On the Metric Dimension of Corona Product of Graphs

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Abstract

For an ordered set $W = \{w_1, w_2, \cdots, w_k\}$ of vertices and a vertex v in a connected graph G, the representation of v with respect to W is the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$ where $d(x, y)$ represents the distance between the vertices x and y. The set W is called a resolving set for G if every vertex of G has a distinct representation. A resolving set containing a minimum number of vertices is called a basis for G. The metric dimension of G, denoted by $\dim(G)$, is the number of vertices in a basis of G. A graph G *corona* H, $G \odot H$, is defined as a graph which formed by taking n copies of graphs H_1, H_2, $\cdots$, H_n of H and connecting i-th vertex of G to the vertices of H_i. In this paper, we determine the metric dimension of corona product graphs $G \odot H$, the lower bound of the metric dimension of $K_1 + H$ and determine some exact values of the metric dimension of $G \odot H$ for some particular graphs H.

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1 Introduction

In this paper we consider finite and simple graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a further reference please see Chartrand and Lesniak [4].

The *distance* $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. The distance is only denoted by $d(x,y)$ if we know the context of the graph G. For an ordered set $W = \{w_1, w_2, \cdots, w_k\} \subseteq V(G)$ of vertices, we refer to the ordered k-tuple $r(v|W) = (d(v,w_1), d(v,w_2), \cdots, d(v,w_k))$ as the *(metric) representation of v with respect to W*. The set W is called a *resolving set* for G if $r(u|W) = r(v|W)$ implies $u = v$ for all $u, v \in G$. A resolving set with minimum cardinality is called a *minimum resolving set* or a *basis*. The *metric dimension* of a graph G, $\dim(G)$, is the number of vertices in a basis for G. To determine whether W is a resolving set for G, we only need to investigate the representations of the vertices in $V(G)\backslash W$, since the representation of each $w_i \in W$ has '0' in the ith-ordinate; and so it is always unique. If $d(u,x) \neq d(v,x)$, we shall say that vertex x *distinguishes* the vertices u and v and the vertices u and v *are distinguished* by x Likewise, if $r(u|S) \neq r(v|S)$, we shall say that the set S *distinguishes* vertices u and v.

The first papers discussing the notion of a (minimum) resolving set were written by Slater [19] and Harary and Melter [8]. Garey and Johnson [7] have proved that the problem of computing the metric dimension for general graphs is NP-complete. The metric dimension of amalgamation of cycle and complete graphs are widely investigated in [11, 12]. Manuel *et al.* [16, 15] determined the metric dimension of graphs which are designed for multiprocessor interconnection networks. Some researchers defined and investigated the family of graphs related to their metric dimension. Hernando *et al.* [9] investigated the extremal problem of the family of connected graphs with metric dimension β and diameter, and Javaid *et al.* [13] for the family of regular graphs with constant metric dimension.

Chartrand *et al.* [5] has characterized all graphs having metric dimensions $1, n-1$, or $n-2$. They also determined the metric dimensions of some well-known families of graphs such as paths, cycles, complete graphs, and trees. Their results can be summarized as follows

Theorem A [5] *Let G be a connected graph of order $n \geq 2$.*

(i) *dim(G) = 1 if and only if $G = P_n$.*

(ii) *dim$(G) = n-1$ if and only if $G = K_n$.*

(iii) *For $n \geq 4$, dim$(G) = n-2$ if and only if $G = K_{r,s}, (r,s \geq 1)$, $G = K_r + \overline{K_s}$, $(r \geq 1, s \geq 2)$, or $G = K_r + (K_1 \cup K_s)$, $(r,s \geq 1)$.*

(iv) *For $n \geq 3$, dim(C_n) = 2.*

(v) *If T is a tree other than a path, then dim(T) $= \sigma(T) - ex(T)$, where $\sigma(T)$ denotes the sum of the terminal degrees of the major vertices of T, and ex(T) denotes the number of the exterior major vertices of T.*

Saenpholphat and Zhang in [17] have discussed the notion of *distance similar* in a graph. The *neighbourhood* $N(v)$ of a vertex v in a graph G is all of vertices in a graph G which adjacent to v. The *closed neighbourhood* $N[v]$ of a vertex v in a graph G is $N(v) \cup v$. Two vertices u and v of a connected graph G are said to be *distance similar* if $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$. They observed the following properties.

