the *l.p.d.e.*s (21)-(23) yields

$$\sigma_{klij}\left[\frac{\partial^r u_i}{\partial x_a\ldots x_c\partial x_j\partial x_k}\right]\;+\;e_{ijl}\left[\frac{\partial^r \phi}{\partial x_a\ldots x_c\partial x_i\partial x_j}\right]\;=\;\rho_o\left[\frac{\partial^r u_l}{\partial x_a\ldots x_c\partial t^2}\right] \quad (24)$$

$$e_{kij}\left[\frac{\partial^r u_j}{\partial x_a\ldots x_c\partial x_i\partial x_k}\right]\;-\;\epsilon_{kj}\left[\frac{\partial^r \phi}{\partial x_a\ldots x_c\partial x_j\partial x_k}\right]\;=\;0 \quad (25)$$

$$-\kappa_{kj}\left[\frac{\partial^r T}{\partial x_a\ldots x_c\partial x_j\partial x_k}\right]\;+\;\kappa^E_{jk}\left[\frac{\partial^r \phi}{\partial x_a\ldots x_c\partial x_j\partial x_k}\right]$$
$$+\,T_0\beta_{kj}\left[\frac{\partial^r u_k}{\partial x_a\ldots x_c\partial x_j\partial t}\right]\;-\;T_0\,\tilde{\omega}_k\left[\frac{\partial^r \phi}{\partial x_a\ldots x_c\partial x_k\partial t}\right]\;=\;0 \quad (26)$$

Now, the compatibility conditions for the jumps (18) apply to each of the functions $(u_i,\,\phi,\,T\,)$ where, in the spatial picture, V must be interpreted as the local speed of propagation w.r.t. the medium. Substituting them in equations (24)-(26) and then multiplying each term by $n_a\ldots n_c$ and summing on the repeated indexes $a,\ldots c$ we have the equations for the jumps

$$\begin{array}{ccccccc} & & (\sigma_{klij}n_jn_k-\rho_oV^2\delta_{li})\lambda_i & + & e_{ilj}n_in_j\,\varphi & = & 0\\ & & e_{kji}n_in_k\,\lambda_j & - & \epsilon_{kj}n_jn_k\,\varphi & = & 0\\ -\kappa_{kj}n_jn_k\,\tau & + & T_0V\beta_{ij}n_j\,\lambda_i & + & (\kappa^E_{jk}n_jn_k-T_0n_k\,\tilde{\omega}_kV)\,\varphi & = & 0\,, \end{array} \quad (27)$$

where

$$\lambda_i\;=\;\left[\frac{\partial^r u_i}{\partial n_r}\right],\qquad \varphi\;=\;\left[\frac{\partial^r \phi}{\partial n_r}\right],\qquad \tau\;=\;\left[\frac{\partial^r T}{\partial n_r}\right]. \quad (28)$$

Note that equations (27) just coincide with equations (14). Q.E.D.

4 Smooth Waves

4.1 Time-Reversible Processes

A *possible (thermo-piezoelectric) process of* $\mathcal{B}$ is defined as a solution

$$(\mathbf{u},\,\phi,\,T) = (\mathbf{u},\,\phi,\,T)\,(\mathbf{x},\,t)$$

of the field equations (11-13).

We say that a possible process of $\mathcal{B}$

$$p := (\,\mathbf{u},\, T,\, \phi\,) \,:\, \mathcal{B} \times [t_o,\, t_1] \to I\!\!R^3 \times I\!\!R \times I\!\!R\,, \qquad p = p(\mathbf{x},\, t)$$

is *time-reversible*, or *reversible in time*, if the process

$$p^- = \left(\mathbf{u}^-,\, \phi^-,\, T^-\right) :\, \mathcal{B}\times[0,\, t_1 - t_o] \to I\!\!R^3\times I\!\!R\times I\!\!R, \quad p^- := p(\mathbf{x},\, r), \quad r := -t+t_1$$

is also possible for $\mathcal{B}$.

