On the Sums of Powers with Positive Integer Exponents

A Historical and Methodological Overview

Thesis of PhD. dissertation

István Molnár

Supervisor:

Dr. József Kosztolányi, Associate Professor

Doctoral School in Mathematics and Computer Science

University of Szeged
Faculty of Natural Sciences and Information Technology
Bolyai Institute

2011
Szeged

I. Choice of subject, objectives, previous research

Two closely related tendencies can be observed in the history of mathematics. On the one hand, as we progress in time, the pursuers of this discipline discover new relations and concrete scientific facts, while on the other, they continually renew and enrich the methodological tools of mathematics. The purpose-driven and adequate use of methods can be especially important for teaching mathematics in secondary and tertiary education. One can solve a problem in more than one way, and for this reason it is important to compare the different ways leading to the solution, to systematize the applied knowledge, and to develop an affinity for ″finding the simplest solution(s)″. The choice of method, the use of different mathematical apparatuses, and the very different ″ways of thinking″ resulting from these can be especially interesting in sums of powers problems.

1. When defining the concept of the dissertation in terms of topics and contents, we focused on researching the relations within one area of mathematics, on examining the applicable methods relating to the chosen problem, and on seeking out possible links with other mathematical areas; all this is complemented by a comprehensive historical analysis of the chosen problem.

2. Our dissertation is about calculating the sums of powers for $S_p = 1^p + 2^p + 3^p + ... + n^p$ (where n and p are natural numbers and $n \geq 1$) with the use of very differing mathematical methods, and hence − as far as possible − it is a comprehensive analysis of the issue: ″problem to be solved″ versus ″applicable methods and mathematical knowledge″. To do this in our work:

   • we give a brief summary of the history of the problem and the developmental aspects of its solution, describing who approximated the problem's solution and to what extent it was solved in the different ages and civilizations of the world,

   • we calculate the S_p sums of powers with seven different methods, one after the other, and in each case not only presenting the results, but also the train of thought leading to the solution,

   • for each method we record some concrete examples of the obtained general formula,

   • with the presentation of some of the methods we make generalizations (for the cases of arithmetic progression, alternating sums), and seek out links to other areas of mathematics, and finally,

   • in the Appendix we present 32 exercises that vividly illustrate the varied practical use of the formulas for sums of powers.

3. Bearing in mind the chosen subject and the coherence of the aims, we formulated the objectives of the research work as follows:

   • the primary objective is to comprehensively discuss the methodology of calculating S_p and at the same time to find new calculation methods: partly relying on the achievements of earlier literature (in some cases supplementing these), and partly publishing the new methodological results that we ourselves have developed,

   • with the most comprehensive presentation of each applied method conceivable, we intend to prove the possibility of using a rich variety of methods, as well as the claim that in solving a problem often widely differing approaches can be appropriate, and that these ways, using very different areas of mathematics, relate to each other in terms of the final result,

   • we would like to give a brief evaluation of the advantages and disadvantages of each method,

   • where it is possible we want to find links and ″bridges″ to other areas of mathematics (Bernoulli and Stirling numbers), and finally,

- as for didactics, it is our objective to ascertain which methods can be used for teaching mathematics in secondary schools, and why and how they can be used, and which ones are more apt to be used at the college and university level.

4. All things considered, the objectives that we have set up belong to the area of applied research (mainly methodological), and our findings – it is our hope – can also be used in teaching practice (in secondary and higher education).

II. Research methods

During our research we used different methods of examination in accordance with our objective. Most of these were mathematical, but the applied methodological apparatuses also feature some other methods as well.

1. The starting point of the examination was the overview of the literature on sums of powers, the interpretation of the concepts used in it, and the synthesis and new contextualization of the findings which have been published.

2. From the wide methodological choice of mathematics we have used two fundamental methods, and also examined the conditions of their applicability. Namely, on the one hand we used methods leading to recursive formulas, while on the other, ones that do not lead to recursion.