Proposition B *Two vertices u and v of a connected graph G are distance similar if and only if (1) $uv \notin E(G)$ and $N(u) = N(v)$ or (2) $uv \in E(G)$ and $N[u] = N[v]$.*

Proposition C *Distance similarity in a connected graph G is an equivalence relation on $V(G)$.*

Proposition D *If U is a distance similar equivalence class of a connected graph G, then U is either independent in G or in $\overline{G}$.*

Proposition E *If U is a distance similar equivalence class in a connected graph G with $|U| = p \geq 2$, then every resolving set of G contains at least $p - 1$ vertices from U.*

Other researchers also considered the metric dimension of the graphs formed by operations of graph such as joint, Cartesian, and composition product of graphs. Caceres *et al.* in [2] stated the results of metric dimension of joint graphs. Caceres *et al.* in [3] investigated the characteristics of Cartesian product of graphs. Saputro *et al.* in [18] determined the metric dimension of Composition product of graphs. Iswadi *et al.* in [10] investigated the metric dimension of corona product $G \odot K_1$ for some particular graph G. In this paper, we continue and determine a general result of the metric dimension of corona product of graphs for any graph G and H. Furthermore, we determine the exact value of the metric dimension of corona product of the graph G with n-ary tree T.

2 Corona Product of Graphs

Let G be a connected graph of order n and H (not necessarily connected) be a graph with $|H| \geq 2$. A graph G *corona* H, $G \odot H$, is defined as a graph which formed by taking n copies of graphs $H_1, H_2, \cdots, H_n$ of H and connecting i-th vertex of G to the vertices of H_i. Throughout this section, we refer to H_i as a i-th copy of H connected to i-th vertex of G in $G \odot H$ for every $i \in \{1, 2, \cdots, n\}$.

We extend the idea of distance similar. Let G be a connected graph. Two vertices u and v in a subgraph H of G are said to be *distance similar with respect to* H if $d(u, x) = d(v, x)$ for all $x \in V(G) - V(H)$. We observed this following fact for the graph of $G \odot H$.

Observation 1. *Let G be a connected graph and H be a graph with order at least 2. Two vertices u, v in H_i is distance similar with respect to H_i.*

We also have a distance property of two vertices x and y in H or in H_i subgraph $G \odot H$. A vertex $u \in G$ is called a *dominant vertex* if $d(u, v) = 1$ for other vertices $v \in G$.

Lemma 1. *Let G be a connected graph and H be a graph with order at least 2. If H contains a dominant vertex v then $d_H(x, y) = d_{G \odot H}(x, y)$, for every x, y in H or in a subgraph H_i of $G \odot H$.*

Proof. Let v be a dominant vertex of H and x, y be in H. If $xy \in E(H)$ then $d_H(x, y) = 1 = d_{G \odot H}(x, y)$. If $xy \notin E(H)$ then $d_H(x, y) = d_H(x, v) + d_H(v, y) = 2 = d_{G \odot H}(x, v) + d_{G \odot H}(v, y) = d_{G \odot H}(x, y)$. Then, $d_H(x, y) = d_{G \odot H}(x, y)$, for every x, y in H. By using similar reason with two previous sentences, we also have a conclusion $d_{G \odot H}(x, y) = d_H(x, y)$, for every x, y in H_i. □

By using the similar reason with the proof of Lemma 1, we can prove this following lemma.

Lemma 2. *Let G be a connected graph and H be a graph with order at least 2. Then $d_{K_1+H}(x, y) = d_{G \odot H}(x, y)$, for every x, y in a subgraph H of $K_1 + H$ or in a subgraph H_i of $G \odot H$.*

By using Observation 1, we have the following lemma.

Lemma 3. *Let G be a connected graph of order n and H be a graph with order at least 2.*

(i) If S is a resolving set of $G \odot H$ then $V(H_i) \cap S \neq \emptyset$ for every $i \in \{1, \dots, n\}$.

(ii) If B is a basis of $G \odot H$ then $V(G) \cap B = \emptyset$.

