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THE METRIC DIMENSION OF AMALGAMATION OF CYCLES

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Abstract

For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the representation of v with respect to W is the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$, where $d(x, y)$ represents the distance between the vertices x and y. The set W is called a resolving set for G if every vertex of G has a distinct representation. A resolving set containing a minimum number of vertices is called a basis for G. The dimension of G, denoted by $\dim(G)$, is the number of vertices

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in a basis of G. Let $\{G_i\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{oi} called a terminal. The amalgamation Amal$\{G_i, v_{oi}\}$ is formed by taking all of the G_i's and identifying their terminals. In this paper, we determine the metric dimension of amalgamation of cycles.

1. Introduction

In this paper, we consider finite, simple, and connected graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. For a further reference please see Chartrand and Lesniak [3].

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest $u - v$ path in G. For an ordered set $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ of vertices, we refer to the ordered k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$ as the (*metric*) *representation of v with respect to W*. The set W is called a *resolving set* for G if $r(u|W) = r(v|W)$ implies $u = v$ for all $u, v \in G$. A resolving set with minimum cardinality is called a *minimum resolving set* or a *basis*. The *metric dimension* of a graph G, $\dim(G)$, is the number of vertices in a basis for G. To determine whether W is a resolving set for G, we only need to investigate the representations of the vertices in $V(G)\backslash W$, since the representation of each $w_i \in W$ has '0' in the ith-ordinate; and so it is always unique.

The initial papers discussing the notion of a (minimum) resolving set were written by Slater in [15] and [16]. Slater introduced the concept of a resolving set for a connected graph G under the term *location set*. He called the cardinality of a minimum resolving set by the *location number* of G. Independently, Harary and Melter [7] introduced the same concept, but used the term *metric dimension* instead.

The problem of finding a resolving set for a given graph can be found in many diverse areas including robotic navigation [12], chemistry [11], or computer science [13]. As described in [12], the navigating agent (a point robot) moves from node to node in a particular 'graph space'. The robot can locate itself by the presence of distinctively labeled "landmark" nodes in the graph. This suggests the problem: for a given graph, what is the smallest number of landmarks needed, and where should they be located, so that the distances to the landmarks uniquely determine the robot's position in the graph?

In general, finding a resolving set for arbitrary graph is a difficult problem. In [6], it is proved that the problem of computing the metric dimension for general graphs is *NP*-complete. Thus, researchers in this area often studied the metric dimension for particular classes of graphs or characterized graphs having certain metric dimension. Some results on the joint graph and cartesian product graph have been obtained by Caceres et al. [1], Khuller et al. [12], and Chartrand et al. [4]. Iswadi et al. obtained some results on the corona product of graphs [8, 9]. Saputro et al. obtained some results on the decomposition product of graphs [19]. Iswadi et al. determined the metric dimension of antipodal and pendant free graph [10]. Further, Saputro et al. found some results on the metric dimension of some type of regular graphs [17, 18].

Chartrand et al. [4] have characterized all graphs having metric dimensions 1, $n-1$, and $n-2$. They also determined the metric dimensions of some well known families of graphs such as paths, cycles, complete graphs, and trees. Chartrand et al. results are written as follows:

Theorem A [4]**.** *Let G be a connected graph of order* $n \geq 2$.

(i) $\dim(G) = 1$ *if and only if* $G = P_n$.

(ii) $\dim(G) = n-1$ *if and only if* $G = K_n$.

(iii) *For* $n \geq 4$, $\dim(G) = n-2$ *if and only if* $G = K_{r,s}$, $(r, s \geq 1)$, $G = K_r + \overline{K_s}$, $(r \geq 1, s \geq 2)$, *or* $G = K_r + (K_1 \cup K_s)$, $(r, s \geq 1)$.

(iv) *For* $n \geq 3$, $\dim(G_n) = 2$.

(v) *If T is a tree other than a path, then* $\dim(T) = \sigma(T) - ex(T)$, *where* $\sigma(T)$ *denotes the sum of the terminal degrees of the major vertices of T, and* $ex(T)$ *denotes the number of the exterior major vertices of T.*

The following identification graph $G = G[G_1, G_2, v_1, v_2]$ definition is from [14].