3. Within the recursive methods we have solved the problem using five different methods. These are the following: matrix (tabulation) (where summing is done in three ways), a special identity leading to recursion, recursion stemming from the binomial theorem, recursion achieved by the use of symmetry, and finally, a recursion formula drawn up by derivation.

4. The second group of mathematical methods includes the methodologies leading to non-recursive formulas. Out of these we have dealt with two. Namely, one of them is the method using differential sequences, while the other one is the final formula obtained through the means of linear algebra.

5. We have also used non-mathematical methods that are common in other disciplines. Besides the already mentioned examination and reinterpretation of the sources, the method of analysis-comparison (the discussion of the advantages and disadvantages of the different methods), and methods serving the purpose of visual representation and illustration appear in our work.

III. Findings

A. Historical overview

1. We have assessed and analyzed the historical aspects of the chosen subject, presenting the historical development of the problem of sums of powers. Summing the same powers of integers and finding the relevant relations have long intrigued mathematicians of different ages. The ancient Greek scientists (Pythagoras, Archimedes, Hypsicles, and Nicomachus) obtained several partial results, mainly through the use of figurate numbers. The summing of arithmetic sequences and the calculation of any term in such a sequence occurred both in ancient and modern Chinese mathematics. Hindu mathematicians, who had a predilection for algebraic methods, were also able to calculate the general term of an arithmetic sequence, and to sum the first *n* squares and cubes. The Arabo-Persian mathematicians had also acquired this knowledge. From mediaeval and early modern European thinkers, Levi ben Gerson, Thomas Harriot, Johann Faulhaber, Pierre de Fermat, and Blaise Pascal can be singled out as those who contributed important partial results to the problem. Calculation of the sum of powers for all exponents was done by Jacob Bernoulli, who succeeded in drawing up a general formula.

B. Recursive methods

We have presented five solutions for the determination of the sum that lead to recursive formulas. In each of these we arrived at the needed recursive fromula in different ways. A common feature of the presented methods is that to calculate the sum for a concrete *p*, it is needed to know the sum/sums of the previous indices.

1. With the matrix (tabulation) method we also relied on matrix arithmetical concepts. Here we used a special *nxn* matrix, namely:

$$M = \begin{bmatrix} 1^{p-1} & 2^{p-1} & 3^{p-1} & ... & n^{p-1} \\ 1^{p-1} & 2^{p-1} & 3^{p-1} & ... & n^{p-1} \\ 1^{p-1} & 2^{p-1} & 3^{p-1} & ... & n^{p-1} \\ --- & --- & --- & --- & --- \\ 1^{p-1} & 2^{p-1} & 3^{p-1} & ... & n^{p-1} \end{bmatrix}$$

First we summed up the elements line by line, then another summing was done with the help of special matrices (upper triangular matrix, lower triangular matrix, diagonal matrix). Drawing up the equation after the intermediate calculations, we obtained the following formula:

$$S_p = (n+1) \cdot S_{p-1} - \sum_{k=1}^{n} S_{k;p-1} .$$

We wrote down the cases of the formula for the first five positive integer powers, some of which are relations well-known from secondary school.

The scheme of this solution can also be applied to the case of the alternating sum $S_p^* = 1^p - 2^p + 3^p - 4^p + ... + (-1)^{n-1} \cdot n^p$. In this case we used another, though similar, special matrix, namely:

$$\overline{M} = \begin{bmatrix} 1^{p-1} & -2^{p-1} & 3^{p-1} & ... & (-1)^{n-1} \cdot n^{p-1} \\ 1^{p-1} & -2^{p-1} & 3^{p-1} & ... & (-1)^{n-1} \cdot n^{p-1} \\ 1^{p-1} & -2^{p-1} & 3^{p-1} & ... & (-1)^{n-1} \cdot n^{p-1} \\ --- & --- & --- & --- & --- \\ 1^{p-1} & -2^{p-1} & 3^{p-1} & ... & (-1)^{n-1} \cdot n^{p-1} \end{bmatrix}$$

The final form of the recursive formula thus gained is:

$$S_p^* = (n+1) \cdot S_{p-1}^* - \sum_{k=1}^{n} S_{k;p-1}^* .$$

We have published the special cases of the formula for the first four positive integer ps.