Proof. (i) Suppose there exists $i \in \{1, \dots, n\}$ such that $V(H_i) \cap S = \emptyset$. Let $x, y \in V(H_i)$. By using Observation 1, $d_{G\odot H}(x, u) = d_{G\odot H}(y, u)$ for every $u \in S$, a contradiction.
(ii) Suppose that $V(G) \cap B \neq \emptyset$. We will show that $S' = B - V(G)$ is a resolving set for $G \odot H$. From (i), it is clear that $S' \neq \emptyset$. Let x, y two different vertices in $G \odot H$. We have four cases:
Case 1: $x, y \in V(H_i)$ for every $i \in \{1, \dots, n\}$. By using (i), there are some $v \in V(H_i) \cap S'$ such that $d(x, v) \neq d(y, v)$.
Case 2: $x \in V(H_i)$ and $y \in V(H_j)$, for every $i \neq j \in \{1, \dots, n\}$. Let $v \in V(H_i) \cap S'$. We have $d(x, v) \leq 2 < 3 \leq d(y, v)$.
Case 3: $x, y \in V(G)$. Let $x = v_i$, for some $i \in \{1, \dots, n\}$ and $v \in V(H_i) \cap S'$. We have $d(x, v) = 1 < d(y, x) + d(x, v) = d(y, v)$.
Case 4: $x \in V(H_i)$ for some $i \in \{1, \dots, n\}$ and $y \in V(G)$. Let $y = v_j$ for some $j \in \{1, \dots, n\}$. There exist $v \in V(H_j) \cap S'$ such that $d(x, v) = d(x, v_i) + d(v_i, v_j) + d(v_j, v) > d(v_j, v) = d(y, v)$.
Then S' is a resolving set for $G \odot H$ where $|S'| < |B|$. We have a contradiction with B is a basis of $G \odot H$. □

The following theorem determine the metric dimension of the graph G *corona* H.

Theorem 1. *Let G be a connected graph, H be a graph with order at least 2. Then*

$$dim(G \odot H) = \begin{cases} |G|dim(H), & \text{if } H \text{ contains a dominant vertex;} \\ |G|dim(K_1 + H), & \text{otherwise.} \end{cases}$$

Proof. Let B be a basis of $G \odot H$. Let H_i be a i-th copy of H connected to i-th vertex of G in $G \odot H$.
Case 1: H contains a dominant vertex.
Suppose that $dim(G \odot H) < |G|dim(H)$. Let $B_i = B \cap V(H_i)$. Since $B \cap V(G) = \emptyset$ (using Lemma 3 (ii)), there exist B_j such that $|B_j| < dim(H)$. It means that every two vertices of H_j can be distinguished by only vertices in B_j. Therefore, B_j is a resolving set for $H_j(\cong H)$, a contradiction. Hence, we have $dim(G \odot H) \geq |G|dim(H)$. Now, we will prove that $dim(G \odot H) \leq |G|dim(H)$. Let W_i be a basis of H_i. Set $S = \bigcup_{i=1}^{n} W_i$. We will show that S is a resolving set of $G \odot H$. Since $S \cap V(G) = \emptyset$, by using the same technique in the proof of

Lemma 3 (ii), we can prove that the set S is a resolving set of $G \odot H$. Hence, $dim(G \odot H) \leq |S| = |\bigcup_{i=1}^{n} W| = |G|dim(H)$.
Case 2: H does not contain a dominant vertex.
This case is proved by a similar way to Case 1, by considering $dim(K_1 + H)$ instead of $dim(H)$ and applying Lemma 2 instead of Lemma 1. To prove $dim(G \odot H) \leq |G|dim(K_1 + H)$, we choose $S' = \bigcup_{i=1}^{n} W_i'$, where W_i' is a basis of $K_1 + H_i$. □

In Theorem 1, the formula of the metric dimension of corona product of graphs depends on the metric dimension of $K_1 + H$. Caceres et.al. [2] stated the lower bound of metric dimension of join graph $G + H$ as follow.

Theorem B [2] *Let G and H be a connected graph. Then*

$$dim(G + K) \geq dim(G) + dim(H).$$

By using this Caceres's result we obtain the following corollary.

Corollary 1. *For any connected graph H, we have*

$$dim(K_1 + H) \geq dim(H) + 1.$$

The lower bound in Corollary 1 is sharp because $H \cong P_2$ fulfills the equality. In [1], Buczkowski et. al. determined the metric dimension of the wheel graph $W_n = K_1 + C_n$. They stated that $\dim(W_3) = 3$, $\dim(W_4) = \dim(W_5) = 2$, $\dim(W_6) = 3$, and if $n \geq 7$, then $\dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$. Caceres et.al. in [2] have determined the metric dimension of the fan graph $F_n = K_1 + P_n$, $\dim(K_1 + P_1)$ $= \dim(P_2) = 1$, $dim(K_1 + P_i) = 2$ for $i \in \{2, 3, 4, 5, 6\}$, and if $n \geq 7$, then $\dim(F_n) = \lfloor \frac{2n+2}{5} \rfloor$.

