Links among Finite Geometries, Graphs and Groups

Thesis

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1 Semiovals

The dissertation is based on four articles of the author [32, 9, 31, 34]. In the first two chapters we discuss our results on *semiovals* which are special types of *semi quadratic sets*. The notion of semi quadratic sets was introduced by Buekenhout in 1973. Since that time a lot of attempts were made to classify all semi quadratic sets, but the problem is still open in general. The classical examples of semiovals arise from polarities (ovals and unitals), and from the theory of blocking sets. The study of semiovals is also motivated by their applications to cryptography.

Definition 1.1. *Let Π be a projective plane of order q. A semioval in Π is a non-empty pointset S with the property that for every point P in S there exists a unique line t_P such that $S \cap t_P = \{P\}$. This line is called the tangent to S at P.*

For planes of small order the complete spectrum of the sizes and the number of projectively non-isomorphic semiovals are known.

A more than 35-year old result of Thas and Hubaut states that if S is a semioval in Π then $q + 1 \leq |S| \leq q\sqrt{q} + 1$ and both bounds are sharp [46], [28].

We present some older results on semiovals with long secants and on the bounds on the size of a semioval we prove that if the semioval is contained in the union of three lines, then there are much better bounds on its size:

Proposition 1.2. [32] *Let S be a semioval in a projective plane Π of order q. If S is contained in the union of three lines then*

$$\frac{3(q-1)}{2} \leq |S| \leq 3(q-1).$$

The main aim of the first two chapters of the dissertation is to characterize the semiovals which are contained in the union of at most three lines.

The case when the semioval is contained in less then three lines is easy. In the case of three lines we have to distinguish two different cases. We completely characterize the semiovals in $\mathrm{PG}(2, q)$ which are contained in three non-concurrent lines. Using additive group theory, results on difference sets and combinatorial arguments and we prove the following:

Theorem 1.3. [32] *Let S be a semioval in $PG(2, q)$ which is contained in the union of three non-concurrent lines. Assume that S is not contained in the union of two lines, thus $L_i \setminus \{P_j, P_k\} \neq \emptyset$ for $\{i, j, k\} = \{1, 2, 3\}$. Then S belongs to one of the following three classes.*

1. *S has a $(q-2)$-secant and two $(t+1)$-secants for a suitable t. A semioval in this class exists if and only if $q=4$ and $t=1$, $q=8$ and $t=4$ or $q=32$ and $t=26$.*

2. *S has two $(q-1)$-secants and a k-secant. Semiovals in this class exist for all $1<k<q$.*

3. *S has three $(q-1-d)$-secants. Semiovals in this class exist if and only if $d|(q-1)$.*

We introduce a possible generalization of semiovals and cite some known results due to B. Csajbók and Gy. Kiss [15] on them.

In the second part of Chapter 1 semiovals contained in three concurrent lines are studied. This case is much more complicated than the previous one. We prove the following bound on the size:

Theorem 1.4. [9] *If a semioval S in Π_q, $q>3$, is contained in the union of three concurrent lines, then $|S| \leq 3\lceil q-\sqrt{q}\rceil$.*

We also show that this bound is sharp:

Example 1.5. [9] *Let $q=s^2$ and let ℓ_1,ℓ_2,ℓ_3 be three concurrent lines in $PG(2,q)$. Choose Baer sublines $\overline{\ell}_1 \subset \ell_1$, $\overline{\ell}_2 \subset \ell_2$, and $\overline{\ell}_3 \subset \ell_3$ in such a way that, for any triple of distinct $i,j,k \in \{1,2,3\}$, the Baer subplane $\mathcal{B}_{j,k} = \langle \overline{\ell_j}, \overline{\ell_k} \rangle$ meets the line ℓ_i only in the common point C. Then $S=(\ell_1 \setminus \overline{\ell}_1) \cup (\ell_2 \setminus \overline{\ell}_2) \cup (\ell_3 \setminus \overline{\ell}_3)$ is a semioval which has $3(q-\sqrt{q})$ points.*

This example has an extra property which suggests us to introduce the concept of *strong semiovals*.