With this method we can obtain the needed result without using matrix arithmetical concepts. For this we did the "staircase-like" summing of the elements of the initial matrix (table). Besides this, we illustrated the process leading to the needed recursion by a "suitable" figure (Figure 1).
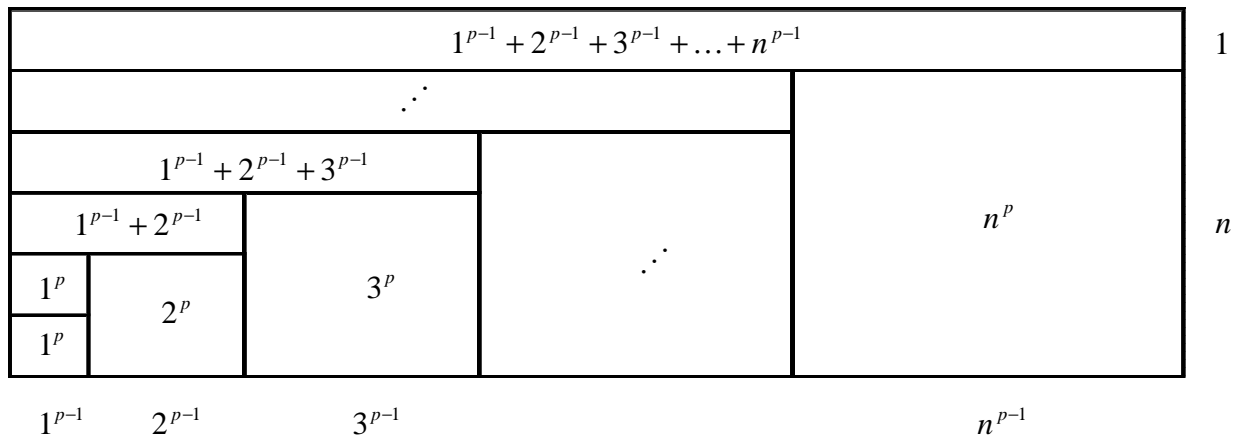

Figure 1

Here we arrived at the needed formula by calculating the area of the rectangle in two ways.

2. In the second case we solved the summing problem by using a special identity as the starting point. Namely:

$$(k+1) \cdot k^p - k \cdot (k-1)^p = (p+1) \cdot k^p - \binom{p}{2} \cdot k^{p-1} + \binom{p}{3} \cdot k^{p-2} - \binom{p}{4} \cdot k^{p-3} + ... + (-1)^{p+1} \cdot \binom{p}{p} \cdot k ,$$

where k is a real number and p is a positive integer.

Substituting k in turns with the positive integers $1,2,...,n$, summing up the obtained relations, and after elimination and rearrangement, we got the following recursive formula:

$$(p+1) \cdot S_p = (n+1)n^p + \binom{p}{2} \cdot S_{p-1} - \binom{p}{3} \cdot S_{p-2} + ... + (-1)^p \cdot \binom{p}{p} \cdot S_1 .$$

Here again we applied the formula for some concrete cases of p.

Similar to the above train of thought, but here starting from two other identities, we could establish relations between the summing formulas (for odd and even indices). We gave general formulas, and then we particularized them for some special cases.

