

# The Resolving Graph of Amalgamation of Cycles

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## Abstract

For an ordered set  $W = \{w_1, w_2, \dots, w_k\}$  of vertices and a vertex  $v$  in a connected graph  $G$ , the representation of  $v$  with respect to  $W$  is the ordered  $k$ -tuple  $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$  where  $d(x, y)$  represents the distance between the vertices  $x$  and  $y$ . The set  $W$  is called a resolving set for  $G$  if every vertex of  $G$  has a distinct representation. A resolving set containing a minimum number of vertices is called a basis for  $G$ . The dimension of  $G$ , denoted by  $\dim(G)$ , is the number of vertices in a basis of  $G$ . A resolving set  $W$  of  $G$  is connected if the subgraph  $\langle W \rangle$  induced by  $W$  is a nontrivial connected subgraph of  $G$ . The connected resolving number is the minimum cardinality of a connected resolving set in a graph  $G$ , denoted by  $cr(G)$ . A *cr-set* of  $G$  is a connected resolving set with cardinality  $cr(G)$ . A connected graph  $H$  is a *resolving graph* if there is a graph  $G$  with a *cr-set*  $W$  such that  $\langle W \rangle = H$ . Let  $\{G_i\}$  be a finite collection of graphs and each  $G_i$  has a fixed vertex  $v_{oi}$  called a terminal. The amalgamation  $\text{Amal}\{G_i, v_{oi}\}$  is formed by taking of all the  $G_i$ 's and identifying their terminals. In this paper, we determine the connected resolving number and characterize the resolving graphs of amalgamation of cycles.

## 1 Introduction

In this paper we consider finite, simple, and connected graphs. The vertex and edge sets of a graph  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively. For a further reference please see Chartrand and Lesniak [3].

The *distance*  $d(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of a shortest  $u - v$  path in  $G$ . For an ordered set  $W = \{w_1, w_2, \dots, w_k\} \subseteq V(G)$ , we refer to the ordered  $k$ -tuple  $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$  as the (*metric*) *representation* of  $v \in V(G)$  with respect to  $W$ . The set  $W$  is called a *resolving set* for  $G$  if  $r(u|W) = r(v|W)$  implies  $u = v$  for all  $u, v \in G$ . A resolving set with minimum cardinality is called a *minimum resolving set* or a *basis*. The *metric dimension* of a graph  $G$ ,  $\dim(G)$ , is the number of vertices in a basis for  $G$ . To determine whether  $W$  is a resolving set for  $G$ , we only need to investigate the representations of the vertices in  $V(G) \setminus W$ , since the representation of each  $w_i \in W$  has '0' in the  $i$ th-ordinate; and so it is always unique.

The first papers discussing the notion of a (minimum) resolving set were written by Slater in [19] and [20]. Slater introduced the concept of a resolving set for a connected graph  $G$  under the term *location set*. He called the cardinality of a minimum resolving set by the *location number* of  $G$ . Independently, Harary and Melter [8] introduced the same concept, but used the term *metric dimension* instead.

In general, finding a resolving set for arbitrary graph is a difficult problem. In [7], it is proved that the problem of computing the metric dimension for general graphs is *NP*-complete. Thus, researchers in this area often studied the metric dimension for particular classes of graphs or characterized graphs having certain metric dimension. Some results on the joint graph and cartesian product graph have been obtained by Caceres et al. [1], Khuller et al. [13], and Chartrand [4]. Iswadi et.al obtained some results on the corona product of graphs [10, 11]. Suhadi et al. obtained some results on the decomposition product of graphs [23]. Iswadi et al. determined the metric dimension of antipodal and pendant free graph [12]. Suhadi et al. found some results on the metric dimension of some type of regular graphs [21, 22]. And, Chartrand et al. [4] has characterized all graphs having metric dimensions 1,  $n - 1$ , and  $n - 2$ . They also determined the metric dimensions of some well known families of graphs such as paths, cycles, complete graphs, and trees.