Proposition 4.1 *Let* $p := (\mathbf{u},\, \phi,\, T)$ *be a thermo-piezoelectric process that is possible for* $\mathcal{B}$.

Then p^- *is possible for* $\mathcal{B}$ *if and only if* p *is solution of the field equations*

$$\sigma_{klij}\, u_{i,\,jk} \,+\, e_{ijl}\, \phi_{,\,ij} \,-\, \beta_{kl}\, T_{,\,k} \,=\, \rho_o\, \ddot{u}_l \qquad (l = 1,\, 2,\, 3)\,, \tag{29}$$

$$e_{kji}\, u_{j,\,ik} \,-\, \epsilon_{kj}\, \phi_{,\,jk} \,+\, \tilde{\omega}_k\, T_{,\,k} \,=\, 0\,, \tag{30}$$

$$-\kappa_{kj}\, T_{,\,jk} \,+\, \kappa^E_{jk}\, \phi_{,\,jk} \,=\, 0\,, \tag{31}$$

$$T_0\, \beta_{kj}\, \dot{u}_{k,\,j} \,+\, \rho_o\, \gamma\, \dot{T} \,-\, T_0\, \tilde{\omega}_k\, \dot{\phi}_{,k} \,=\, 0\,. \tag{32}$$

<u>Proof</u>. The above definition of time-reversed process p^- associated to p implies that if both p and p^- are possible processes, then they solve Eq.s (11-12), p solves Eq.(13), whereas p^- solves the equation

$$-\kappa_{kj}\, T_{,\,jk} \,+\, \kappa^E_{jk}\, \phi_{,\,jk} \,-\, \left(T_0\, \beta_{kj}\, \dot{u}_{k,\,j} \,+\, \rho_o\, \gamma\, \dot{T} \,-\, T_0\, \tilde{\omega}_k\, \dot{\phi}_{,k}\right) \,=\, 0 \tag{33}$$

since

$$\frac{\partial p^-}{\partial t} = -\frac{\partial p}{\partial t}, \qquad \frac{\partial^2 p^-}{\partial t^2} = \frac{\partial p^2}{\partial t^2}\,.$$

The thesis follows since Eq.s (13) and (33) are equivalent to Eq.s (31) and (32). Q.E.D.

4.2 Standing Waves

In considering *standing waves* (or *vibrations*) propagating in a not heat-conducting piezoelectric body one seeks solutions of the type

$$\mathbf{u}(\mathbf{x},\, t) \,=\, \overline{\mathbf{u}}(\mathbf{n}\cdot\mathbf{x})\cos(\omega\, t)\,, \quad \phi(\mathbf{x},\, t) \,=\, \overline{\phi}(\mathbf{n}\cdot\mathbf{x})\cos(\omega\, t)\,, \tag{34}$$

where ω is the *circular frequency* and $\mathbf{n}$ is a unit vector, that may be called the *direction of vibration.* Here, by analogy, for a heat-conducting piezoelectric body we seek solutions $(\mathbf{u},\, \phi,\, T)$ having the form

$$\begin{aligned} \mathbf{u}(\mathbf{x},\, t) &= \overline{\mathbf{u}}(\mathbf{n}\cdot\mathbf{x})\cos(\omega\, t) \\ \phi(\mathbf{x},\, t) &= \overline{\phi}(\mathbf{n}\cdot\mathbf{x})\cos(\omega\, t) \\ T(\mathbf{x},\, t) &= \overline{T}(\mathbf{n}\cdot\mathbf{x})\cos(\omega\, t) \end{aligned} \tag{35}$$

for some $\omega \in I\!\!R$ and unit vector $\mathbf{n}$. The next proposition shows that the solutions of vibration type solve of a system of ordinary differential equations.