Definition B. Let G_1 and G_2 be the nontrivial connected graphs where $v_1 \in G_1$ and $v_2 \in G_2$. An identification graph $G = G[G_1, G_2, v_1, v_2]$ is obtained from G_1 and G_2 by identifying v_1 and v_2 such that $v_1 = v_2$ in *G*.

Poisson and Zhang [14] determined the lower and upper bounds of metric dimension of $G[G_1, G_2, v_1, v_2]$ in terms of $\dim(G_1)$ and $\dim(G_2)$ as stated in the following theorems:

Theorem C. *Let G_1 and G_2 be the nontrivial connected graphs with $v_1 \in G_1$ and $v_2 \in G_2$ and let $G = G[G_1, G_2, v_1, v_2]$. Then*

$$\dim(G) \geq \dim(G_1) + \dim(G_2) - 2.$$

For the upper bound, we define an equivalence class and binary function first. For a set W of vertices of G, define a relation on $V(G)$ with respect to W by uRv if there exists $a \in \mathbb{Z}$ such that $r(v|W) = r(u|W) + (a, a, ..., a)$. It is easy to check that R is an equivalence relation on $V(G)$. Let $[u]_W$ denote the equivalence class of u with respect to W. Then

$$v \in [v]_W \text{ if and only if } r(v|W) = r(u|W) + (a, a, ..., a)$$

for some $a \in \mathbb{Z}$. For a nontrivial connected graph G, define a binary function $f_G : V(G) \to \mathbb{Z}$ with

$$f_G(v) = \begin{cases} \dim(G), & \text{if } v \text{ is not a basis vertex of } G, \\ \dim(G) - 1, & \text{otherwise.} \end{cases}$$

Theorem D. *Let G_1 and G_2 be the nontrivial connected graphs with $v_1 \in G_1$ and $v_2 \in G_2$ and let $G = G[G_1, G_2, v_1, v_2]$. Suppose that G_1 contains a resolving set W_1 such that $[v_1]_{W_1} = \{v_1\}$. Then*

$$\dim(G) \leq |W_1| + f_{G_2}(v_2)$$

$$= \begin{cases} |W_1| + \dim(G_2), & \textit{if } v_2 \textit{ is not a basis vertex of } G_2, \\ |W_1| + \dim(G_2) - 1, & \textit{otherwise.} \end{cases}$$

In particular, if W_1 is a basis for G_1, then

$$\dim(G) \leq \begin{cases} \dim(G_1) + \dim(G_2), & \textit{if } v_2 \textit{ is not a basis vertex of } G_2, \\ \dim(G_1) + \dim(G_2) - 1, & \textit{otherwise.} \end{cases}$$

The following definition of amalgamation of graphs is taken from [2].

Definition E. Let $\{G_i\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{oi} called a *terminal*. The amalgamation Amal$\{G_i, v_{oi}\}$ is formed by taking of all the G_i's and identifying their terminals.

We can consider Definition E as the identification process for all of the members in the collection $\{G_i\}$ consecutively on one identification vertex.

In this paper, we determine the metric dimension of amalgamation of cycles.

2. Results

We could consider amalgamation of cycles on *n*; that is Amal$\{G_i, v_{oi}\}$, where $G_i = C_n$ for all *i*. In this particular amalgamation, the choice of vertex v_{oi} is irrelevant. So, for simplification, we can denote this amalgamation by $(C_n)_t$, where *t* denotes the number of cycles C_n. For $t = 1$, the graphs $(C_n)_1$ are the cycles C_n. For $n = 3$, the graphs $(C_3)_t$ are called the *friendship graphs* or the *Dutch t-windmills* [5].

In this paper, we consider a generalization of $(C_n)_t$, where the cycles under consideration may be of different lengths. We denote this amalgamation by Amal$\{C_{n_i}\}$, $1 \leq i \leq t$, $t \geq 2$. We call every C_{n_i} (including the terminal) in Amal$\{C_{n_i}\}$ as a *leaf* and a path $P_{n_i - 1}$ obtained from C_{n_i} by deleting the terminal as *a nonterminal path*.