The study of finding resolving sets of graphs can also be conducted by considering particular restrictions for the resolving sets. One of the restrictions is connectivity; in [16] Saenpholphat and Zhang introduced the concept of connected resolvability. They defined the following terms. A resolving set  $W$  of  $G$  is *connected* if the subgraph  $\langle W \rangle$  induced by  $W$  is a nontrivial connected subgraph of  $G$ . The *connected resolving number*  $cr(G)$  is the minimum cardinality of a connected resolving set in  $G$ . A *cr-set* of  $G$  is a connected resolving set with cardinality  $cr(G)$ . Since every connected resolving set is a resolving set, then  $\dim(G) \leq cr(G)$  for any connected

graph. A connected graph  $H$  is a *resolving graph* if there is a graph  $G$  with a cr-set  $W$  such that  $\langle W \rangle = H$ . Additionally, they observed the following.

**Observation A** [16] *Let  $G$  be a graph and  $U \subseteq V(G)$ . If  $U$  contains a resolving set of  $G$  as its subset, then  $U$  is also a resolving set of  $G$ .*

**Observation B** [16] *Let  $G$  be a connected graph. Then  $\dim(G) = cr(G)$  if and only if  $G$  contains a connected basis.*

Further properties of connected resolving sets in a graph and its relation with the basis of the graph can be found in [5, 15, 17] and [18].

The following identification graph  $G = G[G_1, G_2, v_1, v_2]$  is defined in [14].

**Definition C** *Let  $G_1$  and  $G_2$  be the non trivial connected graphs where  $v_1 \in G_1$  and  $v_2 \in G_2$ . An identification graph  $G = G[G_1, G_2, v_1, v_2]$  is obtained from  $G_1$  and  $G_2$  by identifying  $v_1$  and  $v_2$  such that  $v_1 = v_2$  in  $G$ .*

Poisson et.al. [14] determined the lower and upper bound of metric dimension of  $G[G_1, G_2, v_1, v_2]$  in terms of  $\dim(G_1)$  and  $\dim(G_2)$  as stated in the following theorems.

**Theorem D** *Let  $G_1$  and  $G_2$  be the non trivial connected graphs with  $v_1 \in G_1$  and  $v_2 \in G_2$  and let  $G = G[G_1, G_2, v_1, v_2]$ . Then*

$$\dim(G) \geq \dim(G_1) + \dim(G_2) - 2.$$

For the upper bound, we define an equivalence class and binary function first. For a set  $W$  of vertices of  $G$ , define a relation on  $V(G)$  with respect to  $W$  by  $uRv$  if there exists  $a \in \mathbb{Z}$  such that  $r(v|W) = r(u|W) + (a, a, \dots, a)$ . It is easy to check that  $R$  is an equivalence relation on  $V(G)$ . Let  $[u]_W$  denote the equivalence class of  $u$  with respect to  $W$ . Then

$$v \in [u]_W \text{ if and only if } r(v|W) = r(u|W) + (a, a, \dots, a)$$

for some  $a \in \mathbb{Z}$ . For a non trivial connected graph  $G$ , define a binary function  $f_G : V(G) \rightarrow \mathbb{Z}$  with

$$f_G(v) = \begin{cases} \dim(G), & \text{if } v \text{ is not a basis vertex of } G; \\ \dim(G) - 1, & \text{otherwise.} \end{cases}$$

**Theorem E** *Let  $G_1$  and  $G_2$  be the non trivial connected graphs with  $v_1 \in G_1$  and  $v_2 \in G_2$  and let  $G = G[G_1, G_2, v_1, v_2]$ . Suppose that  $G_1$  contains a resolving set  $W_1$  such that  $[v_1]_{W_1} = \{v_1\}$ . Then*

$$\begin{aligned} \dim(G) &\leq |W_1| + f_{G_2}(v_2) \\ &= \begin{cases} |W_1| + \dim(G_2), & \text{if } v_2 \text{ is not a basis vertex of } G_2; \\ |W_1| + \dim(G_2) - 1, & \text{otherwise.} \end{cases} \end{aligned}$$

*In particular, if  $W_1$  is a basis for  $G_1$ , then*

$$\dim(G) \leq \begin{cases} \dim(G_1) + \dim(G_2), & \text{if } v_2 \text{ is not a basis vertex of } G_2; \\ \dim(G_1) + \dim(G_2) - 1, & \text{otherwise.} \end{cases}$$

In this paper, we determine the metric dimension, the connected resolving number, and characterize the resolving graphs of amalgamation of cycles.

## 2 Amalgamation of Cycles

The following definition of amalgamation of graphs is taken from [2].