In the case of an odd index the starting relation is ($p \in Z^+$, $a \in R$):

$$[a \cdot (a+1)]^p - [(a-1) \cdot a]^p = 2 \cdot \binom{p}{1} \cdot a^{2p-1} + 2 \cdot \binom{p}{3} \cdot a^{2p-3} + 2 \cdot \binom{p}{5} \cdot a^{2p-5} + ... ,$$

while the general formula is:

$$2^{p-1} \cdot S_1^p = \binom{p}{1} \cdot S_{2p-1} + \binom{p}{3} \cdot S_{2p-3} + \binom{p}{5} \cdot S_{2p-5} + ... \quad .$$

For even indices the starting relation is ($p \in Z^+$, $a \in R$):

$$(2a+1) \cdot [a \cdot (a+1)]^p - (2a-1) \cdot [(a-1) \cdot a]^p = 2 \cdot a^{2p} \cdot \left[\binom{p}{0} + 2 \cdot \binom{p}{1} \right] +$$

$$+ 2 \cdot a^{2p-2} \cdot \left[\binom{p}{2} + 2 \cdot \binom{p}{3} \right] + 2 \cdot a^{2p-4} \cdot \left[\binom{p}{4} + 2 \cdot \binom{p}{5} \right] + ... ,$$

while the general formula is:

$$3 \cdot 2^{p-1} \cdot S_2 \cdot S_1^{p-1} = \left[\binom{p}{0} + 2 \cdot \binom{p}{1} \right] \cdot S_{2p} + \left[\binom{p}{2} + 2 \cdot \binom{p}{3} \right] \cdot S_{2p-2} + \left[\binom{p}{4} + 2 \cdot \binom{p}{5} \right] \cdot S_{2p-4} + ... \quad .$$

3. The method using the binomial theorem also leads to a recursive formula. As the first step here we start from the following relation ($a \in R$, $p \in N$):

$$(a+1)^{p+1} - a^{p+1} = \binom{p+1}{1} \cdot a^p + \binom{p+1}{2} \cdot a^{p-1} + \binom{p+1}{3} \cdot a^{p-2} + ... + \binom{p+1}{p} \cdot a + \binom{p+1}{p+1}.$$

Substituting a in turns with the natural numbers $1,2,...,n$, summing up the obtained relations, and after elimination and rearrangement, with relatively short intermediate calculations, we get that:

$$(p+1) \cdot S_p = (n+1)^{p+1} - \binom{p+1}{2} \cdot S_{p-1} - ... - \binom{p+1}{p} \cdot S_1 - \binom{p+1}{p+1} \cdot (n+1).$$

Here we also drew up three special cases of the obtained relation.

By means of a similar procedure we made a generalization for arithmetic progression, and thus we drew up a recursive formula to calculate the

$$S_p(a,d) = a_1^p + a_2^p + a_3^p + ... + a_n^p = \sum_{i=1}^{n} a_i^p = \sum_{i=1}^{n} [a + (i-1) \cdot d]^p$$

sum, where a is the first term, while d is the difference of the $\{a_n\}_{n \geq 1}$ arithmetic progression.

We started from the relation

$$a_{i+1}^{p+1} - a_i^{p+1} = \binom{p+1}{1} \cdot a_i^p \cdot d + \binom{p+1}{2} \cdot a_i^{p-1} \cdot d^2 + ... + \binom{p+1}{p} \cdot a_i^1 \cdot d^p + \binom{p+1}{p+1} \cdot d^{p+1}$$

and the formula obtained at the end of the calculations is:

$$(p+1) \cdot d \cdot S_p(a,d) = (a+nd)^{p+1} - a^{p+1} - \sum_{k=2}^{p+1} d^k \cdot \binom{p+1}{k} \cdot S_{p+1-k}(a,d).$$

5

By using the formula we generated nine general cases, and some concrete results of these for a (27 in all).