Throughout this paper, we will follow the following notations and labels for cycles, nonterminal path, and vertices in Amal$\{C_{n_i}\}$. For odd n_i, $n_i = 2k_i + 1$, $k_i \geq 1$ and *x* the terminal vertex, we label all vertices in each leaf C_{n_i} such that

$$C_{n_i} = xv_1^i v_2^i \cdots v_{k_i}^i w_{k_i}^i w_{k_i - 1}^i \cdots w_1^i x,$$

this will give the nonterminal path

$$P_{n_i - 1} = v_1^i v_2^i \cdots v_{k_i}^i w_{k_i}^i w_{k_i - 1}^i \cdots w_1^i.$$

For even n_i, $n_i = 2k_i + 2$, $k_i \geq 1$, and *x* the terminal vertex, we define the labels of all vertices in each leaf C_{n_i} as follows:

$$C_{n_i} = xv_1^i v_2^i \cdots v_{k_i}^i u^i w_{k_i}^i w_{k_i - 1}^i \cdots w_1^i x,$$

which leads to the following labeling of the nonterminal path

$$P_{n_i-1} = v_1^i v_2^i \cdots v_{k_i}^i u^i w_{k_i}^i w_{k_i-1}^i \cdots w_1^i.$$

The following four lemmas give us some properties of the members of a resolving set of amalgamation of cycles.

Lemma 1. *Let S be a resolving set of Amal$\{C_{n_i}\}$. Then $|P_{n_i-1} \cap S| \geq 1$, for each i.*

Proof. If S has no vertex in P_{n_j-1}, for some j, then the vertices v_1^j, w_1^j in P_{n_j-1} will have the same distances to S, namely $d(v_1^j, v) = d(w_1^j, v), \forall v \in S$. Therefore, $r(v_1^j | S) = r(w_1^j | S)$, a contradiction. □

Lemma 2. *Let S be a resolving set of Amal$\{C_{n_i}\}$. If n_i is even, $n_i \geq 4$, and $|P_{n_i-1} \cap S| = 1$, then $P_{n_i-1} \cap S \neq \{u^i\}$.*

Proof. Suppose $u_j \in S$, for some j. Since the remaining vertices of S will not be in P_{n_j-1}, $d(v_{k_j}^j, v) = d(w_{k_j}^j, v), \forall v \in S$. Therefore, we have $r(v_{k_j}^j | S) = r(w_{k_j}^j | S)$, a contradiction with S being a resolving set. □

Lemma 3. *Let S be a resolving set of Amal$\{C_{n_i}\}$. For any even n_j, $n_l \geq 4$, $|(P_{n_j-1} \cup P_{n_l-1}) \cap S| \geq 3$.*

Proof. Let $n_j = 2k_j + 2$, $n_l = 2k_l + 2$, and $k_j, k_l \geq 1$. By Lemma 1, $|(P_{n_j-1} \cup P_{n_l-1}) \cap S| \geq 2$. Suppose $|(P_{n_i-1} \cup P_{n_j-1}) \cap S| = 2$. By considering Lemma 2 and the symmetry property, we have $((P_{n_j-1} \cup P_{n_l-1}) \cap S) = \{v_r^j, v_s^l\}$, with $1 \leq r \leq k_j$ and $1 \leq s \leq k_l$. Then, $d(w_1^j, v) = d(w_1^l, v), \forall v \in S$. Therefore, $r(w_1^j | S) = r(w_1^l | S)$, a contradiction; which gives $|(P_{n_j-1} \cup P_{n_l-1}) \cap S| \geq 3$. □

Now, we will determine the metric dimension of amalgamation of cycles Amal$\{C_{n_i}\}$.