These results and the idea of the distance similar of a dominating set in a graph suggest the metric dimension of corona product of any graph G with a complete graph K_n, the graph C_n, or the graph P_n. Since K_n contains a dominant vertex, by using Theorem 1, we have this following corollary.

Corollary 2. *Let K_n be a complete graph. For $n \geq 2$,*

$$dim(G \odot K_n) = |G|(n - 1).$$

Since C_n and P_n do not contain a dominant vertex for $n \geq 7$ then by using Theorem 1, we have this following corollary.

Corollary 3. *Let G be a connected graph and H is isomorphic to C_n or P_n. If $n \geq 7$, then*

$$dim(G \odot H) = |G| \left\lfloor \frac{2m + 2}{5} \right\rfloor$$

For $n = 3, 4, 5$, and 6, $dim(G \odot C_n) = k|G|$, with $k = 3, 2, 2$, and 3,respectively. For $n = 2, 3, 4, 5$, and 6, $dim(G \odot P_n) = q|G|$, with $q = 1, 2, 2, 2$, and 2, respectively.

We have also known the metric dimension of $K_1 + S_n$, where S_n is a star with n pendants. Since the metric dimension of $K_1 + S_n$ is isomorphic to a complete bipartite graph $K_{2,n}$, by using Theorem *A (iii)*, $dim(K_1 + S_n) = n$. Hence, we have the following corollary.

Corollary 4. *Let S_n be a star graph, $n \geq 2$. Then, we have*

$$dim(G \odot S_n) = |G|n.$$

3 Corona Product of a Graph and an n-ary Tree

In the this section, we will determine the metric dimension of a joint graph $K_1 + T$, where T is a n-ary tree. Then by using Theorem 1, we obtain the metric dimension of the corona product of $G \odot T$.

For $T \cong K_2$, the joint graph $K_1 + T \cong C_3$. All vertices in C_3 are the dominant vertices and $dim(C_3) = 2$. For $T \cong S_n$, form the previous section, $dim(K_1 + S_n) = n$.

Proposition 1. *Let T be a tree other than a star. Then, $K_1 + T$ has exactly one dominant vertex and every resolving set S of $K_1 + T$ is a subset of T.*

Proof. Since S_n is the only tree with one dominant vertex then a joint graph $K_1 + T$, where $T \ncong K_2$ or S_n, only contain exactly one dominant vertex, i.e the vertex of K_1, say v. Let S be a resolving set of $K_1 + T$. Since v is the only vertex of $K_1 + T$ at distance 1 to every vertex of T then the representation of v with respect to S is unique. Hence, $v \notin S$. So, $S \subseteq T$. $\square$

A *rooted tree* is a pair (T, r), where T is a tree and $r \in V(T)$ is a distinguished vertex of T called the *root*. In this paper, we simplify the notation of a rooted tree by T. If $xy \in E(T)$ is an edge and the vertex x lies on the unique path from y to the root, we say that x is the *father* of y and y is a *child* of x. A *complete n-ary tree* T is a rooted tree whose every vertex, except the leaves, has exactly n children.

The *i-th level* of an n-ary tree T, denoted by T^i, is the set of vertices in T at distance i from the root vertex. For u in T^i, we said u be on the level i in an n-ary tree T. Then, the level 0, T^0, contains a single vertex r. The set of children of a vertex u in T^{i-1} is denoted by $T^i_{\{u\}}$, and so $T^i = \bigcup_{u\in T^{i-1}} T^i_{\{u\}}$. The set of vertices at distance at most i and at least k from the root r is denoted by $T^i_k = \bigcup_{j=k}^{i} T^j$.