We have also worked out the solution of the problem for the alternating sum of the pth power of the first n positive integers. In this case we started with the relation that

$$(a+2)^{p+1} - a^{p+1} = \binom{p+1}{1} \cdot a^p \cdot 2 + \binom{p+1}{2} \cdot a^{p-1} \cdot 2^2 + ... + \binom{p+1}{p} \cdot a^1 \cdot 2^p + \binom{p+1}{p+1} \cdot 2^{p+1}$$

Here we also substituted a in turn with the natural numbers $1, 2, ..., n$, after which we multiplied the relations by 1 and (-1) in an alternating fashion (multiplying the first one with 1), and then we summed up what we had got in this way. After intermediate calculations, elimination, and rearrangement, the recursive relation is:

$$2 \cdot (p+1) \cdot S_p^* = (-1)^{n-1} \cdot \left[(n+2)^{p+1} - (n+1)^{p+1} - 2^p \right] + 2^p - 1 - \sum_{k=2}^{p} \binom{p+1}{k} \cdot S_{p+1-k}^* \cdot 2^k .$$

Then we drew up two special cases of the alternating sum.

In the following step we made a generalization for the alternating sum of arithmetic progression. Here the initial relation is:

$$a_{i+2}^{p+1} - a_i^{p+1} = \binom{p+1}{1} \cdot a_i^p \cdot (2d)^1 + \binom{p+1}{2} \cdot a_i^{p-1} \cdot (2d)^2 + ... + \binom{p+1}{p} \cdot a_i^1 \cdot (2d)^p + \binom{p+1}{p+1} \cdot (2d)^{p+1} .$$

This time again we substituted a in turns with the natural numbers $1, 2, ..., n$, alternatingly multiplied the obtained relations by 1 and (-1), and then added these up. Having done the calculations, we generated the following:

$$2d \cdot (p+1) \cdot S_p^*(a,d) = (-1)^{n-1} \cdot \left[a_{n+2}^{p+1} - a_{n+1}^{p+1} - d \cdot (2d)^p \right] +$$

$$+ a_2^{p+1} - a_1^{p+1} - d \cdot (2d)^p - \sum_{k=2}^{p} \binom{p+1}{k} \cdot S_{p+1-k}^*(a,d) \cdot (2d)^k .$$

In this case also we gave three general formulas and their 12 concrete cases in all.

4. The recursive relation leading to the calculation of the sums of powers can also be generated with the help of symmetric sums. For these we first defined the concepts of the rising and falling factorial, and then presented a relevant feature of them.
We assigned a sequence to the rising factorial, and established a recursive relation between its terms in the following way:

$$[i]^{k+1} = \frac{1}{k+2} \cdot \left([i]^{k+2} - [i-1]^{k+2} \right), \text{ ahol } k = 0;1;2;... \text{ és } i = 1;2;... \, .$$

Based on the expressed recursive relation, we summed up the terms of the given sequence. With the result of the obtained telescopic sum and with the help of the symmetric sums $s_1, s_2, ..., s_{p-1}$ assigned to the numbers $1, 2, ..., (p-1)$

$$\begin{cases} s_1 = 1 + 2 + 3 + ... + (p-1) \\ s_2 = 1 \cdot 2 + 1 \cdot 3 + ... + (p-2) \cdot (p-1) \\ ------------------ \\ s_{p-1} = 1 \cdot 2 \cdot 3 \cdot ... \cdot (p-1) \end{cases}$$

we arrived at the pursued recursive formula, which takes the form:

$$S_p = \frac{n(n+1)(n+2)...(n+p)}{p+1} - \sum_{i=1}^{p-1} s_i \cdot S_{p-i} .$$

Just as with the previous methods, here we also drew up the special cases of the obtained formula ($p = 3; 4; 5$).

We have introduced the Bell and Stirling numbers, and drew up recursive relations between them. We established a connection, among others, between the rising/falling factorial, the Stirling numbers, and the powers of x with indefinite positive integer exponents. We gave the reciprocal sum of the first n positive integers with the help of the Stirling numbers of the first kind, and then described the Catalan relation.