Theorem 1. *If* $Amal\{C_{n_i}\}$ *is an amalgamation of t cycles that consists of* t_1 *number of odd cycles and* t_2 *number of even cycles, then*

$$\dim(Amal\{C_{n_i}\}) = \begin{cases} t_1, & t_2 = 0, \\ t_1 + 2t_2 - 1, & \text{otherwise.} \end{cases}$$

Proof. Let B be a basis of Amal$\{C_{n_i}\}$. We label the leafs C_{n_i}'s of Amal$\{C_{n_i}\}$ in such a way that C_{n_i} with odd lengths are labeled by $i = 1, \ldots, t_1$ and C_{n_j} with even lengths are labeled by $j = t_1 + 1, \ldots, t_1 + t_2 = t$.

Case 1. For $t_2 = 0$. Thus all n_i's, $1 \le i \le t_1$, are odd, let $n_i = 2k_i + 1$, $k_i \ge 1$. By using Lemma 1, for every resolving set S of Amal$\{C_{n_i}\}$, we will have $|S| \ge t_1$. Hence, for every basis B of Amal$\{C_{n_i}\}$, $|B| \ge t_1$. Choose a set $S = \bigcup_{i=1}^{t_1} S_i$ with $S_i = \{v_{k_i}^i\}$. We will show that S is a resolving set of Amal$\{C_{n_i}\}$. The representations of vertices of Amal$\{C_{n_i}\}$ that is not in S, with respect to S, are

$$r(x|S) = (k_1, \ldots, k_{t_1}),$$

$$r(v_r^i|S) = (k_1 + r, \ldots, \underbrace{k_i - r}_{\text{coord. of } S_i}, \ldots, k_{t_1} + r) \text{ with } 1 \le r \le k_i - 1,$$

and

$$r(w_r^i|S) = (k_1 + r, \ldots, \underbrace{k_i - r + 1}_{\text{coord. of } S_i}, \ldots, k_{t_1} + r) \text{ with } 1 \le r \le k_i.$$

Since the r's are all distinct, all of these representations are distinct. Hence, S is a resolving set of amalgamation Amal$\{C_{n_i}\}$. Since a basis B is a minimum resolving set, $|B| \le t_1$. Therefore, $|B| = t_1$.

Case 2. For $t_2 \ge 1$. Consider an arbitrary resolving set of Amal$\{C_{n_i}\}$. By Lemma 1, every nonterminal path $P_{n_i - 1}$ with n_i odd has at least one vertex of S; and by Lemma 3, every nonterminal path $P_{n_i - 1}$ with n_i even has at least two vertices of S, except for one of them whom can only contain one vertex of S. These lead to $|S| \ge t_1 + 2t_2 - 1$, for every resolving set S; and so $|B| \le t_1 + 2t_2 - 1$.

Next, we will show that $|B| \leq t_1 + 2t_2 - 1$. Choose a set $S = \bigcup_{i=1}^{t} S_i$ with

$$S_i = \{v_{k_i}^i\}, \text{ with } 1 \leq i \leq t_1, n_i = 2k_i + 1, \text{ and } k_i \geq 1,$$

$$S_{t_1+1} = \{v_{k_{t_1+1}}^{t_1+1}\}, \text{ with } n_{t_1+1} = 2k_{t_1+1} + 2, \text{ and } k_{t_1+1} \geq 1,$$

and

$$S_j = \{v_{k_j}^j, u^j\}, \text{ with } t_1 + 2 \leq j \leq t, n_j = 2k_j + 2, \text{ and } k_j \geq 1.$$

The representations of the other vertices of Amal$\{C_{n_i}\}$ with respect to S are

$$r(x|S) = (k_1, \ldots, k_{t_1}, k_{t_1+1}, k_{t_1+2}, k_{t_1+2} + 1, \ldots, k_t, k_t + 1),$$

$$r(v_r^i|S) = (k_1 + r, \ldots, \underbrace{k_i - r}_{\text{coord. of } S_i}, \ldots, k_{t_1+1} + r, k_{t_1+2} + r,$$

$$k_{t_1+2} + r + 1, \ldots, k_t + r, k_t + r + 1),$$

$$\text{with } 1 \leq i \leq t_1 + 1 \text{ and } 1 \leq r \leq k_i - 1,$$

$$r(v_s^j|S) = (k_1 + s, \ldots, k_{t_1+1} + s, k_{t_1+2} + s, k_{t_1+2} + s + 1,$$