We give an algebraic description of semiovals in $\mathrm{PG}(2,q)$ and using this description we study strong semiovals. We prove the following:

Theorem 1.6. [9] *There is no strong semioval in $PG(2,p)$ if p is an odd prime.*

Using classical results from the theory of group factorization we can completely characterize strong semiovals in $\mathrm{PG}(2,p^2)$, p an odd prime:

Theorem 1.7. [9] *If S is a strong semioval in $PG(2,p^2)$, p an odd prime, and S is contained in the union of lines ℓ_1,ℓ_2 and ℓ_3, then $\mathcal{L} \setminus S$ can be described as the point set*

$$\left\{ (-1,a,1),(0,b,1),(1,i,ci+f(c)) \colon\ a,b,c \in GF(p) \right\} \cup \left\{ C \right\}, \tag{1}$$

where $C=(0,1,0)$, $i^2=\varepsilon$ for a non-square element ε of $GF(p)$, $GF(p^2)$ is the extension of $GF(p)$ by i, and eventually, f is a permutation of $GF(p)$.

We consider strong semiovals S satisfying $|S| < 3(q - \sqrt{q})$. For the existence of such a semioval we have the following divisibilty condition.

Theorem 1.8. [9] *If S is a strong semioval of cardinality $|S| = 3(p^m - p^l)$, $m/2 < l < m$, in $PG(2, q)$, $q = p^m$ odd, then*

$$(p-1)(p^{2l-m}-1)^2 \mid (p^{m-l}-1). \tag{2}$$

Another result for strong semiovals gives new necessary condition on the existance of strong semiovals:

Theorem 1.9. [9] *If S is a strong semioval in $PG(2, p^m)$, where p is an odd prime, and*

$$m \leq \begin{cases} (p-1)^2 & p \equiv -1 \pmod 4 \\ 2(p-1)^2 & p \equiv 1 \pmod 4), \end{cases}$$

then $|S| = 3(q - \sqrt{q})$.

These results motivate our final conjecture on the non-existence of strong semiovals different from the above mentioned type.

2 Large Cayley graphs of given degree and diameter

The (Δ, D)-*problem* (or *degree/diameter problem*) is to determine the largest possible number of vertices of a graph which has maximum degree Δ and diameter D.

We recall the old result of Moore:

Theorem 2.1. [27] *Let $n(\Delta, D)$ denote the largest possible number of vertices that a finite simple graph Γ with maximum degree Δ, and diameter at most D can have. If $\Delta > 2$,then:*

$$n(\Delta, D) = \sum_{i=0}^{D} n_i \leq \frac{\Delta(\Delta-1)^D - 2}{\Delta - 2}.$$

We restrict our attention to the class of *linear Cayley graphs.* We present some constructions where the resulting graphs improve the previously known, general lower bounds for vertex-transitive graphs. For small number of vertices these are also compared to the known largest vertex transitive graphs having the same degree and diameter. It turns out that the problem for our case is to look for special pointsets in projective spaces, namely *saturating* *sets*. The graphs in our constructions arise from *comlete arcs, caps* and other objects of finite projective spaces. The following general lower bound for vertex-transitive graphs was previously known:

Theorem 2.2. *For the family of vertex transitive graphs:*

$$n(\Delta, 2) \geq \left\lfloor \frac{\Delta + 2}{2} \right\rfloor \cdot \left\lceil \frac{\Delta + 2}{2} \right\rceil.$$

And our new, improved bounds for specific values of degrees:

Theorem 2.3. [31] *Let* $\Delta = 27 \cdot 2^{m-4} - 1$ *and* $m > 7$. *Then*

$$n(\Delta, 2) \geq \frac{256}{729}(\Delta + 1)^2.$$

If $q > 3$ *is a prime power and* $\Delta = 2q^2 - q - 1$ *then*

$$n(\Delta, 2) > \frac{1}{4}\left(\Delta + \sqrt{\frac{\Delta}{2}} + \frac{5}{4}\right)^2.$$

3 Rose window graphs

The concept of *rose window graphs* was introduced by Wilson [41, 47].

Definition 3.1. *Given natural numbers* $n \geq 3$ *and* $1 \leq a, r \leq n-1$, *the rose window graph* $R_n(a, r)$ *is a quartic graph with vertex set* $\{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$ *and edge set*

$$\{\{x_i, x_{i+1}\} \mid i \in \mathbb{Z}_n\} \cup \{\{y_i, y_{i+r}\} \mid i \in \mathbb{Z}_n\} \cup \{\{x_i, y_i\} \mid i \in \mathbb{Z}_n\} \cup \{\{x_{i+a}, y_i\} \mid i \in \mathbb{Z}_n\}.$$