If all leaves of a complete n-ary tree T are on the same level l then T is called a perfect complete n-ary tree with *depth* l, denoted by $T(n,l)$. The order of $T(n,l)$ is $n^0 + n^1 + \cdots + n^l$, and the number of vertices on level i is $|T^i(n,l)| = n^i$. From now on, we use the term n-ary tree for a perfect complete n-ary. For $n = 1$, $K_1 + T(1,l) \cong K_1 + P_{l+1} = F_{l+1}$ and $\dim(K_1 + T(1,l)) = \left\lfloor \frac{2(l+1)+2}{5} \right\rfloor$. For $l = 1$, $K_1 + T(n,1) \cong K_1 + S_n = K_{2,n}$ and $\dim(K_1 + T(n,1))= n$. So, we will determine the metric dimension of $\dim(K_1 + T(n,l))$ where $T(n,l)$ is an n-ary tree with the *depth* l for $n \geq 2$ and $l \geq 2$.

Lemma 4. *Let S be a resolving set of a graph $K_1 + T(n,l)$ and $i \in \{1,2,\cdots,l\}$. If $S \cap T^{i+1}(n,l) = \emptyset$ then at least $n-1$ vertices of $T^i_{\{u\}}$ must be in S for every u in $T^{i-1}(n,l)$.*

Proof. Suppose that there is a vertex u in $T^{i-1}(n,l)$ such that $|T^i_{\{u\}}(n,l) \cap S| < n-1$. Then there are two vertices x, y in $T^{i-1}(n,l)$ but not in S such that they have the same distance (1 or 2) to every vertex of S, a contradiction. □

Lemma 4 holds for $i = l$ since all vertices u in $T^l(n,l)$ has no children. If $S \cap T^{i+1}(n,l) = \emptyset$ then by using Lemma 4 we have at most one vertex x in $T^i_{\{u\}}(n,l)$ but not in S for every u in $T^{i-1}(n,l)$.

Lemma 5. *If S be a resolving set of a graph $K_1 + T(n,l)$ and $i \in \{1,2,\cdots,l\}$ then at least $n^i - 1$ vertices of $T^{i+1}_{i-1}(n,l)$ must be in S.*

Proof. Suppose that $|T^{i+1}_{i-1}(n,l) \cap S| < n^i - 1$ for some i. Then, we have

$$\begin{aligned} |T^i(n,l) - S| &= |T^i(n,l) - (T^i(n,l) \cap S)| \\ &\geq n^i - (n^i - 2 - |T^{i+1}(n,l) \cap S| - |T^{i-1}(n,l) \cap S|) \\ &= |T^{i+1}(n,l) \cap S| + |T^{i-1}(n,l) \cap S| + 2 \end{aligned}$$

There are two cases:
Case 1: $|T^{i+1}(n,l) \cap S| = 0$. There are two subcases.
Subcase 1.1: $|T^{i-1}(n,l) \cap S| = 0$.
In this case, $|T^i(n,l) - S| \geq 2$. Hence, we have at least two vertices x and y in $T^i(n,l)$ which all of their parents and children are not in S. Then, x and y have the same distance 2 to every vertex of S, a contradiction.
Subcase 1.2: $|T^{i-1}(n,l) \cap S| \neq 0$.

This means $|T^i(n,l) - S| \geq |T^{i-1}(n,l) \cap S| + 2$. Since, by using Lemma 4, we have at most one vertex x in $T^i_{\{u\}}(n,l)$ but not in S for every u in $T^{i-1}(n,l)$ then $|T^{i-1}(n,l) \cap S|$ vertices in $T^{i-1}(n,l) \cap S$ must have at most $|T^{i-1}(n,l) \cap S|$ children in $T^i(n,l) - S$. Then, there are at least two pairs of parent-child ux and vy where u, v in $T^{i-1}(n,l) - S$, x, y in $T^i(n,l) - S$, and $x \in T^i_{\{u\}}(n,l)$, $y \in T^i_{\{v\}}(n,l)$. So, x and y have the same distance 2 to every vertex of S, a contradiction.
Case 2: $|T^{i+1}(n,l) \cap S| \neq 0$. There are two subcases.
Subcase 2.1: $|T^{i-1}(n,l) \cap S| = 0$.
We have $|T^i(n,l) - S| \geq |T^{i+1}(n,l) \cap S| + 2$. Since a vertex w in $T^{i+1} \cap S$ distinguishes two vertices x any y in $T^i(n,l)$ where one of them is the parent of w and the other is not, then $|T^{i+1}(n,l) \cap S|$ vertices of $T^{i+1}(n,l)$ distinguish at most $|T^{i+1}(n,l) \cap S|$ parents in $T^i(n,l) - S$. Hence, we have at least two vertices x and y in $T^i(n,l)$ which all of their parents and children are not in S. Then, x and y have the same distance 2 to every vertex of S, a contradiction.
Subcase 2.2: $|T^{i-1}(n,l) \cap S| \neq 0$.
In this subcase, $|T^i(n,l) - S| \geq |T^{i+1}(n,l) \cap S| + |T^{i-1}(n,l) \cap S| + 2$. By using similar reason to Subcases 1.2 and 2.1, we have $|T^{i-1}(n,l) \cap S|$ vertices in $T^{i-1}(n,l) - S$ must have at most $|T^{i-1}(n,l) \cap S|$ children in $T^i(n,l) - S$ and $|T^{i+1}(n,l) \cap S|$ vertices of $T^{i+1}(n,l)$ distinguish at most $|T^{i+1}(n,l) \cap S|$ parents in $T^i(n,l) - S$. Then, we have at least two vertices x and y in $T^i(n,l)$ which all of their parents and children are not in S. Then, x and y have the same distance 2 to every vertex of S, a contradiction. $\square$