**Definition F** *Let  $\{G_i\}$  be a finite collection of graphs and each  $G_i$  has a fixed vertex  $v_{oi}$  called a terminal. The amalgamation  $\text{Amal}\{G_i, v_{oi}\}$  is formed by taking of all the  $G_i$ 's and identifying their terminals.*

Definition F is a generalization of Definition C. If the collection of graphs in Definition F only consist two graphs then we get Definition F is exactly Definition C.

We could consider amalgamation of cycles; that is  $\text{Amal}\{G_i, v_{oi}\}$  where  $G_i = C_n$  for all  $i$ . In this particular amalgamation, the choice of vertex  $v_{oi}$  is irrelevant. So, for simplification, we can denote this amalgamation by  $(C_n)_t$ , where  $t$  denotes the number of cycle  $C_n$ . For  $t = 1$ , the graph  $(C_n)_1$  is the cycle  $C_n$ . For  $n = 3$ , the graph  $(C_3)_t$  is called the friendship graph or the Dutch  $t$ -windmill [6].

In this paper, we consider a generalization of  $(C_n)_t$  where the cycles under consideration may be of different lengths. We denote this amalgamation by  $\text{Amal}\{C_{n_i}\}$ ,  $1 \leq i \leq t$ ,  $t \geq 2$ . We call every  $C_{n_i}$  (including the

terminal) in  $\text{Amal}\{C_{n_i}\}$  as a *leaf* and a path  $P_{n_i-1}$  obtained from  $C_{n_i}$  by deleting the terminal as a *nonterminal path*.

Throughout this paper, we will follow the following notations and labels for cycles, nonterminal paths, and vertices in  $\text{Amal}\{C_{n_i}\}$ . For odd  $n_i$ ,  $n_i = 2k_i + 1$ ,  $k_i \geq 1$  and for the terminal vertex  $x$ , we label all vertices in each leaf  $C_{n_i}$  such that

$$C_{n_i} = xv_1^i v_2^i \cdots v_{k_i}^i w_{k_i}^i w_{k_i-1}^i \cdots w_1^i x,$$

this will give the nonterminal path

$$P_{n_i-1} = v_1^i v_2^i \cdots v_{k_i}^i w_{k_i}^i w_{k_i-1}^i \cdots w_1^i.$$

For even  $n_i$ ,  $n_i = 2k_i + 2$ ,  $k_i \geq 1$ , and for the terminal vertex  $x$ , we define the labels of all vertices in each leaf  $C_{n_i}$  as follow

$$C_{n_i} = xv_1^i v_2^i \cdots v_{k_i}^i u^i w_{k_i}^i w_{k_i-1}^i \cdots w_1^i x,$$

which leads to the following labeling of the nonterminal path

$$P_{n_i-1} = v_1^i v_2^i \cdots v_{k_i}^i u^i w_{k_i}^i w_{k_i-1}^i \cdots w_1^i.$$

Iswadi et. al [10] characterized the resolving set and determined the metric dimension of amalgamation of cycles  $\text{Amal}\{C_{n_i}\}$  as stated in the following lemma and theorem.

**Lemma G** *Let  $S$  be a resolving set of  $\text{Amal}\{C_{n_i}\}$ . Then  $|P_{n_i-1} \cap S| \geq 1$ , for each  $i$ .*

**Theorem H** *If  $\text{Amal}\{C_{n_i}\}$  is an amalgamation of  $t$  cycles that consists of  $t_1$  number of odd cycles and  $t_2$  number of even cycles, then*

$$\dim(\text{Amal}\{C_{n_i}\}) = \begin{cases} t_1, & t_2 = 0; \\ t_1 + 2t_2 - 1, & \text{otherwise.} \end{cases}$$

One of the natural questions we could pose after proving Theorem H is: Are there any bases other than the bases we constructed in the proof

of Theorem H? Let the number of different basis of  $G$  be denoted by  $\#G$ . Iswadi et. al [9] also determined the number of different basis of  $\text{Amal}\{C_{n_i}\}$  as stated in the following theorem.