$$\ldots, \underbrace{k_j - s, k_j - s + 1}_{\text{coord. of } S_j}, \ldots, k_t + s, k_t + s + 1),$$

$$\text{with } t_1 + 2 \leq j \leq t \text{ and } 1 \leq s \leq k_j - 1,$$

$$r(w_r^i|S) = (k_1 + r, \ldots, \underbrace{k_i - r + 1}_{\text{coord. of } S_i}, \ldots, k_{t_1+1} + r, k_{t_1+2} + r,$$

$$k_{t_1+2} + r + 1, \ldots, k_t + r, k_t + r + 1),$$

$$\text{with } 1 \leq i \leq t_1 + 1 \text{ and } 1 \leq r \leq k_i,$$

$$r(w_s^j|S) = (k_1 + s, \ldots, k_{t_1+1} + s, k_{t_1+2} + s, k_{t_1+2} + s + 1, \ldots,$$

$$\underbrace{k_j - s, k_j - s + 1}_{\text{coord. of } S_j}, \ldots, k_t + s, k_t + s + 1),$$

$$\text{with } t_1 + 2 \leq j \leq t \text{ and } 1 \leq s \leq k_j,$$

and

$$r(w_{k_{t_1+1}}^{t_1+1} \mid S) = (k_1 + k_{t_1} + 1, \ldots, k_{t_1} + k_{t_1} + 1, 1, k_{t_1+2} + k_{t_1} + 1,$$

$$k_{t_1+2} + k_{t_1} + 2, \ldots, k_t + k_{t_1} + 1, k_t + k_{t_1} + 2).$$

By direct inspection, all of these representations are distinct. Therefore, S is a resolving set. Since a basis B is a minimum resolving set S, $|B| \leq t_1 + 2t_2 - 1$. □

We illustrate both of Cases 1 and 2 in Figures 1 and 2. For Case 1, we consider $\text{Amal}\{C_{n_i}\}$ with $i = 3$, $n_1 = 2$, $n_2 = 5$, and $n_3 = 5$. By choosing a basis $B = \{v_1^1, v_2^2, w_1^3\}$, we have the coordinates of all vertices other than the basis vertices as follows:

$$r(x \mid B) = (1, 2, 1), \quad r(w_2^2 \mid B) = (3, 1, 3), \quad r(w_1^1 \mid B) = (1, 3, 2),$$

$$r(v_1^3 \mid B) = (2, 3, 2), \quad r(v_1^2 \mid B) = (2, 1, 2), \quad r(v_2^3 \mid B) = (3, 4, 2),$$

$$r(w_1^2 \mid B) = (2, 2, 2), \quad r(w_2^3 \mid B) = (3, 4, 1).$$

For Case 2, we consider $\text{Amal}\{C_{n_i}\}$ with $i = 3$, $n_1 = 5$, $n_2 = 6$, and $n_3 = 8$. By choosing a basis $B = \{v_2^1, v_2^2, v_2^3, w_3^3\}$, we have the coordinates of all vertices other than the basis vertices in $\text{Amal}\{C_{n_i}\}$ as below:

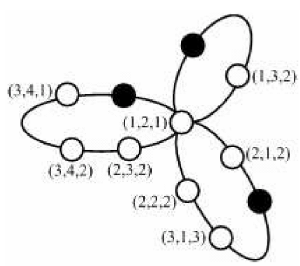

Figure 1. The coordinates of amalgamation $\text{Amal}\{C_{n_i}\}$ with $t_2 = 0$, $t = t_1 = 3$, $n_1 = 3$, $n_2 = 5$, and $n_3 = 5$.

$$r(v_1^1 \mid B) = (1, 3, 3, 4), \quad r(w_1^1 \mid B) = (2, 3, 3, 4), \quad r(w_2^1 \mid B) = (1, 4, 4, 5),$$

$$r(x \mid B) = (2, 2, 2, 3), \quad r(v_1^2 \mid B) = (3, 1, 3, 4), \quad r(u^2 \mid B) = (5, 1, 5, 6),$$

$$r(w_1^2 \mid B) = (3, 3, 3, 4), \quad r(w_2^2 \mid B) = (4, 2, 4, 5), \quad r(v_1^3 \mid B) = (3, 3, 1, 4),$$

$r(v_3^3 \,|\, B) = (5, 5, 1, 2)$, $r(u^3 \,|\, B) = (6, 6, 2, 1)$, $r(w_1^3 \,|\, B) = (3, 3, 3, 2)$,

$r(w_2^3 \,|\, B) = (4, 4, 4, 1)$.