Wilson was primarily interested in embeddings of graphs $R_n(a, r)$ into closed surfaces as *rotary maps*. He gave several examples of such maps, and concluded the paper by a conjecture that the list of parameters n, a, r given there is the complete list of parameters giving rose window graphs which underlie rotary maps. A *map* $\mathcal{M}$ is an embedding of a finite connected graph Γ into a surface so that it divides the surface into simply-connected regions, called the *faces* of $\mathcal{M}$. To each face f there is associated a closed walk of Γ with edges surrounding f, to which we shall also refer as a face of $\mathcal{M}$. An *automorphism* of $\mathcal{M}$ is an automorphism of Γ which preserves its faces. Following [48], $\mathcal{M}$ is called *rotary* if it admits automorphisms R and S with the property that R cyclically permutes the consecutive edges of a face f, and S cyclically permutes the consecutive edges incident to some vertex v of f. In this case the automorphism group $\mathrm{Aut}(\mathcal{M})$ of $\mathcal{M}$ acts transitively on the vertex set, edge set and face set. We remark that the existence of R ensures that the boundary cycle of f is a so called consistent cycle of Γ, for details about this concept see [6, 14, 38].

If a rotary map also contains an automorphism T which 'flips' an edge e of f, and preserves f, then we say that $\mathcal{M}$ is *reflexible*. On the other hand, if no such automorphism T exists, then $\mathcal{M}$ is called *chiral*. One of the central questions regarding maps is the following: which graphs admit an embedding onto some closed surface as a rotary map [7].

Wilson actually posed the following three questions about rose window graphs in [47]:

Question 3.2. [47] *Given natural numbers $n \geq 3$ and $1 \leq a, r \leq n-1$,*

(i) for which n, a and r is $R_n(a,r)$ edge-transitive;
(ii) when $R_n(a,r)$ is edge-transitive, what is the order of its automorphism group;
(iii) for which n, a and r is $R_n(a,r)$ the underlying graph of a rotary map?

Trying to answer the first question Wilson identified the following four families **(a)**-**(d)** of edge-transitive rose window graphs $R_n(a,r)$ given below and conjectured that these graphs exhaust the whole class of edge-transitive rose window graphs [47, Conjecture 11]:

(a) $R_n(2,1)$;
(b) $R_{2m}(m-2, m-1)$;
(c) $R_{12m}(3m+2, 3m-1)$ and $R_{12m}(3m-2, 3m+1)$;
(d) $R_{2m}(2b, r)$, where $b^2 = \pm 1 \pmod{m}$, $2 \leq 2b \leq m$, and $r \in \{1, m-1\}$ is odd.

Kovács, Kutnar and Marušič in [33] confirmed this conjecture.

We could answer the second and third question in [34]. Throughout our work we use some well-known results about *coverings* and *embeddings* of graphs. A graph $\widetilde{\Gamma}$ is called a *covering* of a graph Γ with a *projection* $p\colon \widetilde{\Gamma} \to \Gamma$, if p is a surjection from $V(\widetilde{\Gamma})$ to $V(\Gamma)$ which is locally bijective, that is, $p|_{N(\widetilde{v})} \to N(v)$ is a bijection for any vertex $v \in V(\Gamma)$ and $\widetilde{v} \in p^{-1}(v)$. The graph $\widetilde{\Gamma}$ is also called a *covering graph* and Γ is the *base graph*. A covering $\widetilde{\Gamma}$ of Γ with projection p is said to be *regular* (or *K-covering*) if there is a semiregular subgroup K of $\mathrm{Aut}(\widetilde{\Gamma})$ such that Γ is isomorphic to the quotient $\widetilde{\Gamma}/K$, say by h, and the quotient map $\widetilde{\Gamma} \to \widetilde{\Gamma}/K$ is the composition ph of p and h. If $\widetilde{\Gamma}$ is connected, then K is also called the *covering transformation group*; moreover if K is cyclic then $\widetilde{\Gamma}$ is also called a *cyclic covering* of Γ.