**Theorem I** *If  $\text{Amal}\{C_{n_i}\}$  is an amalgamation of  $t$  cycles that consists of  $t_1$  number of odd cycles and  $t_2$  number of even cycles, then*

$$\#\text{Amal}\{C_{n_i}\} = \begin{cases} 2^{t-1} (\sum_{i=1}^t (n_i - 1) - 2), & t_2 = 0; \\ 2^{t_1} (n_{t_1+1} - 1) \prod_{j=t_1+2}^t (C(n_j - 1, 2) - 2C(k_j, 2)), & \text{otherwise.} \end{cases}$$

where  $C(b, a)$  is the total number of combinations of  $b$  objects taken  $a$ .

### 3 Resolving graph of amalgamation of cycles

By considering all bases of amalgamation of cycles identified in [9], we will show that  $\text{Amal}\{C_{n_i}\}$  contains no connected basis.

**Theorem 1.**  *$\text{Amal}\{C_{n_i}\}$  has no connected basis.*

*Proof.* Let  $x$  be a terminal vertex of  $\text{Amal}\{C_{n_i}\}$ . By direct inspection to all of bases  $B$  of  $\text{Amal}\{C_{n_i}\}$ , we have  $x \notin B$ . It is easy to show that  $x$  is a cut-vertex of  $\text{Amal}\{C_{n_i}\}$ . Since  $x \notin B$ , where  $B$  is a basis of  $\text{Amal}\{C_{n_i}\}$ ,  $x$  is a cut-vertex of  $\text{Amal}\{C_{n_i}\}$ , and every nonterminal path  $P_{n_i-1}$  must contain at least one vertex of every resolving set of  $\text{Amal}\{C_{n_i}\}$  then the subgraph  $\langle B \rangle$  must be disconnected.  $\square$

Next, we will determine the connected resolving number and the resolving graph of an amalgamation of cycles. Since every basis  $B$  of  $\text{Amal}\{C_{n_i}\}$  is unconnected then we must choose a resolving set other than a basis of  $\text{Amal}\{C_{n_i}\}$  to form a connected resolving graph. From the proof of Theorem 1,  $x$  must be contained in any connected resolving set. Hence, the connected resolving set of  $\text{Amal}\{C_{n_i}\}$  is

$$W = S \cup \{x\},$$

where  $S \subseteq \bigcup_{i=1}^t P_{n_i-1}$ .

**Lemma 1.** *Let  $S$  be a connected resolving set and  $P_{n_i-1}$  be a nonterminal path of  $\text{Amal}\{C_{n_i}\}$  with  $n_i \geq 4$ . If  $|P_{n_i-1} \cap S| = 1$  for some  $i$  then  $P_{n_i-1} \cap S = \{v_1^i\}$  or  $\{w_1^i\}$ .*

*Proof.* Let  $x$  be a terminal vertex of  $\text{Amal}\{C_{n_i}\}$ . Let  $u \in P_{n_i-1} \cap S$ . Suppose that  $u \neq v_1^i$  and  $w_1^i$ . Since  $n_i \geq 4$  then  $u$  will be one of  $v_a^i$  with  $2 \leq a \leq k_i$ , or  $w_a^i$  with  $2 \leq a \leq k_i$ , or  $u^i$ . For each case,  $u$  is not adjacent to  $x$ , which is a contradiction with  $S$  being a connected resolving set of  $\text{Amal}\{C_{n_i}\}$ .  $\square$

**Lemma 2.** *Let  $S$  be a connected resolving set of  $\text{Amal}\{C_{n_i}\}$ . Let  $P_{n_i-1}$  and  $P_{n_j-1}$  be a pair of nonterminal paths of  $\text{Amal}\{C_{n_i}\}$  with  $n_i, n_j \geq 4$ . Then  $|(P_{n_i-1} \cup P_{n_j-1}) \cap S| \geq 3$ .*

*Proof.* Let  $x$  be a terminal vertex of  $\text{Amal}\{C_{n_i}\}$ . Since  $S$  is a resolving set of  $\text{Amal}\{C_{n_i}\}$ , by using Lemma H, we have  $|(P_{n_i-1} \cup P_{n_j-1}) \cap S| \geq 2$ . Suppose that  $|(P_{n_i-1} \cup P_{n_j-1}) \cap S| = 2$ . Let  $u \in P_{n_i-1} \cap S$  and  $v \in P_{n_j-1} \cap S$ . By using Lemma 1,  $u = v_1^i$  or  $w_1^i$  and  $v = v_1^j$  or  $w_1^j$ . Without loss of generality, let  $u = v_1^i$  and  $v = v_1^j$ . Since  $n_i, n_j \geq 4$ , then  $w_1^i$  and  $w_1^j$  have the same distance to every  $z \in S$ ,  $d(w_1^i, z) = d(w_1^j, z)$ , a contradiction with  $S$  being a connected resolving set in  $\text{Amal}\{C_{n_i}\}$ . Therefore,  $|(P_{n_i-1} \cup P_{n_j-1}) \cap S| \geq 3$ .  $\square$