5. We have also summed the powers with the same exponent by using differential calculus. With this method we found the needed recursion by deriving a "suitable" function. Here the initial relation is:

$$(e^x -1) \cdot F_n(x) = e^{(n+1) \cdot x} - e^x, \text{ where } F_n(x) = \sum_{l=1}^{n} e^{l \cdot x}, n \in Z^+ .$$

If we derive both sides to rank $(p+1)$, put 0 in the place of x, and then make the necessary calculations, we will arrive at the same result (recursion formula) that we had with the binomial method, that is

$$(p+1) \cdot S_p = (n+1)^{p+1} - \binom{p+1}{2} \cdot S_{p-1} - \binom{p+1}{3} \cdot S_{p-2} - ... - \binom{p+1}{p} \cdot S_1 - \binom{p+1}{p+1} \cdot (n+1)$$

In addition, after this we made a generalization for arithmetic progression. In this case we used the function $F_n(x) = \sum_{l=1}^{n} e^{a_l \cdot x}$, and at the end of the calculations we had the following result:

$$(p+1) \cdot d \cdot S_p(a,d) = (a+nd)^{p+1} - a^{p+1} - \binom{p+1}{2} \cdot d^2 \cdot S_{p-1}(a,d) -$$

$$- \binom{p+1}{3} \cdot d^3 \cdot S_{p-2}(a,d) - ... - \binom{p+1}{p+1} \cdot d^{p+1} \cdot S_0(a,d).$$

We applied the method for the alternating sum of the pth power of the first n positive integers with the help of the function $F_n(x) = \sum_{l=1}^{n} (-1)^{l-1} \cdot e^{l \cdot x}$, which led to:

$$2 \cdot (p+1) \cdot S_p^* = (-1)^{n-1} \cdot \left[(n+2)^{p+1} - (n+1)^{p+1} - 2^p \right] + 2^p - 1 - \sum_{k=2}^{p} \binom{p+1}{k} \cdot S_{p+1-k}^* \cdot 2^k .$$

Then we made a generalization for the alternating sum of arithmetic progression. After drawing up the special function ($F_n(x) = \sum_{l=1}^{n} (-1)^{l-1} \cdot e^{a_l \cdot x}$), derivation, and intermediate calculations, the recursive formula is as follows:

$$2d \cdot (p+1) \cdot S_p^*(a,d) = (-1)^{n-1} \cdot \left[a_{n+2}^{p+1} - a_{n+1}^{p+1} - d \cdot (2d)^p \right] +$$

$$+ a_2^{p+1} - a_1^{p+1} - d \cdot (2d)^p - \sum_{k=2}^{p} \binom{p+1}{k} \cdot S_{p+1-k}^*(a,d) \cdot (2d)^k .$$

C. Non-recursive methods

We have shown two methods that do not lead to a recursive formula for determining the sums of powers. These result in formulas that give the sum of the *p*th power of the first *n* positive natural numbers immediately, so there is no need for the sums corresponding to exponents smaller than *p*.

1. We first summed up with the method that is based on differential sequences. For this we have interpreted the concept of the differential sequence, and then defined what we mean by arithmetic progression of order *p*. We have proved that the *k*th term of any arithmetic progression of order *p* can be given with the *p*th polynomial of *k*. Denoting the sum of the first *n* terms of the arithmetic progression of order *p* with $\sigma_{n;p}$, we have defined $\sigma_{n;p}$ with the help of the first terms (a_1; Δa_1; $\Delta^2 a_1$; …; $\Delta^p a_1$) of the consecutive differential sequences in the following way

$$\sigma_{n;p} = \binom{n}{1} \cdot a_1 + \binom{n}{2} \cdot \Delta a_1 + \binom{n}{3} \cdot \Delta^2 a_1 + ... + \binom{n}{p+1} \cdot \Delta^p a_1.$$

Here again we recorded three particular cases. On top of this, with the help of the formula we gave three general and 12 special cases for arithmetical progression.