Lemma 5 is also hold for $i = l$ since all vertices u in $T^l(n,l)$ has no children. Lemma 4 and 5 give us a procedure to construct a resolving set S of $T(n,l)$ which have a minimal number of vertices. The procedure is done by applying Lemma 4 and 5 from $i = l$ up to $i = 1$ consecutively. The minimal condition of a resolving set S in $T(n,l)$ can be reached if we have as many possible $T^i(n,l)$'s such that $T^i(n,l) \cap S = \emptyset$ and the other levels fulfill Lemma 4 and 5.

Let S be a resolving set of $K_1 + T(n,l)$. By using Proposition 1, we have $S \subseteq T(n,l)$. For $i = l$, since all vertices of $T^l(n,l)$ have no children then, by using Lemma 4 and 5, at least $n^l - 1$ vertices of $T^l_{l-1}(n,l)$ must be in S. These vertices can be distributed in levels $T^l(n,l)$ and $T^{l-1}(n,l)$ such that

$$\begin{aligned} |T^l(n,l) \cap S| &= \underbrace{(n-1) + \cdots + (n-1)}_{n^{l-1} \text{ times}} \\ &= n^l - n^{l-1} \end{aligned}$$

and $|T^{l-1}(n,l) \cap S| = n^{l-1} - 1$ vertices. If we use this distribution, there exists a vertex in $T^l(n,l)$ at distance 2 to every vertex of S. We denote this vertex by $x_{(2,2,\cdots,2)}$.

To reach a minimal condition for S, we can assume that $T^{l-2}(n,l) \cap S = \emptyset$. By using this assumption, we can reapply Lemma 4 and 5 for $i = l-3$. Thus, we have at least $n^{l-3}-1$ vertices of $T^{l-3}_{l-4}(n,l)$ must be in S. Since $x_{(2,2,\cdots,2)}$ is in $T^l(n,l)$ then we must have at least n^{l-3} vertices of $T^{l-2}_{l-4}(n,l)$ must be in S. We then repeat this process up to level 0.

By using this procedure, we can construct a minimal resolving set of a $T(n,l)$. This resolving set will contain $(n^l-1)+n^{l-3}+\cdots+n^i = \sum_{j=0}^{t} n^{l-3j}-1$ vertices, where $l = 3t+i$, $i = 0,1,2$. We will prove that this is indeed the metric dimension of $K_1 + T(n,l)$, where $T(n,l)$ is n-ary tree with a depth l, as stated in the following theorem.

Theorem 2. *For $n, l \geq 2$, $l = 3t+i$, $t \geq 0$, and $i = 0,1,2$, let $T(n,l)$ be a n-ary with a depth l. Then,*

$$dim(K_1 + T(n,l)) = \sum_{j=0}^{t} n^{l-3j} - 1.$$

Proof. We will show that $dim(K_1 + T(n,l)) \geq \sum_{j=0}^{t} n^{l-3j} - 1$. Let S be a resolving set of $K_1 + T(n,l)$. By using Proposition 1, we have $S \subseteq T(n,l)$. Without losing the generalization, we put $x_{2,2,\cdots,2}$ in $T^l(n,l)$. We will show that $|S| \geq (n^l-1)+n^{l-3}+\cdots+n^i = \sum_{j=0}^{t} n^{l-3j}-1$. Suppose that $|S| < \sum_{j=0}^{t} n^{l-3j}-1$. By using Lemma 5, it suffices to show that $|T^{i+1}_{i-1}(n,l) \cap S| = n^i - 1$ for some $i \in \{1,2,\cdots,l-1\}$ is impossible. If $|T^{i+1}_{i-1}(n,l) \cap S| = n^i - 1$ for some $i \in \{1,2,\cdots,l-1\}$ then $|T^i - S| = |T^{i+1}(n,l) \cap S| + |T^{i-1}(n,l) \cap S| + 1$. We have these four possibilities:

(i.) $|T^{i+1}(n,l) \cap S| = 0$ and $|T^{i-1}(n,l) \cap S| = 0$.