**Theorem 2.** *If  $\text{Amal}\{C_{n_i}\}$  is the amalgamation of  $r$  cycles that consists of  $r_1$  number of cycles  $C_3$  and  $r_2$  number of cycles  $C_{n_i}$  with  $n_i \geq 4$ , then*

$$cr(\text{Amal}\{C_{n_i}\}) = \begin{cases} r_1 + 1, & r_2 = 0; \\ r_1 + 2r_2, & \text{otherwise.} \end{cases}$$

*Proof.* Let  $B$  be a connected resolving set with minimum cardinality and  $x$  a terminal vertex of  $\text{Amal}\{C_{n_i}\}$ . Let  $S \subseteq \bigcup_{i=1}^r P_{n_i-1}$  be a set of  $\text{Amal}\{C_{n_i}\}$  and  $W = S \cup \{x\}$ . We label all the leaves  $C_{n_i}$  of  $\text{Amal}\{C_{n_i}\}$  in such a way that  $C_{n_i}$ 's with  $1 \leq i \leq r_1$ , are  $C_3$  and  $C_{n_j}$ 's, with  $r_1 + 1 \leq j \leq r_1 + r_2 = r$ , are cycles with length more than three.

*Case 1.* For  $r_2 = 0$ . Since  $W$  is a resolving set of  $\text{Amal}\{C_{n_i}\}$  then, by using Lemma H, we have  $|\bigcup_{i=1}^r (P_{n_i-1} \cap W)| = |P_{n_1-1} \cap W| + \dots + |P_{n_{r-1}-1} \cap W| \geq r_1$ . Since  $x \in W$  and  $x \notin \bigcup_{i=1}^r P_{n_i-1}$  then  $|W| \geq r_1 + 1$ . Therefore,  $|B| \geq r_1 + 1$ . Choose a set  $W = \bigcup_{i=1}^{r_1} S_i \cup \{x\}$ , where  $S_i = \{w_1^i\}$  with  $1 \leq i \leq r_1$ . The representations of other vertices in  $\text{Amal}\{C_{n_i}\}$  by  $W$  are

$$r(w_1^i|W) = (2, 2, \underbrace{1}_{\text{coord. of } S_i}, 2, 2, 1) \text{ with } 1 \leq i \leq r_1 = r.$$

Therefore, all vertices of  $\text{Amal}\{C_{n_i}\}$  have distinct representations by  $W$ . Hence,  $W$  is a connected resolving set of  $\text{Amal}\{C_{n_i}\}$ . Since  $B$  is a minimum connected resolving set of  $\text{Amal}\{C_{n_i}\}$  then  $|B| \leq r_1 + 1$ . So, we have  $|B| = r_1 + 1$ .

*Case 2.* For  $r_2 \geq 1$ . By using Lemmas H and 2, we have  $|\bigcup_{i=1}^r P_{n_{i-1}} \cap W| = |P_{n_1-1} \cap W| + \dots + |P_{n_{r-1}-1} \cap W| \geq r_1 + 2r_2 - 1$ . By a similar argument as in Case 1, since  $x \in W$  and  $x \notin \bigcup_{i=1}^r P_{n_{i-1}}$  then  $|W| \geq r_1 + 2r_2$ . Therefore, we have  $|B| \geq r_1 + 2r_2$ . Next, we will show that  $|B| \leq r_1 + 2r_2$ . Choose a set  $W = \bigcup_{i=1}^r S_i \cup \{x\}$  with

$$\begin{aligned} S_i &= \{w_1^i\}, \text{ with } 1 \leq i \leq r_1, \\ S_{r_1+1} &= \{w_1^{r_1+1}\}, \\ S_j &= \{v_1^j, w_1^j\}, \text{ with } r_1 + 2 \leq j \leq r. \end{aligned}$$

The representations of the other vertices of  $\text{Amal}\{C_{n_i}\}$  by  $W$  are as follow.