2. The problem can also be solved via linear algebraic knowledge/means. With this method we started from the identity ($a \in R, k \in N$):

$$(a+1)^{k+1} - a^{k+1} = \binom{k+1}{1} \cdot a^k + \binom{k+1}{2} \cdot a^{k-1} + \binom{k+1}{3} \cdot a^{k-2} + ... + \binom{k+1}{k} \cdot a + \binom{k+1}{k+1}.$$

In the relation we substituted a in turns with the numbers $1, 2, ..., n$, summed up the equations on each side, and then rearranged them to get:

$$(n+1)^{k+1} = \binom{k+1}{k+1} \cdot (n+1) + \binom{k+1}{k} \cdot S_1 + \binom{k+1}{k-1} \cdot S_2 + ... + \binom{k+1}{2} \cdot S_{k-1} + \binom{k+1}{1} \cdot S_k.$$

We assigned a special linear system of equations to the obtained relation which we got by substituting in turns the values $0, 1, 2, ..., p$ into *k* (let $X_0 = n+1$):

$$\begin{cases} n+1 = \binom{1}{1} \cdot X_0 \\ (n+1)^2 = \binom{2}{2} \cdot X_0 + \binom{2}{1} \cdot S_1 \\ (n+1)^3 = \binom{3}{3} \cdot X_0 + \binom{3}{2} \cdot S_1 + \binom{3}{1} \cdot S_2 \\ \text{---------------------------------} \\ (n+1)^p = \binom{p}{p} \cdot X_0 + \binom{p}{p-1} \cdot S_1 + \binom{p}{p-2} \cdot S_2 + ... + \binom{p}{2} \cdot S_{p-2} + \binom{p}{1} \cdot S_{p-1} \\ (n+1)^{p+1} = \binom{p+1}{p+1} \cdot X_0 + \binom{p+1}{p} \cdot S_1 + \binom{p+1}{p-1} \cdot S_2 + ... + \binom{p+1}{2} \cdot S_{p-1} + \binom{p+1}{1} \cdot S_p. \end{cases}$$

In the resulting system of equations, made up of $(p+1)$ equations, we have considered the "symbols" $S_1, S_2, ..., S_p$ as unknowns, and proved that the linear system of equations could be solved with Cramer's rule.

Thus we got the following formula for S_p:

$$S_p = \frac{1}{(p+1)!} \cdot \begin{vmatrix} \binom{1}{1} & 0 & 0 & ... & 0 & n+1 \\ \binom{2}{2} & \binom{2}{1} & 0 & ... & 0 & (n+1)^2 \\ \binom{3}{3} & \binom{3}{2} & \binom{3}{1} & ... & 0 & (n+1)^3 \\ --- & --- & --- & --- & --- & --- \\ \binom{p}{p} & \binom{p}{p-1} & \binom{p}{p-2} & ... & \binom{p}{1} & (n+1)^p \\ \binom{p+1}{p+1} & \binom{p+1}{p} & \binom{p+1}{p-1} & ... & \binom{p+1}{2} & (n+1)^{p+1} \end{vmatrix}$$

Once again we calculated the concrete sum in three cases.

Then we generalized the method for arithmetic progression. After the calculations we got the following formula:

$$S_p(a,d) = \frac{1}{(p+1)! \cdot d^{p+1}} \cdot D_p', \text{ where}$$

$$D_p' = \begin{vmatrix} \binom{1}{1} \cdot d & 0 & ... & 0 & (a+nd)^1 - a^1 \\ \binom{2}{2} \cdot d^2 & \binom{2}{1} \cdot d & ... & 0 & (a+nd)^2 - a^2 \\ \binom{3}{3} \cdot d^3 & \binom{3}{2} \cdot d^2 & ... & 0 & (a+nd)^3 - a^3 \\ ----- & ----- & --- & ----- & ------ \\ \binom{p}{p} \cdot d^p & \binom{p}{p-1} \cdot d^{p-1} & ... & \binom{p}{1} \cdot d & (a+nd)^p - a^p \\ \binom{p+1}{p+1} \cdot d^{p+1} & \binom{p+1}{p} \cdot d^p & ... & \binom{p+1}{2} \cdot d^2 & (a+nd)^{p+1} - a^{p+1} \end{vmatrix}$$