Proposition 4.2 *Let the triple of functions* $(\mathbf{u},\, \phi,\, T)$ *be of the form (35) for some* $\omega \in I\!R$ *and unit vector* $\mathbf{n}$. *Then*

$(\mathbf{u},\, \phi,\, T)$ *is a solution of the field equations (11)-(13)*

if and only if

the functions $(\overline{\mathbf{u}},\, \overline{\phi},\, \overline{T})$ *in Eq.s (35) satisfy the system of ordinary differential equations*

$$\sigma_{klij}\, \overline{u}_i^{''}\, n_j n_k \,+\, \rho_o\, \delta_{li}\, \overline{u}_i\, \omega^2 \,+\, e_{ijl}\, \overline{\phi}^{''}\, n_i n_j \,-\, \beta_{kl}\, \overline{T}^{'}\, n_k \,=\, 0 \quad (l = 1,\, 2,\, 3)\,, \tag{36}$$

$$e_{kji}\, \overline{u}_j^{''}\, n_i n_k \,-\, \epsilon_{kj}\, \overline{\phi}^{''}\, n_j n_k \,+\, \tilde{\omega}_k\, \overline{T}^{'}\, n_k \,=\, 0\,, \tag{37}$$

$$(-\kappa_{kj}\, \overline{T}^{''} \,+\, \kappa^E_{jk}\, \overline{\phi}^{''}\,)\, n_j n_k \,=\, 0\,, \tag{38}$$

$$T_0\, \overline{u}_k^{'}\, \beta_{kj}\, n_j \,+\, \rho_o\, \gamma\, \overline{T} \,-\, T_0\, \overline{\phi}^{'}\, \tilde{\omega}_k\, n_k \,=\, 0\,. \tag{39}$$

Proof. Eq.s (35) for each $a,\, b,\, c = 1,\, 2,\, 3$ yield

$$\begin{gathered} u_{a,\,bc} = \overline{u}''_a n_b n_c \cos(\omega\, t), \quad \ddot{u}_a = -\overline{u}_a \omega^2 \cos(\omega\, t), \quad \dot{u}_{a,\,b} = -\overline{u}'_a n_b \omega \sin(\omega\, t), \\ \phi_{,\,bc} = \overline{\phi}'' n_b n_c \cos(\omega\, t), \quad \dot{\phi}_{,\,b} = -\overline{\phi}' n_b \omega \sin(\omega\, t), \\ T_{,\,b} = \overline{T}' n_b \cos(\omega\, t), \quad \dot{T} = -\overline{T} \omega \sin(\omega\, t), \quad \dot{T}_{,\,b} = -\overline{T}' n_b \omega \sin(\omega\, t). \end{gathered} \tag{40}$$

Replacing (35), (40) in (11)-(12) and eliminating the factor $\cos(\omega t)$ yields the *o.d.e.*s (36), (37); replacing (35), (40) in (13) yields the equation

$$\begin{gathered} (-\kappa_{kj}\, \overline{T}^{''}\, n_j n_k \,+\, \kappa^E_{jk}\, \overline{\phi}^{''}\, n_j n_k) \cos(\omega\, t) \,+ \\ +\, (\,-\,T_0\, \beta_{kj}\, \overline{u}_k^{'}\, n_j \,-\, \rho_o\, \gamma\, \overline{T} \,+\, T_0\, \tilde{\omega}_k\, \overline{\phi}^{'}\, n_k\,)\, \omega\, \sin(\omega\, t) = 0\,, \end{gathered} \tag{41}$$

that is equivalent to the equations (38), (39).

Conversely, if a triple of functions $(\overline{\mathbf{u}}, \overline{\phi}, \overline{T}) = (\overline{\mathbf{u}}, \overline{\phi}, \overline{T})(\zeta)$, $\zeta \in I\!R$, are solutions of the equations (36)-(39), then the related triple of functions $(\mathbf{u},\, \phi,\, T)$ defined by (35) satisfies the field equations (11)-(13). Q.E.D.

Corollary 4.1 *Any possible standing wave is reversible in time.*

Proof. $p = (\mathbf{u},\, \phi,\, T)$ in Eq.s (35) is a possible standing wave if and only if p solves Eq.s (36)-(39); the latter are equivalent to Eq.s (29)-(32) under the substitution (35). Q.E.D.