(ii.) $|T^{i+1}(n,l) \cap S| = 0$ and $|T^{i-1}(n,l) \cap S| \neq 0$.

(iii.) $|T^{i+1}(n,l) \cap S| \neq 0$ and $|T^{i-1}(n,l) \cap S| = 0$.

(iV.) $|T^{i+1}(n,l) \cap S| \neq 0$ and $|T^{i-1}(n,l) \cap S| \neq 0$.

By using similar reason to the proof of Lemma 5, for all the above possibilities, we have another vertex $x_{(2,2,\cdots,2)}$ in $T^i(n,l)$ where $i \in \{1,2,\cdots,l-1\}$, a contradiction. Hence, we have $dim(K_1 + T(n,l)) \geq \sum_{j=0}^{t} n^{l-3j} - 1$.

Now, we prove the upper bound. For $l = 3t+i$, $i = 0,1,2$, and $j \in \{0,1,\cdots,t\}$, set W_{l-3j} and W_{l-1-3j} as follow. $W_{l-3j} = T^{l-3j}(n,l)$ except one vertex x in $T^{l-3j}_{\{u\}}(n,l)$ for every u in $T^{l-3j-1}_{\{u\}}(n,l)$ where $j \in \{0,1,\cdots,t\}$,

$W_{l-1} = T^{l-1}(n,l) - \{u\}$, and $W_{l-1-3j} = T^{l-1-3j}$ where $j \in \{1, \cdots, t\}$. Then, we set $W = \bigcup_{j=0}^{t}(W_{l-3j} \cup W_{l-1-3j})$. We have

$$\begin{aligned} |W| &= \sum_{j=0}^{t} |W_{l-3j}| + \sum_{j=0}^{t} |W_{l-1-3j}| \\ &= \sum_{j=0}^{t}(n^{l-3j} - n^{l-1-3j}) + (n^{l-1} - 1) + \sum_{j=1}^{t}(n^{l-1-3j}) \\ &= \sum_{j=0}^{t} n^{l-3j} - 1 \end{aligned}$$

We will prove that W is a resolving set of $K_1 + T(n,l)$. The vertex K_1 has distance 1 to every vertex of W, which is a unique representation with respect to W. Since every vertex in $T^{l-3j}(n,l) - W_{l-3j}$ have distance 1 to their parent in W_{l-1-3j} and 2 to other vertices of W, except for one vertex in $T^{l}(n,l) - W_l$, having a parent in $T^{l-1}(n,l)$. Thus, x have a unique representation with respect to W for every x in $T^{l-3j}(n,l) - W_{l-3j}$. For a vertex in $T^{l}(n,l)$, this vertex has distance 2 to every vertex of S. This is also a unique representation with respect to W. For a vertex in $T^{l-1}(n,l)$, this vertex have distance 1 to each of their children in W_l. For every vertex z in $T^{l-3j-2}(n,l)$ has distance 1 uniquely to every their children in $W_{l-3j-2}(n,l)$. Then, all of vertices in $K_1 + T(n,l)$ have distinct representation with respect to W. Hence, W is a resolving set of $K_1+T(n,l)$. Therefore, $dim(K_1+T(n,l)) \leq \sum_{j=0}^{t} n^{l-3j} - 1$. □

Let B be a basis of graph $K_1 + T(n,l)$, where $T(n,l)$ is a n-ary tree with a depth l, for $n \geq 2$, $l = 3t + i$, $t \geq 0$, and $i = 0, 1, 2$. From Lemma 4 and Theorem 2, we assume that a vertex $x_{(2,2,\cdots,2)}$ in $T^{l}(n,l)$. There are n^l possibilities for the position of $x_{(2,2,\cdots,2)}$ in $T^{l}(n,l)$. But these bases are unique up to isomorphism. The position of $x_{(2,2,\cdots,2)}$ can also be moved to level T^{l-3j}, $j = 1, \cdots, t$. For each of these levels, the basis form a unique basis up to isomorphism. Since there are $t + 1$ ways to put $x_{(2,2,\cdots,2)}$ in $T(n,l)$ then there are $t + 1$ different bases of $K_1 + T$ (up to isomorphism).