For  $1 \leq i \leq r_1$ ,

$$r(v_1^i|W) = (2, \dots, 2, \underbrace{1}_{\text{coord. of } S_i}, 2, \dots, 2, 1) \text{ with } 1 \leq i \leq r_1 + 1.$$

For  $j = r_1 + 1$ ,  $n_j = 2k_j + 1$  with  $k_j \geq 2$ ,

$$\begin{aligned} r(v_{k_j}^{r_1+1}|W) &= (k_j + 1, \dots, k_j + 1, \underbrace{k_j}_{\text{coord. of } S_{r_1+1}}, k_j + 1, \dots, k_j + 1, k_j), \\ r(v_l^{r_1+1}|W) &= (l + 1, \dots, l + 1, \underbrace{l + 1}_{\text{coord. of } S_{r_1+1}}, l + 1, \dots, l + 1, l) \\ &\text{with } 1 \leq l \leq k_j - 1, \text{ and} \\ r(w_l^{r_1+1}|W) &= (l + 1, \dots, l + 1, \underbrace{l - 1}_{\text{coord. of } S_{r_1+1}}, l + 1, \dots, l + 1, l) \\ &\text{with } 2 \leq l \leq k_j. \end{aligned}$$



For  $j = r_1 + 1$ ,  $n_j = 2k_j + 2$  with  $k_j \geq 2$ ,

$$\begin{aligned}
r(u^{r_1+1}|W) &= (k_j + 2, \dots, k_j + 2, \underbrace{k_j}_{\text{coord. of } S_{r_1+1}}, k_j + 2, \dots, k_j + 2, k_j + 1), \\
r(v_l^{r_1+1}|W) &= (l + 1, \dots, l + 1, \underbrace{l + 1}_{\text{coord. of } S_{r_1+1}}, l + 1, \dots, l + 1, l) \\
&\quad \text{with } 1 \leq l \leq k_j, \text{ and} \\
r(w_l^{r_1+1}|W) &= (l + 1, \dots, l + 1, \underbrace{l - 1}_{\text{coord. of } S_{r_1+1}}, l + 1, \dots, l + 1, l) \\
&\quad \text{with } 2 \leq l \leq k_j.
\end{aligned}$$

For  $r_1 + 2 \leq j \leq r$ ,  $n_j = 2k_j + 1$  with  $k_j \geq 2$ ,

$$\begin{aligned}
r(v_{k_j}^j|W) &= (k_j + 1, \dots, k_j + 1, \underbrace{k_j - 1, k_j}_{\text{coord. of } S_j}, k_j + 1, \dots, k_j + 1, k_j), \\
r(w_{k_j}^j|W) &= (k_j + 1, \dots, k_j + 1, \underbrace{k_j, k_j - 1}_{\text{coord. of } S_j}, k_j + 1, \dots, k_j + 1, k_j), \\
r(v_l^j|W) &= (l + 1, \dots, l + 1, \underbrace{l - 1, l + 1}_{\text{coord. by } S_j}, l + 1, \dots, l + 1, l) \\
&\quad \text{with } 1 \leq l \leq k_j - 1, \text{ and} \\
r(w_l^j|W) &= (l + 1, \dots, l + 1, \underbrace{l + 1, l - 1}_{\text{coord. of } S_j}, l + 1, \dots, l + 1, l) \\
&\quad \text{with } 1 \leq l \leq k_j - 1.
\end{aligned}$$

For  $r_1 + 2 \leq j \leq r$ ,  $n_j = 2k_j + 2$  with  $k_j \geq 1$ ,

$$\begin{aligned}
r(u^j|W) &= (k_j + 2, \dots, k_j + 2, \underbrace{k_j, k_j}_{\text{coord. of } S_j}, k_j + 2, \dots, k_j + 2, k_j + 1), \\
r(v_l^j|W) &= (l + 1, \dots, l + 1, \underbrace{l - 1, l + 1}_{\text{coord. of } S_j}, l + 1, \dots, l + 1, l) \\
&\quad \text{with } 1 \leq l \leq k_j, \text{ and} \\
r(w_l^j|W) &= (l + 1, \dots, l + 1, \underbrace{l + 1, l - 1}_{\text{coord. of } S_j}, l + 1, \dots, l + 1, l) \\
&\quad \text{with } 1 \leq l \leq k_j.
\end{aligned}$$