4.3 Plane Progressive Waves

A plane progressive (thermo-piezoelectric) wave propagating in the direction of a unit vector $\mathbf{n}$ with *speed* (or *phase velocity*) V may be represented by

$$\begin{aligned} \mathbf{u} &= \mathbf{u}(\mathbf{n} \cdot \mathbf{x} + Vt) \\ \phi &= \phi(\mathbf{n} \cdot \mathbf{x} + Vt) \\ T &= T(\mathbf{n} \cdot \mathbf{x} + Vt)\,. \end{aligned} \tag{42}$$

Definition 4.1 *The* wavefronts *of the plane progressive wave (42) are the hyperplanes of equation* $\mathbf{n}\cdot\mathbf{x}+Vt=c$ *, with* c *constant.*

Definition 4.2 *We say that* a wavefront π of (42) is characteristic *if at each* $(\mathbf{x},\, t)\in\pi$ *the* 5*−tuple*

$$(\lambda_i,\, \varphi,\, \tau) := (u_i^{''},\, \phi^{''}, T^{''})$$

does not vanish and is a solution of the equations $(14)_{1-3}$.

Proposition 4.3 *Let the triple of functions* $(\mathbf{u},\, \phi,\, T)$ *have the form (42) for some* $V\in I\!R$ *and unit vector* $\mathbf{n}$.

Then $(\mathbf{u},\, \phi,\, T)$ *is a solution of the field equations (11)-(13)*
if and only if

$$\sigma_{klij}\, u_i^{''} n_j n_k - \rho_o\, \delta_{li}\, u_i^{''}\, V^2 + e_{ijl}\, \phi^{''} n_i n_j - \beta_{kl}\, T' n_k \;=\; 0\,, \quad (l=1,\,2,\,3)\,, \tag{43}$$

$$e_{kji}\, u_j^{''} n_i n_k \;-\; \epsilon_{kj}\, \phi^{''} n_j n_k \;+\; \tilde{\omega}_k\, T' n_k \;=\; 0\,, \tag{44}$$

$$-\kappa_{kj}\, T^{''} n_j n_k \;+\; \kappa^E_{jk}\, \phi^{''} n_j n_k \;+\; V\left(T_0\, \beta_{kj}\, u_k^{''} n_j \;+\; \rho_o\, \gamma\, T' \;-\; T_0\, \tilde{\omega}_k\, \phi^{''} n_k\right) \;=\; 0\,. \tag{45}$$

<u>Proof</u>. Eq.s (42) for each $a,\, b,\, c=1,\, 2,\, 3$ yield

$$\begin{aligned} u_{a,\,bc} = u_a'' n_b n_c, \quad \ddot{u}_a = u_a'' V^2, \quad \dot{u}_{a,\,b} = u_a'' n_b V \\ \phi_{,\,bc} = \phi'' n_b n_c, \quad \dot{\phi}_{,\,b} = \phi'' n_b V, \\ T_{,\,b} = T' n_b, \quad \dot{T} = T' V, \quad \dot{T}_{,\,b} = T'' n_b V\,. \end{aligned} \tag{46}$$

The field equations (11)-(13) are equivalent to Eq.s (43)-(45) under the substitution (42). Q.E.D.

Proposition 4.4 *For some* $V\in I\!R$ *and unit vector* $\mathbf{n}$ *let the triple of functions* $(\mathbf{u},\, \phi,\, T)$ *in (42) be a solution of the field equations (43)-(45).*

Then the wavefronts of $(\mathbf{u},\, \phi,\, T)$ *are characteristic if and only if* $(\mathbf{u},\, \phi,\, T)$ *is an isothermal wave.*