A combinatorial description of a K-covering was introduced through a voltage graph by Gross and Tucker [25]. Let Γ be a graph and K be a finite group. By x^{-1} we mean the reverse arc of an arc $x \in A(\Gamma)$. A *voltage assignment* (or, a *K-voltage assignment*) of Γ is a mapping $\zeta\colon A(\Gamma) \to K$ with the property that $\zeta(x^{-1}) = \zeta(x)^{-1}$ for any $x \in A(\Gamma)$. The values of ζ are called *voltages*, and K is the *voltage group*. The *voltage graph* $\Gamma \times_\zeta K$ derived from a voltage assignment $\zeta\colon A(\Gamma) \to K$ has vertex set $V(\Gamma) \times K$, and edges of the form $(u, g)(v, \zeta(x)g)$, where $x = (u, v) \in A(\Gamma)$. Clearly, $\Gamma \times_\zeta K$ is a covering of Γ with the first coordinate projection. By letting K act on $V(\Gamma \times_\zeta K)$ as $(u, g)^{g'} = (u, gg')$, $(u, g) \in V(\Gamma \times_\zeta K)$, $g' \in K$, we obtain a semiregular group of automorphisms of $\Gamma \times_\zeta K$, showing that $\Gamma \times_\zeta K$ can in fact be viewed as a K-covering. Given a spanning tree T of Γ, the voltage assignment ζ is said to be *T-reduced* if the voltages on the tree arcs equal the identity element. In [25] it is shown that every regular covering $\tilde{\Gamma}$ of a graph Γ can be derived from a T-reduced voltage assignment ζ with respect to an arbitrary fixed spanning tree T of Γ.

Let $\tilde{\Gamma}$ be a K-covering of Γ with a projection p. If $\alpha \in \mathrm{Aut}(\Gamma)$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{\Gamma})$ satisfy $\widetilde{\alpha}p = p\alpha$ then we call $\widetilde{\alpha}$ a *lift* of α, and α the *projection* of $\widetilde{\alpha}$. If the covering graph $\widetilde{X}$ is connected then the covering transformation group K is the lift of the trivial subgroup of $\mathrm{Aut}(\Gamma)$. Note that a subgroup $G \leq \mathrm{Aut}(\widetilde{\Gamma})$ projects if and only if the partition of $V(\Gamma)$ into the orbits of K is G-invariant.

The problem of determining whether an automorphism α of Γ lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. We define a function $\bar{\alpha}$ from the set of voltages of fundamental closed walks based at a fixed vertex $v \in V(\Gamma)$ to the voltage group K by $\bar{\alpha}(\zeta(C)) = \zeta(C^\alpha)$, where C ranges over all fundamental closed walk at the base vertex v, and $\zeta(C)$ and $\zeta(C^\alpha)$ are the voltages of C and C^α, respectively. The following two propositions were useful in our work:

Proposition 3.3. [35] *Let $\Gamma \times_\zeta K$ be a connected K-covering. Then an automorphism α of Γ lifts if and only if $\bar{\alpha}$ extends to an automorphism of K.*

Proposition 3.4. [36] *Let $\tilde{\Gamma}_1 = \Gamma \times_\zeta K$ and $\tilde{\Gamma}_2 = \Gamma \times_{\zeta'} K$ be two connected K-coverings of a graph Γ where ζ and ζ' are T-reduced voltage assignments. Then $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are isomorphic if and only if there exist an automorphism $\gamma \in \mathrm{Aut}(K)$ and an automorphism $g \in \mathrm{Aut}(\Gamma)$ such that $\gamma(\zeta(C)) = \zeta'(C^g)$ for every fundamental cycle C with respect to the spanning tree T in Γ.*

The following proposition gives a criterion of embeddings of graphs onto orientable surfaces as rotary maps in terms of their automorphism groups.

Proposition 3.5. [21] *A connected graph* Γ *of valency at least* 3 *underlies a rotary map on an orientable surface if and only if there exists* $K \leq \mathrm{Aut}(\Gamma)$ *satisfying the following properties.*

1. *K is transitive on the set of arcs of* Γ*.*
2. *The vertex stabilizer K_v of a vertex v of* Γ *is cyclic.*

Our main result answers Wilson's third question:

Theorem 3.6. [34] *Let* $\Gamma = R_n(a, r)$ *be a rose window graph underlying a rotary map* $\mathcal{M}$, $1 \leq a, r \leq n/2$*. Then one of the following holds.*

1. $\mathcal{M}$ *is reflexible, and*
   - *(a)* $\Gamma = R_n(2, 1)$, $\gcd(n, 12) > 2$,
   - *(b)* $\Gamma = R_{2m}(m-2, m-1)$, $\gcd(m, 60) > 3$,
   - *(c)* $\Gamma = R_{12m}(3m+2, 3m-1)$ *or* $R_{12m}(3m-2, 3m+1)$, $m \equiv 2 \pmod 4$.
2. $\mathcal{M}$ *is chiral, and* $\Gamma = R_{2m}(2b, r)$, $m > 2$, $2 \leq 2b \leq m$, $b^2 \equiv -1 \pmod m$ *and* $r = 1$*, or* $r = m - 1$ *and m is even.*

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