Thus, all vertices of  $\text{Amal}\{C_{n_i}\}$  have distinct representations by  $W$ . Therefore,  $W$  is a connected resolving set of  $\text{Amal}\{C_{n_i}\}$  and so,  $|B| \leq r_1 + 2r_2$ , which complete the proof.  $\square$

A graph  $G'$  is called a *subdivision* of a graph  $G$  if one or more vertices of degree 2 are inserted into one or more edges of  $G$ . In the following theorem, we state all possible resolving graphs of  $\text{Amal}\{C_{n_i}\}$ .

**Theorem 3.** *The resolving graph  $H$  of  $\text{Amal}\{C_{n_i}\}$  is either a path, a star, or a subdivision of a star.*

*Proof.* Let  $x$  be the terminal vertex of  $\text{Amal}\{C_{n_i}\}$ . Assume  $\text{Amal}\{C_{n_i}\}$  consists of  $r$  cycles with  $r_1$  number of cycles  $C_3$  and  $r_2$  number of cycles  $C_{n_i}$  with  $n_i \geq 4$ . Let  $W$  be a connected resolving set with minimum cardinality of  $\text{Amal}\{C_{n_i}\}$ . Let  $P_{n_i-1}$  and  $P_{n_j-1}$  be the two nonterminal paths of  $\text{Amal}\{C_{n_i}\}$ .

*Case 1.* For  $r = 2$ , there are three subcases; when  $r_1 = 2, r_2 = 0$  or  $r_1 = 1, r_2 = 1$  or  $r_1 = 0, r_2 = 2$ . By using Lemma H, Lemma 1, and Theorem 2, for all of these subcases,  $W = \{x, a, b\}$ , where  $a = v_1^1$  or  $w_1^1$  and  $b = v_1^2$  or  $w_1^2$ . The subgraph  $\langle W \rangle$  is a path  $P_3$  which contains 3 vertices.

*Case 2.* For  $r \geq 3$ , there are two subcases.  
*Claim:* If  $n_i > 4$  and  $|P_{n_i-1} \cap W| = 2$  then  $P_{n_i-1} \cap W = \{v_1^i, w_1^i\}$ . By using Theorem 2, there is one leaf  $C_{n_j}$  such that  $n_j \geq 4$  and  $|P_{n_j-1} \cap W| = 1$ . By using symmetry property, assume that  $P_{n_i-1} \cap W = \{v_1^i, v_2^i\}$  and  $P_{n_j-1} \cap W = \{v_1^j\}$ . Hence,  $d(w_1^i, z) = d(w_1^j, z)$  for all  $z \in W$ , a contradiction with  $W$  being a resolving set. *Subcase 2.1,* there is no leaf  $C_{n_i}$  with  $n_i = 4$ . By using Lemma H, Theorem 2, and the previous claim,  $W = \{x, a_1, \dots, a_{r-1}\}$  where  $d(a_i, x) = 1$  with  $1 \leq i \leq r-1$ . Hence, the subgraph  $\langle W \rangle$  is a star  $S_{r-1}$  which contains  $r$  vertices. *Subcase 2.2,* there are some leaves  $C_{n_i}$  with  $n_i = 4$ . If  $r_2 = 1$ , by using similar reasons with Subcase 2.1, then we also have the subgraph  $\langle W \rangle$  as a star  $S_{r-1}$  which contains  $r$  vertices. If  $r_2 \geq 2$  and  $W = \{x, a_1, \dots, a_{r-1}\}$  is a connected resolving set with minimum cardinality of  $\text{Amal}\{C_{n_i}\}$  then  $d(a_i, x) = 1$  for all  $a_i \in W$  or there are two vertices  $a, b \subseteq W$  such that  $d(a, x) = 1$  and  $d(b, x) = 2$ . The previous gives the subgraph  $\langle W \rangle$  as a star  $S_{r-1}$  which contain  $r$  vertices and the last gives the subgraph  $\langle W \rangle$  as a subdivision of a star  $S_m$  for some  $m$ .  $\square$

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