<u>Proof</u>. Let $(\mathbf{u},\, \phi,\, T)$, having the form (42) for some $V\in I\!R$ and unit vector $\mathbf{n}$, be a solution of (43)-(45); put $(\lambda_i := u_i^{''},\, \phi := \phi^{''},\, \tau := T^{''})$; then by Proposition 4.3 $(\mathbf{u},\, \phi,\, T)$ is solution of (11)-(13) if and only if

$$(\sigma_{klij}\, n_j n_k \;-\; \rho_o\, \delta_{il} V^2)\, \lambda_i \;+\; e_{ijl}\, \phi n_i n_j \;-\; \beta_{kl}\, T' n_k \;=\; 0\,, \quad (l=1,\,2,\,3)\,, \tag{47}$$

$$e_{kji}\, \lambda_j n_i n_k \;-\; \epsilon_{kj}\, \phi n_j n_k \;+\; \tilde{\omega}_k\, T' n_k \;=\; 0\,, \tag{48}$$

$$-\kappa_{kj}\, \tau n_j n_k \;+\; \kappa^E_{jk}\, \phi n_j n_k \;+\; V\,(T_0\, \beta_{kj}\, \lambda_k n_j \;+\; \rho_o\, \gamma\, T' \;-\; T_0\, \tilde{\omega}_k\, \phi n_k) \;=\; 0\,. \tag{49}$$

Hence, confronting the two system of equations (47)-(49) and $(14)_{1-3}$ yields that $(\mathbf{u},\, \phi,\, T)$ is solution of (11)-(13) and its wavefronts are characteristic if and only if $T' = 0$. Q.E.D.

Now we put

$$\begin{aligned} \mathbf{u} &= \mathbf{u}(\mathbf{n}\cdot\mathbf{x} + Vt) \\ \phi &= \phi(\mathbf{n}\cdot\mathbf{x} + Vt) \\ T &= T(\mathbf{n}\cdot\mathbf{x} + Vt) \end{aligned} \tag{50}$$

$$\begin{aligned} \mathbf{u}^- &= \mathbf{u}(\mathbf{n}\cdot\mathbf{x} - Vt) \\ \phi^- &= \phi(\mathbf{n}\cdot\mathbf{x} - Vt) \\ T^- &= T(\mathbf{n}\cdot\mathbf{x} - Vt) \end{aligned} \tag{51}$$

It is easy to prove the following

Proposition 4.5 *Let* $\mathbf{n}$ *be a unit vector and* $V > 0$. *Then both* $p := (\mathbf{u},\, \phi,\, T)$ *and* $p^- := (\mathbf{u}^-,\, \phi^-,\, T^-)$ *are solutions to the field equations (11)-(13) if and only if the equations (52)-(55) below hold.*

$$(\sigma_{klij}\, n_j n_k \,-\, \rho_o\, \delta_{il} V^2)\, u_i^{''} \,+\, e_{ijl}\, \phi^{''} n_i n_j \,-\, \beta_{kl}\, T' n_k \,=\, 0\,, \quad (l = 1,\, 2,\, 3)\,, \tag{52}$$

$$e_{kji}\, u_j^{''} n_i n_k \,-\, \epsilon_{kj}\, \phi^{''} n_j n_k \,+\, \tilde{\omega}_k\, T' n_k \,=\, 0\,, \tag{53}$$

$$-\kappa_{kj}\, T^{''} n_j n_k \,+\, \kappa_{jk}^E\, \phi^{''} n_j n_k \,=\, 0\,, \tag{54}$$

$$T_0\, \beta_{kj}\, u_k^{''} n_j \,+\, \rho_o\, \gamma\, T' \,-\, T_0\, \tilde{\omega}_k\, \phi^{''} n_k \,=\, 0\,. \tag{55}$$

Proof. Let p and p^- be solutions of Eq.s (11)-(13); then by Proposition 4.3 Eq.s (43)-(45) hold and furthermore the latter equation also holds with V replaced with $-V$; hence Eq. (45) splits in Eq.s (54)-(55). Conversely, if Eq.s (52)-(55) hold, then also Eq.s (43)-(45) hold. Q.E.D.