Since a tree which is not isomorphic to K_2 and S_n has no dominant vertices, by using Theorem 1 and 2, we have the following corollary.

Corollary 5. *For $n, l \geq 2$, $l = 3t + i$, $t \geq 0$, and $i = 0, 1, 2$, let G be a connected graph and $T(n,l)$ be a n-ary tree with a depth l. Then,*

$$dim(G \odot T(n,l)) = |G| \left(\sum_{j=0}^{t} n^{l-3j} - 1 \right).$$

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References

[1] P.S. Buczkowski, G. Chartrand, C. Poisson, and P. Zhang, On k-dimensional graphs and their bases, *Period. Math. Hungar.*, **46:1** (2003), pp. 9–15.

[2] J. Cáceres, C. Hernando, M. Mora, M.L. Puertas, I.M. Pelayo and C. Seara, On the metric dimension of some families of graphs, preprint.

[3] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, C. Seara, and D.R. Wood, On the metric dimension of cartesian products of graphs, *Siam J. Discrete Math.*, **21:2** (2007), pp. pp. 423–441.

[4] G. Chartrand, and L. Lesniak, *Graphs and Digraphs*, 3rd ed., Chapman and Hall/CRC, 2000.

[5] G. Chartrand, L. Eroh, M.A. Johnson, and O.R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, **105** (2000), pp. 99–113.

[6] G. Chartrand and P. Zhang, The theory and appllications of resolvability in graphs: a survey, *Congr. Numer.*, **160** (2003), pp. 47–68.

[7] M.R. Garey and D.S. Johnson, *Computers and Intractability: a guide to the theory of NP-completeness*, W.H. Freeman, California, 1979.

[8] F. Harary and R.A. Melter, On the metric dimension of a graph, *Ars. Combin.*, **2** (1976), pp. 191–195.

[9] C. Hernando, M. Mora, I.M. Pelayo, C. Seara, and D.R. Wood, Extremal Graph Theory for Metric Dimension and Diameter, http://arXiv.org/abs/0705.0938v1, (2007), pp. 1–26.

[10] H. Iswadi, E.T. Baskoro, R. Simanjuntak, and A.N.M. Salman, The metric dimensions of graphs with pendant edges, *J. Combin. Math. Combin. Comput.* **65** (2008), pp. 139–145.

[11] H. Iswadi, E.T. Baskoro, R. Simanjuntak, and A.N.M. Salman, Metric dimension of amalgamation of cycles, *Far East Journal of Mathematical Sciences (FJMS)*, **41:1** (2010), pp. 19–31.

[12] H. Iswadi, E.T. Baskoro, R. Simanjuntak, A.N.M. Salman, The resolving graph of amalgamation of cycles, *Utilitas Mathematica*, **83** (2010), pp. 121–132

[13] I. Javaid, M.T. Rahman, and K. Ali, Families of reguler graphs with constant metric dimension, *Util. Math.*, **75** (2008), pp. 21–33.

[14] S. Khuller, B. Raghavachari, and A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.*, **70** (1996), pp. 217–229.

[15] P. Manuel, B. Rajan, I. Rajasingh, and C.M. Mohan, Landmarks in torus networks, *J. Discrete Math. Sci. Cryptogr.*, **9:2** (2006), pp. 263–271.

[16] P. Manuel, B. Rajan, I. Rajasingh, and C.M. Mohan, On minimum metric dimension of honeycomb networks, *J. Discrete Algorithms*, **6** (2008), pp. 20–27.

[17] V. Saenpholphat and P. Zhang, Some results on connected resolvability in graphs, *Congr. Numer.*, **158** (2002), pp. 5–19.

[18] S.W. Saputro and H. Assiyatun and R. Simanjuntak and S. Uttunggadewa and E.T. Baskoro and A.N.M. Salman and M. Bača, The metric dimension of the composition product of graphs, preprint.

[19] P.J. Slater, Leaves of trees, *Congr. Numer.*, **14** (1975), pp. 549–559.