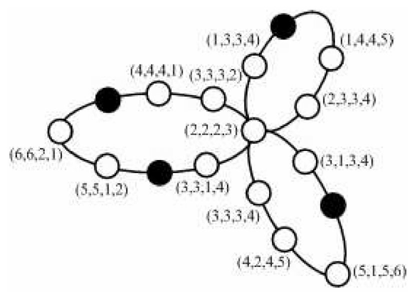

Figure 2. The amalgamation $\text{Amal}\{C_{n_i}\}$ with $t_2 \geq 1$, $t_1 = 1$, $t_2 = 2$, $n_1 = 5$, $n_2 = 6$, and $n_3 = 8$.

One of the natural questions we could pose after proving Theorem 1 is: Are there any basis other than the basis we constructed in the proof of Theorem 1? We will answer the question by identifying all bases of $\text{Amal}\{C_{n_i}\}$. The following lemmas are needed to find such bases:

Lemma 4. *Let B be a basis of Amal* $\{C_{n_i}\}$. *If* $|P_{n_j - 1} \cap B| = 1$ *and* $|P_{n_l - 1} \cap B| = 1$ *for some* $j \neq l$, *then* $P_{n_j - 1} \cap B = \{v_{k_j}^j\}$ *or* $\{w_{k_j}^j\}$ *or* $P_{n_l - 1} \cap B = \{v_{k_l}^l\}$ *or* $\{w_{k_l}^l\}$.

Proof. By Lemma 3, both n_j and n_l cannot be even. Let $P_{n_j - 1} \cap B = \{u\}$ and $P_{n_l - 1} \cap B = \{v\}$. Assume that $u = v_a^j$ with $1 \leq a \leq k_j - 1$ and $v = v_b^l$ with $1 \leq b \leq k_l - 1$. Then, $d(w_1^j, z) = d(w_1^l, z)$, $\forall z \in B$; a contradiction with *B* being a basis. The result follows by using symmetry property. □

Lemma 5. *Let Amal* $\{C_{n_i}\}$ *be an amalgamation of t cycles that consists of* t_1 *number of odd cycles and* t_2 *number of even cycles and B be a basis of Amal* $\{C_{n_i}\}$. *If* n_j *is odd and* $t_2 \geq 1$, *then* $P_{n_j - 1} \cap B = \{v_{k_l}^l\}$ *or* $\{w_{k_l}^l\}$.

Proof. Since n_j is odd, by using Lemma 1 and Theorem 1, $|P_{n_j - 1} \cap B| = 1$.

Since $t_2 \geq 1$, there is at least one leaf C_{n_l} with even vertices such that $|P_{n_j-1} \cap B| = 1$. Let $P_{n_j-1} \cap B = \{u\}$ and $P_{n_l-1} \cap B = \{v\}$. Assume that $u = v_a^j$ with $1 \leq a \leq k_j - 1$ and $v = v_b^l$ with $1 \leq b \leq k_l$. We have $d(w_1^j, z) = d(w_1^l, z)$, $\forall z \in B$. By symmetry property the result follows. □

Lemma 6. *Let B be a basis of Amal* $\{C_{n_i}\}$. *If* n_j *is even and* $P_{n_j-1} \cap B = \{a, b\}$, *then neither* $\{a, b\} \subseteq \{v_1^j, v_2^j, ..., v_{k_j}^j\}$ *nor* $\{a, b\} \subseteq \{w_1^j, w_2^j, ..., w_{k_j}^j\}$.