Corollary 4.2 *A possible plane progressive wave* $(\mathbf{u},\, \phi,\, T)$ *is reversible in time if and only if both the waves* $(\mathbf{u},\, \phi,\, T)$ *and* $(\mathbf{u}^-,\, \phi^-,\, T^-)$ *are possible processes.*

Proof. Eq.s (52)-(55) are equivalent to Eq.s (29)-(32) under the substitution (50) or (51). Hence Proposition 4.1 yields the thesis. Q.E.D.

4.4 Isentropic Plane Waves

Now we consider isentropic plane waves, that is solutions

$$\begin{aligned} p &= (\mathbf{u},\, \phi,\, T) \\ \mathbf{u} &= \mathbf{u}(\mathbf{n}\cdot\mathbf{x} + Vt) \\ \phi &= \phi(\mathbf{n}\cdot\mathbf{x} + Vt) \\ T &= T(\mathbf{n}\cdot\mathbf{x} + Vt) \end{aligned} \tag{56}$$

of the field equations such that (see (4))

$$T_o\, \rho_o\, \dot{\eta} \,=\, \rho_o\, \gamma\, \dot{T} \,+\, T_o\, (\beta_{ij}\, \dot{u}_{i,\,j} \,-\, \tilde{\omega}_k\, \dot{\phi}_{,k}) \,=\, 0\,. \tag{57}$$

By replacing

$$\begin{aligned} \dot{\mathbf{u}} &= \mathbf{u}'(\mathbf{n}\cdot\mathbf{x}+Vt)\,V \\ \dot{\phi} &= \phi'(\mathbf{n}\cdot\mathbf{x}+Vt)\,V \\ \dot{T} &= T'(\mathbf{n}\cdot\mathbf{x}+Vt)\,V \end{aligned} \tag{58}$$

into (57) we obtain

$$T_o\,\rho_o\,\dot{\eta} \;=\; \Big(\rho_o\,\gamma\,T' \;+\; T_o\,(\beta_{ij}\,u_i^{'}n_j \;-\; \tilde{\omega}_k\,\phi^{'}n_k)\Big)(+V) \;=\; 0. \tag{59}$$

Hence

$$\dot{\eta} \;=\; 0 \qquad \Longleftrightarrow \qquad \rho_o\,\gamma\,T' \;+\; T_o\,(\beta_{ij}\,u_i^{'}n_j \;-\; \tilde{\omega}_k\,\phi^{'}n_k) \;=\; 0\,. \tag{60}$$

In particular, *if* $\beta_{ij}\,n_j \;=\; 0 \;=\; \tilde{\omega}_k\,n_k$, *then the plane wave is isentropic if and only if it is isothermal* $\Big(\dot{\eta} \;=\; 0 \Longleftrightarrow T' \;=\; 0\Big)$.

4.5 Isentropic Standing Waves

We can repeat the procedure above for standing waves too. Let $(\mathbf{u},\,\phi,\,T)$ be of the form (35) for some $\omega \in I\!R$ and unit vector $\mathbf{n}$. Hence

$$\begin{aligned} \dot{\mathbf{u}} &= \overline{\mathbf{u}}(\mathbf{n}\cdot\mathbf{x})\,\big[-\omega\sin(\omega t)\big] \\ \dot{\phi} &= \overline{\phi}(\mathbf{n}\cdot\mathbf{x})\,\big[-\omega\sin(\omega t)\big] \\ \dot{T} &= \overline{T}(\mathbf{n}\cdot\mathbf{x})\,\big[-\omega\sin(\omega t)\big] \end{aligned} \tag{61}$$

By replacing (61) into (57) we obtain

$$T_o\,\rho_o\,\dot{\eta} \;=\; \Big(\rho_o\,\gamma\,\overline{T} \;+\; T_o\,(\beta_{ij}\,\overline{u}_i n_j \;-\; \tilde{\omega}_k\,\overline{\phi} n_k)\Big)\,\big[-\omega\sin(\omega t)\big] \tag{62}$$