Another benefit of the method is that it can be applied to calculate the Bernoulli numbers. We have demonstrated this in our work. The $(p+1)$th Bernoulli number can be given by the following formula:

$$B_p = \frac{1}{(p+1)!} \cdot \begin{vmatrix} \binom{1}{1} & 0 & 0 & ... & 0 & 1 \\ \binom{2}{2} & \binom{2}{1} & 0 & ... & 0 & 0 \\ \binom{3}{3} & \binom{3}{2} & \binom{3}{1} & ... & 0 & 0 \\ --- & --- & --- & --- & --- & --- \\ \binom{p}{p} & \binom{p}{p-1} & \binom{p}{p-2} & ... & \binom{p}{1} & 0 \\ \binom{p+1}{p+1} & \binom{p+1}{p} & \binom{p+1}{p-1} & ... & \binom{p+1}{2} & 0 \end{vmatrix}$$

where $p = 0; 1; 2; 3; ...$.

As examples, we have calculated two Bernoulli numbers (B_4 and B_5) using the above formula. In addition to this, we have proved that, with the exception of B_1, all Bernoulli numbers with an odd index equal zero, and that the generation of B_p with another formula, also using a determinant,

$$B_p = \frac{1}{p_{\mathrm{i}}} \cdot \begin{vmatrix} 1_{\mathrm{i}} & 0 & 0 & ... & 0 & 1 \\ 2_{\mathrm{i}} & 1_{\mathrm{i}} & 0 & ... & 0 & 0 \\ 3_{\mathrm{i}} & 2_{\mathrm{i}} & 1_{\mathrm{i}} & ... & 0 & 0 \\ --- & --- & --- & --- & --- & --- \\ p_{\mathrm{i}} & (p-1)_{\mathrm{i}} & (p-2)_{\mathrm{i}} & ... & 2_{\mathrm{i}} & 0 \\ (p+1)_{\mathrm{i}} & p_{\mathrm{i}} & (p-1)_{\mathrm{i}} & ... & 1_{\mathrm{i}} & 0 \end{vmatrix}, \text{ where } n_{\mathrm{i}} = \frac{1}{n!}$$

is equivalent to the one we have given.

At the end of the chapter we highlighted interconnections between the Stirling and Bernoulli numbers, and proved the connection between the sum of the *p*th power of the first *n* positive integers and Bernoulli numbers, which is the following:

$$S_p = \frac{1}{p+1} \cdot \sum_{k=0}^{p} \binom{p+1}{k} \cdot B_k \cdot (n+1)^{p+1-k} .$$

Finally we should mention that for the sake of easy understanding and to enrich the applied methods, we made the text more varied with illustrative proofs (figures) in several chapters and in the Appendix.

IV. Publications on the same subject

Molnár István: *Pozitív racionális szám felbontása egyiptomi törtek összegére*, A Matematika Tanítása 5 (2007), 19-22.

Molnár István: *Az $x^{-n} + y^{-n} = z^{-n}$ egyenletről*, A Matematika Tanítása, 1 (2009), 3-9.

Molnár István: *Egy ötlet: Alkalmazzuk a Catalan-összefüggést!*, POLYGON, XVII/1-2 (2008), 125-133.

Molnár István: *Egy ötlet: A Catalan-összefüggés általánosításai és a kapott formulák alkalmazásai*, POLYGON, XIX/1 (2010), 63-75.

Molnár István: *Some thoughts on a student survey*, Teaching Mathematics and Computer Science, 8/1 (2010), 41-59.

Molnár István: *The sum of the same powers of the first n positive integers and the Bernoulli numbers*, Teaching Mathematics and Computer Science (to appear)

V. References

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