Proof. By Theorem 1, there is at least one leaf C_{n_l} with n_l even and $l \neq j$ such that $|P_{n_l-1} \cap B| = 1$. Let $P_{n_l-1} \cap B = \{u\}$. For the contrary assume that $\{a, b\} \subseteq \{v_1^j, v_2^j, ..., v_{k_j}^j\}$ and $u \in \{v_1^l, v_2^l, ..., v_{k_l}^l\}$. Then, $d(w_1^j, z) = d(w_1^l, z)$, $\forall z \in B$; a contradiction with B being a basis. By using symmetry property, the result follows. □

Now, we are ready to identify all bases of Amal $\{C_{n_i}\}$. We will consider exactly two cases; first, $t_2 = 0$ and second, $t_2 \geq 1$.

Theorem 2. *If Amal* $\{C_{n_i}\}$ *is an amalgamation of t cycles that consists of* t_1 *number of odd cycles and* t_2 *number of even cycles, then*

$$\sharp Amal\{C_{n_i}\} = \begin{cases} 2^{t-1}\left(\sum_{i=1}^{t}(n_i - 1) - 2\right), & t_2 = 0, \\ 2^{t_1}(n_{t_1+1} - 1)\prod_{j=t_1+2}^{t}(C(n_j - 1, 2) - 2C(k_j, 2)), & \text{otherwise}, \end{cases}$$

where $C(b, a)$ *is the total number of combinations of b objects taken a.*

Proof. Let t_1 be a number of odd cycles and t_2 be number of even cycles of Amal $\{C_{n_i}\}$.

Case 1. $t_2 = 0$. By using Lemma 1 and Theorem 1, every nonterminal path P_{n_i-1} will contain only one basis vertex. By using Lemma 4, for every pair of odd leaves P_{n_i-1} and P_{n_j-1}, one of them, say P_{n_i-1}, contains a basis vertex $v_{k_i}^i$ or $w_{k_i}^i$. Then, for $t_1 = 2$, we can identify the basis B of Amal $\{C_{n_i}\}$, $B = \{a, b\}$, where

$a \in \{v_{k_1}^1, w_{k_1}^1\}$ and $b \in V(P_{n_2-1})$ or $a \in V(P_{n_1-1})$ and $b \in \{v_{k_2}^2, w_{k_2}^2\}$. Hence, the number of different bases of Amal$\{C_{n_i}\}$ is $2(n_1 - 1) + 2(n_2 - 1) - 4$. By using similar reason, we can generalize for $t = t_1 > 2$ and we have the number of different bases of Amal$\{C_{n_i}\}$ is $2^{t-1}\left(\sum_{i=1}^{t}(n_i - 1) - 2\right)$.

Case 2. $t_2 \geq 1$. By using Lemma 1, Lemma 3, and Theorem 1, every nonterminal path with even vertices will have one basis vertex and every nonterminal path with odd vertices will have two basis vertices except one only have one basis vertex. We label the nonterminal path P_{n_i-1}'s of Amal$\{C_{n_i}\}$ in such a way that P_{n_i-1} with even lengths are labeled by $i = 1, ..., t_1$, P_{n_i-1} with odd lengths and contains two basis vertices are labeled by $j = t_1 + 2, ..., t_1 + t_2 = t$, and P_{n_l-1} with odd length and contains one basis vertex are labeled by $l = t_1 + 1$. By using Lemma 5, the basis vertices of nonterminal paths with even vertices, having either v_{k_i}'s or w_{k_i}'s. A nonterminal path P_{n_j-1} with odd vertices having one basis vertex can have every $z \in V(P_{n_j-1}) \backslash \{u^j\}$ as its basis vertex. By using Lemma 6, every nonterminal path P_{n_l-1} with odd vertices having two basis vertices cannot have both their basis vertices in either $\{v_1^l, v_2^l, ..., v_{k_l}^l\}$ or $\{w_1^l, w_2^l, ..., w_{k_l}^l\}$. Hence, we can count the number of different bases of Amal$\{C_{n_i}\}$ as follows:

$$2^{t_1}(n_{t_1+1} - 1)\prod_{j=t_1+2}^{t}(C(n_j - 1, 2) - 2C(k_j, 2)),$$

where $C(b, a)$ is the total number of combinations of b objects taken a. □

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