Hence

$$\dot{\eta} \;=\; 0 \qquad \Longleftrightarrow \qquad \rho_o\,\gamma\,\overline{T} \;+\; T_o\,(\beta_{ij}\,\overline{u}_i n_j \;-\; \tilde{\omega}_k\,\overline{\phi} n_k) \;=\; 0\,. \tag{63}$$

In particular,
if $\beta_{ij}\,n_j \;=\; 0 \;=\; \tilde{\omega}_k\,n_k$, *then the standing wave is isentropic if and only if it is isothermal* $\Big(\dot{\eta} \;=\; 0 \Longleftrightarrow \overline{T} \;=\; 0\Big)$.

4.6 Sinusoidal Standing Waves

Next we study a standing wave

$$p := (\mathbf{u},\,\phi,\,T)$$

that is sinusoidal, in the sense that its form (35) is further specified by

$$\begin{aligned} \mathbf{u}(\mathbf{x},\,t) \;&=\; \boldsymbol{\lambda}\sin(k\,\mathbf{n}\cdot\mathbf{x})\cos(\omega\,t) \\ \phi(\mathbf{x},\,t) \;&=\; \phi\sin(k\,\mathbf{n}\cdot\mathbf{x})\cos(\omega\,t) \\ T(\mathbf{x},\,t) \;&=\; \tau\sin(k\,\mathbf{n}\cdot\mathbf{x})\cos(\omega\,t) \end{aligned} \tag{64}$$

for some $\boldsymbol{\lambda} \in I\!R^3$, $\phi,\,\tau \in I\!R$, unit vector $\mathbf{n}$, $k > 0,\,\omega > 0$.

Note that by *prostaferesi formulae* we have

$$\sin u \, \cos v \; = \; \frac{1}{2} \, (\, \sin p \, + \, \sin q \,)$$

where $2u = p + q, \; 2v = p - q$ and $p = u + v, \; q = u - v$.
Hence

$$p = p^{+} + p^{-} \tag{65}$$

is the sum of the two sinusoidal progressive waves

$$p^{+} \, := \, (\mathbf{u}^{+}, \, \phi^{+}, \, T^{+}), \qquad p^{-} \, := \, (\mathbf{u}^{-}, \, \phi^{-}, \, T^{-}), \tag{66}$$

that are defined by

$$\begin{aligned} \mathbf{u}^{\pm}(\mathbf{x}, \, t) \; &= \; \frac{\boldsymbol{\lambda}}{2} \sin(k \, \mathbf{n} \cdot \mathbf{x} \pm \omega \, t) \\ \phi^{\pm}(\mathbf{x}, \, t) \; &= \; \frac{\varphi}{2} \sin(k \, \mathbf{n} \cdot \mathbf{x} \pm \omega \, t) \\ T^{\pm}(\mathbf{x}, \, t) \; &= \; \frac{\tau}{2} \sin(k \, \mathbf{n} \cdot \mathbf{x} \pm \omega \, t) \, . \end{aligned} \tag{67}$$

Next we show that p can propagate if and only if both p^{+} and p^{-} can propagate.

Proposition 4.6 *The sinusoidal standing wave* p*, having the form (64), can propagate if and only if the two sinusoidal plane progressive waves* p^{+} *and* p^{-} *in (66), (67) can propagate.*

Proof. By Proposition 4.2 the wave p can propagate if and only if the equalities (36)-(39) hold with

$$\begin{aligned} \overline{\mathbf{u}}^{'}(\mathbf{x}) \; &= \; k \boldsymbol{\lambda} \cos(k \, \mathbf{n} \cdot \mathbf{x}), \qquad & \overline{\mathbf{u}}^{''}(\mathbf{x}) \; &= \; -k^2 \boldsymbol{\lambda} \sin(k \, \mathbf{n} \cdot \mathbf{x}) \\ \overline{\phi}^{'}(\mathbf{x}) \; &= \; k \varphi \cos(k \, \mathbf{n} \cdot \mathbf{x}), \qquad & \overline{\phi}^{''}(\mathbf{x}) \; &= \; -k^2 \varphi \sin(k \, \mathbf{n} \cdot \mathbf{x}) \\ \overline{T}^{'}(\mathbf{x}) \; &= \; k \tau \cos(k \, \mathbf{n} \cdot \mathbf{x}), \qquad & \overline{T}^{''}(\mathbf{x}) \; &= \; -k^2 \tau \sin(k \, \mathbf{n} \cdot \mathbf{x}) \end{aligned}$$

and by the arbitrariness of $\mathbf{x}$, the equalities (36)-(39) are equivalent to the seven equalities

$$\begin{aligned} \Big(\sigma_{klij} \, n_j n_k \, - \, \rho_o \, \delta_{il} \, V^2 \Big) \, \lambda_i \, + \, \varphi \, e_{ijl} \, n_i n_j \; &= \; 0 \\ \tau \, \beta_{kl} \, n_k \; &= \; 0 \, , \; (l = 1, \, 2, \, 3) \, , \\ e_{kji} \, \lambda_j \, n_i n_k \, - \, \varphi \, \epsilon_{kj} \, n_j n_k \; &= \; 0 \, , \\ \tau \, \tilde{\omega}_k \, n_k \; &= \; 0 \\ (-\kappa_{kj} \, \tau \, + \, \kappa^E_{jk} \, \varphi) \, n_j n_k \; &= \; 0 \, , \\ \lambda_k \, \beta_{kj} \, n_j \, - \, \varphi \, \tilde{\omega}_k \, n_k \; &= \; 0 \, , \\ \rho_o \, \gamma \, \tau \; &= \; 0 \end{aligned} \tag{68}$$

where $V = \omega / k$ is the *phase velocity.*

Conversely, assume that both p^+ and p^- can propagate. Then by Proposition 4.5 equations (52)-(55) hold with

$$\begin{aligned} \mathbf{u}^{''}(\mathbf{x}) &= -\frac{k^2}{2}\boldsymbol{\lambda}\sin(k\,\mathbf{n}\cdot\mathbf{x}+\omega t), & \phi^{''}(\mathbf{x}) &= -\frac{k^2}{2}\phi\sin(k\,\mathbf{n}\cdot\mathbf{x}+\omega t) \\ T^{'}(\mathbf{x}) &= \frac{k}{2}\tau\cos(k\,\mathbf{n}\cdot\mathbf{x}+\omega t), & T^{''}(\mathbf{x}) &= -\frac{k^2}{2}\tau\sin(k\,\mathbf{n}\cdot\mathbf{x}+\omega t) \end{aligned}$$

and by the arbitrariness of $\mathbf{x}$ and t we have that (52)-(55) are equivalent to the seven equalities (68). Q.E.D.

Corollary 4.3 *Any possible sinusoidal wave is reversible in time.*

Proof. In fact, any sinusoidal wave is the sum of two possible standing waves, each reversible in time by Corollary 4.1 Q.E.D.

Note that (68) imply $\tau = 0$, hence they are equivalent to the equations

$$\begin{aligned} \Big(\sigma_{klij}\,n_j n_k \,-\, \rho_o\,\delta_{il}\,V^2\Big)\,\lambda_i \,+\, \varphi\,e_{ijl}\,n_i n_j &= 0 \\ e_{kji}\,\lambda_j\,n_i n_k \,-\, \varphi\,\epsilon_{kj}\,n_j n_k &= 0\,, \\ \kappa^E_{jk}\,\varphi\,n_j n_k &= 0\,, \\ \lambda_k\,\beta_{kj}\,n_j \,-\, \varphi\,\tilde{\omega}_k\,n_k &= 0\,, \\ \tau &= 0 \end{aligned} \tag{69}$$

Hence possible sinusoidal vibrations are characteristic, isothermal, isentropic, and can propagate along material directions that satisfy the conditions $(69)_{2,3,4}$.

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